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On the classification of real simple Lie groups

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CONTENTS

INTRODUCTION

After Killing [1] [1] and Cartan [1] have given a classification of all simple complex Lie groups, the determination of all (essentially anisomorphic) simple real groups may be reduced to the problem of finding different real forms of

[1] The numbers in square brackets refer to the bibliography at the end of this paper (p. 248).

a given complex simple structure. This problem was solved by Cartan in 1914 in his great memoir [2]. In it Cartan does not give a general way to deal with the problem, but considers separate simple complex structures, and operating in each case with specific devices finds all different real forms. In 1929 Cartan [2] established a beautiful theorem giving a general method for the solution of the problem. But although the theorem itself is of a purely algebraical character, Cartan's proof of it is based upon the theory of symmetric Riemann spaces developed by him. In the same memoir Cartan points out how the canonical representation of inner automorphisms in a simple compact Lie group may be used for the application of his theorem to the problem of finding simple real groups.

The absence of a canonical representation of outer automorphisms makes it however impossible for Cartan to apply his second method to some simple complex structures, for instance to the E_6. Lardy [1] filled this gap in 1935—1936, but in a rather round about and complicated way.

The Chapter I of the present paper contains an algebraical proof of Cartan's theorem. We are using here the canonical representation of inner automorphisms with simple elementary divisors in a complex Lie group, established in the preceding paper [1] of the author. Accidentally we find a new proof for the remarkable connection between the complex an the compact semi-simple Lie group (Cartan found this connection starting from his theory of symmetric Riemann spaces). Using further the canonical representation of outer automorphisms [3] we find a similar representation in the compact semi-simple Lie group.

All this gives us the possibility to obtain, in Chapters II and III, with the help of Cartan's theorem, all simple real groups with simple complex structure in a direct and comparatively short way.

For denotations and fundamental conceptions used in the present paper we refer to our preceding paper [4].

CHAPTER I

REAL FORMS OF A COMPLEX SEMI-SIMPLE LIE GROUP

§ 1. The problem

Consider a real infinitesimal Lie group $\mathfrak{R}_1$ of r dimensions. If

$$e_1, e_2, \ldots, e_r \tag{1}$$

is a basis of this group, any infinitesimal element t can be represented in the form

$$t = \sum_{i=1}^{r} \tau_i e_i, \tag{2}$$

where the parameters τ_i may assume arbitrary real values. The operation of commutation, applied to the basis elements, gives

$$[e_i, e_k] = \sum_{s=1}^{r} c_{ik}^{s} e_s \qquad (i, k = 1, \ldots, r). \tag{3}$$

[2] Cartan, [6], p. 27.
[3] Gantmacher, [1], p. 138—143.
[4] Ibid., Introduction.

The c_{ik}^{s} are the structure constants of the given group in the basis (1); for a real group the c_{ik}^{s} are real.

Suppose now that the parameters τ_i in (2) assume all possible complex values. Then the infinitesimal elements t will form a complex Lie group $\mathfrak{R}$ of r complex dimensions. We shall say that the group $\mathfrak{R}$ is obtained from the group $\mathfrak{R}_1$ by the process of „complexing", and shall call the group $\mathfrak{R}_1$ the real form of the group $\mathfrak{R}$.

One and the same complex group $\mathfrak{R}$ may have several different real forms. In fact, there may exist several such bases that the transformation of one of them into another is realized by a non-real linear transformation, while the structure constants of each basis are real. To such bases there will correspond different real forms of the complex group $\mathfrak{R}$.

But beside the process of finding real forms of a given complex Lie group [process A)] there exists another method [process B)] of obtaining real groups from complex ones. If we consider the real and the imaginary parts of the parameters τ_i as real coordinates of a certain vector $\hat{t}$ in the space of $2r$ dimensions and automatically extend to this space the operation of commutation, we obtain a real Lie group $\mathfrak{R}^1$ of $2r$ dimensions. The group $\mathfrak{R}^1$ is uniquely determined by the complex group $\mathfrak{R}$.

Theorem 1. *Applying to all possible complex simple Lie groups the processes* A) *and* B) *of forming real groups we obtain all real simple Lie groups* [1].

Proof. In order to prove our assertion consider an arbitrary real simple Lie group $\mathfrak{R}_1$ of r dimensions. After the process of „complexing" we obtain a complex group $\mathfrak{R}$, which may be non-simple. Accordingly we distinguish two cases:

1) $\mathfrak{R}$ is a simple group. In this case the original simple real group $\mathfrak{R}_1$ is a real form of the complex simple group $\mathfrak{R}$, or, as we shall say, the simple real group $\mathfrak{R}_1$ has a simple complex structure.

