ON RIBAUCOUR TRANSFORMATIONS FOR HYPERSURFACES

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Dedicated to Manfredo P. do Carmo on his 70^{th} birthday

Introduction

The purpose of this paper is to show how a Ribaucour transformation provides families of Dupin hypersurfaces from a given such submanifold. Ribaucour transformations for hypersurfaces, parametrized by lines of curvature, were classically studied by Bianchi [Bi]. They can be used to provide families of surfaces of constant Gaussian curvature from a given such surface. Similarly, by using Ribaucour transformations one may obtain minimal surfaces from a given such surface [Bi]. In this paper, we provide necessary and sufficient conditions for a Ribaucour transformation to associate a Dupin hypersurface into another such submanifold and we apply the theory to special Dupin submanifolds.

Dupin surfaces were first studied by Dupin in 1822 and more recently by many authors [CC],[CeR1],[CeR2],[Ch],[CJ],[M], [N],[P1-P3],[PT],[S],[T] which studied several aspects of Dupin hypersurfaces. The class of Dupin hypersurfaces is invariant under conformal transformations. Moreover, the Dupin property is invariant under Lie transformations [P2]. Therefore, the classification of Dupin hypersurfaces is considered up to these transformations. The local classification of Dupin surfaces in R^3 is well known. However, the classification of Dupin hypersurfaces for higher dimensions is far from complete. Therefore, it is important to study methods which generate such submanifolds.

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In section 1, we fix our notation and we recall the main definitions. In section 2, we revise a characterization of a Ribaucour transformation for hypersurfaces of the Euclidean space in terms of differential equations. Then we provide a necessary and sufficient condition for such a transformation to transform a Dupin hypersurface into another such submanifold. In section 3, we apply the Ribaucour transformation to particular Dupin hypersurfaces generating families of Dupin hypersurfaces.

1. Preliminaries

A *sphere congruence* is an n-parameter family of spheres whose centers lie on an n-dimensional manifold M_0 contained in R^{n+1}. Locally, we may consider M_0 parametrized by $X_0 : U \subset R^n \to R^{n+1}$. For each point $u = (u_1, ..., u_n) \in U$, we consider a sphere centered at $X(u)$ with radius $r(u)$, where r is a differentiable real function. An *involute* of a sphere congruence is an n-dimensional submanifold M of R^{n+1} such that each point of M is tangent to a sphere of the sphere congruence. Two hypersurfaces M and $\tilde{M}$ are said to be associated by a sphere congruence if there is a difeomorphism $\psi : M \to \tilde{M}$ such that at corresponding points p and $\psi(p)$ the manifolds are tangent to the same sphere of the sphere congruence. It follows that the normal lines at corresponding points intersect at an equidistant point on the center manifold. An important special case occurs when ψ preserves lines of curvature.

Let M^n and $\tilde{M}^n$ be orientable hypersurfaces of R^{n+1}. We denote by N and $\tilde{N}$ their Gauss map. We say that M and $\tilde{M}$ are *associated by a Ribaucour transformation*, if and only if, there exists a differentiable function h defined on M and a diffeomorphism $\psi : M \to \tilde{M}$ such that

a) $p + h(p)N(p) = \psi(p) + h(p)\tilde{N}(\psi(p))$, for all $p \in M$.

b) The subset $p + h(p)N(p)$ $p \in M$ is an n-dimensional submanifold.

c) ψ preserves lines of curvature.

We say that M and $\tilde{M}$ are *locally associated by a Ribaucour transformation* if for all $p \in M$ there exists a neighborhood of p in M which is associated by a Ribaucour transformation to an open subset of $\tilde{M}$. Similarly, one may consider the notion of parametrized hypersurfaces *locally associated by a Ribaucour transformation.*

A hypersurface $M^n \subset R^{n+1}$ is a *Dupin submanifold* if its principal curvatures are constant along the corresponding lines of curvature. Whenever the principal curvatures are constant M is a called an *isoparametric* submanifold.

Consider a hypersurface M^n of R^{n+1}. Let e_i $1 \leq i \leq n$ be a an orthonormal frame tangent to M and N a unit normal vector field locally defined. We denote by ω_i the one forms dual to the vector fields e_i and by ω_{ij}, $1 \leq i, j \leq n$ the connection forms determined by

$$d\omega_i = \sum_{j \neq i} \omega_j \wedge \omega_{ji}, \qquad \omega_{ij} + \omega_{ji} = 0.$$

The normal connection $\omega_{in+1} =< de_i, N >$ satisfies $\sum_i \omega_i \wedge \omega_{in+1} = 0$. Hence, $\omega_{in+1} = \sum_j b_{ij}\omega_j$ where $b_{ij} = b_{ji}$. The Gauss equation is given by

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \omega_{in+1} \wedge \omega_{n+1j} \tag{1}$$

and the Codazzi equations are

$$d\omega_{in+1} = \sum_j \omega_{ij} \wedge \omega_{jn+1}.$$

Whenever the hypersurface is parametrized by orthogonal lines of curvature, $X(u_1, ..., u_n)$, the first fundamental form is of the form $I = \sum_i \omega_i^2$, where $\omega_i = a_i du_i$ and a_i are differentiable functions. The principal directions are the vector fields $e_i = X_{,i}/a_i$, where $X_{,i}$ denotes partial derivative with respect to u_i. Then

$$\omega_{ij} = \frac{1}{a_i a_j}\left(-a_{i,j}\omega_i + a_{j,i}\omega_j\right) \tag{2}$$

and the Christoffel symbols are given by

$$\Gamma^i_{ij} = \frac{a_{i,j}}{a_i}, \quad \Gamma^j_{ii} = -\frac{a_i a_{i,j}}{a_j^2}, \quad \Gamma^k_{ij} = 0 \quad \text{for } i, j, k \text{ distinct.} \tag{3}$$

Since the coordinate curves are lines of curvature we have

$$dN(e_i) = \lambda^i e_i, \qquad \omega_{in+1} = -\lambda^i \omega_i. \tag{4}$$

Then the Codazzi equations reduce to

$$d\lambda^i(e_j) = (\lambda^i - \lambda^j)\omega_{ij}(e_i), \qquad i \neq j. \tag{5}$$

Moreover, it follows from the Gauss equation (1), applied to the pair of vector fields e_i, e_s, that

$$\frac{\partial}{\partial u_s}\left(\frac{a_{i,j}}{a_i a_j}\right) = \frac{a_{i,s}}{a_i}\left(\frac{a_{s,j}}{a_s a_j} - \frac{a_{i,j}}{a_i a_j}\right) \qquad \text{for } i,j,s \text{ distinct.} \tag{6}$$

Whenever the submanifold is a Dupin hypersurface we have

$$d\lambda^i(e_i) = 0, \qquad 1 \leq i \leq n.$$

In the following section, by using Ribaucour transformations, we will show how to obtain Dupin hypersurfaces from a given one.

