*Periodica Mathematica Hungarica Vol. 48 (1–2), 2004, pp. 151–164*

# THE SECOND-ORDER ORDINARY DIFFERENTIAL EQUATION: CARTAN, DOUGLAS, BERWALD

M. Crampin (Gent)

**Abstract**

This paper is a re-examination of Cartan's theory of the second-order ordinary differential equation, from a modern perspective; the opportunity is taken to point out some relations between his results and the work of Douglas and Berwald on the geometry of paths.

## 1. Introduction

This paper is devoted to the explication of some aspects of Cartan's theory of the second-order ordinary differential equation in his famous article 'Sur les variétés à connexion projective' [3]; and in particular, to pointing out the relationship between his results and those obtained independently and by different methods by Douglas [5] and Berwald [1].

One of Cartan's main aims in [3] was to give a geometrical interpretation of a result obtained by A. Tresse in 1896, concerned with finding necessary and sufficient conditions for the existence of a so-called point transformation $\bar{x} = \bar{x}(x, y)$, $\bar{y} = \bar{y}(x, y)$ taking the equation

$$\frac{d^2 y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \quad \text{to} \quad \frac{d^2 \bar{y}}{d\bar{x}^2} = 0.$$

These conditions can be expressed in terms of two invariants (strictly speaking, relative invariants) of a second-order ordinary differential equation, $a$ and $b$. If the right-hand side is denoted by $f(x, y, y')$, then $a = -\frac{1}{6} f_{y'y'y'y'}$ (subscripts indicate partial derivatives); the first of Tresse's conditions is that $a = 0$. When this holds we can write $f = A + 3By' + 3Cy'^2 + Dy'^3$, where $A$, $B$, $C$ and $D$ are functions of $x$ and $y$; in such a case $b$ is a linear function of $y'$ whose coefficients depend on $A$, $B$, $C$, $D$ and their first and second partial derivatives with respect to $x$ and $y$. Tresse's second condition is that $b$ should vanish identically. Cartan was able to show that $a$

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*Mathematics subject classification number:* 34A26, 53B10.

*Key words and phrases:* Cartan projective connection, projective differential geometry of sprays, second-order ordinary differential equation.

*0031-5303/2004/$20.00*
*Akadémiai Kiadó, Budapest*

*Kluwer Academic Publishers, Dordrecht*

and $b$ are components of the curvature of a connection associated with the second-order ordinary differential equation, and thereby interpret Tresse's conditions in terms of the properties of the connection.

Several other noteworthy, though less familiar, facts about the invariants $a$ and $b$ appear in Cartan's paper. For example, the final paragraph begins '. . . Nous laissons au lecteur le soin de vérifier l'existence des *invariants intégraux*

$$\int \sqrt[4]{ab}\omega^2, \quad \iiint \sqrt{ab}\omega^1\omega^2\omega_1^2, \quad \iint a^{\frac{1}{8}}b^{\frac{5}{8}}\omega^1\omega^2, \quad \iint a^{\frac{5}{8}}b^{\frac{1}{8}}\omega^2\omega_1^2.\text{'}$$

(Here $\omega^1 = dx$, $\omega^2 = dy - y'dx$, $\omega_1^2 = dy' - f dx$, and Cartan writes exterior products without the wedge sign.) I shall carry out this 'exercise for the reader' later in this paper, and explain where the unexpected powers come from.

Cartan's concept of a connection is very different from Ehresmann's, a point that is made admirably clear by Sharpe in his recent book [9]. So far as I am aware, there has been no attempt to give an account of the part of Cartan's projective connection paper that deals with the general second-order ordinary differential equation from the perspective of Sharpe's book. In addition, the overlap between the results of Cartan and those of Douglas and Berwald has tended to be ignored. This paper tries to put these matters to rights.

I shall begin with the second of these points, by sketching those parts of Douglas's general theory of paths which are relevant to the discussion (Section 2). I shall then give a brief outline of Sharpe's approach to Cartan's theory (Section 3), and exemplify it with a simple account of Cartan's normal projective connection for the affine case (the first part of the projective connection paper) (Section 4). I shall then turn to the second-order ordinary differential equation, and derive the normal projective connection for this situation (Section 5). Finally, I shall establish the existence of the invariant integrals, and mention one or two other interesting consequences of Cartan's analysis.

## 2. Projective geometry of sprays

So far as this paper is concerned, the important point about the general geometry of paths [5] is that it deals with the projective differential geometry of sprays.

I denote by $\tau \colon T^0M \to M$ the tangent bundle of $M$ with the zero section deleted. Coordinates on $T^0M$ will generally be written $(x^i, u^i)$. A spray $\Gamma$ on $T^0M$ is a second-order differential equation field

$$u^i \frac{\partial}{\partial x^i} - 2\Gamma^i \frac{\partial}{\partial u^i}$$

whose coefficients $\Gamma^i$ are homogeneous of degree 2 in the $u^i$; if they are quadratic in the $u^i$ then the spray is affine. Two sprays $\Gamma$, $\hat{\Gamma}$ are projectively equivalent if

$\hat{\Gamma}^i = \Gamma^i + \alpha u^i$, where the function $\alpha$ is homogeneous of degree 1 in the $u^i$. The horizontal distribution associated with a spray is spanned by the vector fields

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial u^j}, \quad \Gamma_i^j = \frac{\partial \Gamma^j}{\partial u^i}.$$

The vertical vector field $\partial/\partial u^i$ will sometimes be denoted by $V_i$. The Berwald connection associated with a spray $\Gamma$ is the connection on the pullback bundle $\tau^* TM \to T^0 M$ with

$$\nabla_{H_i} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k} = \frac{\partial \Gamma_i^k}{\partial u^j} \frac{\partial}{\partial x^k}, \quad \nabla_{V_i} \frac{\partial}{\partial x^j} = 0.$$

