MATH 4290 Geometry and Topology
Lecture Notes Winter 2005

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Contents

Chapter 1

Introduction to Topology

This mini-topic contains basic introduction to topology and topological spaces. It includes definitions, results, examples and problems on: topology, topological spaces, metrics, metric spaces, norm, inner product, normed and inner product spaces; accumulation points, closure, interior and boundary of a set; convergence, complete spaces, compactness.

1.1 Week 1

1.1.1 Sets, operations with and relations on sets, mappings.

We start by presenting the notations used in this text.

- We will denote sets by capital letters, such as A, B, S, K, X, Y; we use small letters for elements in the sets.
- In general, the definition of a set is given in the form:

$$A := \{a \in A \mid a \text{ has the property... }\},$$

  and we read "A is the set of all elements a, where a has the property...".
- We remind that by $a \in A$ we mean "the element a belongs to the set A" and similarly, by $a \notin A$ we mean "the element a does not belong to the set A".
- By $A \subseteq B$ we denote the **inclusion** of the set A in the set B; by $A \subset B$ we denote the **strict inclusion** of the set A in the set B.
- we denote by $\emptyset$ the set with no elements, also called **the empty set**.
- we denote by 2^A the collection of all subsets of a set A (also known as **the power set of** A);
- The following logic symbols will be used throughout the course:

$$\forall \text{ means "for any elements"; } \exists \text{ means "there exists an element";}$$

$$\exists! \text{ means "there exists a unique element";}$$

- We remind the main operations that can be performed with sets together with their symbols:
  -**union:** $\bigcup$
  -**intersection:** $\bigcap$
  -**difference:** $\setminus$
  -**complement:** $\mathbf{C}$ (a particular case of difference).

  The result of each of these operations are also sets, and their definition is given below:

$$\forall A, B,\ A \cup B := \{x \mid x \in A \text{ or } x \in B\};$$

$$\forall A, B,\ A \cap B := \{x \mid x \in A \text{ and } x \in B\};$$

$$\forall A, B,\ A \setminus B := \{x \mid x \in A \text{ and } x \notin B\}$$

$$\forall A \subseteq B,\ \mathbf{C}_B(A) := \{x \mid x \in B \text{ and } x \notin A\}.$$

Definition 1.1.1 *Given any two sets A and B, we can define* **the cartesian product** *of the two sets by*

$$A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}.,$$

i.e., **the set of ordered pairs of elements from A and B.**

We remind briefly here that we denote by $f : A \to B$ a mapping between the sets A and B. A is the **domain** of f and B is the **range**. We denote by

$$f^{-1}(b) := \{a \in A \mid f(a) = b\}, \text{ the pre-image of } b \ .$$

For a quick reference to mappings, composition of mappings, injective (1-to-1), surjective (onto) and bijective mappings, the reader is directed to [1], Ch. 1.

1.1.2 Zorn Lemma.

There are two relations that can exist between the elements of a set; these are:
- **the equivalence relation or (RST)**
- **the order relation or (RAT)**.

We have studied equivalence relations, for example, in MATH 3130 (for a quick reminder and examples please see Algebraic Structures, see [2], Ch. 2, Section 4, page 57 on) and we have seen that they are extremely important in the study of subgroups and ideals. We will only remind here the definition of an RST. Let A be a set with elements $a, b, c, \ldots$.. In what follows, for two arbitrarily chosen elements $a, b \in A$, by $a \sim b$ we shall understand "the element a is in relation with the element b", or "a is equivalent to b".

Definition 1.1.2 *A relation $\sim$ between elements of a set A is an* **equivalence** *(or* RST*) if and only if it satisfies the following three properties:*
1) R:=reflexivity: *$\forall a \in A$, we have that $a \sim a$;*
2) S:=symmetry: *$\forall a, b \in A$, we have that $a \sim b$ implies $b \sim a$;*
3) T:=transitivity: *$\forall a, b, c \in A$, if $a \sim b$ and $b \sim c$, then they imply $a \sim c$.*

We concentrate next on learning some things about RAT relations. An order relation is customary denoted by $\leq$. In what follows, for two arbitrarily chosen elements $a, b \in A$, by $a \leq b$ we shall understand "the element a precedes the element b" or "a is less than b".

Definition 1.1.3 *A relation $\leq$ between elements of a set A is an* **order** *(or* RAT*) if and only if it satisfies the following three properties:*
1) R:=reflexivity: *$\forall a \in A$, we have that $a \leq a$;*
2) A:=antisymmetry: *$\forall a, b \in A$, it is never the case that both $a \leq b$ and $b \leq a$ are true;*
3) T:=transitivity: *$\forall a, b, c \in A$, if $a \leq b$ and $b \leq c$, then they imply $a \leq c$.*

Definition 1.1.4 *A relation $\leq$ is called* **total (or linear) order** *if it satisfies:*

$$\text{if } x \leq y \text{ and } y \leq x, \text{ then } x = y, \text{ and}$$

$$x < y \text{ or } y > x \text{ for any distinct } x, y \in A.$$

A set A together with a total order relation is called a **totally ordered set**.

Definition 1.1.5 *Suppose that $\leq$ is an order relation on a set X. Then an element $x \in X$ is called* **maximal** *if any other element $y \in X$, comparable to x, satisfies $y \leq x$.*

We remind now the definition of the least upper bound of a set. This should be known to the reader from Vector Calculus and/or Real Analysis courses. Here is the definition in the most general context.

Definition 1.1.6 *Suppose that A, X are two sets such that $A \subseteq X$. Suppose that $\leq$ is an order relation on X. Then:*
1) An element $x \in X$ is an **upper bound of** *A if and only if for each $a \in A$, $a \leq x$; notice that if an upper bound exists, it is not necessarily unique.*
2) An element $x \in X$ is a **least upper bound of** *A or* **supremum of** *A if it is an upper bound and is less than or equal to every other upper bound of A.*

Definition 1.1.7 *A set X with a linear order $\leq$ is called* **order-complete** *if any nonempty totally ordered subset $A \subseteq X$ has a supremum.*

Lemma 1.1.1 (Zorn Lemma:) *Let X be an order-complete set. Then X possesses at least one maximal element. In other words, if $x_0 \in X$ is arbitrarily fixed and X is order-complete, then there exists a maximal element in X so that $x_0 \leq m$.*

Proof:[*This proof could be requested for bonus marks*] Since X is order-complete, then every nonempty totally ordered subset A has a supremum. The **Maximum Principle** states that the collection of all nonempty totally ordered subsets $A \subset X$ has a maximal element, say B. This means that any nonempty, totally ordered subset of X is included or is at most equal to B. Obviously then $B \neq \emptyset$ and is totally ordered in X, hence B has a supremum, say $m \in X$. Let $x_0 \in X$ arbitrarily chosen.
Claim: x_0 belongs to one of the nonempty, totally ordered subsets of X.
If the Claim is true, then obviously $x_0 \in B$ and since $m := \sup B$, then we have $x_0 \leq m$.
So all that is left to do is to prove that the Claim is true. Let

$$\mathcal{A} := \{A \mid A \text{ is a nonempty, totally ordered subset of } X\}$$

and suppose there exist at least two elements $x, y \in X \backslash \mathcal{A}$. Then we cannot have $x < y$ or $x > y$, otherwise the set $\{x, y\}$ is a totally ordered subset of X and hence $\{x, y\} \in \mathcal{A}$, which is a contradiction with the assumption $x, y \in X \backslash \mathcal{A}$. The the only possibility is $x = y$, in which case the set $\{x\}$ is a nonempty, totally ordered subset of X, hence $x \in \mathcal{A}$ and the Claim is true. □

1.1.3 Topology, topological space, open and closed sets, neighbourhood.

Definition 1.1.8 *Let X be a set and let τ be a collection of subsets of X (i.e., $\tau \subset 2^X$). Then τ is called* **a topology on** X *if it satisfies the following conditions:*

- $\emptyset \in \tau$ *and* $X \in \tau$;
- $\forall\ D_1, D_2 \in \tau$, *we have that* $D_1 \cap D_2 \in \tau$;
- $\forall\ D_i \in \tau$, $i \in I$, *we have that* $\bigcup\limits_i D_i \in \tau$.

Definition 1.1.9 *The pair (X, τ) is called a* **topological space**.

Remark 1.1.1 It is important to note that a set X can have multiple topologies defined on it, hence multiple structures of topological spaces (see for example Examples 1.1.4 and 1.1.5 in the next section.)

A topology τ divides the subsets of the space into two distinct parts. The subsets $D \in \tau$ are called **open sets relative to** τ. The subsets $\mathbf{C}_X D$ are called **closed relative to** τ. We denote by $\tau^{\mathbf{C}}$ the collection of all closed sets in X, i.e.,

$$\tau^{\mathbf{C}} := \{\mathbf{C}_X D \mid D \in \tau\}.$$

Definition 1.1.10 *Let (X, τ) be a topological space and $x \in X$ an arbitrarily fixed point. A subset $V \subseteq X$ is called a* **neighbourhood of** x *if*

$$\exists\ D \in \tau \text{ so that } x \in D \subseteq V.$$

The point x can have more than one neighbourhood; hence **the set of all neighbourhoods of the point** x *is denoted by* $\mathcal{V}(x)$.

For example, the set $\{a, c\}$ in Example 1.1.4 is a neighbourhood of a in X.

Theorem 1.1.1 below gives the first examples of neighbourhoods of points in a topological spaces.

Theorem 1.1.1 *A set $A \in X$ is open if and only if it contains a neighbourhood of each of its points.*

Proof: ($\rightarrow$:) Let A be open. Then for each $x \in A$, A is a neighbourhood of x (by direct application of Definition 1.1.9).

($\leftarrow$:) Suppose $x \in A$ arbitrarily fixed and A contains a neighbourhood of x, i.e. $\exists V$ s.t. $V \subseteq A$. But this implies (by Definition 1.9) that there exists an open set D so that $x \in D \subseteq V \subseteq A$. Since this is true for any $x \in A$, then $A \subseteq \bigcup\limits_{x\in A} D \subseteq \bigcup\limits_{x\in A} A = A$ which implies $A = \bigcup\limits_{x\in A} D$. Since any union of open sets is open, then we have that A is open. □

Remark 1.1.2 Let X, τ be a topological space and $x_0 \in X$ an arbitrarily chosen point. Then any open set relative to τ that includes x_0 is a neighbourhood of x_0.

1.1.4 Examples and Problems

Example 1.1.1 *The set of integers with the usual order is a totally ordered set.*

Example 1.1.2 *Let X be a set and 2^X the collection of all subsets of this set. Then 2^X with the operation of inclusion of sets is a totally ordered set.*

Example 1.1.3 *Let E a vector space. Then E possesses a maximal linear independent set (any of its algebraic bases).*

Example 1.1.4 *Let the set $X = \{a, b, c\}$ and suppose we choose τ to be the following collection of subsets of X:*

$$\tau = \{\emptyset, \{a\}, \{a, b\}, X\}.$$

Then τ is a topology on X and (X, τ) is a topological space. Now choose $\tilde{\tau}$ to be the following collection of subsets:

$$\tilde{\tau} = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}.$$

Then $\tilde{\tau}$ is also a topology on X.

Example 1.1.5 *Let X be a set. Then the following collections of subsets form topologies on X:*
1) $\tau_d = \{$ all subsets of X $\}$;
2) $\tau_i = \{\emptyset, X\}$. The topology τ_d is also called the **discrete topology**, *while τ_i is called* **the indiscrete** *or* **trivial topology**.

Example 1.1.6 *Let X be a set and τ a topology on X. Let $A \subset X$. Then the collection*

$$\tau_A := \{A \cap D \mid D \in \tau\}$$

is called **the subspace topology**.

Example 1.1.7 *Let (X, τ_X) and (Y, τ_Y) two topological spaces. Then on the cartesian product $X \times Y$ we can define a topology by taking the collection of sets:*

$$\tau_{X \times Y} := \{D \mid D = D_X \times D_Y,\ D_X \in \tau_X,\ D_Y \in \tau_Y\}.$$

The topology $\tau_{X \times Y}$ is called **the product topology**.

Example 1.1.8 *Let X be a set with a total order relation. Suppose that X has more than one element. The sets*

$$(a, b) = \{x \in X \mid a < x < b\}$$

are called order intervals. Then

$$\tau := \{ \text{ the collection of all unions of subsets of the form (a,b)}\}$$

forms a topology on X, called **the order topology**.

An immediate illustration of this example is the following: take $X := \mathbb{R}$ and the usual order between real numbers. Then the order topology on $\mathbb{R}$ is that given by the collection of all unions of open intervals. The order topology on $\mathbb{R}$ is also called **standard topology**.

Problem 1.1.1 *What is the difference between the definition of $A \backslash B$ and that of $\mathbf{C}_B(A)$? When do they coincide?*

Problem 1.1.2 *Let A, B, S three sets. Then show that the following hold:*
1) $A \cap (B \backslash S) = (A \cap B) \backslash S$;
2) if $B \subseteq A$, then $(A \backslash B) \cap S = (A \cap S) \backslash (B \cap S)$.

Problem 1.1.3 *Show that $\leq$ is an order relation on $\mathbb{R}$, where this is the usual order between real numbers. Is this a linear order?*

Problem 1.1.4 *Let $\mathbb{C}$ be the set of complex numbers and let $\leq$ be defined by $z_1 \leq z_2 \in \mathbb{C}$ if and only if $|z_1| < |z_2|$. Is this RAT?*

Problem 1.1.5 *If X, Y are two sets and $f : X \to Y$ a mapping, then show that*
1) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$;
2) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$,
3) $f^{-1}(\mathbb{C}_Y A) = \mathbb{C}_X f^{-1}(A)$
where $A, B \subseteq Y$, $A \neq B$ and $f^{-1}(A)$ is the inverse image of the set A through f.

