SYMMETRY GROUP METHODS FOR FUNDAMENTAL SOLUTIONS AND CHARACTERISTIC FUNCTIONS

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Abstract. This paper uses Lie symmetry group methods to analyse a class of partial differential equations of the form

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + f(x)\frac{\partial u}{\partial x}.$$

It is shown that when the drift function f is a solution of a family of Ricatti equations, then symmetry techniques can be used to find the characteristic functions and transition densities of the corresponding diffusion processes.

1. Introduction

The purpose of this paper is to show how symmetry group methods may be used to compute characteristic functions and fundamental solutions for partial differential equations (PDEs), of the form

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + f(x)\frac{\partial u}{\partial x}, \tag{1.1}$$

when the drift function f is a solution of one of the following three families of Ricatti equations.

$$xf' - f + \frac{1}{2}f^2 = Ax + B \tag{1.2}$$

$$xf' - f + \frac{1}{2}f^2 = Ax^2 + Bx + C \tag{1.3}$$

$$xf' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Bx^2 + Cx - \frac{3}{8} \tag{1.4}$$

A, B and C are arbitrary constants.

Date: September 2002.

Key words and phrases. Lie Symmetry Groups, Green's Functions, Fundamental Solutions, Characteristic Functions, Transition Densities, Symmetry techniques.

We will show that if f is a solution of (1.2), or (1.4), with $B = 0$, then we can obtain the *characteristic function* for the PDE (1.1), from the solution $u = 1$, via a straightforward symmetry group transformation. The characteristic function $U_\lambda(x,t)$ of (1.1) is defined to be

$$U_\lambda(x,t) = \int_0^\infty e^{-\lambda y} p(t,x,y)dy, \tag{1.5}$$

where $p(t,x,y)$ is the *fundamental solution* or *Green's function* of equation (1.1). That is, $U_\lambda(x,t)$ is the Laplace transform of $p(t,x,y)$. The fundamental solution can then be recovered by taking the inverse Laplace transform of U_λ. When f is a solution of (1.3) we are still able to obtain the fundamental solution by symmetry methods, however, this case is more involved, and so we illustrate the procedure by examples. Finally, we will consider the case when f satisfies (1.4) with $B \neq 0$. Here our results are less complete, because we have no explicit solutions of this Ricatti equation.

Our techniques lead to a rich class of PDEs of the form (1.1) for which the fundamental solution may be explicitly computed. It includes as special cases, all the well know examples, such as when the drift function f is affine. In a subsequent paper, we shall introduce a different symmetry based approach to the problem of determining fundamental solutions of (1.1), that provides additional explicit densities.

The problem of computing fundamental solutions for PDEs of the form (1.1), arises for example, when one has to obtain transition densities for certain diffusion processes. Consider a one dimensional generalised square root process, $X = \{X_t, t \in [0,T]\}$, satisfying the Itô stochastic differential equation (SDE)

$$dX_t = f(X_t)dt + \sqrt{2X_t}dW_t, \tag{1.6}$$

for $t \in [0,T]$. Here W is a standard Wiener process, and f is an appropriate drift function. It is well known that the transition density, $p(t,x,y)$ for the process X, is given by the fundamental solution of the PDE

$$\frac{\partial p}{\partial t} = x\frac{\partial^2 p}{\partial x^2} + f(x)\frac{\partial p}{\partial x}. \tag{1.7}$$

See for example Protter [Pro90] or Revuz and Yor [RY98]. For conditions on f guaranteeing the existence of a unique, strong solution of (1.6), see Protter, [Pro90]. Generalised square root processes have important applications, particularly in finance. Several interest rate models involve so called *affine processes*, which are generalised square root processes with drift of the form $f(x) = ax + b$. See the paper by Duffie And Kan, [DK94] for a discussion of this topic. Also, the so called *minimum market model of Platen*, [Pla01] for equity and currency markets involves generalised square root processes.

We will derive fundamental solutions of (1.1) in some illustrative cases, and hence obtain transition densities for generalised square root processes X, satisfying SDE's of the form (1.6). Many of the fundamental solutions that we obtain appear to be new.

The outline of the paper is as follows. In Section 2 we introduce the results we need from the theory of Lie group symmetries. In Sections 3 and 4, we determine the infinitesimal symmetries for the PDE (1.1). Finally, in Sections 5, 6 and 7, we show how these symmetries can be used to obtain characteristic functions and fundamental solutions.

2. Introduction to Symmetry Methods

A *symmetry* of a differential equation is a transformation which maps solutions of the equation to other solutions. More precisely, if $\mathcal{H}_P$ denotes the space of all solutions of the PDE

$$P(x, D^\alpha u) = 0 \tag{2.1}$$

then a symmetry $\mathcal{S}$ is an automorphism of $\mathcal{H}_P$. i.e $\mathcal{S} : \mathcal{H}_P \to \mathcal{H}_P$. Thus $u \in \mathcal{H}_P$ implies that $\mathcal{S}u \in \mathcal{H}_P$.

In the 1880s Lie developed a technique for systematically determining all groups of *point symmetries* for systems of differential equations.[1] Symmetry group methods provide a very powerful tool for the analysis of differential equations. Indeed symmetries often provide the only practical method for finding analytical solutions. The book by Olver [Olv93] gives an excellent modern account of Lie's theory of symmetry groups. Other significant works include Miller [Mil74], Bluman and Kumei [BK89], Olver [Olv95], Hydon, [Hyd00], Stephani [Ste89] and the classic text by Ovsiannikov, [Ovs82]. The papers [Cra95], [Cra00], [Cra94] and [CD01] provide additional information on symmetries and their applications.

The key to calculating group symmetries for differential equations is a theorem of Lie, which we will state below. For the purposes of the current work, we consider a PDE of order n in m variables, defined on a simply connected subset $\Omega \subseteq \mathbb{R}^m$. The PDE takes the form (2.1), where $P(x, y)$ is an analytic function on $\Omega \times \mathbb{R}$,

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_m}}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_m)$, is a multi-index, with $\alpha_i \in \mathbb{N}$ for $i \in \{1, ..., m\}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_m$. The extension of the theory to systems of PDEs is straightforward. Chapter 2 of Olver's book [Olv93], contains

[1] There also exist group symmetries which are more complicated than point symmetries, as well as symmetries which do not have group properties. They are important in many applications, but we do not consider them here.

a detailed and rigorous discussion of the technique which we will now describe.

We begin by considering an arbitrary *vector field*, that is, a first order differential operator of the form

$$\mathbf{v} = \sum_{k=1}^{m} \xi_k(x,u)\frac{\partial}{\partial x_k} + \phi(x,u)\frac{\partial}{\partial u}, \tag{2.2}$$

where $(x,u) \in \Omega\times\mathbb{R}$. The vector field (2.2) is the infinitesimal generator of a one parameter local Lie group, called the *flow* of $\mathbf{v}$, which acts upon elements $(x,u) \in \Omega \times \mathbb{R}$. We call this group $\mathcal{G}$. We require a method which allows us to determine conditions on ξ_k and ϕ, which will ensure that $\mathcal{G}$ is a group of symmetries for (2.1).

We define the *nth prolongation* of $\mathcal{G}$, to be the natural extension of the action of $\mathcal{G}$, from (x,u), to the collection of *all* the derivatives of u, up to order n. That is, the nth prolongation, denoted $\mathrm{pr}^n\mathcal{G}$, acts on $(x,u,u_{x_1},...,u_{x_m...x_m})$, where the order of the highest derivatives is n.

To determine $\mathrm{pr}^n\mathcal{G}$, let $\mathcal{D}^n$ be the *n-jet* mapping defined by

$$\mathcal{D}^n : (x,u) \longmapsto (x,u,u_{x_1},...,u_{x_m...x_m}). \tag{2.3}$$

Then the n-th prolongation must satisfy

$$\mathcal{D}^n \circ \mathcal{G} = \mathrm{pr}^n\mathcal{G} \circ \mathcal{D}^n. \tag{2.4}$$

This condition requires that the chain rule of multi-variable calculus holds.

The infinitesimal generator of $\mathrm{pr}^n\mathcal{G}$, is called the *n-th prolongation* of $\mathbf{v}$, and we denote it by $\mathrm{pr}^n\mathbf{v}$. Using condition (2.4), it is possible to derive an explicit formula for $\mathrm{pr}^n\mathbf{v}$. The details are contained in Chapter 2 of [Olv93].

Theorem 2.1 (Olver)**.** *Let* $\mathbf{v}$ *be a vector field of the form (2.2). Then the n-th prolongation of* $\mathbf{v}$ *is*

$$\mathrm{pr}^n\mathbf{v} = \mathbf{v} + \sum_J \phi^J \frac{\partial}{\partial u_J}, \tag{2.5}$$

where the sum is taken over all multi-indices J, *with* $|J| \leq n$. *The functions* ϕ^J *are given by*

$$\phi^J = \mathfrak{D}_J\left(\phi - \sum_{k=1}^{m}\xi_k u_{x_k}\right) + \sum_{k=1}^{m}\xi_k u_{J,x_k}. \tag{2.6}$$

Here $\mathfrak{D}_J$ *denotes the total derivative operator and* $u_{x_k} = \frac{\partial u}{\partial x_k}$.

We illustrate the notation by considering an example. Let $m = 2$ and label the dependent variables x and t. Let $J = (2,1)$ be a multi-index. Then

$$u_J = u_{xxt} = \frac{\partial^3 u}{\partial x^2 \partial t}, \quad u_{J,x} = u_{xxtx}, \text{ and } \frac{\partial}{\partial u_J} = \frac{\partial}{\partial u_{xxt}}.$$

It is standard to write ϕ^x for ϕ^J when $J = (1,0)$, $\phi^{xx} = \phi^J$ when $J = (2,0)$ etc. So for example, if $J = (2,2)$ then we would write $\phi^J = \phi^{xxtt}$. This is the notation we will use in this paper,

We now state a version of a theorem due to Lie. This is the central result of the theory of Lie group symmetries. It provides necessary and sufficient conditions for a vector field of the form (2.2), to generate symmetries of a specified differential equation. The proof may be found in Chapter 2 of Olver, [Olv93]. See also Lie's original papers in [Lie12].

Theorem 2.2 (Lie). *Let*

$$P(x, D^\alpha u) = 0 \tag{2.7}$$

be an n-th order partial differential equation as defined above. Let $\mathbf{v}$ *be a vector field of the form (2.2). Then* $\mathbf{v}$ *generates a one parameter local group of symmetries of (2.7) if and only if*

$$\mathrm{pr}^n\mathbf{v}[P(x, D^\alpha u)] = 0, \tag{2.8}$$

whenever $P(x, D^\alpha u) = 0$.

Applying Theorem 2.2 to a PDE yields a system of determining equations for the functions ξ_k and ϕ. In most circumstances these determining equations may be solved by inspection. One thus obtains a set of vector fields which generate all point group symmetries. The vector fields satisfying (2.8) are referred to as *infinitesimal symmetries.*

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket. We have the following result, which is also due to Lie. For a proof of Theorem 2.3, see Chapter 2 of Olver, [Olv93].

