Proc. Natl. Acad. Sci. USA
Vol. 74, No. 7, pp. 2655–2658, July 1977
Mathematics

A new class of counterexamples to the integrability problem

(Lie pseudogroup/transitive Lie algebra/Lewy operator/Spencer cohomology)

JACK F. CONN

Department of Mathematics, Princeton University, Princeton, New Jersey 08540

Communicated by D. C. Spencer, April 12, 1977

ABSTRACT For many years, it was believed that the solvability of the integrability problem for a transitive Lie pseudogroup depends only on the local solvability of linear differential operators arising from the abelian quotients in Guillemin's Jordan–Hölder decomposition for the transitive Lie algebra associated to the pseudogroup. We provide an example of a transitive pseudogroup for which the integrability problem is not solvable and whose corresponding Jordan–Hölder sequence has only non-abelian quotients.

Introduction

The first example of a transitive Lie pseudogroup for which the integrability problem is not solvable was given by Guillemin and Sternberg (ref. 1). As in all previously known counterexamples to the integrability problem for transitive pseudogroups, the transitive Lie algebra of formal infinitesimal transformations of this pseudogroup has a Jordan–Hölder sequence in which an abelian quotient appears. The nonsolvability of the integrability problem for this pseudogroup arises from the local nonsolvability of a linear differential operator determined by this abelian quotient. In this paper, we construct a transitive pseudogroup Γ for which the integrability problem is not solvable and whose associated transitive Lie algebra L of formal infinitesimal transformations admits a unique Jordan–Hölder sequence

$$L \supset I \supset \{0\}$$

in which I is a non-abelian minimal closed ideal of L and L/I is a simple non-abelian finite-dimensional Lie algebra. Our construction is based on the example of Guillemin and Sternberg (ref. 1) and, just as in that paper, the nonsolvability of the integrability problem for Γ comes from the local nonsolvability of the linear differential operator of Lewy (ref. 2).

The pseudogroup Γ is composed of local diffeomorphisms of the complex line bundle $X = S^3 \times \mathbf{C}$ over the 3-sphere S^3. If we consider S^3 as a real submanifold of $\mathbf{C}^2$, the Lewy operator is the tangential Cauchy–Riemann operator $\bar{\partial}_b$ on S^3. If z is a complex coordinate on $\mathbf{C}$, the set of all infinitesimal transformations of Γ corresponding to the ideal I of L consists of all real parts of vector fields on X of the form $h(\partial/\partial z)$, in which h is a local complex-valued function on X which is holomorphic in z and satisfies the tangential Cauchy–Riemann equation $\bar{\partial}_b h = 0$. This realization of the ideal I does not correspond to the formal description of I given by the structure theorem of ref. 3; however, at the end of section 2, we do recover this representation of I by means of a formal version of the Lewy extension phenomenon.

If stated in terms of the nonlinear Spencer cohomology of ref. 4, our result says that the nonlinear cohomology $\tilde{H}^1(L)$ of L does not vanish. Moreover, because L/I is finite-dimensional, $\tilde{H}^1(L/I) = 0$ and so corollary 10.1 of ref. 4 implies that the nonlinear cohomology $\tilde{H}^1(L,I)$ of the non-abelian minimal closed ideal I of L does not vanish. We provide therefore a counterexample to conjectures I and II of ref. 4. These statements actually hold for the class of all transitive Lie algebras defined by Eq. 8 of section 2, which includes our Lie algebra L. The vanishing of the nonlinear cohomology of a transitive Lie algebra does not depend only on the quotients appearing in a Jordan–Hölder sequence for this algebra. In fact, there is a transitive Lie algebra L' that admits a Jordan–Hölder sequence whose quotients are isomorphic to those of L and whose nonlinear cohomology $\tilde{H}^1(L')$ vanishes; furthermore, L' has a unique nontrivial closed ideal I', which is isomorphic to I as a topological Lie algebra and satisfies $\tilde{H}^1(L',I') = 0$. Thus, the nonlinear cohomology of a closed ideal of a transitive Lie algebra depends in an essential way on its structure as a module over that Lie algebra, even if the ideal is minimal and non-abelian.

