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Lie group symmetries as integral transforms of fundamental solutions

Mark Craddock *, Kelly A. Lennox

Department of Mathematical Sciences, University of Technology Sydney, PO Box 123, Broadway, New South Wales 2007, Australia

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Abstract

We obtain fundamental solutions for PDEs of the form $u_t = \sigma x^\gamma u_{xx} + f(x)u_x - \mu x^r u$ by showing that if the symmetry group of the PDE is nontrivial, it contains a standard integral transform of the fundamental solution. We show that in this case, the problem of finding a fundamental solution can be reduced to inverting a Laplace transform or some other classical transform.

Keywords: Lie symmetry groups; Fundamental solutions; Transition densities; Short rate models; Zero coupon bond pricing

1. Introduction

In this paper we show how to compute fundamental solutions for every partial differential equation (PDE) of the form

$$u_t = \sigma x^\gamma u_{xx} + f(x)u_x - \mu x^r u, \tag{1.1}$$

which possesses a sufficiently large symmetry group. These PDEs are important in financial mathematics and other areas. We extend a technique due to Craddock and Platen [4], who studied the $\gamma = 1$, $\mu = 0$ case. Their method reduces the problem to the evaluation of a single inverse Laplace transform, which is given as an explicit function of the drift f. Such a method has

* Corresponding author.
E-mail address: mark.craddock@uts.edu.au (M. Craddock).

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many benefits. For example, it is quite direct since it does not require changes of variables or the solution of an ordinary differential equation, and it may be implemented numerically.

Craddock and Platen's analysis relies upon an apparent curiosity. Namely, that for a large class of PDEs, one of the *multipliers* (see Section 2), in the Lie point symmetry group is the Laplace transform of the fundamental solution. This approach is however incomplete, as Craddock and Platen were not able to handle every class of PDEs they studied. We give the complete solution to the problem.

It turns out that what Craddock and Platen discovered is a special case of a more general phenomenon. We prove that for at least one of the vector fields in the $\mathfrak{sl}_2$ part of the Lie symmetry algebra, the multiplier of the symmetry, or a slight modification of the multiplier, is *always* a classical integral transform of the fundamental solution of the PDE. We show that Laplace transforms of the fundamental solutions arise naturally, as do other classical transforms, such as the Whittaker and Hankel transforms. A priori, it is not at all obvious that this should be true. This curious fact suggests a connection between Lie symmetry analysis and harmonic analysis, which should be investigated further.

The method also yields a quite elementary derivation of the so called heat kernel on the Heisenberg group. Most derivations of this heat kernel are far from elementary. See, for example, Gaveau [5] and Jorgensen and Klink [6]. We reduce the problem to inverting a Laplace transform. In fact, we will recover more than just the heat kernel. We obtain a family of solutions containing the heat kernel.

2. Lie symmetries as integral transforms

For the theory of Lie symmetries we refer to Olver's book [10]. For simplicity we consider only a single *linear* equation

$$u_t = P\big(x, u^{(n)}\big), \quad x \in \Omega \subseteq \mathbb{R}, \tag{2.1}$$

with independent variables x and t and dependent variable u. Here $u^{(n)}$ denotes u and its first n derivatives in x. The essence of Lie's method is that we look for vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u, \tag{2.2}$$

which generate one parameter groups preserving solutions of the given PDE. Since every PDE of the form (2.1) has time translation symmetries as well as an infinite-dimensional Lie algebra of symmetries which comes from the superposition of solutions, we call them *trivial* symmetries. When we refer to the dimension of a Lie symmetry algebra, we mean the dimension of the algebra excluding the infinite-dimensional ideal generated by superposition of solutions.

We denote by $\tilde{u}_\epsilon = \rho(\exp \epsilon \mathbf{v})u(x,t)$ the action on solutions generated by $\mathbf{v}$. Typically we have

$$\rho(\exp \epsilon \mathbf{v})u(x,t) = \sigma(x,t;\epsilon)u\big(a_1(x,t;\epsilon), a_2(x,t;\epsilon)\big), \tag{2.3}$$

for some functions σ, a_1 and a_2. We call σ the *multiplier* and a_1 and a_2 the *change of variables* of the symmetry.

Now suppose that (2.1) has a fundamental solution $p(t,x,y)$. Then the function

$$u(x,t)=\int_{\Omega} f(y)p(t,x,y)\,dy, \tag{2.4}$$

solves the initial value problem for (2.1) with appropriate initial data $u(x,0)=f(x)$.

The idea is to connect the solutions (2.3) and (2.4). We take a stationary (time independent) solution $u=u_0(x)$. So in this case

$$\rho(\exp\epsilon\mathbf{v})u_0(x)=\sigma(x,t;\epsilon)u_0\big(a_1(x,t;\epsilon)\big). \tag{2.5}$$

Setting $t=0$ and using (2.4) suggests the relation

$$\int_{\Omega}\sigma(y,0;\epsilon)u_0\big(a_1(y,0;\epsilon)\big)p(t,x,y)\,dy=\sigma(x,t;\epsilon)u_0\big(a_1(x,t;\epsilon)\big). \tag{2.6}$$

Since σ and a_1 are known, we have a family of integral equations for $p(t,x,y)$.

Consider the example of the one-dimensional heat equation $u_t=u_{xx}$. If $u(x,t)$ solves the heat equation, then for ϵ small enough, so does

$$\tilde{u}_\epsilon=e^{-\epsilon x+\epsilon^2 t}u(x-2\epsilon t,t). \tag{2.7}$$

Taking $u_0=1$, Eq. (2.5) gives

$$\int_{-\infty}^{\infty} e^{-\epsilon y}p(t,x-y)\,dy=e^{-\epsilon x+\epsilon^2 t}, \tag{2.8}$$

where $p(t,x)$ is the one-dimensional heat kernel. Thus the multiplier in the symmetry (2.7) is nothing more than the two-sided Laplace transform of $p(t,x-y)$. We can recover $p(t,x-y)$ by inverting (2.8).

A second well-know symmetry of the heat equation is

$$\tilde{u}_\epsilon=\frac{1}{\sqrt{1+4\epsilon t}}\exp\left\{\frac{-\epsilon x^2}{1+4\epsilon t}\right\}u\left(\frac{x}{1+4\epsilon t},\frac{t}{1+4\epsilon t}\right). \tag{2.9}$$

Using $u_0=1$ again we obtain

$$\int_{-\infty}^{\infty} e^{-\epsilon y^2}p(t,x-y)\,dy=\frac{1}{\sqrt{1+4\epsilon t}}\exp\left\{\frac{-\epsilon x^2}{1+4\epsilon t}\right\}. \tag{2.10}$$

Equation (2.10) is an integral transform. It is easy to verify that $p(t,x)=\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$ is a solution of the integral equation (2.10). Directly inverting the transform is more problematic since we do not know of an explicit inversion theorem and the inverse transform will be unique only up to the addition of an arbitrary odd function. However, using the fact that p is positive and setting $z = y^2$ gives

$$\int_{-\infty}^{\infty} e^{-\epsilon y^2} p(t, x-y)\,dy = \int_{0}^{\infty} e^{-\epsilon z} G(t,x,z)\,dz,$$

where $G(t,x,z) = (p(t, x-\sqrt{z}) + p(t, x+\sqrt{z}))/2\sqrt{z}$. Thus the multiplier in the symmetry (2.9) is the Laplace transform of $G(t,x,z)$. Inverting the Laplace transform gives

$$G(t,x,z) = \frac{1}{\sqrt{4\pi t}} \exp\left\{\frac{-(x^2+z)}{4t}\right\} \frac{\cosh(x\sqrt{z}/2t)}{\sqrt{z}}.$$

From this we can recover $p(t,x)$. A more satisfactory method of extracting the fundamental solution from (2.10) is developed in Section 7. Our overall aim is to show that exploiting the relationship (2.6) yields a fundamental solution for any PDE of the form (1.1) with nontrivial symmetry group.

3. Symmetries of generalized bond pricing equations

A zero coupon bond is a security which pays the holder one unit of currency at maturity. A readable account of the theory of bond pricing by PDE methods is contained in the book [7].

Assume that the risk neutral spot rate of interest X satisfies the stochastic differential equation

$$dX_t = f(X_t, t)\,dt + b(X_t, t)\,dW_t, \qquad X_0 = x, \tag{3.1}$$

where W is a standard Wiener process. The function f is known as the drift and b the volatility. Let u be a solution of the PDE

$$u_t = \frac{1}{2} b(x,t)^2 u_{xx} + f(x,t)u_x - xu, \quad x \geqslant 0, \tag{3.2}$$

with initial condition $u(x,0) = 1$. It can be shown that the price at time t of a zero coupon bond maturing at time T, is given by $u(x, T-t)$.

We concentrate here on interest rate models of the form

$$dX_t = f(X_t)\,dt + \sqrt{2\sigma X_t}\,dW_t.$$

This corresponds to $\gamma = 1$ in (1.1). We will illustrate the general principles for the $\gamma = 1$ case. We will proceed by computing the Lie symmetries of the PDE (1.1) with $\gamma = 1$. The case $r = 1$ contains the bond pricing equation for this class of interest rate models. Its symmetries were completely described by Lennox in [9]. The $\mu = 0$ case was analysed by Craddock and Platen in [4]. The case of arbitrary γ in (1.1) is considered in Section 7.

