4. Tom Jenkyns and Eric Muller, Triangular triples from ceilings to floors, *Amer. Math. Monthly* **107** (2000), 634–639.
5. J. H. Jordan, R. Welch and R. J. Wisner, Triangles with integer sides, *Amer. Math. Monthly* **86** (1979), 686–689.
6. Nicholas Krier and Bennet Manvel, Counting integer triangles, this MAGAZINE **71** (1998), 291–295.

A Generalization of Odd and Even Functions

LAWRENCE J. CRONE
American University
Washington, DC 20016-8050
lcrone@american.edu

The familiar concept of odd and even functions can be formulated in terms of the composition of functions. For each real number λ, let M_λ be the function that multiplies by λ. That is, $M_\lambda(x) = \lambda x$. Suppose f_o is an odd function and f_e is an even function. Then $f_o(-x) = -f_o(x)$, and $f_e(-x) = f(x)$, which can be written

$$f_o \circ M_{-1} = M_{-1} \circ f_o, \tag{1}$$

and

$$f_e \circ M_{-1} = f_e. \tag{2}$$

In other words, the odd functions are those that commute with M_{-1}, and the even functions are those that are invariant with respect to M_{-1}. We generalize these concepts by allowing the function M_{-1} in equations (1) and (2) to be replaced by any of an appropriate class of functions.

Algebra of functional composition In this note we will only consider functions that map zero onto zero and which are continuous on an interval of the form $(-\epsilon, \epsilon)$, where ϵ may depend on the function. We are assured that the composition of any two such functions is defined, and that the composite function also has these properties. For convenience we call such functions *continuously fixed at zero.* The functions M_1 and M_0 act as identity and zero functions, respectively, since for each f, $M_1 \circ f = f = f \circ M_1$, and $M_0 \circ f = M_0 = f \circ M_0$. Although functional composition is not commutative, it is associative, and we usually just write $f \circ g \circ h$ rather than $(f \circ g) \circ h$, or $f \circ (g \circ h)$. There is a right distributive law: $(f + g) \circ h = f \circ h + g \circ h$. Functional composition is not in general left distributive, but we do have a special case: $M_\lambda \circ (g + h) = M_\lambda \circ g + M_\lambda \circ h$. We will also need to use the fact that if f is strictly monotonic on an open interval containing zero, then, restricted to that interval, f has a unique functional inverse, which we write f^{-1}. The functional inverse satisfies the equation $f \circ f^{-1} = M_1 = f^{-1} \circ f$. If f and g are invertible, then so is $f \circ g$ and $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. (The basic properties of composition of functions and of functional inverses is covered in most calculus texts. For example see Larson [**1**] and Stewart [**2**].)

Symmetry functions The function M_{-1} which appears in the definition of odd and even functions has two essential properties: $M_{-1} \circ M_{-1} = M_1$, and M_{-1} is monotonically decreasing. We will say that a function continuously fixed at zero is a symmetry function, or simply, a symmetry, if it is its own inverse, and is strictly decreasing in an open interval containing zero. There are many symmetry functions. For example, for each real number α the function $-x/(1+\alpha x)$ is a symmetry. When $\alpha = 0$, this reduces to M_{-1}. Also, for each $\alpha > 0$ the function

$$S(x) = \begin{cases} -\alpha x & x \geq 0 \\ -x/\alpha & x < 0 \end{cases}$$

is a symmetry.

The graphs of symmetry functions are symmetric with respect to the line $y = x$. Symmetry functions are defined by equations that are symmetric in x and y. For example, the function $y = -x/(1+\alpha x)$ is the solution to the symmetric equation $x + y + \alpha xy = 0$. However, not every implicit solution of a symmetric equation is a symmetry. The equation $x^2 + y^2 = 1$ does not define any function that maps zero onto zero. The equation $xy = 0$ has the solution $y = 0$, but that is not a symmetry.