Its curvature can be broken down into two components according as the vector field arguments are taken to be horizontal or vertical. The first is the Berwald curvature

$$\left(\nabla_{V_i} \nabla_{H_j} - \nabla_{H_j} \nabla_{V_i} - \nabla_{[V_i, H_j]}\right) \frac{\partial}{\partial x^k} = B_{kij}^l \frac{\partial}{\partial x^l} \quad \text{where} \quad B_{kij}^l = \frac{\partial \Gamma_{jk}^l}{\partial u^i}.$$

This component of the curvature has no affine counterpart – in fact its vanishing is the necessary and sufficient condition for the spray to be affine. The other component is the Riemann curvature

$$\left(\nabla_{H_i} \nabla_{H_j} - \nabla_{H_j} \nabla_{H_i} - \nabla_{[H_i, H_j]}\right) \frac{\partial}{\partial x^k} = R_{kij}^l \frac{\partial}{\partial x^l}$$

$$\text{where} \quad R_{kij}^l = H_i\left(\Gamma_{jk}^l\right) - H_j\left(\Gamma_{ik}^l\right) + \Gamma_{im}^{\,l} \Gamma_{jk}^m - \Gamma_{jm}^{\,l} \Gamma_{ik}^m.$$

It has the usual symmetries, and reduces to the ordinary curvature tensor when the spray is affine. By taking traces we obtain tensors $B_{ij} = B_{kij}^k$, $R_{ij} = R_{ikj}^k$; by the cyclic identity $R_{kij}^k = R_{ij} - R_{ji}$.

From the basic projective transformation rule we find that

$$\hat{H}_i = H_i - \alpha V_i - \alpha_i u^j V_j, \quad \text{where} \quad \alpha_i = \frac{\partial \alpha}{\partial u^i};$$

$$\hat{\Gamma}_{ij}^k = \Gamma_{ij}^k + \left(\alpha_{ij} u^k + \alpha_i \delta_j^k + \alpha_j \delta_i^k\right), \quad \text{where} \quad \alpha_{ij} = \frac{\partial^2 \alpha}{\partial u^i \partial u^j}.$$

By taking a trace we obtain $\hat{\Gamma}_{ij}^j = \Gamma_{ij}^j + (n+1)\alpha_i$, whence the quantity

$$\Pi_{ij}^k = \Gamma_{ij}^k - \frac{1}{n+1}\left(\Gamma_{il}^l \delta_j^k + \Gamma_{jl}^l \delta_i^k + B_{ij} u^k\right)$$

is projectively invariant. Douglas calls it the fundamental invariant and says in effect that every projective invariant is expressible in terms of it and its partial derivatives. Note that $\Pi_{ij}^j = \Pi_{ji}^j = 0$. On the face of it, if we take

$$\alpha = -\frac{1}{n+1} \Gamma_j^j = -\frac{1}{n+1} \frac{\partial \Gamma^j}{\partial u^j}$$

then the transformed spray has $\Pi_{ij}^k$ for its connection coefficients. However, $\Gamma_j^j$ is not strictly speaking a function: its transformation law under coordinate transformations of the $x^i$ (and the induced transformations of the $u^i$) involves the determinant of the Jacobian of the coordinate transformation. Moreover, the $\Pi^k_{ij}$ are not components of a tensor, nor even of a connection, and this has to be borne in mind when forming projective invariants from it.

I shall work tensorially, but I shall point out the simplifications that arise when one chooses the spray whose connection coefficients with respect to some coordinates are the $\Pi^k_{ij}$; I shall indicate when objects calculated with respect to such a spray are non-tensorial by setting their kernel letters in black-letter. Thus it follows from the fact that the traces of $\Pi^k_{ij}$ vanish that $\mathfrak{B}_{ij} = 0$, and also that $\mathfrak{R}^k_{kij} = 0$, so that $\mathfrak{R}_{ji} = \mathfrak{R}_{ij}$.

An easy calculation leads to the following transformation formula for $B^l_{kij}$:

$$\hat{B}^l_{kij} = B^l_{kij} + \alpha_{ijk}u^l + \alpha_{ij}\delta^l_k + \alpha_{jk}\delta^l_i + \alpha_{ik}\delta^l_j,$$

where $\alpha_{ijk}$ denotes a third partial derivative of $\alpha$. Then by taking a trace $\hat{B}_{ij} = B_{ij} + (n+1)\alpha_{ij}$, whence

$$D^l_{kij} = B^l_{kij} - \frac{1}{n+1}\left(u^l\nabla_{V_k}B_{ij} + B_{ij}\delta^l_k + B_{jk}\delta^l_i + B_{ik}\delta^l_j\right)$$

is a projectively invariant tensor – the Douglas tensor. Since $\mathfrak{B}_{ij} = 0$, $D^l_{kij} = \mathfrak{B}^l_{kij}$. The vanishing of the Douglas tensor is the necessary and sufficient condition for a spray to be projectively equivalent to an affine one.

The projective transformation of the Riemann curvature is given by

$$\hat{R}^l_{kij} = R^l_{kij} + \nabla_{H_i}\alpha^l_{jk} - \nabla_{H_j}\alpha^l_{ik} + (\alpha\alpha_{jk} + \alpha_j\alpha_k)\delta^l_i - (\alpha\alpha_{ik} + \alpha_i\alpha_k)\delta^l_j,$$

where $\alpha^k_{ij}$ is the difference tensor of the connection coefficients. It follows (after a lengthy calculation) that if

$$P^l_{kij} = S^l_{kij} - \frac{1}{n^2-1}\left(Q_{jk}\delta^l_i - Q_{ik}\delta^l_j - (Q_{ij} - Q_{ji})\delta^l_k\right),$$

$$\text{where}\quad S^l_{kij} = R^l_{kij} - \frac{1}{n+1}\left(u^l\nabla_{V_k}(R_{ij} - R_{ji})\right),$$

$S_{ij} = S^k_{ikj}$ and $Q_{ij} = S_{ij} + nS_{ji}$, then $P^l_{kij}$ is a projectively invariant tensor. It is the counterpart of the projective curvature tensor of the affine theory, to which it reduces in the affine case. It has the same symmetries as the Riemann curvature, and in addition all of its traces vanish.