Problem 1.1.6 *Prove DeMorgan formulae:*

1) $X \backslash \bigcap_{i \in I} A_i = \bigcup_{i \in I} (X \backslash A_i)$
2) $X \backslash \bigcup_{i \in I} A_i = \bigcap_{i \in I} (X \backslash A_i)$

Problem 1.1.7 *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Show that τ is a topology on X.*

Problem 1.1.8 *Let X be a set and let*

$$\tau_f = \{U \subseteq X \mid \mathbf{C}_X U \text{ is either finite or is equal to } X\}.$$

Show that τ_f is a topology on X.
(This topology is called **the finite complement topology***).*

Problem 1.1.9 *Let $A = [1, 2] \subset \mathbb{R}$. Write the subspace topology on A, induced by the standard topology of $\mathbb{R}$.*

Problem 1.1.10 *Give examples of product topologies on the sets: $\mathbb{R}^2$, $\mathbb{R}^3$,...,$\mathbb{R}^n$.*

1.2 Week 2

1.2.1 Accumulation points, closure, interior and boundary of a set.

Definition 1.2.1 *Let (X, τ) be a topological space, A a nonempty subset of X and $x \in X$. Then the point x is called* **an accumulation (cluster or limit) point of** *A if and only if*

$$\forall V \in \mathcal{V}(x), \text{ we have that } A \cap (V \backslash \{x\}) \neq \emptyset.$$

We denote by A' **the set of all accumulation points of** *A.*

Remark 1.2.1 It is important to note that not all accumulation points of A actually belong to A; in fact, there are sets A which do not contain (any of) their accumulation point(s) - see Example 1.2.1.

Theorem 1.2.1 *Let (X, τ) be a topological space and $A \subseteq X$. Then $A \bigcup A'$ is always a closed set.*

Proof: [*Sketch*] Suppose $x \in X \backslash (A \bigcup A')$. Then there exists at least one neighbourhood $V \in \mathcal{V}(x)$ such that $A \bigcap V = \emptyset$. Since V is a neighbourhood of each of its points, then no point of V is an accumulation point of A. Hence $A \bigcup A'$ is the complement of an open set, thus is closed. □

Definition 1.2.2 *Let (X, τ) be a topological space and A a nonempty subset of X. Then the smallest closed set in X that contains A is called* **the closure (or τ-closure) of** *A and is denoted by A^c or $\bar{A}$.*

Remark 1.2.2 The following is true: A set $A \subseteq X$ in a topological space (X, τ) is closed if and only if $A = \bar{A}$.

Theorem 1.2.2 below shows the relation between a set, the set of its accumulation points and its closure.

Theorem 1.2.2 *Let (X, τ) be a topological space and A any nonempty subset of X. Then the following holds:*

$$\bar{A} = A \bigcup A'.$$

Proof: To prove this set equality, we need to show that $\bar{A} \subseteq A \bigcup A'$ and also that $\bar{A} \supseteq A \bigcup A'$.
$(\rightarrow)$: Based on Theorem 1.2.1, we know that $A \bigcup A'$ is a closed set and obviously contains A. Since $\bar{A}$ is by Definition 1.2.2 the smallest closed set that contains A, then $\bar{A} \subseteq A \bigcup A'$.
$(\leftarrow)$: Obviously $A \subseteq \bar{A}$ by Definition 1.2.2. (*).

We prove $A' \subset \bar{A}$ by contradiction. Let $x \in A'$ and assume that $x \notin \bar{A}$. Since x is an accumulation point of A, then $\forall V \in \mathcal{V}(x)$, we have $A \cap (V \backslash \{x\}) \neq \emptyset$. (**)

From our assumption that $x \notin \bar{A}$, we get $x \in X \backslash \bar{A}$, which is an open set (since $\bar{A}$ is closed). Thus $X \backslash \bar{A}$ is a neighbourhood of any of its points, and so there exists D open so that

$$x \in D \subseteq X \backslash \bar{A}. \tag{1.1}$$

From set theory we have that the following is true: $A \subseteq \bar{A} \implies X \backslash \bar{A} \subseteq X \backslash A$. From (1.1) we get then

$$x \in D \subseteq X \backslash \bar{A} \subseteq X \backslash A$$

which implies that

$$D \cap A = \emptyset, \text{ hence } (D \backslash \{x\}) \cap A = \emptyset \qquad (***).$$

But (**) and (***) give a contradiction, since every open set like D is a neighbourhood of x. This means that any $x \in A'$ belongs to $\bar{A}$; together with (*), we have $A \cup A' \subseteq \bar{A}$ and the proof is complete. □

Definition 1.2.3 *Let (X, τ) be a topological space, A a nonempty subset of X and $x \in A$. Then x is called* **an interior point of** A *if A is a neighbourhood of x. The set of all interior points is denoted by $\mathring{A}$.*

Theorem 1.2.3 *Let (X, τ) be a topological space and A any nonempty subset of X. The set $\mathring{A}$ is always open and is the largest open subset of A.*

Proof: If $x \in \mathring{A}$ arbitrarily chosen, then A is a neighbourhood of x, i.e., there exists an open set D such that $x \in D \subseteq A$. Since every element in D is also an element of $\mathring{A}$, then $\mathring{A}$ is a neighbourhood of any of its points, so it is open. If V is an open set of A, then A is a neighbourhood of any point of V, hence $V \subset \mathring{A}$. So $\mathring{A}$ is open and contains all open subsets of A, hence it is the largest open subset of A.

□

Definition 1.2.4 *Let (X, τ) be a topological space, A a nonempty subset of X and $x \in A$. Then x is called* **a boundary point of** A

$$\text{if } \forall V \in \mathcal{V}(x) \text{ both } A \bigcap V \neq \emptyset \ \& \ (X \backslash A) \bigcap V \neq \emptyset.$$

The set of all boundary points is denoted by ∂A and is called **the boundary of the set** A.

An easier way to remember the expression of the boundary of a set is through its relation with its closure and its interior.

Theorem 1.2.4 *Let (X, τ) be a topological space, A a nonempty subset of X. Then*

$$\partial A = \bar{A} \backslash \mathring{A}.$$

Proof:[Sketch] We need to show both $\partial A \subseteq \bar{A}\backslash\mathring{A}$ and $\bar{A}\backslash\mathring{A} \subseteq \partial A$. We show here the implication
$(\rightarrow)$: Let $z \in \partial A$ arbitrarily fixed. Then our hypothesis is that for any $V \in \mathcal{V}(z)$, we have both $A \cap V \neq \emptyset$ and $(X\backslash A) \cap V \neq \emptyset$.

We want to show that $z \in \bar{A}\backslash\mathring{A} = (A \cup A')\backslash\mathring{A}$, or equivalently,

$$\{z \in A \text{ or } z \in A'\} \text{ and } \{z \notin \mathring{A}\}.$$

Assume that $z \notin A'$; this implies that there exists at least one $V \in \mathcal{V}(z)$ so that $(V\backslash\{z\}) \cap A = \emptyset$. But from the hypothesis we have that $A \cap V \neq \emptyset$, so we can only deduce that z is the only common element of V and A, so we get that $z \in A$ and obviously $z \in A \cup A'$.

Also, if we assume that $z \notin A$, since $A \cap V \neq \emptyset$ for any $V \in \mathcal{V}(z)$, then $z \in (V\backslash\{z\}) \cap A$ and so $V\backslash\{z\}) \cap A \neq \emptyset$, hence $z \in A'$, which immediately implies $z \in A \cup A'$.

If we assume $z \notin A$ and $z \notin A'$, then we arrive at $A \cap V = \emptyset$ for any $V \in \mathcal{V}(z)$, which contradicts the hypothesis.

So, the conclusion so far is that any $z \in \partial A$ has to belong to $A \cup A'$. TO complete the proof of this part of the theorem, we have to show that any $z \in \partial A$ does not belong to $\mathring{A}$.

We prove this by contradiction, so assume $z \in \mathring{A}$; this means A is a neighbourhood of z and so $\exists D$ open so that

$$x \in D \subseteq A.$$

But this implies $D \cap (X\backslash A) = \emptyset$, which is a contradiction with our hypothesis. So $z \notin \mathring{A}$ and the proof is complete.

$(\leftarrow)$: Homework. □

In general, our perception of topological spaces is that of $\mathbb{R}$ with $\tau_{\mathbb{R}}$. In this topology for example, all singletons, i.e., all sets of one element $\{x\}$, are closed, since their complement is an open set. Also, we know from previous courses that any convergent sequence has a unique limit.

However, to think that these two facts are true in **all** topological spaces is wrong. Take for example the topological space $X = \{a, b, c\}$ and the topology $\{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$. Then the singleton $\{b\}$ is open, and is certainly not closed, since its complement $\{a, c\}$ is neither open nor closed.

So, what are the topological spaces similar (in the sense of having the two properties highlighted above) to $(\mathbb{R}, \tau_{\mathbb{R}})$? The following definiton answer our question.

Definition 1.2.5 *Let (X, τ) be a topological space. The space X is called* **separate** *or* **Hausdorff** *if it satisfies the following property:*

$$\forall\ x, y \in X, x \neq y, \exists\ U \in \mathcal{V}(x),\ \exists V \in \mathcal{V}(y) \text{ s.t. } \bar{U} \cap \bar{V} = \emptyset.$$

1.2.2 Examples and Problems

Example 1.2.1 *a) Consider* $X = \mathbb{R}$ *with the standard topology and* $A := (0, 1]$*. All points of* A *are accumulation points. Moreover,* $x = \{0\}$ *is an accumulation point of* A*, because any neighbourhood* V *of* 0 *in* $\mathbb{R}$*, in the standard topology, contains an open interval* (a, b) *such that* $0 \in (a, b)$*. Obviously, any such* V *has the property that* $(0, 1] \cap (V - \{0\}) \neq \emptyset$*.*

b) Now consider $A = \{\frac{1}{n} \mid n \in \mathbb{Z}_+^*\}$*. Then* 0 *is the only accumulation point of* A*.*

Example 1.2.2

1. *The closure of the set* $\mathbb{Q}$ *is* $\mathbb{R}$*, i.e.,* $\bar{\mathbb{Q}} = \mathbb{R}$*.*
2. *The closure of the set* $\bar{C} = \{0\} \cup (1, 2)$ *is the set* $C = \{0\} \cup [1, 2]$*.*

Example 1.2.3 *Let* $Y = (0, 1] \subset \mathbb{R}$ *with the subspace topology of* $\mathbb{R}$*. The closure of* $A = (0, 1/2)$ *in* $\mathbb{R}$ *is* $[0, 1/2]$ *and the closure of* A *in* Y *is* $[0, 1/2] \cap Y = (0, 1/2]$*.*

Problem 1.2.1 *Show that the set* $\mathbb{R}_+^2$ *is a closed set in the standard topology of* $\mathbb{R}^2$*.*

Problem 1.2.2 *Show that in the discrete topology of any set* X*, every subset is both open and closed.*

Problem 1.2.3 *Using de Morgan's formulae (see Problem 1.1.6, previous week), show that given a topological space* (X, τ)*, the collection of closed subsets of* X *with respect to* τ *have the following properties:*

1. *the sets* ϕ*,* X *are closed;*
2. *arbitrary intersections of closed sets are closed;*
3. *finite unions of closed sets are closed.*

Problem 1.2.4 *Let* $Y = [-1, 1]$ *in* $\mathbb{R}$ *with the subspace topology induced from* $\mathbb{R}$*. By* $| \ |$ *we denote the absolute value. Are the sets*

1. $A = \{x \mid 1/2 < |x| < 1\}$*;*
2. $B = \{x \mid 1/2 \leq |x| < 1\}$

open or closed in Y *with the subspace topology? Are they open or closed in* $\mathbb{R}$ *with the standard topology?*

Problem 1.2.5 *Prove that a subset* A *of a topological space* (X, τ) *is closed if and only if* A *contains its accumulation points.*

Problem 1.2.6 *Prove that for any nonempty subsets* A, B *of a topological space* (X, τ) *the following hold:*
1) $\bar{\bar{A}} = \bar{A}$*;*
2) $\overline{(A \bigcup B)} = \bar{A} \bigcup \bar{B}$*.*

Problem 1.2.7 *Let* (X, τ) *be a topological space and* A *any nonempty subset of* X*. Then the following hold:*
1) a set A *is open if and only if* $A = \mathring{A}$*;*
2) the set of all points of A *which are not points of accumulation of* $X \backslash A$ *is precisely* $\mathring{A}$*.*

Problem 1.2.8 *Let* (X, τ) *be a topological space and* A *any nonempty subset of* X*. Then the following hold:*
1) $\partial A = \bar{A} \bigcap \overline{(X \backslash A)}$*; 2)* $\bar{A} = A \bigcup \partial A$*; 3)* $\mathring{A} = A \backslash \partial A$*.*

Problem 1.2.9 *Find the boundary and the interior of each of the sets in* $\mathbb{R}^2$*, with the standard topology:*

- $A = \{(x, y) \mid y = 0\}$*;*
- $B = \{(x, y) \mid x > 0, y \neq 0\}$*;*
- $C = A \cup B$*;*
- $D = \{(x, y) \mid 0 < x^2 + y^2 \leq 1\}$*.*

1.3 Week 3

1.3.1 Limits, convergence and continuity in topological spaces

We have seen so far that a topology on a set defines what we called a topological structure. We have encountered elements of this structure in previous courses, without perhaps insisting on where they came from. Although a topology is defined by relatively simple rules that have to be satisfied by a certain collection of subsets of the underlying space, notions well-known to us can be defined on such spaces in their most general (topological) way.

The mathematical notions we worked with over the years are limits, convergence, continuity, differentiability, bounded sets/functions, compactness, convexity etc. We will see next that some of them can be defined on topological spaces, while others require more "structure" on the underlying space than just a topology. These extra "structures" can be metrics, norms and inner products. As we go along in the study, we see how some of these structures relate to each other.

*

By a sequence of elements in the topological space (X, τ) we understand a function from the integers $\mathbb{N} := \{0, 1, 2, 3, ..., n...\}$ to the space X, given by $n \mapsto x_n$; sequences in X will be denoted by $\{x_n\}_{n\geq 0}$ or $\{x_n\}_n$.

Definition 1.3.1 *Let (X, τ) be a topological space and $\{x_n\}_n$ a sequence in X and $x \in X$. The sequence $\{x_n\}_n$ is said to have the limit x if $\forall\ V \in \mathcal{V}(x)$, $\exists\ n_0 > 0$ so that $\forall\ n \geq n_0$, we have $x_n \in V$.*

The definition can be also remembered in the following way: the sequence $\{x_n\}_n$ has the limit x, if all the elements of the sequence can be found in any neighbourhood of the point x, except maybe a finite number of them. In other words, if x is a limit of the sequence, the elements x_n of the sequence are "extremely crowded" around x.

