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General Intensity Transformations and Differential Invariants*

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Abstract. We consider the group of invertible image gray-value transformations and propose a generating equation for a complete set of differential gray-value invariants up to any order. Such invariants describe the image's geometrical structure independent of how its gray-values are mapped (contrast or brightness adjustments).

Key words. local gray-value invariants, Cartesian tensors, local isophote structure.

1 Introduction

The incorporation of the notion of *scale* in the analysis of local image structure has led to an operational and well-posed definition of a derivative. To this end, a given image is embedded in a *scale-space*, i.e., a one-parameter family of derived images, intended to represent that image at various resolutions. Such a scale-space is obtained by convolving the image data by a set of Gaussian filters of various widths [1]–[10]. For each fixed-scale representative one can then give an operational definition of a linear derivative by formally applying the appropriate conjugate differential operator to the (smooth) Gaussian kernel and convolving the image data with the resulting derivative kernel. This requires a complete family of Gaussian derivatives in addition to the basic, zeroth-order Gaussian. Note that the preceding definition of an image derivative does not require any infinitesimal limiting procedure; image differentiation, by virtue of scale, is really a matter of *integration* (and this fact, of course, accounts for its well-posed behavior). The mathematical principle underlying well-posed differentiation is well known from the theory of *regular tempered distributions* [11].

Throughout the paper we will be dealing with *local* image properties. To this end, we will set up an arbitrary Cartesian coordinate frame in each point of the image domain, and we will consider local quantities that are invariant against Cartesian coordinate transformations (rotations, reflections,[1] and translations). The reader is assumed to have a basic notion of Cartesian scalars, vectors, and, in general, tensors. These concepts invariably show up in a coordinate-invariant description of physical quantities. See [12], [13] for an easy introduction and [14] for a detailed treatment in the context of local image structure. For an in-depth mathematical exposition of the tensor formalism the reader is referred to [15].

Instead of considering transformations that affect the independent, *spatial* coordinates, in this paper we consider the *group of general intensity transformations* acting on the image's gray-values

$$\mathcal{G} : \mathbb{R} \to \mathbb{R} : L \mapsto \hat{L} = \mathcal{G}(L).$$

In this equation L and $\hat{L}$ denote the gray values before and after the transformation, respectively. Invertibility (or strict monotonicity) is necessary to prevent loss of gray-value information contained in the image. But apart from this natural group requirement, $\mathcal{G}$ may be any nonlinear transformation. Requiring invariance under this group amounts to considering equivalence classes of (locally defined) images that share a common local iso-intensity, or *isophote* structure. Therefore the relevant $\mathcal{G}$-invariant local image properties correspond to geomet-

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rically invariant local properties of isophotes. *The purpose of this paper is to present a generating equation for a complete set of local isophote properties up to any order* (see also [16]–[19]).

Gray-value invariance is of some interest in computer vision and biological vision. It is well known that in human vision a rather general class of (nonlinear) transformations of the retinal irradiance distribution has little effect on perception, or, at least, on recognition. Although gamma corrections may produce results that are more pleasing to the eyes, their primary justification lies in their invariants. It is apparently the case that, at least up to some approximation, many visual tasks can be solved on the basis of such invariants only. This observation should affect the way computer-vision and image-analysis tasks are handled as well. This is our motivation for considering local image properties that are invariant under the group $\mathcal{G}$.

2 Notation and Conventions

In the context of this paper an image is a smooth scalar function defined on a d-dimensional spatial domain of definite resolution, i.e., a fixed-scale section of a given scale-space. The image gray-value or intensity will be denoted by L, and, its value at a particular point $x \in \mathbb{R}^d$ will be denoted by $L(x)$ or, for brevity, by L. We will henceforth assume that all derivatives (including zeroth order) are obtained at a given inner scale, as explained in section 1, but we will not make this scale explicit in our notation.

The Cartesian coordinates of a spatial vector x will be denoted by x_i, where we use a Latin index from the middle of the alphabet, whose values are in the range $1, \ldots, d$. For this type of so-called *spatial indices* we will use the condensed Einstein summation convention, i.e., we will omit the summation symbol whenever such an index occurs twice in a given term. So $t_i u_i$ stands for $\sum_{i=1}^{d} t_i u_i$, etc. This is a useful convention in the context of Cartesian tensor calculus, in which the indices are related to the group of Cartesian coordinate transformations. To be more specific we make the following definitions.

DEFINITION 1 (spatial indices). A spatial index i in a d-dimensional space is a formal index in the range $1, \ldots, d$ that is associated with the ith basis vector of a Cartesian coordinate system and is subject to the Cartesian-tensor transformation law.

DEFINITION 2 (Cartesian-tensor transformation law). If T is a Cartesian n-tensor, the components of which are $T_{i_1 \cdots i_n}$ with respect to a given system of Cartesian coordinates, x_i say, and if $\hat{x}_i = r_{ij} x_j + a_i$ is any Cartesian coordinate transformation, then the components of T with respect to the new coordinate system are given by

$$\hat{T}_{i_1 \cdots i_n} = r_{i_1 j_1} \cdots r_{i_n j_n} T_{j_1 \cdots j_n}.$$

The reader may verify that the vanishing of a tensor in a given coordinate system also implies its vanishing in any other coordinate system. Writing equations in terms of tensors therefore reveals their *manifest invariance* against coordinate transformations. Also, an Einstein summation, or a *contraction* of a pair of indices, in an $(n + 2)$-tensor yields an n-tensor. In particular, a full contraction of all indices (possible for even-rank tensors) yields an *invariant* (i.e., a 0-tensor). This is why spatial indices are referred to as *formal* indices in Definition 1; their intentional use is in the context of tensor calculus, and so an actual realization of a Cartesian coordinate frame is usually of no importance; see also [12]–[15].

Apart from spatial indices, we will introduce *parameter indices* for labeling the various components of a parametric variable ε (these may or may not be connected to a representation of the Cartesian group). For these we will use Greek indices from the beginning of the alphabet, so that ε_α denotes one of the components of ε, with α taken from the range $1, \ldots, p$. These parameters are introduced to parameterize a p-dimensional submanifold of the image domain, and so in general we have $1 \leq p \leq d$. It turns out to be convenient to use the so-called *multi-index* convention for these parametric indices.

DEFINITION 3 (multi-indices). A multi-index $\vec{n}$ of dimension p is an ordered p-tuple $(n_1, \ldots, n_p)$

of nonnegative integers $n_\alpha \in \mathbb{Z}_0^+$, $\alpha = 1, \ldots, p$.

We will henceforth mark a multi-index by a tilde and will assume all multi-indices to be of dimension p unless stated otherwise. The following conventions apply to multi-indices.

DEFINITION 4 (multi-index conventions). Let $\tilde{n}$ be a multi-index of dimension p; then

(1) $|\tilde{n}| \stackrel{\text{def}}{=} \sum_{\alpha=1}^{p} n_\alpha$ (called the order of $\tilde{n}$);

(2) $\tilde{n}! \stackrel{\text{def}}{=} \prod_{\alpha=1}^{p} n_\alpha!$;

(3) for $\varepsilon \in \mathbb{R}^p$, $\varepsilon^{\tilde{n}} \stackrel{\text{def}}{=} \prod_{\alpha=1}^{p} \varepsilon_\alpha^{n_\alpha}$;

(4) for $\varepsilon \in \mathbb{R}^p$, $\frac{\partial^{|\tilde{s}|}}{\partial \varepsilon^{\tilde{s}}} \stackrel{\text{def}}{=} \partial^{s_1+\cdots+s_p}/\partial\varepsilon_1^{s_1}\cdots\partial\varepsilon_p^{s_p}$.

3 Theory

Let us start this section with a useful lemma.

LEMMA 1 (implicit differentiation). Let $L = L(x)$ denote a scalar image on a d-dimensional spatial domain, and let

$$S : \mathbb{R}^p \to \mathbb{R}^d : \varepsilon \mapsto x = \phi(\varepsilon)$$

be a parameterized p-dimensional surface in $\mathbb{R}^d$. Denoting $\partial^n L/\partial x_{i_1}\cdots\partial x_{i_n}$ by $L_{i_1\cdots i_n}$, we have, for all $\tilde{n} = (n_1, \ldots, n_p)$ with $|\tilde{n}| \geq 1$,

$$\frac{1}{\tilde{n}!}\frac{\delta^{|\tilde{n}|}L}{\delta\varepsilon^{\tilde{n}}} = \sum_{q=1}^{|\tilde{n}|}\frac{1}{q!}L_{i_1\cdots i_q}\circ\phi \sum_{|\tilde{s}_1|\geq 1}^{\star} \cdots \sum_{|\tilde{s}_q|\geq 1}^{\star}\prod_{j=1}^{q}\frac{1}{\tilde{s}_j!}\frac{\partial^{|\tilde{s}_j|}\phi_{i_1}}{\partial\varepsilon^{\tilde{s}_j}},$$

in which $\delta f/\delta\varepsilon_\alpha$ denotes the implicit derivative of f with respect to ε_α, i.e.,

$$\frac{\delta f}{\delta\varepsilon_\alpha} \stackrel{\text{def}}{=} \frac{\partial(f\circ\phi)}{\partial\varepsilon_\alpha} = \frac{\partial\phi_i}{\partial\varepsilon_\alpha}\frac{\partial f}{\partial x_i}\circ\phi,$$

and in which the q inner summations marked by "$\star$" are restricted to multi-indices of dimension p that add up to $\tilde{n}$, i.e.,

$$\star : \sum_{k=1}^{q}\tilde{s}_k = \tilde{n},$$

and in which the Einstein summation convention applies to the spatial indices $i_1\cdots i_q$.

