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On the differential geometry of the Euler–Lagrange equations, and the inverse problem of Lagrangian dynamics

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Abstract. The conditions for a system of second-order differential equations to be derivable from a Lagrangian—the conditions of self-adjointness, in the terminology of Santilli and others—are related, in the time-independent case, to the differential geometry of the tangent bundle of configuration space. These conditions are simply expressed in terms of the horizontal distribution which is associated with any vector field representing a system of second-order differential equations. Necessary and sufficient conditions for such a vector field to be derivable from a Lagrangian may be stated as the existence of a two-form with certain properties: it is interesting that it is a deduction, not an assumption, that this two-form is closed and thus defines a symplectic structure. Some other differential geometric properties of Euler–Lagrange second-order differential equations are described.

1. Introduction

In differential geometric terms, the dynamics of a time-independent Lagrangian dynamical system is determined by a vector field on the tangent bundle of a differentiable manifold (the configuration space of the system), of the type known for obvious reasons as a 'second-order differential equation'.

The inverse problem of Lagrangian dynamics is to give necessary and sufficient conditions for a system of second-order differential equations to be the Euler–Lagrange equations of some regular Lagrangian function. Such conditions are to be found in a recent book of Santilli (1978); they have been known for some time, having apparently been first identified by Helmholtz, and having been studied in detail by, for example, Douglas (1941). There has been renewed interest in the inverse problem recently, especially in relation to symmetries and first integrals for second-order differential equations (Sarlet 1981, Schafir 1981).

Any system of second-order differential equations whose coefficients do not depend explicitly on the time may be represented by a suitable vector field on a tangent bundle. The purpose of the present paper is to interpret these conditions for the equations to be derivable from a Lagrangian in terms of differential geometric constructions associated with the second-order differential equation vector field.

The Euler–Lagrange equations are normally expressed, when the second derivatives are put in evidence, and the summation convention is used, in the form

$$g_{ij}\ddot{q}^j + h_i = 0$$

where in the time-independent case g_{ij} and h_i are functions of q^i and $\dot{q}^i$. (Indices i, j, and later k, l, range and sum from 1 to n, the number of degrees of freedom of the system.)

Here

$$g_{ij}=\partial^2 L/\partial\dot q^i\,\partial\dot q^j$$

where L is the Lagrangian function, and the Lagrangian is regular when the matrix (g_{ij}) is non-singular. A system of second-order differential equations of this type is derivable from a Lagrangian provided it is self-adjoint. The conditions for self-adjointness are a system of relations between the functions g_{ij} and h_i and their derivatives, which may be deduced by considering variations of the system of equations.

It is more natural for the task in hand to consider only systems of second-order differential equations solved for the second derivatives:

$$\ddot q^i=f^i(q^j,\dot q^j). \tag{1}$$

Self-adjointness is not invariant under such a change of form. The necessary and sufficient conditions for a system of differential equations in this form to be derivable from a Lagrangian is that there exist a matrix (g_{ij}) of functions of q^i and $\dot q^i$, nowhere singular, such that the equations

$$g_{ij}\ddot q^j-g_{ij}f^j=0$$

are self-adjoint. In terms of the functions f^i, these conditions are given explicitly as follows (Douglas 1941, Santilli 1978): there should exist functions g_{ij} such that

$$\det(g_{ij})\neq 0, \tag{2a}$$

$$g_{ji}=g_{ij}, \tag{2b}$$

$$\partial g_{ij}/\partial\dot q^k=\partial g_{ik}/\partial\dot q^j, \tag{2c}$$

$$\frac{\mathrm d}{\mathrm dt}(g_{ij})+\frac12\frac{\partial f^k}{\partial\dot q^j}g_{ik}+\frac12\frac{\partial f^k}{\partial\dot q^i}g_{kj}=0, \tag{2d}$$

$$g_{ik}\left[\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial f^k}{\partial\dot q^j}\right)-2\frac{\partial f^k}{\partial q^j}-\frac12\frac{\partial f^l}{\partial\dot q^j}\frac{\partial f^k}{\partial\dot q^l}\right]=g_{jk}\left[\frac{\mathrm d}{\mathrm dt}\left(\frac{\partial f^k}{\partial\dot q^i}\right)-2\frac{\partial f^k}{\partial q^i}-\frac12\frac{\partial f^l}{\partial\dot q^i}\frac{\partial f^k}{\partial\dot q^l}\right]. \tag{2e}$$