Since $\mathfrak{R}_{ij}$ is symmetric, $\mathfrak{S}^l_{kij} = \mathfrak{R}^l_{kij}$, whence $\mathfrak{S}_{ij} = \mathfrak{R}_{ij}$, so that

$$P^l_{kij} = \mathfrak{R}^l_{kij} - \frac{1}{n-1}\left(\mathfrak{R}_{jk}\delta^l_i - \mathfrak{R}_{ik}\delta^l_j\right).$$

When $n \geq 3$ the vanishing of both the Douglas and the projective curvature tensors is the necessary and sufficient condition for a spray to be projectively flat. In dimension 2, however, a tensor with the symmetries of the Riemann tensor is determined by its traces, and if they vanish so does the tensor. So the projective curvature tensor is identically zero in dimension 2. Comparison with the affine case (as discussed by Schouten [8] for example) suggests that in order to find a projective invariant in dimension 2 which serves instead of the projective curvature tensor we should consider $\nabla_{H_i} Q_{jk} - \nabla_{H_j} Q_{ik}$, which I will write $Q_{kij}$. This transforms as follows:

$$\hat{Q}_{kij} = Q_{kij} + (n^2 - 1)P^l_{mij}(\alpha_l \delta^m_k + u^m \alpha_{kl}).$$

Thus in dimension 2, $Q_{kij}$ is projectively invariant. It reduces to its counterpart in the affine case, whose vanishing is the necessary and sufficient condition for a 2-dimensional affine spray to be projectively flat.

In dimension 2 $Q_{kij}$ has 2 components; but $u^k Q_{kij}$ is also projectively invariant and has just 1 component, and in the affine case its vanishing is equivalent to the vanishing of $Q_{kij}$. This quantity was originally defined by Berwald [1] so I shall call it the Berwald projective invariant.

A second-order differential equation

$$\frac{d^2y}{dx^2} = f(x, y, y'), \quad y' = \frac{dy}{dx}$$

defines, and is defined by, the projective equivalence class of the 2-dimensional spray with $x^1 = x$, $x^2 = y$, $\Gamma^1 = 0$, $\Gamma^2 = (u^1)^2 f(x^i, y')$, $y' = u^2/u^1$. Formulae for the Douglas tensor and the Berwald invariant for such a spray can be found in Shen's book on spray geometry [10]. Each component of the Douglas tensor for this spray is a multiple of $f_{y'y'y'y'}$, the factor being a simple expression in $u^1$ and $u^2$. Note that the second-order differential equation is projectively equivalent to an affine spray if and only $f$ is cubic in $y'$. The Berwald invariant is a complicated expression in the partial derivatives of $f$ up to the fourth order; when $f_{y'y'y'y'} = 0$ it reduces to Tresse's second invariant.

## 3. Cartan's theory of connections

The Berwald connection is essentially nothing more than its covariant derivative operator, and so belongs to the class of connections introduced by Levi-Civita and formalized by Ehresmann. But Cartan's idea of a connection has nothing to do with covariant differentiation; instead, it is based on Klein's concept of geometry.

For Klein, a geometrical space is a homogeneous space of a Lie group $G$, that is, a manifold $M$ on which $G$ acts transitively (and I shall assume effectively). If $H$ is the stabilizer of some $\xi_0 \in M$, then $M$ can be identified with the coset space $G/H$; $G$ is a right principle $H$-bundle over $M$ with projection $g \mapsto g\xi_0$.

The group $G$, whose Lie algebra I denote by $\mathfrak{g}$, comes equipped with a left-invariant $\mathfrak{g}$-valued 1-form $\mu$, its Maurer-Cartan form, which enjoys the following properties:

- for any $X \in \mathfrak{g}$, $\langle \tilde{X}, \mu \rangle = X$ where $\tilde{X}$ is the left-invariant vector field on $G$ corresponding to $X \in \mathfrak{g}$;
- for each $g \in G$, $\mu_g : T_gG \to \mathfrak{g}$ is an isomorphism;

- for each $g \in G$, $R_g^*\mu = \mathrm{Ad}(g^{-1})\mu$;
- $d\mu + \frac{1}{2}[\mu \wedge \mu] = 0$.

Since $G$ will be a matrix group in the applications, and some formulae are easier to write down for matrix groups, I shall henceforth assume that $G$ is a matrix group. For a matrix group $\mu = g^{-1}dg$.

Given a curve $t \mapsto X(t)$ in $\mathfrak{g}$, and a point $g_0 \in G$, there is a unique curve $t \mapsto g(t)$ in $G$ such that $\langle \dot{g}(t), \mu \rangle = X(t)$ and $g(0) = g_0$; it is a solution of the differential equation $\dot{g} = gX$. The curve $\xi(t) = g(t)\xi_0$ in $M$ is called the development of $X(t)$ into $M$ through $\xi_1 = g_0\xi_0$. If $\gamma$ is a local section of $G$ over some neighbourhood of $\xi_1$ in $M$, then $g(t) = \gamma(\xi(t))h(t)$ for some curve $t \mapsto h(t)$ in $H$; the curves $\xi(t)$ and $h(t)$ satisfy the differential equation

$$h^{-1}\dot{h} + h^{-1}\langle \dot{\xi}, \gamma^*\mu \rangle h = g^{-1}\dot{g} = X.$$

According to Sharpe [9], a Cartan geometry on a manifold $M$ modelled on a Klein geometry $(G, H)$ is a right principle $H$-bundle $\pi : P \to M$ with $\dim P = \dim G = \dim H + \dim M$, and a $\mathfrak{g}$-valued 1-form $\omega$ on $P$, the Cartan connection form, such that

- for $X \in \mathfrak{h}$ (the Lie algebra of $H$), $\langle \tilde{X}, \omega \rangle = X$ where $\tilde{X}$ is the vertical vector field on $P$ generated by $X$ through the action of $H$;
- for each $p \in P$, $\omega_p : T_pP \to \mathfrak{g}$ is an isomorphism;
- for each $h \in H$, $R_h^*\omega = \mathrm{Ad}(h^{-1})\omega$.