As expected, a sequence is called **convergent** if it has a limit.

Proposition 1.3.1 *Let (X, τ) be a topological space and $A \subseteq X$ a closed subset. Then given any convergent sequence $\{x_n\}_n \in A$ of elements from A, its limit also belongs to A.*

Proof: Let $\{x_n\}_n \in A$ such that $x_n \to x^*$, whenever $n \to \infty$. Since x^* is the limit of x_n, then for any neighbourhood $V \in \mathcal{V}(x^*)$, we have that there exists an infinite number of elements of the sequence $\{x_n\}_n \in V$, hence $V - \{x^*\} \cap A \neq \emptyset$. This means that $x^* \in A'$. Since A is closed, we have that

$$A = \bar{A} = A \cup A',$$

so if $x^* \in A'$, then certainly $x^* \in A$. $\square$

Here is the first very interesting fact: the limit of a sequence in a topological space may not be unique! The following result tells us more:

Proposition 1.3.2 *Let (X, τ) be a Hausdorff topological space and $\{x_n\}_n$ a convergent sequence in X. Then the limit is unique.*

Proof: Suppose $\{x_n\}_n$ is convergent to both $x, y \in X$ and that $x \neq y$. Then from the definition of the Hausdorff space, there exist neighbourhoods U, V of x and respectively y such that $\bar{U}$ and $\bar{V}$ are distinct. Since both x, y are limits, from Definition 1.3.1, we have that there exist $n_x > 0$ and $n_y > 0$ so that $\forall\, n \geq max\{n_x, n_y\}$ we have that $x_n \in U$ and $x_n \in V$. But this contradicts the fact that U and V are distinct; the contradiction comes from our supposition that $\{x_n\}_n$ converges to two distinct limits, hence $x = y$ and the limit is unique. $\square$

We now introduce the definition of the limit of a mapping in a topological space.

Definition 1.3.2 *Let (X, τ_X), (Y, τ_Y) be two topological spaces, let $A \subseteq X$, $x_0 \in A'$ and $f : A \to Y$ a mapping. We say that* **f has the limit** y_0 **at the point** x_0, *i.e.,* $\lim\limits_{x \to x_0} f(x) = y_0$, *if*

$$\forall\, V \in \mathcal{V}(y_0),\ \exists U \in \mathcal{V}(x_0)\ s.t.\ f(A \cap (U \backslash \{x_0\})) \subseteq V.$$

Based on this definition, it is intuitively easy to guess the definition of a continuous mapping f at the point x_0, namely:

Definition 1.3.3 *Let (X, τ_X), (Y, τ_Y) be two topological spaces, $A \subseteq X$ and $f : A \to Y$ a mapping, $x_0 \in A \bigcup A'$. We say that* **f is continuous at the point** x_0, *if*

$$\forall\, V \in \mathcal{V}(f(x_0)),\ \exists U \in \mathcal{V}(x_0)\ s.t.\ f(U) \subseteq V.$$

In other words, f is continuous at x_0 if and only if $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Remark 1.3.1 Notice that the definitions of limit of a mapping at a point and that of continuity make sense only for points in $A \bigcup A'$.

In general, to check continuity of a mapping f, one can use Definition 1.3.3 or one can use alternative criteria, such as the ones given by the theorem below.

Theorem 1.3.1 *Let (X, τ_X), (Y, τ_Y) two topological spaces, $f : X \to Y$ and τ_X^c, τ_Y^c be the collection of all closed subsets of X, respectively Y, with respect to the topologies τ_X and τ_Y. Then the following statements are equivalent:*
1) f is continuous on X;
2) $\forall\, G \in \tau_Y$ we have that $f^{-1}(G) \in \tau_X$;
3) $\forall\, F \in \tau_Y^c$, we have that $f^{-1}(F) \in \tau_X^c$;
4) $\forall A \subseteq X,\ f(\bar{A}) \subseteq \overline{f(A)}$;
5) $\forall B \subseteq Y,\ f^{-1}(\bar{B}) \supseteq \overline{f^{-1}(B)}$.

Proof: 1) $\Longrightarrow$ 2) : Let $G \in \tau_Y$ arbitrarily fixed. To show that $f^{-1}(G) \in \tau_X$, we have to show that it is open, i.e., that it is a neighbourhood of each of its points.

We know that f is continuous on X and $f^{-1}(G) \subseteq X$, therefore f is continuous on $f^{-1}(G)$. Let $z \in f^{-1}(G)$ arbitrarily chosen. Then from the continuity of f, for any neighbourhood $V \in \mathcal{V}(f(z))$, there exists a neighbourhood $U \in \mathcal{V}(z)$ such that $f(U) \subset V$.

But since $z \in f^{-1}(G) \implies f(z) \in G$ and G is open in the topology τ_Y, then G is a neighbourhood of each of its points, including a neighbourhood of $f(z)$; hence from the continuity of f, we can choose $V := G \in \mathcal{V}(f(z))$ and so there exists $U \in \mathcal{V}(z)$ so that $f(U) \subset G$. But this is the same as $U \subset f^{-1}(G)$.

So we showed that for each $z \in f^{-1}(G)$, there exists $U \in \mathcal{V}(z)$ such that $U \subset f^{-1}(G)$, hence $f^{-1}(G)$ is open.

2) $\Longrightarrow$ 1) : Conversely, let us suppose that $f^{-1}(G) \in \tau_X$, for any $G \in \tau_Y$. Then for any $x \in X$, and any $V \in \mathcal{V}(f(x))$, there exists $U := f^{-1}(V)$ open and so that $x \in U$ and $f(U) \subset V$. Therefore f is continuous on X.

1) $\Longrightarrow$ 3) : Let $F \in \tau_Y^c$; then $\mathbb{C}_Y F = Y \backslash F$ is an open set. Since f is continuous, then using part 2) above, we get that $f^{-1}(Y \backslash F)$ is an open set in X. According to Problem 1.1.5, part 3), we have that

$$f^{-1}(Y \backslash F) = X \backslash f^{-1}(F) \in \tau_X.$$

In other words, $f^{-1}(F) \in \tau_X^c$. □

We introduce next a fundamental definition that is of use in the Differential Geometry part of the course. We studied in Algebraic Structures the notion of **isomorphism** between algebraic objects, such as groups and rings. An isomorphism is a bijective (1-to-1) correspondence that preserves the algebraic structure involved.

The topological counterpart of isomorphisms are **homeomorphisms**.

Definition 1.3.4 *Let (X, τ_X), (Y, τ_Y) be two topological spaces and $f : X \to Y$. Then f is called a* **homeomorphism** *if f is continuous, bijective and f^{-1} is continuous.*

Thus a homeomorphism is a bijective (1-to-1) correspondence between two topological spaces that preserves the topological structure (i.e., takes open sets in X to open sets in Y and viceversa.)

Lemma 1.3.1 *Let (X, τ_X), (Y, τ_Y) two topological spaces, and let $f : X \to Y$ be injective, continuous with $f^{-1} : f(X) \to X$ also continuous. Then $f : X \to f(X)$ is a homeomorphism. This homeomorphism is called* **imbedding of X into Y**.

Proof: Since f is injective, then we claim that $f : X \to f(X)$ is bijective. To prove the claim, we suppose $f : X \to f(X)$ is not bijective, which implies that $f : X \to f(X)$ is not surjective; this is equivalent to saying that there exists $y \in f(X)$ for which there is no $x \in X$ with $f(x) = y$; but this cannot be, since $y \in f(X)$ and $f(X)$ is, by definition, the subset of Y that contains all images of elements from X through f. So, $f : X \to f(X)$ is bijective and continuous hence is a homeomorphism. □

1.3.2 Examples and Problems

Example 1.3.1 *Give examples of continuous mappings defined on:*
1) $f_1 : \mathbb{R} \to \mathbb{R}$*;*
2) $f_2 : \mathbb{R}^n \to \mathbb{R},\ n = 1, 2, 3...$*;*
3) $f_3 : \mathbb{R}^n \to \mathbb{R}^m$*, for your choice of* $m \neq n \in \{2, 3, 4...\}$*.*

Example 1.3.2 *Here are some examples of homeomorphisms:*
1) $f : \mathbb{R} \to \mathbb{R}$*,* $f(x) = x + 1$*. Then* f *is continuous, is bijective and* $f^{-1}(y) = y - 1$*.*
2) $g : (-1, 1) \to \mathbb{R}$*,* $g(x) = \frac{x}{1-x^2}$*. Then* g *is continuous, bijective and* $g^{-1}(y) = \frac{2y}{1+(1+4y^2)^{1/2}}$*.*

Example 1.3.3 *There are plenty of examples of homeomorphisms in the study of differential geometry of surfaces in* $\mathbb{R}^3$*. One example is the mapping*

$$f : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$$

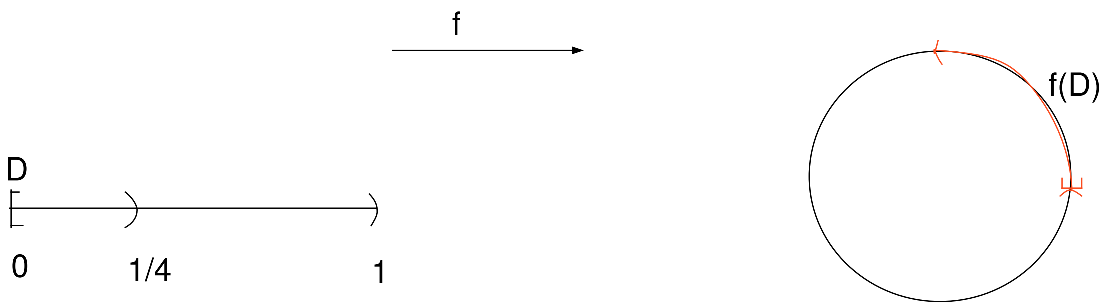
$$f(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v), r \sin u),$$

where if we take $f : [0, 2\pi] \times [0, 2\pi] \to f([0, 2\pi] \times [0, 2\pi])$ *we have that* f *is a homeomorphism. In fact is a famous one, showing that the rectangle* $[0, 2\pi] \times [0, 2\pi]$ *is homeomorphic with a torus, which is a surface in* $\mathbb{R}^3$*, obtained, for example, by rotating a circle in the* (x, z)*-plane around the* z*-axis. The mapping* f *is in fact an imbedding of the torus.*

Not all continuous, bijective mappings are homeomorphisms. Here is an example:

Example 1.3.4 *Let* $f : [0, 1) \to S^1$*, given by* $f(t) = (\cos 2\pi t, \sin 2\pi t)$*. Here* f *is continuous and 1-to-1, but* f^{-1} *is not continuous. Let's see why. Take any open set* D *in* $[0, 1)$ *with the subspace topology given by* $\tau_{\mathbb{R}}$*. If* f^{-1} *is continuous, then by Theorem 1.3.1,* $(f^{-1})^{-1}(D) = f(D)$ *should be open in* S^1 *with the subspace topology given by* $\tau_{\mathbb{R}^2}$*.*

Now take $D := [0, 1/4)$*; then* $f(D)$ *is the first quarter of the circle* S^1*, closed at* $f(0)$ *and open at* $f(1/4)$*, like in the picture below: But there is no open set* $V \in \mathbb{R}^{\nvDash}$ *that contains the*



point $f(0)$ *so that* $V \cap S^1 \subset D$*, so* f^{-1} *is not continuous.*

Problem 1.3.1 *Let* (X, τ_d)*, where* τ_d *is the discrete topology. Is this a Hausdorff space? Motivate your answer.*

Problem 1.3.2 *Prove* Theorem 1.3.1, part 3) $\implies$ 1).

Problem 1.3.3 *Let* $(\mathbb{R}, \tau_{\mathbb{R}})$ *and* $(\{0,1\}, \tau_d)$ *and* $f : \mathbb{R} \to \{0,1\}$ *given by*

$$\begin{cases} 1, & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \in \mathbb{R} \backslash \mathbb{Q}. \end{cases}$$

Is f *continuous? Motivate your answer.*

Problem 1.3.4 *Let* $f, g : (\mathbb{R}, \tau_{\mathbb{R}}) \to (\mathbb{R}, \tau_{\mathbb{R}})$ *be two continuous functions. Then show that the set* $\{x \mid f(x) \geq g(x)\}$ *is closed.*

Problem 1.3.5 *Let* (X, τ) *a Hausdorff topological space. Let* $A \subset X$ *and* $f : A \to X$ *a continuous mapping. Suppose that* f *can be extended to a continuous mapping* $g : \bar{A} \to X$*. Show that* g *is uniquely determined by* f*.*

Problem 1.3.6 *Show that the* **projections on the first and second factor** *of a cartesian product of topological spaces, defined by*

$$\pi_1 : (X, \tau_X) \times (Y, \tau_Y) \to (X, \tau_X),\ \pi_1(x, y) = x$$

and

$$\pi_2 : (X, \tau_X) \times (Y, \tau_Y) \to (Y, \tau_Y),\ \pi_2(x, y) = y,$$

are continuous mappings.

Problem 1.3.7 *Prove the converse of Proposition 1.3.1.*

Problem 1.3.8 *Are the following functions*

$$g_1(x) = \begin{cases} x, & x \leq 0 \\ x/2, & x \geq 0 \end{cases},$$

$$g_2(x) = \begin{cases} x - 2, & x < 0 \\ x + 2, & x \geq 0 \end{cases}$$

homeomorphisms of $\mathbb{R}$*? Motivate your answer.*

1.4 Week 4

1.4.1 Test 1.

Instructor	Student Number	Name(printed)

MATH 4290

TEST 1 February 3, 2005

ALL WORK MUST BE SHOWN PRECISELY AND THOROUGHLY. YOU ARE ALLOWED 60 MINUTES TO COMPLETE THE TEST.

I. Give the definition of a Hausdorff topological space. [1]

If, for any $x, y \in (X, \tau)$, $\exists U \in \mathcal{V}(x)$ and $\exists V \in \mathcal{V}(y)$ such that $\bar{U} \cap \bar{V} = \emptyset$, then (X, τ) is a Hausdorff space.

What does the Hausdorff condition mean for convergence of sequences? [1]

The limit of any convergent sequence in a Hausdorff space is unique.

What does the Hausdorff condition mean for singletons (i.e., one-element sets)? [1]

All singletons are closed.

