New similarity reductions of the Boussinesq equation

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Some new similarity reductions of the Boussinesq equation, which arises in several physical applications including shallow water waves and also is of considerable mathematical interest because it is a soliton equation solvable by inverse scattering, are presented. These new similarity reductions, including some new reductions to the first, second, and fourth Painlevé equations, cannot be obtained using the standard Lie group method for finding group-invariant solutions of partial differential equations; they are determined using a new and direct method that involves no group theoretical techniques.

I. INTRODUCTION

The Boussinesq equation

$$u_{tt} + au_{xx} + b(u^2)_{xx} + cu_{xxxx} = 0, \tag{1.1}$$

where a, b, and c are constants and subscripts denote differentiation, was introduced to Boussinesq in 1871 to describe the propagation of long waves in shallow water[1] (see, also, Ref. 2). The Boussinesq equation also arises in several other physical applications including one-dimensional nonlinear lattice waves,[3,4] vibrations in a nonlinear string,[5] and ion sound waves in a plasma.[6]

It is well known (and was even to Boussinesq) that the Boussinesq equation (1.1) has a bidirectional solitary wave solution

$$u(x,t) = -\frac{3(\gamma^2+a)}{2b} \times \operatorname{sech}^2\left\{\frac{1}{2}\left(\frac{\gamma^2+a}{-c}\right)^{1/2}(x \pm \gamma t) + x_0\right\},$$

where γ and x_0 are constants.

Recently there has been considerable mathematical interest in the Boussinesq equation, primarily because its Cauchy problem (for initial data on the infinite line that decays sufficiently rapidly) is solvable by inverse scattering,[5] through a third-order scattering problem (see, also, Ref. 7).

The inverse scattering method was originally developed by Gardner *et al.*[8] in order to solve the Cauchy problem for the Korteweg–de Vries (KdV) equation. In effect, this method reduces the solution of the nonlinear partial differential equation to that of a linear integral equation, and the partial differential equation is usually then said to be *completely integrable*. Completely integrable partial differential equations generally possess almost all of the following remarkable properties: the existence of multisoliton solutions; an infinite number of independent conservation laws and symmetries, and recursion operators generating them; a bi-Hamiltonian representation; a prolongation structure; a Lax pair; Bäcklund transformations; the Hirota bilinear representation; the Painlevé property, etc. (cf. Ref. 9). However, the precise relationship between these properties has yet to be rigorously established.

In this paper we study similarity reductions of the Boussinesq equation. Without loss of generality we shall assume that $a = 0$, $b = \frac{1}{2}$, and $c = \pm 1$ in Eq. (1.1) since the equation

$$u_{tt} + \tfrac{1}{2}(u^2)_{xx} \pm u_{xxxx} = 0 \tag{1.2}$$

is equivalent to Eq. (1.1) after suitable rescaling and translation of the variables. If the quantities in the equation are to be interpreted as real, then the sign matters and we choose the plus sign from here on only for convenience, and leave the reader the trivial modifications required for the other sign. However, if the quantities are interpreted as complex, then the sign does not matter and our analysis is complete.

The classical method for finding similarity reductions of a given partial differential equation is to use the Lie group method of infinitesimal transformations (sometimes called the method of group-invariant solutions), originally developed by Lie[10] (see Refs. 11–14 for recent descriptions of this method). Though the method is entirely algorithmic, it often involves a large amount of tedious algebra and auxiliary calculations which are virtually unmanageable manually. Recently symbolic manipulation programs have been developed, especially in MACSYMA[15] and REDUCE,[16] in order to facilitate the determination of the associated similarity reductions. (See Ref. 17 for a review of the use of computer algebra to find symmetries of differential equations.)

Bluman and Cole[18] proposed a generalization of Lie's method which they called the "*nonclassical method of group-invariant solutions*," which itself has been generalized by Olver and Rosenau.[19] All these methods determine Lie point transformations of a given partial differential equation, i.e., transformations depending only on the independent and dependent variables.

Noether[20] recognized that Lie's method could be generalized by allowing the transformations to depend upon the derivatives of the dependent variable as well as the independent and dependent variables. The associated symmetries, called *Lie–Bäcklund symmetries*, can also be determined by an algorithmic method (see Refs. 13 and 21).

In a recent paper, Bluman *et al.*[22] introduce an algorithmic method which yields new classes of symmetries of a given partial differential equation that are neither Lie point nor Lie–Bäcklund symmetries.

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A common characteristic of all these methods for finding symmetries and associated similarity reductions of a given partial differential equation is the use of group theory.

In this paper we develop a new method of deriving similarity reductions of partial differential equations and apply it to the Boussinesq equation (1.2). The unusual characteristic of this new method in comparison to the ones mentioned above is that it does not use group theory (though we hope that a group theoretic explanation of the method will be possible in due course[23]). The basic idea is to seek a reduction of a given partial differential equation in the form

$$u(x,t) = U(x,t,w(z(x,t))), \tag{1.3}$$

which is the most general form for a similarity reduction (cf. Bluman and Cole[11]). Substituting this into the partial differential equation and demanding that the result be an ordinary differential equation for $w(z)$ imposes conditions upon U and its derivatives that enable one to solve for U. For the Boussinesq equation (1.2), it turns out to be sufficient to take (1.3) in the special form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)). \tag{1.4}$$

The outline of this paper is as follows: in Sec. II we describe the previously known (classical and nonclassical) similarity reductions of the Boussinesq equation; in Sec. III we present our new method for finding similarity reductions of a given partial differential equation and use it to obtain new similarity reductions of the Boussinesq equation (1.2); in Sec. IV we justify the use of the special form (1.4); and in Sec. V we discuss our results.

II. CLASSICAL AND NONCLASSICAL SIMILARITY REDUCTIONS

First we sketch the derivation of the classical similarity reductions of the Boussinesq equation using Lie group method as given by Bluman and Cole.[11] Consider the one-parameter (ε) Lie group of infinitesimal transformations in (x,t,u) given by

$$\xi = x + \varepsilon X(x,t,u) + O(\varepsilon^2), \tag{2.1a}$$

$$\tau = t + \varepsilon T(x,t,u) + O(\varepsilon^2), \tag{2.1b}$$

$$\eta = u + \varepsilon U(x,t,u) + O(\varepsilon^2), \tag{2.1c}$$

$$\eta_\xi = u_x + \varepsilon U^x + O(\varepsilon^2), \tag{2.2a}$$

$$\eta_{\xi\xi} = u_{xx} + \varepsilon U^{xx} + O(\varepsilon^2), \tag{2.2b}$$

$$\eta_{\xi\xi\xi\xi} = u_{xxxx} + \varepsilon U^{xxxx} + O(\varepsilon^2), \tag{2.2c}$$

$$\eta_{\tau\tau} = u_{tt} + \varepsilon U^{tt} + O(\varepsilon^2), \tag{2.2d}$$

where the functions U^x, U^{xx}, U^{xxxx}, and U^{tt} in (2.2) are determined from Eqs. (2.1) (cf. Ref. 11). The Boussinesq equation (1.2) is invariant under this transformation if

$$\eta_{\tau\tau} + \tfrac{1}{2}(\eta^2)_{\xi\xi} + \eta_{\xi\xi\xi\xi} = 0. \tag{2.3}$$

By (2.1) and (2.2), to first order in ε, this becomes

$$U^{tt} + uU^{xx} + u_{xx}U + 2u_x U^x + U^{xxxx} = 0. \tag{2.4}$$

Conditions on the infinitesimals $X(x,t,u)$, $T(x,t,u)$, and $U(x,t,u)$ are determined by equating coefficients of like derivatives of monomials in u_x and u_t and higher derivatives. Solving these "determining equations" yields the following:

$$X = \alpha x + \beta, \quad T = 2\alpha t + \gamma, \quad U = -2\alpha u, \tag{2.5}$$

where α, β, and γ are arbitrary constants (cf. Refs. 24 and 25). Similarity reductions are then obtained by solving the characteristic equations

$$\frac{dx}{X(x,t,u)} = \frac{dt}{T(x,t,u)} = \frac{du}{U(x,t,u)}.$$

Integration of these ordinary differential equations yields the following cases.

Case (a), $\alpha=0$: This is the traveling wave reduction $u(x,t) = f(z)$, $z = \gamma x - \beta t$, where $f(z)$ satisfies

$$\beta^2 f + \frac{1}{2}\gamma^2 f^2 + \gamma^4 \frac{d^2 f}{dz^2} = Az + B, \tag{2.6}$$

with A and B arbitrary constants of integration. For $\gamma = 0$, this is a form of the first Painlevé equation (cf. Ince[26])

$$\frac{d^2 w}{dz^2} = 6w^2 + z \tag{2.7}$$

(or the Weierstrass elliptic function equation for $A = 0$). This reduction of the Boussinesq equation to the first Painlevé equation is well known in connection with the Painlevé conjecture (cf. Refs. 27 and 28) for soliton equations.