1. Construction of the counterexample

Let X be a manifold of class C^∞ of dimension $2n - 1$, whose tangent bundle we denote by $T(X)$. Let $\mathcal{O}_X$ be the sheaf of complex-valued functions on X. If E is a vector bundle over X, we denote by $\mathcal{E}$ the sheaf of sections of E over X. An almost pseudo-complex structure of codimension one is a complex sub-bundle E'' of rank $n - 1$ (over $\mathbf{C}$) of the complexified tangent bundle $\mathbf{C}T(X)$ of X such that E'' and its complex conjugate have a zero intersection. Let

$$\bar{\partial}_b : \mathcal{O}_X \to \mathcal{E}''^*$$

be the differential operator that is equal to the usual exterior differential operator $d: \mathcal{O}_X \to \mathbf{C}\mathcal{T}(X)^*$ followed by restriction from $\mathbf{C}T(X)$ to $\mathcal{E}''$. Such an almost pseudo-complex structure is a pseudo-complex structure of codimension one if and only if it is induced by an imbedding of X into a complex manifold W with $\dim_{\mathbf{C}} W = n$, in which case E'' is the intersection of $\mathbf{C}T(X)$ with the restriction to X of the bundle of complex vectors of type (0,1) tangent to W.

Let G be the three-dimensional real Lie group $SU(2)$. We choose a basis $\{\eta_1,\eta_2,\eta_3\}$ for the Lie algebra of left-invariant vector fields on G such that the relations

$$[\eta_k,\eta_l] = \eta_m$$

hold for all cyclic permutations (k,l,m) of (1,2,3). The sub-bundle E'' of $T(G)$ generated by the complex vector field $\eta_1 + i\eta_2$ on G is an almost pseudo-complex structure of codimension one on G. Because $\eta_1 + i\eta_2$ is a nowhere-vanishing section of E'' over G, we may identify the differential operator $\bar{\partial}_b$ with the derivation of the sheaf $\mathcal{O}_G$ induced by $\eta_1 + i\eta_2$. Under the well-known identification of G with the sphere $S^3 \subset \mathbf{C}^2$, each left translation of G is the restriction to S^3 of a biholomorphic mapping of $\mathbf{C}^2$ that preserves the sphere. The almost pseudo-complex structure E'' on G coincides with the pseudo-complex structure induced by the imbedding of S^3 in $\mathbf{C}^2$, and the operator $\bar{\partial}_b$ on G coincides essentially with the tangential Cauchy–Riemann operator on S^3, which is the famous locally nonsolvable operator of Lewy (ref. 2).

Let X be the trivial complex line bundle $\pi: G \times \mathbf{C} \to G$ over G, viewed as a five-dimensional real manifold. If ζ is a vector field on G or $\mathbf{C}$, we also denote by ζ the vector field it induces on the product $G \times \mathbf{C}$. Let $z = u + iv$ be a complex coordinate for $\mathbf{C}$. Let $\bar\partial$ and $\bar\partial_b$ be the differential operators on $\mathcal{O}_X$ induced by the vector fields $(\partial/\partial u) + i(\partial/\partial v)$ and $\eta_1 + i\eta_2$ on X, respectively. Let H be the closed subgroup of $Gl(5,\mathbf{R})$ consisting of all 5×5 matrices of the form

$$\begin{pmatrix} \multicolumn{3}{c|}{I_{3\times 3}} & \multicolumn{2}{c}{O_{3\times 2}} \\ \hline a & -b & c & e & -f \\ b & a & d & f & e \end{pmatrix} \qquad [1]$$

in which $a, b, c, d, e, f \in \mathbf{R}$ satisfy $e^2 + f^2 \neq 0$, and $I_{3\times 3}$ denotes the 3×3 identity matrix and $O_{3\times 2}$ the 3×2 matrix all of whose entries are equal to 0. Let B_H be the H-structure on X whose global section is the frame

$$\left(\eta_1, \eta_2, \eta_3, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$$

for $T(X)$ over X.