What does the statement "for every real number r, there exists a sequence of rational numbers convergent to r" mean topologically? [1]

$\bar{\mathbb{Q}} = \mathbb{R}$.

II. *We recall that a set A of a topological space (X, τ_X) is closed if and only if for any convergent sequence with elements in A, i.e., $\{x_n\}_n \in A$, $x_n \to x^*$ as $n \to \infty$, we have $x^* \in A$.*

Let $f : A \to (Y, \tau_Y)$ a continuous mapping such that $A \subseteq X$ is a closed set in the topological space (X, τ_X) and such that Y is a Hausdorff space. Show that the graph of f, i.e., the set

$$graph(f) = \{(x, y) \in A \times Y \mid y = f(x)\}$$

is closed. [4]

Let $\{x_n, y_n\}_n \in graph(f)$ such that $(x_n, y_n) \to (x, y) \in X \times Y$ as $n \to \infty$. Evidently, $y_n = f(x_n)$.

Since A is closed, using the characterization of the closed set A given in the statement, then $x \in A$. Then to prove that $graph(f)$ is closed in the topology of $A \times Y$ we need to show that $(x, y) \in graph(f) \Leftrightarrow y = f(x)$.

But since $y_n = f(x_n)$ for any n, then the following are true:

$$y_n = f(x_n) \overset{\text{by continuity of } f}{\to} f(x) \in Y,$$

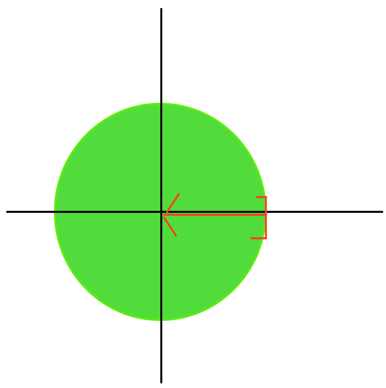
but also

$$y_n \to y \in Y.$$

Since Y is Hausdorff, then necessarily $y = f(x)$ and hence $graph(f)$ is closed.

III. Let $B = \{(x, y) \in \mathbb{R}^2 \mid x = r\cos\theta, y = r\sin\theta, 0 \leq r \leq 1, \theta \in (0, 2\pi)\}$ be a set in the plane with the standard topology $\tau_{\mathbb{R}^2}$.

Sketch the set B. [1]



Find the boundary and the interior points of B. [2]

$$\partial B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 \mid y = 0, 0 < x < 1\}.$$

$$\int B = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\} \backslash \{(x, y) \in \mathbb{R}^2 \mid y = 0, 0 < x \leq 1\}.$$

Find the closure of the set B. [1]

$$\bar{B} = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

IV. Let $X = \{a, b, c, d\}$ and let

$$\tau = \{\emptyset, \{a, b\}, \{a, b, c\}, X\}.$$

Is τ a topology on X? Motivate your answer. [2]

To show that τ is a topology we need to check the 3 conditions from the definition of a topology. Condition 1 is satisfied since $\tau \supset \emptyset,\ X$. Then the following hold:

$$\emptyset \cap D = \emptyset,\ \forall D \in \tau;$$

$$X \cap D = D,\ \forall D \in \tau;$$

$$\{a, b\} \cap \{a, b, c\} = \{a, b\}.$$

The resulting sets of any such intersections belong to τ, so condition 2 is satisfied.

The unions we need to check are those of 2, 3 and 4 sets from τ. Evidently:

$$\emptyset \cup D = D,\ \forall D \in \tau,\ \text{and}$$

$$X \cup D = X,\ \forall D \in \tau,$$

where D can be a union of other two sets in τ. Since also:

$$\{a, b\} \cap \{a, b, c\} = \{a, b, c\},$$

then condition 3 is satisfied and τ is a topology.

Is this topology Hausdorff? Motivate your answer. [2]

No, it is not. If we suppose it were, then fixing a, b in X, we see that any open sets around these 2 points always intersect. Hence we cannot find a pair of neighbourhoods of a, b whose intersection is empty since any neighbourhood contains at leats one open set around each point.

Another solution: Not all singletons are closed, for example {c}. The same argument can be made with the singletons {a} or {b}.

V. Show the following:

$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B);$ [2]

Everyone solved these two questions correctly.

$$X \backslash (\bigcup_{i \in I} A_i) = \bigcap_{i \in I} (X \backslash A_i).$$ [2]

1.5 Week 5

In this chapter we study the relation between a topology and a metric, and consequently, between a topological space and a metric space. The collection of metric spaces is contained in that of topological spaces (in a sense made precise below), but the converse is not true.

The addition of another structure, aside from a topology, will allow us to perform more "analysis" than before, being able to rediscover more of the notions we encountered in other courses, such as bounded sets, complete spaces, isometric mappings etc. We also highlight the new formulations (in metric spaces), for concepts defined so far only in topological spaces. Again, some of them will seem extremely familiar from previous courses.

1.5.1 Metrics, metric spaces, bounded sets, the metric topology

Definition 1.5.1 *Let X a non-empty set. We call* **a metric** *on X, a mapping $d : X \times X \to \mathbb{R}$ with the following properties;*

1. *$d(x, y) > 0$ for all $x \neq y \in X$ and $d(x, y) = 0$ iff $x = y$ (non-degeneracy);*
2. *$d(x, y) = d(y, x)$, for all $x, y \in X$ (symmetry);*
3. *$d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$ (triangle inequality).*

Evidently, a pair (X, d), where X is a set and d a metric on X, is called **a metric space***.*

Let us define the following sets.

Definition 1.5.2 *Let (X, d) be a metric space and $x_0 \in X$, $r \in \mathbb{R}$, with $r > 0$. Then:*

1. **the open ball of radius r and center x_0** *in X is the set*
$$B(x_0, r) := \{x \in X \mid d(x, x_0) < r\};$$
2. **the closed ball of radius r and center x_0***in X is the set*
$$B[x_0, r] := \{x \in X \mid d(x, x_0) \leq r\};$$
3. **the sphere of radius r and center x_0** *in X is the set*
$$S(x_0, r) := \{x \in X \mid d(x, x_0) = r\}.$$

Analogously, we denote by

$$B(A, r) := \{x \in X \mid d(x, A) < r\},$$

$$B[A,r] := \{x \in X \mid d(x,A) \leq r\} \text{ and}$$
$$S(A,r) := \{x \in X \mid d(x,A) = r\},$$

the open ball, closed ball and respectively the sphere of radius r and center A, where A is a non-empty subset of X and where for any arbitrarily fixed $x \in X$ we have

$$d(x,A) := \inf_{a \in A} d(x,a).$$

The relation between the collection of topological spaces and that of metric spaces can be described by the following three remarks.

Remark 1.5.1 *Any metric space (X,τ) is a topological space.*

To see that this is the case, let us consider a metric space (X,d). We define now the following collection of sets:

$$\tau := \{D \subseteq X \mid \forall\ x \in D,\ \exists\ \epsilon > 0 \text{ for which } B(x,\epsilon) \subseteq D\}.$$

It can be shown that this collection τ forms a topology on the set X; since it is induced by the metric on the set X, we shall denote it by τ_m and we shall call it **the metric topology**. Even more importantly, τ_m is always Hausdorff (if $x \neq y \in X$, then choose $\epsilon := \frac{1}{2}d(x,y)$; then according to the triangle inequality, $B(x,\epsilon)$ and $B(y,\epsilon)$ are disjoint neighbourhoods of x and y).

Remark 1.5.2 *Not all topologies on a set X can be defined by a metric.*

See for example the uncountable cartesian product of copies of $\mathbb{R}$.

Remark 1.5.3 *The topological spaces (X,τ) with the property that τ can be given by a metric are called* **metrizable**.

Definition 1.5.3 *Let (X,d_X) be a metric space.*

1. *A subset $A \subset X$ is called* **bounded** *if there exists a number M such that*

$$d_X(a_1,a_2) \leq M, \text{ for every pair } a_1,a_2 \in A.$$

2. *A mapping $f : X \to Y$, where (Y,d_Y) is also a metric space, is called* **bounded** *iff $f(X) := \{y \in Y \mid$ there exists $x \in X$ s.t. $f(x) = y\}$ is a bounded set in the space (Y,d_Y).*

If A is bounded and nonempty, the **diameter** of A is defined to be the number

$$diam(A) := \sup\{d(a_1,a_2) \mid a_1,a_2 \in A\}.$$

The property of a set to be bounded is not a topological property, for it depends on the particular metric d that is used for X. For instance, if X is a metric space with metric d, then there exists a metric $\bar{d}$ that gives the topology of X (same topology as given by d), but with respect to which **every subset of X is bounded**. It is defined as follows:

$$\bar{d} : X \times X \to \mathbb{R},\ \bar{d}(x,y) := \min\{d(x,y), 1\}.$$

1.5.2 Limits, convergence in metric spaces; complete spaces and compactness

We continue the study of continuous functions, this time on metric spaces. We see that the familiar $\epsilon - \delta$ definitions of limits and continuity are rediscovered and we introduce/remind the concepts of Cauchy sequence and complete space.

We also remind the definition of compact sets and we see that compatcness can be defined with the help of sequences, rather than open coverings; the two definitions are equivalent on metric spaces.

Definition 1.5.4 *Let (X, d) be a metric space and $\{x_n\}_n \in X$ a sequence. Then $x_n \to x^*$ if for any $\epsilon > 0$, there exists a rank $n_0 > 0$ so that for any $n \geq n_0$ we have $d(x_n, x^*) < \epsilon$.*

Here is a definition that only makes sense in metric spaces (and not in topological spaces), but it is crucial to defining later what is meant by complete spaces.

Definition 1.5.5 *Let (X, d) be a metric space and $\{x_n\}_n \in X$ a sequence. The sequence $\{x_n\}_n$ is called* **a Cauchy sequence***, or* **a fundamental sequence***, if for any $\epsilon > 0$, there exists a rank $n_0 > 0$ so that for any $m, n \geq n_0$ we have $d(x_m, x_n) < \epsilon$.*

The following result follows immediately from the last two definitions.

Proposition 1.5.1 *Let (X, d) be a metric space and $\{x_n\}_n \in X$ a sequence. If the sequence $\{x_n\}_n$ is convergent, then it is a Cauchy sequence. The converse is not always true (see Problem 1.5.2).*

Proof: Let $\epsilon > 0$ arbitrarily fixed and x^* be the limit of the sequence. Then since $\{x_n\}_n$ is convergent, from Definition 1.4 we have that there exists $n_0 > 0$ so that for any $n \geq n_0$, we have $d(x_n, x^*) < \frac{\epsilon}{2}$.

But now for any $m, n \geq n_0$, the following is true:

$$d(x_n, x_m) \leq d(x_n, x^*) + d(x^*, x_m) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

and the proof is complete. □

When the converse of Proposition 1.6 is true, then the metric space (X, d) has a particular property, that is that the notions of Cauchy sequence and convergent sequence are the same. This can be summarized as follows:

Definition 1.5.6 *A metric space (X, d) in which any Cauchy sequence is convergent is called* **a complete metric space***.*

In a topological space (X, τ), a set $A \subseteq X$ is called **compact** if from any covering of A with open sets, $A \subseteq \bigcup_{i \in I} D_i$, we can extract a finite covering of A, i.e., there exists $\{1, 2, ..., m\} \in I$ such that

$$A \subseteq \bigcup_{i=1}^{m} D_i.$$

A more familiar way to think of compact sets is perhaps the one involving sequences. For this, we need to remind the following definition:

Definition 1.5.7 *Let (X, d) be a metric space and $\{x_n\}_n \in X$ a sequence. If $n_1 < n_2 < n_3 < .... < n_i < ...$ is an increasing sequence of positive integers, then the sequence $\{x_{n_k}\}_{k \in \{1,2,..,n,..\}}$ is called* **a subsequence** *of the original sequence.*

The following theorem gives us another way to check for compact sets in a metric space.

Theorem 1.5.1 *Let (X, d) be a metric space. The the following are equivalent:*

- *X is compact.*
- *Any sequence $\{x_n\}_n \in X$ has a convergent subsequence.*

Proof: For a proof see [3], Chapter 3. □

Even more familiar to the reader: in $\mathbb{R}^n$ or $\mathbb{C}^n$, the compact sets are easily identifiable, due to the following theorem:

Theorem 1.5.2 *A subset $S \subset \mathbb{R}^n$ (or of $\mathbb{C}^n$) is compact iff S is closed and bounded.*

1.5.3 Continuity in metric spaces

Definition 1.5.8 *Let $f : X \to Y$ two metric spaces, with metrics d_X and d_Y respectively. Then f is* **continuous at** *$x_0 \in X$ if for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\text{whenever } y \in X \text{ has the property } d_X(y, x_0) < \delta \text{ then } d_Y(f(y), f(x_0)) < \epsilon.$$

The following notion only makes sense only in metric spaces and is of use for the geometry part of the course.

Definition 1.5.9 *Let $f : X \to Y$ continuous between two metric spaces, with metrics d_X and d_Y respectively. Then f is* **an isometry** *if we have the following*

$$d_Y(f(x, f(y)) = d_X(x, y), \ \forall \ x, y \in X.$$

In other words, an isometry is a mapping that preserves the distance between points, when we pass form one metric space to another. One easy thing to remark here is that any isometry is an injective mapping and $f^{-1} : f(X) \to X$ is also an isometry. In other words, any isometry is a homeomorphism between X and the image of X through f, $f(X)$.

Here is an important definition which is valid in topological spaces, but we give it here in metric spaces. The notion defined below is of wide use in Optimization and Convex Analysis.

Definition 1.5.10 *Let $f : X \to Y$ two metric spaces, with metrics d_X and d_Y respectively. Then f is said to* **have a closed graph** *if the graph of f, i.e. the set*

$$G_f := \{(x, f(x)) \mid x \in X\},$$

is closed in the topological space $(X \times Y, \tau_X \times \tau_Y)$. In metric spaces, f has a **closed graph** *if for any $\{x_n, y_n\}_n \in G_f$, so that $(x_n, y_n) \to (x, y) \in X \times Y$, then $(x, y) \in G_f$.*

Theorem 1.5.3 *Let $f : X \to Y$ two metric spaces, with metrics d_X and d_Y respectively. Then if f is continuous on X, then f has a closed graph.*

Proof: See Test # 1. □

1.5.4 Norm and Inner product spaces

This section offers a very quick overview of 3 other structures for spaces, some of which are more familiar than others: those of vector spaces, normed spaces and inner product spaces. We also show the relation between these new collections of spaces and those of topological and metric spaces.