2. Ribaucour transformations

In this section, we start with a local characterization of a Ribaucour transformation for hypersurfaces of the Euclidean space. This is a classical result (see [Bi]), whenever the hypersurfaces are parametrized by lines of curvature. We then provide a necessary and sufficient condition for such a transformation to associate a Dupin hypersurface to a given one.

Theorem 2.1 *Let M^n be an orientable submanifold of R^{n+1}. Assume e_i, $1 \leq i \leq n$ are orthonormal principal directions, λ^i the corresponding principal curvatures and N is a unit vector field normal to M. A submanifold $\tilde{M}^n$ is locally associated to M by a Ribaucour transformation, if and only if, $\forall\ p \in M$, there exist a parametrization $X : U \subset R^n \to M$ of a neighborhood of p and a differentiable function $h : U \to R$ such that*

$$\tilde{X} = X + h(N - \tilde{N}) \tag{7}$$

is a parametrization of $\tilde{M}$ *and the unit vector field* $\tilde{N}$ *normal to* $\tilde{M}$ *is given by*

$$\tilde{N} = \frac{1}{\Delta+1}\left(\sum_{i=1}^{n} 2Z^i e_i + (\Delta-1)N\right) \tag{8}$$

where

$$Z^i = \frac{dh(e_i)}{1+h\lambda^i} \qquad \Delta = \sum_{i=1}^{n}\left(Z^i\right)^2 \tag{9}$$

and h *satisfies the differential equations*

$$dZ^j(e_i) + Z^i\omega_{ij}(e_i) - Z^iZ^j\lambda^i = 0, \quad 1 \leq i \neq j \leq n. \tag{10}$$

Proof: In order to prove the theorem, we will consider $\tilde{N}$ to be a unit vector field given by

$$\tilde{N} = \sum_{i=1}^{n} b^i e_i + b^{n+1}N, \tag{11}$$

where

$$\sum_{i=1}^{n}(b^i)^2 + (b^{n+1})^2 = 1. \tag{12}$$

We introduce the following notation

$$d\tilde{N}(e_i) = \sum_k L_i^k e_k + L_i^{n+1}N, \tag{13}$$

where for $1 \leq i,k \leq n$

$$L_i^k = db^k(e_i) + b^i\omega_{ik}(e_i) + (\sum_j b^j\omega_{ji}(e_i) + b^{n+1}\lambda^i)\delta_{ik},$$
$$L_i^{n+1} = db^{n+1}(e_i) - b^i\lambda^i.$$

We will later show that the following relations hold

$$b^i = Z^i(1-b^{n+1}), \qquad b^{n+1} = \frac{\Delta-1}{\Delta+1}. \tag{14}$$

In this case, it follows from (9) that

$$\begin{aligned} L_i^k &= \frac{2}{1+\Delta}\left(dZ^k(e_i) + Z^i\omega_{ik}(e_i) + \sum_j Z^j\omega_{ji}(e_i)\delta_{ik}\right) \\ &- 2Z^k\frac{d\Delta(e_i)}{(\Delta+1)^2} + \frac{\Delta-1}{\Delta+1}\lambda^i\delta_{ik}, \end{aligned} \tag{15}$$

$$L_i^{n+1} = \frac{2d\Delta(e_i)}{(\Delta+1)^2} - \frac{2Z^i\lambda^i}{\Delta+1}. \tag{16}$$

We will now prove the theorem. Assume that $\tilde{M}$ is locally associated to M by a Ribaucour transformation. Then by definition there exist local parametrizations X of M, $\tilde{X}$ of $\tilde{M}$ and a function h defined on $U \subset R^n$ such that

$$\tilde{X} + h\,\tilde{N} = X + h\,N,$$

where $\tilde{N}$ is a unit vector field normal to $\tilde{M}$, which may be considered as in (11). Then

$$< d\tilde{X}(e_i), \tilde{N} >= 0, \qquad \text{for all } i,\ 1 \le i \le n. \tag{17}$$

Since

$$d\tilde{X} = dX + dh(N - \tilde{N}) + h(dN - d\tilde{N}) \tag{18}$$

it follows from the relations $dX = \sum_j \omega_j e_j$ and $dN(e_i) = \lambda^i e_i$ that

$$d\tilde{X}(e_i) = (1 + h\lambda^i)e_i + dh(e_i)(N - \tilde{N}) - hd\tilde{N}(e_i). \tag{19}$$

Hence, equation (17) implies

$$(1 + h\lambda^i)b^i + dh(e_i)(b^{n+1} - 1) = 0, \qquad 1 \le i \le n. \tag{20}$$

We claim that the fact that $\tilde{X}$ is a Ribaucour transformation of X implies that $1 + h\lambda^i \neq 0$ for all i. In fact, consider the center manifold $X^0 = X + hN$. Then

$$dX^0(e_i) = (1 + h\lambda^i)e_i + dh(e_i)N.$$

Assume that $(1 + h\lambda^i)(u^0) = 0$ at a point u^0. Then it follows from (20) that $dh(e_i)(b^{n+1} - 1)(u^0) = 0$ and hence $dh(e_i)(u^0) = 0$. Otherwise, $b^{n+1}(u^0) = 1$ implies that $\tilde{X}(u^0) = X(u^0) = X^0(u^0)$. Hence $h(u^0) = 0$, which is a contradiction, since $(1 + h\lambda^i)(u^0) = 0$. Therefore, we have $dh(e_i)(u^0) = 0$ and $dX^0(e_i)(u^0) = 0$, which contradicts the fact that the center manifold X^0 is n-dimensional.

So $1 + h\lambda^i \neq 0$ for all i, therefore we conclude from (20) that the relations (14) hold, where the second equality of (14) follows from the first one, from (12), the notation (9) and the fact that we want $b^{n+1} \neq 1$ (i.e. $\tilde{X} \neq X$).

