Contact Structures and Projection Methods

Pieter Eendebak

Department of Mathematics
Utrecht University

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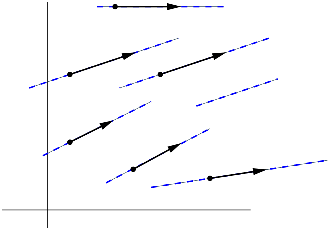
  - Projections of vector fields and distributions
  - The method of Darboux

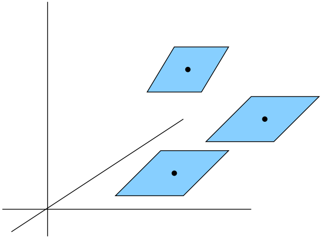
Distributions

Definition

Let M be a smooth manifold. A *distribution* on M is a smooth constant-rank vector subbundle of the tangent bundle TM.

We can think of a rank k distribution as being spanned by k linearly independent vector fields. At each point the distribution gives a k-dimensional linear subspace of the tangent space.





Frobenius theorem

An *integral manifold* U of a distribution $\mathcal{V} \subset TM$ is a submanifold of M such that the tangent space $T_m U \subset \mathcal{V}_m$ for all points in m.

Theorem (Frobenius theorem)

Suppose $\mathcal{V}$ is a rank k distribution such that $[X, Y] \subset \mathcal{V}$ for all $X, Y \subset \mathcal{V}$. Then locally there is a foliation of M by integral manifolds of dimension k.

The k-dimensional integral manifolds are called the *leaves* of $\mathcal{V}$.

Jet bundles

Definition

The k-jet of a map $\phi : \mathbb{R}^n \to \mathbb{R}^s$ at a point x consists of the point x together with the values of the derivatives of ϕ up to order k. The k-th order jet bundle $\mathrm{J}^k(\mathbb{R}^n, \mathbb{R}^s)$ consists of the k-jets of functions $\mathbb{R}^m \to \mathbb{R}^s$.

Example: let $z(x)$ be a function $\mathbb{R} \to \mathbb{R}$. Then the 1-jet of z at a point x is

$$(x, z(x), z'(x)).$$

The jet bundle $\mathrm{J}^1(\mathbb{R}, \mathbb{R})$ is isomorphic to $\mathbb{R}^3$, with coordinates x, z and $p = z'$.

Graphs

The *graph* of the k-jet of a function is a submanifold of the jet bundle.

$$\mathrm{gr}(j^1 z) = \{\, (x, z(x), z'(x)) \in \mathrm{J}^1(\mathbb{R}, \mathbb{R}) \mid x \in \mathbb{R} \,\}.$$

When is a submanifold (locally) the graph of a function?

Tangent space

The graph of the 1-jet of z is parameterized by the map

$$x \mapsto (x, z(x), z'(x)).$$

The tangent space at (x, z, p) is spanned by a single vector:

$$(1, z'(x), z''(x)) = (1, p, z''(x)).$$

The vector is not arbitrary. The tangent space is always contained in the distribution spanned by the two vectors:

$$(1, p, 0) \quad \text{and} \quad (0, 0, 1).$$

This is called the *contact distribution*.

Example: $z(x, y)$

Let $z(x, y)$ be a function $\mathbb{R}^2 \to \mathbb{R}$. Then the 2-jet of z at a point (x, y) is

$$(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) \in \mathrm{J}^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}^7.$$

In this talk we will always denote the independent variables by x, y, the dependent variable by z and the first and second order derivatives will be written as

$$\begin{aligned} p &= \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \\ r &= \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}, \end{aligned}$$

respectively.

The graph of the second order jet of the function is a submanifold of $Q = \mathrm{J}^2(\mathbb{R}^2, \mathbb{R})$.