2) $\mathfrak{R}$ is non-simple. But $\mathfrak{R}$ is at any rate a semi-simple group, since the semi-simplicity, being an implication of the fact that the quadratic form φ is not degenerated [2], is not affected by the process of „complexing". In the case under consideration $\mathfrak{R}$ is a direct sum of two complex conjugated invariant subgroups $\mathfrak{F}$ and $\bar{\mathfrak{F}}$, each of which has $\frac{r}{2}$ complex dimensions. It is easily verified that the process B), applied to any of the groups $\mathfrak{F}$ and $\bar{\mathfrak{F}}$, yields a group $\mathfrak{R}_2$ isomorphic to the original real simple group $\mathfrak{R}_1$.

Indeed, we can choose a basis for $\mathfrak{F}$ and $\bar{\mathfrak{F}}$ respectively in the form

$$e_p = e_p' + ie_p'', \quad \bar{e}_p = e_p' - ie_p'' \qquad (p = 1, 2, \ldots, \tfrac{r}{2}), \tag{4}$$

where

$$e_1', \ldots, e_{\frac{r}{2}}', \; e_1'', \ldots, e_{\frac{r}{2}}''$$

[1] We call an infinitesimal real group simple, if it has no real invariant subgroups different from zero and from itself.

[2] Cartan, [1], p. 51.

may be taken to be a basis of the original real group $\mathfrak{R}_1$. The structure formulae for $\mathfrak{F}$ and $\overline{\mathfrak{F}}$ will be as follows:

$$[e_p, e_q] = \sum_1^{\frac{r}{2}} c^s_{pq} e_s, \quad [\bar{e}_p, \bar{e}_q] = \sum_1^{\frac{r}{2}} \bar{c}^s_{pq} \bar{e}_s \qquad \left(p, q = 1, \ldots, \frac{r}{2}\right). \tag{5}$$

Moreover, since $\mathfrak{F}$ and $\overline{\mathfrak{F}}$ are invariant subgroups in $\mathfrak{R}$,

$$[e_p, \bar{e}_q] = 0 \qquad \left(p, q = 1, \ldots, \frac{r}{2}\right). \tag{6}$$

From (5) and (6) we find the structure formulae for the original group (we put $c^s_{pq} = c^{s'}_{pq} + i c^{s''}_{pq}$, $c^{s'}_{pq}$ and $c^{s''}_{pq}$ are real):

$$\left.\begin{aligned} [e'_p, e'_q] &= \frac{1}{2} \sum_1^{\frac{r}{2}} c^{s'}_{pq} e'_s - \frac{1}{2} \sum_1^{\frac{r}{2}} c^{s''}_{pq} e''_s, \\ [e'''_p, e''_q] &= -\frac{1}{2} \sum_1^{\frac{r}{2}} c^{s'}_{pq} e'_s + \frac{1}{2} \sum_1^{\frac{r}{2}} c^{s''}_{pq} e''_s, \\ [e'_p, e''_q] &= \frac{1}{2} \sum_1^{\frac{r}{2}} c^{s''}_{pq} e'_s + \frac{1}{2} \sum_1^{\frac{r}{2}} c^{s'}_{pq} e''_s \\ &\left(p, q = 1, \ldots, \frac{r}{2}\right). \end{aligned}\right\} \tag{7}$$

On the other hand, if we apply the process B) to $\mathfrak{F}$, we find a real simple group $\mathfrak{R}_2$ with the basis

$$e_1, \ldots, e_{\frac{r}{2}}, \; ie_1, \ldots, ie_{\frac{r}{2}}.$$

If we introduce in the group $\mathfrak{R}_2$ the new basis

$$k'_p = \frac{1}{2} e_p, \quad k''_p = -\frac{1}{2} ie_p \qquad \left(p = 1, \ldots, \frac{r}{2}\right),$$

then, using formulae (5), it is easily verified that the structure constants of the group $\mathfrak{R}_2$ in this basis coincide with the structure constants of the group $\mathfrak{R}_1$ in the basis e'_p, e''_p. The groups $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are thus isomorphic.

Observe that the process B) correlates to every complex simple group $\mathfrak{F}$ of r dimensions a uniquely determined real simple group $\mathfrak{R}_1$ of $2r$ real dimensions having a non-simple (semi-simple) complex structure. The structure constants of the group $\mathfrak{R}_1$ are found in a simple way from the structure constants of $\mathfrak{F}$.

Thus the whole problem is reduced to the process A), namely to the determination of different (i. e. in the real domain anisomorphic) simple groups having a given simple complex structure.