We now compute the pseudogroup Γ of all local diffeomorphisms of X which induce local automorphisms of B_H. If U is an open subset of X, a mapping $f: U \to X$ is given by $f = (\varphi,\psi)$, where $\varphi: U \to G$ and $\psi: U \to \mathbf{C}$. In view of the definition 1 of the group H, it is easily seen that a local diffeomorphism $f = (\varphi, \psi)$ of X defined on an open set U induces a local automorphism of B_H if and only if it satisfies the equations

$$\varphi^* \frac{\partial}{\partial u} = \varphi^* \frac{\partial}{\partial v} = 0, \qquad \varphi^* \eta_j = \eta_j, \qquad [2]$$

for $1 \leq j \leq 3$, and

$$\bar\partial\psi = 0, \qquad \bar\partial_b\psi = 0. \qquad [3]$$

If $a \in G$, we denote by L_a the left-translation of G by a and let $\varphi_a: X \to G$ be the mapping $L_a \circ \pi$. If U is connected, by Frobenius' theorem a mapping $\varphi: U \to G$ satisfies Eqs. 2 if and only if there exists an element a of G such that φ is the restriction to U of φ_a. If $f = (\varphi,\psi)$ is an element of Γ defined on this open set U, it is a local automorphism of the bundle X over a left-translation of G. If $\psi: U \to \mathbf{C}$ satisfies Eqs. 3, then the mapping $f: U \to X$ given by (π,ψ) satisfies Eqs. 2 and 3. Thus, the automorphisms $\tilde\psi_a = \psi_a \times \mathrm{id}$ and $\tilde\tau_w = \mathrm{id} \times \tau_w$ of X, in which τ_w is the translation of $\mathbf{C}$ by $w \in \mathbf{C}$, belong to Γ, and so we see that the pseudogroup Γ is transitive.

Let $\{\xi_1,\xi_2,\xi_3\}$ be a basis for the Lie algebra $\mathcal{G}$ of right-invariant vector fields on G. If ζ is a vector field defined on an open subset U of X, we can write ζ uniquely in the form

$$\zeta = \sum_{j=1}^{3} f_j\xi_j + h_1 \frac{\partial}{\partial u} + h_2 \frac{\partial}{\partial v},$$

where f_1, f_2, f_3, h_1, h_2 are real-valued functions on U. This vector field ζ is an infinitesimal transformation of Γ if and only if

$$\eta_k f_j = 0, \qquad \frac{\partial}{\partial u} f_j = \frac{\partial}{\partial v} f_j = 0, \qquad [4]$$

for $1 \leq j, k \leq 3$ and

$$\bar\partial h = 0, \qquad \bar\partial_b h = 0, \qquad [5]$$

in which $h = h_1 + ih_2$. If U is connected, Eqs. 4 are satisfied if and only if the functions f_j are constant; therefore, an infinitesimal transformation of Γ defined on an open connected subset U of X is a vector field of the form

$$\xi + h_1 \frac{\partial}{\partial u} + h_2 \frac{\partial}{\partial v},$$

in which ξ is the vector field on X induced by a right-invariant vector field on G and the complex-valued function $h = h_1 + ih_2$ satisfies Eqs. 5.

We now construct a second H-structure B'_H on X that is formally equivalent to B_H. Let g_1, g_2 be real-valued functions on the group G and set $\tilde g_i = g_i \circ \pi$, for $i = 1,2$. Let B'_H be the H-structure on X whose global section is the frame

$$\left(\eta_1 + \tilde g_1 \frac{\partial}{\partial u}, \eta_2 + \tilde g_2 \frac{\partial}{\partial u}, \eta_3, \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$$

for $T(X)$ over X. It is easily seen that a local diffeomorphism $f = (\varphi,\psi)$ of X induces a local equivalence of B_H with B'_H if and only if φ satisfies Eqs. 2 for $1 \leq j \leq 3$ and the equations

$$\bar\partial\psi = 0, \qquad \bar\partial_b\psi = (g_1 + ig_2) \circ \varphi \qquad [6]$$