Tangent spaces

The tangent space to the graph of the 2-jet is spanned by two vectors

$$
\begin{aligned}
(x,y) \mapsto & \left(1, 0, \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial x \partial y^2}\right) \\
= & \left(1, 0, p, r, s, \frac{\partial^3 z}{\partial x^3}, \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial x \partial y^2}\right), \\
\text{and} \quad & \left(0, 1, q, s, t, \frac{\partial^3 z}{\partial x^2 \partial y}, \frac{\partial^3 z}{\partial x \partial y^2}, \frac{\partial^3 z}{\partial y^3}\right).
\end{aligned}
$$

These vectors are contained in the *contact distribution* $\mathcal{W}$. The contact distribution is spanned by the vector fields

$$
\begin{aligned}
X &= \partial_x + p\partial_z + r\partial_p + s\partial_q, \\
Y &= \partial_y + q\partial_z + s\partial_p + t\partial_q, \\
R &= \partial_r, \quad S = \partial_s, \quad T = \partial_t.
\end{aligned}
$$

The distribution $\mathcal{W}$ defines the *contact structure* on the jet bundle.

In other words: the graph of the 2-jet of $z(x, y)$ defines an integral manifold of $\mathcal{W}$. The converse is also true.

Theorem

Let $(Q, \mathcal{W})$ be the second order jet bundle of $X = \mathbb{R}^2$ with contact distribution defined previously. Then an integral manifold of $\mathcal{W}$ that is transversal to the projection $Q \to X$ is locally equal to the graph of the 2-jet a function $z(x, y)$.

Partial differential equations

Any second order equation $F(x, y, z, p, q, r, s, t) = 0$ defines a hypersurface $M \subset Q = \mathrm{J}^2(\mathbb{R}^2, \mathbb{R})$. On M we define a contact structure by $\mathcal{V} = \mathcal{W} \cap TM$.

A solution of the partial differential equation $F = 0$ is a function $z(x, y)$ for which the graph of the 2-jet is a submanifold of M. At the same time the graph is an integral manifold of $\mathcal{W}$ and hence of $\mathcal{V}$.

Theorem

Locally, there is a one-to-one correspondence between integral manifolds of $(M, \mathcal{V})$ transversal to the projection $M \to X$ and solutions of the partial differential equation $F = 0$.

Ernest Vessiot

Ernest Vessiot (1865–1952)

- Picard-Vessiot theory (differential Galois theory), ballistics
- Formulation of partial differential equations in terms of distributions (dual to the work of Cartan), Darboux integrable equations

Vessiot theorem

Theorem

Let M be a 7-dimensional manifold with a rank 4 distribution $\mathcal{V}$. Then $(M, \mathcal{V})$ is locally equivalent to the equation manifold of a second order scalar equation in two independent variables if and only if

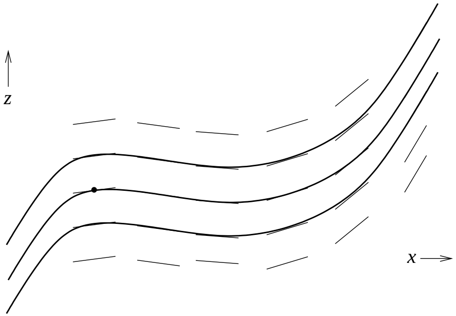
- *For every $m \in M$ the Cauchy characteristic space $C(\mathcal{V})_m$ of $\mathcal{V}$ at m is equal to zero.*
- *For every $m \in M$ the derived bundle $\mathcal{V}'_m$ has rank 6.*
- *For every $m \in M$, $C(\mathcal{V}')_m$ is contained in $\mathcal{V}_m$ and has rank 2.*

Solving an ODE

Example:

$$z' = f(x)$$

First order jet bundle: coordinates $x, z, p = z'$.
Equation manifold: hypersurface in the first order jet bundle defined by $p = f(x)$. Coordinates x, z. Contact distribution: spanned by $\partial_x + f(x)\partial_z$.



Flows

ODE: Go with the flow!
Mathematician: Which one?
ODE: There is only one, now go!

PDE: Go with the flow!
Mathematician: Okay, I'll take these two ...
PDE: **No!** That is not allowed!

Example: incompatible flows

Consider the wave equation $z_{xy} = 0$. The equation manifold has coordinates x, y, z, p, q, r, t. The contact distribution is spanned by the 4 vector fields

$$\begin{aligned} &\partial_x + p\partial_z + r\partial_p, \quad \partial_r, \\ &\partial_y + q\partial_z + t\partial_q, \quad \partial_t. \end{aligned}$$

Suppose we take $V = \partial_x + p\partial_z + r\partial_p$ and $W = \partial_r$. Do we get a surface from the flow of these two vector fields?