Proposition 3.1. *The partial differential equation*

$$u_t = \sigma x u_{xx} + f(x)u_x - \mu x^r u \tag{3.3}$$

with $r \neq -1$ has a nontrivial Lie algebra of symmetries if and only if the drift function f is a solution of one of the following families of Ricatti equations

$$\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\mu\sigma x^{r+1} = Ax + B, \tag{3.4}$$

$$\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\mu\sigma x^{r+1} = \frac{A}{2} x^2 + Bx + C, \tag{3.5}$$

$$\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\mu\sigma x^{r+1} = \frac{A}{2} x^2 + \frac{2}{3} B x^{\frac{3}{2}} + Cx - \frac{3}{8}\sigma^2. \tag{3.6}$$

If $r = -1$ the right-hand side of Eqs. (3.4)–(3.6) *is replaced by* $\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\mu\sigma \ln x$.

The proof of this result is essentially the same as the $\mu = 0$ case given in [4]. The factors of $\frac{1}{2}$ and $\frac{2}{3}$ multiplying the constants A and B in (3.5) and (3.6) are a notational convenience. For our purposes we need only the following facts: when f is a solution of (3.4), then there is an infinitesimal symmetry

$$\mathbf{v}_1 = xt\partial_x + \frac{1}{2} t^2 \partial_t - \frac{1}{2\sigma}\left(x + tf(x) + \frac{1}{2} A t^2\right) u\partial_u. \tag{3.7}$$

When f is a solution of (3.5) there is an infinitesimal symmetry of the form

$$\mathbf{v}_1 = x e^{-\sqrt{A}t}\partial_x - \frac{e^{-\sqrt{A}t}}{\sqrt{A}}\partial_t + \frac{e^{-\sqrt{A}t}}{2\sigma}\left(\sqrt{A}x - f(x) + \frac{B}{\sqrt{A}}\right) u\partial_u. \tag{3.8}$$

When f is a solution of (3.6) with $A = 0$ there is an infinitesimal symmetry

$$\begin{aligned}\mathbf{v}_1 = {} & \left(xt + \frac{B}{12}\sqrt{x} t^3\right)\partial_x + \frac{1}{2} t^2 \partial_t \\ & - \frac{1}{2\sigma}\left(x + \frac{B}{2}\sqrt{x} t^2 + tf(x) - \frac{\sigma B}{24\sqrt{x}} t^3 + \frac{B}{12\sqrt{x}} t^3 f(x) + \frac{C}{2} t^2 + \frac{B^2}{144} t^4\right) u\partial_u.\end{aligned}$$

When f is a solution of (3.6) with $A \neq 0$ then there is an infinitesimal symmetry

$$\begin{aligned}\mathbf{v}_1 = {} & e^{-\sqrt{A}t}\left(x + \frac{2B}{3A}\sqrt{x}\right)\partial_x - \frac{e^{-\sqrt{A}t}}{\sqrt{A}}\partial_t \\ & + e^{-\sqrt{A}t}\left(\frac{\sqrt{A}}{2\sigma} x + \frac{2B\sqrt{x}}{3\sigma\sqrt{A}} - \frac{f(x)}{2\sigma} + \frac{B}{6A\sqrt{x}} - \frac{Bf(x)}{3\sigma A\sqrt{x}} + \frac{C}{2\sigma\sqrt{A}} + \frac{B^2}{9\sigma A^{3/2}}\right) u\partial_u.\end{aligned} \tag{3.9}$$

A complete classification of the infinitesimal symmetries is in [9]. It may be readily checked that the symmetry algebras in all cases are either the Lie algebras of $SL(2,\mathbb{R}) \times \mathbb{R}$ or $SL(2,\mathbb{R}) \rtimes H_3$ where H_3 is the three-dimensional Heisenberg group. The vector fields listed above all come from $\mathfrak{sl}_2$.

For example, consider the case of the Lie symmetry algebra when f satisfies (3.4). Introduce the standard basis for $\mathfrak{sl}_2$ given by

$$k_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad k_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad k_3 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Calculating the commutator table for the Lie symmetries shows that $\mathbf{v_1}$ given by (3.7) is equivalent to the matrix element k_3. The time translation symmetry $\mathbf{v}_2 = \partial_t$ is equivalent to k_1 and the Lie bracket of $\mathbf{v_1}$ and $\mathbf{v}_2$ produces a vector field which is equivalent to the basis vector k_2. Similar comments can be made about the Lie symmetry algebras when the drift f satisfies one of the other classes of Ricatti equations. In general, all the vector fields listed here correspond to k_3.

It is also worth noting that all the known symmetry methods for obtaining fundamental solutions, such as the group invariant solution method of Bluman, Cole and Kumei (cf. [2]) will only work for the drifts described in Proposition 3.1.

4. Laplace transforms of fundamental solutions

The first case to consider is when f satisfies (3.4). The next result generalizes [4, Theorem 4.1]. The case $r = 1$ is in [9].

Theorem 4.1. *Let f be an analytic solution of the Ricatti equation*

$$\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\mu\sigma x^{r+1} = Ax + B. \tag{4.1}$$

Let $u_0(x)$ be an analytic solution of (3.3) *which is independent of t and*

$$U_\lambda(x,t) = \exp\left\{ \frac{1}{2\sigma} \left(F\left(\frac{x}{(1+\sigma\lambda t)^2} \right) - F(x) \right) - \frac{\lambda(x + \frac{A}{2}t^2)}{(1+\sigma\lambda t)} \right\} u_0\left(\frac{x}{(1+\sigma\lambda t)^2} \right), \tag{4.2}$$

where $F'(x) = f(x)/x$. Then

$$U_\lambda(x,t) = \int_0^\infty u_0(y) p_\mu(t,x,y) e^{-\lambda y}\, dy,$$

where $p_\mu(t,x,y)$ is a fundamental solution of (3.3).

Proof. We exponentiate the vector field (3.7) to see that if $u_0(x)$ is a stationary solution of the PDE (3.3) where f satisfies (3.4), then $U_\lambda(x,t)$ given by (4.2) is also a solution. Since u_0 and f are analytic, then for each $t > 0$, U_λ is analytic in $1/\lambda$ and therefore is a Laplace transform in y of $u_0(y)p_\mu(t,x,y)$ for some distribution p_μ. Since $U_\lambda(x,0) = u_0(x)e^{-\lambda x}$ we must have $p(0,x,y) = \delta_x(y)$, the Dirac measure weighted at x. To see that p_μ is a fundamental solution of (3.3), observe that if we integrate a test function $\varphi(\lambda)$ with sufficiently rapid decay against U_λ then the function $u(x,t) = \int_0^\infty U_\lambda(x,t)\varphi(\lambda)\, d\lambda$ is a solution of (3.3). We also have

$$u(x,0)=\int_0^\infty U_\lambda(x,0)\varphi(\lambda)\,d\lambda=\int_0^\infty u_0(x)e^{-\lambda x}\varphi(\lambda)\,d\lambda=u_0(x)\Phi(x),$$

where Φ is the Laplace transform of φ. Next observe that

$$\begin{aligned}\int_0^\infty u_0(y)\Phi(y)p_\mu(t,x,y)\,dy&=\int_0^\infty\int_0^\infty u_0(y)\varphi(\lambda)p_\mu(t,x,y)e^{-\lambda y}\,d\lambda\,dy\\&=\int_0^\infty\int_0^\infty u_0(y)\varphi(\lambda)p_\mu(t,x,y)e^{-\lambda y}\,dy\,d\lambda\\&=\int_0^\infty\varphi(\lambda)U_\lambda(x,t)\,dx=u(x,t).\end{aligned}$$

We know that $u(x,0)=u_0(x)\Phi(x)$. Thus integrating initial data $u_0\Phi$ against p_μ solves the Cauchy problem for (3.3), with this initial data. Hence p_μ is a fundamental solution. □

We will give some financial applications. To obtain drift functions, we set $f(x)=2\sigma xy'(x)/y(x)$. This transforms Eq. (3.4) with $r=1$ to the linear ODE

$$2\sigma^2x^2y''(x)+\left(2\mu\sigma x^2-Ax-B\right)y(x)=0.\tag{4.3}$$

This has general solution

$$y(x)=e^{\frac{-ix\sqrt{\mu}}{\sqrt{\sigma}}}x^{\beta/2}\left(a\,{}_1F_1\left(\alpha,\beta,\frac{2ix\sqrt{\mu}}{\sqrt{\sigma}}\right)+b\Psi\left(\alpha,\beta,\frac{2ix\sqrt{\mu}}{\sqrt{\sigma}}\right)\right),\tag{4.4}$$

where a and b are constants,

$$\alpha=\frac{-\left(iA-2\sqrt{\mu}\sigma^{\frac{3}{2}}-2\sqrt{\mu}\sqrt{1+\frac{2B}{\sigma^2}}\sigma^{\frac{3}{2}}\right)}{4\sqrt{\mu}\sigma^{3/2}},\qquad\beta=1+\sqrt{1+\frac{2B}{\sigma^2}},$$

${}_1F_1$ is Kummer's confluent hypergeometric function and Ψ is Tricomi's confluent hypergeometric function, given by [1, formula (13.1.6)]. This implies that f is analytic.