2. The horizontal distribution of a second-order differential equation

A vector field Γ on the tangent bundle $\pi: T(M)\to M$ of a differentiable manifold M (of dimension n) is called a second-order differential equation if for each point (q,u) of $T(M)$ (where $u\in T_q(M)$)

$$\pi_*\Gamma_{(q,u)}=u.$$

Such a vector field Γ has the local coordinate expression

$$u^i\,\partial/\partial q^i+f^i(q^j,u^j)\,\partial/\partial u^i$$

and its integral curves satisfy

$$\dot q^i=u^i,\qquad \dot u^i=f^i(q^j,u^j).$$

Consequently, the projections of its integral curves onto M are solutions of the differential equations (1),

$$\ddot q^i=f^i(q^j,\dot q^j).$$

Any second-order differential equation vector field may be used to construct a horizontal distribution on $T(M)$ (Crampin 1971, Yano and Ishihara 1973). A horizontal distribution is an assignment to each point (q, u) of $T(M)$ of a vector subspace of $T_{(q,u)}(T(M))$ which is complementary to the subspace of tangent vectors to the fibres (the vertical subspace). This generalises the construction of the horizontal distribution arising from a connection, with which it coincides when Γ is the geodesic second-order differential equation: one could say that each second-order differential equation gives rise to a nonlinear (that is, not necessarily linear) connection. This horizontal distribution plays an important part in what follows, so it is necessary to describe its construction and some of its properties; to do this, it is necessary to mention some standard constructions on tangent bundles.

To construct the vertical lift, X^{v}, of a vector field X on a differential manifold M to its tangent bundle $T(M)$, one exploits that fact that any tangent space to the fibre of $T(M)$ over any point q of M is naturally isomorphic to $T_q(M)$, to locate a vertical copy of X_q at each point (q, u). In local coordinates, the vertical lift of X (whose components are X^i) is

$$X^i\, \partial/\partial u^i.$$

It is easy to verify that the bracket of two vertical lifts is zero. The complete lift, X^{c}, of a vector field X on M, to $T(M)$, is obtained by extending the action of the one-parameter group generated by X on M, to $T(M)$, by incorporating its tangent maps. The local coordinate expression of the complete lift is

$$X^i \frac{\partial}{\partial q^i} + u^j \frac{\partial X^i}{\partial q^j} \frac{\partial}{\partial u^i}.$$

Whereas the complete lift of a vector field depends on the first partial derivatives of its components, its vertical lift depends only on their values: the vertical lift can therefore be defined as well for a tangent vector at a point of M as for a vector field on M. This fact allows one to define a type (1, 1) tensor field S on $T(M)$ whose effect is to map each tangent vector ζ at a point (q, u) of $T(M)$ to a vertical tangent vector, by first projecting it into $T_q(M)$ and then lifting the result vertically to (q, u): thus

$$S_{(q,u)}(\zeta) = (\pi_* \zeta)^{\mathrm{v}}_{(q,u)}. \tag{3}$$

In coordinates, one finds that

$$S(\partial/\partial q^i) = \partial/\partial u^i, \qquad S(\partial/\partial u^i) = 0. \tag{4}$$

Finally, the dilation vector field Δ on $T(M)$ is the vertical vector field generated by the one-parameter group of dilations of tangent vectors $(q, u) \to (q, \mathrm{e}^s u)$. The coordinate expression for Δ is

$$u^i\, \partial/\partial u^i.$$

Suppose given a second-order differential equation Γ on $T(M)$. For any vector field X on M, the value of the vector field

$$\tfrac{1}{2}([X^{\mathrm{v}}, \Gamma] + X^{\mathrm{c}})$$

on $T(M)$ at a point (q, u) depends on X, like the vertical lift, only through its value at q. One may thus define, for any vector $w \in T_q(M)$, a vector $w^{\mathrm{h}}_{(q,u)}$ in $T_{(q,u)}(T(M))$, by