Theorem 2.3 (Lie). *Let*

$$P(x, D^\alpha)u = 0$$

be a differential equation defined on $M = \Omega \times \mathbb{R}^n$*. The set of all infinitesimal symmetries form a Lie algebra of vector fields on* M*. Moreover, if this Lie algebra is finite dimensional, the symmetry group of the system is a local Lie group of transformations acting on* M*.*

2.1. **The One Dimensional Heat equation.** As an illustrative example of the application of Theorem 2.2, we consider the one dimensional heat equation,

$$u_{xx} = u_t. \tag{2.9}$$

This example was originally studied by Lie. To compute the symmetries of (2.9), we set

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u},$$

and compute the second prolongation of $\mathbf{v}$. According to Theorem 2.2, $\mathbf{v}$ generates symmetries of the heat equation, if and only if

$$\mathrm{pr}^2\mathbf{v}[u_{xx} - u_t] = 0 \tag{2.10}$$

whenever $u_{xx} - u_t = 0$. The general form of the second prolongation of $\mathbf{v}$ is

$$\mathrm{pr}^2\mathbf{v} = \mathbf{v} + \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} + \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}}, \tag{2.11}$$

and hence the condition, (2.10) implies that,

$$\phi^t = \phi^{xx} \tag{2.12}$$

The functions ϕ^t and ϕ^{xx} can be explicitly computed from the prolongation formula in Theorem 2.1. This gives a set of defining equations for ξ, τ and ϕ, which may readily be solved. The full details of the calculation are in Olver's book [Olv93], p120ff. From ξ, τ and ϕ we may determine a basis for the Lie algebra of infinitesimal symmetries. A basis for the Lie algebra of symmetries of the one dimensional heat equation is,

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \quad \mathbf{v}_2 = \frac{\partial}{\partial t}, \quad \mathbf{v}_3 = u\frac{\partial}{\partial u}, \quad \mathbf{v}_4 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \frac{1}{2}u\frac{\partial}{\partial u},$$
$$\mathbf{v}_5 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \quad \mathbf{v}_6 = 4xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (x^2 + 2t)u\frac{\partial}{\partial u}.$$

In addition, there are infinitely many infinitesimal symmetries of the form $\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u}$, where $\beta(x,t)$ is an arbitrary solution of the heat equation. The existence of these symmetries reflects the fact that adding two solutions of the heat equation yields a third solution. These *trivial symmetries* are usually ignored. We note however, that there are circumstances in the study of non linear PDEs where such symmetries are important. See the book by Bluman and Kumei [BK89] for a discussion of this topic. In this paper, we shall only be interested in *nontrivial symmetries.*

The process of obtaining the group transformation which is generated by a given infinitesimal symmetry is known as *exponentiating* the vector field. To exponentiate an infinitesimal symmetry, $\mathbf{v}_k$, we solve the system of first order ordinary differential equations (ODEs),

$$\frac{d\tilde{x}}{d\epsilon} = \xi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{2.13}$$
$$\frac{d\tilde{t}}{d\epsilon} = \tau(\tilde{x}, \tilde{t}, \tilde{u}) \tag{2.14}$$
$$\frac{d\tilde{u}}{d\epsilon} = \phi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{2.15}$$

subject to the initial conditions

$$\tilde{x}(0) = x, \quad \tilde{t}(0) = t, \quad \tilde{u}(0) = u.$$

If $u(x,t)$ is a solution of the heat equation we will express the action of the symmetry generated by $\mathbf{v}_k$ on u by writing

$$\tilde{u}(x,t) = \rho(\exp(\epsilon \mathbf{v}_k))u(x,t) \tag{2.16}$$

Here $\tilde{u}(x,t)$ is the new solution obtained from u by the action of the symmetry generator $\mathbf{v}_k$, and $\rho(\exp(\epsilon \mathbf{v}_k))u(x,t)$ is the action of the local group generated by $\mathbf{v}_k$, on u.[2] The real number ϵ is the *group parameter.*

Exponentiating the infinitesimal symmetries of the one dimensional heat equation, produces the following symmetry transformations

$$\rho(\exp(\epsilon \mathbf{v}_1))u(x,t) = u(x-\epsilon, t) \tag{2.17}$$

$$\rho(\exp(\epsilon \mathbf{v}_2))u(x,t) = u(x, t-\epsilon) \tag{2.18}$$

$$\rho(\exp(\epsilon \mathbf{v}_3))u(x,t) = e^{\epsilon}u(x,t) \tag{2.19}$$

$$\rho(\exp(\epsilon \mathbf{v}_4))u(x,t) = e^{-\frac{1}{2}\epsilon}u(e^{\epsilon}x, e^{2\epsilon}t) \tag{2.20}$$

$$\rho(\exp(\epsilon \mathbf{v}_5))u(x,t) = e^{-\epsilon x+\epsilon^2 t}u(x-2\epsilon t, t) \tag{2.21}$$

$$\rho(\exp(\epsilon \mathbf{v}_6))u(x,t) = \frac{1}{\sqrt{1+4\epsilon t}}\exp\left\{\frac{-\epsilon x^2}{1+4\epsilon t}\right\}u\left(\frac{x}{1+4\epsilon t}, \frac{t}{1+4\epsilon t}\right) \tag{2.22}$$

The significance of (2.17)-(2.22), is that whenever $u(x,t)$ is a solution of the one dimensional heat equation, and ϵ is sufficiently small, then the right hand side of (2.17),...,(2.22) will also be a solution. The restriction that ϵ be 'sufficiently small' may be dropped if the solution space of the heat equation is restricted in an appropriate way. The papers [Cra95] and [Cra00] contain the technical details.

As an application, consider the symmetry (2.22). Since $u(x,t) = 1$ is a solution of (2.9), then by symmetry so is

$$\tilde{u}(x,t) = \frac{1}{\sqrt{1+4\epsilon t}}\exp\left\{\frac{-\epsilon x^2}{1+4\epsilon t}\right\}. \tag{2.23}$$

In (2.23), let $t \to t - 1/4\epsilon$. and set $\epsilon = \pi$. In this way, we obtain the fundamental solution of the heat equation,

$$k(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}. \tag{2.24}$$

from the constant solution, $u = 1$, by simple group transformation.

It is natural to ask whether we can obtain fundamental solutions for other PDEs by symmetry? In a recent paper, [CD01], Craddock and

[2] This notation is chosen to reflect the fact that exponentiating a vector field produces a local representation of the underlying Lie group. See the papers [Cra95] and [Cra00] for a discussion of the connection between group symmetries and group representation theory.

Dooley, have shown that for the heat equation on a nilpotent Lie group, there always exists a symmetry which maps the constant solution to the fundamental solution. Craddock and Dooley also investigated a class of heat equations with drift on the real line,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x)\frac{\partial u}{\partial x}. \tag{2.25}$$

They showed that the fundamental solution of (2.25) can always be obtained from the constant solution by a symmetry transformation, whenever the drift function f is a solution of any one of five families of Ricatti equations. This immediately leads to a rich class of PDEs, whose fundamental solutions can be explicitly computed by symmetry. It also motives the remainder of this paper.

3. The Equations Defining the Infinitesimal Symmetries.

In the next two sections we will determine all possible Lie symmetry algebras for PDEs of the form (1.1). As described in Section 2, we look for vector fields of the form

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{3.1}$$

We observe that (1.1) is linear, and further that it is first order in t and second order in x. It is a simple exercise to show that in this case ξ and τ cannot depend upon u, and τ must be a function of t only. Furthermore, since (1.1) is second order, we need the second prolongation of $\mathbf{v}$. This was given by equation (2.11). If we apply (2.11) to equation (1.1), then by Theorem 2.2, we see that $\mathbf{v}$ generates symmetries of (1.1), if and only if

$$\phi^t = x\phi^{xx} + f(x)\phi^x + (u_{xx} + f'(x)u_x)\xi. \tag{3.2}$$

To proceed further, we calculate ϕ^t, ϕ^x and ϕ^{xx} by means of equation (2.6) and apply the results to (3.2). We then obtain the system of defining equations that ξ, τ and ϕ must satisfy in order for $\mathbf{v}$ to generate a symmetry. This leads to the equation

$$\begin{aligned}
&\phi_t - \xi_t u_x + (\phi_u - \tau_t)(xu_{xx} + f(x)u_x) = \\
&\quad x(\phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x + \phi_{uu}u_x^2 + (\phi_u - 2\xi_x)u_{xx}) + (u_{xx} + f'(x)u_x)\xi \\
&\quad + f(x)(\phi_x + (\phi_u - \xi_x)u_x)).
\end{aligned} \tag{3.3}$$

Here, subscripts denote partial differentiation.

From (3.3) we can read off individual equations for ξ, τ and ϕ by equating the coefficients of the derivatives of u. First, from the terms involving the zeroth derivatives of u, we see that

$$\phi_t = x\phi_{xx} + f(x)\phi_x. \tag{3.4}$$

From the coefficients of u_x, we get

$$-\xi_t + f(x)(\phi_u - \tau_t) = x(2\phi_{xu} - \xi_{xx}) + f(x)(\phi_u - \xi_x) + f'(x)\xi. \tag{3.5}$$

The coefficients of u_{xx} give

$$x(\phi_u - \tau_t) = x(\phi_u - 2\xi_x) + \xi. \tag{3.6}$$

And finally, examining the terms involving u_x^2, we see that we must have

$$\phi_{uu} = 0. \tag{3.7}$$

The solution of these equations is elementary. We first consider (3.6). Since τ is independent of x, we may solve the equation for ξ by determining the appropriate integrating factor. We readily obtain

$$\xi = x\tau_t + \sqrt{x}\rho(t). \tag{3.8}$$

Here, the arbitrary function ρ depends upon t alone. This immediately allows us to write

$$\xi_t = x\tau_{tt} + \sqrt{x}\rho_t, \tag{3.9}$$

$$\xi_x = \tau_t + \frac{1}{2}x^{-\frac{1}{2}}\rho, \tag{3.10}$$

and

$$\xi_{xx} = -\frac{1}{4}x^{-\frac{3}{2}}\rho. \tag{3.11}$$

Equation (3.7) implies that ϕ must be linear in u. Thus

$$\phi(x,t,u) = \alpha(x,t)u + \beta(x,t), \tag{3.12}$$

for some functions α and β. On the other hand, equation (3.4) requires that

$$\alpha_t = x\alpha_{xx} + f(x)\alpha_x, \tag{3.13}$$

and

$$\beta_t = x\beta_{xx} + f(x)\beta_x. \tag{3.14}$$

We can say no more about β other than that it is an arbitrary solution of the original equation (1.1). From (3.12) and (3.5) we get

$$\begin{aligned} -\xi_t - f(x)\tau_t = 2x\alpha_x + x(\frac{1}{4}x^{\frac{-3}{2}})\rho - f(x)(\tau_t + \frac{1}{2\sqrt{x}}\rho) \\ + f'(x)(x\tau_t + \sqrt{x}\rho), \end{aligned} \tag{3.15}$$

which upon rearrangement gives

$$\alpha_x = -\frac{1}{2}\tau_{tt} - \frac{1}{2\sqrt{x}}\rho_t - \frac{1}{8}x^{-\frac{3}{2}}\rho + \frac{1}{2}(\frac{f(x)}{2x\sqrt{x}} - \frac{f'(x)}{\sqrt{x}})\rho - \frac{1}{2}f'(x)\tau_t. \tag{3.16}$$