Proof 1. The proof of Lemma 1 follows from a straightforward application of the chain rule for differentiation in combination with Leibniz's product rule. These will be considered familiar.

Lemma 1 gives the $\tilde{n}$th derivative of the image L, taken along the surface S. In principle, one needs to know only the values of L on that surface in order to calculate it. But note that we have expressed this surface derivative in terms of *measurable* properties, viz., the image's partial derivatives $L_{i_1\cdots i_q}$, which are defined on a full neighborhood of each surface point. As explained in section 1, there is an easy operational method for extracting these partial derivatives in a well-posed way, viz., through convolution with Gaussian derivatives.

A special case of Lemma 1 is obtained if we consider a $(d-1)$-dimensional surface in $\mathbb{R}^d$ with a Monge-patch parameterization.

PROPOSITION 1 (implicit differentiation along a $(d-1)$-dimensional Monge patch). Let $L = L(x)$ denote a scalar image on a d-dimensional spatial domain, and let

$$S : \mathbb{R}^{d-1} \to \mathbb{R}^d : u \mapsto x = \phi(u)$$

be a parameterized $(d-1)$-dimensional Monge patch, with $\phi : \mathbb{R}^{d-1} \to \mathbb{R}^d$ given by

$$\begin{cases}\phi_a(u) = u_a & (a = 1, \ldots, d-1),\\ \phi_d(u) = w(u)\end{cases}$$

for some smooth function $w : \mathbb{R}^{d-1} \to \mathbb{R}$. Then, using the multi-index notation for the $(d-1)$-dimensional parameter space, we have

$$\frac{1}{\tilde{n}!}\frac{\delta^{|\tilde{n}|}L}{\delta u^{\tilde{n}}} = \sum_{q=1}^{|\tilde{n}|}\sum_{k=0}^{q}\sum_{|\tilde{\sigma}_{q-k}|=q-k}^{\star}\frac{1}{k!}\frac{1}{\tilde{\sigma}_{q-k}!}\frac{\partial^q L}{\partial w^k\partial u^{\tilde{\sigma}_{q-k}}} \times \sum_{|\tilde{s}_1|\geq 1}^{\star}\cdots\sum_{|\tilde{s}_k|\geq 1}^{\star}\prod_{j=1}^{k}\frac{1}{\tilde{s}_j!}\frac{\partial^{|\tilde{s}_j|}w}{\partial u^{\tilde{s}_j}},$$

in which $\delta f/\delta u_a$ denotes the implicit derivative of f with respect to u_a, i.e.,

$$\frac{\delta f}{\delta u_a} \stackrel{\text{def}}{=} \frac{\partial (f \circ \phi)}{\partial u_a} = \frac{\partial f}{\partial u_a} \circ \phi + \frac{\partial w}{\partial u_a} \frac{\partial f}{\partial w} \circ \phi,$$

and in which the $k + 1$ multi-index summations are subject to the constraint

$$\star : \vec{\sigma}_{q-k} + \sum_{j=1}^{k} \vec{s}_j = \vec{n}.$$

By convention, if $k = 0$, the j-sum vanishes identically and the k inner summations over $\vec{s}_j$ evaluate to unity.

Note the following:

(1) All ($q, k = 0$) terms vanish identically, except for the term ($q = |\vec{n}|, k = 0$), for which the $\vec{\sigma}_{|\vec{n}|}$-sum contributes by one effective term only, viz., the one for which $\vec{\sigma}_{|\vec{n}|} = \vec{n}$, yielding $(1/\vec{n}!)\partial^{|\vec{n}|}L/\partial u^{\vec{n}}$.
(2) The ($q = 1, k = 1$) term also contributes by only one multi-index in the innermost sum, viz., the one with $\vec{s}_1 = \vec{n}(\vec{\sigma}_0 = \vec{0})$, yielding the term $(1/\vec{n}!)\ (\partial L/\partial w)(\partial^{|n|}w/\partial u^{\vec{n}})$.

Proof 2. We use Lemma 1 and insert $\phi_a(u) = u_a$ for $a = 1, \ldots, d-1$ and $\phi_d(u) = w(u)$. We then evaluate the Einstein summation, write $x_a = u_a$ for $a = 1, \ldots, d-1$, and write $x_d = w$, using the following observation:

$$\frac{\partial^{|\vec{s}_j|}\phi_{a_j}}{\partial u^{\vec{s}_j}} = \delta_{\vec{s}_j \vec{a}_j}.$$

In this equation the right-hand side is 1 if $\vec{s}_j = \vec{a}_j$ and is 0 otherwise, where $\vec{a}_j$ is the $(d-1)$-dimensional multi-index whose αth entry is given by $\delta_{a_j\alpha}(\alpha = 1, \ldots, d-1)$. Furthermore,

$$\frac{\partial^{|\vec{s}_j|}\phi_d}{\partial u^{\vec{s}_j}} = \frac{\partial^{|\vec{s}_j|}w}{\partial u^{\vec{s}_j}}.$$

With $\vec{\sigma}_{q-k}$ defined as $\sum_{j=1}^{q-k}\vec{a}_j$, the result then follows after some elementary arithmetic.

EXAMPLE 1 (some lowest-order results). Consider an image defined on a 3D space. Let u, v be the Monge-patch parameters for a 2D surface $S : (u, v) \mapsto (u, v, w(u, v))$, defined on a full neighborhood of (0, 0), with a smooth height function $w(u, v)$. Then, in terms of the Cartesian coordinates (u, v, w) (defined in the straightforward way), we have the following variations up to third order:

$$\frac{\delta L}{\delta u} = L_u + L_w \frac{\partial w}{\partial u},$$

$$\frac{\delta L}{\delta v} = L_v + L_w \frac{\partial w}{\partial v},$$

$$\frac{\delta^2 L}{\delta u^2} = L_{uu} + 2L_{uw}\frac{\partial w}{\partial u} + L_{ww}\left(\frac{\partial w}{\partial u}\right)^2 + L_w \frac{\partial^2 w}{\partial u^2},$$

$$\frac{\delta^2 L}{\delta u \delta v} = L_{uv} + L_{uw}\frac{\partial w}{\partial v} + L_{vw}\frac{\partial w}{\partial u} + L_{ww}\frac{\partial w}{\partial u}\frac{\partial w}{\partial v} + L_w \frac{\partial^2 w}{\partial u \partial v},$$

$$\frac{\delta^2 L}{\delta v^2} = L_{vv} + 2L_{vw}\frac{\partial w}{\partial v} + L_{ww}\left(\frac{\partial w}{\partial v}\right)^2 + L_w \frac{\partial^2 w}{\partial v^2},$$

$$\frac{\delta^3 L}{\delta u^3} = L_{uuu} + 3L_{uuw}\frac{\partial w}{\partial u} + 3L_{uww}\left(\frac{\partial w}{\partial u}\right)^2 + L_{www}\left(\frac{\partial w}{\partial u}\right)^3 + 3L_{uw}\frac{\partial^2 w}{\partial u^2} + 3L_{ww}\frac{\partial^2 w}{\partial u^2}\frac{\partial w}{\partial u} + L_w \frac{\partial^3 w}{\partial u^3},$$

$$\frac{\delta^3 L}{\delta u^2 \delta v} = L_{uuv} + L_{uuw}\frac{\partial w}{\partial v} + 2L_{uvw}\frac{\partial w}{\partial u} + 2L_{uww}\frac{\partial w}{\partial u}\frac{\partial w}{\partial v} + 2L_{uw}\frac{\partial^2 w}{\partial u \partial v} + L_{vww}\left(\frac{\partial w}{\partial u}\right)^2 + L_{www}\left(\frac{\partial w}{\partial u}\right)^2\frac{\partial w}{\partial v} + 2L_{ww}\frac{\partial^2 w}{\partial u \partial v}\frac{\partial w}{\partial u} + L_{vw}\frac{\partial^2 w}{\partial u^2} + L_{ww}\frac{\partial^2 w}{\partial u^2}\frac{\partial w}{\partial v} + L_w \frac{\partial^3 w}{\partial u^2 \partial v},$$

$$\frac{\delta^3 L}{\delta u \delta v^2} = L_{uvv} + L_{vvw}\frac{\partial w}{\partial u} + 2L_{uvw}\frac{\partial w}{\partial v} + 2L_{vww}\frac{\partial w}{\partial u}\frac{\partial w}{\partial v} + 2L_{vw}\frac{\partial^2 w}{\partial u \partial v} + L_{uww}\left(\frac{\partial w}{\partial v}\right)^2 + L_{www}\left(\frac{\partial w}{\partial v}\right)^2\frac{\partial w}{\partial u} + 2L_{ww}\frac{\partial^2 w}{\partial u \partial v}\frac{\partial w}{\partial v} + L_{uw}\frac{\partial^2 w}{\partial v^2}$$

$$
\begin{aligned}
&+ L_{ww} \frac{\partial^2 w}{\partial v^2} \frac{\partial w}{\partial u} + L_w \frac{\partial^3 w}{\partial u \partial v^2}, \\
\frac{\delta^3 L}{\delta v^3} &= L_{vvv} + 3L_{vvw} \frac{\partial w}{\partial v} + 3L_{vww} \left(\frac{\partial w}{\partial v}\right)^2 \\
&+ L_{www} \left(\frac{\partial w}{\partial v}\right)^3 + 3L_{vw} \frac{\partial^2 w}{\partial v^2} \\
&+ 3L_{ww} \frac{\partial^2 w}{\partial v^2} \frac{\partial w}{\partial v} + L_w \frac{\partial^3 w}{\partial v^3}, \\
&\vdots
\end{aligned}
$$

The following proposition shows the effect of a gray-value transformation on the image's differential structure.