To obtain a stationary solution u_0 we solve

$$\sigma xu_{xx}+f(x)u_x-\mu xu=0.\tag{4.5}$$

We set $u=\tilde{u}(x)e^{\int\varphi(x)\,dx}$ with $\varphi(x)=-\frac{1}{2\sigma x}f(x)$. An easy calculation shows that $\tilde{u}$ satisfies the ODE

$$2\sigma^2x^2\tilde{u}_{xx}-(Ax+B)\tilde{u}=0.\tag{4.6}$$

Equation (4.6) is solved in terms of Bessel functions. Hence u_0 is also analytic.

Example 4.1. If $A=0$, then

$$y(x)=a\sqrt{x}J_\eta\big(x\sqrt{\mu/\sigma}\big)+b\sqrt{x}Y_\eta\big(x\sqrt{\mu/\sigma}\big). \tag{4.7}$$

Here $\eta=\frac{1}{2}\sqrt{1+2B/\sigma^2}$, J_η, Y_η are Bessel functions of the first and second kinds and a and b are arbitrary constants. If $B=0$, $a=0$ and $b=1$ we obtain the drift function $f(x)=2x\sqrt{\mu\sigma}\cot(x\sqrt{\frac{\mu}{\sigma}})$.

A stationary solution of the corresponding bond pricing equation is $u_0(x)=\csc(x\sqrt{\frac{\mu}{\sigma}})$. Applying Theorem 4.1 we obtain

$$U_\lambda(x,t)=\exp\left\{\frac{-\lambda x}{1+t\lambda\sigma}\right\}\csc\left(x\sqrt{\frac{\mu}{\sigma}}\right).$$

Computing the inverse Laplace transform gives

$$p_\mu(t,x,y)=\exp\left\{\frac{-(x+y)}{\sigma t}\right\}\left(\frac{\sqrt{x}}{\sqrt{y}\sigma t}I_1\left(\frac{2\sqrt{xy}}{\sigma t}\right)+\delta(y)\right)\frac{\sin\big(y\sqrt{\frac{\mu}{\sigma}}\big)}{\sin\big(x\sqrt{\frac{\mu}{\sigma}}\big)}.$$

The bond price $B(x,t,T)$ when the risk neutral spot rate of interest x follows the SDE $dX_t=2X_t\sqrt{\sigma}\cot(\frac{X_t}{\sqrt{\sigma}})dt+\sqrt{2\sigma X_t}\,dW_t$, is then

$$B(x,t,T)=\int_0^\infty p_1(T-t,x,y)\,dy=e^{\{\frac{-x(T-t)}{1+\sigma(T-t)^2}\}}\frac{\sin\big(\frac{x}{\sqrt{\sigma}+\sigma^{3/2}(T-t)^2}\big)}{\sin\big(\frac{x}{\sqrt{\sigma}}\big)}.$$

With the fundamental solution we can also price options on bonds, interest rate swaps, interest rate caps and many other instruments. For example, a European call/put option is the right to buy/sell an underlying asset for a fixed *strike* price at some date in the future. The price of a call option with strike E on the zero coupon bond B is simply

$$C(B,x,t,T)=\int_0^\infty \max\big(B(y,t,T)-E,0\big)p_1(T-t,x,y)\,dy.$$

Example 4.2. Setting $A=0$, $B=4\sigma^2$, $a=1$ and $b=1$ in the general solution (4.4) we obtain the drift function

$$f(x)=\frac{-2\sigma}{g(x)}\left(g(x)+\frac{\mu x^2}{\sqrt{\sigma}}\left(\sin\left(x\sqrt{\frac{\mu}{\sigma}}\right)-\cos\left(x\sqrt{\frac{\mu}{\sigma}}\right)\right)\right),$$

where $g(x)=(x\sqrt{\mu}+\sqrt{\sigma})\cos(x\sqrt{\frac{\mu}{\sigma}})+(x\sqrt{\mu}-\sqrt{\sigma})\sin(x\sqrt{\frac{\mu}{\sigma}})$.

Omitting the details, we obtain via Theorem 4.1 a fundamental solution of $u_t = \sigma x u_{xx} + f(x)u_x - \mu x u$. It is

$$p_\mu(t,x,y) = \frac{1}{\sigma t}\exp\left\{-\frac{x+y}{\sigma t}\right\}\left(\frac{x}{y}\right)^{\frac{3}{2}} I_3\left(\frac{2\sqrt{xy}}{\sigma t}\right)\frac{g(y)}{g(x)}.$$

The corresponding bond price may now readily be obtained.

Many such examples are possible. The drifts which arise can be very complex, and so allow for very different kinds of interest rate dynamics.

4.0.1. Stochastic volatility models

Consider the problem of pricing a European option under stochastic volatility. The time evolution of the underlying asset S_t is modelled by the SDE $dS_t = rS\,dt + \sqrt{v_t}\,dW_t^1$ and the volatility v_t satisfies $dv_t = f(v_t)\,dt + a\sqrt{v_t}\,dW_t^2$.

We assume that the two Wiener processes W^1, W^2 are uncorrelated. The value of an option at time t with expiry T and payoff g is denoted $V(S,v,t)$. It can be shown that V must satisfy the PDE

$$V_t + \frac{1}{2}vS^2V_{SS} + rSV_S + \frac{1}{2}a^2vV_{vv} + f(v)V_v - rV = 0, \tag{4.8}$$

subject to $V(S,v,T) = g(S,v)$. For a discussion of stochastic volatility models see the book by Joshi [7]. Letting $S = \ln x$, $t \to T - t$ and $\widehat{V}(\xi,v,t) = \int_{-\infty}^{\infty} V(x,v,t)e^{-i\xi x}\,dx$ leads to the PDE

$$U_t = \frac{1}{2}a^2vU_{vv} + f(v)U_v - \mu vU, \tag{4.9}$$

with $\alpha = r(i\xi - 1)$, $\mu = \frac{1}{2}(\xi^2 + i\xi)$ and $U = e^{-\alpha t}\widehat{V}$. If $p_\mu(t,v,y)$ is the fundamental solution of (4.9), then

$$\widehat{V}(\xi,v,t) = e^{\alpha(T-t)}\int_0^\infty \hat{g}(\xi,y)p_\mu(T-t,v,y)\,dy.$$

From which we can determine V. Consequently, for our models, the problem of pricing an option under stochastic volatility is reduced to the evaluation of an integral.

5. Whittaker transforms of fundamental solutions

Let us consider the vector field $\mathbf{v}_1$ from (3.8). Craddock and Platen were only able to handle some special cases for this symmetry. Here we present the complete solution to the problem.

Exponentiating $\mathbf{v}_1$ and replacing ϵ with $-\epsilon$ we see that if $u_0(x)$ is a stationary solution of Eq. (3.3), then with $F'(x) = f(x)/x$,

$$U_{-\epsilon}(x,t) = e^{-\frac{Bt}{2\sigma}}\exp\left\{\frac{-\sqrt{A}x\epsilon}{2\sigma(e^{\sqrt{A}t} - \epsilon)} + \frac{1}{2\sigma}\left(F\left(\frac{xe^{\sqrt{A}t}}{e^{\sqrt{A}t} - \epsilon}\right) - F(x)\right)\right\}$$
$$\times \left(e^{\sqrt{A}t} - \epsilon\right)^{\frac{B}{2\sigma\sqrt{A}}} u_0\left(\frac{xe^{\sqrt{A}t}}{(e^{\sqrt{A}t} - \epsilon)}\right) \tag{5.1}$$

is also a solution of (3.3). We assume that u_0 is not left invariant by this symmetry. If it is, then the analysis we present fails. However if (5.1) leaves u_0 invariant, we may use a second $\mathfrak{sl}_2$ symmetry, generated by

$$\mathbf{v}_2 = xe^{\sqrt{A}t}\partial_x + \frac{e^{\sqrt{A}t}}{\sqrt{A}}\partial_t - \frac{e^{\sqrt{A}t}}{2\sigma}\left(\sqrt{A}x + f(x) + \frac{B}{\sqrt{A}}\right)u\partial_u.$$

This symmetry will not fix u_0. The analysis we present here can be repeated for that symmetry and the results are essentially identical.

We assume that $A > 0$. The case $A < 0$ can be handled by a slight modification of our argument. We write $U_{-\epsilon}(x,t)$ as an integral of its initial value against our fundamental solution. Then we have

$$\int_0^\infty U_{-\epsilon}(y,0)p_\mu(t,x,y)\,dy = U_{-\epsilon}(x,t). \tag{5.2}$$

To identify this as a known integral transform we observe that the solution u_0 may be written as $u_0(x) = \tilde{u}_0(x)e^{-F(x)/2\sigma}$ where $\tilde{u}_0$ satisfies the ODE $2\sigma^2x^2\tilde{u}_0''(x) - (\frac{1}{2}Ax^2 + Bx + C) \times \tilde{u}_0(x) = 0$. This has solution

$$\tilde{u}_0(x) = e^{\frac{-\sqrt{A}x}{2\sigma}}x^{\frac{\beta}{2}}\left(a_1F_1\left(\alpha,\beta,\frac{\sqrt{A}x}{\sigma}\right) + b\Psi\left(\alpha,\beta,\frac{\sqrt{A}x}{\sigma}\right)\right), \tag{5.3}$$

where $\alpha = \frac{B}{2\sigma\sqrt{A}} + \frac{1}{2}\beta$, $\beta = 1 + \sqrt{1 + \frac{2C}{\sigma^2}}$. With $a = 0$, $b = 1$, this gives

$$U_{-\epsilon}(x,t) = e^{-\frac{Bt}{2\sigma}}\left(e^{\sqrt{A}t} - \epsilon\right)^{\frac{B}{2\sigma\sqrt{A}} - \frac{\beta}{2}}x^{\frac{\beta}{2}}e^{-\frac{F(x)}{2\sigma}}\exp\left\{\frac{-\sqrt{A}x(e^{\sqrt{A}t} + \epsilon)}{2\sigma(e^{\sqrt{A}t} - \epsilon)}\right\}$$
$$\times\Psi\left(\alpha,\beta,\frac{\sqrt{A}xe^{\sqrt{A}t}}{\sigma(e^{\sqrt{A}t} - \epsilon)}\right). \tag{5.4}$$