No!

$$[V, W] = [\partial_x + p\partial_z + r\partial_p, \partial_r] = -\partial_p$$

The Lie brackets are the first order obstruction to the existence of integral manifolds.

Structure theory: hyperbolic equations

We have a 7-dimensional manifold M with a rank 4 distribution $\mathcal{V}$. We can use the Lie brackets to analyze the geometry of the system.

- The contact distribution is the direct sum of two rank 2 distributions

  $$\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$$

  At every point the tangent space of an integral manifold has 1-dimensional intersection with both $\mathcal{V}_+$ and $\mathcal{V}_-$.
- The characteristic systems satisfy $[\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V}$.

Geometry of partial differential equations

	PDE	Contact structure
framework	local coordinates	geometric structures
system	system of PDE's	distribution
solutions	functions	integral manifolds
differentiation	partial derivatives	Lie brackets

Projection of vector fields

Let $\pi : M \to B$ be a smooth map. Suppose X is a vector field on M. We say the vector field X *projects* to B if for all points x and y with $\pi(x) = \pi(y)$:

$$T_x\pi(X) = T_y\pi(Y).$$

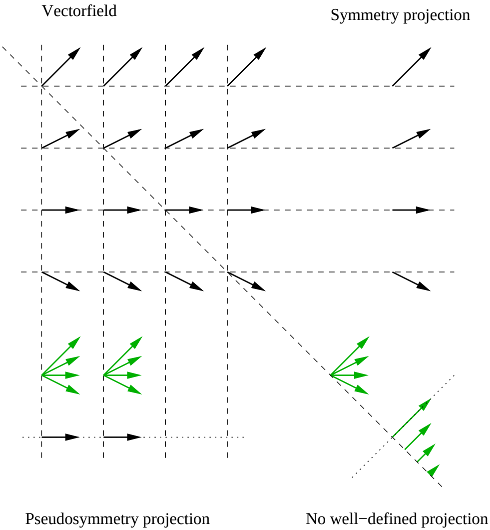
In other words, $T_x\pi(X)$ depends only on the point $\pi(x)$, and not on x itself.

Let $\mathcal{V}$ be a distribution on M. We say the distribution *projects* if

$$T_x\pi(\mathcal{V}_x) \subset T_{\pi(x)}B$$

is independent of the point x.

Think geometrically!



The projection method

Suppose we have a system of two rank 2 characteristic distributions $\mathcal{V}_+, \mathcal{V}_-$ on a manifold M such that $[\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$. Suppose we want to find integral manifolds.

1. Find a projection $\pi : M \to B$ that projects the two characteristic distributions.
2. Find solutions of the system $(B, T\pi\mathcal{V}_+, T\pi\mathcal{V}_-)$.
3. Lift the solutions to solutions of the original system.

Examples of such systems: second order equations (dimension 7), first order systems of 2 equations for 2 functions in 2 variables (dimension 6), Monge-Ampère equations (dimension 5), pseudoholomorphic curves (dimension 4).

Gaston Darboux

Gaston Darboux (1842–1917)

- Darboux integral (integration theory), Darboux theorem (symplectic geometry)
- The method of Darboux to integrate second order partial differential equations

The method of Darboux

We look for projections $M \to \mathbb{R}^2 \times \mathbb{R}^2$ such that the projections of $\mathcal{V}_+$ and $\mathcal{V}_-$ are equal to the tangent spaces to the components of $\mathbb{R}^2 \times \mathbb{R}^2$.

1. Find a projection: find invariants of the characteristic systems.
2. Solve the projected system: hyperbolic holomorphic curves.
3. Lift hyperbolic holomorphic curves to solutions of the equation.

The method of Darboux 1:

If each characteristic system has at least two invariants the equation is Darboux integrable. We can use the invariants as coordinates on the base manifold B.
Suppose I^1, I^2 are invariants of $\mathcal{V}_-$ and I^3, I^4 invariants of $\mathcal{V}_+$.
Define the *Darboux projection*

$$\pi : M \to \mathbb{R}^2 \times \mathbb{R}^2 : m \mapsto (I^1(m), I^2(m), I^3(m), I^4(m)).$$

Then the characteristic systems are mapped to the tangent spaces of the components of $\mathbb{R}^2 \times \mathbb{R}^2$.