Case (b), $\alpha \neq 0$: This is the scaling reduction

$$u(x,t) = \frac{g(z)}{[t + \gamma/(2\alpha)]}, \quad z = \frac{(x + \beta/\alpha)}{[t + \gamma/(2\alpha)]^{1/2}}, \tag{2.8}$$

where $g(z)$ satisfies

$$\frac{z^2}{4}\frac{d^2 g}{dz^2} + \frac{7z}{4}\frac{dg}{dz} + 2g + g\frac{d^2 g}{dz^2} + \left(\frac{dg}{dz}\right)^2 + \frac{d^4 g}{dz^4} = 0. \tag{2.9}$$

This can be solved in terms of solutions of the fourth Painlevé equation

$$\frac{d^2 w}{dz^2} = \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - a)w + \frac{b}{w}, \tag{2.10}$$

where a and b are arbitrary constants[29] (see also Appendix A).

However, there also exist similarity reductions of the Boussinesq equation that *cannot* be obtained by the classical Lie group method. As noted by several authors,[19,24,25,29] the Boussinesq equation (1.2) possesses the similarity reduction

$$u(x,t) = f(z) - 4\lambda^2 t^2, \quad z = x + \lambda t^2, \tag{2.11}$$

where λ is a constant and $f(z)$ satisfies

$$\frac{d^3 f}{dz^3} + f\frac{df}{dz} + 2\lambda f = 8\lambda^2 z + A, \tag{2.12}$$

with A a constant of integration. If, in (2.12), we make the transformation

$$f(z) = \eta(\xi) + 2\lambda\xi, \quad z = \xi - A/(8\lambda^2),$$

then $\eta(\xi)$ satisfies

$$\frac{d^3\eta}{d\xi^3} + \eta\frac{d\eta}{d\xi} + 2\lambda\left(\xi\frac{d\eta}{d\xi} + 2\eta\right) = 0. \tag{2.13}$$

Solutions of Eq. (2.13) are known to be related through a one-to-one transformation to solutions of the second Painlevé equation

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$$\frac{d^2w}{dz^2} = 2w^3 + zw + a, \tag{2.14}$$

where a is an arbitrary constant[30]—see, also, Appendix A. [We remark that this equation also arises from the scaling reduction

$$u(x,t) = (-3\lambda t)^{-2/3}\eta(\xi), \quad \xi = x/(-3\lambda t)^{-1/3}$$

of the KdV equation

$$u_t + uu_x + u_{xxx} = 0$$

—see Ref. 27.]

The infinitesimals that give rise to the similarity reduction (2.11) of the Boussinesq equation are

$$X(x,t,u) = 2\lambda t, \quad T(x,t,u) = -1, \quad U(x,t,u) = 8\lambda^2 t, \tag{2.15}$$

which are clearly not a special case of (2.5). Since Eqs. (2.15) describe a Lie point transformation of the Boussinesq equation, Rosenau and Schwarzmeier[25] suggest it can be obtained using the nonclassical method of Bluman and Cole[18] (see, also, Ref. 11). This method involves more algebra and calculations than the classical Lie method; in fact, Olver and Rosenau[19] suggest that for some partial differential equations, the determining equations for these nonclassical symmetries might be too difficult to solve explicitly. The principal reason for this is that although the determining equations for the infinitesimals X, T, and U in the classical method are a linear system of equations (in X, T, and U), in the nonclassical method, they are a nonlinear system. Furthermore, for some equations, such as the linear heat equation, it is well known that the nonclassical method does not appear to yield any more similarity reductions than the classical Lie method does[18] (see, also, Ref. 31).

III. NEW SIMILARITY REDUCTIONS

In this section we seek reductions of the Boussinesq equation (1.2) in the form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)), \tag{3.1}$$

where $\alpha(x,t)$, $\beta(x,t)$, and $z(x,t)$ are to be determined. [We shall show in Sec. IV why it is sufficient to seek a similarity reduction of the Boussinesq equation (1.2) in the form (3.1) rather than the more general form (1.3).]

Substituting (3.1) into (1.2) and collecting coefficients of monomials of w and its derivatives yields

$$\begin{aligned}\beta z_x^4 w'''' &+ [6\beta z_x^2 z_{xx} + 4\beta_x z_x^3]w''' + [\beta(3z_{xx}^2 + 4z_x z_{xxx}) + 12\beta_x z_x z_{xx} + 6\beta_{xx} z_x^2 + \alpha\beta z_x^2 + \beta z_t^2]w'' \\ &+ [\beta z_{xxxx} + 4\beta_x z_{xxx} + 6\beta_{xx} z_{xx} + 4\beta_{xxx} z_x + 2\alpha_x \beta z_x + 2\alpha\beta_x z_x + \alpha\beta z_{xx} + 2\beta_t z_t + \beta z_{tt}]w' \\ &+ [\beta_{xxxx} + 2\alpha_x\beta_x + \alpha\beta_{xx} + \alpha_{xx}\beta + \beta_{tt}]w + \beta^2 z_x^2 ww'' + \beta[4\beta_x z_x + \beta z_{xx}]ww' \\ &+ \beta^2 z_x^2 (w')^2 + [\beta_x^2 + \beta\beta_{xx}]w^2 + [\alpha_{tt} + \alpha\alpha_{xx} + \alpha_x^2 + \alpha_{xxxx}] = 0,\end{aligned} \tag{3.2}$$

where $' := d/dz$. In order that this equation be an ordinary differential equation for $w(z)$ the ratios of coefficients of different derivatives and powers of $w(z)$ have to be functions of z only. This gives a set of conditions for $\alpha(x,t)$, $\beta(x,t)$, and $z(x,t)$ for which any solution will yield a similarity reduction.

Remark 1: We use the coefficient of w'''' (i.e., βz_x^4) as the normalizing coefficient and therefore require that the other coefficients be of the form $\beta z_x^4 \Gamma(z)$, where Γ is a function of z to be determined.

Remark 2: We reserve uppercase greek letters for undetermined functions of z so that after performing operations (differentiation, integration, exponentiation, rescaling, etc.) the result can be denoted by the same letter [e.g., the derivative of $\Gamma(z)$ will be called $\Gamma(z)$].

Remark 3: There are three freedoms in the determination of α, β, z and w we can exploit, without loss of generality, that are valuable in keeping the method manageable: (i) if $\alpha(x,t)$ has the form $\alpha = \alpha_0(x,t) + \beta(x,t)\Omega(z)$, then we can take $\Omega \equiv 0$ [by substituting $w(z) \to w(z) - \Omega(z)$]; (ii) if $\beta(x,t)$ has the form $\beta = \beta_0(x,t)\Omega(z)$, then we can take $\Omega \equiv 1$ [by substituting $w(z) \to w(z)/\Omega(z)$]; and (iii) if $z(x,t)$ is determined by an equation of the form $\Omega(z) = z_0(x,t)$, where $\Omega(z)$ is any invertible function, then we can take $\Omega(z) = z$ [by substituting $z \to \Omega^{-1}(z)$].

We shall now proceed to determine the general similarity reductions of the Boussinesq equation using this method.

The coefficients of ww'' and $(w')^2$ yield the common constraint

$$\beta z_x^4 \Gamma(z) = \beta^2 z_x^2,$$

where $\Gamma(z)$ is a function to be determined. Hence, using the freedom mentioned in Remark 3(ii) above, we choose

$$\beta = z_x^2. \tag{3.3}$$

The coefficient of w''' yields

$$\beta z_x^4 \Gamma(z) = 4\beta_x z_x^3 + 6\beta z_x^2 z_{xx},$$

where $\Gamma(z)$ is another function to be determined. Hence using (3.3) and rescaling Γ, we have

$$z_x \Gamma(z) + z_{xx}/z_x = 0,$$

which upon integration gives

$$\Gamma(t) + \ln z_x = \Theta(t),$$

where $\Theta(t)$ is a function of integration. Exponentiated this becomes

$$z_x \Gamma(z) = \Theta(t) \tag{3.4}$$

(recall Remark 2). Integrating again gives

$$\Gamma(z) = x\Theta(t) + \Sigma(t),$$

with $\Sigma(t)$ is another function of integration. By Remark 3(iii), we have

$$z = x\theta(t) + \sigma(t), \tag{3.5}$$

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where $\theta(t)$ and $\sigma(t)$ are to be determined. From Eqs. (3.3) and (3.5), we have

$$\beta = \theta^2(t). \tag{3.6}$$

The coefficient of w'' yields

$$\beta z_x^4 \Gamma(z) = \beta(3z_{xx}^2 + 4z_x z_{xxx}) + 12\beta_x z_x z_{xx} + 6\beta_{xx} z_x^2 + \beta(\alpha z_x^2 + z_t^2),$$

where $\Gamma(z)$ is to be determined, and by Eqs. (3.5) and (3.6) this simplifies to

$$\theta^4 \Gamma(z) = \alpha\theta^2 + \left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt} \right)^2.$$

