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# CORRECTION TO "HARMONICALLY IMMERSED SURFACES OF $\mathbf{R}^n$"

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Formulas (1.20) through (1.23) of §1 of [1] are incorrect. The correct formulas given below require a revision of the results of Theorems 1.1 and 1.2 and their remarks, while Proposition 1.1 and its Corollary 1.1 actually become strengthened. §2 (with appropriate renormalizations given below) and §3 remain correct as given.

Everything is correct through (1.19), which we take as our starting point here. We follow the notation and index conventions of the paper. A unitary frame along $\gamma_f$ (see (1.9)) is given by $(u_j, u_\alpha)$ where

$$u_1 = \frac{1}{r}(R_1^j + iR_2^j)e_j, \quad r = \|R\| = \left[\sum (R_j^i)^2\right]^{1/2}, \tag{1}$$

$u_1, u_2$ is a special unitary basis of $\operatorname{span}_{\mathbf{C}}\{e_1, e_2\}$, and $u_\alpha = e_\alpha$. Thus $\gamma_f = [u_1]$, and

$$du_1 = -\left(\frac{dr}{r} + i\varphi_1^2\right)u_1 + \frac{1}{r}\left(R_{1j}^A + iR_{2j}^A\right)e_A\varphi^j, \tag{2}$$

which we find useful to write as

$$du_1 = -\left(\frac{dr}{r} + i\varphi_1^2\right)u_1 + \frac{1}{2r}\tau(f)\varphi + \frac{1}{r}L\bar{\varphi}, \tag{3}$$

where the tension of $f$ is $\tau(f) = (R_{11}^A + R_{22}^A)e_A$, and $L = L(R^A)e_A$, where $L(R^A) = (R_{11}^A - R_{22}^A)/2 - iR_{12}^A$. Thus we can rewrite (1.19) as

$$d\gamma_f = \frac{1}{2r}p*_{u_1}\tau(f)\varphi + \frac{1}{r}p*_{u_1}L\bar{\varphi}, \tag{4}$$

and, since $\gamma_f^* dF^2 = \langle du_1, du_1\rangle - \langle du_1, u_1\rangle\langle u_1, du_1\rangle = \|\operatorname{Proj}_{u_1^\perp} du_1\|^2$, we obtain as a correction to (1.20)

$$\begin{aligned}\gamma_f^* dF^2 = {} & (\|\operatorname{Proj}_{u_1^\perp}\tau(f)\|^2/4r^2 + \|\operatorname{Proj}_{u_1^\perp}L\|^2/r^2)\varphi\bar{\varphi} \\ & + \frac{1}{2r^2}\langle \operatorname{Proj}_{u_1^\perp}\tau(f), \operatorname{Proj}_{u_1^\perp}L\rangle\varphi\varphi + \frac{1}{2r^2}\langle \operatorname{Proj}_{u_1^\perp}L, \operatorname{Proj}_{u_1^\perp}\tau(f)\rangle\overline{\varphi\varphi},\end{aligned} \tag{5}$$

where $\operatorname{Proj}_{u_1^\perp}$ is orthogonal projection (with respect to the Hermitian metric of $\mathbf{C}^n$) onto the orthogonal complement of $u_1$.

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**Theorem 1.2** (corrected). *Let* $f\colon (M, g) \to \mathbf{R}^n$ *be a harmonic immersion. Then* $\gamma_f\colon (M, g) \to \mathbf{C}P^n$ *is anti-holomorphic and* $\gamma_f^* dF^2 = -\mathscr{A} g$, *where*

$$(6)\qquad \mathscr{A} = -\frac{1}{4r^2}\|\operatorname{Proj}_{u_1^\perp} \tau(f)\|^2 - \frac{1}{r^2}\|\operatorname{Proj}_{u_1^\perp} L\|^2.$$