Therefore, with $\eta = \frac{B}{2\sigma\sqrt{A}} - \frac{1}{2}\beta$, Eq. (5.2) reads

$$U_{-\epsilon}(x,t) = (1-\epsilon)^\eta\int_0^\infty y^{\frac{\beta}{2}}e^{-\frac{F(y)}{2\sigma}}\exp\left\{-\frac{\sqrt{A}y(1+\epsilon)}{2\sigma(1-\epsilon)}\right\}\Psi\left(\alpha,\beta,\frac{\sqrt{A}y}{\sigma(1-\epsilon)}\right)p_\mu(t,x,y)\,dy.$$

Upon setting $\lambda = \frac{\sqrt{A}}{\sigma(1-\epsilon)}$ this becomes

$$\int_0^\infty e^{-\lambda y}(\lambda y)^{\frac{\beta}{2}}\Psi(\alpha,\beta,\lambda y)h_\mu(t,x,y)\,dy = \lambda^{\frac{B}{2\sigma\sqrt{A}}}U_{\sqrt{A}/(\sigma\lambda)-1}(x,t),$$

where $h_\mu(t,x,y) = (\frac{\sqrt{A}}{\sigma})^\eta e^{\frac{\sqrt{A}y - F(y)}{2\sigma}}p_\mu(t,x,y)$.

Let $W_{k+1/2,\nu}(z) = e^{-z/2} z^{1/2+\nu} \Psi(\nu - k, 1 + 2\nu, z)$ be the second Whittaker function (see [1, formula (13.1.33)]). If $1 + 2\nu = \beta$, $\nu - k = \alpha$, then the previous integral becomes

$$\int_0^\infty e^{-\frac{\lambda y}{2}} W_{k+\frac{1}{2},\nu}(\lambda y) h_\mu(t, x, y)\, dy = \lambda^{\frac{B}{2\sigma\sqrt{A}}} U_{\sqrt{A}/(\sigma\lambda)-1}(x, t).$$

We may write this as

$$\int_0^\infty e^{-\frac{\lambda y}{2}} (\lambda y)^{-k-\frac{1}{2}} W_{k+1/2,\nu}(\lambda y)(\lambda y)^{k+\frac{1}{2}} h_\mu(t, x, y)\, dy = \lambda^{\frac{B}{2\sigma\sqrt{A}}} U_{\sqrt{A}/(\sigma\lambda)-1}(x, t).$$

If $\tilde{h}_\mu(t, x, y) = y^{k+\frac{1}{2}} h_\mu(t, x, y)$, then this becomes

$$\int_0^\infty e^{-\frac{\lambda y}{2}} (\lambda y)^{-k-\frac{1}{2}} W_{k+1/2,\nu} \tilde{h}_\mu(t, x, y)\, dy = \lambda^{\frac{B}{\sigma\sqrt{A}}} U_{\sqrt{A}/(\sigma\lambda)-1}(x, t). \tag{5.5}$$

The final integral in (5.5) is the so-called *Whittaker* transform of $\tilde{h}_\mu$.

Definition 5.1 *(The Whittaker transform).* The Whittaker transform of a suitable function ϕ is defined by

$$(\mathbf{W}_{k,\nu}\phi)(\lambda) = \Phi(\lambda) = \int_0^\infty (\lambda y)^{-k-1/2} e^{-\lambda y/2} W_{k+1/2,\nu}(\lambda y)\phi(y)\, dy. \tag{5.6}$$

An inversion theorem for this transform is known. For a suitable constant ρ we have

$$\phi(y) = \frac{1}{2\pi i} \frac{\Gamma(1 + \nu - k)}{\Gamma(1 + 2\nu)} \int_{\rho-i\infty}^{\rho+i\infty} (\lambda y)^{-k-1/2} e^{\lambda y/2} M_{k-1/2,\nu}(\lambda y)\Phi(\lambda)\, d\lambda.$$

The function $M_{k-1/2,\nu}$ is the first Whittaker function given by [1, formula (13.1.32)]. The integral is taken in the principal value sense. For a discussion of the transform, see [3, p. 110]. Note that Brychkov and Prudnikov call this the *Meijer transform*. There does not seem to be a general naming convention in the literature regarding these transforms. Unfortunately, theorems guaranteeing that a given function is a Whittaker transform seem to be difficult to prove, so our next result is not as strong as Theorem 4.1.

Theorem 5.2. *Let f be a solution of* (3.5) *for $\mu \neq 0$. Let*

$$\eta = \frac{B}{2\sigma\sqrt{A}} - \frac{1}{2}\beta, \qquad \nu = \frac{1}{2}\sqrt{1 + 2C/\sigma^2} \quad \textit{and} \quad k + \frac{1}{2} = -\frac{B}{2\sigma\sqrt{A}}.$$

Let $U_{\sqrt{A}/(\lambda\sigma)-1}(x, t)$ be given by (5.1). *Suppose that $\lambda^{\frac{B}{\sigma\sqrt{A}}} U_{\sqrt{A}/(\lambda\sigma)-1}(x, t)$ is the Whittaker transform of a function $\tilde{h}_\mu(t, x, y)$. Then*

$$\tilde{h}_\mu(t,x,y) = \left(\frac{\sqrt{A}}{\sigma}\right)^\eta y^{k+\frac{1}{2}} e^{\frac{\sqrt{A}y - F(y)}{2\sigma}} p_\mu(t,x,y),$$

where p_μ *is a fundamental solution of the PDE* (3.3).

Note. If $\mu = 0$ this theorem is still valid if we simply replace the stationary solution given with $u_0 = 1$. It is also easy to show that if $A \to 0$ then this result reduces to Theorem 4.1.

Proof of Theorem 5.2. From the initial value for $\lambda^{\frac{B}{\sigma\sqrt{A}}} U_{\sqrt{A}/(\lambda\sigma)-1}(x,t)$, it is clear that its Whittaker transform must be of the form $\tilde{h}_\mu(t,x,y)$, where $p(0,x,y) = \delta_x(y)$. To show that p_μ is a fundamental solution, we use the same argument as in the proof of Theorem 4.1. We need to know that the Whittaker transform has the right operational properties, such as $\int_0^\infty f(x)(\mathbf{W}_{k,\nu} g)(x)\,dx = \int_0^\infty (\mathbf{W}_{k,\nu} f)(x) g(x)\,dx$. This is the case. For a discussion of this and other operational properties of the Whittaker transform see [8, Chapter 7]. □

Note. Fortunately, it is possible to explicitly prove that for a wide range of parameters we do indeed have a Whittaker transform. For example, if $\mathrm{Re}(\nu - k) > -1$ then $\tilde{h}_\mu$ can be shown to be a Whittaker transform. Two other special cases are presented now.

5.1. Some special cases

5.1.1. The case $\frac{B}{2\sigma\sqrt{A}} + \frac{1}{2} = \frac{1}{2}\sqrt{1 + \frac{2C}{\sigma^2}}$

Set $\nu = \frac{1}{2}\sqrt{1 + \frac{2C}{\sigma^2}}$. Since $\Psi(2\nu, 1+2\nu, x) = x^{-2\nu}$, then for this choice of parameters the problem reduces to inverting a Laplace transform. We consider only the case $A > 0$. We obtain after a straightforward calculation

$$\begin{aligned}
&\int_0^\infty \exp\left\{\frac{-\sqrt{A}y(1+\epsilon)}{2\sigma(1-\epsilon)} - \frac{F(y)}{2\sigma}\right\} y^{\frac{1}{2}-\nu}(1-\epsilon)^{-1+2\nu} p_\mu(t,x,y)\,dy \\
&\quad = e^{\sqrt{A}t}\left(e^{\sqrt{A}t} - \epsilon\right)^{-1+2\nu} x^{\frac{1}{2}-\nu} \exp\left\{\frac{-\sqrt{A}(e^{\sqrt{A}t} + \epsilon)}{2\sigma(e^{\sqrt{A}t} - \epsilon)} - \frac{F(x)}{2\sigma}\right\}.
\end{aligned} \tag{5.7}$$

We set $\lambda = \frac{\sqrt{A}(1+\epsilon)}{2\sigma(1-\epsilon)}$. After some further calculations this reduces to

$$\begin{aligned}
&\int_0^\infty e^{-\lambda y} y^{\frac{1}{2}-\nu} e^{-\frac{F(y)}{2\sigma}} p_\mu(t,x,y)\,dy \\
&\quad = e^{-\frac{F(x)}{2\sigma}} \left(\frac{\sqrt{A}}{\sigma(e^{\sqrt{A}t} - 1)}\right)^{1-2\nu} e^{\sqrt{A}t} \\
&\qquad \times \exp\left\{\frac{-\sqrt{A}x}{2\sigma\tanh\left(\frac{\sqrt{A}t}{2}\right)} + \frac{m}{\lambda + \frac{\sqrt{A}}{2\sigma}\coth\left(\frac{\sqrt{A}t}{2}\right)}\right\}\left(\lambda + \frac{\sqrt{A}}{2\sigma}\coth\left(\frac{\sqrt{A}t}{2}\right)\right)^{-1+2\nu},
\end{aligned}$$

where $m = \frac{Ax}{2\sigma}\operatorname{cosech}^2\left(\frac{\sqrt{A}t}{2}\right)$. Inversion of this Laplace transform is straightforward. There are two cases.

