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On New Exact Solutions of the Eikonal Equation

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Some new classes of exact solutions of the investigated equation have been found.

The relativistic eikonal equation is fundamental in theoretical and mathematical physics. Here we consider the equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} \equiv \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \quad (1)$$

In [1] it has been shown that the maximal local invariance group of the equation (1) is the conformal group $C(1,4)$ of the 5-dimensional Poincaré–Minkowski space. Using special ansatzes multiparameter families of exact solutions of the eikonal equation were constructed [1, 2, 3, 4].

It is well known that the conformal group $C(1,4)$ contains the generalized Poincaré group $P(1,4)$ as a subgroup. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. For the investigation of the equation (1) we have used the continuous subgroups [5, 6, 7, 8, 9] of the group $P(1,4)$. Earlier using the subgroup structure of the group $P(1,4)$, we have constructed ansatzes which reduce the equation (1) to differential equations with fewer independent variables. The corresponding symmetry reduction has been done. Among the reduced equations there are one-, two-, and three-dimensional ones. For some of the reduced equations we have found its exact solutions. On this base some classes of exact solutions of the eikonal equation have been constructed. The part of the results obtained can be found in [10, 11, 12].

The present paper is devoted to the construction of new exact solutions of the investigated equation. In order to find these solutions we have solved some other reduced equations. Using the solutions of these reduced equations, we have obtained some new classes of exact solutions of the eikonal equation.

At first, we present some new exact solutions of the investigated equation which have been obtained on the base of solutions of one-dimensional reduced equations.

1. $\alpha \ln\left(\left(\alpha^2 + x_0^2 - x_1^2 - x_2^2 - u^2\right)^{1/2} + \varepsilon\alpha\right) - \varepsilon\left(\alpha^2 + x_0^2 - x_1^2 - x_2^2 - u^2\right)^{1/2}$
   $- x_3 - \alpha\ln(x_0 + u) = 0, \quad \varepsilon = \pm 1;$
2. $\alpha \ln\left(\left(\alpha^2 + x_0^2 - x_3^2 - u^2\right) + \varepsilon\alpha\right) = \varepsilon\left(\alpha^2 + x_0^2 - x_3^2 - u^2\right) + x_2$
   $+ \alpha\ln(x_0 + u) + c, \quad \varepsilon = \pm 1;$
3. $\frac{1}{2}(x_0+u)^4 - \left(x_0 + \frac{1}{2}(\beta - 1) - c\right)(x_0+u)^3 + \left[\frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + (\beta - 1)x_0\right.$
   $\left. -\left(c + \frac{1}{2}\right)\beta + c\right](x_0+u)^2 + \left[\beta\left(x_0 - \frac{x_1^2}{2} - \frac{x_2^2}{2}\right) + \frac{x_1^2}{2} + \frac{x_3^2}{2} - c\beta\right](x_0+u) - \beta\frac{x_1^2}{2} = 0;$
4. $\frac{1}{2}(x_0+u)^4 - \left(x_0 + \frac{1}{2}(\beta + 1) - c\right)(x_0+u)^3 + \left[\frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + (\beta + 1)x_0\right.$
   $\left. -\left(c - \frac{1}{2}\right)\beta - c\right](x_0+u)^2 - \left[\beta\left(x_0 + \frac{x_1^2}{2} + \frac{x_2^2}{2}\right) + \frac{x_1^2}{2} + \frac{x_3^2}{2} - c\beta\right](x_0+u) + \beta\frac{x_1^2}{2} = 0;$

5. $\frac{1}{2}(x_0+u)^3-\left(x_0+\frac{k}{2}+c\right)(x_0+u)^2+\left(\frac{x_1^2}{2}+\frac{x_2^2}{2}+\frac{x_3^2}{2}+kx_0+ck\right)(x_0+u)-k\frac{x_3^2}{2}=0.$

Now, we give some new exact solutions of the eikonal equation which have been constructed on the base of solutions of two-dimensional reduced equations.

1. $u=i\varepsilon\sqrt{c_1^2+1}\sqrt{x_1^2+x_2^2}+c_1x_3+c_2, \quad \varepsilon=\pm 1;$
2. $u=\varepsilon x_0\sqrt{c_1^2+1}+c_1x_3+c_2, \quad \varepsilon=\pm 1;$
3. $u=\varepsilon x_0\sqrt{c_1^2+1}+c_1\sqrt{x_1^2+x_2^2}+c_2, \quad \varepsilon=\pm 1;$
4. $u=\varepsilon x_0\sqrt{c_1^2+1}+c_1\sqrt{x_1^2+x_2^2+x_3^2}+c_2, \quad \varepsilon=\pm 1;$
5. $u=i\varepsilon x_2\sqrt{c_1^2+1}+c_1x_3+c_2, \quad \varepsilon=\pm 1;$
6. $u^2=x_0^2-\left(\left(x_1^2+x_2^2\right)^{1/2}+c_1\right)^2-(x_3+c_2)^2;$
7. $u^2=x_0^2-(x_1+c_1)^2-(x_2+c_2)^2-x_3^2;$
8. $u^2=(x_0+c_1)^2-\left(\left(x_1^2+x_2^2\right)^{1/2}+c_2\right)^2-x_3^2;$
9. $(x_0-u+c_1)(x_0+u)=(x_2+c_2)^2+x_3^2;$
10. $(x_0-u+c_1)(x_0+u)=(x_3+c_2)^2+x_1^2+x_2^2;$
11. $(x_0-u+c_1)(x_0+u)=\left(\left(x_1^2+x_2^2\right)^{1/2}+c_2\right)^2+x_3^2;$
12. $$\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}+\frac{x_3}{\varepsilon(x_0+u)}=f(x_0+u)$$
$$+i\varepsilon\left[\sqrt{1+\frac{x_1^2+x_2^2}{(x_0+u)^2}}+\ln\left(\frac{\sqrt{x_1^2+x_2^2}}{x_0+u+\sqrt{(x_0+u)^2+x_1^2+x_2^2}}\right)\right], \quad \varepsilon=\pm 1,$$

where f is an arbitrary smooth function;