Most of these spaces form the object of study for courses of functional analysis, which is why we do not detail them here, but rather we define them and place them in the largest (topological) context.

From the Linear Algebra, or Applied Matrix Algebra or Algebraic Structures, we assume the reader is familiar with the concept of **vector spaces**. For a reminder, please see [2]. Here, we denote by E a generic vector space over the field $\mathbb{K}$ and we shall write, in short, E a $\mathbb{K} - VS$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. We shall see that there exist spaces (sets) that admit a topological structure, as well as an algebraic structure. This combination allows the introduction of new concepts and new results. These new spaces do not form the object of a topology course, but rather the object of a Functional Analysis course.

In this section we shall just define them and show their topological structure and give some examples.

Definition 1.5.11 *Let E be a $\mathbb{K}$-VS. Let $p : E \to \mathbb{R}$ with the properties:*

1. *$p(x) > 0$ for any $x \neq 0 \in E$;*

2. *$p(\lambda x) = |\lambda| p(x)$, for any $x \in E$ and $\lambda \in \mathbb{K}$;*

3. *$p(x+y) \leq p(x) + p(y)$, for all $x, y \in E$.*

Then p is called a norm on E and the space (E, p) is called a normed space.

We note that any norm on E gives a metric on E, by defining:

$$d_p(x, y) := p(x - y), \forall\ x, y \in E.$$

So any normed space is a metric space, hence, is a topological space. However, not all metrics on a metric space are coming from a norm.

A normed space that is complete in the distance d_p given by p, is called a **complete normed space** or equivalently **a Banach space**.

Definition 1.5.12 *Let E be a $\mathbb{K} - SV$. A mapping $<,>: E \times E \to \mathbb{K}$ is called* **an inner product** *on E if it satisfies the following properties:*

1. *$< x, x >> 0$, for any $x \neq 0 \in E$;*

2. *$< x, y >= \overline{< y, x >}$, for all $x, y \in E$;*

3. *$< \alpha x + \beta y, z >= \alpha < x, z > + \beta < y, z >$, for all $x, y, z \in E$ and $\alpha, \beta \in \mathbb{K}$.*

The space $(E, <,>)$ is called **a pre-Hilbert space**.

We note that from any $<,>$ on E we can define a norm on E, by

$$p_{<,>}(x) := \sqrt{< x, x >},\ \forall\ x \in E.$$

So any pre-Hilbert space is a normed space; the converse is not true, in the sense that there are normed spaces that are not pre-Hilbert.

Keeping in mind that any norm p can define a metric on E, we have

$$d_p(x, y) := p_{<,>}(x - y) = \sqrt{< x - y, x - y >},\ \forall\ x, y \in E,$$

hence we can talk about complete pre-Hilbert spaces; a pre-Hilbert space that is complete in the metric d_p is called **a Hilbert space**.

To summarize the relations between the various types of spaces we have encountered/studied, we use the diagrams

$$pre - H\ spaces \subsetneqq normed\ spaces \subsetneqq metric\ spaces \subsetneqq topological\ spaces,$$

and

$$Hilbert\ spaces \subsetneqq Banach\ spaces \subsetneqq complete\ metric\ spaces.$$

1.5.5 Examples and Problems

Example 1.5.1 *Here are a couple of examples of metrics.*

1. *Let* X *and* $d(x,y) = 1$ *if* $x \neq y$ *and* 0 *if* $x = y$*;*
2. *Let* $X = \mathbb{R}^n$ *with* $d_2(x,y) := \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$*;* d_2 *is called the Euclidean distance.*
3. *Let* $\mathbb{R}^n$ *with* $d(x,y) = \max_{i=1,..,n}\{|x_i - y_i|\}$*;*
4. *Let* $S^2 \subset \mathbb{R}^3$ *and* $d_R(u,v) =$ *arc of minimal length joining* u *and* v*.*

Example 1.5.2 $\mathbb{R}^n$ *has all the structures mentioned above, i.e., it is a topological space, a metric space, a normed space and a Hilbert space.*

Example 1.5.3 *Example of Hilbert spaces, other than* $\mathbb{R}^n$*, are the* L^2*-spaces of integrable functions.*

Problem 1.5.1 *Show that all the mappings in Example 1.5.1 are metrics. For which one do we have* $\tau_m = \tau_d$*?*

Problem 1.5.2 *Let* $X = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous }\}$ *and* $d : X \times X \to \mathbb{R}$ *so that* $d(f,g) = \int_a^b |f(x) - g(x)|dx$*. Then*

1. *Show that d is a metric on* X*;*
2. *Construct a Cauchy sequence in* X*;*
3. *Show that* (X,d) *is not complete.*

Problem 1.5.3 *What are the definitions of* $<,>$*,* $p_{<,>}$*, and* d_p *which give* $\tau_m = \tau_{\mathbb{R}}$*?*

Problem 1.5.4 *Using the fact that* $(\mathbb{R}, \tau_{\mathbb{R}})$ *is Hausdorff show that the function* $f(x) = \cos\frac{1}{x}$ *is not continuous at* $x = 0$*.*

Problem 1.5.5 *Show that all compact subsets of* $\mathbb{R}$ *are of the form* $[a,b]$*,* $a, b \in \mathbb{R}$*.*

Problem 1.5.6 *Let* $X = \{f : [a,b] \to \mathbb{R} \mid f \text{ is integrable}\}$ *be a set and we define the mapping*

$$< f, g >= \int_a^b f(x)g(x)dx, \ \forall f, g \in X.$$

Is $<,>$ *an inner product on* X*? (Hint: check first whether* X *has a structure of* $\mathbb{R}$*-vector space).*

Problem 1.5.7 *Let* $f : (X, d_X) \to (Y, d_Y)$ *a continuous mapping and* $C \subset X$ *a compact set. Is* $f(C)$ *compact in* Y*?*

Problem 1.5.8 *Let* $(\mathbb{C}, d_{\mathbb{C}})$*, where* $d_{\mathbb{C}}(z_1, z_2) = |z_1 - z_2|$*. Show that if* $f : (\mathbb{R}^2, d_2) \to (\mathbb{C}, d_{\mathbb{C}})$ *is given by* $f(x, y) = x + iy$*, then* f *is an isometry.*

Problem 1.5.9 *Using the definition of a compact set in a topological space, show that:*

1. *Any finite set is compact in* $\mathbb{R}$*;*
2. *The set of positive integers is not compact.*

Chapter 2

Geometry of curves and surfaces

2.1 Week 6

For this part of the course [1] was used as reference.

2.1.1 Elements of the theory of curves

For the rest of this text, we shall denote by I a finite time interval in $\mathbb{R}$, and by $\alpha : I \to \mathbb{R}^3$ a parametrized curve on R^3, i.e. a mapping assigning to each $t \in I$, the triplet $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$, or in the Vector Calculus usual notation, $(x(t), y(t), z(t)) \in R^3$. The set $Im(\alpha) \in R^3$ is called **the trace of the curve** α. It is possible that two curves have the same trace (see Example 2.1.1 below).

Definition 2.1.1 *Let $\alpha : I \to \mathbb{R}^3$ be a curve. Then*

1. *the curve is called* **differentiable** *or* **smooth** *if derivatives of any order exist and are continuous;*
2. *the curve is called* **regular** *if $\alpha'(t) \neq 0$, for all $t \in I$;*
3. *a point $t_0 \in I$ is called* **a singular point of** α *if $\alpha'(t_0) = 0$.*

Remark 2.1.1 For each $t \in I$ so that $\alpha'(t) \neq 0$, there exists a well-defined straight line containing $\alpha(t)$ and $\alpha'(t)$, called **the tangent line**.

In fact, the vector $\alpha'(t) = (x'(t), y'(t), z'(t))$ is called **the velocity vector** to the curve.

We shall introduce now the following definition:

Definition 2.1.2 *Let $\alpha : I \to \mathbb{R}^3$ be a parametrized, differentiable curve. Given $t \in I$, we call* **the arc length of** α **from** $t_0 \in I$ *the function:*

$$s(t) := \int_{t_0}^{t} |\alpha'(t)| dt, \text{ where}$$

$$|\alpha'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} \text{ is the norm of } \alpha'(t).$$

Obviously, from Real Analysis we know that

$$\frac{d}{dt}s(t) = \frac{d}{dt}\left(\int_{t_0}^{t} |\alpha'(t)|dt\right) = |\alpha'(t)|.$$

Since α is regular, then $\frac{d}{dt}s(t)$ always exists and is positive.

Let us make some interesting and useful remarks.

Remark 2.1.2

- It is possible that t is the arc length measured from some point $t_0 \in I$. Then this means that

$$s(t) = t - t_0 \implies \frac{d}{dt}s(t) = 1 = |\alpha'(t)|, \text{ i.e.}$$

  the velocity vector has constant norm 1.

- Conversely, if $|\alpha'(t)| = 1$, then

$$s(t) := \int_{t_0}^{t} |\alpha'(t)|dt = t - t_0$$

  and we say that α is parametrized by arc length, i.e. by

$$s(t) = t - t_0 \implies t := s(t) + t_0.$$

Such curves are called **parametrized by arc length**.

We remind here that we denote by $< \cdot, \cdot >$ the inner product of two vectors in R^n; its definition is

$$< u, v >= \sum_{i=1}^{n} u_i v_i.$$

The properties of the inner product on R^n are considered known, as they were presented in both Vector Calculus and Linear Algebra classes.

In the context of the theory of curves, the following is a property of the inner product: If $\alpha(t) = (x_1(t), y_1(t), z_1(t))$ and $\beta(t) = (x_2(t), y_2(t), z_2(t))$ are two parametrized differentiable curves, then we can define their inner product as follows

$$\text{for each } t \in I, < \alpha(t), \beta(t) >:= x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t),$$

and the function $t \mapsto < \alpha(t), \beta(t) >$ is a differentiable function with

$$\frac{d}{dt} < \alpha(t), \beta(t) >:= \alpha'(t)\beta(t) + \alpha(t)\beta'(t). \tag{2.1}$$

We remind next the definition of the cross product of two vectors in $\mathbb{R}^3$:

$$u \wedge v := \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} e_1 - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} e_2 + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} e_3,$$

where $\{e_i\}_{i=1,..,3}$ are the standard unit vectors in $\mathbb{R}^3$.

The relation between the inner product and the cross product in $\mathbb{R}^3$ is given by the following formula:

$$< u \wedge v, w >:= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}, \ \forall u, v, w \in \mathbb{R}^3. \tag{2.2}$$

Here is a reminder of the properties of the cross product: for all $u, v, w \in \mathbb{R}^3$ and all $a, b \in \mathbb{R}$ we have

- $u \wedge v = -v \wedge u$;
- $(au + bv) \wedge w = (au \wedge w) + (bv \wedge w)$;
- $u \wedge v = 0$ iff u, v are linearly dependent;
- $< u \wedge v, u >=< u \wedge v, v >= 0$;
- the cross product is not associative.

Remark 2.1.3 For any two vectors $u, v \in \mathbb{R}^3$, the vector $u \wedge v$ is normal to the plane generated by the vectors u and v. Moreover, the norm of the vector $u \wedge v$ is given by

$$< u \wedge v, u \wedge v >= |u \wedge v|^2 > 0.$$

Based on the remark above and on relation (2.2), then $\{u, v, u \wedge v\}$ is always a basis of $\mathbb{R}^3$.

Proposition 2.1.1 *The following relation holds for any four vectors in* $\mathbb{R}^3$*:*

$$< u \wedge v, x \wedge y >= \begin{vmatrix} < u, x > & < v, x > \\ < u, y > & < v, y > \end{vmatrix}. \tag{2.3}$$

Proof: See Problem 2.1.1 below. □

In our context, the cross product has the following property:
If $\alpha(t) = (x_1(t), y_1(t), z_1(t))$ and $\beta(t) = (x_2(t), y_2(t), z_2(t))$ are two parametrized differentiable curves, then the function $t \mapsto \alpha(t) \wedge \beta(t)$ is a differentiable function with

$$\frac{d}{dt}[\alpha(t) \wedge \beta(t)] := \alpha'(t) \wedge \beta(t) + \alpha(t) \wedge \beta'(t). \tag{2.4}$$

2.1.2 Fundamental theorem of the local theory of curves

From this point on, the study of curves is conducted for curves parametrized by arc length. We can think of a space curve as being obtained from a straight line by bending and twisting it. The bending of the curve is called curvature; the twisting of the curve will be called torsion. By the **study of local geometry** of such curves we shall understand the study of those vectors which characterize the curve in a neighbourhood of the given point.

*

Let $\alpha : I \to \mathbb{R}^3$ a differentiable curve, parametrized by arc length. This implies that $|\alpha'(s)| = 1$, for any $s \in I$ and so

$$< \alpha'(s), \alpha'(s) >= |\alpha'(s)|^2 = 1,$$

and by differentiation we get

$$0 = \frac{d}{ds} < \alpha'(s), \alpha'(s) >= 2 < \alpha'(s), \alpha''(s) >,$$

which means that the vector $\alpha''(s)$ is orthogonal on $\alpha'(t)$.

Definition 2.1.3 *The number $k(s) := |\alpha''(s)|$ is called* **the curvature of α at** *s.*

Quick check The curvature of the straight line is 0.
To see this, let $\alpha(s) = us + v$, with $u, v \in \mathbb{R}^3$, $|u| = 1$. Then $\alpha''(s) = 0$.

In general, the curvature measures how rapidly the curve pulls away from the tangent line at s, in a neighbourhood of s.

Obviously, we can always normalize the vector $\alpha''(s)$ and so we always have that

$$\alpha''(s) = |\alpha''(s)| n(s),$$

where $n(s) := \frac{\alpha''(s)}{|\alpha''(s)|}$.

Definition 2.1.4 *The vector $n(s)$ is called* **the normal vector** *to the curve at the point s.*

Definition 2.1.5 *The plane formed by the unit tangent vector $\alpha'(s)$ and the normal vector $n(s)$ is called* **the osculating plane** *at s.*

We note that at points where the curvature $k(s) = 0$ the normal vector and the osculating plane are not defined.