$$w^{\mathrm{h}}_{(q,u)} = \tfrac{1}{2}([W^{\mathrm{v}}, \Gamma] + W^{\mathrm{c}})_{(q,u)}$$

where W is any vector field on M such that $W_q = w$. It happens that $w^{\mathrm{h}}_{(q,u)}$ projects onto w; and that the subset

$$\{w^{\mathrm{h}}_{(q,u)} \mid w \in T_q(M)\}$$

of $T_{(q,u)}((T(M))$ is a subspace complementary to the vertical subspace. This subspace is accordingly called the horizontal subspace of $T_{(q,u)}((T(M))$ determined by Γ. The vector $w^{\mathrm{h}}_{(q,u)}$ is called the horizontal lift of w to (q, u). Any vector field X on M has a horizontal lift X^{h} to $T(M)$ given by

$$X^{\mathrm{h}} = \tfrac{1}{2}([X^{\mathrm{v}}, \Gamma] + X^{\mathrm{c}}).$$

Each vector or vector field on $T(M)$ has horizontal and vertical components. For example, since

$$[X^{\mathrm{v}}, \Gamma] = X^{\mathrm{h}} + (X^{\mathrm{h}} - X^{\mathrm{c}})$$

and $X^{\mathrm{h}} - X^{\mathrm{c}}$ is vertical (both vector fields project on X), the horizontal component of $[X^{\mathrm{v}}, \Gamma]$ is X^{h}.

As well as determining a nonlinear connection, the horizontal distribution may also be used to define an almost complex structure on $T(M)$, that is, a type $(1, 1)$ tensor field J satisfying $J^2 = -1$ (where 1 is the identity $(1, 1)$ tensor field, or Kronecker delta). The tensor field J is given explicitly as follows: for $w \in T_q(M)$,

$$J_{(q,u)}(w^{\mathrm{h}}_{(q,u)}) = w^{\mathrm{v}}_{(q,u)}, \qquad J_{(q,u)}(w^{\mathrm{v}}_{(q,u)}) = -w^{\mathrm{h}}_{(q,u)}.$$

Note that J agrees with S on horizontal vectors.

Suppose that in terms of local coordinates

$$\Gamma = u^i\, \partial/\partial q^i + f^i\, \partial/\partial u^i.$$

Then the horizontal distribution is spanned by the local vector fields

$$H_i = \frac{\partial}{\partial q^i} + \frac{1}{2}\frac{\partial f^j}{\partial u^i}\frac{\partial}{\partial u^j};$$

these are the horizontal lifts of the coordinate vector fields on M. The vector fields $\{H_i, \partial/\partial u^i\}$ form a basis of local vector fields, or a frame, for $T(M)$; the dual basis of one-forms is $\{\mathrm{d}q^i, \theta^i\}$, where

$$\theta^i = \mathrm{d}u^i - \tfrac{1}{2}(\partial f^i/\partial q^j)\, \mathrm{d}q^j. \tag{5}$$

The Lie derivatives of these one-forms by Γ, which will be of interest later, are

$$L_\Gamma\, \mathrm{d}q^i = \mathrm{d}u^i = \tfrac{1}{2}(\partial f^i/\partial u^j)\, \mathrm{d}q^j + \theta^i, \tag{6}$$

$$\begin{aligned} L_\Gamma \theta^i &= -\tfrac{1}{2}\Gamma(\partial f^i/\partial u^j)\, \mathrm{d}q^j - \tfrac{1}{2}(\partial f^i/\partial u^j)\, \mathrm{d}u^j + \mathrm{d}f^i \\ &= -\tfrac{1}{2}A^i_j\, \mathrm{d}q^j + \tfrac{1}{2}(\partial f^i/\partial u^j)\theta^j, \end{aligned} \tag{7}$$

where

$$A^i_j = \Gamma\left(\frac{\partial f^i}{\partial u^j}\right) - 2\frac{\partial f^i}{\partial q^j} - \frac{1}{2}\frac{\partial f^i}{\partial u^k}\frac{\partial f^k}{\partial u^j}.$$

These are recognisable as the quantities appearing in equations (2*e*), when it is recalled that operating with Γ is equivalent to taking the total time derivative.