We will show that the differential equation which is satisfied by h, is a consequence of the property

$$< d\tilde{N}(e_i), d\tilde{X}(e_j) >= 0, \qquad \text{for } i \neq j.$$

Since $\tilde{X}$ preserves lines of curvature, we have

$$< d\tilde{X}(e_i), d\tilde{X}(e_j) >=< d\tilde{N}(e_j), d\tilde{X}(e_i) >=< d\tilde{N}(e_j), d\tilde{N}(e_i) >= 0 \quad \text{for } i \neq j.$$

Hence, using the notation (13) and equation (19), we get

$$\begin{aligned} < d\tilde{N}(e_i), d\tilde{X}(e_j) > &= < d\tilde{N}(e_i), (1+h\lambda^j)e_j + dh(e_j)N >, \\ &= L_i^j(1+h\lambda^j) + L_i^{n+1} dh(e_j) = 0, \qquad \text{for } i \neq j. \end{aligned}$$

By using the relations (15) and (16) in the last equality, we conclude that equation (10) holds.

Conversely, assume h is a solution of (10), then we define the functions Z^i and Δ by (9), b^i and b^{n+1} by (14). It follows from (15), (16) and (10) that

$$L_i^k + Z^k L_i^{n+1} = 0, \qquad i \neq k. \tag{21}$$

Moreover, from (15) we have for all $i, 1 \leq i \leq n$,

$$L_i^i = \frac{2}{\Delta+1}\left(dZ^i(e_i) + \sum_k Z^k \omega_{ki}(e_i) - 2Z^i \frac{d\Delta(e_i)}{\Delta+1} + (\Delta-1)\frac{\lambda^i}{2} \right). \tag{22}$$

Therefore, using the definition of Δ, (10) and (16), it follows from a straightforward computation that

$$Z^i L_i^i + \left((Z^i)^2 - \frac{\Delta+1}{2} \right) L_i^{n+1} = 0. \tag{23}$$

We consider $\tilde{N}$ and $\tilde{X}$ as in (11) and (7) respectively. We need to show that $\tilde{X}$ is associated to X by a Ribaucour transformation. We first observe that $\tilde{N}$ is a unit vector field. In fact,

$$\sum_i (b^i)^2 + (b^{n+1})^2 = (1-b^{n+1})^2\Delta + (b^{n+1})^2 = 1,$$

where the last equality follows by using the expression of b^{n+1}.

We next verify that $\tilde{N}$ is normal to $\tilde{X}$. From the definition of $\tilde{X}$, we have that $d\tilde{X}(e_i)$ is given by (19). Hence, using the fact that $|\tilde{N}| = 1$, we conclude that

$$\begin{aligned} < d\tilde{X}(e_i), \tilde{N} > &= (1+h\lambda^i)b^i + dh(e_i)(b^{n+1}-1) \\ &= (-(1+h\lambda^i)Z^i + dh(e_i))(b^{n+1}-1) = 0. \end{aligned}$$

In order to show that the principal directions are preserved, we first show that

$$< d\tilde{N}(e_i), d\tilde{N}(e_j) >= 0 \qquad i \neq j.$$

This follows from

$$\begin{aligned}
&< d\tilde{N}(e_i), d\tilde{N}(e_j) > \\
&= L_i^i L_j^i + L_i^j L_j^j + \sum_{k\neq i, k\neq j} L_i^k L_j^k + L_i^{n+1} L_j^{n+1} \\
&= -Z^i L_i^i L_j^{n+1} - Z^j L_j^j L_i^{n+1} + \sum_{k\neq i, k\neq j} (Z^k)^2 L_i^{n+1} L_j^{n+1} + L_i^{n+1} L_j^{n+1} \\
&= \left((Z^i)^2 + (Z^j)^2 - (\Delta + 1) + \sum_{k\neq i, k\neq j} (Z^k)^2 + 1 \right) L_i^{n+1} L_j^{n+1} = 0,
\end{aligned}$$

where in the two last equalities we have used (23) and the definition of Δ. Now, using the above equality and equations (18) and (21), we conclude that for $i \neq j$

$$\begin{aligned}
&< d\tilde{N}(e_i), d\tilde{X}(e_j) > \\
&=< d\tilde{N}(e_i), (1 + h\lambda^j)(e_j) + dh(e_j)N > \\
&= L_i^j (1 + h\lambda^j) + L_i^{n+1} dh(e_j) = 0.
\end{aligned}$$

Finally, we conclude that the images of the vector fields e_i, e_j by $d\tilde{X}$ are orthogonal for all $i \neq j$. In fact,

$$\begin{aligned}
&< d\tilde{X}(e_i), d\tilde{X}(e_j) >= (1 + h\lambda^i) < e_i, d\tilde{X}(e_j) > + dh(e_i) < N, d\tilde{X}(e_j) > \\
&= -(1 + h\lambda^i)(dh(e_j)b^i + hL_j^i) + dh(e_i) \left(dh(e_j)(1 - b^{n+1}) - hL_j^{n+1} \right) = 0,
\end{aligned}$$

where the last equality follows from the definition of b^i and equation (21). Moreover, generically $\tilde{X}$ is an n-dimensional manifold, since

$$|d\tilde{X}(e_i)|^2 = \sum_k \left((1 + h\lambda^i)\delta_{ik} - dh(e_i) b^k - hL_i^k \right)^2 + \left(dh(e_i)(1 - b^{n+1}) - hL_i^{n+1} \right)^2. \tag{24}$$

□

We observe that, whenever the manifold M is parametrized by orthogonal lines of curvature $X(u_1, ..., u_n)$, equation (10) in classical notation is written as

$$h_{,ij} - \frac{1 + h\lambda^i}{1 + h\lambda^j} \Gamma_{ij}^j h_{,j} - \frac{1 + h\lambda^j}{1 + h\lambda^i} \Gamma_{ij}^i h_{,i} - \left(\frac{\lambda^j}{1 + h\lambda^j} + \frac{\lambda^i}{1 + h\lambda^i} \right) h_{,i} h_{,j} = 0, i \neq j,$$

where $h_{,i}$ and $h_{,ij}$ denote the partial derivative of h with respect to u_i and to $u_i\, u_j$ respectively. This equation is easily obtained by substituting in (10) the expression of Z^j given by (9), equations (2), (3) and using Codazzi equation (5).

Our next result will show how to linearize the problem of obtaining the function h.