PROPOSITION 2 (transformation of spatial derivatives). Let $L = L(x)$ denote a scalar image on a d-dimensional spatial domain, and let $\hat{L} = \hat{L}(x)$ denote the scalar image obtained by applying a gray-value transformation $\hat{L} = \mathcal{G}(L)$ to L. Then we have, for all $\vec{n} = (n_1, \ldots, n_d)$ with $|\vec{n}| \geq 1$,

$$
\frac{1}{\vec{n}!} \frac{\partial^{|\vec{n}|} \hat{L}}{\partial x^{\vec{n}}} = \sum_{q=1}^{|\vec{n}|} \frac{1}{q!} \mathcal{G}^{(q)}(L) \times \sum_{|\vec{s}_1| \geq 1}^{\star} \cdots \sum_{|\vec{s}_q| \geq 1}^{\star} \prod_{j=1}^{q} \frac{1}{\vec{s}_j!} \frac{\partial^{|\vec{s}_j|} L}{\partial x^{\vec{s}_j}},
$$

in which the q inner summations are restricted to multi-indices of dimension d that add up to $\vec{n}$, i.e.,

$$
\star : \sum_{k=1}^{q} \vec{s}_k = \vec{n},
$$

and in which $\mathcal{G}^{(q)}(L)$ denotes the qth derivative of $\mathcal{G}(L)$.

Proof 3. Proposition 2 can be regarded as a special case of Lemma 1 by making some formal substitutions. The idea is to consider the image $L = L(x)$ as a Monge patch in a $(d+1)$-dimensional space, for which the spatial variable x is identified with the parameter ε. To this end, replace d in Lemma 1 by $D = d + 1$, p by $P = d$, $x \in \mathbb{R}^d$, and $\varepsilon \in \mathbb{R}^p$ by $X = (x; y) \in \mathbb{R}^D$ and $E \in \mathbb{R}^P$, respectively, and, accordingly, each spatial index $i_k = 1, \ldots, d$ by a corresponding index $I_k = 1, \ldots, D$. Consequently, each derivative $\partial/\partial x_i$ is replaced by $\partial/\partial X_I = (\partial/\partial x_i; \partial/\partial y)$. Moreover, replace $\phi : \mathbb{R}^p \to \mathbb{R}^d : \varepsilon \mapsto \phi(\varepsilon)$ by the Monge-patch parameterization $\Phi : \mathbb{R}^P \to \mathbb{R}^D : E \mapsto \Phi(E) = (E; L(E))$, and, finally, replace $L : \mathbb{R}^d \to \mathbb{R} : x \mapsto L(x)$ by $\Lambda : \mathbb{R}^D \to \mathbb{R} : X \mapsto \Lambda(X) = \mathcal{G}(y)$. With these formal substitutions and the observation that the Einstein summation yields effective terms only for index values $I_k = D$, $k = 1, \ldots, q$, corresponding to y-derivatives in X-space (by virtue of the x-independence of $\Lambda(X) = \mathcal{G}(y)$), the proof of Proposition 2 follows from Lemma 1.

Proposition 2 relates the image's spatial derivatives before and after a gray-value transformation. It states that, in general, nth-order derivative is affected by a gray-value transformation in a way that depends not only on the transformation details (i.e., the $\mathcal{G}^{(q)}(L)$, with $q = 1, \ldots, n$), but also on all image derivatives of orders less than or equal to n. In other words, only the *set* of all derivatives up to nth order (inclusive) transforms in a closed way. In particular, one can therefore expect an nth-order differential gray-value invariant to entail all spatial derivatives of orders $1, \ldots, n$. The notion of a *local jet of order n*, notation $J^n[L](x)$, captures this nth-order local structure. It is defined as the equivalence class of functions with spatial contact of order n at a given base point. The following is a possible operational definition.

DEFINITION 5 (local jet for gray-value images). Let $L = L(x)$ be a scalar image defined on a d-dimensional spatial domain, and let $L_{i_1 \cdots i_k}$ be the tensor of rank k formed by the kth-order partial derivatives $\partial^k L / \partial x_{i_1} \cdots \partial x_{i_k}$ (with $i_j = 1, \ldots, d$ for each $j = 1, \ldots, k$). Then the image's local jet of order n at base point P can be represented by the set of all k-tensors $L_{i_1 \cdots i_k}$ up to rank n (inclusive), evaluated at base point P:

$$
J^n[L](P) = \{L_{i_1 \cdots i_k}(P) \mid k = 0, \ldots, n\}.
$$

Note that this definition is independent of the choice of a Cartesian frame since it relies on Cartesian tensors only. This Cartesian representation will turn out to be useful here, but it should be noted that there are many equivalent, non-Cartesian representations (see [6]). In the mathematical literature, one often finds the zeroth order excluded from the definition (cf. [20]).

The preceding local-jet representation accounts for a complete description of local image structure up to nth order. This is obvious from Taylor's expansion, which expresses the image's gray values in a full neighborhood of a given base point, x say, in terms of the image derivatives at x; so if $\delta x \in \mathbb{R}^d$, then

$$L(x + \delta x) = \sum_{k=0}^{n} \frac{1}{k!} L_{i_1 \cdots i_k}(x) \delta x_{i_1} \cdots \delta x_{i_k} + \mathcal{O}(\|\delta x\|^{n+1}).$$

The concept of a local jet for an image remains meaningful even when invariance under gray-value transformations is required. This follows from the fact that if $L_1, L_2 \in J^n[L](P)$, then also $\hat{L}_1, \hat{L}_2 \in J^n[\hat{L}](P)$.

Recall that multi-indices merely label components, but they are not related to a representation of a transformation group, as opposed to Cartesian (or spatial) indices. To make its invariance against Cartesian coordinate transformations manifest, we have to recast Proposition 2 into covariant form.

PROPOSITION 3 (transformation of spatial derivatives in covariant form). Let $L = L(x)$ denote a scalar image on a d-dimensional spatial domain, and let $\hat{L} = \hat{L}(x)$ denote the scalar image obtained by applying a gray-value transformation $\hat{L} = \mathcal{G}(L)$ to L. Then, using a spatial index to denote a derivative with respect to the associated spatial variable, we have, for all $n \geq 1$,

$$\frac{1}{n!} \hat{L}_{i_1 \cdots i_n} = \sum_{q=1}^{n} \frac{1}{q!} \mathcal{G}^{(q)}(L) \times \sum_{l_1 \geq 1}^{\star} \cdots \sum_{l_q \geq 1}^{\star} \mathcal{S} \circ \Lambda \left[\prod_{k=1}^{q} \frac{1}{l_k!} L_{i_1^k \cdots i_{l_k}^k} \right],$$

in which the q inner summations are restricted to indices that add up to n, i.e.,

$$\star : \sum_{m=1}^{q} l_m = n,$$

$\mathcal{S}$ denotes the index symmetrization operator, and Λ is the lexicographical index-ordering operator that "unravels" or "flattens" a doubly indexed list $\{a_l^k\}_{l=1,\ldots,l_k}^{k=1,\ldots,q}$ into a singly indexed list of type $\{a_i\}_{i=1,\ldots,n}$ with $n = \sum_{m=1}^{q} l_m$. In this procedure the upper index k takes lexicographical precedence over the lower index l. In other words, Λ concatenates the q rows labeled by k, the kth row of which has l_k entries:

$$\Lambda\left[\{a_l^k\}_{l=1,\ldots,l_k}^{k=1,\ldots,q}\right] = \{a_i\}_{i=1,\ldots,n} \text{ with } i = l + \sum_{m=1}^{k-1} l_m.$$

Proof 4. The proof of Proposition 3 again relies entirely on the chain rule and Leibniz's product rule, and it follows by induction with respect to n.

It is instructive to write out some lowest-order results.