We can immediately integrate this to obtain

$$\alpha = -\frac{1}{2}x\tau_{tt} - \sqrt{x}\rho_t + \frac{1}{2\sqrt{x}}(\frac{1}{2} - f(x))\rho - \frac{1}{2}f(x)\tau_t + \sigma(t), \tag{3.17}$$

for some function σ, of t only. We now see that

$$\alpha_t = -\frac{1}{2}x\tau_{ttt} - \sqrt{x}\rho_{tt} + \frac{1}{2\sqrt{x}}(\frac{1}{2} - f(x))\rho_t - \frac{1}{2}f(x)\tau_{tt} + \sigma_t, \tag{3.18}$$

and

$$\alpha_{xx} = \frac{1}{4}x^{-\frac{3}{2}}\rho_t + \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{(\frac{1}{2} - f(x))}{\sqrt{x}}\right)\rho - \frac{1}{2}f''(x)\tau_t. \tag{3.19}$$

Finally, we substitute these into equation (3.13) to derive the equation

$$\begin{aligned}-\frac{1}{2}x\tau_{ttt} - \sqrt{x}\rho_{tt} + \frac{1}{2\sqrt{x}}\left(\frac{1}{2} - f(x)\right)\rho_t - \frac{1}{2}f(x)\tau_{tt} + \sigma_t = \\ x\left(\frac{1}{4}x^{-\frac{3}{2}}\rho_t + \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{(\frac{1}{2} - f(x))}{\sqrt{x}}\right)\rho - \frac{1}{2}f''(x)\tau_t\right) \\ + f(x)\left(-\frac{1}{2}\tau_{tt} - \frac{1}{2}x^{-\frac{1}{2}}\rho_t + \frac{1}{2}\frac{d}{dx}\left(\frac{(\frac{1}{2} - f(x))}{\sqrt{x}}\right)\rho - \frac{1}{2}f'(x)\tau_t\right).\end{aligned} \tag{3.20}$$

Performing the obvious cancellations and collecting terms, we arrive at the final defining equation

$$\begin{aligned}-\frac{1}{2}x\tau_{ttt} - \sqrt{x}\rho_{tt} + \sigma_t = -\frac{1}{2}\left(xf'' + ff'\right)\tau_t \\ + \left[\frac{3 + 8(xf' - f + \frac{1}{2}f^2) - 8x(xf'' + ff')}{16x^{\frac{3}{2}}}\right]\rho.\end{aligned} \tag{3.21}$$

Equation (3.21) determines τ, ρ and σ for every choice of C^2 drift function f. It fixes the final structure of the symmetry group. To proceed further, it is necessary to specify the form of f. We shall do this in the next section.

4. Computing the Infinitesimal Symmetries

Our aim is to use symmetry transformations to obtain fundamental solutions of (1.1) from trivial solutions. This should only be possible if the Lie algebra of infinitesimal symmetries contains a vector field whose action transforms a solution in the t variable. If the action of the symmetry transformation were trivial in t, it could not transform a solution which is constant in t, to one which is nonconstant, such as the fundamental solution.

This motivates us to look for vector fields where the coefficient of $\frac{\partial}{\partial t}$ is nonconstant. Examining equation (3.21), we see that the coefficients of τ_t and ρ depend upon the drift function f. To determine conditions on f in order that τ be nonconstant, we must equate the appropriate terms in x. Thus it is the drift term f which constrains the dimension of the Lie algebra of symmetries of (1.1).

For convenience, we have split our analysis into four cases

4.1. **Case 1.** Let

$$xf'' + ff' = A,$$

where A is a constant. Then integration by parts gives

$$xf' - f + \frac{1}{2}f^2 = Ax + B. \tag{4.1}$$

From (3.21) we see that

$$-\frac{1}{2}x\tau_{ttt} - \sqrt{x}\rho_{tt} + \sigma_t = -\frac{1}{2}A\tau_t + \left(\frac{3+8B}{16x^{\frac{3}{2}}}\right)\rho. \tag{4.2}$$

There are two obvious subcases.

4.1.1. *Subcase 1a.* If $3 + 8B \neq 0$, then we must have

$$\tau_{ttt} = 0, \quad \rho = 0, \quad \sigma_t = -\frac{1}{2}A\tau_t. \tag{4.3}$$

Integration yields, $\tau = c_1 + 2c_3t + 4c_4t^2$, $\sigma = -c_4At - 2c_6At^2 + c_2$, for some arbitrary constants $c_1, ..., c_4$. From this and (3.8) we see that

$$\xi = 2c_3x + 8c_4xt, \tag{4.4}$$

and

$$\alpha = -4c_4x - \frac{1}{2}f(x)(2c_3 + 8c_4t) - c_3At - 2c_4At^2 + c_2. \tag{4.5}$$

Recall that a vector field generating a symmetry of (1.1) is chosen to have the form (2.2). Our choice of basis for the Lie algebra of symmetries is determined by the numbering of the constants appearing in the expressions for ϕ, τ and ξ. Obviously there are other equivalent choices that we could have made.

Because the Lie algebra contains vector fields of the form $\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u}$, in which β is any solution of (3.14), it is clear that the Lie algebra is infinite dimensional. We also have a four dimensional Lie subalgebra of symmetries, arising from the functions ξ, τ and α. A basis for this Lie subalgebra of point symmetries is

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \tag{4.6}$$

$$\mathbf{v}_2 = u\frac{\partial}{\partial u}, \tag{4.7}$$

$$\mathbf{v}_3 = 2x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - (f(x) + At)u\frac{\partial}{\partial u}, \tag{4.8}$$

$$\mathbf{v}_4 = 8xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left(4x + 4f(x)t + 2At^2\right)u\frac{\partial}{\partial u}. \tag{4.9}$$

The symmetries generated by these vector field may be determined by solving the system of ODEs given in (2.13). Here we observe that the vector field $\mathbf{v}_1$ generates translations in time. That is, if $u(x,t)$ is a solution of (1.1), then so is $u(x, t+\epsilon)$. The vector field $\mathbf{v}_2$ implies that if we multiply a solution by a constant, then the result is another solution. $\mathbf{v}_3$ generates scaling symmetries in the x, t and u variables. We will consider the vector field $\mathbf{v}_4$ in the following section. Finally, the vector fields of the form $\mathbf{v}_\beta$, show that if u is a solution of (1.1) and β is another solution, then $u + \beta$ is also a solution. These symmetries are straightforward consequences of the linearity of equation (1.1) and the fact that the coefficients of the equation are constant in time.

We point out one more interesting feature. Since the Lie algebra of symmetries is closed under Lie brackets, then we may easily obtain new symmetries. For example, $[\mathbf{v}_4, \mathbf{v}_\beta]$ produces the new infinitesimal symmetry

$$[\mathbf{v}_4, \mathbf{v}_\beta] = \left(8xt\beta_x + 4t^2\beta_t + (4x + 4f(x)t + 2At^2)\right)\beta\frac{\partial}{\partial u}. \tag{4.10}$$

This allows us to conclude that if β is any solution of (1.1), then so is $8xt\beta_x + 4t^2\beta_t + (4x + 4f(x)t + 2At^2)\beta$. We may of course compute other such symmetries.

4.1.2. *Subcase 1b.* If $3 + 8B = 0$, then

$$\tau_{ttt} = 0, \quad \rho_{tt} = 0, \quad \sigma_t = -\frac{1}{2}A\tau_t. \tag{4.11}$$

Thus from (4.1)

$$xf' - f + \frac{1}{2}f^2 = Ax - \frac{3}{8}. \tag{4.12}$$

From (4.11) we obtain

$$\tau = c_2 + 2c_4t + 4c_6t^2, \quad \rho = c_1 + 2c_5t, \quad \sigma = \sigma = -c_4At - 2c_6At^2 + c_3. \tag{4.13}$$

Combining this with (3.8) gives

$$\xi = x(2c_4 + 8c_6t) + \sqrt{x}(c_1 + 2c_5t), \tag{4.14}$$

and

$$\begin{aligned}\alpha = &-4c_6x - (2c_5)\sqrt{x} - \frac{1}{2\sqrt{x}}\left(\frac{1}{2} - f(x)\right)(c_1 + 2c_5t)\\ &-\frac{1}{2}f(x)(2c_4 + 8c_6t) - c_4At - 2c_6At^2 + c_3.\end{aligned} \tag{4.15}$$

We thus have a six dimensional Lie subalgebra of symmetries, plus the infinite dimensional ideal generated by the vector fields of the form $\mathbf{v}_\beta$.

A basis for the six dimensional subalgebra is seen to be

$$\mathbf{v}_1 = \sqrt{x}\frac{\partial}{\partial x} - \frac{1}{2\sqrt{x}}\left(\frac{1}{2} - f(x)\right)u\frac{\partial}{\partial u}, \tag{4.16}$$

$$\mathbf{v}_2 = \frac{\partial}{\partial t}, \tag{4.17}$$

$$\mathbf{v}_3 = u\frac{\partial}{\partial u}, \tag{4.18}$$

$$\mathbf{v}_4 = 2x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - (f(x) + At)u\frac{\partial}{\partial u}, \tag{4.19}$$

$$\mathbf{v}_5 = 2\sqrt{x}t\frac{\partial}{\partial x} - (2\sqrt{x} - \frac{1}{\sqrt{x}}\left(\frac{1}{2} - f(x)\right)tu\frac{\partial}{\partial u}, \tag{4.20}$$

$$\mathbf{v}_6 = 8xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (4x + 4f(x)t + 2At^2)u\frac{\partial}{\partial u}. \tag{4.21}$$

4.2. **Case 2.** Let

$$xf'' + ff' = Ax + B,$$

$A \neq 0, B$ constants. Integration by parts then yields the Ricatti equation

$$xf' - f + \frac{1}{2}f^2 = \frac{1}{2}Ax^2 + Bx + D. \tag{4.22}$$

Consequently, the final determining equation (3.21) reads

$$-\frac{1}{2}x\tau_{ttt} - \sqrt{x}\rho_{tt} + \sigma_t = -\frac{1}{2}(Ax + B)\tau_t + \left(\frac{3 + 8D - 4Ax^2}{16x^{\frac{3}{2}}}\right)\rho. \tag{4.23}$$

Again we have two subcases:

4.2.1. *Subcase 2a.* If $3 + 8D \neq 0$, then

$$\tau_{ttt} = A\tau_t \quad \rho = 0, \quad \sigma_t = -\frac{B}{2}\tau_t.$$

By calculations similar to Case 1, we see that a basis for the Lie algebra of symmetries is

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \tag{4.24}$$

$$\mathbf{v}_2 = u\frac{\partial}{\partial u}, \tag{4.25}$$

$$\mathbf{v}_3 = x\sqrt{A}e^{\sqrt{A}t}\frac{\partial}{\partial x} + e^{\sqrt{A}t}\frac{\partial}{\partial t} - \frac{1}{2}(Ax + \sqrt{A}f(x) + B)e^{\sqrt{A}t}u\frac{\partial}{\partial u}, \tag{4.26}$$

$$\mathbf{v}_4 = -x\sqrt{A}e^{-\sqrt{A}t}\frac{\partial}{\partial x} + e^{-\sqrt{A}t}\frac{\partial}{\partial t} - \frac{1}{2}(Ax - \sqrt{A}f(x) + B)e^{-\sqrt{A}t}u\frac{\partial}{\partial u}, \tag{4.27}$$

$$\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u}, \tag{4.28}$$

where β is an arbitrary solution of equation (1.1).