Proposition 2.2. *Suppose that h is a nonvanishing function which satisfies equation (10) then*

$$\psi = \frac{1}{h}\sum_{i=1}^{n} Z^i \omega_i$$

is a closed 1-form and there exists a nonvanishing function Ω defined on a simply connected domain such that

$$d\Omega(e_i) = \frac{\Omega}{h} Z^i.$$

Proof: By considering the exterior differentiation of ψ, we get

$$d\psi = \sum_{i \neq j} \frac{1}{h}\left[dZ^i(e_j) - Z^i \frac{dh(e_j)}{h} - Z^j \omega_{ij}(e_j)\right] \omega_j \wedge \omega_i.$$

As a consequence of (10) we conclude that the coefficients of $\omega_j \wedge \omega_i$ for $j < i$ vanish. Hence, on any simply connected domain, there exists a differentiable function Ω such that

$$d(\log \Omega) = \psi.$$

Therefore, $d\Omega(e_i)/\Omega = \psi(e_i)$, which concludes the proof. □

Based on the previous result, for each nonvanishing function h, which is a solution of (10), we consider Ω as given by Proposition 2.2 and we define

$$\Omega^i = d\Omega(e_i), \qquad W = \frac{\Omega}{h}. \tag{25}$$

With this notation, (10) is given by

$$d\Omega^i(e_j) \;=\; \Omega^j \omega_{ij}(e_j), \quad \text{for } i \neq j, \tag{26}$$

$$d\Omega = \sum_{i=1}^{n} \Omega^i \omega_i, \tag{27}$$

$$dW = -\sum_{i=1}^{n} \Omega^i \lambda^i \omega_i \tag{28}$$

and

$$h = \Omega / W. \tag{29}$$

Hence, we have the following

Proposition 2.3 *A nonvanishing function h is a solution of (10) defined on a simply connected doamin, if and only if, $h = \Omega/W$ where Ω and W are nonvanishing functions which satisfy (26)-(28).*

With the above notation , it follows that

$$dh(e_i) = \frac{\Omega^i}{W}(1 + \Omega\lambda^i/W) \qquad \text{and} \qquad 1 + h\lambda^i = 1 + \Omega\lambda^i/W. \tag{30}$$

Hence,

$$Z^i = \frac{\Omega^i}{W} \qquad \Delta = \frac{1}{W^2}\sum_j (\Omega^j)^2. \tag{31}$$

Therefore, Theorem 2.1 can be rewritten as follows

Theorem 2.4. *Let M^n be a hypersurface of R^{n+1} parametrized by $X : U \subset R^n \to M$. Assume e_i, $1 \leq i \leq n$ are the principal directions, λ^i the corresponding principal curvatures and N is a unit vector field normal to M. A submanifold $\tilde{M}^n$ is locally associated to M, by a Ribaucour transformation, if and only if, there exist differentiable functions $W, \Omega, \Omega^i : V \subset U \to R$, which satisfy (26)-(28) and $\tilde{X} : V \subset R^n \to \tilde{M}$, is a parametrization of $\tilde{M}$ given by*

$$\tilde{X} = X - \frac{2\Omega}{\sum_i (\Omega^i)^2 + W^2}\left(\sum_i \Omega^i e_i - WN\right). \tag{32}$$

Remark 2.5. We observe that $d\Omega^i = \sum_{k=1}^{n} d\Omega^i(e_k)\omega_k$. Therefore, (26) is equivalent to

$$d\Omega^i \wedge \omega_i - \sum_{j\neq i} \Omega^j \omega_{ij}(e_j)\omega_j \wedge \omega_i = 0. \tag{33}$$

In classical notation, whenever M is parametrized by lines of curvature $X(u_1, ..., u_n)$, the system of equations (26), (27), (28) is written as

$$\frac{\partial \Omega^i}{\partial u_j} = \Omega^j \frac{1}{a_i} \frac{\partial a_j}{\partial u_i}, \quad i \neq j, \tag{34}$$

$$\frac{\partial \Omega}{\partial u_i} = a_i \Omega^i, \tag{35}$$

$$\frac{\partial W}{\partial u_i} = -a_i \Omega^i \lambda^i, \tag{36}$$

where the functions a_i correspond to the metric of the manifold M, i.e. $ds^2 = \sum_i a_i^2 \, du_i^2$. It is a straightforward computation to see that the compatibility condition of equation (34) is given by (6).

The following result shows that for each solution Ω^i, $1 \leq i \leq n$, of (26), there exists a 2-parameter family of solutions of the system (27), (28).

Proposition 2.6. *Equation (26) is the integrability condition of the system of equations (27), (28) for Ω and W.*

Proof: Consider the ideal $\mathcal{I}$ generated by the 1-forms

$$\begin{aligned} \alpha &= d\Omega - \sum_i \Omega^i \omega_i, \\ \beta &= dW + \sum_i \Omega^i \lambda^i \omega_i. \end{aligned}$$

We will show that if (26) holds then $\mathcal{I}$ is closed under exterior differentiation. In fact,

$$\begin{aligned} d\alpha &= -\sum_i d\Omega^i \wedge \omega_i - \sum_{i \neq j} \Omega^i \omega_j \wedge \omega_{ji} \\ &= -\sum_i d\Omega^i \wedge \omega_i - \sum_{i \neq j} \Omega^j \wedge \omega_{ji}(e_j) \omega_j \wedge \omega_i = 0 \end{aligned}$$

where the last equality follows from Remark 2.5.

Similarly,

$$\begin{aligned} d\beta &= \sum_i d\Omega^i \lambda^i \wedge \omega_i + \sum_i \Omega^i d\lambda^i \wedge \omega_i + \sum_{i \neq j} \Omega^i \lambda^i \omega_j \wedge \omega_{ji} \\ &= \sum_i d\Omega^i \lambda^i \wedge \omega_i + \sum_i \Omega^i d\lambda^i(e_j) \omega_j \wedge \omega_i + \sum_{i \neq j} \Omega^i \lambda^i \omega_{ji}(e_i) \omega_j \wedge \omega_i. \end{aligned}$$

Using Codazzi equation (5) we conclude that

$$d\beta = \sum_i \lambda^i (d\Omega^i \wedge \omega_i - \sum_{j\neq i} \Omega^j \omega_{ij}(e_j)\omega_j \wedge \omega_i) = 0,$$

as a consequence of equation (33).

□

From now on, whenever we say that two hypersurfaces are locally associated by a Ribaucour transformation we are assuming that there are functions where Ω^i, Ω and W locally defined, satisfying the system (26)-(28). Moreover, we observe that since the normal lines at corresponding points intersect at a distance $h = \Omega/W$, it follows from (30) that $dh(e_i) \neq 0$ if and only if $\Omega^i \neq 0$.