Hence by Remark 3(i) above

$$\alpha = -\frac{1}{\theta^2(t)} \left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt} \right)^2. \tag{3.7}$$

Let us see how Eq. (3.2) looks with the simplifications as determined so far, viz. (3.5)–(3.7):

$$\theta^6\{w'''' + ww'' + (w')^2\} + \theta^2\left(x\frac{d^2\theta}{dt^2} + \frac{d^2\sigma}{dt^2} \right) w' + 2\theta\frac{d^2\theta}{dt^2} w - \frac{d^2}{dt^2}\left[\left\{ \frac{1}{\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt} \right) \right\}^2 \right] + \frac{6}{\theta^4}\left[\frac{d\theta}{dt}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt} \right) \right]^2 = 0. \tag{3.8}$$

We continue to make this an ordinary differential equation for $w(z)$. Then the remaining coefficients yield

$$\theta^6\gamma_1(z) = \theta^2\left(x\frac{d^2\theta}{dt^2} + \frac{d^2\sigma}{dt^2} \right), \tag{3.9}$$

$$\theta^6\gamma_2(z) = 2\theta\frac{d^2\theta}{dt^2}, \tag{3.10}$$

$$\theta^6\gamma_3(z) = -\frac{d^2}{dt^2}\left[\left\{ \frac{1}{\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt} \right) \right\}^2 \right] + \frac{6}{\theta^4}\left[\frac{d\theta}{dt}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt} \right) \right]^2, \tag{3.11}$$

with $\gamma_1(z)$, $\gamma_2(z)$, and $\gamma_3(z)$ to be determined. First, since $z = x\theta(t) + \sigma(t)$ and the right-hand side of Eq. (3.9) is linear in x, consequently $\gamma_1(z) = Az + B$, where A and B are constants, and so

$$\theta^4[A(x\theta + \sigma) + B] = x\frac{d^2\theta}{dt^2} + \frac{d^2\sigma}{dt^2}. \tag{3.12}$$

Equating coefficients of powers of x gives

$$\frac{d^2\theta}{dt^2} = A\theta^5, \tag{3.13}$$

$$\frac{d^2\sigma}{dt^2} = \theta^4(A\sigma + B). \tag{3.14}$$

It is then easily seen from Eqs. (3.10) and (3.11) that

$$\gamma_2(z) = 2A, \quad \gamma_3(z) = -2(Az+B)^2.$$

[The Boussinesq equation is special in that, having satisfied Eq. (3.9), Eqs. (3.10) and (3.11) are satisfied automatically; slight modifications of the equation would not have significantly affected the application of the method until this point when further restrictions, on $\theta(t)$ and $\sigma(t)$, would arise from (3.10) and (3.11), severely limiting the set of similarity reductions.]

We conclude that the general similarity reduction of the Boussinesq equation (1.2) is given by

$$u(x,t) = \theta^2(t)w(z) - \frac{1}{\theta^2(t)}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt} \right)^2, \tag{3.15a}$$

$$z(x,t) = x\theta(t) + \sigma(t), \tag{3.15b}$$

where $\theta(t)$ and $\sigma(t)$ satisfy Eqs. (3.13) and (3.14), and $w(z)$ satisfies

$$w'''' + ww'' + (w')^2 + (Az+B)w' + 2Aw = 2(Az+B)^2. \tag{3.16}$$

It can be shown that of all the equations of the form

$$w'''' + ww'' + (w')^2 + f(z)w' + g(z)w = h(z),$$

with $f(z)$, $g(z)$, and $h(z)$ analytic, (3.16) is the most general one having the Painlevé property, that is, having no solutions with movable singularities except poles. In general, (3.16) is equivalent to the fourth Painlevé equation; but, when $A = 0$, it is equivalent to the second Painlevé equation, and, when $B = 0$ as well, it is equivalent to either first Painlevé equation of the Weierstrass elliptic function equation—see Appendix A for details. We remark that it is *not* essential to our method that all ordinary differential equations arising from similarity reductions are equivalent to one of the Painlevé equations (or more generally possess the Painlevé property). The Boussinesq equation is a completely integrable soliton equation for which the Painlevé conjecture[28] asserts that every ordinary differential equation arising from a similarity reduction is necessarily of the Painlevé type, in agreement with our results.

Henceforth, new symbols appearing in an equation obtained by integration are generally understood to be arbitrary constants. Furthermore, whenever we set a constant to be a specific value without further explanation, it is implied that this is easily seen to be without loss of generality.

There are three cases to consider.

Case 1. $A=0$, $B=0$: In this case, the general solutions of Eqs. (3.13) and (3.14) are

$$\theta(t) = a_1 t + a_0, \quad \sigma(t) = b_1 t + b_0,$$

and the similarity reduction of the Boussinesq equation is

$$u(x,t) = (a_1 t + a_0)^2 w(z) - \left(\frac{a_1 x + b_1}{a_1 t + a_0} \right)^2, \tag{3.17a}$$

$$z = x(a_1 t + a_0) + b_1 t + b_0, \tag{3.17b}$$

where $w(z)$ satisfies

$$w'' + \tfrac{1}{2}w^2 = c_1 z + c_0. \tag{3.17c}$$

Equation (3.17c) is the same as Eq. (2.6) and so, as we remarked in Sec. II, it is equivalent to either the first Painlevé equation (2.7) or the Weierstrass elliptic function equation. We note also that the traveling wave reduction arises as the special case of (3.17) where $a_1 = 0$ and $b_1 \neq 0$. However, if $a_1 = 0$, then we set $a_1 = 1$, $a_0 = b_1 = b_0 = 0$, and obtain the similarity reduction

$$u(x,t) = t^2 w(z) - x^2/t^2, \quad z = xt, \tag{3.18}$$

where $w(z)$ satisfies Eq. (3.17c). This is a new reduction of the Boussinesq equation to the first Painlevé equation.

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With z and w as invariants, Eqs. (3.18) define the point transformation group

$$(x,t,u)\to(\gamma^{-1}x,\gamma t,\gamma^2 u+(\gamma^2-\gamma^{-4})x^2/t^2).$$

The infinitesimals associated with this are

$$X=-x,\quad T=t,\quad U=2u+6x^2/t^2, \tag{3.19}$$

which clearly are not a special case of the infinitesimals obtained by the classical Lie group method [cf. (2.5)].

Case 2. $A=0$, $B\neq 0$: In this case the general solution of Eqs. (3.13) and (3.14) are

$$\theta(t)=a_1t+a_0,$$

$$\sigma(t)=\begin{cases}\frac{1}{30}Ba_1^{-2}(a_1t+a_0)^6+b_1t+b_0, & \text{if } a_1\neq 0,\\ \frac{1}{2}Ba_0^2t^2+b_1t+b_0, & \text{if } a_1=0.\end{cases}$$

Case (a). $a_1=0$: The similarity reduction of the Boussinesq equation is

$$u(x,t)=a_0^2w(z)-(Ba_0^2t+b_1)^2/a_0^2, \tag{3.20a}$$

$$z=a_0x+\tfrac{1}{2}Ba_0^2t^2+b_1t+b_0, \tag{3.20b}$$

where $w(z)$ satisfies

$$w'''+ww'+Bw=2B^2z+c_0. \tag{3.21}$$

Equation (3.21) is the same as Eq. (2.12) and so, as remarked in Sec. II, it is equivalent to the second Painlevé equation (2.14)—see, also, Appendix A. We set $a_0=1$, $b_1=b_0=0$, in (3.20), in which case it just reduces to the "nonclassical" similarity reduction (2.11) (cf. Refs. 19, 24, 25, and 29).

Case (b). $a_1\neq 0$: The similarity reduction of the Boussinesq equation is

$$u(x,t)=(a_1t+a_0)^2w(z)-\left(\frac{a_1^2x+\frac{1}{5}B(a_1t+a_0)^5+a_1b_1}{a_1(a_1t+a_0)}\right)^2, \tag{3.20a$'$}$$

$$z=x(a_1t+a_0)+[B/30a_1^2](a_1t+a_0)^6+b_1t+b_0, \tag{3.20b$'$}$$

where $w(z)$ satisfies (3.21). We set $a_1=1$, $a_0=b_1=b_0=0$, and obtain

$$u(x,t)=t^2w(z)-(x+\lambda t^5)^2/t^2,\quad z=xt+\tfrac{1}{6}\lambda t^6, \tag{3.22}$$

where $w(z)$ satisfies (3.21) (we have also set $B=5\lambda$). This is another new reduction of the Boussinesq equation; this time to the second Painlevé equation (2.14). The infinitesimals associated with the transformation group defined by (3.22) are

$$X=-(x+\lambda t^5),\quad T=t,$$
$$U=2u+2(x+\lambda t^5)(3x-2\lambda t^5)/t^2. \tag{3.23}$$

[We note that if $\lambda=0$ in (3.22) and (3.23), they reduce to (3.18) and (3.19).]