The curvature $\Omega$ of a Cartan connection is the $\mathfrak{g}$-valued 2-form

$$\Omega = d\omega + \tfrac{1}{2}[\omega \wedge \omega], \quad \text{or} \quad \Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j.$$

Its vanishing is the necessary and sufficient condition for the Cartan geometry to be locally diffeomorphic to the Klein geometry on which it is modelled. The curvature satisfies the Bianchi identitiy $d\Omega = [\Omega \wedge \omega]$. The torsion of the Cartan connection is the $\mathfrak{g}/\mathfrak{h}$-valued 2-form $\rho(\Omega)$ (where $\rho : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the projection).

If $t \mapsto p(t)$ is a curve in $P$ then $t \mapsto \langle \dot{p}(t), \omega_{p(t)} \rangle$ is a curve in $\mathfrak{g}$ which can be developed into $G/H$; its development is $\xi(t) = g(t)\xi_0$ where $\langle \dot{g}, \mu \rangle = \langle \dot{p}, \omega \rangle$. But $\mu$ and $\omega$ transform identically under the action of $H$, which means that $\xi(t)$ depends only on $\pi(p(t))$; it is called the development of $x(t)$ into $G/H$. If the Klein geometry contains straight lines, a curve in $M$ is called a geodesic of the Cartan geometry if its development through any point is a straight line.

The definition of a Cartan geometry given above is conceptually the most satisfactory but not the best to work with. For calculational purposes it is better to choose a gauge (as indeed Cartan always did). By a gauge I mean a local section $\sigma$ of $P$. The gauged Cartan connection form is $\sigma^*\omega$, a $\mathfrak{g}$-valued local 1-form on $M$; $\sigma^*\omega|_x$ is an injective map $T_xM \to \mathfrak{g}$, and $\rho \circ \sigma^*\omega|_x : T_xM \to \mathfrak{g}/\mathfrak{h}$ is an isomorphism. If $\hat{\sigma}$ is another gauge then on the intersection of their domains $\hat{\sigma}(x) = \sigma(x)h(x)$ for some $H$-valued function $h$; then

$$\hat{\sigma}^*\omega = h^{-1}(\sigma^*\omega)h + h^{-1}dh, \quad \hat{\sigma}^*\Omega = h^{-1}(\sigma^*\Omega)h.$$

I shall use a gauge henceforth, and ignore the $\sigma^*$.

The differential equation for a development, when expressed in a gauge, is

$$h^{-1}\dot{h} + h^{-1}\langle\dot{\xi}, \mu\rangle h = \langle\dot{x}, \omega\rangle;$$

here $\mu$ is the gauged Maurer-Cartan form. This comprises $\dim \mathfrak{g}$ equations for $\dim \mathfrak{g}/\mathfrak{h}$ unknowns $\xi$ and $\dim \mathfrak{h}$ unknowns $h$.

Cartan's method of analyzing connections is first to fix the gauge by imposing conditions on the connection form, and then fix the connection by imposing gauge-invariant conditions on the curvature. The first part of the procedure leads to a standard gauge. The second part, which is carried out in the standard gauge, leads to the so-called normal connection for that geometry. Very often, the first step leading to a normal connection is to make the torsion zero. I shall exemplify this procedure in the next section.

## 4. Cartan's normal projective connection for affine sprays

We now consider Cartan geometries modelled on projective geometry. Projective space $\mathrm{RP}^n$ consists of lines through the origin in $\mathrm{R}^{n+1}$ (coordinates $(\xi^0, \xi^1, \ldots \ldots, \xi^n)$). In order to avoid some complications I shall restrict $n$ to be even. Then the group $\mathrm{SL}(n+1) = G$ acts transitively and effectively on $\mathrm{RP}^n$, and $H$, the stabilizer of the $\xi^0$-axis, is the subgroup of $\mathrm{SL}(n+1)$ consisting of matrices with zeros below the diagonal in the first column.

If $\omega$ is a connection form and $\tilde{\omega} = h^{-1}\omega h + h^{-1}dh$, where $h \in H$,

$$\omega = \begin{bmatrix} \omega_0^0 & \omega_i^0 \\ \omega_0^i & \omega_j^i \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} h_0^0 & h_i^0 \\ 0 & h_j^i \end{bmatrix}, \quad i, j = 1, 2, \ldots, n$$

then

$$\tilde{\omega}_0^0 = \omega_0^0 - h_j^0 \bar{h}_i^j \omega_0^i + \bar{h}_0^0 dh_0^0, \quad \tilde{\omega}_0^i = h_0^0 \bar{h}_j^i \omega_0^j$$

where the overbar signifies (an element of) the inverse matrix ($n \times n$ or $1 \times 1$ as the case may be). The $\omega_0^i$ must be linearly independent, so if $\omega_0^i = \omega_{0j}^i dx^j$ the equations $h_0^0 \bar{h}_k^i \omega_{0j}^k = \delta_j^i$ have a unique solution for $h_0^0$ and $h_j^i$ (again assuming $n+1$ odd):

$$h_0^0 = (\det(\omega_{0j}^i))^{1/(n+1)}, \quad h_j^i = (\det(\omega_{0j}^i))^{1/(n+1)} \omega_{0j}^i.$$