Theorem 2.7. *Assume $\tilde{M}$ is locally associated to M by a Ribaucour transformation. Let e_i, $1 \leq i \leq n$ be the principal directions and λ^i the corresponding principal curvatures of M. Then the principal curvatures of $\tilde{M}$ for each $1 \leq i \leq n$ are given by*

$$\tilde{\lambda}^i = \frac{dS(e_i)W + \Omega^i \lambda^i S}{\Omega^i S - \Omega dS(e_i)} \qquad \text{if } \Omega^i \neq 0, \tag{37}$$

$$\tilde{\lambda}^i = \frac{WT^i + \lambda^i S}{S - \Omega T^i} \qquad \text{if } \Omega^i \equiv 0, \tag{38}$$

where Ω^i, Ω and W satisfy the system (26)-(28) and

$$S = W^2 + \sum_j (\Omega^j)^2, \qquad T^i = 2(\sum_k \Omega^k \omega_{ki}(e_i) - W\lambda^i). \tag{39}$$

Proof: Let X and $\tilde{X}$ be parametrizations of M and $\tilde{M}$ associated by a Ribaucour transformation. The principal curvatures of $\tilde{M}$ are given by

$$\tilde{\lambda}^i = \frac{< d\tilde{N}(e_i), d\tilde{X}(e_i) >}{< d\tilde{X}(e_i), d\tilde{X}(e_i) >}. \tag{40}$$

Since $\tilde{X}$ is associated to X by a Ribaucour transformation, we have

$$d\tilde{X} = dX + dh(N - \tilde{N}) + h(dN - d\tilde{N}),$$

where $h = \Omega/W$ and $\tilde{N}$ is given by (8) Now, $d\tilde{N}(e_i) = \tilde{\lambda}^i d\tilde{X}(e_i)$ therefore, from the last equation we get

$$(1 + h\tilde{\lambda}^i)d\tilde{X}(e_i) = (1 + h\lambda^i)e_i + dh(e_i)(N - \tilde{N}). \tag{41}$$

Hence, using (11), we obtain

$$\begin{aligned}&(1 + h\tilde{\lambda}^i)^2 < d\tilde{X}(e_i), d\tilde{X}(e_i) > \\ &= (1 + h\lambda^i)^2 + 2(dh(e_i))^2 - 2dh(e_i)[(1 + h\lambda_i)b^i + dh(e_i)b^{n+1}].\end{aligned}$$

It follows from (14) and (9) that

$$(1 + h\tilde{\lambda}^i)^2 < d\tilde{X}(e_i), d\tilde{X}(e_i) >= (1 + h\lambda^i)^2. \tag{42}$$

On the other hand, using (13) and (41), we have

$$< d\tilde{N}(e_i), d\tilde{X}(e_i) >= \frac{1}{1 + h\tilde{\lambda}^i}\left[(1 + h\lambda^i)L_i^i + dh(e_i)L_i^{n+1}\right]. \tag{43}$$

Assume $\Omega^i \neq 0$ i.e. $dh(e_i) \neq 0$, then it follows from (23) that

$$L_i^i + Z^i L_i^{n+1} = \frac{\Delta + 1}{2Z^i} L_i^{n+1},$$

hence

$$< d\tilde{N}(e_i), d\tilde{X}(e_i) >= \frac{(1 + h\lambda^i)(\Delta + 1)}{2Z^i(1 + h\tilde{\lambda}^i)} L_i^{n+1}.$$

We conclude, using (40) and (42), that $\tilde{\lambda}^i$ is given by

$$\tilde{\lambda}^i = \frac{(\Delta + 1)L_i^{n+1}}{2dh(e_i) - h(\Delta + 1)L_i^{n+1}} \quad \text{if } \Omega^i \neq 0, \tag{44}$$

where L_i^{n+1} is defined by (16).

Observe that $\Delta + 1 = S/W^2$, where S is defined by (39). Therefore

$$(\Delta + 1)L_i^{n+1} = 2\frac{dS(e_i)}{S} - 2\frac{dW(e_i)}{W}.$$

Hence, it follows from (27), (28) and from the fact that $h = \Omega/W$ that $\tilde{\lambda}^i$ is given by (37).

If $\Omega^i \equiv 0$, i.e. $dh(e_i) \equiv 0$, then $Z^i = 0$ and hence, it follows from (10) that $dZ^j(e_i) = 0$ for all $j \neq i$. Therefore, we get from (9) that $d\Delta(e_i) = 0$. Using equations (15) and (16) we conclude that $L_i^{n+1} = L_i^k = 0$ for all $k \neq i$. Moreover, L_i^i is given by (46). Now it follows from (24) that

$$|d\tilde{X}(e_i)|^2 = (1 + h\lambda^i - hL_i^i)^2.$$

Since $d\tilde{N}(e_i) = L_i^i e_i$, we get

$$< d\tilde{X}(e_i), d\tilde{N}(e_i) >= (1 + h\lambda_i - hL_i^i)L_i^i.$$

Therefore, from (40) we obtain that

$$\tilde{\lambda}^i = \frac{L_i^i}{1 + h\lambda^i - hL_i^i} \quad \text{if } \Omega^i \equiv 0, \tag{45}$$

where

$$L_i^i = \frac{1}{\Delta + 1}\left(2\sum_k Z^k \omega_{ki}(e_i) + (\Delta - 1)\lambda^i\right) \quad \text{when } \Omega^i \equiv 0. \tag{46}$$

Using (9) we have that $\Delta + 1 = S/W^2$, $\Delta - 1 = (S - 2W^2)/W^2$, and

$$L_i^i = \frac{WT^i + \lambda^i S}{S},$$

where T^i is defined by (39) Moreover,

$$1 + h\lambda^i - hL_i^i = \frac{S - \Omega T^i}{S}.$$

Therefore, we conclude from (45) that (38) holds.

□

Theorem 2.8 *Let M^n be a Dupin submanifold of R^{n+1} whose principal curvatures and corresponding principal directions are given by λ^i and e_i, $1 \leq i \leq n$, respectively. Let $\tilde{M}$ be a hypersurface of R^{n+1} locally associated to M by a Ribaucour transformation. Then $\tilde{M}$ is a Dupin submanifold, if and only if, the functions Ω^i, Ω and W satisfies the following additional condition for each i $1 \leq i \leq n$,*

i) $d\left(\dfrac{dS(e_i)}{\Omega^i}\right)(e_i) = 0,$ *whenever* $\Omega^i \neq 0;$

ii) $d\left(\dfrac{T^i}{S}\right)(e_i) = 0,$ *whenever* $\Omega^i \equiv 0,$

where S *and* T^i *are given by (39).*

Proof: Assume that $\Omega^i \neq 0$, i.e. h depends on u_i. Since M is a Dupin submanifold, it follows from (37) that $d\tilde{\lambda}^i(e_i) = 0$, if and only if,

$$d\left(\frac{dS(e_i)W + \Omega^i\lambda^i S}{\Omega^i S - \Omega dS(e_i)}\right)(e_i) = 0.$$

Using (27) and (28), this equation is equivalent to

$$S(W + \Omega\lambda^i)\left(\Omega^i d(dS(e_i))(e_i) - dS(e_i)d\Omega^i(e_i)\right) = 0.$$

Since $S(W + \Omega\lambda^i) \neq 0$, we conclude that i) must be satisfied.