Example 2 (some lowest-order results). If $\hat{L} = \mathcal{G}(L)$, then up to order four we have the following relationship between the derivatives before and after a gray-value transformation:

$$\begin{aligned}
\hat{L}_i &= L_i \mathcal{G}'(L), \\
\hat{L}_{ij} &= L_{ij} \mathcal{G}'(L) + L_i L_j \mathcal{G}''(L), \\
\hat{L}_{ijk} &= L_{ijk} \mathcal{G}'(L) \\
&\quad + [L_{ij} L_k + L_{jk} L_i + L_{ki} L_j] \mathcal{G}''(L) \\
&\quad + L_i L_j L_k \mathcal{G}'''(L), \\
\hat{L}_{ijkl} &= L_{ijkl} \mathcal{G}'(L) \\
&\quad + [L_{ijk} L_l + L_{ijl} L_k + L_{jkl} L_i + L_{ij} L_{kl} \\
&\quad + L_{ik} L_{jl} + L_{il} L_{jk}] \mathcal{G}''(L) \\
&\quad + [L_{ij} L_k L_l + L_{ik} L_j L_l + L_{il} L_j L_k \\
&\quad + L_{jk} L_i L_l + L_{jl} L_i L_k + L_{kl} L_i L_j] \mathcal{G}'''(L) \\
&\quad + L_i L_j L_k L_l \mathcal{G}''''(L), \\
&\vdots
\end{aligned}$$

Proposition 3 shows that the tensorial derivatives in Definition 5 are not invariant against general intensity transformations nor do they transform in a simple (covariant) way. There is an apparent redundancy in the image derivatives with respect to this group of transformations. Because this redundancy is rather complex, according to Proposition 3, it is far from trivial to form differential gray-value invariants based on the local-jet components $L_{i_1 \cdots i_k}$.

Image derivatives directly relate to the structure of the intensity profile of the image, i.e., the form of the graph $(x; y = L(x)) \in \mathbb{R}^d \times \mathbb{R}$. Consequently, their relation to the graph's y-independent ($\mathcal{G}$-invariant) geometrical structure, i.e., the underlying structure common to all members of the equivalence class of images related by one-to-one gray-value maps, is rather indirect.

Proposition 3 also shows that things are much simpler for the two-parameter affine subgroup of first order gray-value transformations $\Gamma(L; \lambda, \mu) = \lambda + e^{\mu} L$ (offset and linear scaling of gray values), in which case each derivative $L_{i_1 \cdots i_k} (k > 0)$ merely scales by a constant factor e^{μ} (hence becomes a relative invariant of unit weight, or even an absolute invariant under the one-parameter subgroup $\Gamma(L; \lambda, 0) = \lambda + L$). In that particular case, the local-jet representation of Definition 5 clearly has its merits.[2] Because of their (relative) invariant nature, it is very easy to form differential invariants with respect to the affine subgroup $\Gamma(L; \lambda, \mu)$ on the basis of the tensors $L_{i_1 \cdots i_k}$; simply combine these into any Cartesian invariant, and the result is a (relative) invariant under the affine gray-value transformation group. If absolute invariants are required, one can always pair relative invariants of the same weight and consider their ratios. See [14] for details.

The general case, in which arbitrary gray-value transformations are admitted, is not so simple. Clearly, the relevant geometry in this context is that of the image's iso-intensity, or *isophote* picture (or, equivalently, of its dual objects, the gradient integral curves, or *flow lines*). In that case, the representation of Definition 5 is not very convenient for reasons already explained. On the other hand, Definition 5 is very attractive for its *operational* nature (its components can be obtained directly by mere linear correlations), and so it would be convenient to have a similar local-jet construct that explicitly captures the image's $\mathcal{G}$-invariant isophote structure and whose components can be expressed in terms of those in Definition 5.

In fact, the essential parts for such a construct are readily available from the previous theory.

DEFINITION 6 (local jet for isophotes). Let $L = L(x)$ denote a scalar image on a d-dimensional spatial domain, and let P be a regular point, i.e., a point at which the image gradient does not vanish. By a suitable choice of Cartesian coordinates we may assume that $x_P = 0 \in \mathbb{R}^d$ are the coordinates of P. Let

$$S : \mathbb{R}^{d-1} \to \mathbb{R}^d : u \mapsto x = \phi(u)$$

be the Monge-patch parameterization of the isophote (hyper) surface through P, with $\phi : \mathbb{R}^{d-1} \to \mathbb{R}^d$ given by

$$\begin{cases} \phi_a(u) = u_a & (a = 1, \ldots, d-1), \\ \phi_d(u) = w(u). \end{cases}$$

Furthermore, let $w_{a_1 \cdots a_k}$ be the tensor of rank k formed by the kth-order partial derivatives of the Monge patch, i.e., $\partial^k w / \partial u_{a_1} \cdots \partial u_{a_k}$ (with $a_j = 1, \ldots, d-1$ for each $j = 1, \ldots, k$). Then the local jet of order n for the image's isophote I at base point P can be represented by the set of all k-tensors $w_{a_1 \cdots a_k}$ up to rank n (inclusive), evaluated at base point P:

$$J^n[I](P) = \{w_{a_1 \cdots a_k}(P) \mid k = 1, \ldots, n\}.$$

The function $w : \mathbb{R}^{d-1} \to \mathbb{R}$ can be interpreted as the height map for the isophote at the point $x = (u; 0) \in \mathbb{R}^d$ near P. So the local-jet definition is actually the same as for the gray-value case of Definition 5, with the gray-value map on the d-dimensional spatial domain replaced by this height map on the isophote's tangent plane.

The coordinates of a Cartesian frame established in this way will henceforth be denoted by $(u_a; w)$, $a = 1, \ldots, d-1$, in which the positive w axis is aligned with the image gradient. This is always possible at a regular point. Note that there is a residual parameterization freedom in the isophote's tangent plane, since only the w axis is fixed. Instead of the Cartesian symmetry group $E(d) = SO(d) \times T(d)$ of rotations ($SO(d)$) and translations ($T(d)$), we are left with the subgroup $E(d-1)$ of coordinate transformations in the tangent plane (the tensor indices a_j in Definition 6 transform according to this subgroup).

An admissible constraint on the set of all possible local frames is called a *gauge condition*. The preceding partial w gauge will be referred to as the "gradient gauge."

Completeness of the representation of Definition 6 follows from the observation that if $\delta u \in \mathbb{R}^{d-1}$ is a small vector in the tangent plane of the isophote at base point $P : u = 0$, we have:

$$w(\delta u) = \sum_{k=1}^{n} \frac{1}{k!} w_{a_1 \cdots a_k}(0) \delta u_{a_1} \cdots \delta u_{a_k} + \mathcal{O}(\|\delta u\|^{n+1}).$$

It is understood that the Einstein summation now applies to the a_j indices.

We have arrived at a proper definition of local $\mathcal{G}$-invariant image structure (Definition 6), but it remains to be made operational by establishing its relation to Definition 5. To this end, we may apply Proposition 1 to the image's isophotes. The $d - 1$ Monge-patch parameters correspond to the nongauged Cartesian coordinates on the $(d - 1)$-dimensional isophote's tangent plane.

To make the $E(d - 1)$ symmetry of Proposition 1 manifest, we can write it as a tensor equation on the isophote's $(d - 1)$-dimensional tangent space.

PROPOSITION 4 (implicit differentiation along isophotes in covariant form). Let $L = L(x)$ denote a scalar image on a d-dimensional spatial domain, and let I_P be an isophote passing through a regular point P. Let TI_P be the $(d-1)$-dimensional tangent (hyper) plane to this isophote at point P, and let $u_a = 1, \ldots, d - 1$ be any set of Cartesian coordinates on TI_P. Moreover, let $w_{a_1 \cdots a_k}$ be defined as in Definition 6, and define the k-tensor $L^{(j)}_{a_1} \cdots \partial u_{a_k}$ as $\partial^{j+k} L / \partial w^j \partial u_{a_1} \cdots \partial u_{a_k}$, i.e., the $(j + k)$th-order derivative with respect to the gradient gauged $(u; w)$ coordinate system in $\mathbb{R}^d$. Finally, let $\delta_{a_1 \cdots a_n} L$ denote the nth-order variation $\delta^n L / \delta u_{a_1} \cdots \delta u_{a_n}$. Then, using $E(d - 1)$-tensor notation for TI_P, we have

$$\frac{1}{n!} \delta_{a_1 \cdots a_n} L = \mathcal{S} \circ \Lambda \left[\sum_{q=1}^{n} \sum_{k=0}^{q} \frac{1}{k!} \frac{1}{(q-k)!} L^{(k)}_{a_1^0 \cdots a_{q-k}^0} \times \sum_{l_1 \geq 1}^{\star} \cdots \sum_{l_k \geq 1}^{\star} \prod_{m=1}^{k} \frac{1}{l_m!} w_{a_1^m \cdots a_{l_m}^m} \right],$$

in which, the k l_m summations $(m = 1, \ldots, k)$ are subject to the constraint

$$\star : q - k + \sum_{m=1}^{k} l_m = n,$$

Λ is the lexicographical index-ordering operator as introduced in Proposition 3, and $\mathcal{S}$ denotes spatial index symmetrization. By convention, if $k = 0$, the m-sum vanishes identically and the k inner summations over l_m evaluate to unity.

