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Symmetry group methods for fundamental solutions

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Abstract

This paper uses Lie symmetry group methods to study PDEs of the form $u_t = xu_{xx} + f(x)u_x$. We show that when the drift function f is a solution of a family of Ricatti equations, then symmetry techniques can be used to find a fundamental solution.

Keywords: Lie symmetry groups; Fundamental solutions; Characteristic solutions; Transition densities

1. Introduction

The purpose of this paper is to use symmetry group methods to compute fundamental solutions for a class of partial differential equations (PDEs), of the form,

$$u_t = xu_{xx} + f(x)u_x, \quad x \geqslant 0. \tag{1.1}$$

By a fundamental solution, we mean a kernel function $p(t, x, y)$ such that $u(x, t) = \int_0^\infty \varphi(y)p(t, x, y)\,dy$ is a solution of the Cauchy problem for (1.1) with $u(x, 0) = \varphi(x)$.

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We will show that such a fundamental solution can often be obtained by Lie symmetry group methods when the drift function f is a solution of one of the following three families of Ricatti equations:

$$xf' - f + \tfrac{1}{2}f^2 = Ax + B, \tag{1.2}$$

$$xf' - f + \tfrac{1}{2}f^2 = Ax^2 + Bx + C, \tag{1.3}$$

$$xf' - f + \tfrac{1}{2}f^2 = Ax^{\frac{3}{2}} + Bx^2 + Cx - \tfrac{3}{8}, \tag{1.4}$$

where A, B and C are arbitrary constants.

The problem of computing fundamental solutions for PDEs of the form (1.1), arises in the study of one-dimensional *generalized square root* or GSR processes. Let $X = \{X_t, t \in [0, T]\}$ satisfy the Itô stochastic differential equation (SDE),

$$dX_t = f(X_t)\,dt + \sqrt{2X_t}\,dW_t \tag{1.5}$$

for $t \in [0, T]$. Here W is a standard Wiener process, and f is a drift function. It is well known that the transition density, $p(t, x, y)$ for X, is given by the fundamental solution of the PDE (1.1). See for example Revuz and Yor [16].

GSR processes have applications in finance and other areas. Certain GSR processes exhibit a property known as mean reversion, making them ideal for modelling interest rates. For example, the GSR process with drift equal to $f(x) = a - bx$ is used by Cox, Ingersoll and Ross (CIR) to model bond prices. (See the book [8]). Longstaff by contrast, models bond prices with a GSR process in which the drift $f(x) = a - b\sqrt{x}$. (See [12] for Longstaff's analysis and a comparison with the CIR model). The transition densities for both the CIR model and the Longstaff model can be obtained by our methods.

Another application of GSR processes is to the modelling of inflation rates. Typically a central bank would like to keep inflation within a certain band, say (α, β). Such inflation rates can be modelled by GSR processes in which the drift has discontinuities at $x = \alpha, x = \beta$. An instance of such a process is given by our Example 4.8. GSR processes also play a fundamental role in the *minimum market model* of Platen, [15], which models equity and currency markets. In our Example 4.2, we present a simple application of our methods to the pricing of an option on a commodity.

2. Symmetry methods and fundamental solutions

Symmetry group methods provide a natural approach to the problem of finding fundamental solutions of PDEs. The book by Olver [14] gives an excellent modern account of Lie's theory of symmetry groups. See also the books by Miller [13] and Bluman and Kumei [1] and the papers [2–4]. Lie himself considered symmetries of higher order PDEs in [9,10]. The book [11] is based on Lie's papers.

For illustration, consider the one-dimensional heat equation. Lie showed that if $u(x,t)$ is a solution of the equation $u_{xx} = u_t$, then so is

$$\tilde{u}_\varepsilon(x,t) = \frac{1}{\sqrt{1+4\varepsilon t}} \exp\left\{\frac{-\varepsilon x^2}{1+4\varepsilon t}\right\} u\left(\frac{x}{1+4\varepsilon t}, \frac{t}{1+4\varepsilon t}\right) \tag{2.1}$$

for ε sufficiently small (see for example [14]). In (2.1), let $u = 1$, $t \to t - \frac{1}{4}\varepsilon$, and set $\varepsilon = \pi$. In this way, we obtain the fundamental solution of the heat equation, $k(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$, from the constant solution, $u = 1$, by simple group transformation.

It is natural to ask whether we can obtain fundamental solutions for other PDEs by similar means? In a recent paper, Craddock and Dooley [5] showed that for the heat equation on a nilpotent Lie group, there is always a symmetry which maps the constant solution to the fundamental solution. They also studied the equation

$$u_t = u_{xx} + f(x)u_x. \tag{2.2}$$

The fundamental solution of (2.2) can be obtained from the solution $u = 1$ by a symmetry transformation, whenever the drift function f is a solution of any one of five families of Ricatti equations. This immediately leads to a rich class of PDEs, whose fundamental solutions can be explicitly computed. It also motivates the remainder of this paper.

In the current work, our approach is to obtain an integral transform of the fundamental solution by a symmetry transformation. In Section 4, we show that if the drift function f satisfies (1.2), then we can obtain the solution

$$U_\lambda(x,t) = \int_0^\infty e^{-\lambda y} p(t,x,y)\, dy, \tag{2.3}$$

of (1.1) from the trivial solution $u = 1$ by symmetry. This is the Laplace transform of $p(t,x,y)$. So the fundamental solution can then be recovered by taking the inverse Laplace transform of U_λ. We shall call U_λ a *characteristic solution* for (1.1).

In Section 5, we treat the case when f is a solution of (1.3). Here, we do not immediately obtain a characteristic solution. Rather, we obtain an integral transform of the fundamental solution which is more complicated. However, this typically reduces to a Laplace transform. We will present some illustrative examples.

In Section 6, we treat the case when f satisfies (1.4). For the subcase, when $B = 0$, we are again able to derive a characteristic solution from a trivial solution by a symmetry transformation. If f satisfies (1.4) with $B \neq 0$, we can also obtain an integral transform of $p(t,x,y)$.

Our techniques lead to a rich class of PDEs with explicitly computable fundamental solutions. Many of the fundamental solutions that our methods give appear to be new. We also recover all the well-known examples, such as when the drift function f is affine.