If h is independent of the variable u_i, i.e. $\Omega_i \equiv 0$, then it follows from (26), (27) and (28) that $d\Omega(e_i) = dW(e_i) = 0$ and $d\Omega^k(e_i) = 0$ for all $k \neq i$. Moreover, we have seen in Theorem 2.7 that $\tilde{\lambda}^i$ is given by (38). Hence, $d\tilde{\lambda}^i(e_i) = 0$ if and only if,

$$(W + \Omega\lambda^i)\left(SdT^i(e_i) - T^i dS(e_i)\right) = 0.$$

Therefore, we conclude that ii) holds.

□

3. Applications

In this section we will generate families of Dupin hypersurfaces by applying Theorem 2.8 to a hyperplane, a torus, $S^1 \times R^{n-1}$ and $S^2 \times R^{n-2}$. These examples will show that the transformation we are using to generate Dupin hypersurfaces is not a Lie transformation.

Proposition 3.1. *Consider the hyperplane in the Euclidean space* R^{n+1}, *parametrized by* $X(u_1, ..., u_n) = (u_1, ..., u_n, 0)$. $\tilde{X}$ *is a parametrized Dupin* *hypersurface locally associated to X by a Ribaucour transformation, if and only if,*

$$\tilde{X} = X - \frac{2(\sum_{j=1}^{n} f_j)}{\sum_j (f_j')^2 + c^2}(f_1', f_2', ..., f_n', -c), \tag{47}$$

where

$$f_i = c_{i2}u_i^2 + c_{i1}u_i + c_{i0} \tag{48}$$

and $c \neq 0$, c_{i2}, c_{i1}, c_{i0}, $\in R$.

Proof. Since the principal curvatures of X are $\lambda^i = 0$ and the metric $g_{ij} = a_i\delta_{ij} = 1$, for $1 \leq i, j \leq n$, it follows from equations (34)-(36) that

$$\Omega = \sum_{i=1}^{n} f_i, \qquad W = c \neq 0, \qquad h = \Omega/c, \quad \text{and } \Omega^i = f_i',$$

where $f_i(u_i)$ are differentiable functions. For any such functions, the submanifold $\tilde{X}$ given by (32) is a Ribaucour transformation of the hyperplane. In order to obtain Dupin submanifolds associated to M, we consider the expressions

$$S = c^2 + \sum_{i=1}^{n}(f_i')^2 \quad \text{and} \quad T^i = 0$$

defined by (39).

It follows from Theorem 2.7, that for each i, $1 \leq i \leq n$, such that f_i is constant, we get $\tilde{\lambda}^i = 0$. For each i such that f_i is not constant, it follows from Theorem 2.8, that $\tilde{X}$ is a Dupin submanifold, if and only if,

$$d\left(\frac{dS(e_i)}{\Omega^i}\right)(e_i) = 0.$$

Since $dS(e_i)/\Omega^i = 2f_i''$, we conclude that the condition above is equivalent to requiring that (48) is satisfied.

If $c_{i2} = 0$, and $c_{i1} \neq 0$, then from (37) we get $\tilde{\lambda}^i = 0$. If $c_{i2} \neq 0$, then

$$\tilde{\lambda}^i = \frac{4cc_{i2}}{\left(\sum_{j\neq i}(f_j')^2 - 4c_{i2}\sum_{j\neq i} f_j + A_i\right)},$$

where the constant $A_i = c^2 + c_{i1}^2 - 4c_{i2}c_{i0}$.

Moreover, the transformed Dupin submanifold (see equation (32)) is given by (47). □

Remark. In Proposition 3.1, one observes that the Dupin hypersurface $\tilde{X}$ has the following properties:

a) Generically, whenever the coefficients $c_{i2} \neq 0$ are distinct, then the curvatures $\tilde{\lambda}^i$ have multiplicity one.

b) If $c_{i2} = 0$ for k distinct indices i, then $\tilde{X}$ has a zero principal curvature of multiplicity k. In particular, if $c_{i2} = 0$ for all i, then $\tilde{X}$ is an open subset of a hyperplane.

c) If $c_{i2} = B \neq 0$ for all i, then all $\tilde{\lambda}^i$ are equal to a non zero constant. Hence, $\tilde{X}$ is an open subset of a sphere.

d) If for k distinct indices we have $c_{i_1 2} = c_{i_2 2} = \ldots = c_{i_k 2} \neq 0$, then we get a principal curvature of multiplicity k, namely $\tilde{\lambda}^{i_1}, \ldots, \tilde{\lambda}^{i_k}$ are equal to a function which is independent of $u_{i_1}, \ldots, u_{i_k}$.

Proposition 3.2. *Consider the torus in R^3, parametrized by*

$$X(u_1, u_2) = ((a + r\cos u_2)\cos u_1, (a + r\cos u_2)\sin u_1, r\sin u_2).$$

$\tilde{X}$ is a parametrized Dupin surface locally associated to X by a Ribaucour transformation, if and only if, $\tilde{X}$ is given by (32) where

$$\Omega^1 = f_1', \qquad \Omega^2 = f_2' - f_1 \sin u_2$$

$$\Omega = (a + r\cos u_2)f_1 + rf_2 + A, \qquad W = -f_1\cos u_2 - f_2 + B$$

$$f_i = a_i \cos u_i + b_i \sin u_i + c_i \qquad i = 1, 2, \tag{49}$$

where a_i, b_i, c_i, A and B are real constants such that if $\Omega^2 \equiv 0$ then $\Omega^1 \equiv 0$.

Proof. The principal curvatures of the torus are

$$\lambda^1 = \frac{\cos u_2}{a + r\cos u_2}, \quad \lambda^2 = \frac{1}{r}.$$

The coefficients of the metric are given by

$$a_1 = a + r\cos u_2 \qquad a_2 = r.$$

It follows from equations (34)-(36) that

$$\Omega^1 = f_1'(u_1), \quad \Omega^2 = -f_1 \sin u_2 + f_2'(u_2)$$

where f_1 and f_2 are differentiable functions of u_1 and u_2 respectively. Moreover,

$$\Omega = (a + r\cos u_2) f_1 + r f_2 + A \qquad W = -\cos u_2 f_1 - f_2 + B,$$

where A and B are constants.