Case 3. $A\neq 0$: In this case we can set $B=0$ in Eq. (3.14). Multiplying Eq. (3.13) by $d\theta/dt$ and integrating gives

$$\left(\frac{d\theta}{dt}\right)^2=\frac{1}{3}A\theta^6+A_0, \tag{3.24}$$

where A_0 is a constant. There are two possibilites.

Case (a). $A_0=0$: Equation (3.24) has the solution

$$\theta(t)=c_0(t+t_0)^{-1/2}, \tag{3.25}$$

with $c_0^4=3/(4A)$. Substituting this into Eq. (3.14) and solving yields

$$\sigma(t)=c_1(t+t_0)^{3/2}+c_2(t+t_0)^{-1/2}.$$

Therefore we may set $t_0=0$, $c_0=1$, and $c_2=0$, and obtain the similarity reduction

$$u(x,t)=t^{-1}w(z)-\tfrac{1}{4}t^{-2}(x-3c_1t^2)^2,$$
$$z=xt^{-1/2}+c_1t^{3/2}, \tag{3.26}$$

where $w(z)$ satisfies

$$w''''+ww''+(w')^2+\tfrac{3}{4}zw'+\tfrac{3}{2}w=\tfrac{9}{8}z^2. \tag{3.27}$$

Note that the scaling reduction (2.8) arises as the special case of (3.26) with $c_1=0$. If $c_1\neq 0$, this is a new similarity reduction, namely, to the fourth Painlevé equation, since if in (3.27) we make the transformation $w(z)=g(z)+z^2/4$, then $g(z)$ satisfies Eq. (2.9) and therefore Eq. (3.27) is also equivalent to the fourth Painlevé equation (2.10)—see, also, Appendix A.

Case (b). $A\neq 0$: Equation (3.24) can be solved in terms of Jacobian elliptic functions (cf. Ref. 32). Furthermore we may set

$$A_0=k^2,\quad A=(k^2+1)/3k^2, \tag{3.28}$$

where k is a constant to be chosen. For this choice of constants, the transformation

$$\theta^2(t)=1/[\eta^2(t)-A] \tag{3.29}$$

reduces (3.24) to the normal form

$$\left(\frac{d\eta}{dt}\right)^2=(1-\eta^2)(1-k^2\eta^2), \tag{3.30}$$

provided that

$$k^2=\tfrac{1}{2}(1\pm i\sqrt{3}) \tag{3.31}$$

(which we may assume without loss of generality). The solution of (3.30) is the Jacobian elliptic function $\mathrm{sn}(t+t_0;k)$, and so

$$\theta(t)=\{\mathrm{sn}^2(t+t_0;k)-(k^2+1)/3k^2\}^{-1/2}. \tag{3.32}$$

Equation (3.14) becomes

$$\frac{d^2\sigma}{dt^2}=\frac{k^2+1}{3k^2}\theta^4\sigma,$$

which has the solution

$$\sigma(t)=[C\{[(2-k^2)/3k^2]t-k^{-2}E(t+t_0;k)\}+D]\theta(t), \tag{3.33}$$

where $E(t+t_0;k)$ is the elliptic integral of the second kind given by

$$E(t+t_0;k)=\int_0^{t+t_0}[1-k^2\,\mathrm{sn}^2(s;k)]ds$$

and C and D are arbitrary constants—we set $D=0$.

Therefore we have the following similarity reduction:

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$$u(x,t) = (\mathrm{sn}^2(t+t_0;k) - A)^{-1} w(z) - [\, C(\mathrm{sn}^2(t+t_0;k) - A) - \{x + C([(2-k^2)/3k^2]t - k^{-2}E(t+t_0;k))\} \times [\mathrm{sn}(t+t_0;k)\sqrt{(1-\mathrm{sn}^2(t+t_0;k))(1-k^2\,\mathrm{sn}^2(t+t_0;k))/(\mathrm{sn}^2(t+t_0;k)-A)}\,]^2, \tag{3.34a}$$

with

$$z = [x + C([(2-k^2)/3k^2]t - k^{-2}E(t+t_0;k))] \times (\mathrm{sn}^2(t+t_0;k) - A)^{-1/2}, \tag{3.34b}$$

and

$$k^2 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \quad A = \frac{k^2+1}{3k^2} = \frac{1}{2} \mp \frac{i}{2\sqrt{3}}, \tag{3.34c}$$

where $w(z)$ satisfies

$$w'''' + ww'' + (w')^2 + Azw' + 2Aw = 2A^2z^2. \tag{3.34d}$$

This is another new similarity reduction, again to the fourth Painlevé equation (2.10).

As for the other new similarity reductions given above [(3.18) and (3.22)], we can write down the infinitesimals associated with the transformation groups defined by (3.26) and (3.34). Again they are not special cases of those obtained by the classical Lie group method.

In all three cases we have obtained new similarity reductions of the Boussinesq equation more general than those previously obtained (though, interestingly, the resulting ordinary differential equations are the same). As mentioned above, these similarity reductions are associated with Lie point transformations (since they depend only on the independent and dependent variables and not upon the derivatives of the dependent variable). It remains an open question as to whether all these new similarity reductions and their associated transformations can be obtained using any of the other generalizations of the classical Lie method, such as the nonclassical method of Bluman and Cole[18] (cf. Ref. 23), and the method developed by Bluman *et al.*[22] However, even if theoretically they can be obtained by either of these methods it seems that our method is somewhat simpler to implement; in fact, it appears to be simpler than calculating the classical Lie point symmetries manually.

It can be shown that for the similarity reductions of the Boussinesq equation that cannot be obtained using the classical Lie group method, the associated group transformation does not map the Boussinesq equation into itself, whereas the similarity reductions obtained by the classical Lie group method do. For example, consider the similarity reduction

$$u(x,t) = t^2 w(z) - (x+\lambda t^5)^2/t^2, \quad z = xt + \tfrac{1}{6}\lambda t^6. \tag{3.22}$$

The one-parameter (γ) group associated with this similarity reduction is given by

$$x \to \gamma^{-1}x + \tfrac{1}{6}\lambda\gamma^{-1}(1-\gamma^6)t^5, \tag{3.35a}$$

$$t \to \gamma t, \tag{3.35b}$$

$$u \to \gamma^2 u + \gamma^2(1-\gamma^{-6}) \times \{x^2/t^2 + \tfrac{1}{3}\lambda xt^3 + \tfrac{1}{36}\lambda^2 t^8(1-25\gamma^{-6})\}. \tag{3.35c}$$

This group maps solutions of the Boussinesq equation (1.2) into solutions of

$$u_{tt} + uu_{xx} + u_x^2 + u_{xxxx} = (\gamma^6 - 1)t^{-2}\Phi, \tag{3.36a}$$

where

$$\Phi := (x^2 + \tfrac{1}{3}\lambda xt^5 - \tfrac{2}{3}\lambda^2 t^{10})u_{xx} + 4(x+\lambda t^5)u_x + 2u + \tfrac{5}{3}\lambda t^6 u_{xt} - t^2 u_{tt} + 6x^2/t^2 + 2\lambda xt^3 - \tfrac{112}{3}\lambda^2 t^8. \tag{3.36b}$$

If u is the similarity reduction (3.22), then it is easily seen that $\Phi \equiv 0$, i.e., the group (3.35) maps the Boussinesq equation (1.2) into the "perturbed Boussinesq equation" (3.36a), but (3.36b) is identically zero. Therefore the perturbed equation is identical to the Boussinesq equation when u is given by (3.22).