Definition 2.1.6 *Let $\alpha : I \to \mathbb{R}^3$ parametrized by arc length. A point $s \in I$ is called* **singular of order 1** *if $\alpha''(s) = 0$.*

From here on, we consider only curves without singular points of order 1.

Let us denote by $t(s) := \alpha'(s)$ the unit tangent vector at s. Then obviously $\alpha''(s) = t'(s) = n(s)k(s)$. The vector

$$b(s) := t(s) \wedge n(s)$$

is normal to the osculating plane.

Definition 2.1.7 *The vector $b(s)$ is called* **the binormal vector** *of α at s.*

We can immediately conclude, from formula (2.3) of previous section that

$$|b(s)|^2 = 1.$$

In general, $|b'(s)|$ measures the rate of change of the neighbouring osculating planes with the osculating plane at s, i.e. $b'(s)$ measures how rapidly the curve pulls away from the osculating plane at s, in a neighbourhood of s.

We notice that

$$b'(s) = \frac{d}{ds}b(s) = \frac{d}{ds}(t(s) \wedge n(s)) \overset{(2.4)}{=} t'(s) \wedge n(s) + t(s) \wedge n'(s).$$

But remember that $t'(s) = \alpha''(s)$ which in turn is collinear with $n(s)$, hence

$$b'(s) = t(s) \wedge n'(s),$$

so $b'(s)$ is normal to $t(s)$; this means that it should be proportional to the normal $n(s)$. This observation leads to the following

Definition 2.1.8 *Let $\alpha : I \to \mathbb{R}^3$ parametrized by arc length such that $\alpha''(s) \neq 0$, for all $s \in I$. The number $\tau(s)$ given by $b'(s) = \tau(s)n(s)$, for each $s \in I$, is called* **the torsion of α at** *s.*

Remark 2.1.4

- Note that if α is a plane curve (i.e., is completely contained in a plane), then its torsion will be identically zero. Conversely, if $\tau(s) = 0$ for all s and $k(s) \neq 0$, for all s, then the curve is a plane curve.
- The torsion can be positive, zero, or negative; the curvature can only be zero or positive.

Definition 2.1.9

1. *The unit vectors $\{t(s), n(s), b(s)\}$ are called* **the Frenet trihedron**
2. *The following formulae*

$$\begin{cases} t'(s) = k(s)n(s) \\ n'(s) = -k(s)t(s) - \tau(s)b(s) \\ b'(s) = \tau(s)n(s) \end{cases}$$

are called **the Frenet formulae**.

3. *The* (t, b)*-plane is called* **the rectifying plane** *and the* (n, b)*-plane is called* **the normal plane***.*

4. *The number* $R(s) = 1/k(s)$ *is called* **the radius of curvature at** s*.*

We have seen so far that given an arc length parametrized curve α in space, we can completely describe the curve, in a neighbourhood of every point by computing its curvature and its torsion. The following theorem assures us that given two functions, one of which has positive values, there always exists a space curve whose curvature and torsion correspond to the two given functions.

Theorem 2.1.1 (Fundamental theorem of the local theory of curves) *Given differentiable functions* $k(s) > 0$ *and* $\tau(s)$, $s \in I$, *there exists a regular parametrized curve* $\alpha : I \to \mathbb{R}^3$ *such that* s *is the arc length,* $k(s)$ *is the curvature and* $\tau(s)$ *is the torsion of* α*. Moreover, any other curve* $\bar{\alpha}$*, satisfying the same conditions, differs from* α *by a rigid motion (a rotation and a translation).*

The proof of this theorem is fairly long and can be found in [1], Appendix to Chapter 4.

2.1.3 Examples and Problems

Example 2.1.1 *The curves* $\alpha_1 : [0, 2\pi] \to R^2$ *and* $\alpha_2 : [0, 2\pi] \to R^2$ *given by*

$$\alpha_(t) := (\sin t, \cos t) \text{ and } \alpha_2(t) := (\sin 2t, \cos 2t)$$

have the same trace.

Example 2.1.2 *Let* $\alpha : [0, \pi] \to \mathbb{R}^2$ *and* $\beta : [-1, 1] \to \mathbb{R}^2$ *two parametrized curves, given by*

$$\alpha(t) = (\cos t, \sin t), \ \ \beta(t) = (t, \sqrt{1 - t^2}).$$

We see that they have the same trace, i.e., $Im(\alpha) = Im(\beta) =$ *the upper semicircle of radius 1. However,* α *is parametrized by arc length while* β *is not.*

Example 2.1.3 *The circle of radius* r*, given by the parametrized curve* $\alpha : [0, 2\pi] \to \mathbb{R}^2$*,* $\alpha(t) = (r\cos t, r\sin t)$ *has the constant curvature* $k(s) = 1/r$ *and the radius of curvature* r*.*

Problem 2.1.1 *Prove Proposition 2.1.1.*

Problem 2.1.2 *Find the arc length parametrization of the circle of radius* $r > 0$ *given by* $\alpha : [0, 2\pi] \to \mathbb{R}^2$*,* $\alpha(t) = (r\cos t, r\sin t)$*.*

Problem 2.1.3 *Consider the map*

$$\alpha(t) = \begin{cases} (t, 0, e^{1-/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t < 0 \\ (0,0,0) & t = 0 \end{cases}$$

Prove that α is a differentiable curve. Prove that α is regular for all t and that the curvature $k(t) \neq 0$ for all $t \neq 0, t \neq \pm\sqrt{\frac{2}{3}}$ and $k(0) = 0$.

Problem 2.1.4 *Determine the angle of intersection of the two planes $5x + 3y + 2z - 4 = 0$ and $3x + 4y - 7z = 0$.*

Problem 2.1.5 *Show that the equation of a plane passing through three noncolinear points $p_i = (x_i, y_i, z_i)$, $i = 1, .., 3$ is given by*

$$< (p - p_1) \wedge (p - p_2), (p - p_3) >= 0,$$

where $p = (x, y, z)$ is a generic point of the plane.

Problem 2.1.6 *Let $u(t) = (x_1(t), y_1(t), z_1(t))$ and $v(t) = (x_2(t), y_2(t), z_2(t))$ be two differentiable curves defined from (a, b) into $\mathbb{R}^3$. If their derivatives satisfy:*

$$u'(t) = au(t) + bv(t), \;\; v'(t) = cu(t) - av(t),$$

where a, b, c are constants, show that $u(t) \wedge v(t)$ is a constant vector.

Problem 2.1.7 *Given the parametrized curve (helix)*

$$\alpha(s) = (a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}), \;\; s \in \mathbb{R},$$

where $c^2 = a^2 + b^2$,

1. *show that α is parametrized by arc length;*
2. *determine the curvature and the torsion;*
3. *determine the osculating plane.*

2.2 Week 7

2.2.1 The isoperimetric inequality

In this section we discuss the oldest and one of the most important properties of plane closed curves.

Definition 2.2.1 • *A* **closed plane curve** *is a regular parametrized curve* $\alpha : [a, b] \to \mathbb{R}^2$ *such that* α *and all its derivatives agree at* a *and* b*, i.e.,*

$$\alpha(a) = \alpha(b),\ \alpha'(a) = \alpha'(b),\ \alpha''(a) = \alpha''(b), \ldots$$

- *The curve* α *is* **simple** *if* t *has no further self intersections, i.e., if* $t_1, t_2 \in [a, b)$*, with* $t_1 \neq t_2$*, then* $\alpha(t_1) \neq \alpha(t_2)$*.*

We remind the reader that the image of a curve is called trace and we denote by $C = tr(\alpha)$. We make the following assumption: *we assume that a simple closed curve* C *in the plane bounds a region of this plane that is called* **the interior of** C.
We assume further that the parameter of a simple closed curve can be so chosen that if one is going along the curve in the direction of increasing parameters, then the interior of the curve remains always to the left (or go along the curve in the trigonometric sense). Such a curve will be called **positively oriented**.

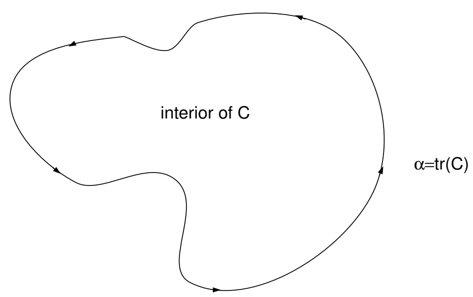


Figure 2.1: A positively oriented curve

The isoperimetric inequality answers the following question:
Of all simple closed curves in the plane with a given length l**, which one bounds the largest area?**
The Greeks knew the answer to this question, namely the curve is the circle, but a proof of this result came much later. The proof we present here was given in 1939 by E. Schmidt.

Before we proceed to formulate and prove the theorem, we need to remind ourselves the following fact:

Theorem 2.2.1 (Green's Theorem in the plane) *Let $\alpha : [a, b] \to \mathbb{R}^2$ be a closed, positively oriented, plane curve given by $\alpha(t) = (x(t), y(t))$, with $C = tr(\alpha)$ and $R = interior(C)$. Let $p(x, y), q(x, y)$ two real functions with continuous partial derivatives p_x, p_y, q_x, q_y. Then:*

$$\int_R (q_x - p_y)dxdy = \int_C (p\frac{dx}{dt} + q\frac{dy}{dt})dt,$$

where in the second integral we restricted p, q at α and the integral is taken between $t = a$ and $t = b$.

Now suppose in Green's theorem, we take $p(x, y) = -y$ and $q(x, y) = x$. Then applying the theorem we get that

$$A(R) = \int\int_R dxdy = \frac{1}{2}\int_a^b (x(t)\frac{dy(t)}{dt} - y(t)\frac{dx(t)}{dt})dt,$$

where by $A(R)$ we denoted the area of the interior of C. By integration by parts in the last equality we get

$$A(R) = \frac{1}{2}\left[x(t)y(t) \mid_a^b -2\int_a^b y(t)\frac{dx(t)}{dt}dt\right] = -\int_a^b y(t)\frac{dx(t)}{dt}dt = \int_a^b x(t)\frac{dy(t)}{dt}. \qquad (2.5)$$

Theorem 2.2.2 (The isoperimetric inequality) *Let α be a simple, closed curve of length l, let $C = tr(\alpha)$ and let A be the area of the region bounded by C. Then*

$$l^2 - 4\pi A \geq 0, \qquad (2.6)$$

and equality holds if and only if C is a circle.

Proof: We consider the following picture (See Figure 2.2 below), where the system of axes is centered at the center of the circle of radius $r > 0$ such that $2r$ is the distance between the lines L, L'. We consider the circle to be parametrized by $\alpha_1 : [0, l] \to \mathbb{R}^2$ and by the way we chose the axes, we can see that the curves α and α_1 have the property that their first component is the same, i.e.,

$$\alpha(s) = (x(s), y(s)) \text{ and } \alpha_1(s) = (x(s), y_1(s)), s \in [0, l].$$

We can certainly compute the areas given by the interior of the two curves, using (2.5), to be

$$A = \int_0^l xy'ds \text{ and } A_1 = \pi r^2 = -\int_0^l y_1x'ds.$$

Then

$$A + \pi r^2 = \int_0^l (xy' - y_1x')ds \leq \int_0^l \sqrt{(xy' - y_1x')^2}ds \qquad (2.7)$$

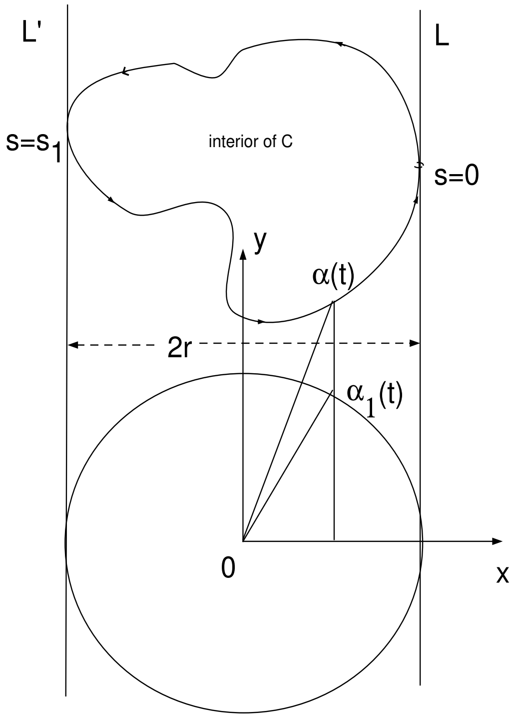


Figure 2.2: The curve versus a circle

$$\leq \int_0^l \sqrt{(x^2+(y_1)^2)(x'^2+y'^2)}ds = r\int_0^l \sqrt{x^2+y_1^2}ds = lr.$$

This means that

$$A+\pi r^2 \leq lr.$$

But it is always the case that the geometric mean is less than the arithmetic mean for any two positive numbers and they are equal when the numbers are equal. This further gives:

$$\sqrt{A}\sqrt{\pi r^2} \leq \frac{1}{2}(A+\pi r^2) \leq lr \implies 4\pi r^2 A \leq l^2 r^2, \tag{2.8}$$

and the proof of (2.6) is complete.

Now, if equality in (2.6) holds, then equality holds also in (2.8) and so $A = \pi r^2$. Thus $l = 2\pi r$ and r does not depend on the orientation of C. Furthermore, we have equality in (2.7) and from that we get

$$(xy' - y_1x')^2 = (x^2+(y_1)^2)(x'^2+y'^2) \implies xx' + y_1y' = 0 \implies$$

$$\frac{x}{y'} = \frac{y_1}{x'} = \frac{\sqrt{x^2+y^2}}{\sqrt{y'^2+x'^2}} = \pm r.$$

Thus $x = \pm r y'$; since r does not depend on the orientation of C, we can interchange x and y and get $y = \pm r x'$. Thus $x^2 + y^2 = r^2(x'^2 + y'^2) = r^2$ (since $C = tr(\alpha)$ and α is parametrized by arc length).

So C is a circle of radius r. □

Remark 2.2.1 The isoperimetric inequality also holds for piecewise C^1-differentiable closed curves.