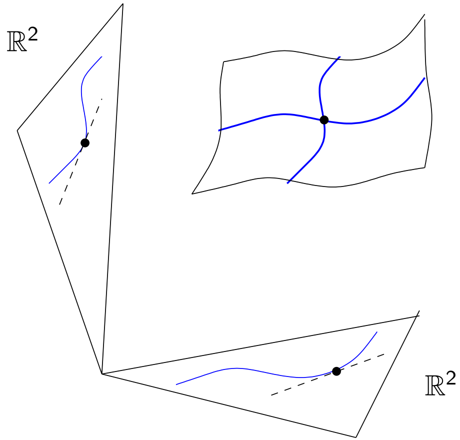
The method of Darboux 2: hyperbolic holomorphic curves

For a complex structure J a *holomorphic curve* is a real 2-dimensional manifold for which the tangent space is J-invariant. Construction for $\mathbb{C}^2$: holomorphic curves are given by the graphs of complex-differentiable functions $\mathbb{C} \to \mathbb{C}$.

We have direct product $B = \mathbb{R}^2 \times \mathbb{R}^2$. Define $K : TB \to TB$ by $K = \mathrm{id}$ on the first component and $K = -\mathrm{id}$ on the second component. The integral manifolds we are looking for are K-invariant. Since $K^2 = \mathrm{id}$ we call K a *hyperbolic structure* and the integral manifolds *hyperbolic holomorphic curves*.

Construction for $\mathbb{R}^2 \times \mathbb{R}^2$: choose two curves ϕ_1, ϕ_2 in $\mathbb{R}^2$. Then the direct product $\tilde{S} = \phi_1 \times \phi_2 \subset \mathbb{R}^2 \times \mathbb{R}^2$ defines a hyperbolic holomorphic curve.

The method of Darboux 2: constructing hyperbolic pseudoholomorphic curves



The method of Darboux 3: lifting solutions

Let $\tilde{S}$ be a solution of the system on B. Let $S = \pi^{-1}(\tilde{S})$ and on S we define $\mathcal{W} = \mathcal{V} \cap TS$. Since $\mathcal{V}$ is transversal to π we have $\operatorname{rank} \mathcal{W} = \dim \tilde{S} = 2$.

For any pair of vector fields $X, Y \subset \mathcal{W}$ we have

- $T\pi([X, Y]) = [T\pi(X), T\pi(Y)] \subset T\tilde{S}$. Hence $[X, Y] \subset TS$.
- Assume $X \subset \mathcal{W}_+ = \mathcal{V}_+ \cap TS$, $Y \subset \mathcal{W}_- = \mathcal{V}_- \cap TS$. Then $[X, Y] \subset [\mathcal{V}_+, \mathcal{V}_-] \subset \mathcal{V}$.
  Since $\mathcal{W}$ has rank 2 this shows that $[X, Y] \subset \mathcal{V}$ for all $X, Y \subset \mathcal{V}$.

Together: $[X, Y] \subset \mathcal{W} = TS \cap \mathcal{V}$.

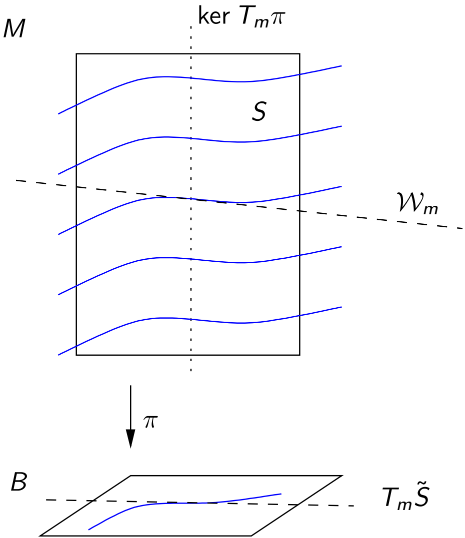
The method of Darboux 3: lifting solutions

So for this distribution there are ‘no obstructions’ to find integral manifolds. The distribution $\mathcal{W}$ is ‘integrable’ and by the Frobenius theorem there exists a local foliation of M by 2-dimensional integral manifolds of $\mathcal{W}$ and hence of $\mathcal{V}$.