If $\nu \neq \frac{1}{2}$ then

$$p_\mu(t,x,y) = 4^\nu \sqrt{\frac{x}{y}} \frac{\sqrt{A} e^{(\frac{1}{2}+\nu)\sqrt{A}t}}{4\sigma \sinh(\frac{\sqrt{A}t}{2})} e^{\frac{F(y)-F(x)}{2\sigma}} \exp\left\{-\frac{\sqrt{A}(x+y)}{2\sigma \tanh(\frac{\sqrt{A}t}{2})}\right\} I_{-2\nu}\left(\frac{2\sqrt{Axy}}{\sigma \sinh(\frac{\sqrt{A}t}{2})}\right).$$

If $\nu = \frac{1}{2}$ then the fundamental solution is given by

$$p_\mu(t,x,y) = e^{\frac{F(y)-F(x)}{2\sigma}} \exp\left\{-\frac{\sqrt{A}(x+y)}{2\sigma \tanh(\frac{\sqrt{A}t}{2})}\right\} \left(\frac{\sqrt{A}}{2\sigma \sinh(\frac{\sqrt{A}t}{2})} \sqrt{\frac{x}{y}} I_1\left(\frac{2\sqrt{Axy}}{\sigma \sinh(\frac{\sqrt{A}t}{2})}\right) + \delta(y)\right). \tag{5.8}$$

5.1.2. The case $B = 0$

If $B = 0$ then we make use of the fact that

$$(\lambda x)^{\frac{1}{2}+\nu} e^{-\lambda x} \Psi(1/2+\nu, 1+2\nu, \lambda x) = 2^{-\nu} \sqrt{\lambda x/\pi} K_\nu(\lambda x), \tag{5.9}$$

where K_ν is a modified Bessel function. This implies that

$$U_{\sigma\lambda/\sqrt{A}-1}(x,t) = \left(\frac{\sqrt{A}}{\sigma}\right)^\eta \frac{2^{-\nu}}{\sqrt{\pi}} \int_0^\infty \sqrt{\lambda y} K_\nu(\lambda y) e^{\frac{\sqrt{A}y}{2\sigma} - \frac{F(y)}{2\sigma}} p_\mu(t,x,y)\, dy.$$

This so-called K transform can be inverted. We have

$$\left(\frac{\sqrt{A}}{\sigma}\right)^\eta \frac{2^{-\nu}}{\sqrt{\pi}} e^{\frac{\sqrt{A}y - F(y)}{2\sigma}} p_\mu(t,x,y) = \int_{\rho-i\infty}^{\rho+i\infty} (\lambda y)^{\frac{1}{2}} I_\nu(\lambda y) U_{\sigma\lambda/\sqrt{A}-1}(x,t)\, d\lambda,$$

for a suitable constant ρ. Here I_ν is the modified Bessel function of the first kind. Notice that for $A < 0$ we obtain the Hankel transform of the fundamental solution.

5.1.3. The case $a = 1$, $b = 0$

If we had taken $a = 1, b = 0$ in (5.3), we would arrive at the so-called ${}_1F_1$ transform, described in [3, p. 115]. We leave the details of this calculation to the interested reader.

6. The final class of Ricatti equations

Now suppose that the drift function f satisfies the Ricatti equation $\sigma x f' - \sigma f + \frac{1}{2} f^2 + 2\mu\sigma x^{r+1} = \frac{A}{2} x^2 + \frac{2}{3} B x^{\frac{3}{2}} + Cx - \frac{3}{8}\sigma^2$. When $A = 0$ we obtain the Laplace transform of the fundamental solution. The result here generalizes Theorem 6.1 of Craddock and Platen [4]. The proof is identical to that of Theorem 4.1 above and we omit it.

Theorem 6.1. *Let f be a solution of the Ricatti equation,*

$$\sigma x f' - f + \frac{1}{2} f^2 + 2\mu\sigma x^{r+1} = A x^{\frac{3}{2}} + Cx - \frac{3}{8}\sigma^2. \tag{6.1}$$

Let $u_0(x)$ be an analytic solution of (3.3) *which is independent of t. Let*

$$F'(x)=f(x)/x, \qquad H(\lambda,x,t)=\frac{(12(1+\lambda\sigma t)\sqrt{x}-A\lambda(\sigma t)^3)^2}{144(1+\lambda\sigma t)^4} \quad \textit{and}$$

$$G\left(\lambda,x,\frac{t}{\sigma}\right)=-\frac{\lambda(x+\frac{1}{2}Ct^2)}{1+\lambda t}-\frac{\frac{2}{3}At^2\sqrt{x}(3+\lambda t)}{(1+\lambda t)^2}+\frac{A^2t^4(2\lambda t(3+\frac{1}{2}\lambda t)-3)}{108(1+\lambda t)^3}.$$

Then for $\lambda \geqslant 0$, $U_\lambda(x,t)=\int_0^\infty u_0(y)p_\mu(t,x,y)e^{-\lambda y}\,dy$, where $p_\mu(t,x,y)$ is a fundamental solution of (3.3) *and*

$$U_\lambda(x,t)=\sqrt{\frac{\sqrt{x}(1+\lambda\sigma t)}{\sqrt{x}(1+\lambda\sigma t)-\frac{A\lambda}{12}(\sigma t)^3}}\exp\bigl\{G(\lambda,x,t)\bigr\}u_0\bigl(H(\lambda,x,t)\bigr)$$
$$\times\exp\left\{-\frac{1}{2\sigma}\bigl(F(x)-F\bigl(H(\lambda,x,t)\bigr)\bigr)\right\}. \tag{6.2}$$

The case when $A\neq 0$ in (3.6) turns out to be similar to that of the previous section, though because of the nature of the symmetries, it is perhaps not quite as elegant.

Let $D=\frac{2B}{3A}$, $E=\frac{1}{\sqrt{A}}(\frac{C}{2\sigma}+\frac{B^2}{9\sigma A})$ and $F'(x)=f(x)/x$. For simplicity we will begin with the $\mu=0$ case and again assume that $A>0$. The cases $A<0$ and $\mu\neq 0$ can be handled similarly.

To solve the Ricatti equation (3.6) we set

$$y=2\sqrt{x} \quad \text{and} \quad \frac{h'(y)}{h(y)}=\frac{1}{\sigma y}\left(f\left(\frac{y^2}{4}\right)-\frac{\sigma}{2}\right),$$

then we obtain a second order ODE for h which is solvable in terms of confluent hypergeometric functions. Omitting the rather straightforward details this leads to

$$\exp\left\{\frac{1}{2\sigma}F(x)\right\}=x^{\frac{1}{4}}e^{-\frac{\sqrt{A}(x+2D\sqrt{x})}{2\sigma}}$$
$$\times\left(c_1\Psi\left(-\frac{\alpha}{2},\frac{1}{2},\frac{\sqrt{A}(D+\sqrt{x})^2}{\sigma}\right)+c_{2\,1}F_1\left(\beta,\frac{1}{2},\frac{\sqrt{A}(D+\sqrt{x})^2}{\sigma}\right)\right), \tag{6.3}$$

where $\alpha=-\frac{1}{2}+\frac{2B^2-9CA}{9\sigma A^{3/2}}$ and $\beta=\frac{-4B^2+9(2AC+\sigma A^{3/2})}{36\sigma A^{3/2}}$ and c_1 and c_2 are arbitrary constants. We will take the solution corresponding to $c_2=0$, $c_1=1$.

We exponentiate the infinitesimal symmetry (3.9). Let $u(x,t)$ be a solution of (3.3), then the following is also a solution:

$$\rho\big(\exp(\epsilon \mathbf{v}_1)\big)u(x,t) = \big(1+\epsilon e^{-\sqrt{A}t}\big)^{-E}\sqrt{\frac{\sqrt{x}e^{\sqrt{A}t/2}}{e^{\frac{\sqrt{A}t}{2}}(D+\sqrt{x}) - D\sqrt{e^{\sqrt{A}t}+\epsilon}}}$$
$$\times \exp\left\{\frac{\sqrt{A}\epsilon(\sqrt{x}+D)^2}{2\sigma(e^{\sqrt{A}t}+\epsilon)} + \frac{1}{2\sigma}F\left(\left(\frac{e^{\frac{\sqrt{A}t}{2}}(D+\sqrt{x})}{\sqrt{e^{\sqrt{A}t}+\epsilon}} - D\right)^2\right)\right\}$$
$$\times \exp\left\{-\frac{1}{2\sigma}F(x)\right\}u\left(\left(\frac{e^{\frac{\sqrt{A}t}{2}}(D+\sqrt{x})}{\sqrt{e^{\sqrt{A}t}+\epsilon}} - D\right)^2, \ln\big(e^{\sqrt{A}t}+\epsilon\big)\right). \tag{6.4}$$

Since we have assumed that $\mu = 0$ we may take a stationary solution $u_0(x) = 1$. After cancellations, this leads to the following solution of (3.3)

$$U_\epsilon(x,t) = \big(1+\epsilon e^{-\sqrt{A}t}\big)^{-E} e^{\frac{\sqrt{A}t}{4}} \exp\left\{-\frac{\sqrt{A}(\sqrt{x}+D)^2}{2\sigma}\left(\frac{e^{\sqrt{A}t}-\epsilon}{e^{\sqrt{A}t}+\epsilon}\right)\right\}$$
$$\times \Psi\left(-\frac{\alpha}{2}, \frac{1}{2}, \frac{\sqrt{A}(\sqrt{x}+D)^2}{\sigma(e^{\sqrt{A}t}+\epsilon)}\right)\Big/ h_\alpha(x), \tag{6.5}$$

with $h_\alpha(x) = \exp\{-\frac{\sqrt{A}(\sqrt{x}+D)^2}{2\sigma}\}\Psi(-\frac{\alpha}{2}, \frac{1}{2}, \frac{\sqrt{A}}{\sigma}(\sqrt{x}+D)^2)$.