are satisfied. Because Γ is transitive, to prove that B_H and B'_H are formally equivalent, it suffices to find for all $x = (a,z) \in X$ a formal equivalence of B_H with B'_H with source $(e,0)$ and target x, in which e is the identity element of G. In fact, because the differential operator $\bar\partial_b$ on G is formally solvable, we may choose a complex-valued function u on G satisfying $u(e) = z$ and whose infinite jet at e is a formal solution of the differential equation

$$\bar\partial_b u = (g_1 + ig_2) \circ L_a \qquad [7]$$

on G. If $\psi: X \to \mathbf{C}$ sends (b,w) into $[b, w + u(b)]$, then $f = (\varphi_a, \psi)$ is an automorphism of X whose infinite jet at $(e,0)$ is a formal equivalence of B_H with B'_H.

Let $a \in G$. Because the operator $\bar\partial_b$ on G is not locally solvable at e, we may choose the functions g_1, g_2 on G such that Eq. 7 has no solution u on any neighborhood of e. If a local diffeomorphism $f = (\varphi,\psi)$ defined on a neighborhood of $(e,z) \in X$ satisfies $\pi f(e,z) = a$ and induces a local equivalence of B_H with B'_H, then $\varphi = \varphi_a$ on a neighborhood of (e,z) and the function u on G, defined by $u(b) = \psi(b,z)$, for $b \in G$, is a solution of Eq. 7 on a neighborhood of e. We conclude that:

THEOREM. *Let* a $\in$ G. *There exist functions* g_1, g_2 *on* G *such that the* H*-structures* B_H *and* B'_H *are formally equivalent and such that there are no local equivalences* f *of* B_H *with* B'_H *defined on a neighborhood of* (e,z) $\in$ X *satisfying* $\pi f(e,z) = a$.

2. Formal infinitesimal transformations

We endow the field of real numbers $\mathbf{R}$ and the field of complex numbers $\mathbf{C}$ with the discrete topology. A transitive Lie algebra L over $\mathbf{R}$ (or $\mathbf{C}$) is a linearly compact topological Lie algebra over $\mathbf{R}$ (or $\mathbf{C}$) that possesses a fundamental subalgebra, that is, an open subalgebra L^0 containing no ideals of L other than $\{0\}$. For a more detailed exposition of transitive Lie algebras, the reader may wish to refer to ref. 3. Let A be a closed subalgebra of a transitive Lie algebra L. We inductively define closed subalgebras $D_L{}^k(A)$ of L, for $k \geq 0$, by

$$D_L{}^0(A) = A, \qquad D_L{}^{k+1}(A) = \{\xi \in D_L{}^k(A) \mid [L,\xi] \subset D_L{}^k(A)\};$$

the closed subalgebra

$$D_L{}^\infty(A) = \bigcap_{k\geq 0} D_L{}^k(A)$$

of L is an ideal of L in A and contains all ideals of L in A. When applied to a fundamental subalgebra L^0 of L, this construction yields a family of open subalgebras $D_L{}^k(L^0)$ that form a fundamental system of neighborhoods of 0 in L. Let R be a real transitive Lie algebra; assume that R is non-abelian and simple, i.e., having no nontrivial ideals. The commutator field K_R of R is the space of all endomorphisms u of R satisfying

$$u([\xi,\eta]) = [\xi,u(\eta)],$$

for all $\xi, \eta \in R$. This space K_R is a subalgebra of the algebra of continuous linear endomorphisms of R and, according to proposition 4.4 of ref. 3, is actually a field isomorphic to $\mathbf{R}$ or $\mathbf{C}$. We shall suppose in the sequel that K_R is isomorphic to $\mathbf{C}$; we may thus consider R as a transitive Lie algebra over $\mathbf{C}$. Let R^0 be a fundamental $\mathbf{C}$-subalgebra of R; we write $R^{-1} = R$ and $R^k = D_R{}^kR^0$, for $k \geq 1$.