With this choice, $\tilde{\omega}_0^i = dx^i$. Then if we define $h_i^0$ by $h_i^0 dx^i = h_0^0 \omega_0^0 + dh_0^0$, we will have $\tilde{\omega}_0^0 = 0$. Therefore, for any Cartan projective connection there is a unique choice of gauge such that

$$\omega = \begin{bmatrix} 0 & \omega_i^0 \\ dx^i & \omega_j^i \end{bmatrix}.$$

The curvature $\Omega$ of such a connection has $\Omega_0^i = -\omega_{jk}^i dx^j \wedge dx^k$ where $\omega_j^i = \omega_{jk}^i dx^k$; so the connection has zero torsion if and only if $\omega_{jk}^i$ is symmetric in its lower indices. Since $\omega$ takes its values in $\mathfrak{sl}(n+1)$, the Lie algebra of $\mathrm{SL}(n+1)$, it has zero trace, so $\omega^i_{ij} = 0$.

Now projective space certainly contains straight lines, so any Cartan projective geometry has geodesics; I now proceed to find them. There is an obvious local section $\gamma$ of $\mathrm{SL}(n+1)$ over $\mathrm{RP}^n$:

$$\gamma(\xi^1, \xi^2, \ldots, \xi^n) = \begin{bmatrix} 1 & 0 \\ \xi^i & \delta^i_j \end{bmatrix};$$

the Maurer-Cartan form in this gauge is

$$\mu = \begin{bmatrix} 0 & 0 \\ \xi^i & 0 \end{bmatrix}.$$

The development equations are

$$h^{-1}\dot{h} + h^{-1}\begin{bmatrix} 0 & 0 \\ \dot{\xi}^i & 0 \end{bmatrix} h = \begin{bmatrix} 0 & \omega^0_{ij}\dot{x}^j \\ \dot{x}^i & \omega^i_{jk}\dot{x}^k \end{bmatrix}.$$

These give

$$\begin{aligned} \dot{x}^i &= h^0_0 \bar{h}^i_j \dot{\xi}^j \\ \omega^i_{jk}\dot{x}^k &= \bar{h}^i_k\big(\dot{h}^k_j + \dot{\xi}^k h^0_j\big) \\ 0 &= \bar{h}^0_0 \dot{h}^0_0 - h^0_j \bar{h}^j_k \dot{\xi}^k, \end{aligned}$$

and another (vector) equation which is of no interest. It follows that

$$\ddot{x}^i + \omega^i_{jk}\dot{x}^j\dot{x}^k = 2\bar{h}^0_0(h^0_j\dot{x}^j)\dot{x}^i \qquad (\mathrm{mod}\ \ddot{\xi}^j),$$

so $x(t)$ is a geodesic in the sense of Cartan if and only if it is a geodesic (in the ordinary sense) of the projective class of affine sprays with coefficients $\omega^i_{jk}$. But $\omega^i_{ij} = 0$, so these coefficients are collectively the fundamental invariant of this class of affine sprays, and we may set $\omega^i_{jk} = \Pi^i_{jk}$.

I now assume that we are given an affine connection, up to projective equivalence, and that the Cartan connection is adapted to it as just described. I seek to determine the remaining elements of the connection by further conditions on the curvature, so as to fix them uniquely and therefore obtain a normal connection. First, $\Omega^0_0 = -\omega^0_{ij}dx^i \wedge dx^j$; thus if we take $\omega^0_{ij}$ to be symmetric we will have $\Omega^0_0 = 0$. Then

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j + \omega^i_0 \wedge \omega^0_j = \tfrac{1}{2}\left(\mathfrak{R}^i_{jkl} + \delta^i_k\omega^0_{jl} - \delta^i_l\omega^0_{jk}\right) dx^k \wedge dx^l.$$

Thus if $\Omega^i_j = \frac{1}{2}\Omega^i_{jkl}dx^k \wedge dx^l$, with $\Omega^i_{jkl}$ skew in $k$ and $l$,

$$\Omega^i_{jkl} = \mathfrak{R}^i_{jkl} + \delta^i_k\omega^0_{jl} - \delta^i_l\omega^0_{jk}.$$

So $\Omega^i_{jkl}$ can be made trace-free by choosing $(n-1)\omega^0_{jk} = -\mathfrak{R}_{jk}$, in which case $\Omega^i_{jkl} = P^i_{jkl}$ ($= 0$ for $n = 2$). That is to say, given a projective equivalence class of affine connections, on any coordinate patch there is a unique $\mathfrak{sl}(n+1)$-valued torsion-free Cartan projective connection whose geodesics are those of the given projective class, such that $\Omega^0_0 = 0$ and $\Omega^k_{ikj} = 0$. It is called the normal projective connection, and in the standard gauge it is given by

$$\omega = \begin{bmatrix} 0 & -\frac{1}{n-1}\mathfrak{R}_{ij}dx^j \\ dx^i & \Pi^i_{jk}dx^k \end{bmatrix}.$$

The curvature of the normal projective connection is

$$\Omega = \begin{bmatrix} 0 & \frac{1}{n-1}\mathfrak{R}_{i[j|k]}dx^j \wedge dx^k \\ 0 & \frac{1}{2}P^i_{jkl}dx^k \wedge dx^l \end{bmatrix},$$

where the brackets in the suffix indicate skew-symmetrization and the solidus 'covariant differentiation' with respect to the fundamental invariant. In case $n = 2$ the normal projective connection satisfies, and is determined by, the condition $\Omega^i_j = 0$.