The method that we describe in this paper is one of two symmetry-based techniques which allow us to construct fundamental solutions for these partial differential equations. In a subsequent paper, we shall describe a second approach. This method yields entirely new fundamental solutions.

3. The infinitesimal symmetries

In this section, we present a complete list of *all* possible Lie symmetry algebras for PDEs of the form (1.1). Recall that if the vector field

$$\mathbf{v} = \xi(x,t,u)\,\frac{\partial}{\partial x} + \tau(x,t,u)\,\frac{\partial}{\partial t} + \phi(x,t,u)\,\frac{\partial}{\partial u}, \tag{3.1}$$

generates a symmetry of (1.1), then $\mathbf{v}$ must satisfy Lie's condition:

$$\mathrm{pr}^2\,\mathbf{v}[u_t - xu_{xx} - f(x)u_x] = 0, \text{ whenever } u_t = xu_{xx} + f(x)u_x.$$

Here $\mathrm{pr}^2\,\mathbf{v}$ denotes the second prolongation of $\mathbf{v}$. If $\mathbf{v}$ generates a symmetry of (1.1), then standard symmetry group calculations lead to the following conditions on the coefficients ξ, τ and ϕ. The full details are given in [6]. Let $\beta(x,t)$ be an arbitrary solution of (1.1). Then

$$\begin{aligned}\xi &= x\tau_t + \sqrt{x}\rho(t), \quad \phi(x,t,u) = \alpha(x,t)u + \beta(x,t),\\ \alpha &= -\frac{1}{2}x\tau_{tt} - \sqrt{x}\rho_t + \frac{1}{2\sqrt{x}}\left(\frac{1}{2} - f(x)\right)\rho - \frac{1}{2}f(x)\tau_t + \sigma(t)\end{aligned}$$

for some functions ρ and σ. Now the function τ depends only on t. Set $\mathfrak{L}f = xf' - f + \frac{1}{2}f^2$. Then

$$-\frac{1}{2}x\tau_{ttt} - \sqrt{x}\rho_{tt} + \sigma_t = -\frac{1}{2}\frac{d}{dx}\left(\mathfrak{L}f\right)\tau_t + \left[\frac{3 + 8\mathfrak{L}f - 8x\,\frac{d}{dx}(\mathfrak{L}f)}{16x^{\frac{3}{2}}}\right]\rho.$$

These equations fix ξ, τ, ρ, σ and ϕ for every C^2 drift function *f*.

We now list the infinitesimal point symmetries of (1.1). In the following, we set $g(x) = \frac{1}{2\sqrt{x}}\left(\frac{1}{2} - f(x)\right)$. As usual, we write ∂_u for $\frac{\partial}{\partial u}$ etc.

Case 1: Let $xf' - f + \frac{1}{2}f^2 = Ax + B$, where A and B are constants.

Subcase 1a: If $3 + 8B = 0$, then a basis for the Lie algebra is:

$$\begin{aligned}\mathbf{v}_\beta &= \beta(x,t)\partial_u, \quad \mathbf{v}_1 = \sqrt{x}\partial_x - g(x)u\partial_u, \quad \mathbf{v}_2 = \partial_t, \quad \mathbf{v}_3 = u\partial_u,\\ \mathbf{v}_4 &= 2x\partial_x + 2t\partial_t - (f(x) + At)u\partial_u, \quad \mathbf{v}_5 = \sqrt{x}t\partial_x - (\sqrt{x} - g(x)t)u\partial_u,\\ \mathbf{v}_6 &= 8xt\partial_x + 4t^2\partial_t - (4x + 4f(x)t + 2At^2)u\partial_u.\end{aligned}$$

Subcase 1b: If $3+8B \neq 0$, then the Lie algebra is spanned by the vector fields $\{\mathbf{v}_\beta, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_6\}$, from Subcase 1a.

Case 2: Let $xf' - f + \frac{1}{2}f^2 = \frac{1}{2}Ax^2 + Bx + C$, where A, B and C are constants. The extra $\frac{1}{2}$ in the coefficient of x^2 is convenient here.

Subcase 2a: If $A > 0$, and $3+8C = 0$, a basis for the Lie algebra is:

$$\mathbf{v}_\beta = \beta(x,t)\partial_u \ \mathbf{v}_1 = \sqrt{x}e^{\frac{1}{2}\sqrt{A}t}\partial_x - \frac{1}{2}\left(\sqrt{Ax} - 2g(x)\right)e^{\frac{1}{2}\sqrt{A}t}u\partial_u, \ \mathbf{v}_2 = \partial_t,$$

$$\mathbf{v}_3 = u\partial_u, \mathbf{v}_4 = x\sqrt{A}e^{\sqrt{A}t}\partial_x + e^{\sqrt{A}t}\partial_t - \frac{1}{2}(Ax + \sqrt{A}f(x) + B)e^{\sqrt{A}t}u\partial_u,$$

$$\mathbf{v}_5 = \sqrt{x}e^{-\frac{1}{2}\sqrt{A}t}\partial_x + \frac{1}{2}\left(\sqrt{Ax} + 2g(x)\right)e^{-\frac{1}{2}\sqrt{A}t}u\partial_u,$$

$$\mathbf{v}_6 = -x\sqrt{A}e^{-\sqrt{A}t}\partial_x + e^{-\sqrt{A}t}\partial_t - \frac{1}{2}(Ax - \sqrt{A}f(x) + B)e^{-\sqrt{A}t}u\partial_u.$$

Subcase 2b: If $3+8C \neq 0$, then the Lie algebra is spanned by the vector fields $\{\mathbf{v}_\beta, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_6\}$, from the list in Subcase 2a.

Subcase 2c: If $A < 0$, then a basis for the Lie algebra of symmetries may be obtained from the real and imaginary parts of the vector fields in Subcases 2a and 2b.

Case 3: Let $xf' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Bx^2 + Cx + D$. If $3+8D \neq 0$, then the Lie algebra of symmetries is spanned by $\mathbf{v}_\beta = \beta\partial_u$, $\mathbf{v}_1 = \partial_t$, and $\mathbf{v}_2 = u\partial_u$. If $3+8D = 0$, there are three subcases.