*Remarks.*

1. $\nabla df = \frac{1}{2}\tau(f)\varphi\bar{\varphi} + \frac{1}{2}L\varphi\varphi + \frac{1}{2}\overline{L\varphi\varphi}$.
2. Suppose $f$ is an isometry. We may then assume $\varphi^i = f^*\theta^i$, so $R = I$. Thus $r = \sqrt{2}$, $\frac{1}{2}\tau(f) = H$ (=mean curvature vector), $\nabla df = \mathrm{II}$ (=second fundamental form), and $\mathscr{A} = -\frac{1}{2}\|H\|^2 - \frac{1}{2}\|L\|^2 = -\frac{1}{4}\|\mathrm{II}\|^2 = -\|H\|^2 + K/2$, where $K$ is the Gaussian curvature of $g$. If, in addition, $f$ is harmonic, then $\mathscr{A} = K/2$.

We add the index convention $2 \le a, b, c \le n$, and let $X_a = p_{*u_1} u_a$, a unitary frame field along $\gamma_f$. If we set $\tau(f) = \tau^A u_A$, $L = L^A u_A$, then formula (4) can be rewritten as

$$(7)\qquad d\gamma_f = \frac{1}{2r}\tau^a X_a \varphi + \frac{1}{r} L^a X_a \bar{\varphi} \equiv B_1^a \varphi X_a + B_{\bar{1}}^a \bar{\varphi} X_a.$$

Then $\nabla d\gamma_f = (B_{11}^a \varphi\varphi + 2B_{1\bar{1}}^a \varphi\bar{\varphi} + B_{\bar{1}\bar{1}}^a \overline{\varphi\varphi}) X_a$ where

$$(8)\qquad B_{1\bar{1}}^a = \frac{1}{2r}\left\{\tau_{\bar{1}}^a - \frac{1}{r}(L^1\tau^a + \tau^1 L^a)\right\},$$

and $\tau_{\bar{1}}^a$ is given by

$$\nabla\tau(f) = (\tau_1^A \varphi + \tau_{\bar{1}}^A \bar{\varphi}) u_A,$$

and since $\nabla\tau(f)$ is real, we have $\tau_{\bar{1}}^A u_A = \overline{\tau_1^A u_A}$. Hence the tension of $\gamma_f$, which is $\tau(\gamma_f) = 4B_{1\bar{1}}^a X_a$, is

$$(9)\qquad \tau(\gamma_f) = \frac{2}{r} p_{*u_1}\left\{\tau_{\bar{1}}^A u_A - \frac{1}{r}(L^1\tau(f) + \tau^1 L)\right\}.$$

Formula (9) reaffirms the assertion of Theorem 1.2 (corrected) that if $f$ is harmonic then $\gamma_f$ is harmonic.

**Theorem 1.1** (corrected). *Let* $f\colon (M, g) \to \mathbf{R}^n$ *be an affine immersion (i.e.,* $T = 0$ *in (1.11)). Then* $\gamma_f\colon (M, g) \to \mathbf{C}P^{n-1}$ *is harmonic if and only if* $\operatorname{Proj}_{u_1^\perp} \nabla\tau(f) \equiv 0$ *modulo* $\varphi$.

*Proof.* Recall that $f$ affine means that the matrices $R^j = (R_{ik}^j) = 0$. It is easily verified that $f$ is affine if and only if $\tau^j = 0 = L^j$. Hence, if $f$ is affine then (9) becomes

$$\tau(\gamma_f) = \frac{2}{r} p_{*u_1} \tau_{\bar{1}}^A u_A.$$

As $p_{*u_1}$ maps $u_1^\perp$ isomorphically onto the tangent space of $\mathbf{C}P^{n-1}$ at $[u_1]$, it follows that $\tau(\gamma_f) = 0$ iff $\tau_{\bar{1}}^a u_a = 0$, which proves the theorem. □

*Remark.* If $f$ is an isometry, then $\tau(f)$ $(= 2H)$, $\nabla\tau(f)$ and $L$ are all normal, which implies that $f$ is affine and $\mathrm{Proj}_{u_1^\perp} \nabla\tau(f) = \nabla\tau(f)$. Hence $\gamma_f$ is harmonic if and only if $f$ has parallel mean curvature vector. The Ruh-Vilms theorem says that the usual Gauss map $\Gamma_f \colon M \to Q_{n-2}$ (=complex quadric) is harmonic if and only if $f$ has parallel mean curvature vector. If $q \colon Q_{n-2} \to \mathbf{C}P^{n-1}$ is the inclusion map, then $\gamma_f = q \circ \Gamma_f$.