Proof 5. Proposition 4 follows simply by reformatting the equivalent Proposition 1, switching from multi-indices to tensor indices.

In this tensorial form, the analog of Example 1 becomes as follows.

EXAMPLE 3 (some lowest-order results). Let $L_{a_1 \cdots a_n w \cdots \leftarrow j \rightarrow \cdots w}$ denote $L^{(j)}_{a_1 \cdots a_n}$. Then the variations of Proposition 4 up to order three are given by (compare this tensorial notation with Example 1):

$$\begin{aligned}
\delta_a L &= L_a + L_w w_a, \\
\delta_{ab} L &= L_{ab} + L_{aw} w_b + L_{bw} w_a \\
&\quad + L_{ww} w_a w_b + L_w w_{ab}, \\
\delta_{abc} L &= L_{abc} + L_{abw} w_c + L_{acw} w_b + L_{bcw} w_a \\
&\quad + L_{aww} w_b w_c + L_{bww} w_a w_c + L_{cww} w_a w_b \\
&\quad + L_{aw} w_{bc} + L_{bw} w_{ac} + L_{cw} w_{ab} \\
&\quad + L_{ww} w_{ac} w_b + L_{ww} w_{bc} w_a + L_{ww} w_{ab} w_c \\
&\quad + L_{www} w_a w_b w_c + L_w w_{abc}, \\
&\;\vdots
\end{aligned}$$

The crucial observation that finally establishes the relation between Definition 5 and Definition 6 is trivial.

OBSERVATION 1. All variations $\delta_{a_1 \cdots a_n} L$ on $TI_P (n \geq 1)$ defined in Proposition 4 vanish identically.

This leads to the following *generating equation* for the components of Definition 6.

PROPOSITION 5 (generating equation). Let $L = L(x)$ be a scalar image, let P be a regular point, and let I_P be the isophote passing through P. If $\delta_n L$ is the system of all kth-order variations of L on TI_P for $k = 1, \ldots, n$, then the $\mathcal{G}$-invariant local n-jet components $w_{a_1 \cdots a_k}(k = 1, \ldots, n)$ are determined by the generating equation

$$\delta_n L = 0.$$

In turn, this leads to the following inductive scheme for $w_{a_1 \cdots a_n}$.

PROPOSITION 6 (inductive scheme for local isophote jet). The components $\partial^{|\vec{k}|} w / \partial u^{\vec{k}} (|\vec{k}| = 1, \ldots, |\vec{n}|)$ are determined inductively by

$$\frac{\partial^{|\vec{n}|} w}{\partial u^{\vec{n}}} = -L_w^{-1} \left(\frac{\partial^{|\vec{n}|} L}{\partial u^{\vec{n}}} + \sum_{q=1}^{|\vec{n}|} \sum_{\substack{k=1 \\ (q,k)\neq(1,1)}}^{q} \sum_{|\vec{\sigma}_{q-k}|=q-k}^{\star} \times \frac{1}{k!} \frac{1}{\vec{\sigma}_{q-k}!} \frac{\partial^q L}{\partial w^k \partial u^{\vec{\sigma}_{q-k}}} \times \sum_{|\vec{s}_1|\geq 1}^{\star} \cdots \sum_{|\vec{s}_k|\geq 1}^{\star} \prod_{j=1}^{k} \frac{1}{\vec{s}_j!} \frac{\partial^{|\vec{s}_j|} w}{\partial u^{\vec{s}_j}} \right),$$

with the usual constraint on the $k + 1$ multi-index summations

$$\star : \vec{\sigma}_{q-k} + \sum_{j=1}^{k} \vec{s}_j = \vec{n}$$

and the convention that if $k = 0$, the j-sum vanishes identically and the k inner summations over $\vec{s}_j$ evaluate to unity. Alternatively, in covariant $E(d-1)$ form, the k-tensors $w_{a_1 \cdots a_k}(k = 1, \ldots, n)$ are determined inductively by

$$w_{a_1 \cdots a_n} = -L_w^{-1} \left(L_{a_1 \cdots a_n} + \mathcal{S} \circ \mathcal{A} \left[\sum_{q=1}^{n} \sum_{\substack{k=1 \\ (q,k)\neq(1,1)}}^{q} \times \frac{1}{k!} \frac{1}{(q-k)!} L^{(k)}_{u^0_{a_1} \cdots u^0_{a_{q-k}}} \times \sum_{l_1 \geq 1}^{\star} \cdots \sum_{l_k \geq 1}^{\star} \prod_{m=1}^{k} \times \frac{1}{l_m!} w_{a^m_1 \cdots a^m_{l_m}} \right] \right),$$

in which the k l_m summations ($m = 1, \ldots, k$) are subject to the constraint

$$\star : q - k + \sum_{m=1}^{k} l_m = n$$

and in which $\mathcal{S}$ and $\mathcal{A}$ are defined as before.

Proof 6. Set the left-hand side of the equation in Proposition 4 (or in the corresponding Proposition 1) equal to zero, and solve for the ($q = 1, k = 1$) term, i.e., the only term containing the nth-order ($|\vec{n}|$th-order) derivative $w_{a_1 \cdots a_n} (\partial^{|\vec{n}|} w / \partial u^{\vec{n}})$, multiplied by the factor L_w. The ($q = n, k = 0$) term, the only effective one in the ($q, k = 0$) sum, has been made explicit here. See the remarks after Proposition 1.

Clearly, the local isophote jet is ill defined at singular points where $L_w = 0$. In a generic image, however, almost all points are regular. Proposition 6 shows how to obtain the isophote local jet in any given regular point P in terms of $E(d-1)$ tensors $w_{a_1 \cdots a_n}$ defined on the isophote's tangent bundle. On the basis of this one can construct Cartesian differential invariants in the usual way (see [9], [14]), which are also invariant under the group $\mathcal{G}$ of general gray-value transformations.

There remains a problem of practical interest, viz., how to evaluate those $E(d-1)$-invariants on the isophote's tangent bundle in terms of $E(d)$-invariants on the image domain, so that one does not need to perform any actual transformation to an explicit gradient-gauged $(u; w)$ coordinate frame to derive $\mathcal{G}$-invariants. One way of achieving this is to extend the gradient gauge by imposing a geometrically significant gauge on the u coordinates, thereby removing all residual gauge degrees of freedom, and to relate the gauged $(u;w)$-coordinate derivatives to Cartesian invariants on the image domain. One may set up such a gauge system in a regular point by complementing the gradient vector to any independent set of d vectors by suitable contractions of tensor products based on local image derivatives. This set can then be orthogonalized, e.g., by running the Gram–Schmidt orthogonalization process from standard linear algebra.

It should be noted that there are several possibilities for choosing a geometrically independent set of vectors in addition to the gradient, each of which may lead to a different gauge system. The following example shows one such possibility for using the incomplete gradient gauge to initialize an inductive scheme for an unambiguous, full gauge.

EXAMPLE 4 (inductive scheme for full gauge). Consider the following set of d vectors in $\mathbb{R}^d$:

$$\begin{cases} v_i^{(0)} = L_i, \\ v_i^{(a)} = L_{ij} v_j^{(a-1)} \qquad (a = 1, \ldots, d-1). \end{cases}$$

Then the following system is orthogonal (Gram–Schmidt):

$$\begin{cases} u_i^{(0)} = v_i^{(0)}, \\ u_i^{(a)} = v_i^{(a)} - \sum_{l=0}^{a-1} \frac{v_j^{(a)} u_j^{(l)}}{u_n^{(l)} u_n^{(l)}} u_i^{(l)} \quad (a = 1, \ldots, d-1). \end{cases}$$

In particular, if the $v^{(k)}$ are linearly independent $(k = 0, \ldots, d-1)$, then the $u^{(k)}$ provide an orthogonal basis for $\mathbb{R}^d$. In that case, the coordinate derivatives are given by the following Cartesian-invariant, directional derivatives:

$$\begin{cases} \frac{\partial}{\partial w} = \frac{u_i^{(0)}}{\sqrt{u_j^{(0)} u_j^{(0)}}} \frac{\partial}{\partial x_i}, \\ \frac{\partial}{\partial u_a} = \frac{u_i^{(a)}}{\sqrt{u_j^{(a)} u_j^{(a)}}} \frac{\partial}{\partial x_i} \qquad (a = 1, \ldots, d-1). \end{cases}$$

One may wonder whether the gauge condition in Example 4 is admissible, i.e., whether the $u^{(k)}$ are independent $(k = 0, \ldots, d-1)$. The admissibility of the gauge for a given point P generally depends on the choice of the system $v^{(a)}$ $(a = 1, \ldots, d-1)$ in relation to the local geometry of the isophote at P. In this case, if one of the eigenvectors of the Hessian L_{ij} is proportional to the gradient at a given point P, then the vectors $v^{(a)}$, and hence $u^{(a)}$ $(a = 1, \ldots, d-1)$, are not linearly independent in P and the gauge is not admissible. This is most easily verified by diagonalizing the Hessian. (The scalars $v^{(k)} \cdot v^{(0)}$ $(k = 0, \ldots, d-1)$ are part of a set of so-called *irreducible invariants*; see [14] for an explanation and a proof.)