S-odd and S-even functions For each symmetry S, we say that a function is S-odd if it satisfies the condition

$$f \circ S = S \circ f \tag{3}$$

and that a function is S-even if it satisfies

$$f \circ S = f. \tag{4}$$

The S-odd and S-even functions corresponding to the symmetry $S = M_{-1}$ are the ordinary odd and even functions. More interesting is the symmetry

$$S(x) = \begin{cases} -2x & x \geq 0 \\ -x/2 & x < 0 \end{cases}.$$

For every real β, the function

$$f_e(x) = \begin{cases} \beta x & x \geq 0 \\ -\beta x/2 & x < 0 \end{cases}$$

is S-even. The situation is a little more complicated when we look for corresponding S-odd functions. For positive β,

$$f_o(x) = \beta x$$

and

$$g_o(x) = \begin{cases} -\beta x & x \geq 0 \\ -\beta x/4 & x < 0 \end{cases}$$

are S-odd.

The function M_0 is both S-odd and S-even for every symmetry. The function M_1 is S-odd for every symmetry, but is never S-even. For any symmetry, S itself is always S-odd, and $M_1 + S$ is always S-even. The function $M_1 - S$ turns out to be extremely interesting, and is central to the discussion following equation (15). There are many S-odd and S-even functions for every symmetry. In fact we will show that for each symmetry, there is a one-to-one correspondence between the S-odd functions and the odd functions, and between the S-even and the even functions. We will derive formulas that give all the S-odd and S-even functions for arbitrary symmetries. The example section below shows the computation of an S-odd and an S-even function in a more complicated case.

The sets of S-odd and S-even functions share many of the properties of the odd and even functions. For instance, if S is any symmetry then:

1. If f and g are S-odd, then $f \circ g$ is S-odd.
2. If f is S-even and g is S-odd, then $f \circ g$ is S-even.
3. If f is any function and g is S-even, then $f \circ g$ is S-even.
4. If f and g are S-even, then fg is S-even.
5. If f and g are S-even, then $f + g$ is S-even.
6. If f is S-even and α is any real number, then αf is S-even

The proofs are all elementary, and none of them depend on the symmetry properties of S. These are properties of functions that are commuting and invariant with respect to any function. There is another fact about odd and even functions that does not generalize so easily. Every function can be written in a unique way as a sum of an odd function and an even function: $f = f_o + f_e$. We say that f_o is the odd part of f and f_e is the even part of f. It is not hard to see that $f_o(x) = (f(x) - f(-x))/2$, and $f_e(x) = (f(x) + f(-x))/2$. Perhaps the most exciting aspect of this study is the fact that this result generalizes to S-odd and S-even functions for an arbitrary symmetry S

THEOREM 1. *Let S be a symmetry, and let f be continuously fixed at zero. Then f can be written in a unique way as a sum of an S-odd function and an S-even function.*

Proof. Suppose f has the decomposition

$$f = f_o + f_e, \tag{5}$$

where f_o is S-odd and f_e is S-even. Then

$$f \circ S = S \circ f_o + f_e. \tag{6}$$

Subtracting equation (6) from equation (5) gives

$$f - f \circ S = (M_1 - S) \circ f_o. \tag{7}$$

The function $M_1 - S$ is monotonic increasing at zero. Therefore we may compose both sides of equation (7) from the left with $(M_1 - S)^{-1}$ to solve for f_o:

$$f_o = (M_1 - S)^{-1} \circ (f - f \circ S). \tag{8}$$

From equation (5) we have

$$f_e = f - (M_1 - S)^{-1} \circ (f - f \circ S). \tag{9}$$

This shows that if the decomposition exists, it is unique, and the S-odd and S-even parts must be given by equations (8) and (9). It remains to demonstrate that equations (8) and (9) give S-odd and S-even functions, respectively. We first establish several identities, which will prove useful. Since $(f - f \circ S) \circ S = (f \circ S - f)$, we have

$$(f - f \circ S) \circ S = M_{-1} \circ (f - f \circ S). \tag{10}$$

This equation remains true if f is replaced by M_1.