Subcase 3a: If $B = 0$, then a basis for the Lie algebra consists of $\mathbf{v}_\beta = \beta(x,t)\partial_u$ and

$$\mathbf{v}_1 = \sqrt{x}\partial_x - \left(\frac{At}{6} - g(x)\right)u\partial_u, \ \mathbf{v}_2 = \partial_t, \ \mathbf{v}_4 = \left(2x + \frac{A}{2}\sqrt{x}t^2\right)\partial_x$$

$$+2t\partial_t - \left((C + A\sqrt{x})t + \frac{A^2}{36}t^3 - \frac{A}{2}g(x)t^2 + f(x)\right)u\partial_u, \quad \mathbf{v}_3 = u\partial_u,$$

$$V_5 = t\sqrt{x}\partial_x - \left(\frac{A}{12}t^2 + \sqrt{x} - g(x)t\right)u\partial_u, \quad \mathbf{v}_6 = \left(8xt + \frac{2A}{3}\sqrt{x}t^3\right)\partial_x$$

$$+4t^2\partial_t - \left(4x + 2Ct^2 + 4f(x)t + \frac{A^2}{36}t^4 + 2A\sqrt{x}t^2 - \frac{2}{3}Ag(x)t^3\right)u\partial_u.$$

Subcase 3b: If $B > 0$, set $\gamma = \frac{2A}{3\sqrt{B}}$ and $\kappa = \frac{2A^2+9BC}{18B}$. Then a basis consists of $\mathbf{v}_\beta = \beta(x,t)\partial_u$ and

$$\mathbf{v}_1 = \sqrt{x}e^{\frac{1}{2}\sqrt{B}t}\partial_x - \left(\frac{1}{2}\sqrt{B}\sqrt{x} - g(x) + \frac{\gamma}{2}\right)e^{\frac{1}{2}\sqrt{B}t}u\partial_u,$$

$$\mathbf{v}_2 = \sqrt{x}e^{-\frac{1}{2}\sqrt{B}t}\partial_x + \left(\frac{1}{2}\sqrt{B}\sqrt{x} + g(x) + \frac{\gamma}{2}\right)e^{-\frac{1}{2}\sqrt{B}t}u\partial_u,$$

$$\mathbf{v}_3 = u\partial_u, \mathbf{v}_4 = \partial_t, \mathbf{v}_5 = \left(\gamma\sqrt{x} + \sqrt{B}x\right)e^{\sqrt{B}t}\partial_x + e^{\sqrt{B}t}\partial_t$$
$$- \left(\frac{B}{2}x + \frac{2A}{3}\sqrt{x} + \frac{\sqrt{B}}{2}f(x) - \frac{\gamma}{2}g(x) + \kappa\right)e^{\sqrt{B}t}u\partial_u,$$

$$\mathbf{v}_6 = -\left(\gamma\sqrt{x} + \sqrt{B}x\right)e^{-\sqrt{B}t}\partial_x + e^{-\sqrt{B}t}\partial_t$$
$$- \left(\frac{B}{2}x + \frac{2A}{3}\sqrt{x} - \frac{\sqrt{B}}{2}f(x) + \frac{1}{2}\gamma g(x) + \kappa\right)e^{-\sqrt{B}t}u\partial_u.$$

Subcase 3c: If $B < 0$, then a basis may be obtained from the real and imaginary parts of the vector fields in Subcase 3b.

Case 4: If f does not satisfy any of the Ricatti equations of Cases 1–3, the symmetry algebra has basis: $\mathbf{v}_\beta = \beta\partial_u$, $\mathbf{v}_1 = \partial_t$ and $\mathbf{v}_2 = u\partial_u$.

4. Fundamental solutions and characteristic solutions

We will now exploit the symmetries obtained in Section 3 to compute characteristic solutions and fundamental solutions for PDEs of the form (1.1). We will first consider the case where the drift function f is a solution of the Ricatti equation (1.2).

The key observation is that the characteristic solution is a solution of (1.1), with the initial condition $u(x, 0) = e^{-\lambda x}$. By symmetry, we can obtain a solution with this initial data from a solution with initial data $u(x, 0) = 1$. We illustrate by an example.

Example 4.1. Let $f(x) = \alpha$, $\alpha > 0$. Consider the PDE

$$u_t = xu_{xx} + \alpha u_x. \tag{4.1}$$

From Case 1 of Section 4 (4.1) has an infinitesimal symmetry

$$\mathbf{v}_6 = 8xt\partial_x + 4t^2\partial_t - (4x + 4\alpha t)u\partial_u.$$

Recall that the group action generated by a vector field of the form (3.1), is obtained by solving the first-order system of ODEs,

$$\tilde{x}'(\varepsilon) = \xi(\tilde{x}, \tilde{t}, \tilde{u}), \quad \tilde{t}'(\varepsilon) = \tau(\tilde{u}, \tilde{t}, \tilde{u}), \quad \tilde{u}'(\varepsilon) = \phi(\tilde{x}, \tilde{t}, \tilde{u}) \tag{4.2}$$

subject to the initial conditions $\tilde{x}(0) = x$, $\tilde{t}(0) = t$ and $\tilde{u}(0) = u$. From this we obtain the new solution, $\tilde{u}_\varepsilon(x, t) = \tilde{u}(\tilde{x}, \tilde{t})$.

Solving these equations for $\mathbf{v}_6$, gives

$$\tilde{u}_\varepsilon(x,t) = \frac{1}{(1+4\varepsilon t)^\alpha} \exp\left\{\frac{-4\varepsilon x}{1+4\varepsilon t}\right\} u\left(\frac{x}{(1+4\varepsilon t)^2}, \frac{t}{1+4\varepsilon t}\right). \tag{4.3}$$

Thus if u is any solution of (4.1), then (4.3) is also a solution, at least for ε sufficiently small. We set $\lambda = 4\varepsilon$, and take $u = 1$. By symmetry,

$$U_\lambda(x,t) = \frac{1}{(1+\lambda t)^\alpha} \exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}, \tag{4.4}$$

is also a solution of (4.1). This is known to be a characteristic solution of (4.1). To invert it, let $\mathcal{L}$ denote Laplace transformation in λ. Then

$$\mathcal{L}^{-1}\left(\frac{1}{\lambda^\mu} e^{\frac{k}{\lambda}}\right) = \left(\frac{y}{k}\right)^{\frac{\mu-1}{2}} I_{\mu-1}(2\sqrt{ky}), \quad \mu > 0, \tag{4.5}$$

where I_ν is a modified Bessel function of the first kind with order ν.