3. The Lagrangian case

Associated with a Lagrangian function L on $T(M)$ there is an exact two-form

$$\omega = \mathrm{d}(\mathrm{d}L \circ S).$$

This two-form has maximal rank when L is regular. The function

$$E = \Delta(L) - L$$

is the energy associated with L; the vector field Γ defined by

$$\Gamma \lrcorner \omega = -\mathrm{d}E \tag{8}$$

is a second-order differential equation which represents the Euler–Lagrange equations for L. It follows from (8) that

$$L_\Gamma \omega = 0.$$

The two-form ω defines a symplectic structure on $T(M)$. This fact, and the properties of Γ with respect to ω, closely parallels the situation in Hamiltonian mechanics; in fact ω is the pullback to $T(M)$ of the canonical two-form Ω on $T^*(M)$ by the Legendre map. However, the crucial difference between the Lagrangian and Hamiltonian symplectic structures is that the latter is an intrinsic structure of the cotangent bundle, being given by the canonical two-form, while the former involves the Lagrangian function in its construction. Whereas the necessary and sufficient condition for a vector field Z on $T^*(M)$ to represent Hamilton's equations for some (at least locally defined) Hamiltonian function is that $L_Z\Omega = 0$, the necessary and sufficient conditions for a second-order differential equation Γ on $T(M)$ to represent the Euler–Lagrange equations for some (at least locally defined) Lagrangian function must include conditions establishing the existence of a symplectic structure with appropriate properties.

It is in fact enough to ask for rather less than the closure of the maximal rank two-form defining the symplectic structure. What is required appears in the following theorem. Its statement must be prefaced by the reminder that a subspace of a tangent space to a differential manifold is called a Lagrangian subspace for a two-form if the two-form vanishes on each pair of vectors from the subspace.

Theorem. Let Γ be a second-order differential equation on the tangent bundle of a differentiable manifold M. Necessary and sufficient conditions for Γ to be derivable from a regular Lagrangian are that there exist on $T(M)$ a two-form ω, of maximal rank, for which $L_\Gamma\omega = 0$, and such that all vertical subspaces are Lagrangian both for ω and for $H \lrcorner \mathrm{d}\omega$ where H is any horizontal vector.

Proof. In the case of a second-order differential equation derived from a regular Lagrangian L, the two-form $\omega = \mathrm{d}(\mathrm{d}L \circ S)$ has maximal rank, satisfies $L_\Gamma\omega = 0$, and certainly satisfies the condition on $\mathrm{d}\omega$ since $\mathrm{d}\omega = 0$ (Crampin 1971). It remains to be shown that vertical subspaces are Lagrangian for ω. For any vector fields X, Y on M, their vertical lifts satisfy $S(X^\mathrm{v}) = S(Y^\mathrm{v}) = 0$, by (3) or (4), and $[X^\mathrm{v}, Y^\mathrm{v}] = 0$. Thus

$$\omega(X^\mathrm{v}, Y^\mathrm{v}) = X^\mathrm{v}(S(Y^\mathrm{v})(L)) - Y^\mathrm{v}(S(X^\mathrm{v})(L)) - S([X^\mathrm{v}, Y^\mathrm{v}])(L) = 0.$$

This suffices to show that vertical subspaces are Lagrangian.

Suppose, for the converse, that ω is a two-form with the given properties. It will be shown first that ω is actually closed. Note first that for any vector fields X, Y, Z on M,

$$\mathrm{d}\omega(X^{\mathrm{v}}, Y^{\mathrm{v}}, Z^{\mathrm{v}}) = 0$$

since ω vanishes on pairs of vertical vectors, and the bracket of vertical lifts is zero. By hypothesis,

$$\mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{v}}, Z^{\mathrm{v}}) = 0.$$