Let V, W be linearly compact topological vector spaces over $\mathbf{C}$ whose topological duals we denote by V^*, W^*. We define the completed tensor product $V \hat{\otimes}_{\mathbf{C}} W$ to be the linearly compact topological vector space that is the topological dual of the algebraic tensor product $V^* \otimes_{\mathbf{C}} W^*$ endowed with the discrete topology. There is a natural injective mapping

$$V \otimes_{\mathbf{C}} W \rightarrow V \hat{\otimes}_{\mathbf{C}} W,$$

which allows us to identify $V \otimes_{\mathbf{C}} W$ with a dense subspace of $V \hat{\otimes}_{\mathbf{C}} W$. If W is a linearly compact topological associative algebra over $\mathbf{C}$, there is a unique structure of topological Lie algebra on $R \hat{\otimes}_{\mathbf{C}} W$ that extends the standard Lie algebra structure on $R \otimes_{\mathbf{C}} W$.

Let F denote the ring of jets of infinite order at e of complex-valued functions on G. Then F is a local ring whose maximal ideal F^0 consists of the jets of functions on G that vanish at e. Set $F^{-1} = F$. If F^k is the $(k+1)$-rst power of F^0, the ideals $\{F^k\}_{k\geq 0}$ form a fundamental system of neighborhoods of 0 for the Krull topology on F. The ring F, endowed with this topology, is a linearly compact topological algebra over $\mathbf{C}$. The algebra F is the topological direct sum of F^0 and its closed subspace V consisting of the jets of constant functions on G. Let T be the Lie algebra of jets of infinite order at e of *real* vector fields on G and let T^k be the subalgebra of T of jets of vector fields that vanish to order k at e. By taking the subalgebras $\{T^k\}_{k\geq 0}$ to be a fundamental system of neighborhoods of 0 in T, we obtain a structure of real transitive Lie algebra T such that T^0 is a fundamental subalgebra of T. An element of T induces a continuous derivation of F, and belongs to T^k if and only if the derivation maps F into F^k. The action of $\zeta \in T$ on $R \otimes_{\mathbf{C}} F$ determined by

$$\zeta \cdot (\eta \otimes f) = \eta \otimes (\zeta \cdot f),$$

for $\eta \in R, f \in F$, extends uniquely to a continuous derivation of $R \hat{\otimes}_{\mathbf{C}} F$ and gives us a structure of topological T-module on $R \hat{\otimes}_{\mathbf{C}} F$. The semi-direct product

$$S = (R \hat{\otimes}_{\mathbf{C}} F) \oplus T$$

is a linearly compact topological Lie algebra over $\mathbf{R}$. The closed subalgebra

$$S^0 = (R^0 \hat{\otimes}_{\mathbf{C}} F + R \hat{\otimes}_{\mathbf{C}} F^0) \oplus T^0$$

of S is clearly of finite codimension in S and so is open. It is easily seen that

$$D_S{}^k(S^0) \subset \Big(\sum_{l+m=k-1} R^l \hat{\otimes} F^m\Big) \oplus T^k,$$

for $k \geq 0$; hence $D_S{}^\infty(S^0) = \{0\}$. It follows that S^0 is a fundamental subalgebra of S and that S is a transitive Lie algebra.

If η is a vector field on G, we denote by $\tilde{\eta}$ the jet of infinite order of η at e. The homomorphism of Lie algebras $\mathcal{G} \rightarrow T$ sending η into $\tilde{\eta}$ is injective and we denote by M its image. The subspace H of F consisting of all formal solutions at e of the equation $\bar{\partial}_b u = 0$ is a closed subalgebra of F. Because H is equal to the algebra of all elements $f \in F$ satisfying the equation

$$(\tilde{\eta}_1 + i\tilde{\eta}_2)f = 0,$$

and because the right-invariant vector fields on G commute with the left-invariant vector fields, the subspace H of F is stable under the action of the subalgebra M of L. If H^0 is the closed ideal $H \cap F^0$ of H, it is easily seen that the only element of M leaving H^0 invariant is the zero element. Moreover the algebra H is the topological direct sum of its subspaces H^0 and V. We see that

$$L = (R \hat{\otimes}_{\mathbf{C}} H) \oplus M \qquad [8]$$

is a closed subalgebra of S and that

$$I = R \hat{\otimes}_{\mathbf{C}} H$$

is a closed ideal of L. The quotient L/I is isomorphic to the simple non-abelian Lie algebra $\mathcal{G}$.