In order to understand why the perturbation Φ must vanish identically, consider the infinitesimals

$$X = -(x+\lambda t^5), \quad T = t, \qquad U = 2u + 2(x+\lambda t^5)(3x - 2\lambda t^5)/t^2, \tag{3.23}$$

for the similarity reduction (3.22). The similarity reduction necessarily satisfies the invariant surface condition

$$X(x,t,u)u_x + T(x,t,u)u_t = U(x,t,u),$$

i.e.,

$$\psi := (x+\lambda t^5)u_x - tu_t + 2u + 6x^2/t^2 + 2\lambda xt^3 - 4\lambda^2 t^8 = 0. \tag{3.37}$$

It is easily shown that

$$\Phi = (x - \tfrac{2}{3}\lambda t^5)\psi_x + t\psi_t + \psi. \tag{3.38}$$

IV. JUSTIFICATION OF THE SPECIAL FORM (1.4)

We show here that it is sufficient to seek a similarity reduction of the Boussinesq equation (1.2) in the special form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)), \tag{4.1}$$

rather than the more general form

$$u(x,t) = U(x,t,w(z(x,t))). \tag{4.2}$$

Substituting (4.2) into (1.2) yields

$$[U_{tt} + 2U_{tw}w'z_t + U_{ww}(w')^2z_t^2 + U_w(w''z_t + w'z_{tt})] + U[U_{xx} + 2U_{xw}w'z_x + U_{ww}(w')^2z_x^2 + U_w(w''z_x + w'z_{xx})] + U_x^2 + 2U_xU_ww'z_x + U_w^2(w')^2z_x^2 + U_{xxxx} + 4U_{xxxw}w'z_x + 6U_{xxww}(w')^2z_x^2 + 4U_{xwww}(w')^3z_x^3 + U_{wwww}(w')^4z_x^4 + 6U_{xxw}(w'z_{xx} + w''z_x^2) + 12U_{xww}[w'w''z_x^3 + (w')^2z_xz_{xx}]$$

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$$+ 6U_{www}[(w')^2w''z_x^4 + (w')^3z_x^2z_{xx}] + 4U_{xw}(w'''z_x^3 + 3w''z_xz_{xx} + w'z_{xxx})$$
$$+ U_{ww}[\{4w'w''' + 3(w'')^2\}z_x^4 + 18w'w''z_x^2z_{xx} + (w')^2(4z_xz_{xxx} + 3z_{xx}^2)]$$
$$+ U_w[w''''z_x^4 + 6w'''z_x^2z_{xx} + w''(4z_x z_{xxx} + 3z_{xx}^2) + w'z_{xxxx}] = 0. \quad (4.3)$$

For this to be an ordinary differential equation in $w(z)$, the ratios of different derivatives of $w(z)$ must be functions of w and z. Using the coefficient of w'''' (i.e., $U_wz_x^4$) as the normalizing coefficient, the coefficients of $w'w'''$ and $(w'')^2$ require that

$$U_wz_x^4\Gamma(w,z) = U_{ww}z_x^4, \quad (4.4)$$

where $\Gamma(w,z)$ is a function to be determined. Hence

$$\Gamma(w,z) = U_{ww}/U_w ,$$

which after two integrations yields

$$U(x,t,w) = \Theta(x,t)\Gamma(w,z) + \Phi(x,t), \quad (4.5)$$

with $\Theta(x,t)$ and $\Phi(x,t)$ arbitrary functions (cf. Remarks 2 and 3 in Sec. III). Therefore it is sufficient to seek similarity reductions of the Boussinesq equation (1.2) in the form (4.1).

Therefore, if we seek a similarity reduction of the Boussinesq equation in the general form (4.2), we are naturally led to the special form (4.1). Although, for many partial differential equations such as the Boussinesq equation, it is sufficient to seek similarity reductions in the special form (4.2), for some others it may be necessary to transform the dependent variable before using (4.1); however, the assumption (4.2) leads naturally to the required transformation.

For example, consider the Harry–Dym equation (cf. Ref. 33).

$$u_t + 2(u^{-1/2})_{xxx} = 0, \quad (4.6)$$

which can be solved by inverse scattering[34] (see, also, Ref. 12) and is related to the Korteweg–de Vries and modified Korteweg–de Vries equations through hodograph transformations.[35] Let us seek a similarity reduction in the form (4.2). Substitution yields

$$U_t + U_ww'z_t - \tfrac{15}{4}U^{-7/2}(U_x + U_ww'z_x)^3$$
$$+ \tfrac{9}{2}U^{-5/2}(U_x + U_ww'z_x)[U_{xx} + 2U_{xw}w'z_x + U_{ww}(w')^2z_x^2 + U_w(w''z_x^2 + w'z_{xx})]$$
$$- U^{-3/2}[U_{xxx} + 3U_{xxw}w'z_x + 3U_{xww}(w')^2z_x^2 + U_{www}(w')^3z_x^3 + 3U_{xw}(w''z_x^2 + w'z_{xx})$$
$$+ 3U_{ww}\{w'w''z_x^3 + (w')^2z_xz_{xx}\} + U_w(w'''z_x^3 + 3w''z_xz_{xx} + w'z_{xxx})] = 0. \quad (4.7)$$

Using the coefficient of w''' (i.e., $U^{-3/2}U_wz_x^3$) as the normalizing coefficient, the coefficient of $w'w''$ requires that

$$U^{-3/2}U_wz_x^3\Gamma(w,z) = \tfrac{3}{2}U^{-5/2}U_w^2z_x^3 - U^{-3/2}U_{ww}z_x^3 ,$$

that is,

$$\Gamma(w,z) = -\tfrac{3}{2}U_w/U + U_{ww}/U_w , \quad (4.8)$$

where $\Gamma(w,z)$ is a function to be determined. Integrating twice yields

$$U^{-1/2}(x,t) = \Theta(x,t)\Gamma(w,z) + \Phi(x,t) , \quad (4.9)$$

with $\Theta(x,t)$ and $\Phi(x,t)$ arbitrary functions (cf. Remark 2 in Sec. III). Hence it is sufficient to seek similarity reductions of the Harry–Dym equation (4.6) in the form

$$u^{-1/2}(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)) .$$

Alternatively we could first make the transformation $v = u^{-1/2}$ and then seek similarity reductions in the form (4.1). Obvious as this transformation is, our method leads to it systematically.

V. DISCUSSION

In this paper we have developed a direct method for determining similarity reductions of a given partial differential equation. However, there are a number of open questions our method poses. First, what is the relationship (if any) between our method and other generalizations of the classical Lie method, such as those of Bluman and Cole[18] (cf. Ref. 23), Olver and Rosenau,[19] and Bluman *et al.*[22]? In their generalization of the method of Bluman and Cole,[18] Olver and Rosenau[19] showed that in order to determine a group-invariant solution to a given partial differential equation, one could try *any* group of infinitesimal transformations whatsoever. Generally, for any specific group and any specific equation, there will be *no* solutions of the equation invariant under the group, and so the question becomes how does one determine *a priori* which groups will give meaningful similarity reductions? One possibility is that by seeking a reduction of a certain form (as done in this paper), one is naturally led to the appropriate group (i.e., the requirement that the similarity reduction reduce the partial differential equation to an ordinary differential equation is equivalent to the *side conditions* in the terminology of Olver and Rosenau[18]).

Second, what kind of "*symmetries*" of the Boussinesq equation are those we have obtained that are not found using the classcial Lie method? (They are "*weak symmetries*" in the terminology of Olver and Rosenau.[19]) As shown in Sec. III, the associated group of infinitesimal transformations does *not* map solutions of the Boussinesq equation into other solutions of the Boussinesq equation, but rather into solutions of other equations.

The idea of making the ansatz that a similarity reduction of a given partial differential equation have a particular form has been suggested previously in the literature. For example, (i), Gilding[36] seeks solutions of the porous media equation

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$$u_t = (u^m)_{xx}\,, \quad m > 1,$$

in the form

$$u(x,t) = \mu(t) f(z), \quad z = \rho(t)[x + \lambda(t)];$$

and (ii), Fushchlich, in a series of papers with various coauthors,[37] has obtained exact solutions of several nonlinear relativistic and nonlinear wave equations (including the nonlinear Dirac, Klein–Gordon, Maxwell, and Schrödinger equations) in three spatial and one temporal dimension, using their symmetry properties and seeking solutions in the form

$$u(x_0,x_1,x_2,x_3) = A(x_0,x_1,x_2,x_3)\,w(z_1,z_2,z_3) + B(x_0,x_1,x_2,x_3)\,,$$

where

$$z = (z_1(x_0,x_1,x_2,x_3), z_2(x_0,x_1,x_2,x_3), z_3(x_0,x_1,x_2,x_3))$$

are the new independent variables, $w(z_1,z_2,z_3)$ the new dependent variable, and $A(x_0,x_1,x_2,x_3)$ and $B(x_0,x_1,x_2,x_3)$ are determined.

We have applied the method to several other integrable equations including Burgers' equation

$$u_t + uu_x + u_{xx} = 0\,, \tag{5.1}$$

which can be mapped into the linear heat equation through the Cole–Hopf transformation[38]; the Korteweg–de Vries equation

$$u_t + uu_x + u_{xxx} = 0\,, \tag{5.2}$$

which can be solved by inverse scattering[8]; and the modified Korteweg–de Vries equation

$$u_t + u^2u_x + u_{xxx} = 0\,, \tag{5.3}$$

which also can be solved by inverse scattering.[39] However, for these three equations, the similarity reductions obtained are precisely the same as those obtained using the classical Lie method of infinitesimal transformations (for further details see Appendices B, C, and D, respectively, which also provide further examples of the application of our method).

There is much current interest in the mathematically and physically significant determination of similarity reductions of given partial differential equations. (In addition to the references mentioned above, the interested reader might also consult Refs. 40–43, and the references therein.) Our method is a practical and direct one for finding similarity reductions; it has generated similarity reductions that, to the best of our knowledge, are previously unknown. It seems probable that the method can be generalized to higher-order equations with more independent and dependent variables.