$$(M_1 - S) \circ S = M_{-1} \circ (M_1 - S). \tag{11}$$

The inverse of this equation will also be useful.

$$S \circ (M_1 - S)^{-1} = (M_1 - S)^{-1} \circ M_{-1}. \tag{12}$$

■

Proof that f_o given by equation (8) is S-odd:

$$\begin{aligned} f_o \circ S &= (M_1 - S)^{-1} \circ (f - f \circ S) \circ S \\ &= (M_1 - S)^{-1} \circ M_{-1} \circ (f - f \circ S) && \text{by equation (10)} \\ &= S \circ (M_1 - S)^{-1} \circ (f - f \circ S) && \text{by equation (12)} \\ &= S \circ f_o. \end{aligned}$$

■

Proof that f_e given by equation (9) is S-even: First note that by equation (8) $(M_1 - S) \circ f_o = f - f \circ S$, and therefore $f \circ S = f - (M_1 - S) \circ f_o$. this gives

$$\begin{aligned} f_e \circ S &= (f - f_o) \circ S \\ &= f \circ S - f_o \circ S \\ &= f - (M_1 - S) \circ f_o - f_o \circ S \\ &= f - f_o \\ &= f_e \end{aligned}$$

■

EXAMPLE. We leave it as an exercise to show that if $S = M_{-1}$, then f_o and f_e are the ordinary odd and even parts of f.

For a less familiar example, we now determine the S-odd and S-even parts of the function $M_2(x) = 2x$ with respect to the symmetry $S(x) = -x/(1+x)$. We will need to know the inverse function $(M_1 - S)^{-1}(x)$, so we first compute

$$x = (M_1 - S)(y) = y + y/(1+y) = (y^2 + 2y)/(1+y),$$

and solve for y:

$$\begin{aligned} x + xy &= y^2 + 2y \\ y^2 - 2(x/2 - 1)y - x &= 0. \end{aligned}$$

The quadratic formula gives

$$\begin{aligned} y = (M_1 - S)^{-1}(x) &= (x/2 - 1) + \sqrt{(x/2-1)^2 + x} \\ &= (x/2 - 1) + \sqrt{x^2/4 + 1}. \end{aligned} \tag{13}$$

We need to take the "+" sign in the quadratic formula to make zero map to zero. The next step is to compute

$$(M_2 - M_2 \circ S)(x) = 2x - 2(-x/(1+x)) = 2(x + x/(1+x)). \tag{14}$$

Composing equations (13) and (14) gives

$$f_o(x) = x + \frac{x}{1+x} - 1 + \sqrt{\left(x + \frac{x}{1+x}\right)^2 + 1}.$$

Finally, since $f_e = M_2 - f_o$, we have

$$f_e(x) = x - \frac{x}{1+x} + 1 - \sqrt{\left(x + \frac{x}{1+x}\right)^2 + 1}.$$

The reader is invited to verify algebraically that these functions are indeed S-odd and S-even. This can also be verified graphically. The closure of the L-shaped region in FIGURE 1, and the rectangle in FIGURE 2 demonstrate that equations (3) and (4) hold at x_0.