Elementary properties of Laplace transforms and (4.5) now give,

$$\begin{aligned} p(t,x,y) &= \mathcal{L}^{-1}\left(\frac{1}{(1+\lambda t)^\alpha} \exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}\right) \\ &= \frac{1}{t}\left(\frac{x}{y}\right)^{\frac{1-\alpha}{2}} I_{\alpha-1}\left(\frac{2\sqrt{xy}}{t}\right) \exp\left\{-\frac{(x+y)}{t}\right\}. \end{aligned} \tag{4.6}$$

This is the fundamental solution of (4.1) given in [16].

This example shows that it is sometimes possible to obtain a characteristic solution of (1.1) by a symmetry transformation. In fact, we can easily establish the following theorem.

Theorem 4.1. *Let f be a solution of the Ricatti equation*

$$xf' - f + \tfrac{1}{2} f^2 = Ax + B. \tag{4.7}$$

Let

$$U_\lambda(x,t) = \exp\left\{-\frac{\lambda(x + \frac{1}{2}At^2)}{1+\lambda t} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{(1+\lambda t)^2}\right)\right)\right\}, \tag{4.8}$$

where $F'(x) = f(x)/x$ *and* $\lambda \geqslant 0$. *Then* U_λ *is a characteristic solution of* (1.1). *That is,* U_λ *is the Laplace transform of the fundamental solution* $p(t, x, y)$ *of* (1.1).

Proof. Clearly $U_\lambda(x, 0) = e^{-\lambda x}$. Since $xf' - f + \frac{1}{2}f^2 = Ax + B$, then, from Case 1 of Section 3, Eq. (1.1) has an infinitesimal symmetry of the form $\mathbf{v}_6 = 8xt\partial_x + 4t^2\partial_t - \left(4x + 4f(x)t + 2At^2\right)u\partial_u$.

Exponentiating $\mathbf{v}_6$, we see that if u is a solution of (1.1), then so is

$$\tilde{u}_\varepsilon(x, t) = \exp\left\{-\frac{(4\varepsilon x + 2A\varepsilon t^2)}{1 + 4\varepsilon t} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{(1 + 4\varepsilon t)^2}\right)\right)\right\} \times u\left(\frac{x}{(1 + 4\varepsilon t)^2}, \frac{t}{1 + 4\varepsilon t}\right),$$

where $F'(x) = f(x)/x$. Taking $u = 1$, and setting $\lambda = 4\varepsilon$, we see that (4.8) is a solution of (1.1) for all $\lambda \geqslant 0$.

First, let us assume that U_λ is a Laplace transform. Now define $\rho(y) = \int_0^\infty \phi(\lambda) e^{-\lambda y}\, d\lambda$, where ϕ is a distribution with the property that $\int_0^\infty \phi(\lambda)U_\lambda(x, t)\, d\lambda$ is absolutely convergent. Differentiation under the integral sign shows that $u(x, t) = \int_0^\infty \phi(\lambda) U_\lambda(x, t)\, d\lambda$ is a solution of (1.1) with $u(x, 0) = \rho(x)$. Now let $p(t, x, y)$ be the inverse Laplace transform of U_λ. Then by Fubini's theorem

$$\begin{aligned}\int_0^\infty \rho(y)p(t, x, y)\, dy &= \int_0^\infty \left(\int_0^\infty \phi(\lambda)e^{-\lambda y} p(t, x, y)\, d\lambda\right) dy \\ &= \int_0^\infty \phi(\lambda)\left(\int_0^\infty e^{-\lambda y} p(t, x, y)\, dy\right) d\lambda \\ &= \int_0^\infty \phi(\lambda)U_\lambda(x, t)\, d\lambda = u(x, t).\end{aligned}$$

Hence $u(x, t) = \int_0^\infty \rho(y)p(t, x, y)$ and $u(x, 0) = \rho(x)$. Next, observe that $\int_0^\infty p(t, x, y) = U_0(x, t) = 1$, as expected. Consequently, $p(t, x, y)$ is a fundamental solution of (1.1).

Now, we prove that $U_\lambda(x, t)$ is the Laplace transform of a generalised function $p(t, x, y)$. The case when $A = 0$ clearly yields a Laplace transform, so we assume that $A \neq 0$.

It is well known that a function $K(\lambda)$ is a Laplace transform if it can be written in the form $K(\lambda) = G(\lambda)H(\lambda)$, where both G and H are Laplace transforms. (See for example the book by Widder [17]). Now U_λ is the product of $H_\lambda(x, t) = \exp\left\{\frac{1}{2}F\left(\frac{x}{(1+\lambda t)^2}\right)\right\}$ and $G_\lambda(x, t) = \exp\left\{-\frac{\lambda(x+\frac{1}{2}At^2)}{1+\lambda t} - \frac{1}{2}F(x)\right\}$. G_λ is well known to be a Laplace transform. We therefore have to show that H_λ is a Laplace transform.

Under the change of variables $f = 2xy'/y$, the Ricatti equation $xf' - f + \frac{1}{2}f^2 = h(x)$, becomes the second-order linear ODE,

$$2x^2y''(x) - h(x)y(x) = 0. \tag{4.9}$$

The general solution of (4.9) for $h(x) = Ax + B$, is

$$y(x) = c_1x^{\frac{1}{2}}I_{\sqrt{1+2B}}\left(\sqrt{2Ax}\right) + c_2x^{\frac{1}{2}}I_{-\sqrt{1+2B}}\left(\sqrt{2Ax}\right). \tag{4.10}$$