## 5. Cartan's normal projective connection for a second-order differential equation

In order to develop a connection theory for general second-order differential equations, not just those of affine type, it is necessary to use a different model geometry. Cartan deals only with a single differential equation, and I shall do the same; but the relevant model geometry is easy to describe in an arbitrary number of dimensions. As a space it is $PT(\mathrm{RP}^n)$, the projective tangent bundle of $\mathrm{RP}^n$. Each point of $PT(\mathrm{RP}^n)$ consists of a line through the origin in $\mathrm{R}^{n+1}$ and a 2-plane containing the line. The group $\mathrm{SL}(n+1)$ acts transitively on $PT(\mathrm{RP}^n)$, and effectively for $n$ even (which I assume to be the case, as before). The stabilizer of the point consisting of the $\xi^0$-axis and the $\xi^0\xi^1$-plane is the subgroup $H$ of $\mathrm{SL}(n+1)$ of matrices with zeros below the main diagonal in the first and second columns; so we can identify $PT(\mathrm{RP}^n)$ with $\mathrm{SL}(n+1)/H$. In the case of interest, $n = 2$, $H$ is just the group of unideterminantal upper triangular $3 \times 3$ matrices, so I shall denote it by $T$.

A 'manifold of elements with projective connection' is a Cartan geometry on the projective tangent bundle $PTM$ of a 2-dimensional manifold $M$, modelled on $PT(\mathrm{RP}^2) = \mathrm{SL}(3)/T$, in which the projective tangent bundle structures are compatible in the following sense. First, note that any curve in $M$ has a natural lift to $PTM$ obtained by adjoining to each point on it its tangent line at that point. The compatibility conditions are that the development into $PT(\mathrm{RP}^2)$ of a vertical curve in $PTM$ is vertical, and the development into $PT(\mathrm{RP}^2)$ of a lifted curve in $PTM$ is a lifted curve.

Before proceeding, it will be useful to calculate the effects of a gauge transformation on the strictly lower triangular terms in a connection form $\omega$. Suppose that

$$\omega = \begin{bmatrix} * & * & * \\ u & * & * \\ w & v & * \end{bmatrix} \quad \text{and} \quad h = \begin{bmatrix} A & D & F \\ 0 & B & E \\ 0 & 0 & C \end{bmatrix} \quad \text{with } ABC = 1.$$

Since $h^{-1}dh$ is upper triangular it has no effect on the relevant terms, and

$$h^{-1}\omega h = \begin{bmatrix} * & * & * \\ AB^{-1}u - A^2Ew & * & * \\ AC^{-1}w & BC^{-1}v + C^{-1}Dw & * \end{bmatrix}.$$

The effect of a change of gauge on a curvature form which is strictly upper triangular will also be useful: if

$$\Omega = \begin{bmatrix} 0 & U & * \\ 0 & 0 & V \\ 0 & 0 & 0 \end{bmatrix} \quad \text{then} \quad h^{-1}\Omega h = \begin{bmatrix} 0 & A^{-1}BU & * \\ 0 & 0 & B^{-1}CV \\ 0 & 0 & 0 \end{bmatrix}.$$

We may introduce local coordinates on $PTM$ by taking local coordinates $(x, y)$ on $M$, and by noting that every equivalence class of tangent vectors

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

for which $u \neq 0$ has a unique representative of the form

$$\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y};$$

then $(x, y, y')$ are local coordinates on $PTM$. With respect to these coordinates, a curve in $PTM$ is vertical if its tangent vector is annihilated by $dx$ and $dy$, and a curve in $PTM$ is a natural lift if its tangent vector is annihilated by the so-called contact form $dy - y'dx$. It is easy to see that

$$(\xi, \eta, \eta') \mapsto \begin{bmatrix} 1 & 0 & 0 \\ \xi & 1 & 0 \\ \eta & \eta' & 1 \end{bmatrix}$$

is a local section of $\mathrm{SL}(3) \to PT(\mathrm{RP}^2)$, and that the corresponding gauged Maurer-Cartan form is

$$\begin{bmatrix} 0 & 0 & 0 \\ d\xi & 0 & 0 \\ d\eta - \eta' d\xi & d\eta' & 0 \end{bmatrix}.$$

Let us write the connection form $\omega$ on $PTM$ as

$$\omega = \begin{bmatrix} \omega_0 & \omega_1 & \omega_2 \\ \omega^1 & \omega_1^1 & \omega_2^1 \\ \omega^2 & \omega_1^2 & \omega_2^2 \end{bmatrix}.$$

The development equations for a curve $\sigma$ in $PTM$ give

$$a\dot{\xi} - b(\dot{\xi} - \eta'\dot{\eta}) = \langle \dot{\sigma}, \omega^1 \rangle, \quad c(\dot{\xi} - \eta'\dot{\eta}) = \langle \dot{\sigma}, \omega^2 \rangle$$

for some functions $a(t)$, $b(t)$, $c(t)$. The compatibility conditions therefore require that if $\sigma$ is vertical $\langle \dot{\sigma}, \omega^1 \rangle = \langle \dot{\sigma}, \omega^2 \rangle = 0$, while if $\sigma$ is a lift $\langle \dot{\sigma}, \omega^2 \rangle = 0$. It follows that $\omega^1 = \alpha dx + \beta dy$, $\omega^2 = \gamma(dy - y'dx)$ for some functions $\alpha$, $\beta$ and $\gamma$ on $PTM$. Then by a change of gauge with

$$A = ((\alpha + y'\beta)\gamma)^{-1/3}, \quad B = (\alpha + y'\beta)A, \quad C = \gamma A, \quad E = \beta\gamma A$$

we can make $\omega^1 = dx$ and $\omega^2 = dy - y'dx$. We can further choose $D$ such that the $dy$ component of $\omega^2_1$ vanishes, so that $\omega^2_1 = k(dy' - fdx)$ for some functions $k$ and $f$ on $PTM$; $k$ must be nonzero since the forms $\omega^1$, $\omega^2$ and $\omega^2_1$ must be linearly independent. The remaining gauge freedom is just

$$h = \begin{bmatrix} 1 & 0 & F \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and with such $h$,

$$h^{-1}\omega h + h^{-1}dh = \begin{bmatrix} \omega_0 - F\omega^2 & * & * \\ \omega^1 & \omega^1_1 & * \\ \omega^2 & \omega^2_1 & \omega^2_2 + F\omega^2 \end{bmatrix}.$$

We may therefore choose the gauge for any projective connection on a manifold of elements so that $\omega^1 = dx$, $\omega^2 = dy - y'dx = \theta$, $\omega^2_1 = k(dy' - fdx) = k\phi$, and $\omega_0 - \omega^1_1$ does not contain $\theta$. This is the standard gauge for a projective connection on a manifold of elements.