Before we turn to some applications, it is useful to relax the strict $\mathcal{G}$-invariance requirement a bit and admit relative tensors and invariants as well. Relative $\mathcal{G}$-invariants acquire a factor that is some power of the Jacobian of the group transformation, i.e., some power of $\mathcal{G}'(L)$.

PROPOSITION 7 (relative $\mathcal{G}$-tensors). Let $w_{a_1 \cdots a_n}$ be the $\mathcal{G}$-invariant Cartesian tensors defined in Definition 6. Then the Cartesian tensors $w^{[p]}_{a_1 \cdots a_n}$ defined by

$$w^{[p]}_{a_1 \cdots a_n} = L_w^p w_{a_1 \cdots a_n}$$

are relative $\mathcal{G}$-invariants of weight p. This means that if $\hat{L} = \mathcal{G}(L)$ is a gray-value transformation and $\hat{w}^{[p]}_{a_1 \cdots a_n}$ is the $\mathcal{G}$-transform of $w^{[p]}_{a_1 \cdots a_n}$, then one has

$$\hat{w}^{[p]}_{a_1 \cdots a_n} = (\mathcal{G}'(L))^p w^{[p]}_{a_1 \cdots a_n}.$$

Proof 7. This follows from the fact that L_w is a relative $\mathcal{G}$-invariant of unit weight: $\hat{L}_w = \mathcal{G}'(L) L_w$; see also Example 2.

Although not invariant, these relative invariants are in some sense close to invariant because they all transform with some power of one and the same factor. Consequently, it is easy to combine them into absolute invariants by taking appropriate products and ratios. Or, if one refrains from that but considers only relative invariants of equal weight, I and J say, then all *relations* of the type $I = J$ are invariant in the absolute sense.

4 Applications

To illustrate the general method for generating gray-value invariants, we consider the lowest-order cases in 2D and 3D images.

4.1 Isophote Curvature and Corner Detection in 2D

In the 2D case the isophotes are generally 1D curves and the index $a = 1$ in Proposition 4 refers to a single Cartesian coordinate for parameterizing the isophote's local tangent; we will write v instead of u_1.

First of all, we need to specify a full gauge compatible with the partial gradient gauge. In this case this is trivial; there is no residual (rotational) freedom for the v axis once we have made the w axis coincident with the gradient direction (hence we can manage without the general scheme of Example 4). Let us define the basis vectors $\hat{v}$, $\hat{w}$ as follows:

$$\hat{v}_i = \frac{\varepsilon_{ij} L_j}{\sqrt{L_k L_k}},$$
$$\hat{w}_i = \frac{\delta_{ij} L_j}{\sqrt{L_k L_k}},$$

with ε_{ij} the 2D, antisymmetric Lévi–Civita tensor defined by $\varepsilon_{12} = -\varepsilon_{21} = 1$ and $\varepsilon_{11} = \varepsilon_{22} = 0$. The derivatives here and in what follows are to be evaluated at the location of interest. The gauge is related to an arbitrary Cartesian coordinate frame by the following identities:

$$\frac{\partial}{\partial v} = \frac{\varepsilon_{ij} L_j}{\sqrt{L_k L_k}} \frac{\partial}{\partial x_i},$$
$$\frac{\partial}{\partial w} = \frac{\delta_{ij} L_j}{\sqrt{L_k L_k}} \frac{\partial}{\partial x_i}.$$

Working out Proposition 4 (or Example 3) to order three within this gauge yields the following result (a self-explanatory notation is used): by virtue of the gradient gauge one has $w' = 0$ and

$$w'' = -\frac{L_{vv}}{L_w} \stackrel{\text{def}}{=} \kappa,$$
$$w''' = -\frac{L_{vvv}}{L_w} + 3\frac{L_{vv}}{L_w}\frac{L_{vw}}{L_w} \stackrel{\text{def}}{=} \lambda \stackrel{\text{def}}{=} \nu + 3\kappa\mu,$$

with κ, μ, and ν given by

$$\kappa = \frac{\varepsilon_{ij}\varepsilon_{kl} L_i L_{jk} L_l}{(L_m L_m)^{3/2}},$$
$$\mu = \frac{\varepsilon_{ij} L_j L_k L_{ik}}{(L_m L_m)^{3/2}},$$
$$\nu = -\frac{\varepsilon_{il}\varepsilon_{jm}\varepsilon_{kn} L_l L_m L_n L_{ijk}}{(L_p L_p)^2}$$

(one may use the identity $\varepsilon_{ij}\varepsilon_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ to substitute for double ε-tensors; see [14] for a general result in d dimensions). Note that the third-order derivative w''' is expressed in terms of two second-order gray-value invariants κ, μ and one third-order gray-value invariant ν (see Proposition 3 or Example 2 for a verification of this statement). Clearly, $w'' = \kappa$ is the *isophote curvature* [5], [21]. Ignoring the genuine third-order property ν, we may wonder about the geometrical significance of the other second-order gray-value invariant μ.

To appreciate the geometrical significance of μ, note that our generating scheme for gray-value invariants is based on the isophote picture; the structure of the orthogonal trajectories of the isophotes, i.e., the gradient integral curves (or *flow lines*, for short), has been deemphasized since these *dual* objects are implicitly defined by the isophotes. Hence one may foresee a relationship between isophote properties captured in the derivatives of w and properties of flow lines.

For example, $\mu = L_{vw}/L_w$, an apparent gray-value invariant itself (and a Cartesian relative, or pseudoinvariant), is the *flow-line curvature*. This is easily demonstrated. A flow line is a gradient integral curve by definition, and so it can be parameterized by $x_i(s)$, the unit tangent vector of which is aligned with the gradient $\dot{x}_i = \hat{w}_i$. Then the absolute value of the flow-line curvature $|\kappa_f|$ equals $\sqrt{\ddot{x}_i \ddot{x}_i}$, or $\kappa_f^2 = (L_i L_{ij} L_{jk} L_k L_l L_l - (L_i L_{ij} L_j)^2)(L_m L_m)^{-3}$. But it is easily verified that this is just μ^2. The Cartesian pseudocharacter of μ accounts for the orientation of the flow-line-curvature vector relative to the v axis (which is chosen so as to give the (v, w) frame the standard orientation). This orientation reverses upon reflections of the coordinate axes (because $\hat{v}$ flips). Taking this orientation into account, we can thus define $\kappa_f = \mu$.

Isophote curvature has been used successfully in corner-detection algorithms. If a corner is defined as a point at which both the isophote curvature κ and the gradient magnitude L_w (the latter of which, of course, is not a gray-value invariant) are large, it makes sense to consider their trade-off in the form of a *relative gray-value invariant*, as proposed in Proposition 7. This leads to the Cartesian invariants $\kappa^{[p]} = \kappa L_w^p$, with $p > 0$. The smallest value of p that turns this into a *polynomial invariant* is $p = 3$. Moreover, this particular choice yields a *relative affine invariant*, which means that it is invariant under the special linear group of spatial transformations $SL(2)$. Since it is a shear-invariant combination of gradient magnitude and isophote curvature, one could call this invariant "edge-curvature strength." Figure 1 shows the invariant $\kappa^{[3]} = L_i\varepsilon_{ij}L_{jk}\varepsilon_{kl}L_l$, or

$$\begin{aligned}\kappa^{[3]} &= L_iL_{ij}L_j - L_iL_iL_{jj} \\ &= -L_x^2L_{yy} + 2L_xL_yL_{xy} - L_y^2L_{xx} \\ &= -L_{vv}L_w^2\end{aligned}$$

for a synthetic, noise-perturbed gray-value image. The derivatives have been calculated for a couple of resolutions by convolutions with the appropriate Gaussian derivative kernels. Figure 1 also shows the third-order invariant $\lambda^{[6]} = w'''L_w^6$, which apparently trades off edge strength against the directional derivative of isophote curvature in the isophote direction. One could call it "inflection strength." Its zero crossings correspond to extrema of isophote curvature. It is a relative gray-value invariant of weight 6 and also a relative Cartesian invariant; it is sensitive to the orientation of the Cartesian coordinate frame. In arbitrary Cartesian coordinate systems it is given by

$$\begin{aligned}\lambda^{[6]} &= -\varepsilon_{il}\varepsilon_{jm}\varepsilon_{kn}L_lL_mL_n \\ &\quad\times(L_{ijk}L_pL_p + 3L_{ij}L_{kp}L_p) \\ &= -3(L_x^2L_{yy} - 2L_xL_yL_{xy} + L_y^2L_{xx}) \\ &\quad\times(L_xL_yL_{yy} - L_y^2L_{xy} + L_x^2L_{xy} - L_xL_yL_{xx} \\ &\quad+(L_x^2 + L_y^2)(L_x^3L_{yyy} - 3L_x^2L_yL_{xyy} \\ &\quad+3L_xL_y^2L_{xxy} - L_y^3L_{xxx}) \\ &= -L_{vvv}L_w^5 + 3L_{vv}L_{vw}L_w^4.\end{aligned}$$

The reader is referred to [22] for an in-depth study of $\kappa^{[3]}$ and to [23] for an application of isophote and flow-line curvature to medical imaging.