Our corrections give a stronger version of Proposition 1.1.

**Proposition 1.1** (corrected). *If $f \colon (M, g) \to \mathbf{R}^n$ is a harmonic immersion, then $\mathscr{A}$ is of analytic type (i.e., if not identically zero it has isolated zeros of well-defined multiplicities).*

*Proof.* Let $p \in M$, and let $U$, $z$ be a local complex coordinate about $p$. For $U$ sufficiently small there exists a holomorphic map $\mathscr{F} \colon U \to \mathbf{C}$ such that $f$ is its real part. That is, $\mathscr{F} = f + ih$, for some harmonic map $h \colon U \to \mathbf{R}^n$. Then $\gamma_f = [\overline{\mathscr{F}'}]$, where $\mathscr{F}' = \frac{d\mathscr{F}}{dz}$. In fact, if $\varphi = \varphi^1 + i\varphi^2 = \lambda\, dz$, then $\frac{\partial f}{\partial \bar z} = \frac{\lambda}{2}(R_1^j + iR_2^j)e_j$ and $\overline{\mathscr{F}'} = \frac{d\overline{\mathscr{F}}}{d\bar z} = 2\frac{\partial f}{\partial \bar z}$.

Now $u_1 = \frac{2}{\lambda r}\frac{\partial f}{\partial \bar z} = \frac{1}{\lambda r}\overline{\mathscr{F}'}$, so $du_1 = Cu_1 + \frac{1}{\lambda r}\overline{\mathscr{F}''}\, d\bar z$, for some function $C$. Hence $d\gamma_f = (p_{*u_1}\frac{1}{\lambda r}\overline{\mathscr{F}''})\, d\bar z$ and thus $\mathscr{A} = -\|\mathrm{Proj}_{u_1^\perp}\overline{\mathscr{F}''}\|^2/r^2|\lambda|^4$ since

$$\gamma_f^* dF^2 = \|\mathrm{Proj}_{u_1^\perp}\overline{\mathscr{F}''}\bar\varphi/r|\lambda|^2\|^2 = \frac{1}{r^2|\lambda|^4}\|\mathrm{Proj}_{u_1^\perp}\overline{\mathscr{F}''}\|^2\varphi\bar\varphi$$

and by Theorem 1.2. Then $\mathscr{A}(p) = 0$ if and only if $\overline{\mathscr{F}'} \wedge \overline{\mathscr{F}''}(p) = 0$. As $\mathscr{F}' \wedge \mathscr{F}''$ is a vector valued holomorphic function, it is either identically zero or it has isolated zeros of well-defined multiplicities, and these are the zeros and multiplicities of $\mathscr{A}$. □

Corollary 1.1 of [1] is true even with the affine hypothesis dropped, since that hypothesis is not required in the corrected version of Proposition 1.1.

This completes the corrections of §1. In §2, the inequality (2.1) must be changed to

(2.1) revised $$K - 2\mathscr{A} \geq 0.$$

In the line following (2.8) we must revise the definition of $\tilde g$ to be $\tilde g = v^2 g$, in which case $\tilde g \geq \varepsilon^2 g$ and its curvature becomes

$$\widetilde{K} = (K - \tfrac{1}{2}\Delta \log v^2)/v^2,$$

so that

(2.9) revised $$\widetilde{K} = (k - 2\mathscr{A})/v^2.$$

Finally, the second line of the first remark at the end of this section requires the correction $\mathscr{A} = K/2$. No corrections are required in §3.

## REFERENCES

1. G. R. Jensen and M. Rigoli, *Harmonically immersed surfaces of* $\mathbf{R}^n$, Trans. Amer. Math. Soc. **307** (1988), 363–372.

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