In order to use Theorem 2.8, we consider the functions S and T^i defined by (39). Then

$$S = \sum_{i=1}^{2} f_i^2 + \sum_{i=1}^{2} (f_i')^2 - 2f_1(f_2' \sin u_2 - f_2 \cos u_2) - 2B(f_1 \cos u_2 + f_2) + B^2.$$

and

$$T^1 = 2(-\Omega^2 \sin u^2 - W\lambda^1) \qquad T^2 = -2W\lambda^2.$$

If $\Omega^i \neq 0$ $i = 1, 2$, then requiring $dS(e_i)/\Omega^i$ to be independent of u_i is equivalent to having f_i satisfying (49).

If $\Omega^1 \equiv 0$ then $f_1(u_1) = c_1$, Ω^2, W and S do not depend on u_1. Therefore, the condition ii) of Theorem 2.8 for $i = 1$ is trivially satisfied.

If we consider $\Omega^2 = 0$ then we necessarily have $f_1 = c_1$ and $f_2 = -c_1 \cos u_2 + c_2$ where c_1 and $c_2 \neq B$ are real constants. In this case, the associated surface is parallel to the torus and its principal curvatures are

$$\tilde{\lambda}^i = \frac{-\lambda^i}{1 + 2h\lambda^i}.$$

where $h = (ac_1 + rc_2 + A)/(B - c_2)$.

□

Proposition 3.3 *We consider the submanifold $M^n = S^1 \times R^{n-1}$ in the Euclidean space R^{n+1}, parametrized by*

$$X(u_1, ..., u_n) = (\cos u_1, \sin u_1, u_2, ..., u_n).$$

Then $\tilde{X}$ is a parametrized Dupin hypersurface locally associated to X by a Ribaucour transformation, if and only if,

$$\tilde{X} = X - \frac{2\sum_{j=1}^{n} f_j}{\sum_j (f_j')^2 + (c - f_1)^2} \left(((c - f_1)\sin u_1)', ((c - f_1)\cos u_1)', f_2', f_3', \ldots, f_n' \right) \tag{50}$$

where

$$\begin{aligned} f_1 &= a_1 \cos u_1 + b_1 \sin u_1 + c + c_1, \\ f_i &= c_{i2} u_i^2 + c_{i1} u_i + c_{i0} \qquad \text{if } i \geq 2. \end{aligned} \tag{51}$$

Proof. The principal curvatures and the metric for the manifold M are $\lambda_1 = 1$, $\lambda^i = 0$, $i \geq 2$ and $a_j = 1$, for $1 \leq j \leq n$. Hence, it follows from equations (34)-(36) that

$$\Omega^j = f_j', \quad \Omega = \sum_{j=1}^{n} f_j, \qquad W = -f_1 + c \qquad \text{and} \qquad h = \Omega/(-f_1 + c)$$

where $f_j(u_j)$ are differentiable functions of u_j.

For any such functions the parametrized submanifold $\tilde{X}$ defined by (32) is a Ribaucour transformation of the cylinder. In order $\tilde{X}$ to be a Dupin submanifold we consider the expressions defined by (39)

$$S = (-f_1 + c)^2 + \sum_{j=1}^{n} (f_j')^2$$

and

$$T^i = \begin{cases} 2(f_1 - c) & \text{if } i = 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

We will first assume that $\Omega^i \neq 0$, i.e. f_i is a non constant function of u_i, then

$$\frac{dS(e_i)}{\Omega^i} = \begin{cases} 2(f_1 - c) + 2f_1'' & \text{if } i = 1, \\ 2f_i'' & \text{if } i \geq 2. \end{cases}$$

Hence, condition i) of Theorem 2.8 is satisfied if and only if f_1 and f_j are given by (51). Moreover, the associated principal curvature is given by

$$\tilde{\lambda}^i = \begin{cases} \dfrac{2c_1 W + S}{S - 2\Omega c_1} & \text{if } i = 1, \\[2ex] \dfrac{4c_{i2} W}{S - 4\Omega c_{i2}}. & \text{if } i \geq 2. \end{cases}$$

If $\Omega^i = 0$ for some i, i.e. $f_i = c_i$ is a constant, $c_1 \neq c$ if $i = 1$. Then the condition ii) in Theorem 2.8 is trivially verified and

$$\tilde{\lambda}^i = \begin{cases} \dfrac{-(c-c_1)^2 + \sum_{j\geq 2}(f_j')^2}{(c-c_1)(c+c_1+2\sum_{j\geq 2} f_j) + \sum_{j\geq 2}(f_j')^2} & \text{if } i = 1, \\ 0 & \text{if } i \geq 2. \end{cases}$$

We conclude that in both cases the functions $f_1, ..., f_n$ are given by (51). From (32) we get the associated Dupin submanifold which is given by (50).

□

Remark. In Proposition 3.3, the Dupin hypersurface $\tilde{X}$ has the following properties:

a) If all the functions f_i are constant, then $\tilde{X}$ is a hypersurface parallel to $S^1 \times R^{n-1}$.

b) If $f_1 = c_1 \neq c$ and for all $j \geq 2$, f_j is a linear function given by $f_j = c_{j0} + (c - c_1)u_j/\sqrt{n-1}$, then $\tilde{X}$ is an open subset of a hyperplane since all its principal curvatures vanish.

c) If for k indices $2 \leq i_1, ..., i_k \leq n$, the functions f_{i_k} are linear in u_{i_k}, then $\tilde{X}$ has a zero principal curvature of multiplicity k.

d) If for k distinct indices we have $c_{i_1 2} = c_{i_2 2} = ... = c_{i_k 2} \neq 0$, then we get a nonzero principal curvature of multiplicity k, corresponding to $\tilde{\lambda}^{i_1} = ... = \tilde{\lambda}^{i_k}$.

e) In the generic case all principal curvatures have multiplicity one.