From this point the analysis is similar to that of the previous section for the case of the Ricatti equation (3.5). We wish to interpret Eq. (2.6) as a known integral transform. We set $\lambda = \frac{\sqrt{A}}{\sigma(1+\epsilon)}$. Thus (2.6) becomes

$$\int_0^\infty e^{-\lambda(\sqrt{y}+D)^2}\Psi\left(-\frac{\alpha}{2}, \frac{1}{2}, \lambda(\sqrt{y}+D)^2\right) g_\alpha(y) p_\mu(t,x,y)\,dy = \overline{U}_\lambda(x,t),$$

where $g_\alpha(y) = (\frac{\sqrt{A}}{\sigma})^E \exp\{\frac{\sqrt{A}}{2\sigma}(\sqrt{y}+D)^2\}/h_\alpha(y)$ and $\overline{U}_\lambda(x,t) = \lambda^{-E}U_{\sqrt{A}/(\sigma\lambda)-1}(x,t)$. Upon setting $z = (\sqrt{y}+D)^2$ we have

$$\int_0^\infty e^{-\lambda z}\Psi\left(-\frac{\alpha}{2}, \frac{1}{2}, \lambda z\right) H\big(\sqrt{z}-|D|\big)\tilde{g}_\alpha(z)\tilde{p}_\mu(t,x,z)\,d\omega(z) = \overline{U}_\lambda(x,t), \tag{6.6}$$

where H is the Heaviside step function and $\tilde{g}_\alpha$ and $\tilde{p}_\mu$ are the obvious transformed functions and $d\omega(z) = (\frac{\sqrt{z}-|D|}{\sqrt{z}})\,dz$. If we set $\nu = -1/4$, $k = \alpha/2 - 1/4$ then (6.6) becomes

$$\int_0^\infty e^{-\frac{\lambda z}{2}}(\lambda z)^{-k-\frac{1}{2}} W_{k+\frac{1}{2},\nu}(\lambda z) G_\alpha(z)\tilde{p}_\mu(t,x,z)\,dz = \lambda^{-k-\frac{1}{4}}\overline{U}_\lambda(x,t),$$

where $G_\alpha(z) = H(\sqrt{z} - D)\tilde{g}_\alpha(z)(\sqrt{z} - D)z^{k-\frac{1}{4}}$. We can thus apply the inversion formula for the Whittaker transform to recover the fundamental solution. Again, the proof of the next result follows the same pattern as for Theorem 4.1.

Theorem 6.2. *Let f be a solution of the Ricatti equation* (3.6). *Let $U_\epsilon(x,t)$ be as given by* (6.5). *Further let $G_\alpha(z)$ be as defined above and $k = -\alpha/2 - 1/4$, $\nu = -1/4$. Suppose that $\overline{U}_{\sqrt{A}/(\lambda\sigma)-1}(x,t) = \lambda^{-k-\frac{1}{4}}(\sigma\lambda/\sqrt{A})^{-E}U_{\sqrt{A}/(\sigma\lambda)-1}(x,t)$ is a Whittaker transform of a function $G_\alpha(z)p(t,x,z)$. Then*

$$G_\alpha(z)p(t,x,z) = \int_{\rho-i\infty}^{\rho+i\infty} (\lambda y)^{-k-1/2} e^{\lambda y/2} M_{k-1/2,\nu}(\lambda y)\overline{U}_{\sqrt{A}/(\lambda\sigma)-1}(x,t)\,d\lambda, \tag{6.7}$$

and $\tilde{p}(t,x,(\sqrt{y}+D)^2)$ is a fundamental solution of (1.1) *for the given drift.*

Note. This result can easily be extended to the $\mu \neq 0$ case. Just replace $U_\epsilon(x,t)$ in the integral (6.7) by $U_\epsilon(x,t)$ multiplied by the appropriate stationary solution u_0 of the PDE, as per the symmetry (6.4).

Example 6.1. Suppose that $\mu = 0$ and $(2B^2 - 9CA)/9\sigma A^{\frac{3}{2}} = \frac{3}{2}$. Further assume that $E > 0$. It is straightforward to invert the transform for these values and we obtain the fundamental solution

$$\begin{aligned} p(t,x,y) = {} & \frac{2^{\frac{1}{2}-E}}{\sqrt{y}}(\sqrt{y}+D)\left(\frac{\sqrt{x}+D}{\sqrt{y}+D}\right)^{\frac{1}{2}-E} \exp\left\{\frac{\sqrt{A}Et}{2} + G(x,y,t)\right\} \\ & \times I_{E-\frac{1}{2}}\left(\frac{\sqrt{A}(\sqrt{x}+D)(\sqrt{y}+D)}{\sigma\sinh(\frac{\sqrt{A}t}{2})}\right)\left(1 - e^{-\sqrt{A}t}\right)^{-1} H\left(\sqrt{y}+D-|D|\right), \end{aligned}$$

where $G(x,y,t) = \frac{-\sqrt{A}([(\sqrt{y}+D)^2 - (\sqrt{x}+D)^2] + ((\sqrt{x}+D)^2 + (\sqrt{y}+D)^2)\coth(\frac{\sqrt{A}t}{2}))}{2\sigma}$.

7. The case of arbitrary γ

We have shown that for every class of Ricatti equation in Proposition 3.1, there is an $\mathfrak{sl}_2$ symmetry whose action comes from a well-known integral transform of the fundamental solution. Similar results can be established for PDEs of the form (1.1) with $\gamma \neq 1$. There are various special cases which need to be considered in order to present a complete picture, but this can be done by following the method described here. Rather than embarking on an exhaustive treatment, we will content ourselves with an illustrative result for the general case and some examples.

Theorem 7.1. *Suppose that $g(x) = x^{1-\gamma}f(x)$, $\gamma \neq 2$, $r + 2 - \gamma \neq 0$ and that g satisfies the Ricatti equation $\sigma xg' - \sigma g + \frac{1}{2}g^2 + 2\sigma\mu x^{r+2-\gamma} = 2\sigma Ax^{2-\gamma} + B$, where A and B are constants. Then Eq.* (1.1) *has a symmetry of the form*

$$U_\epsilon(x,t)=\frac{1}{(1+4\epsilon t)^{(1-\gamma)/(2-\gamma)}}\exp\left\{\frac{-4\epsilon(x^{2-\gamma}+\sigma(2-\gamma)^2At^2)}{\sigma(2-\gamma)^2(1+4\epsilon t)}\right\}$$
$$\times\exp\left\{\frac{1}{2\sigma}\left(F\left(\frac{x}{(1+4\epsilon t)^{2/(2-\gamma)}}\right)-F(x)\right)\right\}u\left(\frac{x}{(1+4\epsilon t)^{2/(2-\gamma)}},\frac{t}{1+4\epsilon t}\right),$$

where $F'(x)=f(x)/x^\gamma$. Suppose further that the natural domain of (1.1) *is $x\geqslant 0$. Let $\overline{U}_\epsilon$ be the value of U_ϵ when $u=u_0$ is a stationary solution of the PDE* (1.1)*. Then*

$$\int_0^\infty e^{-\lambda y^{2-\gamma}}u_0(y)p_\mu(t,x,y)\,dy=\overline{U}_{\frac{1}{4}\lambda\sigma(2-\gamma)}(x,t),\tag{7.1}$$

where $p_\mu(t,x,y)$ is a fundamental solution. If the equation is defined on $\mathbb{R}$ then

$$\int_{-\infty}^\infty e^{-\lambda y^{2-\gamma}}u_0(y)p_\mu(t,x,y)\,dy=\overline{U}_{\frac{1}{4}\lambda\sigma(2-\gamma)}(x,t).\tag{7.2}$$

Proof. A straightforward application of Lie's prolongation algorithm shows that when g satisfies the given Ricatti equation, there is an infinitesimal symmetry of the form

$$\mathbf{v}=\frac{8xt}{2-\gamma}\partial_x+4t^2\partial_t-\left(\frac{4x^{2-\gamma}}{\sigma(2-\gamma)^2}+\frac{4x^{1-\gamma}tf(x)}{\sigma(2-\gamma)}+\frac{4(1-\gamma)}{2-\gamma}t+4At^2\right)u\partial_u.$$

Exponentiating the symmetry gives U_ϵ. The remainder of the proof is similar to that of Theorem 4.1. □

The integral transform in (7.1) can obviously be reduced to a Laplace transform. We present an example.