4.2 Principal Isophote Curvatures in 3D

In the 3D case the isophotes are generally 2D surfaces and the index $a = 1, 2$ in Proposition 4

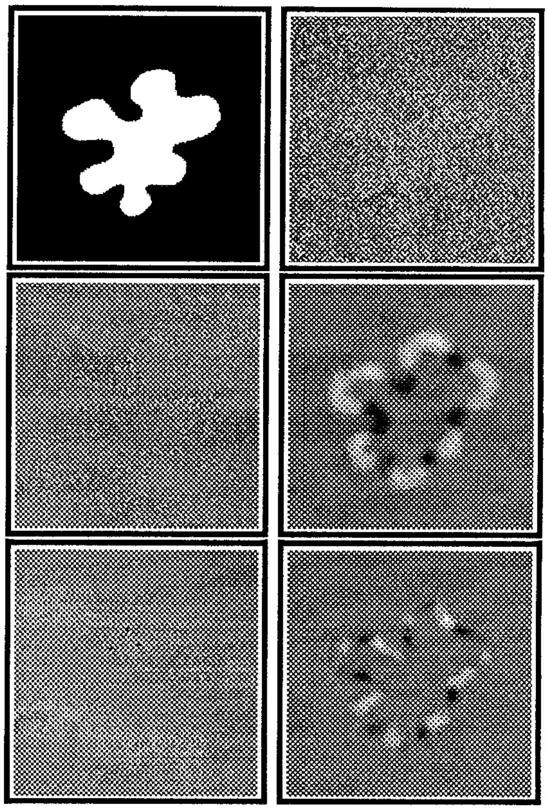

Fig. 1. Top row: an artificially created binary "amoeba" image (first image, intensity difference 100 arbitrary units) perturbed by additive pixel uncorrelated Gaussian noise with a standard deviation of 100 units (second image). Middle row: the invariant $\kappa^{[3]} = -L_{vv}L_w^2 = L_iL_{ij}L_j - L_iL_iL_{jj}$ applied to the perturbed image at two different scales, $\ln\sigma = 1/2$ and $\ln\sigma = 9/4$, respectively (σ taken relative to pixel scale). This invariant expresses a shear-invariant trade-off between edge strength (the factor L_w^3) and isophote curvature ($-L_{vv}/L_w$). In the small-scale case the invariant responds to a trade-off of small-scale gradients and isophote curvatures (as it should), which are primarily determined by the significant small-scale noise. In the large-scale case the positive and negative arcs corresponding to the curved edged of the large-scale amoeba structure are apparent. Bottom row: the invariant $-\lambda^{[6]} = L_{vvv}L_w^5 - 3L_{vv}L_{vw}L_w^4 = \varepsilon_{il}\varepsilon_{jm}\varepsilon_{kn}L_lL_mL_n(L_{ijk}L_pL_p + 3L_{ij}L_{kp}L_p)$ calculated for the amoeba image. This third-order invariant measures a trade-off between gradient magnitude (the factor L_w^6) and rate of change of isophote curvature along the isophote (the remainder) and therefore could be called "inflection strength." Note that its extrema correspond to points of inflection, and its zero crossings correspond to curvature extrema on the boundary. Scales: same as previously.

refers to a pair of Cartesian coordinates for parameterizing the isophote's local tangent plane; we will write (u, v) instead of (u_1, u_2).

In addition to the gradient gauge, which is now a truly partial gauge, we now have a nontrivial, residual $E(2)$-gauge group corresponding to Cartesian transformations of the (u, v) plane (rotations around the w axis and translations of the origin within the plane). We need to set up a Cartesian frame according to some admissible, geometry-induced full gauge. Once we have this, we may derive gray-value invariants from their generating equation simply by susbstituting the directional (u, v, w) derivatives from this gauge.

We will focus on the lowest-order isophote properties, the *principal curvatures*. In particular, we address the question of how to evaluate them in an arbitrary Cartesian coordinate system (x, y, z). There are several ways to achieve this.

One way (which always works) is to systematically carry out the inductive scheme of Example 4. According to that example one may set up a Cartesian coordinate frame at each point P of the image domain by orthogonalizing the three vectors $v_i^{(0)} = L_i$, $v_i^{(1)} = L_{ij}L_j$, and $v_i^{(2)} = L_{ij}L_{jk}L_k$, provided they are linearly independent. This is the case if P is a regular point and if the eigenvectors of the Hessian are not proportional to the gradient. However, although this brute force method will work, the reader may verify that applying the Gram–Schmidt procedure of Example 4 turns out to be rather cumbersome.

Instead, one may proceed more intelligently in this still relatively simple 3D case by using the following three vectors (ε_{ijk} is the antisymmetric Lévi–Civita pseudotensor in 3D, normalized by fixing $\varepsilon_{123} = 1$):

$$u_i^{(0)} = L_i,$$
$$u_i^{(1)} = \varepsilon_{ijk}L_jL_{kl}L_l,$$
$$u_i^{(2)} = \varepsilon_{ijk}L_j\varepsilon_{klm}L_lL_{mn}L_n.$$

This triple is already orthogonal by construction. One may then evaluate the tensors $w_{a_1\ldots a_n}$ of Proposition 6 in this particular gauge by substituting the directional derivatives as indicated in Example 4. The $w_{a_1\ldots a_n}$ resulting from this gauge form a complete set of gray-value invariants.

Both methods are based on an *explicit* construction of a Cartesian basis. For the lowest-order gray-value invariants there is an even simpler way, which does not seem to require such a construction at all. All we actually need is its *implicit* definition, i.e., an admissible, full gauge. To see how it works, recall (from elementary linear algebra) that the residual $E(2)$-transformation group in the partial gradient gauge is just what it takes to diagonalize the Hessian $w_{u_a u_b}$ (again we tacitly assume all quantities to be evaluated at the spatial location of interest). So let us set $w_{uv} = 0$ at the point of interest; this, together with the gradient gauge $w_u = w_v = 0$, establishes an admissible, full gauge, at least up to (discrete) reflections.

In this frame Proposition 4 (or Example 3) yields

$$w_{uu} = -\frac{L_{uu}}{L_w} \stackrel{\text{def}}{=} \kappa_u,$$
$$w_{vv} = -\frac{L_{vv}}{L_w} \stackrel{\text{def}}{=} \kappa_v$$

up to second order and

$$w_{uuu} = -\frac{L_{uuu}}{L_w} + 3\frac{L_{uu}}{L_w}\frac{L_{uw}}{L_w},$$
$$w_{uuv} = -\frac{L_{uuv}}{L_w} + \frac{L_{uu}}{L_w}\frac{L_{vw}}{L_w},$$
$$w_{uvv} = -\frac{L_{uvv}}{L_w} + \frac{L_{vv}}{L_w}\frac{L_{uw}}{L_w},$$
$$w_{vvv} = -\frac{L_{vvv}}{L_w} + 3\frac{L_{vv}}{L_w}\frac{L_{vw}}{L_w}.$$

up to third order.

Although we do not have the *principal directions* as such, we do know (by our implicit gauge condition $w_{uv} = 0$) that u and v are the Cartesian coordinates corresponding to these. We may replace the principal curvatures w_{uu} and w_{vv} by the (u, v)-symmetric combinations M and G, given by

$$M = \frac{1}{2}(w_{uu} + W_{vv}) = -\frac{1}{2}\frac{L_{uu} + L_{vv}}{L_w},$$
$$G = w_{uu}w_{vv} = \frac{L_{uu}L_{vv}}{L_w^2},$$

i.e., the *mean curvature* and the *gaussian curvature*, respectively [24]–[26]. Then all it takes to find their expression in arbitrary Cartesian coordinate systems is to write down a manifest Cartesian invariant that reduces to the preceding expressions for M and G when subjected to the gauge $w_u = w_v = w_{uv} = 0$. One may find these in one step (consider only the few simplest possibilities that are consistent with the order and zero homogeneity of M and G):

$$M = \frac{1}{2}\frac{L_i L_{ij} L_j - L_i L_i L_{jj}}{(L_k L_k)^{3/2}},$$
$$G = \frac{1}{2}\frac{\varepsilon_{ijk}\varepsilon_{lmn} L_i L_l L_{jm} L_{kn}}{(L_p L_p)^2}.$$

To see that this is correct, insert the (u, v, w) gauge (which implies $L_u = L_v = L_{uv} = 0$) in the right-hand side and observe that the expressions simplify to the previous definitions of M and G. By manifest invariance, this identity then holds in *any* other Cartesian coordinate system! The method used here demonstrates the power of tensor calculus. Choosing an explicit (x, y, z) coordinate system, we obtain

$$M = \tfrac{1}{2}(2L_x L_y L_{xy} - L_x^2 L_{yy} - L_y^2 L_{xx} + \text{cycl.}(x, y, z))/(L_x^2 + L_y^2 + L_z^2)^{3/2},$$
$$G = (-L_x^2 L_{yz}^2 + L_x^2 L_{yy} L_{zz} + 2L_x L_y L_{xz} L_{yz} - 2L_x L_z L_{xz} L_{yy} + \text{cycl.}(x, y, z))/(L_x^2 + L_y^2 + L_z^2)^2,$$

in which cycl.(x, y, z) stands for all terms obtained from the previous ones by the two cyclic permutations of the coordinates $(x, y, z) \to (y, z, x) \to (z, x, y)$.