From (4.10), solutions of $xf' - f + \frac{1}{2}f^2 = Ax + B$, can be obtained. Now $F'(x) = f(x)/x$. Hence $\frac{1}{2}F(x) = \int y'(x)/y(x)\,dx = \ln y(x)$. Thus $H_\lambda(x,t) = y(\frac{x}{(1+\lambda t)^2})$. By (4.10), H_λ is a Laplace transform if $\frac{\sqrt{x}}{1+\lambda t}I_{\pm\sqrt{1+2B}}(\frac{\sqrt{2Ax}}{1+\lambda t})$ is a Laplace transform. By elementary properties of Laplace transforms it is sufficient to show that $\frac{\sqrt{x}/t}{\lambda}I_{\pm\sqrt{1+2B}}(\frac{\sqrt{2Ax}/t}{\lambda})$ is a Laplace transform. But

$$\frac{1}{\lambda}I_{\pm\sqrt{1+2B}}\left(\frac{\sqrt{2Ax}}{t\lambda}\right) = \left(\frac{\sqrt{2Ax}}{2t}\right)^{\pm\sqrt{1+2B}}\sum_{n=0}^{\infty}\frac{\left(\frac{Ax}{2t^2}\right)^n/\lambda^{1+2n\pm\sqrt{1+2B}}}{n!\Gamma(1+n\pm\sqrt{1+2B})} \tag{4.11}$$

with the series being absolutely convergent for $\lambda > 0$. Therefore, by Theorem 30.2 of [7] the left-hand side of (4.11) has inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{1}{\lambda}I_{\pm\sqrt{1+2B}}\left(\frac{\sqrt{2Ax}}{t\lambda}\right)\right)$$
$$= \left(\frac{\sqrt{2Ax}}{2t}y\right)^{\pm\sqrt{1+2B}}\sum_{n=0}^{\infty}\frac{\left(\frac{Ax}{2t^2}\right)^n y^{2n}}{n!\Gamma(1+n\pm\sqrt{1+2B})\Gamma(1+2n\pm\sqrt{1+2B})},$$

U_λ is therefore a Laplace Transform and hence it is a characteristic solution. □

We shall now consider some applications of this theorem.

Example 4.2. We solve the PDE,

$$u_t = xu_{xx} + \left(\frac{ax}{1+\frac{1}{2}ax}\right)u_x, \quad a > 0. \tag{4.12}$$

The drift satisfies $xf' - f + \frac{1}{2}f^2 = 0$. Applying Theorem 4.1, we obtain $F(x) = 2\ln(1+ax)$. Hence, the characteristic solution for (4.12) is

$$U_\lambda(x,t) = \left(\frac{(1+\lambda t)^2 + \frac{1}{2}ax}{(1+\lambda t)^2(1+\frac{1}{2}ax)}\right)\exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}. \tag{4.13}$$

This gives the fundamental solution

$$\begin{aligned} p(t,x,y) &= \frac{1}{1+\frac{1}{2}ax}\,\mathcal{L}^{-1}\left(\left(\frac{\frac{1}{2}ax}{(1+\lambda t)^2}+1\right)\exp\left\{\frac{-\lambda x}{1+\lambda t}\right\}\right) \\ &= \frac{e^{-\frac{(x+y)}{t}}}{(1+\frac{1}{2}ax)t}\left[\left(\sqrt{\frac{x}{y}}+\frac{a\sqrt{xy}}{2}\right)I_1\left(\frac{2\sqrt{xy}}{t}\right)+t\delta(y)\right], \end{aligned} \tag{4.14}$$

in which δ is the Dirac delta function.

Since $\int_0^\infty p(t,x,y)\,dy = 1$, if we interpret $\frac{e^{-\frac{x}{t}}}{(1+\frac{1}{2}ax)t}$ as the probability of absorption at the origin, then $p(t,x,y)$ may be viewed as the transition density for the GSR process X_t, satisfying the SDE with bounded drift

$$dX_t = \frac{aX_t}{1+\frac{1}{2}aX_t}\,dt + \sqrt{2X_t}\,dW_t. \tag{4.15}$$

GSR processes with bounded drift are of interest in the modelling of price dynamics for commodities such as oil. As an application, let us price a *European call option* on a commodity whose discounted price X_t at time t satisfies (4.15). Such a call option gives the holder the right to buy the commodity for an agreed price of K dollars, at a future time T. According to standard option pricing theory, (cf. [8]), the price $c_T(x,t)$ of the option, at time $T-t$, when the commodity price is x, is given by the solution of (4.12) with initial data $c_T(x,0) = \max(x-K,0)$. Hence the price is given by the integral

$$c_T(x,t) = \int_K^\infty \frac{(y-K)e^{-\frac{(x+y)}{T-t}}}{(1+\frac{1}{2}ax)(T-t)}\left(\sqrt{\frac{x}{y}}+\frac{a\sqrt{xy}}{2}\right)I_1\left(\frac{2\sqrt{xy}}{T-t}\right)dy.$$

This integral would typically be evaluated numerically. We could of course perform similar modelling with the other processes which appear in this paper.

The next few examples illustrate some applications of Theorem 4.1 where we leave the details to the reader.

Example 4.3. For the PDE

$$u_t = xu_{xx} + \left(\frac{(1+3\sqrt{x})}{2(1+\sqrt{x})}\right)u_x.$$

Theorem 4.1 gives the characteristic solution

$$U_\lambda(x,t) = \left(\frac{1}{(1+\lambda t)^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{(1+\lambda t)^{\frac{3}{2}}}\right)\frac{\exp\left\{-\frac{\lambda x}{(1+\lambda t)}\right\}}{(1+\sqrt{x})}.$$

Inverting gives the fundamental solution

$$p(t,x,y) = \frac{\cosh\left(\frac{2\sqrt{xy}}{t}\right)}{\sqrt{\pi y t}(1+\sqrt{x})}\left(1+\sqrt{y}\tanh\left(\frac{2\sqrt{xy}}{t}\right)\right)\exp\left\{\frac{-(x+y)}{t}\right\}.$$

Example 4.4. The equation

$$u_t = xu_{xx} + \left(1+\mu\tanh\left(\mu+\frac{1}{2}\mu\ln x\right)\right)u_x, \quad \mu = \frac{1}{2}\sqrt{\frac{5}{2}} \tag{4.16}$$

has characteristic solution

$$U_\lambda(x,t) = \frac{\cosh\left(\frac{\sqrt{5}}{8}\left(2+\ln\left(\frac{x}{(1+\lambda t)^2}\right)\right)\right)}{(1+\lambda t)\cosh\left(\frac{\sqrt{5}}{8}(2+\ln x)\right)}\exp\left\{-\frac{\lambda x}{(1+\lambda t)}\right\}.$$