Theorem

There is a one-to-one correspondence between hyperbolic pseudoholomorphic curves for the structure on B and 3-dimensional families of integral manifolds of $\mathcal{V}$ on M.

Geometric picture of lifting



Example: wave equation

Consider the wave equation $s = 0$. We can use x, y, z, p, q, r, t as coordinates for the equation manifold. The Monge systems are given by

$$\begin{aligned} \mathcal{V}_+ &= \mathrm{span}(\partial_x + p\partial_z + r\partial_p, \partial_r), \\ \mathcal{V}_- &= \mathrm{span}(\partial_y + q\partial_z + t\partial_q, \partial_t). \end{aligned}$$

The invariants of $\mathcal{V}_-$ are x, p, r and the invariants of $\mathcal{V}_+$ are y, q, t.

We make the projection $m \mapsto (x, r, y, t)$.
On $B = \mathbb{R}^2 \times \mathbb{R}^2$ we have coordinates (x, r, y, t). We take two curves $r = \phi(x)$ and $t = \psi(y)$. The direct product is a hyperbolic holomorphic curve.

Contact distribution for the wave equation

Lifting this surface yields

$$
\begin{aligned}
&z_{xx} = \phi(x) \Rightarrow \\
&z_x = \int^x \phi(x) + D(y) \Rightarrow \\
&z = \iint^x \phi(x) + C(y) + D(y)x \Rightarrow \\
&\dots
\end{aligned}
$$

Together

$$z(x, y) = z_0 + \iint^x \phi(x) + \iint^y \psi(y) = z_0 + A(x) + B(y).$$

Example: $s = p/(y-x)$

Let us consider the second order scalar partial differential equation $s = p/(y-x)$. The Monge systems are given by

$$\mathcal{V}_+ = \mathrm{span}\left(\partial_x + p\partial_z + r\partial_p + \frac{p}{y-x}\partial_q, \partial_r\right),$$
$$\mathcal{V}_- = \mathrm{span}\left(\partial_y + q\partial_z + \frac{p}{y-x}\partial_p + t\partial_q + \left(\frac{r(y-x)+p}{(y-x)^2}\right)\partial_r, \partial_t\right).$$

Invariants. The distribution $\mathcal{V}_+$ has invariants y and $\tau = t$. The distribution $\mathcal{V}_-$ has invariants x, $p/(y-x)$ and $\rho = r/(y-x) + p/(y-x)^2$.

Projection. We make the projection

$$\pi : M \to \mathbb{R}^2 \times \mathbb{R}^2 : (x, y, z, p, q, r, t) \mapsto (x, \rho), (y, \tau).$$

On $\mathbb{R}^2 \times \mathbb{R}^2$ we use (x, ρ) and (y, τ) as coordinates.

Example: $s = p/(y-x)$

Lifting. Choose two arbitrary functions ϕ, ψ. In $\mathbb{R}^2 \times \mathbb{R}^2$ we define a holomorphic curve for the direct product structure by

$$\rho = \phi(x), \quad \tau = \psi(y).$$

On the inverse image of the curve under π we have a rank 2 integrable distribution.

Integration yields the general solution of the equation:

$$z(x,y) = A(y) + B(x) + B'(x)(y-x).$$

$$p = B'(x) + B''(x)(y-x) - B'(x) = B''(x)(y-x)$$
$$s = B''(x)$$

Some results

- All equations solvable by the method of Darboux are examples of projections.
- Symmetry reductions (Sophus Lie) are examples of projections.
- There are examples of projections that are neither a symmetry reduction nor a Darboux projection.
- Some Bäcklund transformations (KdV equation, Sine-Gordon equation) can be formulated as projections through pseudosymmetries.

Messages to take home

- Contact structures can be used to give a geometric description of partial differential equations
- Whenever symmetries are used, maybe pseudosymmetries can be used as well (for making projections, or maybe something completely different)

More: *Contact Structures of Partial Differential Equations*, http://www.math.uu.nl/Research/Projects/Contact-Structures/