Proposition 3.4 *Consider the submanifold $M^n = S^2 \times R^{n-2}$, parametrized by*

$$X(u_1, ..., u_n) = (\sin u_1 \cos u_2, \sin u_1 \sin u_2, \cos u_1, u_3, ..., u_n).$$

Then $\tilde{X}$ is a parametrized Dupin hypersurface locally associated to X, if and only if, $\tilde{X}$ is given by (32) where

$$\Omega^1 = \cos u_1 f_2 + f_1', \qquad \Omega^i = f_i' \quad \text{for } i \geq 2, \tag{52}$$

$$\Omega = \sin u_1 f_2 + \sum_{i\neq 2} f_i, \qquad W = -(\sin u_1 f_2 + f_1), \tag{53}$$

and

$$f_i = a_i \cos u_i + b_i \sin u_i + c_i \quad 1 \leq i \leq 2, \tag{54}$$

$$f_j = c_{j2}u_j^2 + c_{j1}u_j + c_{j0} \quad 3 \leq j \leq n. \tag{55}$$

Proof. The principal curvatures of M are $\lambda^1 = \lambda^2 = 1$ and $\lambda^i = 0$ for $3 \leq i \leq n$. The first fundamental form of M is given by $ds^2 = \sum_j a_j^2 du_j^2$ where, $a_2 = \sin u_1$ and $a_i = 1$ for $i \neq 2$. It follows from (34), (35) and (36) that Ω^j, $1 \leq j \leq n$, Ω and W are given by (52) and (53), where $f_i(u_i)$ are differentiable functions.

In order to get the are Dupin submanifolds $\tilde{X}$ locally associated to X by a Ribaucour transformation, we consider the following expressions:

$$S = f_1^2 + f_2^2 + 2f_2(f_1 \sin u_1 + f_1' \cos u_1) + \sum_{j=1}^{n} (f_j')^2$$

$$T^i = \begin{cases} -2W & \text{if } i = 1 \\ -2W + 2\cos u_1 \Omega^1 & \text{if } i = 2 \\ 0 & \text{if } i \geq 3 \end{cases}$$

We will first consider the cases in which all functions Ω^i are non zero. Since $\Omega^i \neq 0$, we conclude from Theorem 2.8 that the functions f_i are given by (54) and (55).

Now we assume that a function Ω^i vanishes identically. If $\Omega^i \equiv 0$ for some $i \geq 3$, then $T^i = 0$, and $\tilde{\lambda}^i = 0$.

Assume $\Omega^1 \equiv 0$, then this is equivalent to $f_2 = c_2$ and $f_1 = -c_2 \sin u_1 + c_1$, where $c_1 \neq 0$. In this case, we necessarily have $\Omega^2 \equiv 0$. Hence, condition ii) of Theorem 2.8 must be satisfied for $i = 1, 2$. Since the expression of S reduces to

$$S = c_1^2 + \sum_{j\geq 3} (f_j')^2$$

and $T^1 = T^2 = 2c_1$, the condition is trivially verified. Moreover,

$$\tilde{\lambda}^1 = \tilde{\lambda}^2 = \frac{-c_1^2 + \sum_{j\geq 3}(f_j')^2}{-c_1^2 + \sum_{j\geq 3}(f_j')^2 - 2c_1 \sum_{j\geq 3} f_j}.$$

In the previous case, we have seen that $\Omega^1 \equiv 0$ implies that $\Omega^2 \equiv 0$. However, we may have $\Omega^2 \equiv 0$ without necessarily having $\Omega^1 \equiv 0$. In this case we have $f_2 = c_2$ and $f_1' \neq -c_2 \cos u_1$. From the fact that T^2 and S are independent of u_2, it follows that condition ii) of Theorem 2.8 is trivially satisfied.

□

Remark. The family of Dupin hypersurfaces $\tilde{X}$ of Proposition 3.4 have the following properties:

a) For generic choices of the constants involved in the functions (54) and (55), we get $\tilde{\lambda}^i$ to be independent of u_i and of multiplicity one.

b) If $a_1 = a_2 = b_2 = 0$, $b_1 = -c_2$ and $c_1 \neq 0$ (i.e. $\Omega^1 = \Omega^2 = 0$), then $\tilde{X}$ has a principal curvature of multiplicity 2, namely $\tilde{\lambda}^1 = \tilde{\lambda}^2$ is a function independent of u_1 and u_2.

c) If $c_{j2} = 0$ for some $j \geq 3$, then $\tilde{\lambda}^j = 0$.

d) Under the same conditions as in b), if in addition we require $c_{j2} = 0$ for all $j \geq 3$ and the constants c_{j1} to satisfy $\sum_{j\geq 3} c_{j1}^2 - c_1^2 = 0$, then $\tilde{X}$ is an open subset of a hyperplane.

e) If for k distinct indices $j_1, ...j_k \geq 3$ we have $c_{j_1 2} = ... = c_{j_k 2}$ then $\tilde{X}$ has a principal curvature of multiplicity k, namely $\tilde{\lambda}^{j_1} = ... = \tilde{\lambda}^{j_k}$ is a function independent of $u_{j_1}, ..., u_{j_k}$.

f) If $c_1 = 0$ and $c_2 + b_1 \neq 0$ then $\tilde{\lambda}^1 = 1$ and $\tilde{\lambda}^2 \neq 1$.

g) If $c_1 = 0$, $c_2 + b_1 = 0$ and $a_2^2 + b_2^2 \neq 0$ then $\tilde{\lambda}^1 = \tilde{\lambda}^2 = 1$.

h) Under the conditions of g), if in addition we require $c_{j2} = 0$ for all $j \geq 3$, $c_1 \neq 0$ and $a_1^2 - c_1^2 + a_2^2 + b_2^2 + \sum_{j\geq 3} c_{j1}^2 = 0$ then the principal curvatures for $\tilde{X}$ are $\tilde{\lambda}^1 = \tilde{\lambda}^2 = 1$, and $\tilde{\lambda}^j = 0$ for $j \geq 3$.

i) Under the conditions of g) if in addition we require $c_{j2} = C \neq 0$ for all $j \geq 3$, and $a_1^2 + a_2^2 + b_2^2 + \sum_{j\geq 3} c_{j1}^2 - 4C \sum_{j\geq 3} c_{j0} = 0$ then $\tilde{X}$ is an open subset of a unit sphere.

We observe that a Ribaucour transformation, which transforms a Dupin submanifold into another Dupin submanifold, is not a Lie transformation [] and it does not necessarily preserve the property of being a proper Dupin submanifold. A Dupin hypersurface whose principal curvatures have constant multiplicity is called a *proper Dupin submanifold.* It is easy to see that in Proposition 3.4 we can choose $\tilde{X}$ to be a non proper Dupin submanifold associated to $S^2 \times R^{n-2}$ by a Ribaucour transformation. In fact, by choosing $c_{32} = 0$ and $c_{j2} \neq 0 \, \forall j \geq 4$, in equation (55), we get generically $\tilde{\lambda}^i$ of order one except on the submanifold $\sum_{j\geq 4}(f_j')^2 = c_1^2 - c_{31}^2$ where $\tilde{\lambda}^1 = \tilde{\lambda}^2 = \tilde{\lambda}^3 = 0$.

Transformations for Dupin submanifolds with higher codimension will be treated in another paper.

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