From which we obtain

$$p(t,x,y) = \left(\frac{x}{y}\right)^{\frac{\mu}{2}}\left[I_{-\mu}\left(\frac{2\sqrt{xy}}{t}\right) + e^{2\mu}y^\mu I_\mu\left(\frac{2\sqrt{xy}}{t}\right)\right]\frac{\exp\left\{\frac{-(x+y)}{t}\right\}}{(1+e^{2\mu}x^\mu)t}.$$

Example 4.5. For

$$u_t = xu_{xx} + \left(\frac{1}{2}+\sqrt{x}\right)u_x. \tag{4.17}$$

we have,

$$U_\lambda(x,t) = \frac{1}{\sqrt{1+\lambda t}}\exp\left\{\frac{-\lambda(t+2\sqrt{x})^2}{4(1+\lambda t)}\right\}$$

and

$$p(t,x,y)=\frac{e^{-\sqrt{x}}}{\sqrt{\pi y t}}\cosh\left(\frac{(t+2\sqrt{x})\sqrt{y}}{t}\right)\exp\left\{-\frac{(x+y)}{t}-\frac{1}{4}t\right\}.$$

Example 4.6. If

$$u_t=xu_{xx}+\left(\frac{1}{2}+\sqrt{x}\tanh(\sqrt{x})\right)u_x, \tag{4.18}$$

then Theorem 4.1 gives the characteristic solution

$$U_\lambda(x,t)=\frac{\cosh\left(\frac{\sqrt{x}}{\sqrt{1+\lambda t}}\right)}{\cosh(\sqrt{x})\sqrt{1+\lambda t}}\exp\left\{\frac{-\lambda(x+\frac{1}{4}t^2)}{1+\lambda t}\right\}.$$

Inverting the Laplace transform gives

$$p(t,x,y)=\frac{\cosh\left(\frac{2\sqrt{xy}}{t}\right)}{\sqrt{\pi y t}}\frac{\cosh(\sqrt{y})}{\cosh(\sqrt{x})}\exp\left\{-\frac{(x+y)}{t}-\frac{1}{4}t\right\}.$$

Example 4.7. The PDE

$$u_t=xu_{xx}+\left(\frac{1}{2}+\sqrt{x}\coth(\sqrt{x})\right)u_x \tag{4.19}$$

has characteristic solution

$$U_\lambda(x,t)=\frac{\sinh\left(\frac{\sqrt{x}}{1+\lambda t}\right)}{\sinh(\sqrt{x})\sqrt{1+\lambda t}}\exp\left\{-\frac{\lambda(x+\frac{1}{4}t^2)}{1+\lambda t}\right\}.$$

From which we obtain the fundamental solution

$$p(t,x,y)=\frac{\sinh\left(\frac{2\sqrt{xy}}{t}\right)}{\sqrt{\pi y t}}\frac{\sinh(\sqrt{y})}{\sinh(\sqrt{x})}\exp\left\{-\frac{(x+y)}{t}-\frac{1}{4}t\right\}.$$

Example 4.8. Consider the PDE

$$u_t=xu_{xx}+(1+\cot\left(\ln\sqrt{x}\right))u_x. \tag{4.20}$$

The drift is discontinuous at the points $x = e^{2n\pi}$, $n \in \mathbb{Z}$. Nevertheless, Theorem 4.1 gives a characteristic solution of (4.20) as

$$U_\lambda(x,t) = \frac{\operatorname{cosec}\left(\ln\sqrt{x}\right)}{2i(1+\lambda t)}\left[\frac{x^{\frac{i}{2}}}{(1+\lambda t)^i} - \left(\frac{x^{\frac{i}{2}}}{(1+\lambda t)^i}\right)^{-1}\right]\exp\left\{\frac{-\lambda x}{1+\lambda t}\right\},$$

where $i = \sqrt{-1}$. We can invert this Laplace transform. We get,

$$p(t,x,y) = \frac{e^{-\frac{(x+y)}{t}}}{2it\ \sin\left(\ln\sqrt{x}\right)}\left(y^{\frac{i}{2}} I_i\left(\frac{2\sqrt{xy}}{t}\right) - y^{-\frac{i}{2}} I_{-i}\left(\frac{2\sqrt{xy}}{t}\right)\right).$$

To show that this fundamental solution is real valued, recall that,

$$I_v(z) = \left(\frac{1}{2}z\right)^v \sum_{k=0}^{\infty} \frac{(\frac{1}{4}z^2)^k}{k!\Gamma(v+k+1)} \tag{4.21}$$

and $\Gamma(\bar{z}) = \overline{\Gamma(z)}$. Expanding the series for $I_{\pm i}$ and collecting terms, leads to the expression,

$$p(t,x,y) = \frac{e^{-\frac{(x+y)}{t}}}{t\ \sin\left(\ln\sqrt{x}\right)} \sum_{k=0}^{\infty}\left(\frac{xy}{t^2}\right)^k \left\{a_k \sin\left(\ln\frac{\sqrt{xy}}{t}\right) + b_k \cos\left(\ln\frac{\sqrt{xy}}{t}\right)\right\},$$

where, $a_k = \operatorname{Re}\left(\frac{1}{k!\Gamma(k+1+i)}\right)$, $b_k = \operatorname{Im}\left(\frac{1}{k!\Gamma(k+1+i)}\right)$. Consequently, $p(t,x,y)$ is real valued.

5. The Ricatti equation $xf' - f + \frac{1}{2}f^2 = \frac{1}{2}Ax^2 + Bx + C$

We now consider the case when f satisfies the Ricatti equation (1.3) with $A > 0$. The case when $A < 0$ can be treated by similar methods. To compute fundamental solutions, we require the following result.