It follows that for any vector field W on $T(M)$,

$$\mathrm{d}\omega(W, Y^{\mathrm{v}}, Z^{\mathrm{v}}) = 0. \tag{9}$$

Now

$$L_\Gamma(\mathrm{d}\omega) = \mathrm{d}(L_\Gamma\omega) = 0,$$

and so from the fact that $\Gamma(\mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{v}}, Z^{\mathrm{v}})) = 0$ it follows that

$$\mathrm{d}\omega([\Gamma, X^{\mathrm{h}}], Y^{\mathrm{v}}, X^{\mathrm{v}}) + \mathrm{d}\omega(X^{\mathrm{h}}, [\Gamma, Y^{\mathrm{v}}], Z^{\mathrm{v}}) + \mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{v}}, [\Gamma, Z^{\mathrm{v}}]) = 0.$$

The first term is zero by virtue of (9); only the horizontal components of $[\Gamma, Y^{\mathrm{v}}]$ and $[\Gamma, Z^{\mathrm{v}}]$ contribute to the remaining terms, and these are $-Y^{\mathrm{h}}$ and $-Z^{\mathrm{h}}$ respectively. Thus

$$\mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{h}}, Z^{\mathrm{v}}) = \mathrm{d}\omega(X^{\mathrm{h}}, Z^{\mathrm{h}}, Y^{\mathrm{v}}).$$

But then

$$\begin{aligned}\mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{h}}, Z^{\mathrm{v}}) &= \mathrm{d}\omega(X^{\mathrm{h}}, Z^{\mathrm{h}}, Y^{\mathrm{v}}) = -\mathrm{d}\omega(Z^{\mathrm{h}}, X^{\mathrm{h}}, Y^{\mathrm{v}}) \\ &= -\mathrm{d}\omega(Z^{\mathrm{h}}, Y^{\mathrm{h}}, X^{\mathrm{v}}) = \mathrm{d}\omega(Y^{\mathrm{h}}, Z^{\mathrm{h}}, X^{\mathrm{v}}) \\ &= \mathrm{d}\omega(Y^{\mathrm{h}}, X^{\mathrm{h}}, Z^{\mathrm{v}}) = -\mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{h}}, Z^{\mathrm{v}})\end{aligned}$$

and so

$$\mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{h}}, Z^{\mathrm{v}}) = 0. \tag{10}$$

It is clear at this stage that $\mathrm{d}\omega$ vanishes except possibly when all three of its arguments are horizontal. But applying Γ again to equation (10), one finds that

$$\mathrm{d}\omega([\Gamma, X^{\mathrm{h}}], Y^{\mathrm{h}}, X^{\mathrm{v}}) + \mathrm{d}\omega(X^{\mathrm{h}}, [\Gamma, Y^{\mathrm{h}}], Z^{\mathrm{v}}) + \mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{h}}, [\Gamma, Z^{\mathrm{v}}]) = 0$$

whence

$$\mathrm{d}\omega(X^{\mathrm{h}}, Y^{\mathrm{h}}, Z^{\mathrm{h}}) = 0.$$

It follows that $\mathrm{d}\omega = 0$.

Thus, locally at least, ω is exact: say $\omega = \mathrm{d}\psi$ for some one-form ψ defined on some region R of $T(M)$. The restriction of $\mathrm{d}\psi$ to vertical subspaces is zero, since they are Lagrangian for ω; thus the restriction of ψ to each fibre is exact, and there is a function F on R such that $\psi(X^{\mathrm{v}}) = X^{\mathrm{v}}(F)$ for any vector field X on $\pi(R)$. Set $\phi = \psi - \mathrm{d}F$: then $\mathrm{d}\phi = \omega$ and $\phi(X^{\mathrm{v}}) = 0$.