13. $$\frac{1}{2}\arcsin\frac{x_3}{\sqrt{x_3^2+u^2}}+\frac{1}{e}\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}$$
$$=\frac{1}{e}\left[\sqrt{c_1e^2\left(x_1^2+x_2^2\right)^2-1}+\arctan\left(\frac{1}{\sqrt{c_1e^2\left(x_1^2+x_2^2\right)^2-1}}\right)\right]$$
$$+i\frac{\varepsilon}{2}\left[\sqrt{4c_1\left(x_3^2+u^2\right)+1}+\ln\left(\frac{\sqrt{x_3^2+u^2}}{1+\sqrt{4c_1\left(x_3^2+u^2\right)^2+1}}\right)\right]+c_2, \quad \varepsilon=\pm 1, \quad e\neq 0;$$
14. $$\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}-\frac{1}{e}\operatorname{arccosh}\frac{x_0}{\sqrt{x_0^2-u^2}}$$
$$=\frac{\varepsilon}{e}\left[\sqrt{c_1e^2\left(x_0^2-u^2\right)+1}+\ln\left(\frac{\sqrt{c_1}e\sqrt{x_0^2-u^2}}{1+\sqrt{c_1e^2\left(x_0^2-u^2\right)+1}}\right)\right]$$
$$-\sqrt{c_1\left(x_1^2+x_2^2\right)-1}-\arctan\left(\frac{1}{\sqrt{c_1\left(x_1^2+x_2^2\right)-1}}\right)+c_2, \quad \varepsilon=\pm 1, \quad e\neq 0;$$

15. $\arcsin \dfrac{x_2}{\sqrt{x_1^2+x_2^2}} - \dfrac{1}{d}\ln(x_0+u) = \sqrt{c_1\left(x_1^2+x_2^2\right)^2-1}$

$+\arctan\left(\dfrac{1}{\sqrt{c_1\left(x_1^2+x_2^2\right)^2-1}}\right) + \dfrac{\varepsilon}{d}\left[\sqrt{1-d^2\left(u^2+x_3^2-x_0^2\right)c_1}\right.$

$\left. -\operatorname{arctanh}\left(\dfrac{1}{\sqrt{1-d^2\left(u^2+x_3^2-x_0^2\right)c_1}}\right) - \dfrac{\varepsilon}{2}\ln\left(u^2+x_3^2-x_0^2\right)\right] + c_2,\ \varepsilon=\pm1,\ d\neq 0;$

16. $u = c_1x_3 + i\left[\sqrt{c_1^2\left(x_1^2+x_2^2\right)+1} - \operatorname{arctanh}\left(\dfrac{1}{\sqrt{c_1^2\left(x_1^2+x_2^2\right)+1}}\right)\right]$

$-\left(\varepsilon\arcsin\dfrac{x_1}{\sqrt{x_1^2+x_2^2}} + x_0\right) + c_2, \quad \varepsilon=\pm1;$

17. $u = c_1x_3 + \dfrac{i}{2}\left[\sqrt{4\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+d_4^2}\right.$

$\left. -d_4\operatorname{arctanh}\left(\dfrac{d_4}{\sqrt{4\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+d_4^2}}\right)\right] - \dfrac{d_4}{2}\arcsin\dfrac{x_1}{\sqrt{x_1^2+x_2^2}} + c_2;$

18. $\varepsilon x_1(x_0+u) = i\varepsilon x_2\sqrt{(x_0+u)^2+1} - x_3 + f(x_0+u), \quad \varepsilon=\pm1,$

where f is an arbitrary smooth function;

19. $\dfrac{\alpha}{\mu}\operatorname{arccosh}\dfrac{x_0}{\sqrt{x_0^2-u^2}} - \arcsin\dfrac{x_1}{\sqrt{x_1^2+x_2^2}} = \sqrt{c_1\left(x_1^2+x_2^2\right)-1}$

$-\arctan\left(\sqrt{c_1\left(x_1^2+x_2^2\right)-1}\right) + \dfrac{1}{\mu}\sqrt{\left(c_1\mu^2+1\right)\left(x_0^2-u^2\right)+\alpha^2}$

$-\dfrac{\alpha}{\mu}\operatorname{arctanh}\left(\sqrt{\dfrac{\left(c_1\mu^2+1\right)\left(x_0^2-u^2\right)}{\alpha^2}+1}\right) - \dfrac{x_3}{\mu} + c_2, \quad \alpha,\mu\neq 0;$

20. $\dfrac{\alpha}{\mu}\arcsin\dfrac{x_2}{\sqrt{x_1^2+x_2^2}} - \arcsin\dfrac{x_3}{\sqrt{x_3^2+u^2}} = \sqrt{c_1\left(x_1^2+x_2^2\right)-\dfrac{\alpha^2}{\mu^2}}$

$-\dfrac{\alpha}{\mu}\arctan\left(\sqrt{\dfrac{c_1\mu^2\left(x_1^2+x_2^2\right)}{\alpha^2}-1}\right) + \sqrt{\dfrac{\left(1-c_1\mu^2\right)\left(x_3^2+u^2\right)}{\mu^2}-1}$

$-\arctan\left(\sqrt{\dfrac{\left(1-c_1\mu^2\right)\left(x_3^2+u^2\right)}{\mu^2}-1}\right) + \dfrac{x_0}{\mu} - c_2, \quad \alpha,\mu\neq 0.$

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