§ 2. The real forms of a complex semi-simple Lie group

Every real semi-simple group gives after the process of „complexing" again a semi-simple group. On the other hand, every complex semi-simple group can be obtained, after Cartan-Weyl, by the process of „complexing" from a compact real group, for which the form φ is negative definite.

Hence all real semi-simple groups having a given complex structure may be obtained in the following way:

1°. We start from a real compact semi-simple infinitesimal group $\mathfrak{R}_0$, in which, for an appropriate choice of the real basis

$$e_1, e_2, \ldots, e_n$$

(which is supposed to be fixed throughout the paper) and any element $t = \sum \tau_i e_i$ [1]:

$$-\varphi(t, t) = \sum \tau_i^2. \tag{8}$$

Applying the process of „complexing" to the group $\mathfrak{R}_0$, we obtain the group $\mathfrak{R}$.

2°. We look now for all linear transformations P in $\mathfrak{R}$, which transform our basis e_i into a new basis $g_i = Pe_i$, in which the structure constants c^s_{ik} of the „complexed" group,

$$[g_i, g_k] = \sum c^s_{ik} g_s, \tag{9}$$

will be real. To each such basis g_i there corresponds a real semi-simple infinitesimal group. In this way we obtain up to an isomorphism all real forms of the complex group $\mathfrak{R}$.

3°. Among the so obtained real groups we choose a complete system of anisomorphic groups.

Every linear transformation P, which transforms the basis e_i into the basis g_i, is determined by a matrix (p_{ik}) such that

$$Pe_i = \sum p_{ki} e_k. \tag{10}$$

Two questions arise in connection with what has been said above:

1. Which are the linear transformations P realizing the transition to real groups (see 2°)?

2. In which case do two linear transformations P and P_1 lead to two isomorphic real groups (see 3°)?

Let us answer the first question. Consider the complex conjugated matrices (p_{ik}) and $(\bar{p}_{ik})$. The transformations P and $\bar{P}$ [2] defined by them transform the basis e_i into respectively the bases g_i and h_i, and the structure constants c^s_{ik} and d^s_{ik} in these bases will be conjugated:

$$d^s_{ik} = \bar{c}^s_{ik}. \tag{11}$$

If P realizes the transition to a real group, the c^s_{ik} are real and $d^s_{ik} = c^s_{ik}$. Then, since P^{-1} transforms the basis g_i into the basis e_i, the product $\bar{P}P^{-1}$ transforms the basis g_i into h_i and hence preserves the structure constants c^s_{ik}. But a linear transformation A transforming a basis into a basis and preserving the structure constants is an automorphism of the given group. Thus

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[1] All variable indices occuring in this section run from 1 to r, where r is the dimensionality of the group.

[2] According to this denotation we shall call two linear transformations (complexly) conjugated, if in the basis e_i they are characterized by (complexly) conjugated matrices.

Theorem 2. *A linear transformation P transforms a real compact group $\mathfrak{R}_0$ into a real group then and only then, when*

$$\bar{P}P^{-1}=A, \tag{12}$$

where A is an automorphism of the complex infinitesimal group $\mathfrak{R}$.

This is the answer to the first question.

Let now P and P_1 transform the basis e_i into respectively

$$g_i=Pe_i, \quad k_i=P_1e_i, \tag{13}$$

and suppose that to these bases correspond isomorphic real groups with the structure constants

$$c_{ik}^s \text{ and } d_{ik}^s. \tag{14}$$

This means that there exists a basis l_i, connected with k_i by real relations

$$l_i=\sum r_{ji}k_j \quad (r_{ji} \text{ — real numbers}), \tag{15}$$

for which the structure constants are the same c_{ik}^s as for g_i. Then $l_i=P_1\left(\sum r_{ji}e_j\right)$. Let us now define a real transformation R ($R=\bar{R}$, see footnote [2]) by the equations

$$Re_i=\sum r_{ji}e_j \; ^3. \tag{16}$$

Then

$$l_i=P_1Re_i. \tag{17}$$

Since P^{-1} transforms the basis g_i into e_i, the transformation P_1RP^{-1} transforms the basis g_i into l_i and preserves the structure constants c_{ik}^s. Consequently

$$P_1RP^{-1}=A_1 \quad (A_1 \text{ is an automorphism}), \tag{18}$$

or

$$P=AP_1R, \text{ where } A=A_1^{-1}. \tag{19}$$

Hence

Theorem 3. *Two linear transformations P and P_1 satisfying each the relation* (12) *transform the compact group $\mathfrak{R}_0$ into two isomorphic real groups then and only then, when*

$$P=AP_1R, \tag{20}$$

where R is an arbitrary real transformation and A is an arbitrary automorphism of the „complexed" semi-simple Lie group $\mathfrak{R}$.