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APPENDIX A: REDUCTION TO PAINLEVÉ EQUATIONS

In this appendix it is shown that of all the equations of the form

$$w'''' + ww'' + (w')^2 + f(z)w' + g(z) = h(z)\,, \tag{A1}$$

with $f(z)$, $g(z)$, and $h(z)$ analytic, the most general one having the Painlevé property, that is, having no solutions with movable singularities except poles, is given by

$$w'''' + ww'' + (w')^2 + (Az+B)w' + 2Aw = 2(Ax+B)^2\,, \tag{A2}$$

where A and B are arbitary constants. To show this we follow Ablowitz *et al.*[28] in seeking a solution of Eq. (A2) in the Laurent series form

$$w(z) = \sum_{j=0}^{\infty} w_j (z-z_0)^{j+p}\,, \tag{A3}$$

with z_0 an arbitrary constant, $w_0 \neq 0$ and $w_j, j \geqslant 0$, constants to be determined. Leading-order analysis shows that

$$w_0 = -12, \quad p = -2. \tag{A4}$$

Substituting into (A1) and equating coefficients of powers yields for $j \geqslant 1$ the recursion relation

$$(j+1)(j-4)(j-5)(j-6)w_j + \frac{1}{2}(j-4)(j-5)\sum_{k=1}^{j-1} w_k w_{j-k} = -\sum_{k=0}^{j-3} f_k(j-k-5)w_{j-k-3} - \sum_{k=0}^{j-4} g_k w_{j-k-4} + h_{j-6}\,, \tag{A5a}$$

where

$$f(z) = \sum_{k=0}^{\infty} f_k (z-z_0)^k, \text{ etc.} \tag{A5b}$$

(defining $w_j = 0$ for $j < 0$, etc.). This determines w_j for $j \geqslant 1$ except for $j = 4,5,6$, which are the so-called resonances. For each resonance there is a compatibility condition that must be identically satisfied for Eq. (A1) to have a solution in the form (A3). From Eq. (A5) we obtain

$$w_1 = 0, \quad w_2 = 0, \quad w_3 = f_0\,. \tag{A6}$$

The compatibility conditions for $j = 4$ and $j = 5$ are

$$g_0 = 2f_1, \quad g_1 = 2f_2\,,$$

respectively. Since z_0 is arbitrary, necessarily

$$g(z) = 2\frac{df}{dz}, \quad \frac{dg}{dz} = \frac{d^2f}{dz^2}\,. \tag{A7}$$

These hold simultaneously if and only if

$$\frac{d^2f}{dz^2} = 0\,,$$

i.e.,

$$f(z) = Az + B\,, \quad g(z) = 2A\,, \tag{A8a}$$

with A and B arbitrary constants. The compatibility condition for $j = 6$ is

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$$h_0 = 2f_0^2 .$$

Thus

$$h(z) = 2(Az + B)^2 . \tag{A8b}$$

Unless $f(z)$, $g(z)$, and $h(z)$ are as given in Eqs. (A8), the compatibility conditions are violated and so Eq. (A1) has the Painlevé property only if it has the special form (A2).

In order to complete the proof that Eq. (A2) has the Painlevé property, we show that no solution of it has a movable essential singularity by reducing it to known such equations.

Case (a). $A=0$, $B=0$: Integrating Eq. (A2) twice yields

$$\frac{d^2w}{dz^2} + \frac{1}{2} w^2 = c_1 z + c_0 . \tag{A9}$$

If $c_1 = 0$, $w(z)$ is a Weierstrass elliptic function (cf. Ref. 32); otherwise (A9) is the first Painlevé equation (cf. Ref. 26). In either case, all solutions possess the Painlevé property (in fact, are meromorphic); hence no solution of Eq. (A9) has a movable essential singularity.

Case (b). $A=0$, $B\neq 0$: Integrating Eq. (A2) once yields

$$\frac{d^3w}{dz^3} + w\frac{dw}{dz} + Bw = 2B^2 z + c_2 . \tag{A10}$$

Then make the transformation

$$w(z) = B^{2/3}W(Z) + Bz + c_2/2B,$$
$$Z = -(B^{1/3}z + \tfrac{1}{2} c_2 B^{-5/3}) , \tag{A11}$$

which produces

$$\frac{d^3W}{dZ^3} + W\frac{dW}{dZ} - \left(2W + Z\frac{dW}{dZ}\right) = 0 . \tag{A12}$$

Whitham (see Refs. 27 and 30) noted that solutions of this equation are related to solutions of the second Painlevé equation

$$\frac{d^2V}{dZ^2} = 2V^3 + ZV + \alpha , \tag{A13}$$

with α an arbitrary constant. Actually, as shown by Fokas and Ablowitz,[30] there is a one-to-one correspondence between solutions of (A12) and (A13) given by

$$W(Z) = -6(V'(Z) + V^2(Z)) , \tag{A14a}$$

$$V(Z) = [W'(Z) + 6a]/[2W(Z) - 6Z] , \tag{A14b}$$

where $' := d/dZ$. [Equation (A14a) is just the scaling, or self-similar, reduction of the Miura transformation[44] relating solutions of the modified Korteweg–de Vries equation (5.3) to solutions of the Korteweg–de Vries equation (5.2).] All solutions of the second Painlevé equation possess the Painlevé property (in fact, are meromorphic); hence no solution of Eq. (A10) has a movable essential singularity.

Case (c). $A\neq 0$: The transformation

$$w \rightarrow \left(\frac{4A}{3}\right)^{1/2} w, \quad z \rightarrow \left(\frac{3}{4A}\right)^{1/4} z - \frac{B}{A},$$

takes (A2) to the form

$$w'''' + w'' + (w')^2 + \tfrac{3}{4} zw' + \tfrac{3}{2} w = \tfrac{9}{8} z^2 . \tag{A15}$$

Hirota and Satsuma[45] show that there is a "Miura-type" transformation relating solutions of the modified Boussinesq equation

$$q_{tt} - q_t q_{xx} - \tfrac{1}{2} q_x^2 q_{xx} + q_{xxxx} = 0 , \tag{A16}$$

to solutions of the Boussinesq equation (1.2) (see, also, Refs. 29 and 46). The Bäcklund transformation

$$v_x(x,t) = -q_t + \sqrt{3} q_{xx} - \tfrac{1}{2} q_x^2 , \tag{A17a}$$

$$v_t(x,t) = \sqrt{3} q_{xt} + q_{xxx} - q_x q_t - \tfrac{1}{6} q_x^3 + \delta , \tag{A17b}$$

where δ is a constant, is easily seen to take a solution q of the modified Boussinesq equation (A16) to a solution v of the potential Boussinesq equation

$$v_{tt} + v_x v_{xx} + v_{xxxx} = 0; \tag{A18}$$

furthermore $u = v_x$ is a solution of the Boussinesq equation (1.2). The modified Boussinesq equation (A16) has the similarity solution (cf. Ref. 29)

$$q(x,t) = -\gamma \ln t + p(z), \quad z = xt^{-1/2} , \tag{A19}$$

where $p(z)$ satisfies

$$\gamma + \frac{3z}{4}p' + \frac{z^2}{4}p'' + \left(\gamma + \frac{1}{2}zp'\right)p''$$
$$- \frac{1}{2}(p')^2 p'' + p'''' = 0 ,$$

with $' := d/dz$; and if we now make the transformation

$$p'(z) = -3^{3/4}Q(Z) - z, \quad Z = 3^{1/4}z/2 , \tag{A20}$$

then $Q(Z)$ satisfies the fourth Painlevé equation

$$\frac{d^2Q}{dZ^2} = \frac{1}{2Q}\left(\frac{dQ}{dZ}\right)^2 + \frac{3}{2}Q^3 + 4ZQ^2$$
$$+ 2(Z^2 - \alpha)Q + \frac{\beta}{Q} , \tag{A21}$$

with $\alpha = 8\gamma/(9\sqrt{3})$ and β an arbitrary constant (see, also, Ref. 46). The Boussinesq equation (1.2) and the potential Boussinesq equation (A18) possess the similarity reductions

$$u(x,t) = t^{-1}w(z) - x^2/4t^2 , \quad z = xt^{-1/2} , \tag{A22a}$$

$$v(x,t) = t^{-1/2}r(z) , \quad z = xt^{-1/2} , \tag{A22b}$$

where $w(z)$ satisfies Eq. (A15) and $r(z)$ satisfies

$$r'''' + r'r'' - \tfrac{1}{2}(r + zr') = 0 . \tag{A23}$$

Therefore, Eqs. (A17)–(A22) show that if $Q(Z)$ is a solution of the fourth Painlevé equation, then

$$w(z) := -\frac{3\sqrt{3}}{2}\left(\frac{dQ}{dZ} + Q^2(Z) + 2ZQ(Z) + 3Z^2\right)$$
$$+ \frac{9\sqrt{3}}{8}\alpha - \sqrt{3} , \tag{A24a}$$

$$Z := 3^{1/4}z/2 , \tag{A24b}$$

is a solution of Eq. (A15).