PROPOSITION. *The topological Lie algebra* L *is transitive, and* I *is the unique nontrivial closed ideal of* L *and is non-abelian.*

Proof: Because $M + T^0 = T$ and H contains V, we have

$$L + S^0 = S.$$

From theorem 13.2 of ref. 5, it follows that L is a transitive Lie algebra and $L^0 = L \cap S^0$ is a fundamental subalgebra of L. As $M \cap T^0 = \{0\}$, we have

$$L^0 = R^0 \hat{\otimes}_{\mathbf{C}} H + R \hat{\otimes}_{\mathbf{C}} H^0.$$

Because R is non-abelian and V is contained in H, the ideal I is non-abelian. If J is the closed ideal $R \hat{\otimes}_{\mathbf{C}} H^0$ of I, the quotient I/J is isomorphic to R and so is simple; hence J is a closed maximal ideal of I. We have

$$D_L{}^\infty(J) \subset D_L{}^\infty(L^0) = \{0\},$$

and so, by the corollary to proposition 6.4 of ref. 3, I is a minimal closed ideal of L. Let I' be the commutator ideal of I in L; because I' is closed, the properties of I that we have just verified imply that $I' \cap I = \{0\}$. If $\zeta \in I'$, we may write ζ uniquely as $\zeta = \xi + \eta$, where $\xi \in I$ and $\eta \in M$. Because J is an ideal of I, the mapping $ad(\xi)$ leaves J invariant; therefore so does $ad(\eta)$. This implies that the action of η on H preserves the subspace H^0; we conclude that $\eta = 0$ and that $\zeta \in I' \cap I$. Hence $\zeta = 0$ and $I' = \{0\}$. Let I'' be a closed ideal of L; because I is a minimal closed ideal, the closed ideal $I'' \cap I$ of L must be equal to $\{0\}$ or I. If $I'' \cap I = \{0\}$, we have

$$[I'',I] \subset I'' \cap I = \{0\};$$

because $I' = \{0\}$, we deduce that $I'' = \{0\}$. On the other hand, if I is contained in I'', the quotient I''/I is an ideal of L/I. Because L/I is simple, the ideal I'' must be equal to L or I, concluding the proof.

From the proposition, we deduce that

$$L \supset I \supset \{0\}$$

is the unique Jordan–Hölder sequence for L in the sense of ref. 3 and its quotients are non-abelian.

If R is the transitive Lie algebra of formal holomorphic vector fields on $\mathbf{C}$ at the origin 0 of $\mathbf{C}$, it is easily seen from Eqs. 4 and 5 that there is an isomorphism of transitive Lie algebras from

L to the Lie algebra of formal infinitesimal transformations of Γ at a point $x \in X$, under which the image of I is the ideal of all π-vertical formal infinitesimal transformations of Γ at x.

Finally, we outline an explicit construction of the representation of I as specified in the theorem of Guillemin (theorem 7.2 of ref. 3) on the structure of non-abelian minimal closed ideals of transitive Lie algebras.

We identify the group G with the sphere $S^3 \subset \mathbf{C}^2$ in such a way that the identity element of G corresponds to a point x_0 of $\mathbf{C}^2$ and that each left-translation of G is the restriction to S^3 of a biholomorphic mapping of $\mathbf{C}^2$ that preserves the sphere. Let $\mathcal{H}$ be the ring of all formal holomorphic functions on $\mathbf{C}^2$ at x_0, i.e., the space of all formal solutions at x_0 of the equation $\bar{\partial} u = 0$ for a complex-valued function u on $\mathbf{C}^2$. Then $\mathcal{H}$ is a local ring whose maximal ideal $\mathcal{H}^0$ consists of all formal functions of $\mathcal{H}$ which vanish at x_0; endowed with the Krull topology, $\mathcal{H}$ is a linearly compact topological algebra over $\mathbf{C}$. The restriction mapping from jets of functions on $\mathbf{C}^2$ at x_0 to jets of functions on S^3 at x_0 induces a continuous mapping $\rho:\mathcal{H} \rightarrow H$ of topological algebras; a formal analogue of the Lewy extension phenomenon asserts that ρ is an isomorphism, and so we obtain the inverse λ of ρ. The complex Lie algebra $\mathcal{T}$ of formal holomorphic vector fields on $\mathbf{C}^2$ at x_0 is the transitive Lie algebra of continuous derivations of $\mathcal{H}$; its subalgebra $\mathcal{T}^0$ consisting of all formal vector fields vanishing at x_0 is fundamental. To each right-invariant vector field on G corresponds a holomorphic vector field on $\mathbf{C}^2$ that is tangent to S^3. We therefore obtain a continuous mapping $\mu:M \rightarrow \mathcal{T}$ of real topological Lie algebras such that