4.2.2. *Subcase 2b.* If $3+8D=0$, then

$$xf'-f+\frac{1}{2}f^2=\frac{1}{2}Ax^2+Bx-\frac{3}{8}, \tag{4.29}$$

and

$$\tau_{ttt}=A\tau_t \quad \rho_{tt}=\frac{A}{4}\rho, \quad \sigma_t=-\frac{B}{2}\tau_t.$$

Proceeding in the same way as before leads to the following basis for the Lie algebra of infinitesimal symmetries.

$$\mathbf{v}_1=\sqrt{x}e^{\frac{1}{2}\sqrt{A}t}\frac{\partial}{\partial x}-\frac{1}{2}\left(\sqrt{x}-\frac{1}{2\sqrt{x}}(\frac{1}{2}-f(x))\right)e^{\frac{1}{2}\sqrt{A}t}u\frac{\partial}{\partial u}, \tag{4.30}$$

$$\mathbf{v}_2=\frac{\partial}{\partial t} \tag{4.31}$$

$$\mathbf{v}_3=u\frac{\partial}{\partial u}, \tag{4.32}$$

$$\mathbf{v}_4=x\sqrt{A}e^{\sqrt{A}t}\frac{\partial}{\partial x}+e^{\sqrt{A}t}\frac{\partial}{\partial t}-\frac{1}{2}(Ax+\sqrt{A}f(x)+B)e^{\sqrt{A}t}u\frac{\partial}{\partial u}, \tag{4.33}$$

$$\mathbf{v}_5=\sqrt{x}e^{-\frac{1}{2}\sqrt{A}t}+\frac{1}{2}\left(\sqrt{x}+\frac{1}{2\sqrt{x}}(\frac{1}{2}-f(x))\right)e^{-\frac{1}{2}\sqrt{A}t}u\frac{\partial}{\partial u}, \tag{4.34}$$

$$\mathbf{v}_6=-x\sqrt{A}e^{-\sqrt{A}t}\frac{\partial}{\partial x}+e^{-\sqrt{A}t}\frac{\partial}{\partial t}-\frac{1}{2}(Ax-\sqrt{A}f(x)+B)e^{-\sqrt{A}t}u\frac{\partial}{\partial u} \tag{4.35}$$

$$\mathbf{v}_\beta=\beta(x,t)\frac{\partial}{\partial u}, \tag{4.36}$$

where β is an arbitrary solution of equation (1.1).

4.3. **Case 3.** Let $xf''+ff'=A\sqrt{x}+Bx+C$. Then we have

$$xf'-f+\frac{1}{2}f^2=\frac{2}{3}Ax^{\frac{3}{2}}+\frac{1}{2}Bx^2+Cx+D. \tag{4.37}$$

Consequently, equation (3.21) reads

$$-\frac{1}{2}x\tau_{ttt}-\sqrt{x}\rho_{tt}+\sigma_t=-\frac{1}{2}(A\sqrt{x}+Bx+C)\tau_t + \left(\frac{3+8D-\frac{8}{3}Ax^{\frac{3}{2}}-4Bx^2}{16x^{\frac{3}{2}}}\right)\rho.$$

If $3+8D\neq 0$, then $\rho=0$. This implies $\sigma_t=\tau_t=0$. Hence τ and σ are constants.

In the case where $3+8D=0$, then

$$\tau_{ttt}=B\tau_t \quad \rho_{tt}=\frac{A}{2}\tau_t+\frac{B}{4}\rho, \quad \sigma_t=-\frac{C}{2}\tau_t-\frac{A}{6}\rho.$$

The cases, $B=0$, and $B\neq 0$ are different.

4.3.1. *Subcase 3a.* If $B = 0$ then a basis for the Lie algebra of infinitesimal symmetries is

$$\mathbf{v}_1 = \sqrt{x}\frac{\partial}{\partial x} - \left(\frac{A}{6}t - \left(\frac{\frac{1}{2} - f(x)}{2\sqrt{x}}\right)\right) u\frac{\partial}{\partial u} \tag{4.38}$$

$$\mathbf{v}_2 = \frac{\partial}{\partial t} \tag{4.39}$$

$$\mathbf{v}_3 = u\frac{\partial}{\partial u} \tag{4.40}$$

$$\begin{aligned}\mathbf{v}_4 = &\left(2x + \frac{A}{2}\sqrt{x}t^2\right)\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} \\ &- \left((C + A\sqrt{x})t + \frac{A^2}{36}t^3 - \frac{A(\frac{1}{2} - f(x))t^2}{4\sqrt{x}} + f(x)\right) u\frac{\partial}{\partial u}\end{aligned} \tag{4.41}$$

$$\mathbf{v}_5 = \sqrt{x}t\frac{\partial}{\partial x} - \left(\frac{A}{12}t^2 + \sqrt{x} - \frac{(\frac{1}{2} - f(x))}{2\sqrt{x}}t\right) u\frac{\partial}{\partial u} \tag{4.42}$$

$$\begin{aligned}\mathbf{v}_6 = &\left(8xt + \frac{2A}{3}\sqrt{x}t^3\right)\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \\ &\left(4x + 2Ct^2 + 4f(x)t + \frac{A^2}{36}t^4 + 2A\sqrt{x}t^2 - \frac{A(\frac{1}{2} - f(x))}{3\sqrt{x}}t^3\right) u\frac{\partial}{\partial u}\end{aligned} \tag{4.43}$$

$$\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u}, \tag{4.44}$$

where β is an arbitrary solution of equation (1.1).

4.3.2. *Subcase 3b.* In the case when $B \neq 0$, the calculations are similar. A basis for the Lie algebra of infinitesimal symmetries is

$$\mathbf{v}_1 = \sqrt{x}e^{\frac{1}{2}\sqrt{B}t}\frac{\partial}{\partial x} - \left(\frac{1}{2}\sqrt{B}\sqrt{x} - \frac{(\frac{1}{2} - f(x))}{2\sqrt{x}} + \frac{A}{3\sqrt{B}}\right) e^{\frac{1}{2}\sqrt{B}t}u\frac{\partial}{\partial u} \tag{4.45}$$

$$\mathbf{v}_2 = \sqrt{x}e^{-\frac{1}{2}\sqrt{B}t}\frac{\partial}{\partial x} + \left(\frac{1}{2}\sqrt{B}\sqrt{x} + \frac{(\frac{1}{2} - f(x))}{2\sqrt{x}} + \frac{A}{3\sqrt{B}}\right) e^{-\frac{1}{2}\sqrt{B}t}u\frac{\partial}{\partial u} \tag{4.46}$$

$$\mathbf{v}_3 = u\frac{\partial}{\partial u}, \tag{4.47}$$

$$\mathbf{v}_4 = \frac{\partial}{\partial t}, \tag{4.48}$$

$$\mathbf{v}_5 = \left(\frac{2A}{3\sqrt{B}}\sqrt{x} + \sqrt{B}x\right) e^{\sqrt{B}t}\frac{\partial}{\partial x} + e^{\sqrt{B}t}\frac{\partial}{\partial t}$$
$$- \left(\frac{B}{2}x + \frac{2A}{3}\sqrt{x} + \frac{\sqrt{B}}{2}f(x) - \frac{A(\frac{1}{2} - f(x))}{3\sqrt{B}\sqrt{x}} + \frac{2A^2 + 9BC}{18B}\right)$$
$$\times e^{\sqrt{B}t}u\frac{\partial}{\partial u}, \tag{4.49}$$

$$\mathbf{v}_6 = -\left(\frac{2A}{3\sqrt{B}}\sqrt{x} + \sqrt{B}x\right) e^{-\sqrt{B}t}\frac{\partial}{\partial x} + e^{-\sqrt{B}t}\frac{\partial}{\partial t}$$
$$- \left(\frac{B}{2}x + \frac{2A}{3}\sqrt{x} - \frac{\sqrt{B}}{2}f(x) + \frac{A(\frac{1}{2} - f(x))}{3\sqrt{B}\sqrt{x}} + \frac{2A^2 + 9BC}{18B}\right)$$
$$\times e^{-\sqrt{B}t}u\frac{\partial}{\partial u}, \tag{4.50}$$

$$\mathbf{v}_\beta = \beta(x,t)\frac{\partial}{\partial u}, \tag{4.51}$$

where β is an arbitrary solution of equation (1.1).

4.4. **Case 4.** The final case we must consider is when the drift f does not satisfy any of the Ricatti equations of Cases 1 through 3. Here, we must have $\tau_{ttt} = \tau_t = 0, \quad \rho = 0, \quad \sigma_t = 0$. Thus the symmetry algebra is two dimensional. A basis is

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \quad \mathbf{v}_2 = u\frac{\partial}{\partial u}$$

Therefore, if the drift does not satisfy one of the Ricatti equations (1.2), (1.3) or (1.4), then only the only possible symmetries are translation in t and scaling in the u variable.

This completes our determination of the Lie algebra of symmetries for equations of the form (1.1). There are no other possibilities. In the next section we will show how to use the symmetries determined here to construct fundamental solutions of (1.1) for different choices of f.

5. Fundamental Solutions and Characteristic Functions

We will now exploit the symmetries found in the previous section to compute characteristic functions and fundamental solutions for PDEs of the form (1.1). We will first consider the case where the drift function f is a solution of the Ricatti equation (1.2). We will consider the PDEs associated with the Ricatti equations, (1.3) and (1.4) in the following sections.

In this section we introduce a method for explicitly computing fundamental solutions, which involves taking the inverse Laplace transform of the characteristic function. One immediate result of the symmetry analysis in the previous section is that we may often obtain the characteristic function of (1.1) from a trivial solution by a straightforward group transformation. We illustrate with an example, before stating a theorem.

Example 5.1. The simplest case is $f(x) = \alpha$, where α is constant. In this case we have $xf'' + ff' = 0$, so $xf' - f + \frac{1}{2}f^2 = B$. We are thus considering the PDE,

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x}, \tag{5.1}$$

which is fundamental to the theory of Bessel processes. See chapter 11 of Revuz and Yor, [RY98] for a detailed discussion of Bessel processes. From Case 1 of Section 4, we see that a basis for the Lie algebra of symmetries of (5.1) is

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t},\\
\mathbf{v}_2 &= u\frac{\partial}{\partial u},\\
\mathbf{v}_3 &= 2x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - \alpha u\frac{\partial}{\partial u},\\
\mathbf{v}_4 &= 8xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - (4x + 4\alpha t)u\frac{\partial}{\partial u},\\
\mathbf{v}_\beta &= \beta(x,t)\frac{\partial}{\partial u}.
\end{aligned}$$

We are interested here in $\mathbf{v}_4$. We compute the action of the one parameter local Lie group generated by $\mathbf{v}_4$, by solving the system of ODEs defined in (2.13). From this we obtain

$$\begin{aligned}
\rho(\exp(\epsilon\mathbf{v}_4))u(x,t) = \exp&\left\{\frac{-4\epsilon x}{1+4\epsilon t} - \frac{\alpha}{2}\left(\ln(x) - \ln(\frac{x}{(1+4\epsilon t)^2})\right)\right\}\\
&\times u\left(\frac{x}{(1+4\epsilon t)^2}, \frac{t}{1+4\epsilon t}\right).
\end{aligned} \tag{5.2}$$

Thus if u is any solution of (5.1), then (5.2) is also a solution, at least for ϵ sufficiently small. We set $\lambda = 4\epsilon$, and consider the solution $u = 1$. Then by symmetry,

$$U_\lambda(x,t) = (1+\lambda t)^{-\alpha}\exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}, \tag{5.3}$$

is also a solution of (5.1). This is well known to be the characteristic function for (5.1). It is the Laplace transform of $p(t,x,y)$. This transform can be inverted using the fundamental identity

$$\mathcal{L}^{-1}\left(\frac{1}{\lambda^\mu}e^{\frac{k}{\lambda}}\right) = \left(\frac{y}{k}\right)^{\frac{\mu-1}{2}} I_{\mu-1}(2\sqrt{ky}), \tag{5.4}$$

where I_ν is a modified Bessel function of the first kind with order ν. See [AS72], Chapter 9 for properties of modified Bessel functions.