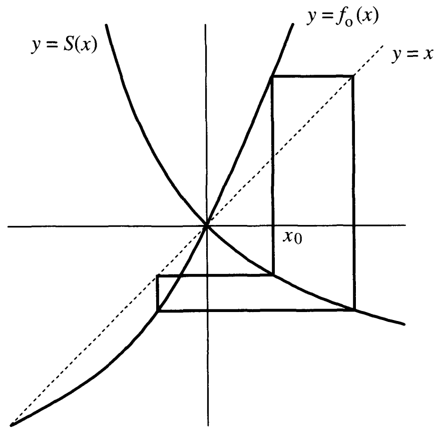

Figure 1 The L-shaped region closes because f_o is S-odd

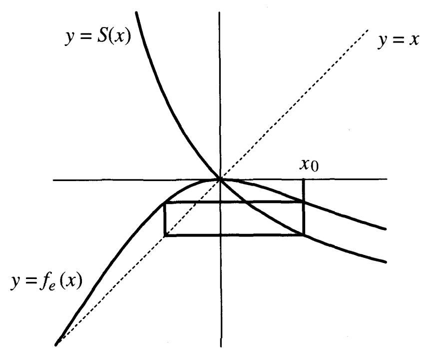

Figure 2 The rectangle closes because f_e is S-even

It is only slightly messier to compute the S-odd and S-even parts of M_λ with respect to $S_\alpha(x) = -x/(1+\alpha x)$. In this case the S-odd and S-even parts are given by

$$f_o(x) = \frac{\alpha\lambda(x + \frac{x}{1+\alpha x}) - 2 + \sqrt{\alpha^2\lambda^2(x + \frac{x}{1+\alpha x})^2 + 4}}{2\alpha},$$

and

$$f_e(x) = \frac{\alpha\lambda(x - \frac{x}{1+\alpha x}) + 2 - \sqrt{\alpha^2\lambda^2(x + \frac{x}{1+\alpha x})^2 + 4}}{2\alpha}.$$

Notice that the computations involved are relatively straightforward, with the possible exception of the computation of the inverse of $(M_1 - S)$. In particular, it would not be hard to determine the S-odd and S-even parts of $f(x) = \sin x$ or $\ln(x+1)$ with respect to $S(x) = -x/(1+x)$.

Conjugation There is a wealth of information hidden in equation (11). Multiplying equation (11) on the left by $(M_1 - S)^{-1}$ gives

$$S = (M_1 - S)^{-1} \circ M_{-1} \circ (M_1 - S). \tag{15}$$

We see that every symmetry is of the form $h^{-1} \circ M_{-1} \circ h$, where h is an invertible function. We say that an invertible function h determines a conjugation, which maps the set of functions continuously fixed at zero into itself. That is, if f is any such function, then the congugate of f is $\hat{f}$, where

$$\hat{f} = h^{-1} \circ f \circ h. \tag{16}$$

This map is one-to-one and onto, and the inverse map is the conjugation determined by h^{-1}. Conjugations have the nice property that the conjugate of the composition of two functions is the composition of their conjugates. If h is invertible, and f and g are arbitrary, then

$$h^{-1} \circ (f \circ g) \circ h = (h^{-1} \circ f \circ h) \circ (h^{-1} \circ g \circ h). \tag{17}$$

Conjugation is an essential tool in the study of the commutativity of functions. In particular, we state two simple corollaries of equation 17:

COROLLARY 1. *f commutes with g if and only if $h^{-1} \circ f \circ h$ commutes with $h^{-1} \circ g \circ h$.*

COROLLARY 2. *f is invariant with respect to g if and only if $h^{-1} \circ f \circ h$ is invariant with respect to $h^{-1} \circ g \circ h$.*

Our first application of these ideas is to characterize symmetry functions.

THEOREM 2. *A function S is a symmetry function if and only if there is a function h that is strictly monotonic on an open interval containing zero such that*

$$S = h^{-1} \circ M_{-1} \circ h.$$

Proof. Equation (15) shows that S has the form $h^{-1} \circ M_{-1} \circ h$. If we have $S = h^{-1} \circ M_{-1} \circ h$, then

$$S \circ S = h^{-1} \circ M_{-1} \circ h \circ h^{-1} \circ M_{-1} \circ h = h^{-1} \circ M_{-1} \circ M_{-1} \circ h = h^{-1} \circ h = M_1.$$

Because h is strictly monotonic and M_{-1} is strictly decreasing, S is strictly decreasing.