Since $L_\Gamma\omega = 0$ and vertical subspaces are Lagrangian for ω,

$$\omega([\Gamma, X^{\mathrm{v}}], Y^{\mathrm{v}}) + \omega(X^{\mathrm{v}}, [\Gamma, Y^{\mathrm{v}}]) = 0.$$

Again, only the horizontal components of the brackets signify, and so

$$\omega(X^{\mathrm{h}}, Y^{\mathrm{v}}) = \omega(Y^{\mathrm{h}}, X^{\mathrm{v}}).$$

Thus for vector fields X, Y on $\pi(R)$,

$$\begin{aligned}\omega(X^{\mathrm{h}}, Y^{\mathrm{v}}) &= X^{\mathrm{h}}(\phi(Y^{\mathrm{v}})) - Y^{\mathrm{v}}(\phi(X^{\mathrm{h}})) - \phi([X^{\mathrm{h}}, Y^{\mathrm{v}}]) \\ &= -Y^{\mathrm{v}}(\phi(X^{\mathrm{h}})) = \omega(Y^{\mathrm{h}}, X^{\mathrm{v}}) = -X^{\mathrm{v}}(\phi(Y^{\mathrm{h}}))\end{aligned}$$

since $[X^{\mathrm{h}}, Y^{\mathrm{v}}]$ is vertical. Set $\Phi = \phi \circ J$, where J is the almost complex structure determined by Γ: then $X^{\mathrm{v}}(\Phi(Y^{\mathrm{v}})) = Y^{\mathrm{v}}(\Phi(X^{\mathrm{v}}))$. As before, the restriction of $\mathrm{d}\Phi$ to vertical subspaces is zero, and as before, there is a function K on R such that $\Phi(X^{\mathrm{v}}) = -X^{\mathrm{v}}(K)$. Thus $\phi(X^{\mathrm{v}}) = 0$ and $\phi(X^{\mathrm{h}}) = \mathrm{d}K(X^{\mathrm{v}})$, which is to say that $\phi = \mathrm{d}K \circ S$. The function K is determined up to the addition of a function constant along the fibres of $T(M)$.

Now ω is constructed as if from the Lagrangian K, and is of maximal rank, so the equation $\Gamma_K \lrcorner\, \omega = -\mathrm{d}E_K$ (where $E_K = \Delta(K) - K$ is the energy corresponding to K) determines a vector field Γ_K which is a second-order differential equation. On the other hand, since $L_\Gamma \omega = \mathrm{d}(\Gamma \lrcorner\, \omega) = 0$, there is at least locally some function E such that $\Gamma \lrcorner\, \omega = -\mathrm{d}E$. Since Γ_K and Γ are both second-order differential equations, $\Gamma_K - \Gamma$ is vertical; and since $(\Gamma_K - \Gamma) \lrcorner\, \omega = \mathrm{d}(E - E_K)$, and ω vanishes on pairs of vertical vectors, any vertical derivative of $E - E_K$ is zero. Thus $E - E_K$ is constant on fibres, and so $E - E_K = \pi^* G$ for some function G defined on some region of M. Set $L = K - \pi^* G$: then $\mathrm{d}L \circ S = \mathrm{d}K \circ S = \phi$, and $\Delta(L) - L = \Delta(K) - K + \pi^* G = E$. Thus Γ is derived from the (locally defined) Lagrangian L.

4. The self-adjointness conditions

The conditions satisfied by the two-form ω will now be expressed in terms of coordinates. Suppose that

$$\omega = a_{ij}\, \mathrm{d}q^i \wedge \mathrm{d}q^j + g_{ij}\, \mathrm{d}q^i \wedge \theta^j$$

where $a_{ij} + a_{ji} = 0$; the one-forms θ^i are those introduced in equation (5) and chosen so that $\{\mathrm{d}q^i, \theta^i\}$ is the basis of one-forms dual to $\{H_i, \partial/\partial u^i\}$. The absence of terms in $\theta^i \wedge \theta^j$ from the expression for ω is a consequence of the condition that vertical subspaces are Lagrangian for ω. Then, using equations (6) and (7), one finds that

$$\begin{aligned}L_\Gamma \omega = {} & (\Gamma(a_{ij}) + a_{ik}\, \partial f^k/\partial u^j - \tfrac{1}{2} g_{ik} A^k_j)\, \mathrm{d}q^i \wedge \mathrm{d}q^j \\ & + (2a_{ij} + \Gamma(g_{ij}) + \tfrac{1}{2} g_{ik}\, \partial f^k/\partial u^j + \tfrac{1}{2} g_{kj}\, \partial f^k/\partial u^i)\, \mathrm{d}q^i \wedge \theta^j + g_{ij} \theta^i \wedge \theta^j.\end{aligned}$$