We then obtain the well known transition density of Bessel processes

$$p(t,x,y) = \mathcal{L}^{-1}\left((1+\lambda t)^{-\alpha}\exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}\right)$$
$$= \frac{1}{t^2}\left(\frac{x}{y}\right)^{\frac{1-\alpha}{2}} I_{\alpha-1}\left(\frac{2\sqrt{xy}}{t}\right)\exp\left\{-\frac{(x+y)}{t}\right\}. \tag{5.5}$$

This example shows that it is possible to obtain the characteristic function for the PDE (1.1) by a straightforward symmetry transformation. In fact, we can obtain characteristic functions, and hence fundamental solutions, for a wide class of equations by the same procedure. The key is that the characteristic function can be viewed as a solution of (1.1), with the initial condition

$$u(x,0) = e^{-\lambda x} \tag{5.6}$$

By symmetry, we can obtain a solution satisfying (5.6) from a solution with initial data $u(x,0)=1$.

Theorem 5.1. *Let f be a solution of the Ricatti equation*

$$xf' - f + \frac{1}{2}f^2 = Ax + B \tag{5.7}$$

Then the characteristic function $U_\lambda(x,t)$ for the PDE (1.1) is given by

$$U_\lambda(x,t) = \exp\left\{-\frac{\lambda(x+\frac{1}{2}At^2)}{1+\lambda t} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{(1+\lambda t)^2}\right)\right)\right\} \tag{5.8}$$

where $F'(x) = f(x)/x$

Proof. Clearly $U_\lambda(x,0) = e^{-\lambda x}$. Now, since $xf' - f + \frac{1}{2}f^2 = Ax + B$, then, from Case 1 of Section 4, equation (1.1) has an infinitesimal symmetry of the form

$$\mathbf{v} = 8xt\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left(4x + 4f(x)t + 2At^2\right)u\frac{\partial}{\partial u}. \tag{5.9}$$

The exponentiation of $\mathbf{v}$ shows that if u is a solution of equation (1.1) with $xf' - f + \frac{1}{2}f^2 = Ax + B$, then so is

$$\tilde{u}_\epsilon(x,t) = \exp\left\{-\frac{(4\epsilon x + 2A\epsilon t^2)}{1+4\epsilon t} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{(1+4\epsilon t)^2}\right)\right)\right\}$$
$$\times u\left(\frac{x}{(1+4\epsilon t)^2}, \frac{t}{1+4\epsilon t}\right), \tag{5.10}$$

where $F'(x) = f(x)/x$. Taking $u=1$, and setting $\lambda = 4\epsilon$, we obtain the characteristic function (5.8). □

Let us make the following observation. If we substitute the expression (5.8) equation (1.1), then, after some manipulations, we see that if $U_\lambda(x,t)$ is a solution, we must have

$$\begin{aligned}
&-2A\lambda xt(2+\lambda t)+2(1+\lambda t)^2\Bigg(xf'-f+\frac{1}{2}f^2\\
&-\left(\frac{x}{(1+\lambda t)^2}f'\left(\frac{x}{(1+\lambda t)^2}\right)-f\left(\frac{x}{(1+\lambda t)^2}\right)+\frac{1}{2}f^2\left(\frac{x}{(1+\lambda t)^2}\right)\right)\Bigg)\\
&=-2A\lambda xt(2+\lambda t)+2(1+\lambda t)^2\left(g(x)-g\left(\frac{x}{(1+\lambda t)^2}\right)\right)=0,
\end{aligned} \tag{5.11}$$

where $xf'-f+\frac{1}{2}f^2=g(x)$. This immediately implies that we must have

$$g(x)-g\left(\frac{x}{(1+\lambda t)^2}\right)=\frac{Ax\lambda t(2+\lambda t)}{(1+\lambda t)^2}. \tag{5.12}$$

It is clear that $g(x)=Ax+B$ is a solution of this functional equation, as we expect.

5.1. **Solving the Ricatti Equations.** Before presenting our examples, we consider the problem of solving Ricatti equations of the form

$$xf'-f+\frac{1}{2}f^2=g(x) \tag{5.13}$$

Equation (5.13) can be transformed into a second order linear equation by the change of variable $f=2xy'/y$. Under this change of variables, equation (5.13) becomes

$$2x^2y''(x)-g(x)y(x)=0. \tag{5.14}$$

The equation (5.14) can be solved by standard techniques for a wide range of functions $g(x)$. In Theorem 5.1 we have, $g(x)=Ax+B$. The general solution of (5.14) for this choice of g is

$$y(x)=c_1x^{\frac{1}{2}}I_{\sqrt{1+2B}}\left(\sqrt{2Ax}\right)+c_2x^{\frac{1}{2}}I_{-\sqrt{1+2B}}\left(\sqrt{2Ax}\right). \tag{5.15}$$

From (5.15), all solutions of $xf'-f+1/2f^2=Ax+B$, can be obtained.

A natural question to consider is what functions are covered by Theorem 5.1? It is well known that Bessel functions are related to many different functions. Airy functions, spheroidal wave functions, and many other important functions which arise in mathematical physics are actually special cases of Bessel functions. (See for example Watson's treatise on Bessel functions, [Wat22], for an exhaustive study). For example, whenever

$$\sqrt{1+2B}=\frac{2n+1}{2},\quad n\in\mathbb{N}$$

the solutions of (1.2) will be given by either hyperbolic functions, by functions of the form $r(\sqrt{x})/s(\sqrt{x})$, where r and s are polynomials, or a combination of both. Moreover, for such a choice of B, the resulting characteristic function can always be explicitly inverted.

Investigation of the full range fundamental solutions which may be obtained by our methods is beyond the scope of this paper. We shall content ourselves with some examples to illustrate how Theorem 5.1 is used. The fundamental solutions and transition densities obtained in the following examples appear to be new.

Example 5.2. We consider drift functions of the form

$$f(x) = \frac{ax}{1+\frac{1}{2}ax}, \quad a, x > 0$$

Since, $xf' - f + \frac{1}{2}f^2 = 0$, by Theorem 5.1 the characteristic function for the PDE

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \frac{ax}{1+\frac{1}{2}ax}\frac{\partial u}{\partial x}, \tag{5.16}$$

is

$$U_\lambda(x,t) = \left(\frac{(1+\lambda t)^2 + \frac{1}{2}ax}{(1+\lambda t)^2(1+\frac{1}{2}ax)}\right)\exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}. \tag{5.17}$$

It is now an easy matter to recover $p(t,x,y)$, by inversion of the Laplace transform.

We have

$$p(t,x,y) = \mathcal{L}^{-1}\left(\left(\frac{(1+\lambda t)^2 + \frac{1}{2}ax}{(1+\lambda t)^2(1+\frac{1}{2}ax)}\right)\exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}\right), \tag{5.18}$$

where $\mathcal{L}$ denotes Laplace transform. Inversion of the transform is straightforward with the aide of the relation (5.4), and standard properties of the Laplace transform.

After some manipulation, we arrive at the expression

$$\begin{aligned} p(t,x,y) &= \frac{e^{-\frac{x+y}{t}}}{1+\frac{1}{2}ax}\mathcal{L}^{-1}\left(\exp\left\{\frac{x/t^2}{\lambda}\right\} + \frac{ax}{2t^2\lambda^2}\exp\left\{\frac{x/t^2}{\lambda}\right\}\right) \\ &= \frac{e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t}\left[\left(\sqrt{\frac{x}{y}} + \frac{a\sqrt{xy}}{2}\right)I_1\left(\frac{2\sqrt{xy}}{t}\right) + t\delta(y)\right], \end{aligned} \tag{5.19}$$

in which δ is the Dirac delta function. Consequently

$$\begin{aligned} u(x,t) &= \int_0^\infty \frac{\varphi(y)e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t}\left[\left(\sqrt{\frac{x}{y}} + \frac{a\sqrt{xy}}{2}\right)I_1\left(\frac{2\sqrt{xy}}{t}\right) + t\delta(y)\right]dy \\ &= \frac{\varphi(0)e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)} + \int_0^\infty \frac{\varphi(y)e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t}\left(\sqrt{\frac{x}{y}} + \frac{a\sqrt{xy}}{2}\right)I_1\left(\frac{2\sqrt{xy}}{t}\right)dy, \end{aligned} \tag{5.20}$$

is a solution of the PDE (5.16), with initial data $u(x,0) = \varphi(x)$. Differentiation under the integral sign shows that (5.20) satisfies (5.16). A more involved calculation shows that the solution u satisfies

$$\lim_{t\to 0} u(x,t) = \varphi(x) \tag{5.21}$$

Furthermore, it is not difficult to show that

$$\int_0^\infty \frac{e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t}\left(\sqrt{\frac{x}{y}} + \frac{a\sqrt{xy}}{2}\right) I_1\left(\frac{2\sqrt{xy}}{t}\right) dy = 1 - \frac{e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)}$$

and hence

$$\int_0^\infty p(t,x,y)dy = \frac{e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)} + 1 - \frac{e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)} = 1. \tag{5.22}$$

If we interpret $\frac{e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)}$ as the probability of absorption at the origin, then $p(t,x,y)$ may be interpreted as the transition density for the generalised square root process X, satisfying the SDE

$$dX_t = \frac{aX_t}{1+\frac{1}{2}aX_t}dt + \sqrt{2X_t}dW_t. \tag{5.23}$$

Example 5.3. Consider the drift function

$$f(x) = \frac{(1+3\sqrt{x})}{2(1+\sqrt{x})}.$$

For this choice of f we have $xf' - f + \frac{1}{2}f^2 = -\frac{3}{8}$. Thus, by Theorem 5.1 the characteristic function for the PDE

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \frac{(1+3\sqrt{x})}{2(1+\sqrt{x})}\frac{\partial u}{\partial x}, \tag{5.24}$$

is

$$U_\lambda(x,t) = \left(\left(\frac{x}{(1+\lambda t)^2}\right)^{\frac{1}{4}} + \left(\frac{x}{(1+\lambda t)^2}\right)^{\frac{3}{4}}\right)\frac{\exp\left\{-\frac{\lambda x}{(1+\lambda t)}\right\}}{(1+\sqrt{x})x^{\frac{1}{4}}}. \tag{5.25}$$