2.2.2 Problems

Problem 2.2.1 *Is there a simple closed curve in the plane with length equal to 6 feet and bounding an area of 3 square feet?*

Problem 2.2.2 *Let α be a simple, closed, plane curve. Assume that the curvature $k(s)$ satisfies $0 \leq k(s) \leq c$, where $c > 0$ is a constant (thus α is less curved than a circle of radius $1/c$). Prove that*

$$\text{length of } \alpha \geq \frac{2\pi}{c}.$$

Problem 2.2.3 *Compute the curvature of the ellipse $x(t) = a\cos t$, $y(t) = b\sin t$, $t \in [0, 2\pi]$, $a \neq b$. How much is its torsion?*

Problem 2.2.4 *One often gives a plane curve in polar coordinates $\rho = \rho(\theta)$, $a \leq \theta \leq b$. Then show that the arc length is equal to*

$$\int_a^b \sqrt{\rho^2 + (\frac{d\rho}{d\theta})^2} d\theta.$$

2.3 Week 8

2.3.1 Review

2.3.2 Test 2.

Instructor | Student Number | Name(printed)

MATH 4290

TEST 2 March 10, 2005

ALL WORK MUST BE SHOWN PRECISELY AND THOROUGHLY. YOU ARE ALLOWED 60 MINUTES TO COMPLETE THE TEST.

I. Give the definition of a metric space. [1]

See Notes, Definition 1.5.1.

Give the definition of the metric topology. [1]

See Notes page 29.

Let X and $d(x, y) = 1$ if $x \neq y$ and 0 if $x = y$. Show that d is a metric. [2]

Everybody answered this question correctly.

Show that the topology given by the metric d above coincides with the discrete topology of X. [3]

Since $\tau_m = \{D \subseteq X \mid \forall x \in D, \exists \epsilon > 0 \text{ s.t. } B(x, \epsilon) \subseteq D\}$ *and* $B(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$, *then it is enough to notice that for the choice of* $\epsilon = 1/2$ *any subset* $D \subseteq X$ *can be made into an open set since it contains* $B(x, \frac{1}{2}) = \{x\}$ *around any of its points* x. *Therefore* τ_m *contains all subsets of* X, *thus* $\tau_m = \tau_d$.

II. Let $\alpha : [a, b] \to \mathbb{R}^3$ a differentiable parametrized curve so that $\alpha'(t) = (1, \tan t, 0)$.
Find the arc length parametrization of α. [2] Find the curvature and the torsion of α. [5]

This question was discarded in most papers.

III. Consider the curve

$$\alpha(t) = \begin{cases} (t, 0, e^{1-/t^2}), & t > 0 \\ (t, e^{-1/t^2}, 0), & t < 0 \\ (0,0,0) & t = 0 \end{cases}$$

Prove that α is regular for all $t \neq 0$ and that the curvature $k(t) \neq 0$ for all $t \neq 0, t \neq \pm\sqrt{\frac{2}{3}}$ and $k(0) = 0$. [6]

Everyone answered this part of the question correctly.

Is α regular of degree 1? [1]

No, α is not regular of degree 1, since $\alpha''(t) = 0$ when $t = 0$.

IV. Give the definition of an inner product **on a vector space over R**. [1]

Let E be a $\mathbb{R} - SV$. A mapping $<,>: E \times E \to \mathbb{R}$ is called **an inner product** *on E if it satisfies the following properties:*

1. $< x, x >> 0$, *for any* $x \neq 0 \in E$;
2. $< x, y >=< y, x >$, *for all* $x, y \in E$;
3. $< \alpha x + \beta y, z >= \alpha < x, z > +\beta < y, z >$, *for all* $x, y, z \in E$ *and* $\alpha, \beta \in \mathbb{R}$.

How does the definition of an inner product modify if we work with a **vector space over** $\mathbb{C}$? (Hint: think of property #2). [1]

Property # 2 becomes

$$< x, y >= \overline{< y, x >}, \forall x, y \in E.$$

Let $X = \{f : [a, b] \to \mathbb{R} \mid f \text{ is integrable}\}$. We know that X is a real vector space. Show that the mapping

$$< f, g >= \int_a^b f(x)g(x)dx, \ \forall f, g \in X$$

defines an inner product on X.
Write the norm given by this inner product on X, i.e. $p_{<,>}$. [2]

$p_{<,>}(f) = \sqrt{< f, f >} = \sqrt{\int_a^b f(x)^2 dx}$.

Write the distance given by the above norm on X. [2]

$d(f,g) = p_{<,>}(f-g) = \sqrt{\int_a^b [f(x)-g(x)]^2 dx}.$

V. State the Isoperimetric Inequality Theorem. [1]

See Notes, Theorem 2.2.2.

Consider the ellipse $x(t) = a\cos t$, $y(t) = b\sin t$, $t \in [0, 2\pi]$, $a \neq b$. Is this a closed, simple curve? Justify your answer. [2]

Yes, it is closed and simple. Everyone answered and motivated their answers correctly.

Compute the area of the region R encompassed by this ellipse, keeping in mind that

$$A(R) = -\int_a^b y(t)\frac{dx(t)}{dt}dt = \int_a^b x(t)\frac{dy(t)}{dt}dt.$$

[4]

Here we get

$$\int_0^{2\pi} x(t)\frac{dy(t)}{dt}dt = \int_0^{2\pi} ab\cos^2 t dt = \frac{ab}{2}\int_0^{2\pi}[\cos 2t + 1]dt$$

using the formula for cos *of a double angle:* $\cos 2t = \cos^2 t - \sin^2 t = 2\cos^2 t - 1$. *Then by integration we get* $A(R) = ab\pi$.

What is the minimal length the ellipse can achieve? Justify your answer. [1]

From the isoperimetric inequality, we have that $l^2 - 4\pi A(R) \geq 0$ *and* l *is minimal when* $l = \sqrt{4\pi A(R)} = \sqrt{4\pi^2 ab} = 2\pi\sqrt{ab}$, *hence if the ellipse is a circle of radius* $\sqrt{ab}$.

Chapter 3

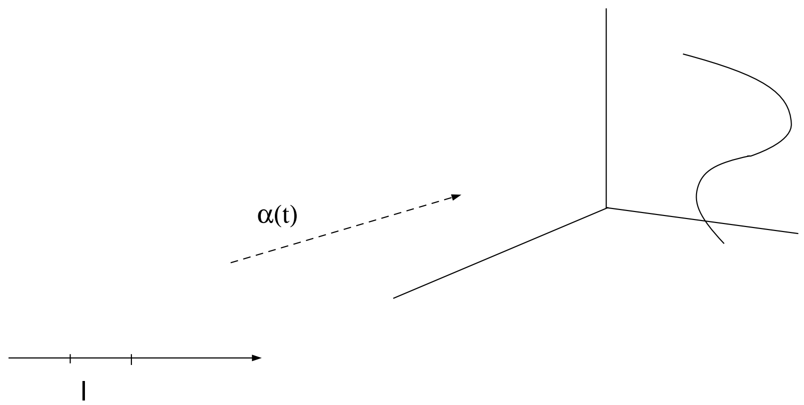
Differential Geometry of Surfaces

3.1 Week 9

We start by introducing the notion of **regular surface in** $\mathbb{R}^3$. In short, a regular surface in space is a obtained by taking pieces of a plane, deforming them and arranging them in such a way that the resulting figure has no sharp points or edges, or self-intersections and so that it makes sense to speak of a tangent plane at points of the figure.

3.1.1 Regular surface. Inverse images of regular values.

When we studied the geometry of curves in $\mathbb{R}^2$ and $\mathbb{R}^3$, we defined them by means of parametrizations, or differentiable mappings α from an interval I of $\mathbb{R}$ into $\mathbb{R}^2$ of $\mathbb{R}^3$. We called the image of such parametrizations the trace of α. Obviously, the trace of α is a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. So, we can define a curve either as a differentiable map (α) or as a set $(tr(\alpha))$ in $\mathbb{R}^2$ or $\mathbb{R}^3$.



In the same way, we can define a regular surface in $\mathbb{R}^3$ either as a differentiable map (we shall see below) or as a subset of $\mathbb{R}^3$.

We start with the definition of a regular surface as a subset of $\mathbb{R}^3$.

Definition 3.1.1 *A subset $S \subset \mathbb{R}^3$ is a* **regular surface** *if, for each $p \in S$, there exist a neighbourhood $V \subset \mathbb{R}^3$ and a map $\alpha : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \mathbb{R}^3$ such that:*

1. *α is differentiable, i.e.*

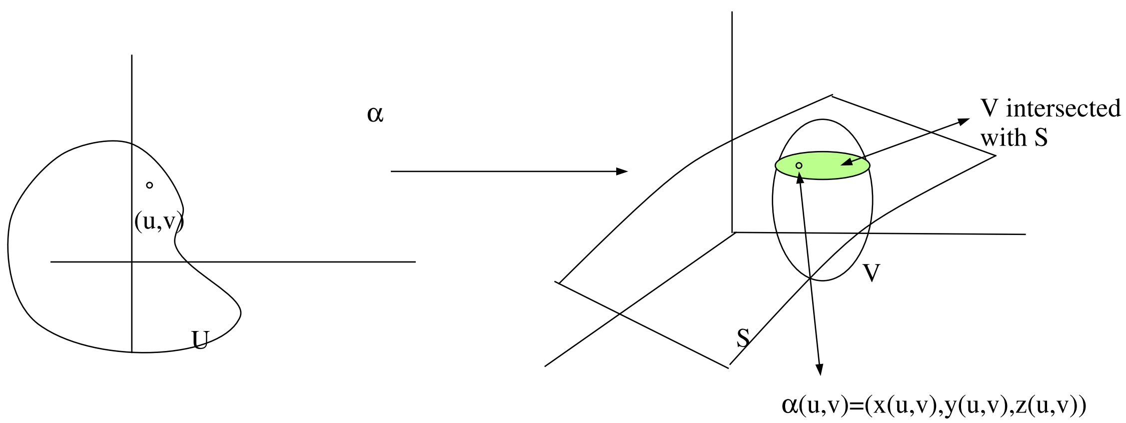
$$\alpha(u, v) = (x(u, v), y(u, v), z(u, v)), \ (u, v) \in U,$$

the functions $x(u, v), y(u, v), z(u, v)$ have continuous partial derivatives of all orders in U;

2. *α is a homeomorphism, i.e. α is continuous, $\alpha^{-1} : V \cap S \to U$ exists and is also continuous;*

3. *For each $(u, v) \in U$, the differential $d\alpha_{(u,v)}$ is one-to-one.*

The definition can be better understood on the following image:



Condition number 3 from Definition 3.1.1 can be expressed in the more familiar form: *for each $(u, v) \in U$, the column vectors of the linear map*

$$d\alpha_{u,v} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

are linearly independent.

Another equivalent formulation of the same condition is the following:
for each $(u,v) \in U$, one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} := \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}, \ \frac{\partial(y,z)}{\partial(u,v)}, \ \frac{\partial(x,z)}{\partial(u,v)}$$

is different from 0.

Example 3.1.1 Intuitively, the sphere $S^2 := \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$ seems to fit the requirements of a regular surface. Let us show that S^2 is indeed such a surface, using Definition 3.1.1.

We are supposed to take open pieces U of $\mathbb{R}^2$, deform them and put them together so that they cover the sphere; moreover, each such correspondence between our open sets U and pieces of S^2 should satisfy the conditions of the Definition 3.1.1.

Suppose $U := \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ and take

$$\alpha_1 : U \to \mathbb{R}^3, \text{ given by } \alpha_1(x,y) = (x,y,\sqrt{1-(x^2+y^2)}).$$

Note that $\alpha(U)$ is the upper hemisphere, without the equator.

Since $x^2+y^2 < 1$, then $\sqrt{1-(x^2+y^2)}$ has continuous partial derivatives of all orders. Thus α_1 is differentiable. Obviously, $\frac{\partial(x,y)}{\partial(x,y)} = 1$ in this case and so condition 3 is satisfied. For condition 2, we define $\alpha_1^{-1} : \alpha(U) \to U$, so that $\alpha_1^{-1}(x,y,\sqrt{1-(x^2+y^2)}) = \pi(x,y,\sqrt{1-(x^2+y^2)}) = (x,y)$, where π is the projection on the third factor of $\mathbb{R}^3$ (see Problem 1.3.6, Week 3) and so α_1^{-1} exists and is continuous, hence α_1 is a homeomorphism on its image (see also Definition 1.3.4 and Lemma 1.3.1 Week 3).

To show that S^2 is a regular surface, we need to cover it all with parametrizations of the type of α_1. We therefore need 6 of them and they are defined as follows:

- $U = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 < 1\}$ and

$$\alpha_2 : U \to \mathbb{R}^3, \text{ given by } \alpha_2(x,y) = (x,y,-\sqrt{1-(x^2+y^2)});$$

- $V = \{(x,z) \in \mathbb{R}^2 \mid x^2+z^2 < 1\}$ and

$$\alpha_3 : V \to \mathbb{R}^3, \text{ given by } \alpha_3(x,z) = (x,\sqrt{1-(x^2+z^2)},z);$$

$$\alpha_4 : V \to \mathbb{R}^3, \text{ given by } \alpha_4(x,z) = (x,-\sqrt{1-(x^2+z^2)},z);$$

- $W = \{(y,z) \in \mathbb{R}^2 \mid y^2 + z^2 < 1\}$ and

$$\alpha_5 : W \to \mathbb{R}^3, \text{ given by } \alpha_5(y,z) = (\sqrt{1-(y^2+z^2)}, y, z);$$

$$\alpha_6 : W \to \mathbb{R}^3, \text{ given by } \alpha_6(y,z) = (-\sqrt{1-(y^2+z^2)}, y, z).$$

Now, for each point $p \in S^2$, we have found neighbourhoods in $\mathbb{R}^3$ and parametrizations of open sets in $\mathbb{R}^2$ satisfying Definition 3.1.1, so S^2 is a regular surface.

Obviously, the use of Definition 3.1.1 is not easy, so we are looking for simpler ways to show that a surface is regular. This can be done via the two Propositions below.

Proposition 3.1.1 *If $f : U \to \mathbb{R}$ is a differentiable function on an open subset U of $\mathbb{R}^2$, then the graph of f, i.e.*

$$graph(f) = \{(x,y,z) \in \mathbb{R}^3 \mid z = f(x,y)\},$$

is a regular surface.