A geodesic of this projective connection is a curve whose development satisfies $\dot{\eta} - \eta'\dot{\xi} = 0$ and $\dot{\eta}' = 0$; that is, a geodesic is a curve whose tangents are annihilated by both $\theta$ and $\phi$, and is therefore a solution of the second-order differential equation

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right).$$

To put it another way, the geodesics are the base integral curves of the vector field

$$\Gamma = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + f\frac{\partial}{\partial y'},$$

the 'second-order differential equation field' corresponding to the projective connection.

It may be shown, following Cartan, that given a second-order differential equation there is a unique $\mathfrak{sl}(3)$-valued projective connection $\omega$ whose geodesics are its solution curves, whose curvature form $\Omega$ is strictly upper triangular with $\Omega^1_2$ a multiple of $\theta \wedge \phi$. Note that the form of $\Omega$ is unchanged by a gauge transformation (though of course its entries will in general be changed). This connection is called the normal projective connection of $\Gamma$. In the standard gauge $\omega$ is given explicitly in terms of the basis of 1-forms $\{dx, \theta, \phi\}$ and $\Gamma$ by

$$\begin{bmatrix} -\frac{1}{3}f_{y'}dx + \frac{1}{6}f_{y'y'}\theta & f_y dx + \lambda\theta + \frac{1}{2}f_{y'y'}\phi & \mu dx + \nu\theta - \frac{1}{6}f_{y'y'y'}\phi \\ dx & \frac{2}{3}f_{y'}dx + \frac{1}{6}f_{y'y'}\theta & \frac{1}{6}f_{y'y'}\theta \\ \theta & \phi & -\frac{1}{3}f_{y'}dx - \frac{1}{3}f_{y'y'}\theta \end{bmatrix},$$

the coefficients $\lambda$, $\mu$ and $\nu$ being given by

$$\lambda = \tfrac{2}{3} f_{yy'} - \tfrac{1}{6}\Gamma(f_{y'y'}), \quad \mu = \tfrac{1}{3}(f_{yy'} - \Gamma(f_{y'y'})), \quad \nu = -\tfrac{1}{6}(f_{y'} f_{y'y'} + \Gamma(f_{y'y'y'})).$$

The curvature is

$$\Omega = \begin{bmatrix} 0 & bdx \wedge \theta & hdx \wedge \theta + k\theta \wedge \phi \\ 0 & 0 & a\theta \wedge \phi \\ 0 & 0 & 0 \end{bmatrix}$$

with $a = -\frac{1}{6} f_{y'y'y'y'}$; Cartan says 'it is pointless to calculate $b$, $h$ and $k$ explicitly', but in fact $b$ turns out to be the Berwald projective invariant of the spray determined by $\Gamma$. By the Bianchi identity, $h$ and $k$ are given in terms of $a$ and $b$ by

$$k = \tfrac{1}{3} f_{y'} a - \Gamma(a), \quad h = b_{y'}.$$

## 6. The invariant integrals

I now show that $a$ and $b$ are relative invariants – that is, that they get multiplied by certain factors under a coordinate transformation on $M$, so that if either vanishes in one coordinate system (on $M$) it does so in all.

Let $\hat{\omega}$ be the normal projective connection corresponding to the given differential equation $\Gamma$, expressed in terms of coordinates $\hat{x}$, $\hat{y}$, $\hat{y}'$, where $\hat{x} = \hat{x}(x, y)$, $\hat{y} = \hat{y}(x, y)$ is a coordinate transformation on the base manifold $M$, which is extended to a coordinate transformation on $PTM$. Since $\hat{\omega}$ is a projective connection associated with $\Gamma$, when expressed in terms of $x$, $y$ and $y'$ it is gauge equivalent to a projective connection $\omega$ in standard gauge; since the curvature form $\hat{\Omega}$ of $\hat{\omega}$ is strictly upper triangular, so is that of $\omega$.

Set $\hat{x}_x\hat{y}_y - \hat{x}_y\hat{y}_x = J \neq 0$ (the Jacobian determinant), and set $\Xi = \hat{x}_x + y'\hat{x}_y$. The induced transformation of $y'$ is $y' \mapsto \Xi^{-1}(\hat{y}_x + y'\hat{y}_y)$. Contact 1-forms are relative invariants of coordinate transformations on the base; in fact $\hat{\theta} = J\Xi^{-1}\theta$. Furthermore $d\hat{x} = \Xi dx + \hat{x}_y\theta = \Xi dx \pmod{\theta}$. Finally, $\hat{\phi}$ vanishes on $\Gamma$, so must take the form $\hat{\phi} = r\theta + s\phi$ for some functions $r$ and $s$; but

$$\langle \partial/\partial y', \theta\rangle = 0, \quad \langle \partial/\partial y', \phi\rangle = 1, \quad \langle \partial/\partial y', \hat{\phi}\rangle = \frac{\partial \hat{y}'}{\partial y'},$$

whence $s = J\Xi^{-2}$, and $\hat{\phi} = J\Xi^{-2}\phi \pmod{\theta}$. Thus $\hat{\theta}\wedge\hat{\phi} \propto \theta\wedge\phi$, so the curvature of $\omega$ is of precisely the same form as the curvature of the normal projective connection in the original coordinates; but the normal projective connection in the standard gauge is uniquely determined by the fact that its curvature is strictly upper triangular with $\Omega^1_2$ a multiple of $\theta \wedge \phi$, so $\omega$ is the normal projective connection in the original coordinates.