Inverting the Laplace transform gives the fundamental solution

$$p(t,x,y) = \frac{e^{\frac{-(x+y)}{t}}}{\sqrt{\pi y t}(1+\sqrt{x})}\left(\cosh\left(\frac{2\sqrt{xy}}{t}\right) + \sqrt{y}\sinh\left(\frac{2\sqrt{xy}}{t}\right)\right). \tag{5.26}$$

It can be shown that $p(t,x,y)$ is integrable at $y=0$, and that

$$\int_0^\infty p(t,x,y)dy = 1. \tag{5.27}$$

As an example, let us compute a solution of (5.24) with initial data $u(x,0) = x$, which is continuous at the origin. The integration is straightforward, and we obtain

$$u(x,t) = \int_0^\infty y p(t,x,y) dy = x + \frac{t\ (1+3\sqrt{x})}{2\ (1+\sqrt{x})}. \tag{5.28}$$

It is clear that u is a solution of equation (5.24), and further,

$$\lim_{t\to 0} u(x,t) = x,$$

as required. The SDE for the corresponding generalized square root process X is of the form

$$dX_t = \frac{(1+3\sqrt{X_t})}{2(1+\sqrt{X_t})}\,dt + \sqrt{2X_t}\,dW_t. \tag{5.29}$$

Example 5.4. Consider the PDE

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \left[\left(1 + \frac{\sqrt{10}}{4}\tanh\left(\frac{\sqrt{10}}{4} + \frac{\sqrt{10}}{8}\ln x\right)\right)\right]\frac{\partial u}{\partial x}. \tag{5.30}$$

The drift function satisfies

$$xf' - f + \frac{1}{2}f^2 = -\frac{3}{16}.$$

By Theorem 5.1, the characteristic function of (5.30) is

$$U_\lambda(x,t) = \frac{\cosh\left(\frac{1}{4}\sqrt{\frac{5}{4}}\left(2 + \ln\left(\frac{x}{(1+\lambda t)^2}\right)\right)\right)}{(1+\lambda t)\cosh\left(\frac{1}{4}\sqrt{\frac{5}{4}}\,(2+\ln x)\right)}\exp\left\{-\frac{\lambda x}{(1+\lambda t)}\right\}. \tag{5.31}$$

Again this Laplace transform is easily inverted. The kernel is

$$\begin{aligned} p(t,x,y) = \frac{\left(\frac{x}{y}\right)^{\frac{1}{4}\sqrt{\frac{5}{2}}} e^{\frac{-(x+y)}{t}}}{(1+e^{\sqrt{\frac{5}{2}}}x^{\frac{1}{2}\sqrt{\frac{5}{2}}})t}&\left[I_{-\frac{1}{2}\sqrt{\frac{5}{2}}}\left(\frac{2\sqrt{xy}}{t}\right)\right. \\ &\left. + e^{\sqrt{\frac{5}{2}}}y^{\frac{1}{2}\sqrt{\frac{5}{2}}}I_{\frac{1}{2}\sqrt{\frac{5}{2}}}\left(\frac{2\sqrt{xy}}{t}\right)\right]. \end{aligned} \tag{5.32}$$

Once more we can show that $\int_0^\infty p(t,x,y)dy = 1$. Hence (5.32) is the transition density for the generalised square root process which satisfies the SDE

$$dX_t = \left(\left(1 + \frac{\sqrt{10}}{4}\tanh\left(\frac{\sqrt{10}}{4} + \frac{\sqrt{10}}{8}\ln X_t\right)\right)\right)dt + \sqrt{2X_t}dW_t. \tag{5.33}$$

Example 5.5. Let us now consider three separate problems arising from the equation

$$xf' - f + \frac{1}{2}f^2 = \frac{1}{2}x - \frac{3}{8}. \tag{5.34}$$

We exhibit three different solutions to this Ricatti equation. These are,

$$f^1(x) = \frac{1}{2} + \sqrt{x}, \tag{5.35}$$

$$f^2(x) = \frac{1}{2} + \sqrt{x}\tanh(\sqrt{x}), \tag{5.36}$$

and

$$f^3(x) = \frac{1}{2} + \sqrt{x}\coth(\sqrt{x}). \tag{5.37}$$

We shall solve the corresponding PDE for each of these drift functions in turn.

First, the equation arising from f^1 is

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2} + \sqrt{x}\right)\frac{\partial u}{\partial x}. \tag{5.38}$$

By Theorem 5.1 the characteristic function for (5.38) is

$$U^1_\lambda(x,t) = \frac{1}{\sqrt{1+\lambda t}}\exp\left\{-\frac{\lambda(t+2\sqrt{x})^2}{4(1+\lambda t)}\right\}. \tag{5.39}$$

As in the preceding examples, the inversion of the Laplace transform is straightforward. Inverting the transform gives the density

$$p^1(t,x,y) = \frac{1}{\sqrt{\pi y t}}e^{-\sqrt{x}}\cosh\left(\frac{(t+2\sqrt{x})\sqrt{y}}{t}\right)\exp\left\{-\frac{(x+y)}{t} - \frac{1}{4}t\right\} \tag{5.40}$$

Next we solve the PDE coming from f^2,

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2} + \sqrt{x}\tanh(\sqrt{x})\right)\frac{\partial u}{\partial x}. \tag{5.41}$$

By Theorem 5.1 the characteristic function for (5.41) is

$$U^2_\lambda(x,t) = \frac{1}{\cosh(\sqrt{x})\sqrt{1+\lambda t}}\cosh\left(\frac{\sqrt{x}}{\sqrt{1+\lambda t}}\right)\exp\left\{\frac{-\lambda(x+\frac{1}{4}t^2)}{1+\lambda t}\right\}. \tag{5.42}$$

Inverting the Laplace transform leads to the fundamental solution

$$p^2(t,x,y) = \frac{1}{\sqrt{\pi y t}}\frac{\cosh(\sqrt{y})}{\cosh(\sqrt{x})}\cosh\left(\frac{2\sqrt{xy}}{t}\right)\exp\left\{-\frac{(x+y)}{t} - \frac{1}{4}t\right\}. \tag{5.43}$$

Finally, we consider the PDE

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2} + \sqrt{x}\coth(\sqrt{x})\right)\frac{\partial u}{\partial x}. \tag{5.44}$$

From Theorem 5.1, the characteristic function for (5.44) is

$$U_\lambda^3(x,t) = \frac{1}{\sinh(\sqrt{x})\sqrt{1+\lambda t}} \sinh\left(\frac{\sqrt{x}}{1+\lambda t}\right) \exp\left\{-\frac{\lambda(x+\frac{1}{4}t^2)}{1+\lambda t}\right\}. \tag{5.45}$$

Inversion of the Laplace transform leads to

$$p^3(t,x,y) = \frac{1}{\sqrt{\pi y t}} \frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})} \sinh\left(\frac{2\sqrt{xy}}{t}\right) \exp\left\{-\frac{(x+y)}{t} - \frac{1}{4}t\right\}. \tag{5.46}$$

For each of these cases, it is easy to verify that

$$\int_0^\infty p^i(t,x,y)dy = 1, \qquad i = 1,2,3$$

It is also easy to generate solutions of these PDEs with, say, polynomial initial data. For example,

$$u(x,t) = \int_0^\infty y p^2(t,x,y)dy = x + \sqrt{x}\tanh(\sqrt{x})t + \frac{1}{4}t^2 + \frac{1}{2}t. \tag{5.47}$$

is a solution of the PDE (5.41), with initial data, $u(x,0) = x$.

Example 5.6. We consider now an example of a drift function which possesses discontinuities. The equation

$$xf' - f + \frac{1}{2}f^2 = -1,$$

has a solution

$$f(x) = 1 + \cot\left(\ln\sqrt{x}\right).$$

This solution is discontinuous at points of the form $x = e^{4n\pi}$, $n \in \{0,1,2...\}$. Nevertheless, by applying Theorem 5.1, we can obtain a characteristic function and fundamental solution for the PDE

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \left(1 + \cot\left(\ln\sqrt{x}\right)\right)\frac{\partial u}{\partial x}. \tag{5.48}$$

Applying equation (5.8), we arrive at the characteristic function

$$U_\lambda(x,t) = \operatorname{cosec}\left(\ln\sqrt{x}\right)\left[x^{\frac{i}{2}}(1+\lambda t)^{-i} - x^{-\frac{i}{2}}(1+\lambda t)^{i}\right]\frac{\exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}}{2i(1+\lambda t)}, \tag{5.49}$$

where $i = \sqrt{-1}$. Inversion of the Laplace transform gives the fundamental solution

$$p(t,x,y) = \operatorname{cosec}\left(\ln\sqrt{x}\right)\frac{e^{-\frac{(x+y)}{t}}}{2it}\left(y^{\frac{i}{2}}I_i\left(\frac{2\sqrt{xy}}{t}\right) - y^{-\frac{i}{2}}I_{-i}\left(\frac{2\sqrt{xy}}{t}\right)\right). \tag{5.50}$$

Although the right hand side of (5.50) involves i, it is in fact real valued. To see this we use the series expansion

$$I_\nu(z) = (\frac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}. \tag{5.51}$$

Recall that $\Gamma(\bar{z}) = \overline{\Gamma(z)}$. Expanding the series for I_ν and collecting terms, leads to the expression,

$$p(t,x,y) = \frac{1}{t}\text{cosec}\left(\ln\sqrt{x}\right) e^{-\frac{(x+y)}{t}} \times \sum_{k=0}^{\infty} \left(\frac{xy}{t^2}\right)^k \left\{ a_k \sin\left(\ln\sqrt{\frac{xy^2}{t^2}}\right) + b_k \cos\left(\ln\sqrt{\frac{xy^2}{t^2}}\right)\right\}. \tag{5.52}$$

Where,

$$a_k = \text{Re}\left(\frac{1}{k!\Gamma(k+1+i)}\right), \quad b_k = \text{Im}\left(\frac{1}{k!\Gamma(k+1+i)}\right)$$

Consequently the function (5.50) is real valued. Further, using standard integrals of Bessel functions, (see chapter 10 of Abramowitz and Stegun, [AS72], or Chapter 13 of Watson, [Wat22]), and (5.50), we obtain

$$\begin{aligned}
\int_0^\infty p(t,x,y)dy &= \frac{\left(-\left(\frac{1}{t}\right)^{2\,i} + \left(\frac{x}{t^2}\right)^i\right)\ \text{cosec}(\ln\sqrt{x})}{2\,i\left(\frac{1}{t}\right)^i\left(\frac{x}{t^2}\right)^{\frac{i}{2}}} \\
&= \frac{\left(-\left(\frac{1}{t}\right)^{2\,i} + \left(\frac{x}{t^2}\right)^i\right)}{2\,i\left(\frac{1}{t}\right)^i\left(\frac{x}{t^2}\right)^{\frac{i}{2}}}\frac{2ix^{\frac{i}{2}}}{x^i-1} \\
&= 1.
\end{aligned} \tag{5.53}$$

An interesting SDE with discontinuous drift function arises for the corresponding generalised square root process. We have

$$dX_t = \left(1 + \cot\left(\ln\sqrt{X}\right)\right) dt + \sqrt{2X_t}dW_t. \tag{5.54}$$

In a subsequent paper we will study the generalised square root process which is defined by solutions of (5.54).

It should be clear that Theorem 5.1 is easy to apply. In fact, we point out that the process of determining the characteristic function and the associated fundamental solutions/transition densities may readily be automated using a symbolic manipulation package. In this way it is possible to quickly determine exact solutions for a range of problems which are not covered by standard techniques.