$$\rho(\mu(\xi) \cdot g) = \xi \cdot \rho g,$$

for $\xi \in M, g \in \mathcal{H}$; the image $\mathcal{M}$ of μ is a *real* subalgebra of $\mathcal{T}$ isomorphic to M. In a manner analogous to the construction of S, we endow $R \hat{\otimes}_{\mathbf{C}} \mathcal{H}$ with a structure of a topological $\mathcal{T}$-module; the semi-direct product

$$\mathcal{S} = (R \hat{\otimes}_{\mathbf{C}} \mathcal{H}) \oplus \mathcal{T}$$

is a transitive Lie algebra (over $\mathbf{C}$), whose subalgebra

$$\mathcal{S}^0 = (R^0 \hat{\otimes} \mathcal{H} + R \hat{\otimes}_{\mathbf{C}} \mathcal{H}^0) \oplus \mathcal{T}^0$$

is fundamental. The real subalgebra

$$\mathcal{L} = (R \hat{\otimes}_{\mathbf{C}} \mathcal{H}) \oplus \mathcal{M}$$

of $\mathcal{S}$ is closed and the unique linear mapping $\varphi:L \rightarrow \mathcal{L}$ satisfying

$$\varphi(\xi \otimes f) = \xi \otimes \lambda f, \qquad \varphi(\eta) = \mu(\eta),$$

for all $\xi \in R, f \in H, \eta \in M$ is an isomorphism of topological Lie algebras. The image under φ of the non-abelian minimal closed ideal I is $R \hat{\otimes}_{\mathbf{C}} \mathcal{H}$, proving theorem 7.2 of ref. 3 for the closed ideal I of L. However, we remark that because $\mathcal{M}$ is a three-dimensional *real* subalgebra of $\mathcal{T}$, the Lie algebra $\mathcal{L}$ is not a transitive subalgebra of $(\mathcal{S}, \mathcal{S}^0)$, in the sense that $\mathcal{L} + \mathcal{S}^0 \neq \mathcal{S}$.

The author is indebted to H. Goldschmidt and D. C. Spencer for suggestions concerning the content and presentation of this paper. The preparation of this manuscript was supported by National Science Foundation Grant MCS 76-23465.

The costs of publication of this article were defrayed in part by the payment of page charges from funds made available to support the research which is the subject of the article. This article must therefore be hereby marked "advertisement" in accordance with 18 U. S. C. §1734 solely to indicate this fact.

1. Guillemin, V. W. & Sternberg, S. (1967) "The Lewy counterexample and the local equivalence problem for G-structures," *J. Differential Geometry* **1**, 127–131.
2. Lewy, H. (1957) "An example of a smooth linear partial differential equation without solution," *Annals of Mathematics* **66**, 155–158.
3. Guillemin, V. W. (1968) "A Jordan-Hölder decomposition for a certain class of infinite dimensional Lie algebras," *J. Differential Geometry* **2**, 313–345.
4. Goldschmidt, H. & Spencer, D. (1976) "On the non-linear cohomology of Lie equations. I, II," *Acta Mathematica* **136**, 103–239.
5. Goldschmidt, H. (1976) "Sur la structure des equations de Lie: III. La cohomologie de Spencer," *J. Differential Geometry* **11**, 167–223.