Proposition 5.1. *Let f be a solution of* (1.3) *and u be a solution of the corresponding* PDE (1.1). *Let the vector fields* $\mathbf{v}_4$ *and* $\mathbf{v}_6$ *be as given in Subcase* 2a. *Then, for ε sufficiently small, the following are also solutions of* (1.1):

$$\rho(\exp(\varepsilon\mathbf{v}_4)u(x,t)) = U_\varepsilon^1(x,t)u\left(\frac{x}{1+\varepsilon\sqrt{A}e^{\sqrt{A}t}}, \frac{1}{\sqrt{A}}\ln\left(\frac{e^{\sqrt{A}t}}{1+\varepsilon\sqrt{A}e^{\sqrt{A}t}}\right)\right),$$

$$\rho(\exp(\varepsilon\mathbf{v}_6)u(x,t)) = U_\varepsilon^2(x,t)u\left(\frac{xe^{\sqrt{A}t}}{e^{\sqrt{A}t}-\varepsilon\sqrt{A}}, \frac{\ln(e^{\sqrt{A}t}-\varepsilon\sqrt{A})}{\sqrt{A}}\right),$$

where $\rho(\exp(\varepsilon \mathbf{v}_i))u$ *is the symmetry obtained from* $\mathbf{v}_i$, $F'(x) = f(x)/x$ *and*

$$U_\varepsilon^1(x,t) = \left(1 + \varepsilon\sqrt{A}e^{\sqrt{A}t}\right)^{\frac{-B}{2\sqrt{A}}} \times \exp\left\{\frac{-\varepsilon A e^{\sqrt{A}t}x}{2(1+\varepsilon\sqrt{A}e^{\sqrt{A}t})} - \frac{1}{2}\left(F(x) - F\left(\frac{x}{1+\varepsilon\sqrt{A}e^{\sqrt{A}t}}\right)\right)\right\}, \tag{5.1}$$

$$U_\varepsilon^2(x,t) = e^{-\frac{B}{2}t}\left(e^{\sqrt{A}t} - \varepsilon\sqrt{A}\right)^{\frac{B}{2\sqrt{A}}} \times \exp\left\{\frac{-\varepsilon A x}{2(e^{\sqrt{A}t} - \sqrt{A}\varepsilon)} - \frac{1}{2}\left(F(x) - F\left(\frac{xe^{\sqrt{A}t}}{e^{\sqrt{A}t} - \varepsilon\sqrt{A}}\right)\right)\right\}. \tag{5.2}$$

Proof. The proof simply requires us to solve the system of ODEs, (4.2), which correspond to $\mathbf{v}_4$ and $\mathbf{v}_6$. □

Since $u = 1$ is a solution of equation (1.1), so are, U_ε^1 and U_ε^2. Although neither of them is the characteristic solution, we can write

$$U_\varepsilon^i(x,t) = \int_0^\infty U_\varepsilon^i(y,0)\,p(t,x,y)\,dy. \tag{5.3}$$

In principle we can recover the fundamental solution by inverting (5.3). In practice, this typically reduces to inverting a Laplace transform. We shall not state a theorem. Instead we will illustrate the process by example.

Example 5.1. A special case of a PDE which arises in interest rate modelling is

$$u_t = xu_{xx} + \left(\frac{3}{2} - x\right)u_x. \tag{5.4}$$

Applying Proposition 5.1 and setting $\varepsilon = \lambda/(1+\lambda)$, we see that,

$$\tilde{U}_\lambda(x,t) = \left(\frac{(1+\lambda)e^t}{(1+\lambda)e^t - \lambda}\right)^{\frac{3}{2}} \exp\left\{\frac{-\lambda x}{(1+\lambda)e^t - \lambda}\right\} \tag{5.5}$$

is a solution of (5.4). This is a scalar multiple of the characteristic solution. Multiplying $\tilde{U}_\lambda$ by $1/(1+\lambda)^{\frac{3}{2}}$ gives the characteristic solution

$$U_\lambda(x,t) = \left(\frac{e^t}{(1+\lambda)e^t - \lambda}\right)^{\frac{3}{2}} \exp\left\{\frac{-\lambda x}{(1+\lambda)e^t - \lambda}\right\}. \tag{5.6}$$

Inverting the Laplace transform, we obtain the fundamental solution for (5.4).

$$p(t,x,y) = \left(\frac{e^t}{e^t-1}\right)^{\frac{3}{2}} \mathcal{I}_{\frac{1}{2}}\left(\frac{2\sqrt{xye^t}}{e^t-1}\right) \exp\left\{-\frac{(x+y)}{e^t-1}\right\}, \tag{5.7}$$

where $\mathcal{I}_{\nu-\frac{1}{2}}(z) = 2^{\nu-\frac{1}{2}}\Gamma(\nu+\frac{1}{2})z^{-\nu+\frac{1}{2}}I_{\nu-\frac{1}{2}}(z)$.

By the same approach we may find the fundamental solution when the drift takes the form $f(x) = a - bx$. Cox et al. have used these fundamental solutions to derive bond prices. See [8].

Example 5.2. Next, we consider the PDE

$$u_t = xu_{xx} + x\,\coth\left(\frac{x}{2}\right)u_x. \tag{5.8}$$

Here $xf' - f + \frac{1}{2}f^2 = \frac{1}{2}x^2$. By Proposition 5.1, Eq. (5.8) has a solution,

$$u_\varepsilon(x,t) = \frac{\sinh\left(\frac{xe^t}{2(e^t-\varepsilon)}\right)}{\sinh\left(\frac{x}{2}\right)} \exp\left\{\frac{-\varepsilon x}{2(e^t-\varepsilon)}\right\}. \tag{5.9}$$

Observe that,

$$u_\varepsilon(x,0) = \frac{1}{2}\left(\frac{e^{\frac{x}{2}}}{\sinh\left(\frac{x}{2}\right)} - \frac{1}{\sinh\left(\frac{x}{2}\right)} \exp\left\{\frac{-(1+\varepsilon)x}{2(1-\varepsilon)}\right\}\right). \tag{5.10}$$

Furthermore, notice that $g(x) = \frac{e^{\frac{x}{2}}}{\sinh(\frac{x}{2})}$ is a stationary solution of (5.8). We therefore look for a fundamental solution $p(t,x,y)$ with the property that,

$$\int_0^\infty \frac{e^{\frac{y}{2}}}{\sinh\left(\frac{y}{2}\right)} p(t,x,y)\,dy = \frac{e^{\frac{x}{2}}}{\sinh\left(\frac{x}{2}\right)}. \tag{5.11}$$