6. The Ricatti Equation $xf' - f + \frac{1}{2}f^2 = \frac{1}{2}Ax^2 + Bx + C$

Next we consider the case when the drift function satisfies the Ricatti equation (1.3). From Section 4, Case 2, it is clear that equation (1.1) has infinitesimal symmetries of the form

$$\mathbf{v}_3 = x\sqrt{A}e^{\sqrt{A}t}\frac{\partial}{\partial x} + e^{\sqrt{A}t}\frac{\partial}{\partial t} - \frac{1}{2}(Ax + \sqrt{A}f(x) + B)e^{\sqrt{A}t}u\frac{\partial}{\partial u}$$
$$\mathbf{v}_4 = -x\sqrt{A}e^{-\sqrt{A}t}\frac{\partial}{\partial x} + e^{-\sqrt{A}t}\frac{\partial}{\partial t} - \frac{1}{2}(Ax - \sqrt{A}f(x) + B)e^{-\sqrt{A}t}u\frac{\partial}{\partial u}$$

In order to compute fundamental solutions, we require the corresponding group actions. This is given by our next result.

Proposition 6.1. *Let f be a solution of (1.3) and u be a solution of (1.1). Then, for ϵ sufficiently small, the following functions are also solutions of (1.1).*

$$\rho(\exp(\epsilon\mathbf{v}_3))u(x,t) =$$
$$\left(1+\epsilon\sqrt{A}e^{\sqrt{A}t}\right)^{\frac{B}{2\sqrt{A}}} u\left(\frac{x}{1+\epsilon\sqrt{A}e^{\sqrt{A}t}}, \frac{1}{\sqrt{A}}\ln\left(\frac{e^{\sqrt{A}t}}{1+\epsilon\sqrt{A}e^{\sqrt{A}t}}\right)\right)$$
$$\times\exp\left\{\frac{-\epsilon A e^{\sqrt{A}t}x}{2(1+\epsilon\sqrt{A}e^{\sqrt{A}t})} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{1+\epsilon\sqrt{A}e^{\sqrt{A}}}\right)\right)\right\} \tag{6.1}$$

and

$$\rho(\exp(\epsilon\mathbf{v}_4))u(x,t) =$$
$$e^{-\frac{B}{2}t}\left(e^{\sqrt{A}t} - \epsilon\sqrt{A}\right)^{\frac{B}{2\sqrt{A}}} u\left(\frac{xe^{\sqrt{A}t}}{e^{\sqrt{A}t}-\epsilon\sqrt{A}}, \frac{\ln(e^{\sqrt{A}t}-\epsilon\sqrt{A})}{\sqrt{A}}\right)$$
$$\times\exp\left\{\frac{-\epsilon Ax}{2(e^{\sqrt{A}t}-\sqrt{A}\epsilon)} - \frac{1}{2}\left(F(x) - F\left(\frac{xe^{\sqrt{A}t}}{e^{\sqrt{A}t}-\epsilon\sqrt{A}}\right)\right)\right\}, \tag{6.2}$$

In which $F'(x) = f(x)/x$.

Proof. The proof is straightforward. It simply requires us to solve the system of ODES, (2.13), which correspond to $\mathbf{v}_3$ and $\mathbf{v}_4$. □

Since $u = 1$ is a solution of equation (1.1), then by Proposition 6.1, so are,

$$U^1_\epsilon(x,t) = \left(1+\epsilon\sqrt{A}e^{\sqrt{A}t}\right)^{\frac{B}{2\sqrt{A}}}$$
$$\times\exp\left\{\frac{-\epsilon A e^{\sqrt{A}t}x}{2(1+\epsilon\sqrt{A}e^{\sqrt{A}t})} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{1+\epsilon\sqrt{A}e^{\sqrt{A}}}\right)\right)\right\}, \tag{6.3}$$

and

$$U_\epsilon^2(x,t) = e^{-\frac{B}{2}t}\left(e^{\sqrt{A}t} - \epsilon\sqrt{A}\right)^{\frac{B}{2\sqrt{A}}} \times \exp\left\{\frac{-\epsilon A x}{2(e^{\sqrt{A}t} - \sqrt{A}\epsilon)} - \frac{1}{2}\left(F(x) - F\left(\frac{x e^{\sqrt{A}t}}{e^{\sqrt{A}t} - \epsilon\sqrt{A}}\right)\right)\right\}. \tag{6.4}$$

Neither of these two solutions can be immediately identified with the characteristic function of (1.1). However it is often possible to derive the fundamental solution from them. We illustrate the method with examples.

Example 6.1. Consider the PDE

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \left(\frac{3}{2} - x\right)\frac{\partial u}{\partial x} \tag{6.5}$$

Applying equation (6.4), and setting $\epsilon = \lambda/(1+\lambda)$, we see that

$$\tilde{U}_\lambda(x,t) = \left(\frac{(1+\lambda)e^t}{(1+\lambda)e^t - \lambda}\right)^{\frac{3}{2}} \exp\left\{\frac{-\lambda x}{(1+\lambda)e^t - \lambda}\right\} \tag{6.6}$$

is a solution of (6.5). Next, we use the fact that multiplication of solutions of (1.1) yields a new solution. We multiply $\tilde{U}_\lambda$ by $1/(1+\lambda)^{\frac{3}{2}}$ to obtain

$$U_\lambda(x,t) = \left(\frac{e^t}{(1+\lambda)e^t - \lambda}\right)^{\frac{3}{2}} \exp\left\{\frac{-\lambda x}{(1+\lambda)e^t - \lambda}\right\}. \tag{6.7}$$

which is well known to be the characteristic function for (6.5). See for example Revuz and Yor, [RY98]. Inverting the Laplace transform, we obtain the fundamental solution for (6.5). It is

$$p(t,x,y) = \left(\frac{e^t}{e^t - 1}\right)^{\frac{3}{2}} \exp\left\{-\frac{(x+y)}{e^t - 1}\right\} \mathcal{I}_{\frac{1}{2}}\left(\frac{2\sqrt{xye^t}}{e^t - 1}\right) \tag{6.8}$$

where

$$\mathcal{I}_{\nu-\frac{1}{2}}(z) = 2^{\nu-\frac{1}{2}}\Gamma(\nu + \frac{1}{2})z^{-\nu+\frac{1}{2}} I_{\nu-\frac{1}{2}}(z) \tag{6.9}$$

Example 6.2. Consider the drift function $f(x) = x\coth\left(\frac{x}{2}\right)$. Here

$$xf' - f + \frac{1}{2}f^2 = \frac{1}{2}x^2.$$

By (6.4) of Proposition 6.1, the equation

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + x\coth\left(\frac{x}{2}\right)\frac{\partial u}{\partial x} \tag{6.10}$$

has a solution

$$u_\epsilon(x,t) = \frac{\sinh\left(\frac{xe^t}{2(e^t-\epsilon)}\right)}{\sinh\left(\frac{x}{2}\right)} \exp\left\{\frac{-\epsilon x}{2(e^t-\epsilon)}\right\}. \tag{6.11}$$

From this we can derive the fundamental solution $p(t,x,y)$ of (6.10). Observe that

$$u_\epsilon(x,0) = \frac{1}{2}\left(\frac{e^{\frac{x}{2}}}{\sinh\left(\frac{x}{2}\right)} - \frac{1}{\sinh\left(\frac{x}{2}\right)}\exp\left\{\frac{-(1+\epsilon)x}{2(1-\epsilon)}\right\}\right). \tag{6.12}$$

Furthermore, we note that $g(x) = \frac{e^{\frac{x}{2}}}{\sinh\left(\frac{x}{2}\right)}$ is a stationary solution of the equation (6.10). We therefore look for a fundamental solution $p(t,x,y)$ with the property that

$$\int_0^\infty \frac{e^{\frac{y}{2}}}{\sinh\left(\frac{y}{2}\right)} p(t,x,y)dy = \frac{e^{\frac{x}{2}}}{\sinh\left(\frac{x}{2}\right)}. \tag{6.13}$$

We introduce the new parameter $\lambda = \frac{1+\epsilon}{2(1-\epsilon)}$. The solution u_ϵ becomes

$$u_\lambda(x,t) = \frac{\sinh\left(\frac{(2\lambda+1)xe^t}{2((2\lambda+1)e^t-(2\lambda-1))}\right)}{\sinh\left(\frac{x}{2}\right)} \exp\left\{\frac{-(2\lambda-1)x}{2((2\lambda+1)e^t-(2\lambda-1))}\right\}. \tag{6.14}$$

By (6.13), we may write (6.14) as

$$\begin{aligned} u_\lambda(x,t) &= \frac{1}{2}\int_0^\infty \left(\frac{e^{\frac{y}{2}}}{\sinh(\frac{y}{2})} - \frac{1}{\sinh(\frac{y}{2})}e^{-\lambda y}\right) p(t,x,y)dy \\ &= \frac{e^{\frac{x}{2}}}{2\sinh(\frac{x}{2})} - \frac{1}{2}\mathcal{L}\left(\frac{1}{\sinh\left(\frac{y}{2}\right)}p(t,x,y)\right), \end{aligned} \tag{6.15}$$

where $\mathcal{L}$ denotes Laplace transform as before.

Rearranging equation (6.15), we obtain the fundamental solution

$$\begin{aligned} p(t,x,y) &= \frac{\sinh\left(\frac{y}{2}\right)}{\sinh\left(\frac{x}{2}\right)}\mathcal{L}^{-1}\left(\exp\left\{\frac{-(2\lambda(1+e^t)+e^t-1)x}{2((2\lambda+1)e^t-(2\lambda-1))}\right\}\right) \\ &= \frac{\sinh(\frac{y}{2})}{\sinh(\frac{x}{2})}\exp\left\{-\frac{(e^t+1)(x+y)}{2(e^t-1)}\right\}\left[\frac{e^{\frac{1}{2}t}}{e^t-1}\sqrt{\frac{x}{y}}I_1\left(\frac{2\sqrt{xye^t}}{e^t-1}\right)+\delta(y)\right], \end{aligned} \tag{6.16}$$

where I_1 is a modified Bessel function of the first kind of order 1, and δ is the Dirac delta function. The reader may check that the same fundamental solution is obtained if we start with the solution arising from equation (6.3) in Proposition 6.1, as indeed we should.

Obtaining the fundamental solution in these examples is more involved than was the case for Theorem 5.1. For examples coming from (1.3), the procedure which yield the characteristic equation differs from case to case. This makes the formulation of a theorem equivalent to

Theorem 5.1 difficult. However, it should also be clear that applying Proposition 6.1 often allows us to derive the fundamental solution for equation (1.1), when the drift satisfies (1.3). We will consider this case in more detail in a subsequent paper.

7. The Ricatti Equation $xf' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Bx^2 + Cx - \frac{3}{8}$

The last case which we must consider is when the drift function f is a solution of the third Ricatti equation (1.4). There are two subcases here. $B = 0$, and $B \neq 0$, In the case $B = 0$, we can obtain the characteristic function by symmetry directly as we did in Theorem 5.1. Recall that when f is a solution of (1.4), and $B = 0$, the PDE (1.1) has an infinitesimal symmetry of the form

$$\mathbf{v}_6 = \left(8xt + \frac{2A}{3}\sqrt{x}t^3\right)\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \left(4x + 2Ct^2 + 4f(x)t + \frac{A^2}{36}t^4 + 2A\sqrt{x}t^2 - \frac{A(\frac{1}{2} - f(x))}{3\sqrt{x}}t^3\right)u\frac{\partial}{\partial u}. \tag{7.1}$$

The group action generated by this symmetry allows us to determine the characteristic function for (1.1). We have the following result.