Example 7.1. Let $\gamma=0$. The Ricatti equation in Theorem 7.1 is easily solved by the change of variables $g=2x\sigma y'/y$. One solution gives the drift $f(x)=\frac{-\sigma}{2x}+2\sqrt{x}\sqrt{\mu}\sqrt{\sigma}\cot(\frac{2x^{\frac{3}{2}}\sqrt{\mu}}{3\sqrt{\sigma}})$. A stationary solution of the corresponding PDE is $u_0(x)=x^{\frac{3}{2}}\csc(\frac{2x^{\frac{3}{2}}\sqrt{\mu}}{3\sqrt{\sigma}})$. Applying Theorem 7.1 gives

$$U_\epsilon(x,t)=\frac{x^{3/2}}{(1+4\epsilon t)^{7/4}}\exp\left\{\frac{-\epsilon x^2}{\sigma(1+4\epsilon t)}\right\}\csc\left(\frac{2x^{\frac{3}{2}}\sqrt{\mu}}{3\sqrt{\sigma}}\right).$$

The substitution $z=y^2$ in (7.1) turns it into a Laplace transform. Taking the inverse leads to the fundamental solution

$$p(t,x,y)=\frac{1}{2\sigma t}x^{\frac{3}{4}}y^{\frac{1}{4}}\exp\left\{\frac{-(x^2+y^2)}{4\sigma t}\right\}I_{\frac{3}{4}}\left(\frac{xy}{2\sigma t}\right)\frac{\sin(\frac{2y^{3/2}\sqrt{\mu}}{3\sqrt{\sigma}})}{\sin(\frac{2x^{3/2}\sqrt{\mu}}{3\sqrt{\sigma}})}.$$

Extracting the fundamental solution from (7.2) is not quite so straightforward. Fortunately there is an effective way of doing this. The basic approach is to integrate the symmetries with respect to the group parameter.

The heat equation has two stationary solutions $u_0(x) = 1$ and $u_1(x) = x$. From these we obtain the symmetry solutions

$$U_\epsilon^0(x,t) = \frac{1}{\sqrt{1+4\epsilon t}} e^{\frac{-\epsilon x^2}{1+4\epsilon t}}, \qquad U_\epsilon^1(x,t) = \frac{x}{(1+4\epsilon t)^{\frac{3}{2}}} e^{\frac{-\epsilon x^2}{1+4\epsilon t}}. \tag{7.3}$$

Now let $\varphi(\epsilon)$ and $\psi(\epsilon)$ have sufficient decay to guarantee the convergence of the integrals in

$$u(x,t) = \int_0^\infty \varphi(\epsilon) U_\epsilon^0(x,t)\, d\epsilon + \int_0^\infty \psi(\epsilon) U_\epsilon^1(x,t)\, d\epsilon. \tag{7.4}$$

The function $u(x,t)$ is a solution of the heat equation with $u(x,0) = \Phi(x^2) + x\Psi(x^2)$, where Φ and Ψ are the Laplace transforms of φ and ψ. The solutions U_ϵ^0 and U_ϵ^1 may be represented as Laplace transforms

$$U_\epsilon^0(x,t) = \frac{1}{\sqrt{4t}} e^{-\frac{x^2}{4t}} \int_0^\infty e^{-\epsilon z} e^{-\frac{z}{4t}} \frac{1}{\sqrt{\pi z}} \cosh\left(\frac{\sqrt{z}x}{2t}\right) dz, \tag{7.5}$$

$$U_\epsilon^1(x,t) = \frac{1}{\sqrt{4t}} e^{-\frac{x^2}{4t}} \int_0^\infty e^{-\epsilon z} e^{-\frac{z}{4t}} \frac{1}{\sqrt{\pi}} \sinh\left(\frac{\sqrt{z}x}{2t}\right) dz. \tag{7.6}$$

Using these in (7.4) and reversing the order of integration gives

$$u(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2+z}{4t}} \left[\Phi(z) \frac{\cosh\left(\frac{\sqrt{z}x}{2t}\right)}{\sqrt{z}} + \Psi(z) \sinh\left(\frac{\sqrt{z}x}{2t}\right)\right] dz.$$

Letting $z = y^2$ this becomes after simplification,

$$\begin{aligned} u(x,t) &= \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2+y^2}{4t}} \left[e^{\frac{xy}{2t}} \left(\Phi\left(y^2\right) + y\Psi\left(y^2\right)\right) + e^{-\frac{xy}{2t}} \left(\Phi\left(y^2\right) - y\Psi\left(y^2\right)\right)\right] dy \\ &= \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \left(\Phi\left(y^2\right) + y\Psi\left(y^2\right)\right) dy + \int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x+y)^2}{4t}} \left(\Phi\left(y^2\right) - y\Psi\left(y^2\right)\right) dy. \end{aligned} \tag{7.7}$$

Letting $y \to -y$ in the second integral gives

$$u(x,t) = \int_{-\infty}^\infty \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} \left(\Phi\left(y^2\right) + y\Psi\left(y^2\right)\right) dy. \tag{7.8}$$

Since $u(x,0)=\Phi(x^2)+x\Psi(x^2)$ this implies that $p(t,x-y)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}$ which is correct. Let us now use this procedure to determine the fundamental solution for another PDE.

Example 7.2. Consider the equation $u_t=u_{xx}+2\tanh(x)u_x$. This is defined for all $x\in\mathbb{R}$. So we are in the situation of (7.2) in Theorem 7.1. Stationary solutions are $u_0(x)=1$ and $u_1(x)=\tanh x$. From the symmetry in Theorem 7.1 we obtain the two symmetry solutions

$$U_\epsilon^0(x,t)=\frac{1}{\sqrt{1+4\epsilon t}}\exp\left\{\frac{-\epsilon(x^2+4t^2)}{1+4\epsilon t}\right\}\frac{\cosh(\frac{x}{1+4\epsilon t})}{\cosh x}, \tag{7.9}$$

$$U_\epsilon^1(x,t)=\frac{1}{\sqrt{1+4\epsilon t}}\exp\left\{\frac{-\epsilon(x^2+4t^2)}{1+4\epsilon t}\right\}\frac{\sinh(\frac{x}{1+4\epsilon t})}{\cosh x}. \tag{7.10}$$

The solution $u(x,t)=\int_0^\infty\varphi(\epsilon)U_\epsilon^0(x,t)\,d\epsilon+\int_0^\infty\psi(\epsilon)U_\epsilon^1(x,t)\,d\epsilon$ satisfies the initial condition $u(x,0)=\Phi(x^2)+\tanh x\Psi(x^2)$, where Φ and Ψ are the Laplace transforms of φ and ψ. Now

$$U_\epsilon^0(x,t)=\int_0^\infty\frac{e^{-t-\epsilon z-\frac{x^2+z}{4t}}}{\cosh x}\frac{1}{2\sqrt{4\pi zt}}\left[\cosh\left(\frac{\sqrt{z}(x+2t)}{2t}\right)+\cosh\left(\frac{\sqrt{z}(x-2t)}{2t}\right)\right]dz\quad\text{and}$$

$$U_\epsilon^1(x,t)=\int_0^\infty\frac{e^{-t-\epsilon z-\frac{x^2+z}{4t}}}{\cosh x}\frac{1}{2\sqrt{4\pi zt}}\left[\cosh\left(\frac{\sqrt{z}(x+2t)}{2t}\right)-\cosh\left(\frac{\sqrt{z}(x-2t)}{2t}\right)\right]dz.$$

We use these representations in the integrals defining u and reverse the order of integration. Then we let $z=y^2$ and expand the hyperbolic cosines to get

$$\begin{aligned}u(x,t)&=\int_0^\infty\frac{1}{\sqrt{\pi t}}e^{-\frac{x^2+y^2}{4t}-t}\Phi\left(y^2\right)\cosh y\cosh\left(\frac{xy}{2t}\right)\operatorname{sech}x\,dy\\&\quad+\int_0^\infty\frac{1}{\sqrt{\pi t}}e^{-\frac{x^2+y^2}{4t}-t}\Psi\left(y^2\right)\sinh y\sinh\left(\frac{xy}{2t}\right)\operatorname{sech}x\,dy\\&=\int_0^\infty e^{-t}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\frac{\cosh y}{\cosh x}\left(\Phi\left(y^2\right)+\tanh y\Psi\left(y^2\right)\right)dy\\&\quad+\int_0^\infty e^{-t}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x+y)^2}{4t}}\frac{\cosh y}{\cosh x}\left(\Phi\left(y^2\right)-\tanh y\Psi\left(y^2\right)\right)dy.\end{aligned}$$

The replacement $y\to -y$ in the second integral leads to

$$u(x,t)=\int_{-\infty}^\infty e^{-t}\frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}\frac{\cosh y}{\cosh x}\left(\Phi\left(y^2\right)+\tanh y\Psi\left(y^2\right)\right)dy.$$

This implies that the desired fundamental solution is

$$p(t,x,y) = e^{-t}\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{(x-y)^2}{4t}\right\}\frac{\cosh y}{\cosh x}. \tag{7.11}$$

That this is the fundamental solution can be readily checked. It is also straightforward to verify that

$$\int_{-\infty}^{\infty} e^{-\epsilon y^2} e^{-t}\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{(x-y)^2}{4t}\right\}\frac{\cosh y}{\cosh x}\,dy = U_\epsilon^0(x,t), \tag{7.12}$$

as required by Theorem 7.1.