Equating separately to zero the coefficients of linearly independent terms, to find the consequences of $L_\Gamma \omega$ being zero, one finds that

$$g_{ji} = g_{ij} \tag{2b}$$

and therefore

$$\begin{aligned}&a_{ij} = 0, \\ &\Gamma(g_{ij}) + \tfrac{1}{2} g_{ik}\, \partial f^k/\partial u^j + \tfrac{1}{2} g_{kj}\, \partial f^k/\partial u^i = 0,\end{aligned} \tag{2d}$$

since these are respectively the skew-symmetric and symmetric parts of the coefficient of $\mathrm{d}q^i \wedge \theta^j$; and finally that

$$g_{ik} A^k_j = g_{jk} A^k_i. \tag{2e}$$

Thus

$$\omega = g_{ij}\, dq^i \wedge \theta^j. \qquad (11)$$

Then ω is of maximal rank if and only if (g_{ij}) is non-singular, that is,

$$\det(g_{ij}) \neq 0. \qquad (2a)$$

The only terms of interest in $d\omega$ are those involving $dq^i \wedge \theta^j \wedge \theta^k$: the coefficient of such a term is easily shown to be $\partial g_{ij}/\partial u^k$, and so the condition on $d\omega$, which is equivalent to the vanishing of these terms, amounts to

$$\partial g_{ij}/\partial u^k = \partial g_{ik}/\partial u^j. \qquad (2c)$$

The self-adjointness conditions are thus equivalent to the conditions on the two-form ω stated in the theorem. The theorem is a geometrical proof of the necessity and sufficiency of the existence of g_{ij} for which the self-adjointness conditions are satisfied for the existence of a Lagrangian.

5. Further geometry of ω

The expression (11) for ω in terms of $\{dq^i, \theta^j\}$ shows that as well as vertical subspaces, horizontal subspaces are also Lagrangian for it. It follows that for any vector fields T, U on $T(M)$,

$$\omega(J(T), J(U)) = \omega(T, U)$$

where J is the almost complex structure defined by Γ. In fact, taking T, U to be successively X^{v}, Y^{v}; X^{v}, Y^{h}; X^{h}, Y^{h} one finds that

$$\omega(J(X^{\mathrm{v}}), J(Y^{\mathrm{v}})) = \omega(-X^{\mathrm{h}}, -Y^{\mathrm{h}}) = 0 = \omega(X^{\mathrm{v}}, Y^{\mathrm{v}}),$$
$$\omega(J(X^{\mathrm{v}}), J(Y^{\mathrm{h}})) = \omega(-X^{\mathrm{h}}, Y^{\mathrm{v}}) = -\omega(Y^{\mathrm{h}}, X^{\mathrm{v}}) = \omega(X^{\mathrm{v}}, Y^{\mathrm{h}}),$$
$$\omega(J(X^{\mathrm{h}}), J(Y^{\mathrm{h}})) = \omega(X^{\mathrm{v}}, Y^{\mathrm{v}}) = 0 = \omega(X^{\mathrm{h}}, Y^{\mathrm{h}}).$$

If, further, one defines a type (0, 2) tensor field g on $T(M)$ by

$$g(T, U) = \omega(T, J(U))$$

then g is a pseudo-Hermitian metric on $T(M)$: that is to say, g is symmetric and non-degenerate, and satisfies

$$g(J(T), J(U)) = g(T, U);$$

however, g is not necessarily positive definite. The two-form ω is the fundamental two-form of g; since ω is closed, when g is positive definite it is an almost Kaehler metric (Kobayashi and Nomizu 1969). The vertical and horizontal subspaces are orthogonal with respect to g; one finds that

$$g(X^{\mathrm{v}}, Y^{\mathrm{v}}) = g(X^{\mathrm{h}}, Y^{\mathrm{h}}) = \omega(X^{\mathrm{h}}, Y^{\mathrm{v}});$$

the components of g are the g_{ij}, that is, the second derivatives of the Lagrangian with respect to the velocities.

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