Theorem 7.1. *Let f be a solution of the Ricatti equation*

$$xf' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Cx - \frac{3}{8}. \tag{7.2}$$

Then the characteristic function $U_\lambda(x,t)$ for the corresponding PDE (1.1) is given by

$$U_\lambda(x,t) = \sqrt{\frac{\sqrt{x}(1+\lambda t)}{\sqrt{x}(1+\lambda t) - \frac{A\lambda}{12}t^3}} \times \exp\left\{\frac{1}{2}\left(F(x) - F\left(\left(\frac{\sqrt{x}}{1+\lambda t} - \frac{A\lambda t^4}{12(1+\lambda t)^2}\right)^2\right)\right)\right\} \times \exp\left\{-\frac{\lambda(x+\frac{1}{2}Ct^2)}{1+\lambda t} - \frac{\frac{2}{3}At^2\sqrt{x}(3+\lambda t)}{(1+\lambda t)^2} + \frac{A^2t^4(2\lambda t(3+\frac{1}{2}\lambda t) - 3)}{108(1+\lambda t)^3}\right\}, \tag{7.3}$$

where $F'(x) = f(x)/x$

Proof. The idea of the proof is the same as for Theorem 5.1. First we observe that $U_\lambda(x,0) = e^{-\lambda x}$. In order to show that (7.3) is the characteristic function we exponentiate the infinitesimal symmetry (7.1). The only difficulty here is solving the equation

$$\frac{d\tilde{x}}{d\epsilon} = 8\tilde{x}\tilde{t} + \frac{2A}{3}\sqrt{\tilde{x}}\tilde{t}^3. \tag{7.4}$$

This is facilitated by making the change of variables $\sqrt{\tilde{x}} = y$. Under this change of variables, equation (7.4) becomes first order linear. This leads to

$$\sqrt{\tilde{x}} = \frac{\sqrt{x}}{1-4\epsilon t} + \frac{A\epsilon t^3}{3(1-4\epsilon t)^2}. \tag{7.5}$$

The rest of the calculation is straightforward. We see by symmetry, that if u is a solution of (1.1), then so is

$$U_\epsilon(x,t) = \sqrt{\frac{\sqrt{x}(1+4\epsilon t)}{\sqrt{x}(1+4\epsilon t) - \frac{A\epsilon}{3}t^3}}\, u\left(\left(\frac{\sqrt{x}}{1+4\epsilon t} - \frac{A\epsilon t^3}{3(1+4\epsilon t)^2}\right)^2, \frac{t}{1+4\epsilon t}\right)$$
$$\times \exp\left\{\frac{1}{2}\left(F(x) - F\left(\left(\frac{\sqrt{x}}{1+4\epsilon t} - \frac{A\epsilon t^4}{3(1+4\epsilon t)^2}\right)^2\right)\right)\right\}$$
$$\times \exp\left\{-\frac{4\epsilon(x+\frac{1}{2}Ct^2)}{1+4\epsilon t} - \frac{\frac{2}{3}At^2\sqrt{x}(3+4\epsilon t)}{(1+4\epsilon t)^2} + \frac{A^2t^4(8\epsilon t(3+2\epsilon t)-3)}{108(1+4\epsilon t)^3}\right\}. \tag{7.6}$$

Taking $u = 1$, and setting $\lambda = 4\epsilon$ gives the result. □

It should be clear that if we take $A = 0$, in (7.3), then it reduces to equation (5.8). In order to apply Theorem 7.1 we need solutions of (1.4). As per the comments following Theorem 5.1, we can transform (1.4) to the linear equation,

$$2x^2y''(x) - (Ax^{3/2} + Cx - \frac{3}{8})y(x) = 0. \tag{7.7}$$

The general solution of (7.7), is easily found to be

$$y(x) = x^{\frac{1}{4}}\left(a_1 Ai\left(3^{\frac{2}{3}}\frac{(2C+\frac{4A}{3}\sqrt{x})}{(2^4A^2)^{\frac{1}{3}}}\right) + a_2 Bi\left(3^{\frac{2}{3}}\frac{(2C+\frac{4A}{3}\sqrt{x})}{(2^4A^2)^{\frac{1}{3}}}\right)\right), \tag{7.8}$$

where Ai and Bi are the first and second kind Airy functions, and a_1 and a_2 are arbitrary constants. Setting $f = 2xy/y'$ gives solutions of (1.4).

Taking $A = \frac{4}{3}, C = 0, a_1 = 1$ and $a_2 = 0$, gives the solution

$$f(x) = \frac{1}{2} + \frac{\sqrt{x}Ai'(\sqrt{x})}{Ai(\sqrt{x})}. \tag{7.9}$$

Since

$$F(x) = \int \frac{f(x)}{x}dx = \frac{1}{2}\left(\ln(x) + 4\ln\left(Ai(\sqrt{x})\right)\right), \tag{7.10}$$

an application of Theorem 7.1 allows us to determine the characteristic function for

$$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \left(\frac{1}{2} + \frac{\sqrt{x}Ai'(\sqrt{x})}{Ai(\sqrt{x})}\right)\frac{\partial u}{\partial x}. \tag{7.11}$$

However at this stage we are unable to invert the Laplace transform. It should however be possible to invert the transform numerically. See the paper by Craddock, Heath and Platen, [CHP00] on the numerical inversion of Laplace transforms and the references therein.

The last case we have to consider is the case when the drift is a solution of (1.4) and $B \neq 0$. Recall that when f was a solution of (1.4), for $B \neq 0$, then (1.1) has two infinitesimal symmetries of the form

$$\begin{aligned}
\mathbf{v}_5 &= \left(\frac{2A}{3\sqrt{B}}\sqrt{x} + \sqrt{B}x\right) e^{\sqrt{B}t}\frac{\partial}{\partial x} + e^{\sqrt{B}t}\frac{\partial}{\partial t} \\
&\quad - \left(\frac{B}{2}x + \frac{2A}{3}\sqrt{x} + \frac{\sqrt{B}}{2}f(x) - \frac{A(\frac{1}{2} - f(x))}{3\sqrt{B}\sqrt{x}} + \alpha\right) e^{\sqrt{B}t}u\frac{\partial}{\partial u}, \\
\mathbf{v}_6 &= -\left(\frac{2A}{3\sqrt{B}}\sqrt{x} + \sqrt{B}x\right) e^{-\sqrt{B}t}\frac{\partial}{\partial x} + e^{-\sqrt{B}t}\frac{\partial}{\partial t} \\
&\quad - \left(\frac{B}{2}x + \frac{2A}{3}\sqrt{x} - \frac{\sqrt{B}}{2}f(x) + \frac{A(\frac{1}{2} - f(x))}{3\sqrt{B}\sqrt{x}} + \alpha\right) e^{-\sqrt{B}t}u\frac{\partial}{\partial u}
\end{aligned}$$

Where $\alpha = \frac{2A^2+9BC}{18B}$. At present we are unable to determine any characteristic functions for (1.1) because we have not yet found any explicit solutions of (1.4) for $B \neq 0$. Nevertheless, for completeness, we present the group symmetries which are generated by $\mathbf{v}_5$ and $\mathbf{v}_6$.

Proposition 7.2. *Let f be a solution of (1.4) and u be a solution of (1.1). Then, for ϵ sufficiently small, the following functions are also solutions of (1.1).*

$$\begin{aligned}
&\rho(\exp(\epsilon\mathbf{v}_5))u(x,t) \\
&= \left(1 + \epsilon\sqrt{B}e^{\sqrt{B}t}\right)^{\frac{2A^2+9B\alpha}{9B^{3/2}}} \sqrt{\frac{3B\sqrt{x} + 2A(1 - \sqrt{1 + \epsilon\sqrt{B}e^{\sqrt{B}t}})}{3B\sqrt{x}}} \\
&\times \exp\left\{-\frac{\epsilon e^{\sqrt{B}t}(2A + 3B\sqrt{x})^2}{18B(1 + \epsilon\sqrt{B}e^{\sqrt{B}t})}\right\} \\
&\times \exp\left\{-\frac{1}{2}\left(F(x) - F\left(\left(\frac{\sqrt{x}}{1 + \epsilon\sqrt{B}e^{\sqrt{B}t}} - D\right)^2\right)\right)\right\} \\
&\times u\left(\left(\frac{\sqrt{x}}{1 + \epsilon\sqrt{B}e^{\sqrt{B}t}} - D\right)^2, \frac{1}{\sqrt{B}}\ln\left(\frac{e^{\sqrt{B}t}}{1 + \epsilon\sqrt{B}e^{\sqrt{B}t}}\right)\right) \qquad (7.12)
\end{aligned}$$

$$
\begin{aligned}
&\rho(\exp(\epsilon \mathbf{v}_6))u(x,t)\\
&= \left(e^{\sqrt{B}t} - \epsilon\sqrt{B}\right)^{\frac{9B\alpha - 2A^2}{9B^{3/2}}} \sqrt{\frac{3Be^{\frac{\sqrt{B}}{2}t} + 2A(e^{\frac{\sqrt{B}}{2}t} - \sqrt{e^{\sqrt{B}t} - \epsilon\sqrt{B}})}{3B\sqrt{x}e^{\frac{\sqrt{B}}{2}t}}}\\
&\times \exp\left\{-\frac{\epsilon(2A + 3B\sqrt{x})^2}{18B(e^{\sqrt{B}t} - \epsilon\sqrt{B})}\right\}\\
&\times \exp\left\{-\frac{1}{2}\left(F(x) - F\left(\left(\frac{(\sqrt{x} + \frac{2A}{3B})e^{\frac{\sqrt{B}2}{t}}}{\sqrt{e^{\sqrt{B}t} - \epsilon\sqrt{B}}} - D\right)^2\right)\right)\right\}\\
&\times u\left(\left(\frac{(\sqrt{x} + \frac{2A}{3B})e^{\frac{\sqrt{B}2}{t}}}{\sqrt{e^{\sqrt{B}t} - \epsilon\sqrt{B}}} - D\right)^2, \frac{\ln(e^{\sqrt{B}t} - \epsilon\sqrt{B})}{\sqrt{B}}\right), \qquad (7.13)
\end{aligned}
$$

where $F'(x) = f(x)/x$, $D = \frac{2A}{3B}$ *and* $\alpha = \frac{1}{9}A^2 + \frac{1}{2}BC$.

Proof. As for our previous cases, the proof simply involves solving the system of ODEs (2.13) □

Taking $u = 1$ immediately allows us to write down solutions of (1.1) for any f that is a solution of (1.4). Our experience with the previous cases strongly suggests that if we can obtain solutions to (1.4) with $B \neq 0$, then we would be able to determine the corresponding characteristic function and fundamental solutions. We also note that the solutions of (1.1) obtained by letting $u = 1$ in equations (7.12) and (7.13), are closely related to the fundamental solution. In a forthcoming paper we will show how to extend these solutions to directly obtain the fundamental solution.

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