What all this shows is that from any solution of the fourth Painlevé equation we can obtain a solution of (A15). To obtain the converse we substitute the similarity reductions (A19) and (A22) into the Bäcklund transformation (A17) and easily see that if $r(z)$ is a solution of (A23), then

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$$Q(Z) := -3^{-3/4}\left(\frac{\frac{3}{2}(r - zr') + \sqrt{3}r'' - z^3 - 2\sqrt{3}z}{r' + z^2 + 2\gamma - \sqrt{3}}\right),$$
$$Z = 3^{1/4}z/2, \qquad (A25)$$

satisfies the fourth Painlevé equation (A21); furthermore solutions of Eqs. (A15) and (A23) are related by

$$w(z) = \frac{dr}{dz} + \frac{z^2}{4}. \qquad (A26)$$

Equations (A24)–(A26) provide a one-to-one relationship between solutions of Eq. (A15) and solutions of the fourth Painlevé equation. All solutions of the fourth Painlevé equation possess the Painlevé property, i.e., have no movable essential singularities (in fact, are meromorphic). Therefore no solution of Eq. (A15) has a movable essential singularity.

We remark that there is also a direct method to show that no solution of Eq. (A2) has a movable essential singularity. Making the transformation

$$w(z) = v'(z) - (Az + B)^2/A, \qquad (A26')$$

we obtain a fifth-order equation easily integrated twice to yield

$$v''' + \frac{1}{2}(v')^2 - \frac{Az + B}{A}\,[(Az + B)v' - Av] = c_1 z + c_2. \qquad (A27)$$

Multiplying by v'' and integrating again yields

$$\tfrac{1}{2}(v'')^2 + \tfrac{1}{6}(v')^3 - (1/2A)[(Az + B)v' - Av]^2$$
$$= (c_1/A)[(Az + B)v' - Av] + c_2 v' + c_3. \qquad (A28)$$

This is equivalent (through rescaling and translation of the variables) to an equation given by Chazy,[47]

$$(y'')^2 + 4(y')^3 + (zy' - y)^2 + \alpha y' + \beta = 0, \qquad (A29)$$

with α and β constants. According to Chazy, this is "an algebraic transformation of the fourth Painlevé equation" [Eq. (A29) is sometimes referred to as Chazy IV, cf. Refs. 29 and 48]. Furthermore, as shown by Chazy,[47] for any solution of (A29), $\exp\{\int^z y(s)ds\}$ is analytic except at the points $0, \infty$. Hence we conclude that no solution of Eq. (A26), and hence also of Eq. (A2), has a movable essential singularity.

APPENDIX B: BURGERS' EQUATION

In this appendix we outline how to determine the similarity reductions of Burgers' equation

$$u_t + uu_x + u_{xx} = 0, \qquad (B1)$$

using the method developed in this paper. As with the Boussinesq equation (1.2), it suffices to seek similarity reductions in the special form

$$u(x,t) = \alpha(x,t) + \beta(x,t)w(z(x,t)). \qquad (B2)$$

Substituting (B2) and (B1) and collecting coefficients yields

$$\beta z_x^2 w'' + (2\beta_x z_x + \beta z_{xx} + \beta z_t + \alpha\beta z_x)w'$$
$$+ (\beta_{xx} + \beta_t + \alpha\beta_x + \alpha_x\beta)w$$
$$+ \beta^2 z_x ww' + \beta\beta_x w^2 + \alpha_{xx} + \alpha_t + \alpha\alpha_x = 0. \qquad (B3)$$

We use the coefficient of w'' as the normalizing coefficient. For this to be an ordinary differential equation, from the coefficient of ww' we get

$$\beta z_x^2 \Gamma(z) = \beta^2 z_x,$$

where $\Gamma(z)$ is to be determined. Using the freedom in Remark 3(i) in Sec. III, we take

$$\beta = z_x. \qquad (B4)$$

The coefficient of w^2 gives

$$\beta z_x^2 \Gamma(z) = \beta\beta_x,$$

where $\Gamma(z)$ is to be determined. Using (B4), integrating twice, and using the freedoms in Remark 2 and 3(iii), we have

$$z = x\theta(t) + \sigma(t), \quad \beta = \theta(t), \qquad (B5)$$

where $\theta(t)$ and $\sigma(t)$ are to be determined. Equation (B3) simplifies to

$$\theta^3(w'' + ww') + \theta\left\{\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right) + \alpha\theta\right\}w'$$
$$+ \left\{\frac{d\theta}{dt} + \alpha_x\theta\right\}w + \alpha_{xx} + \alpha_t + \alpha\alpha_x = 0. \qquad (B6)$$

This is an ordinary differential equation for $w(z)$ provided that

$$\alpha = -\frac{1}{\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right), \qquad (B7)$$

$$\theta\frac{d^2\theta}{dt^2} - 2\left(\frac{d\theta}{dt}\right)^2 = A^2\theta^6, \qquad (B8)$$

$$\theta\frac{d^2\sigma}{dt^2} - 2\frac{d\theta}{dt}\frac{d\sigma}{dt} = \theta^5(A^2\sigma + 2B), \qquad (B9)$$

with A and B arbitrary constants. Multiplying (B8) by $2\theta^{-2}\,d\theta/dt$ and integrating gives

$$\left(\frac{d\theta}{dt}\right)^2 = A^2\theta^6 + C^2\theta^4, \qquad (B10)$$

with C an arbitrary constant.

Therefore the general similarity reduction of Burgers' equation (B1) is given by

$$u(x,t) = \theta(t)w(z) - \frac{1}{\theta}\left(x\frac{d\theta}{dt} + \frac{d\sigma}{dt}\right),$$
$$z = x\theta(t) + \sigma(t),$$

where $\theta(t)$ and $\sigma(t)$ satisfy (B9) and (B10).

There are four cases to consider.

Case 1. $A=0$, $C=0$: Here the solutions are

$$\theta(t) = \theta_0, \quad \sigma(t) = Bt^2 + c_1 t + c_2.$$

We set $\theta_0 = 1$ and obtain the similarity reduction,

$$u(x,t) = w(z) - 2Bt - c_1, \quad z = x + Bt^2 + c_1 + c_2. \qquad (B11)$$

Case 2. $A\neq 0$, $C=0$: We set $A = -\frac{1}{2}$ and $B = 0$. Then

$$\theta(t) = (t - t_0)^{-1/2},$$
$$\sigma(t) = c_3(t - t_0)^{1/2} + c_4(t - t_0)^{-1/2}.$$

Setting $t_0 = 1$, $c_4 = 0$, we obtain

$$u(x,t) = t^{-1/2}w(z) + x/2t - \tfrac{1}{2}c_3. \qquad (B12)$$

Case 3. $A=0$, $C\neq 0$: We set $C = -1$. Then

$$\theta(t) = (t - t_0)^{-1},$$
$$\sigma(t) = B(t - t_0)^{-2} + c_5(t - t_0)^{-1} + c_6.$$

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Setting $t_0=1$, $c_5=0$, and $c_6=0$, we obtain

$$u(x,t)=t^{-1}w(z)+\frac{x}{t}+\frac{2B}{t^2},\quad z=\frac{x}{t}+\frac{B}{t^2}. \tag{B13}$$

Case 4. $A\neq 0$, $C\neq 0$: We set $A^2=-1$, $B=0$, $C^2=1$. Then

$$\theta(t)=(t^2\pm 1)^{-1/2},\quad \sigma(t)=c_7t+c_8(t^2\pm 1)^{-1/2}.$$

Setting $c_8=0$, we obtain

$$u(x,t)=(t^2\pm 1)^{-1/2}w(z)+\frac{xt-c_7}{t^2\pm 1},\quad z=\frac{x+c_7t}{t^2\pm 1}. \tag{B14}$$

The infinitesimals for Burgers' equation obtained using the classical Lie method are

$$X=\alpha x+\beta t+\gamma xt+\delta, \tag{B15a}$$

$$T=2\alpha t+\gamma t^2+\kappa, \tag{B15b}$$

$$U=-\alpha u+\gamma(x-tu)+\beta, \tag{B15c}$$

with α, β, γ, δ, and κ arbitrary constants (cf. Ref. 49). It is easily shown that all the similarity reductions obtained by our method (B11)–(B14) for Bergers' equation (B1) can also be obtained from these infinitesimals (cf. Ref. 49).