The conjugating function h in Theorem 2 is not unique. If k is any odd function and $g = k \circ h$, then $g^{-1} \circ M_{-1} \circ g = h^{-1} \circ M_{-1} \circ h$.

Theorem 2 gives us a recipe for constructing symmetries. Theorem 3 gives us recipes for the construction of all the S-odd and S-even functions corresponding to a given symmetry. ■

THEOREM 3. *Let $S = h^{-1} \circ M_{-1} \circ h$ be a symmetry, and let f be arbitrary. Then f is S-odd if and only if $h \circ f \circ h^{-1}$ is odd, and f is S-even if and only if $h \circ f \circ h^{-1}$ is even.*

Proof. If $S = h^{-1} \circ M_{-1} \circ h$, then $M_{-1} = h \circ S \circ h^{-1}$. The result follows from Corollaries 1 and 2. ■

These are practical recipes in the sense that, given h, we can compute S and the S-odd and S-even functions. If we start with a symmetry S, on the other hand, we know that $M_1 - S$ is an appropriate choice for h. Note that Theorem 5 establishes a one-to-one correspondence between the S-odd functions and the odd functions, and between the S-even functions and the even functions.

As an example of the power of Theorem 3, consider the following observations. We remarked above that M_0 is both S-odd and S-even for every symmetry S. It is easy to check that M_0 is the only function that is both odd and even. Since every conjugate of M_0 is M_0, the only function that is S-odd and S-even for any symmetry is M_0. It is also easy to see that no even function can be strictly monotonic in every open interval containing zero. By Theorem 3 this is true for S-even functions with respect to any symmetry. The odd functions x, $-x$, and $x^2\sin(1/x)$ demonstrate that S-odd functions can be monotonically increasing, decreasing, or neither near zero.

We might suspect that conjugation gives an easier way to determine the S-odd and S-even parts of a function f. Let $h = (M_1 - S)^{-1}$. Then $\hat{S} = h^{-1} \circ S \circ h = M_{-1}$. The conjugate of f, $\hat{f} = h^{-1} \circ f \circ h$, has a unique decomposition into odd and even parts: $\hat{f} = \hat{f}_o + \hat{f}_e$. The inverse conjugates f_o and f_e of $\hat{f}_o$ and $\hat{f}_e$ are S-odd and S-even, respectively. But unless $S = M_{-1}$ it is not true that $f = f_o + f_e$.

We conclude with a rather surprising theorem.

THEOREM 4. *Let S be a symmetry. Then the S-odd parts of the functions M_λ for all real λ commute with each other.*

Proof. By equation (8) the S-odd part of M_λ is $(M_1 - S)^{-1} \circ M_\lambda \circ (M_1 - S)$. The result follows from Corollary 1 since $M_\lambda \circ M_\alpha = M_{\lambda\alpha} = M_\alpha \circ M_\lambda$. ∎

REFERENCES

1. Roland E. Larson, Robert P. Hostetler, and Bruce H. Edwards, *Calculus of a Single Variable, 6th Edition*, Houghton Mifflin Company, 1998.
2. James Stewart, *Calculus, 4th Edition*, Brooks/Cole Publishing Company, 1999.

The Number of 2 by 2 Matrices over $\mathbb{Z}/p\mathbb{Z}$ with Eigenvalues in the Same Field

GREGOR OLŠAVSKÝ
Penn State Erie, The Behrend College
Erie, PA 16563
gxo1@psu.edu

In this note, I count the number of 2 by 2 matrices over the field of integers mod p (where p is an odd prime), with the added restriction that all eigenvalues must also belong to $\mathbb{Z}/p\mathbb{Z}$. One consequence of the count is that, as p gets larger, the number of such matrices approaches half the total number of 2 by 2 matrices over $\mathbb{Z}/p\mathbb{Z}$. We use some easy matrix theory, as well as a simple result from group theory about counting conjugacy classes [**1**].

Given any 2 by 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$