Definition 3.1.2 *Given a differentiable map $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, then a point $p = (x,y,z) \in U$ is called* **a critical point** *of f if $f_x = f_y = f_z = 0$ at p.*

Definition 3.1.3 *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable map and p a critical point of f, then the image $f(p) \in \mathbb{R}$ is called* **a critical value**. *Finally, any point of $\mathbb{R}$ which is not a critical value of f is called* **a regular value**.

With the definitions above, we have the following result.

Proposition 3.1.2 *If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable map and$a \in f(U)$ is a regular value of f, then*

$$f^{-1}(a) = \{(x,y,z \in U \mid f(x,y,z) = a)\}$$

is a regular surface in $\mathbb{R}^3$.

Example 3.1.2 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface, as the inverse image of the regular value 0 via the mapping $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$. 0 is a regular value since $f_x = f_y = f_z = 0$ only happens for the point $(0,0,0)$ which does not belong to $f^{-1}(0)$. In the particular case when $a = b = c$, then the ellipsoid becomes the sphere S^2.

Definition 3.1.4 *A regular surface S is called* **connected** *if any two of its points can be joined by a continuous curve in S.*

Both the ellipsoid and the sphere are connected surfaces.

Example 3.1.3 The hyperboloid of two sheets $-x^2 - y^2 + z^2 = 1$ is a regular surface given by $S = f^{-1}(0)$, where $f(x, y, z) = -x^2 - y^2 + z^2 - 1$. This surface is not connected (sketch the surface at home).

The next result is a local converse result of Proposition 3.1.1.

Proposition 3.1.3 *Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists a neighbourhood V of p in S such that V is the graph of a differentiable function which has one of of the following three forms: $z = f(x, y)$, $y = g(x, z)$ or $x = h(y, z)$.*

Proposition 3.1.3 is very useful in proving that there are geometric shapes in $\mathbb{R}^3$ which are not regular surfaces. Here is one example.

Example 3.1.4 The one-sheeted cone C given by $z = \sqrt{x^2 + y^2}$ is not a regular surface. Note that we cannot conclude this just by noticing that the "natural parametrization"

$$(x, y) \to (x, y, \sqrt{x^2 + y^2})$$

is not differentiable. There could be other parametrizations which satisfy Definition 3.1.1. However, is we use Proposition 3.1.3, we could rigorously prove the claim.

Assume by contradiction that C is a regular surface. Then by Proposition 3.1.3, then it would be, in a neighbourhood of the point (0,0,0), the graph of a differentiable function having one of the forms $z = f(x, y)$, $y = g(x, z)$ or $x = h(y, z)$. These forms give

$$z = \sqrt{x^2 + y^2}, y = \pm\sqrt{(z^2 - x^2)}, x = \pm\sqrt{z^2 - y^2}.$$

In the neighbourhood of (0,0,0), neither of these functions are differentiable, hence C is not a regular surface.

3.1.2 The differential of a differentiable mapping

In order to understand better Definition 3.1.1, condition 3, and the material in the coming weeks, we will now make precise the notion of differential of a mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, the way we compute such a map and its properties.

We begin first by noticing the following:

Remark 3.1.1 *For any $p \in U \subset \mathbb{R}^n$, U open and $w \in \mathbb{R}^n$, we can always find a differentiable curve $\gamma : (-\epsilon, \epsilon) \to U$ so that $\gamma(0) = p$ and $\gamma'(0) = w$.*
To see this, define $\gamma(t) = p + tw$.

Definition 3.1.5 *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable mapping. To each $p \in U$, we associate a linear mapping $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ which is called the differential of F at p and is defined as follows:*
let $w \in \mathbb{R}^n$ and $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ be a differential curve with $\gamma(0) = p$ and $\gamma'(0) = w$; then the curve $\beta = F \circ \gamma : (-\epsilon, \epsilon) \to \mathbb{R}^m$ is also differentiable and

$$dF_p(w) = \beta'(0).$$

We can show that the definition of dF_p does not depend on γ or w, and is in fact a linear map. In order to deduce a more simple expression of dF, let us look at Definition 3.2.5 again for the case when $n = 2$ and $m = 3$.

Suppose that $p := (u, v)$, that $\gamma(t) = (u(t), v(t))$, $F(u, v) = (x(u, v), y(u, v), z(u, v))$. Then obviously

$$\beta(t) = F(\gamma(t)) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t))).$$

Then using the Chain Rule when taking derivatives and evaluating at $t = 0$, we get the following:

$$\beta'(0) = \frac{d}{dt}(F(\gamma(t))) \mid_{t=0} = dF(\gamma(t)) \mid_{t=0} \frac{d}{dt}(\gamma(t)) \mid_{t=0} = dF_p(w).$$

On the other hand,

$$\beta'(0) = \frac{d}{dt}(F(\gamma(t))) \mid_{t=0} = \frac{d}{dt}((x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))) \mid_{t=0}$$

$$= (\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}, \frac{\partial y}{\partial u}\frac{du}{dt} + \frac{\partial y}{\partial v}\frac{dv}{dt}, \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt}) \mid_{t=0}$$

$$= \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \mid_p \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} \mid_{t=0}$$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \mid_p w.$$

From the two expressions of $\beta'(0)$ we deduce that

$$dF_p = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} \mid_p$$

and is therefore a linear mapping.

Remark 3.1.2 *Note that a similar computation can be carried out for arbitrary dimensions n, m and the result will be*

$$dF_p = \left((\frac{\partial f_i}{\partial x_j}) \mid_p \right)_{i=1..,nj=1,..,m}.$$

When $n = m$, the matrix of dF_p is square and its determinant is called the Jacobian determinant.

Example 3.1.5 *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, $F(u, v) = (u^2 - v^2, 2uv)$. Then*

$$dF_{p=(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} 2u & -2v \\ 2u & 2v \end{pmatrix}.$$

Then if $p = (1, 1)$, we get

$$dF_p = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$$

and if $w = (2, 3)$ we get

$$dF_p(w) = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}.$$

3.1.3 Examples and Problems

Example 3.1.6 *Show that the cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$ is a regular surface. Find parametrizations of the cylinder whose coordinate neighbourhoods are covering it.*

Example 3.1.7 *Let $f(x, y, z) = z^2$. Show that 0 is not a regular value of f, however, $f^{-1}(0)$ is a regular surface.*

Problem 3.1.1 *Find a parametrization of the hyperboloid with two sheets $\{(x, y, z) \in \mathbb{R}^3 \mid -x^2 - y^2 + z^2 = 1\}$.*

Problem 3.1.2

Problem 3.1.3 *Let $f(x, y, z) = (x + y + z - 1)^2$.*
a) Find the critical points and the critical values of f.
b) For what values of c is the set $f(x, y, z) = c$ a regular surface?

Problem 3.1.4 *Let V be an open set in the xy-plane. Show that the set*

$$(x, y, z) \in \mathbb{R}^3 \mid z = 0 \text{ and } (x, y) \in V$$

is a regular surface.

Problem 3.1.5 *Show that* $\alpha : U \in \mathbb{R}^2 \to \mathbb{R}^3$ *given by*

$$\alpha(u, v) = (a \sin u \cos v, b \sin u \sin v, ccosu), \;\; a, b, c \neq 0$$

where $0 < u < \pi, 0 < v < 2\pi$*, is a parametrization for the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Problem 3.1.6 *Let* $\alpha(u, v)$ *as given in Definition 3.1.1. Verify that the differential* $d\alpha_{(u,v)}$ *is one-to-one if and only if*

$$\frac{\partial \alpha}{\partial u} \wedge \frac{\partial \alpha}{\partial v} \neq 0.$$

3.3 Week 11

We continue the study of regular surfaces in analogy to that of regular parametrized curves. We have seen that for such a curve, we can always define the tangent vector to the curve at any point. Similarly, we should be able to define a tangent plane to a regular surface at any point of this surface.

3.3.1 The tangent plane to a regular surface

Let S be a regular surface in $\mathbb{R}^3$. We begin by giving the following definition.

Definition 3.3.1 *By* **a tangent vector to** S **at** $p \in S$ *we mean the tangent vector* $\gamma'(0)$ *of a differentiable parametrized curve* $\gamma : (-\epsilon, \epsilon) \to S$*, with* $\gamma(0) = p$*.*

The tangent vectors given by Definition 3.3.1 above are characterized by the following:

Proposition 3.3.1 *Let* S *be a regular surface in* $\mathbb{R}^3$*. Let* $\alpha : U \subset \mathbb{R}^2 \to S$ *be some parametrization of* S *and let* $(u, v) \in U$*. Then the vector subspace of dimension 2 (the plane)*

$$d\alpha_{(u,v)}(\mathbb{R}^2) \subset \mathbb{R}^3$$

coincides with the set of tangent vectors to S *at the point* $\alpha(u, v)$*.*

Proof: For a proof of this result, the reader is directed to [1], Chapter 2. □

Remark 3.3.1 *The plane* $d\alpha_{(u,v)}$ *does not depend on the parametrization* α *and will be called* **the tangent plane to** S **at** $p = \alpha(u, v)$*. It is usually denoted by* T_pS*.*

Example 3.3.1 *Let* $C = \{(x, y, z) \mid x^2 + y^2 = 1\}$ *be the cylinder of radius 1. We saw that* C *is a regular surface (Week 9). Let us compute* T_pC*, where* $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 3)$*.*

Using Proposition 3.3.1, we need to use a parametrization of C around the given point p. We choose

$$\alpha : (-1,1) \times \mathbb{R} \to C,\ \alpha(y,z) = (\sqrt{1-y^2}, y, z).$$

Evidently $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 3)$ belongs to $\alpha((-1,1)\times\mathbb{R})$.

Now from Week 10, we compute the differential of the parametrization α as follows:

$$d\alpha_{(y,z)} = \begin{pmatrix} \frac{-y}{\sqrt{1-y^2}} & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The point (y,z) so that $\alpha(y,z) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 3)$ is $(y,z) = (\frac{1}{\sqrt{2}}, 3)$, so

$$d\alpha_{(\frac{1}{\sqrt{2}},3)} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the plane in $\mathbb{R}^3$ given by $d\alpha_{(\frac{1}{\sqrt{2}},3)}(\mathbb{R}^2)$ is computed as follows:

$$d\alpha_{(\frac{1}{\sqrt{2}},3)}(\begin{pmatrix} a \\ b \end{pmatrix}) = \begin{pmatrix} -a \\ a \\ b \end{pmatrix},$$

so

$$T_{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},3)}C = \{(x,y,z) \in \mathbb{R}^3 \mid x = -y\}.$$

We have seen that in order to check whether a surface S is regular, we can employ easier criteria, like those in Propositions 3.1.1 and 3.1.2 of Week 9.

Based on these propositions, we have easier ways to compute the tangent planes to a regular surface S, provided S is the graph of a 2-dimensional vector function or is the inverse image of a regular value of a 3-dimensional vector function.

Proposition 3.3.2 *Suppose S is the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = z$. Then the tangent plane to S at a point $p = (x_0, y_0, z_0)$ has the equation*

$$z = f(x_0,y_0) + \frac{\partial f}{\partial x}\mid_{(x_0,y_0,z_0)} (x - x_0) + \frac{\partial f}{\partial y}\mid_{(x_0,y_0,z_0)} (y - y_0)$$

Proposition 3.3.3 *Suppose 0 is a regular value of $f : \mathbb{R}^3 \to \mathbb{R}$ and $S = f^{-1}(0)$. Then the tangent plane to S at a point $p = (x_0, y_0, z_0)$ has the equation*

$$\frac{\partial f}{\partial x}\mid_{(x_0,y_0,z_0)} (x - x_0) + \frac{\partial f}{\partial y}\mid_{(x_0,y_0,z_0)} (y - y_0) + \frac{\partial f}{\partial z}\mid_{(x_0,y_0,z_0)} (z - z_0).$$

These last results should be more than familiar to the reader from Vector Calculus.

3.3.2 The normal vector.

Let S be a regular surface. Then at each $p \in S$ we saw we can compute T_pS. At the same point p, there are two unit vectors that are perpendicular on T_pS. Each of them is called **a normal unit vector** at p. The straight line passing through p containing the normal unit vector is called **the normal line** to p. By fixing a parametrization, we can make a definite choice of a normal vector as follows:

Definition 3.3.2 *Let $\alpha : U \to S$ be a parametrization of S at $p \in S$. Then the normal vector at each point $q = \alpha(U)$ is given by the rule*

$$N(q) := \frac{\frac{\partial\alpha}{\partial u} \wedge \frac{\partial\alpha}{\partial v}}{|\frac{\partial\alpha}{\partial u} \wedge \frac{\partial\alpha}{\partial v}|}(q).$$

Thus we obtain a differentiable map $N : \alpha(U) \to \mathbb{R}^3$, called **the Gauss map**.

3.3.3 Examples and Problems

Problem 3.3.1 *Using the parametrization of S^2 from Week 9, compute the tangent spaces T_pS^2 at $p = (0, 0, 1)$ and at $(0, 0, -1)$.*

Problem 3.3.2 *Determine the tangent planes of the paraboloid $x^2+y^2-z^2 = 1$ at the points $(x, y, 0)$ and show that they are parallel to the z-axis.*

Problem 3.3.3 *Show that the tangent planes of a surface given by $z = xf(\frac{y}{x})$, $x \neq 0$, where f is differentiable, all pass through the origin (0,0,0).*

Problem 3.3.4 *The torus T is a surface generated by a rotating circle S^1 of radius r about a straight line belonging to the plane of the circle and at a distance $a > r$ away from the center of the circle.*
a) Show that T is given by the equation $z^2 = r^2 - (\sqrt{x^2 + y^2} - a)^2$.
b) Show that T is a regular surface.
c) Write the equation of the tangent planes to T, using Prop. 3.3.3 above.

Problem 3.3.5 *Compute the normal vector $N(q)$ for the regular surface with parametrization $\alpha(u, v) = (v \cos u, v \sin u, au)$, $a \neq 0$.*

3.4 Week 12

3.4.1 Review

Bibliography

[1] DoCARMO, M. P., **Riemannian Geometry of Curves and Surfaces**, Prentice Hall, 1976.

[2] HERSTEIN, I. N., **Abstract Algebra**, 3rd Edition, John Wiley & Sons (1999)

[3] MUNKRES, J. R., **Topology**, Second edition, Prentice Hall (1975, 2000)