APPENDIX C: KORTEWEG–dE VRIES EQUATION

In this appendix we outline how to determine the similarity reductions of the Korteweg–de Vries equation

$$u_t+uu_x+u_{xxx}=0, \tag{C1}$$

using the method developed in this paper. It suffices to assume the special form

$$u(x,t)=\alpha(x,t)+\beta(x,t)w(z(x,t)). \tag{C2}$$

Substituting and collecting coefficients yields

$$\begin{aligned}&\beta z_x^3w'''+(3\beta_xz_x^2+3\beta z_xz_{xx})w''\\&\quad+(3\beta_{xx}z_x+3\beta_xz_{xx}+\beta z_{xxx}+\beta z_t+\alpha\beta z_x)w'\\&\quad+(\beta_{xxx}+\beta_t+\alpha\beta_x+\alpha_x\beta)w+\beta^2z_xww'+\beta\beta_xw^2\\&\quad+\alpha_{xxx}+\alpha_t+\alpha\alpha_x=0.\end{aligned} \tag{C3}$$

We use the coefficient of w''' as the normalizing coefficient. For this to be an ordinary differential equation, from the coefficient of ww' we get

$$\beta z_x^3\Gamma(z)=\beta^2z_x,$$

where $\Gamma(z)$ is to be determined. Using the freedom in Remark 3(i) in Sec. III,

$$\beta=z_x^2. \tag{C4}$$

The coefficient of w^2 gives

$$\beta z_x^3\Gamma(z)=\beta\beta_x$$

where $\Gamma(z)$ is to be determined. Using (C4), integrating twice, and using the freedoms in Remarks 2 and 3(iii), we have

$$z=x\theta(t)+\sigma(t),\quad \beta=\theta^2(t), \tag{C5}$$

where $\theta(t)$ and $\sigma(t)$ are to be determined. Equation (C3) simplifies to

$$\begin{aligned}&\theta^5(w'''+ww')+\theta^2\left\{\left(x\frac{d\theta}{dt}+\frac{d\sigma}{dt}\right)+\alpha\theta\right\}w'\\&\quad+\left\{2\theta\frac{d\theta}{dt}+\alpha_x\theta^2\right\}w+\alpha_{xxx}+\alpha_t+\alpha\alpha_x=0.\end{aligned} \tag{C6}$$

The conditions for this to be an ordinary differential equation give successively, from the coefficients of w', w, and 1,

$$\alpha=-\frac{1}{\theta}\left(x\frac{d\theta}{dt}+\frac{d\sigma}{dt}\right), \tag{C7}$$

$$\frac{d\theta}{dt}=A\theta^3, \tag{C8}$$

$$\theta\frac{d^2\sigma}{dt^2}-2\frac{d\theta}{dt}\frac{d\sigma}{dt}=2\theta^6(A^2\sigma+B), \tag{C9}$$

with B another arbitrary constant.

Therefore the general similarity reduction of the Kortweg–de Vries equation (C1) is

$$u(x,t)=\theta^2(t)w(z)-\frac{1}{\theta}\left(x\frac{d\theta}{dt}+\frac{d\sigma}{dt}\right),$$

$$z=x\theta(t)+\sigma(t),$$

where $\theta(t)$ and $\sigma(t)$ satisfy (C8) and (C9).

There are two cases to consider.

Case 1. $A\neq 0$: We set $A=-\frac{1}{3}$, $B=0$. Thus

$$\theta=(t-t_0)^{-1/3},\quad \sigma(t)=c_1(t-t_0)^{2/3}+c_2(t-t_0)^{-1/3}.$$

We set $t_0=0$, $c_2=0$ and obtain the similarity reduction

$$u(x,t)=t^{-2/3}w(z)+\frac{x}{3t}-\frac{2}{3}c_1,\quad z=\frac{x+c_1t}{t^{1/3}}. \tag{C10}$$

Case 2. $A=0$. We set $\theta=1$, and then

$$\sigma(t)=Bt^2+c_3t+c_4.$$

Now set $c_4=0$ and obtain the similarity reduction

$$u(x,t)=w(z)-2Bt-c_3,\quad z=x+Bt^2+c_1t. \tag{C10$'$}$$

The infinitesimals for the Korteweg–de Vries equation obtained using the classical Lie method are

$$X=\alpha x+\beta t+\gamma,\quad T=3\alpha t+\delta,\qquad U=-2\alpha u+\beta, \tag{C11}$$

with α, β, γ, and δ arbitrary constants (cf. Ref. 16, p. 129, and Refs. 41–43). It is easily shown that both the similarity reductions (C10) and (C11) for the Korteweg–de Vries equation (C1) can be obtained from these infinitesimals (cf. Ref. 16, p. 196, and Ref. 43).

APPENDIX D: MODIFIED KORTEWEG–dE VRIES EQUATION

In this appendix we outline how to determine similarity reductions of the modified Korteweg–de Vries equation

$$u_t+u^2u_x+u_{xxx}=0, \tag{D1}$$

using the method developed in this paper. It suffices to assume

$$u(x,t)=\alpha(x,t)+\beta(x,t)w(z(x,t)). \tag{D2}$$

Substituting and collecting coefficients yields

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$$\beta z_x^3 w''' + (3\beta_x z_x^2 + 3\beta z_x z_{xx})w'' + (3\beta_{xx} z_x + 3\beta_x z_{xx} + \beta z_{xxx} + \beta z_t + \alpha^2 \beta z_x)w' + (\beta_{xxx} + \beta_t + \alpha^2 \beta_x + 2\alpha\alpha_x \beta)w + \beta^3 z_x w^2 w' + \beta^2 \beta_x w^3 + 2\alpha\beta^2 z_x w w' + (2\alpha\beta\beta_x + \alpha_x \beta^2)w^2 + \alpha_{xxx} + \alpha_t + \alpha\alpha_x = 0 . \quad \text{(D3)}$$

We use the coefficient of w''' as the normalizing coefficient. For this to be an ordinary differential equation for $w(z)$, from the coefficient of $w^2 w'$ we get

$$\beta z_x^3 \Gamma(z) = \beta^3 z_x ,$$

where $\Gamma(z)$ is to be determined. Using the freedom in Remark 3(i) of Sec. III,

$$\beta = z_x . \quad \text{(D4)}$$

The coefficient of w^3 gives

$$\beta z_x^3 \Gamma(z) = \beta^2 \beta_x ,$$

where $\Gamma(z)$ is to be determined. Using (D4), integrating twice, and using the freedoms in Remarks 2 and 3(iii), we have

$$z = x\theta(t) + \sigma(t), \quad \beta = \theta(t) , \quad \text{(D5)}$$

where $\theta(t)$ and $\sigma(t)$ are to be determined. The coefficient of ww' gives

$$\beta z_x^3 \Gamma(z) = 2\alpha\beta^2 z_x ,$$

where $\Gamma(z)$ is to be determined. Using (D4) and the freedom in Remark 3(i), we have

$$\alpha \equiv 0. \quad \text{(D6)}$$

Equation (D3) simplies to

$$\theta^4 (w''' + ww') + \theta\left(x \frac{d\theta}{dt} + \frac{d\sigma}{dt}\right)w' + \frac{d\theta}{dt} w = 0 . \quad \text{(D7)}$$

This is an ordinary differential equation for $w(z)$ provided that

$$\frac{d\theta}{dt} = A\theta^4 , \quad \text{(D8a)}$$

$$\frac{d\sigma}{dt} = \theta^3 (A\sigma + B) , \quad \text{(D8b)}$$

where A and B are arbitrary constants.

Therefore the general similarity reduction of the modified Kortweg–de Vries equation is

$$u(x,t) = \theta(t) w(z), \quad z = x\theta(t) + \sigma(t) ,$$

where $\theta(t)$ and $\sigma(t)$ satisfy Eqs. (D8).

There are two cases to consider.

Case 1. $A \neq 0$: We set $A = -\frac{1}{3}$, $B = 0$. Hence

$$\theta(t) = (t - t_0)^{-1/3}, \quad \sigma(t) = c_1 (t - t_0)^{-1/3} . \quad \text{(D9)}$$

Setting $t_0 = 0$, $c_1 = 0$, we obtain the similarity reduction

$$u(x,t) = t^{-1/3} w(z), \quad z = xt^{-1/3} . \quad \text{(D10)}$$

Case 2. $A = 0$: Solving (D8),

$$\theta(t) = c_2, \quad \sigma(t) = Bt + c_3 .$$

Setting $c_2 = 1$, $c_3 = 0$, we obtain the similarity reduction

$$u(x,t) = w(z), \quad z = x + Bt. \quad \text{(D11)}$$

The infinitesimals for the modified Korteweg–de Vries equation obtained using the classical Lie method are

$$X = \alpha x + \beta, \quad T = 3\alpha t + \gamma, \quad U = -2\alpha u , \quad \text{(D12)}$$

with α, β, and γ arbitrary constants (cf. Ref. 42). It is easily shown that both the similarity reductions (D10) and (D11) for the modified Korteweg–de Vries equation (D1) can be obtained from these infinitesimals (cf. 42).

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