Using this procedure, the reader may check that

$$p(t,x,y) = e^{-t}\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{(x-y)^2}{4t}\right\}\frac{\sinh y}{\sinh x} \tag{7.13}$$

is a fundamental solution for $u_t = u_{xx} + 2\coth(x)u_x$ valid for $x \neq 0$. Other examples can be computed the same way.

One may prove results similar to Theorem 7.1 which cover every possible drift for which there is a nontrivial Lie algebra of symmetries. For example, if

$$\sigma x g' - \sigma g + \frac{1}{2}g^2 + 2\sigma\mu x^{r+2-\gamma} = \frac{Ax^{4-2\gamma}}{4-2\gamma} + \frac{Bx^{2-\gamma}}{2-\gamma} + C, \quad \gamma \neq 2,$$

then we have a Whittaker-type transform giving the fundamental solution. There are also theorems analogous to Theorems 6.1 and 6.2. We leave these and the $\gamma = 2$ case to the reader. We will conclude by relating the $\gamma = 0$ case to an important equation in nilpotent harmonic analysis.

8. The heat equation on the Heisenberg group

We choose a basis for the Heisenberg Lie algebra, by setting $X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$ and $Z = \frac{\partial}{\partial z}$. The sub-Laplacian is then $X^2 + Y^2$ and so the heat equation on the Heisenberg group may be expressed as

$$U_t = U_{xx} + U_{yy} + 2yU_{xz} - 2xU_{yz} + \left(x^2 + y^2\right)U_{zz}. \tag{8.1}$$

This possesses few symmetries, but it is invariant under rotations, so the heat kernel itself should also be invariant under rotations. We therefore make the change of variables $r = \sqrt{x^2 + y^2}$. This reduces the heat equation to

$$U_t = U_{rr} + \frac{1}{r}U_r + r^2 U_{zz}, \quad r \geqslant 0. \tag{8.2}$$

Taking the Fourier transform in z gives

$$u_t = u_{rr} + \frac{1}{r}u_r - \lambda^2 r^2 u, \quad \text{where} \tag{8.3}$$

$$u(r,t;\lambda) = \int_{-\infty}^{\infty} U(r,z,t)e^{-i\lambda z}\,d\lambda. \tag{8.4}$$

This PDE does have useful symmetries. However instead of considering (8.3), we will study

$$u_t = u_{rr} + \frac{1}{r}u_r - \big(\lambda^2 r^2 - 2|\lambda|\big)u. \tag{8.5}$$

If u is a solution of (8.5), then $w = e^{-2|\lambda|t}u(r,t)$ is a solution of (8.3). We consider (8.5) because it has a stationary solution $u_0(r) = e^{-\frac{|\lambda|}{2}r^2}$. Equation (8.3) has Bessel functions as stationary solutions. These lead to Hankel type transforms, which are harder to invert than Laplace transforms.

Proposition 8.1. *A basis for the Lie algebra of symmetries of* (8.5) *is given by*

$$\mathbf{v}_1 = 2|\lambda|re^{4|\lambda|t}\partial_r + e^{4|\lambda|t}\partial_t - 2\lambda^2 r^2 e^{4|\lambda|t}u\partial_u,$$
$$\mathbf{v}_2 = -2|\lambda|re^{-4|\lambda|t}\partial_r + e^{-4|\lambda|t}\partial_t - 2\big(\lambda^2 r^2 - 2|\lambda|\big)e^{-4|\lambda|t}u\partial_u, \qquad \mathbf{v}_3 = \partial_t, \qquad \mathbf{v}_4 = u\partial_u,$$

and there is an infinite-dimensional ideal spanned by vector fields of the form $\mathbf{v}_\beta = \beta(r,t)\partial_u$*, where* β *is an arbitrary solution of* (8.5).

We are only interested here in $\mathbf{v}_2$. If $u_0(r) = e^{-\frac{|\lambda|r^2}{2}}$ then an elementary calculation shows that

$$\rho\big(\exp(\epsilon\mathbf{v}_2)\big)u_0(r) = \frac{e^{4|\lambda|t}}{e^{4|\lambda|t} - 4|\lambda|\epsilon}\exp\left\{\frac{-|\lambda|r^2}{2}\left(\frac{e^{4|\lambda|t} + 4|\lambda|\epsilon}{e^{4|\lambda|t} - 4|\lambda|\epsilon}\right)\right\}.$$

From this solution we can recover the heat kernel for both (8.5) and (8.3), as well as the heat kernel for the Heisenberg group. The main result is the following.

Theorem 8.2. *The heat kernel for the PDE* (8.5) *is*

$$p(t,r,\xi) = \frac{|\lambda|\xi e^{2|\lambda|t}}{\sinh(2|\lambda|t)}\exp\left\{\frac{-|\lambda|(r^2+\xi^2)}{2\tanh(2|\lambda|t)}\right\}I_0\left(\frac{|\lambda|r\xi}{\sinh(2|\lambda|t)}\right),$$

where I_ν *is the modified Bessel function of the first kind.*

Proof. If $p(t,r,\xi)$ is the heat kernel, then given a solution $u(r,t)$ we must have $u(r,t) = \int_0^\infty u(\xi,0)p(t,r,\xi)\,d\xi$. Applying this to the solution $\rho(\exp\epsilon\mathbf{v}_2)u_0(r)$, we obtain

$$\int_0^\infty \frac{p(t,r,\xi)}{1-4|\lambda|\epsilon}\exp\left\{\frac{-|\lambda|\xi^2}{2}\left(\frac{1+4|\lambda|\epsilon}{1-4|\lambda|\epsilon}\right)\right\}d\xi = \frac{e^{4|\lambda|t}}{e^{4|\lambda|t}-4|\lambda|\epsilon}\exp\left\{\frac{-|\lambda|r^2}{2}\left(\frac{e^{4|\lambda|t}+4|\lambda|\epsilon}{e^{4|\lambda|t}-4|\lambda|\epsilon}\right)\right\}.$$

We set $s = \frac{|\lambda|}{2}(\frac{1+4|\lambda|\epsilon}{1-4|\lambda|\epsilon})$. After some simplification this gives

$$\int_0^\infty e^{-s\xi^2} p(t,r,\xi)\,d\xi = \frac{|\lambda| e^{2|\lambda|t}}{2\sinh(2|\lambda|t)} \exp\left\{\frac{-|\lambda|r^2}{2\tanh(2|\lambda|t)}\right\} \times \frac{1}{s+\frac{|\lambda|}{2}\coth(2|\lambda|t)} \exp\left\{\frac{k}{s+\frac{|\lambda|}{2}\coth(2|\lambda|t)}\right\}, \tag{8.6}$$

where $k = \lambda^2 r^2 \operatorname{cosech}^2(2|\lambda|t)/4$. Setting $\eta = \xi^2$ converts this into a Laplace transform. The right-hand side of (8.6) can be inverted. With η replaced by ξ^2 this gives

$$p(t,r,\xi) = \frac{|\lambda|\xi e^{2|\lambda|t}}{\sinh(2|\lambda|t)} \exp\left\{\frac{-|\lambda|(r^2+\xi^2)}{2\tanh(2|\lambda|t)}\right\} I_0\left(\frac{|\lambda|r\xi}{\sinh(2|\lambda|t)}\right), \tag{8.7}$$

which is the desired result. □

For suitable φ the following integral defines solutions of (8.5),

$$u(r,t) = \int_0^\infty \varphi(\xi) \frac{|\lambda|\xi e^{2|\lambda|t}}{\sinh(2|\lambda|t)} \exp\left\{\frac{-|\lambda|(r^2+\xi^2)}{2\tanh(2|\lambda|t)}\right\} I_0\left(\frac{|\lambda|r\xi}{\sinh(2|\lambda|t)}\right) d\xi.$$

Since $u(r,t) = \int_0^\infty \varphi(\xi) e^{-2|\lambda|t} p(t,r,\xi)\,d\xi$ is a solution of (8.3) for suitable initial data φ, we can obtain solutions of the heat equation on H_3 by setting

$$U(r,z,t) = \int_{-\infty}^\infty \frac{|\lambda|}{2\pi\sinh(2|\lambda|t)} \exp\left\{\frac{-|\lambda|r^2}{2\tanh(2|\lambda|t)}\right\} e^{i\lambda z} K(r,\lambda,t)\,d\lambda, \tag{8.8}$$

where the kernel K is given by

$$K(r,\lambda,t) = \int_0^\infty \xi \exp\left\{\frac{-|\lambda|\xi^2}{2\tanh(2|\lambda|t)}\right\} I_0\left(\frac{|\lambda|r\xi}{\sinh(2|\lambda|t)}\right) \varphi(\xi)\,d\xi. \tag{8.9}$$

An obvious choice is to take $K = 1$. This is achieved by setting $\xi\varphi(\xi) = \delta(\xi)$, which gives the following solution of (8.1)

$$h(r,z,t) = \int_{-\infty}^\infty \frac{|\lambda|}{2\pi\sinh(2|\lambda|t)} \exp\left\{\frac{-|\lambda|r^2}{2\tanh(2|\lambda|t)}\right\} e^{i\lambda z}\,d\lambda. \tag{8.10}$$

This is, up to scalar multiple, the heat kernel for H_3 obtained by Gaveau and others. We have thus obtained a family of solutions of the heat equation which contains the heat kernel.

The natural question to ask is whether the methods used here can be applied to other heat equations on nilpotent Lie groups? There are reasons to believe that this is in fact the case. This question will be addressed in future work.

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