We have $\omega = h^{-1}\hat{\omega}h + h^{-1}dh$ where (in the notation of the preceeding section) $AB^{-1} = \Xi^{-1}$, $AC^{-1} = J^{-1}\Xi$ and $BC^{-1} = J^{-1}\Xi^2$. Then $\hat{\Omega} = h\Omega h^{-1}$, or

$$\hat{a}\hat{\theta} \wedge \hat{\phi} = J^{-1}\Xi^2 a\theta \wedge \phi, \quad \hat{b}d\hat{x} \wedge \hat{\theta} = \Xi^{-1} bdx \wedge \theta,$$

so that

$$\hat{a} = J^{-3}\Xi^5 a, \quad \hat{b} = (J\Xi)^{-1} b.$$

This confirms that $a$ and $b$ are relative invariants. Knowing their transformation laws, one can explain the remark about the existence of invariant integrals. Note for example that $\hat{a}^5\hat{b} = (J^{-2}\Xi^3)^8 a^5 b$ and $\hat{\theta} \wedge \hat{\phi} = (J^2\Xi^{-3})\theta \wedge \phi$; thus $\hat{a}^{\frac{5}{8}}\hat{b}^{\frac{1}{8}}\hat{\theta} \wedge \hat{\phi} = a^{\frac{5}{8}} b^{\frac{1}{8}} \theta \wedge \phi$. The others work in a similar fashion.

If both $a$ and $b$ vanish, then by the Bianchi identity the curvature vanishes and the structure is locally diffeomorphic to $PT(\mathrm{RP}^2)$ with the straight line geodesic spray. The vanishing of $a$ alone has the following equivalent consequences: $f$ is cubic in $y'$; the Douglas tensor vanishes; the differential equation is projectively affine. In such a case the connection should reduce to a projective connection of the first kind (the kind considered in Section 4), and in particular it should give rise to a connection form on the base manifold $M$. It is instructive to see how this arises. The requisite condition is that for any vector field $V$ vertical with respect to the projection $PTM \to M$, $\mathcal{L}_V\omega$ should be infinitesimally gauge equivalent to $\omega$ by a gauge transformation of the first kind. That is to say, there should be a function $K$ taking its values in $\mathfrak{h}$, the Lie algebra of this gauge group, such that $\mathcal{L}_V\omega = [\omega, K] + dK$. Now

$$\mathcal{L}_V\omega = V \lrcorner\, d\omega + d\langle V, \omega\rangle = V \lrcorner\, \Omega + [\omega, \langle V, \omega\rangle] + d\langle V, \omega\rangle,$$

and $\langle V, \omega\rangle$ takes its values in $\mathfrak{h}$. But

$$V \lrcorner\, \Omega = \begin{bmatrix} 0 & 0 & -k\theta \\ 0 & 0 & -a\theta \\ 0 & 0 & 0 \end{bmatrix},$$

so if $a = 0$, $V \lrcorner\, \Omega = 0$ and $\omega$ satisfies the requisite condition with $K = \langle V, \omega\rangle$.

The significance of $b$ vanishing alone can be explained as follows. If we set

$$\varpi = \begin{bmatrix} -\omega_4^2 & \omega_2^1 & -\omega_2 \\ \omega_1^2 & -\omega_1^1 & \omega_1 \\ -\omega^2 & \omega^1 & -\omega_0 \end{bmatrix} = \begin{bmatrix} * & * & * \\ \phi & * & * \\ -\theta & dx & * \end{bmatrix},$$

then the curvature $\Pi$ of $\varpi$ (given by $\Pi = d\varpi + [\varpi \wedge \varpi]$) takes the form

$$\Pi = \begin{bmatrix} 0 & a\theta \wedge \phi & * \\ 0 & 0 & bdx \wedge \theta \\ 0 & 0 & 0 \end{bmatrix};$$

the positions of $a$ and $b$ are interchanged. If $b = 0$ then by a similar argument to the one already used, for any vector field $W$ which is a multiple of the second-order differential equation field $\Gamma$, $\mathcal{L}_W\varpi = [\varpi, L] + dL$ where $L = \langle \varpi, W\rangle$ is an $\mathfrak{h}$-valued function, and therefore $\varpi$ reduces to a normal projective connection of the first kind on $N = PTM/\langle\Gamma\rangle$, the quotient of $PTM$ by the flow of $\Gamma$. There is an unsuspected duality here, which will be clearer if we write $S$ for the manifold of elements. Then $S$ is doubly fibred: we have the original fibration $S \to M$, but also the fibration $S \to N$ in which the differential equation field becomes vertical, and the original vertical fibres become the integral paths of a new second-order differential equation field. Then $b = 0$ is the condition for the new second-order differential equation field to be projectively affine. In the underlying Klein geometry, this interchange of roles is just the usual duality transformation of projective geometry. This duality is discussed in general terms by Bryant [2], and in more detail in [4]; see also [6], [7].

ACKNOWLEDGEMENTS. I am a Guest Professor at Ghent University: I am grateful to the university, and in particular to the Department of Mathematical Physics and Astronomy, for its hospitality. I am also grateful to Pawel Nurowski for several enlightening conversations.

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*(Received: March 31, 2004)*

M. CRAMPIN
DEPARTMENT OF MATHEMATICAL PHYSICS AND ASTRONOMY
GHENT UNIVERSITY
KRIJGSLAAN 281, B–9000 GENT
BELGIUM
*Address for correspondence:*
65 MOUNT PLEASANT, ASPLEY
GUISE, BEDS MK17 8JX
UK
E-MAIL: Crampin@btinternet.com