As before, we may give up strict gray-value invariance and consider relative gray-value invariants according to Proposition 7. So let us define the Cartesian invariants $M^{[p]} = ML_w^p$ and $G^{[p]} = GL_w^p$, with $p > 0$.

Figure 2 shows a visualization of a 3D image of a torus. Figures 3 and 4 show the invariants $M^{[3]}$ and $G^{[4]}$ calculated for the torus image. They are presented on a slice-by-slice basis, where the torus is sliced along a plane parallel to its symmetry axis.

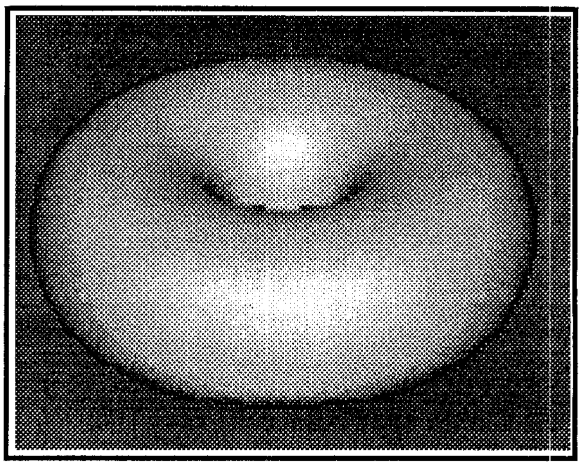

Fig. 2. Visualization of a 3-D, 128 × 128 × 64-voxel image of a torus. The torus is defined by the parametric equation $(x, y, z) = ((R+r\sin\phi)\cos\theta, (R+r\sin\phi)\sin\theta, r\cos\phi)$, with $(\theta, \phi) \in [0, 2\pi)$ and $R > r$. The two defining radii r and R are 18 and 32 units relative to voxel size, respectively. The interior and exterior of the torus have uniform gray-values L_i and L_e, respectively, with $L_i - L_e = \Delta L > 0$.

5 Conclusion and Discussion

We have considered the group of gray-value transformations and have proposed a generating equation for a complete set of local gray-value invariants up to any order for gray-value images of arbitrary spatial dimensions d. The method is based on the gray-value-invariant geometrical structure of the typographical isophote picture. Within the partial gradient gauge, the solution comprises a set of gray-value-invariant tensor fields transforming under the Euclidean group $E(d-1)$. These tensors parameterize the isophote's local-jet bundle. Various complete families of gray-value invariants can be derived from these tensors by imposing geometrically meaningful gauge conditions, in addition to the gradient gauge, for removing the residual $E(d-1)$-gauge degree of freedom.

We have illustrated the general scheme by examples in 2D and in 3D. These examples are simple enough to serve as a clear geometrical illustration of the general principle. Higher-order invariants are increasingly difficult to interpret geometrically. Nevertheless, completeness of the scheme guarantees that one does not end

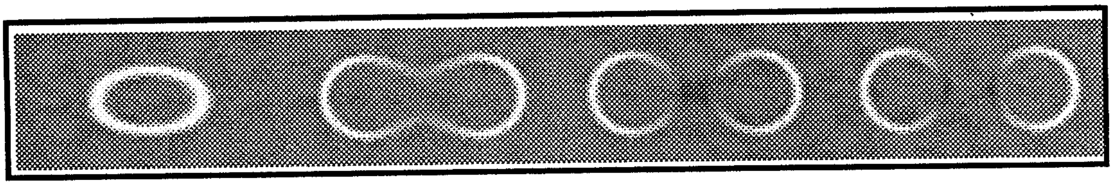

Fig. 3. Four typical slices showing the relative gray-value invariant $M^{[3]}$ for the torus image. Background gray corresponds to $M^{[3]} = 0$. The major part of the torus in this example has positive mean curvature. In this example there is a slightly negative mean curvature at the inside (because $r < R < 2\pi$), reaching an absolute minimum on the circle $(x, y, z) = ((R-r)\cos\theta, (R-r)\sin\theta, 0)(0 \le \theta < 2\pi)$; in this example $M_{\min} = (R-2r)/(2r(R-r)) = -1/126$ in inverse voxel units. The maximum value is reached at the outside, i.e., on the circle $(x, y, z) = ((R+r)\cos\theta, (R+r)\sin\theta, 0)(0 \le \theta < 2\pi)$, viz., $M_{\max} = (R+2r)/(2r(R+r)) = 17/450$ inverse voxel units. Scale: $\sigma = 2.25$ voxel units.

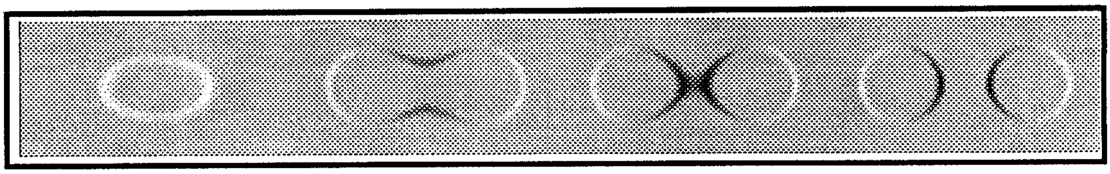

Fig. 4. Four typical slices showing the relative gray-value invariant $G^{[4]}$ for the torus image. Background gray corresponds to $G^{[4]} = 0$. The image clearly shows the elliptic points on the outside of the torus $(0 < \phi < \pi)$, the hyperbolic points on the inside $(\pi < \phi < 2\pi)$, and the parabolic points separating the elliptic and hyperbolic parts of the surface $(\phi = 0, \pi)$. Scale: $\sigma = 2.25$ voxel units.

up with a mere ad hoc set of gray-value invariants; it gives a *complete* account of the image's gray-value-invariant structure.

By duality, local isophote properties arising from our generating scheme are compounded with properties of the gradient-integral curves, or flow lines. We have given a simple example of this in the case. Despite the dual relationship, which has been our motivation for disregarding the flow line picture, it may still be interesting to derive the explicit relationship between these dual sets. We have not addressed this issue here.

An important problem that remains to be solved is the effect of gray-value transformations on *scale-space*; although a gray-value transformation preserves the isophote picture on each fixed level of scale, it does have a nontrivial effect on the isophote *hierarchy* across scales, i.e., the structure of the iso-intensity surfaces in the $(d+1)$-dimensional scale-space domain.

Finally, note that the general group of gray-value transformations has nontrivial subgroups that may be of particular interest to certain applications. For these subgroups, the sets of invariants may be larger than those for the general group.

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We are indebted to J. Blom, R. van Maarseveen, and M. van Eert for their contributions to the software development and A. Salden for his in-depth study of isophote surfaces in 3D. We also thank A. van Doorn for useful comments and suggestions.

Notes

1. When reflections are included, we will allow for so-called pseudoinvariants, i.e., quantities invariant up to a minus sign, which show up under a mirror transformation. The term *invariant* will then be used to denote both absolute invariants and pseudoinvariants.
2. Note that for the λ-parameterized groups it makes sense to exclude the exceptionally transforming zeroth-order component from the definition of a local jet, whereas for the pure scaling and trivial subgroups, $\Gamma(L; 0, \mu) = e^{\mu}L$ and $\Gamma(L; 0, 0) = L$, there is no reason to do so.

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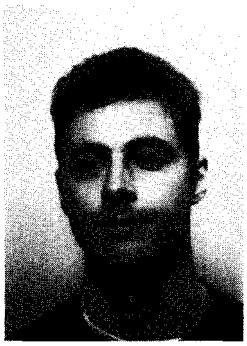

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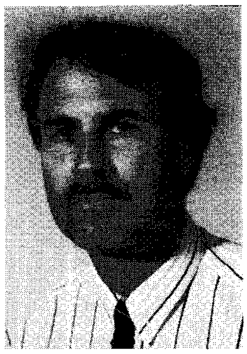

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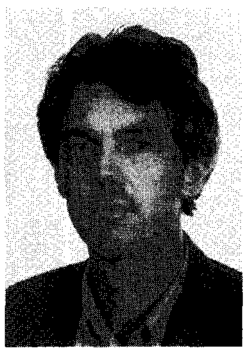

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