We introduce the new parameter $\lambda = \frac{1+\varepsilon}{2(1-\varepsilon)}$. The solution u_ε becomes,

$$u_\lambda(x,t) = \frac{\sinh\left(\frac{(2\lambda+1)xe^t}{2((2\lambda+1)e^t-(2\lambda-1))}\right)}{\sinh\left(\frac{x}{2}\right)} \exp\left\{\frac{-(2\lambda-1)x}{2((2\lambda+1)e^t-(2\lambda-1))}\right\}.$$

By (5.11), we may write,

$$\begin{aligned} u_\lambda(x,t) &= \frac{1}{2}\int_0^\infty \left(\frac{e^{\frac{y}{2}}}{\sinh(\frac{y}{2})} - \frac{1}{\sinh(\frac{y}{2})}e^{-\lambda y}\right) p(t,x,y)\,dy \\ &= \frac{e^{\frac{x}{2}}}{2\sinh(\frac{x}{2})} - \frac{1}{2}\mathcal{L}\left(\frac{1}{\sinh\left(\frac{y}{2}\right)}\, p(t,x,y)\right). \end{aligned} \tag{5.12}$$

Equating (5.12) with the explicit expression for u_λ above, we obtain,

$$\begin{aligned} p(t,x,y) &= \frac{\sinh\left(\frac{y}{2}\right)}{\sinh\left(\frac{x}{2}\right)}\mathcal{L}^{-1}\left(\exp\left\{\frac{-(2\lambda(1+e^t)+e^t-1)x}{2((2\lambda+1)e^t-(2\lambda-1))}\right\}\right) \\ &= \frac{\sinh(\frac{y}{2})}{\sinh(\frac{x}{2})}\exp\left\{-\frac{(x+y)}{2\tanh\frac{t}{2}}\right\}\left[\frac{e^{\frac{1}{2}t}}{e^t-1}\sqrt{\frac{x}{y}}\,I_1\left(\frac{\sqrt{xy}}{\sinh\frac{t}{2}}\right)+\delta(y)\right]. \end{aligned}$$

The same procedure can be used to show that if $f(x) = x\tanh(\frac{x}{2})$, the fundamental solution is

$$p(t,x,y) = \frac{\cosh(\frac{y}{2})}{\cosh(\frac{x}{2})}\exp\left\{-\frac{(x+y)}{2\tanh\frac{t}{2}}\right\}\left[\frac{e^{\frac{1}{2}t}}{e^t-1}\sqrt{\frac{x}{y}}I_1\left(\frac{\sqrt{xy}}{\sinh\frac{t}{2}}\right)+\delta(y)\right].$$

Standard integrals of Bessel functions give $\int_0^\infty p(t,x,y)\,dy = 1$ for both cases. Other examples can be treated similarly.

We should note here that it is possible to derive the fundamental solution, without having to invert any transform. Consider the solutions U_ε^1 and U_ε^2. If we take $\varepsilon = 1$ we can identify $U_\varepsilon^1|_{\varepsilon=1}$ or $U_\varepsilon^2|_{\varepsilon=1}$ with multiples of the fundamental solution at $y = 0$. From this we can derive $p(t,x,y)$. Similar comments apply to the PDEs associated with the Ricatti equations (1.2) and (1.4). We will discuss this method in a subsequent paper.

6. The Ricatti equation $xf' - f + \frac{1}{2}f^2 = Ax^{\frac{3}{2}} + Bx^2 + Cx - \frac{3}{8}$

The last case is when the drift function f is a solution of the third Ricatti equation (1.4). If $B = 0$, we can obtain characteristic solutions by symmetry directly as in Theorem 4.1.

Theorem 6.1. *Let f be a solution of the Ricatti equation,*

$$xf' - f + \tfrac{1}{2} f^2 = Ax^{\frac{3}{2}} + Cx - \tfrac{3}{8}. \tag{6.1}$$

Let

$$U_\lambda(x,t) = \sqrt{\frac{\sqrt{x}(1+\lambda t)}{\sqrt{x}(1+\lambda t) - \frac{A\lambda}{12}t^3}}\, \exp\left\{G(\lambda,x,t)\right\} \times \exp\left\{-\frac{1}{2}\left(F(x) - F\left(\frac{(12(1+\lambda t)\sqrt{x} - A\lambda t^3)^2}{144(1+\lambda t)^4}\right)\right)\right\}, \tag{6.2}$$

where $F'(x) = f(x)/x$,

$$G(\lambda,x,t) = -\frac{\lambda(x+\frac{1}{2}Ct^2)}{1+\lambda t} - \frac{\frac{2}{3}At^2\sqrt{x}(3+\lambda t)}{(1+\lambda t)^2} + \frac{A^2t^4(2\lambda t(3+\frac{1}{2}\lambda t)-3)}{108(1+\lambda t)^3}$$

and $\lambda \geqslant 0$. *Then* U_λ *is a characteristic solution of* (1.1).

Proof. The proof is similar to that of Theorem 4.1 and we omit it. □

Note that if we take $A = 0$, in (6.2), it reduces to Eq. (4.8).

When $B = 0$, Eq. (1.4) can be solved in terms of Airy functions. We can thus generate characteristic solutions for (1.1). Unfortunately, we have not been able to explicitly invert the resulting Laplace transforms.

Finally, we discuss the case when f satisfies (1.4) with $B \neq 0$. Solutions of (1.4) can be obtained in the following way. If we set

$$\frac{h'(y)}{h(y)} = \frac{1}{y}\left(f\left(\frac{y^2}{4}\right) - \frac{1}{2}\right), \text{ then } 2h''(y) - \left(\frac{1}{4}By^2 + \frac{1}{2}Ay + C\right)h(y) = 0.$$

This ODE for h is easily solved in terms of hypergeometric functions.

It is not difficult to exponentiate the infinitesimal symmetries from Subcases 3b and 3c. As in the previous cases, by starting with the trivial solution $u = 1$, and applying the symmetry transformations generated by $\mathbf{v}_5$ and $\mathbf{v}_6$, we obtain integral transforms of the fundamental solution. However, we have not yet been able to invert any of the transforms that we have obtained. We are continuing to study this problem.

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