§ 3. Cartan's theorem

In what follows we shall denote by the same letter the linear transformation and the matrix characterizing it in the original basis e_i.

Every automorphism A leaves the form φ invariant:

$$\varphi(At, At)=\varphi(t, t). \tag{21}$$

[3] We may point out that from the equation (15) it does not follow that $l_i=Rk_i$, since the matrix (r_{ij}) corresponds to the linear transformation R only in the basis e_i.

But in our case $\varphi = -\sum \tau_i^2$. Consequently A is an orthogonal (in general complex) matrix.

Consider those A, for which the equation

$$\bar{P}P^{-1} = A \tag{22}$$

has solutions P. Evidently they satisfy the condition

$$A\bar{A} = E, \tag{23}$$

where E is the unit matrix, i. e.

$$\bar{A} = A^{-1}. \tag{24}$$

On the other hand, A is a complex orthogonal matrix. Consequently

$$A' = A^{-1}\ ^{1}. \tag{25}$$

Hence

$$A' = \bar{A} = A^{-1}, \tag{26}$$

or A is simultaneously Hermitian and orthogonal.

In connection with this fact we have to analyse the structure of matrices, which are simultaneously Hermitian and orthogonal.

T h e o r e m 4. *If a matrix A is simultaneously Hermitian and orthogonal, then*

$$A = Se^{i\Phi}, \tag{27}$$

where S is a real symmetrical orthogonal matrix ($S^2 = E$), Φ is a real skew-symmetrical matrix and S and Φ are commutable:

$$S\Phi = \Phi S. \tag{28}$$

P r o o f. Put $A = F + iK$, F and K being real. Since A is Hermitian, $\bar{A} = A'$, and consequently

$$F = F', \ K = -K', \tag{29}$$

i. e. F is symmetrical and K is skew-symmetrical. On the other hand, since

$$A\bar{A} = E,$$

we have

$$F^2 + K^2 + i(KF - FK) = E,$$

whence

$$F^2 + K^2 = E, \quad FK = KF. \tag{30}$$

The symmetrical matrix F and the skew-symmetrical K are thus commutable). Therefore we can reduce them to the canonical form by means of one and the same real orthogonal transformation. To this end we first reduce F by a real orthogonal transformation Q to the diagonal form. Then

$$F^* = QFQ^{-1} = \{f_1^* E_1, f_2^* E_2, \ldots, f_\mu^* E_\mu\}, \tag{31}$$

where $f_\gamma^* \neq f_\delta^*$ ($\gamma \neq \delta$, $\gamma, \delta = 1, 2, \ldots, \mu$) and $E_1, E_2, \ldots, E_\mu$ are unit matrices). Observe that $K^* = QKQ^{-1}$ is again a real skew-symmetrical matrix.

[1] By A' we denote the transposed matrix of A.

The matrices F^* and K^* are commutable. Therefore K^* consists of μ „cells“ situated along the main diagonal (all other elements of K^* are equal to zero):

$$K^* = \{K_1, K_2, \ldots, K_\mu\}. \tag{32}$$

The skew-symmetrical matrix K_γ may be now reduced to the canonical form $\widehat{K}_\gamma$ by a real orthogonal transformation O_γ. The corresponding transformations of the matrices $f^*_\gamma E_\gamma$ by O_γ do not change the matrix F^*. Thus we can reduce, by a simultaneous orthogonal transformation O, the matrices F and K to the following canonical form:

$$\left.\begin{aligned} \widehat{F} = F^* = \{\widehat{f}_1, \widehat{f}_2, \ldots, \widehat{f}_n\}, \\ \widehat{K} = \left\{\begin{pmatrix} 0 & -k_1 \\ k_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -k_\nu \\ k_\nu & 0 \end{pmatrix}, 0, 0, \ldots, 0\right\}. \end{aligned}\right\} \tag{33}$$

Since the matrices $\widehat{F}$ and $\widehat{K}$ are commutable,

$$\widehat{f}_1 = \widehat{f}_2 = f_1, \ldots, \widehat{f}_{2\nu-1} = \widehat{f}_{2\nu} = f_\nu. \tag{34}$$

On the other hand,

$$\widehat{F}^2 + \widehat{K}^2 = E, \tag{35}$$

whence

$$f_1^2 - k_1^2 = \ldots = f_\nu^2 - k_\nu^2 = 1, \quad f^2_{2\nu+1} = \ldots = 1. \tag{36}$$

Therefore

$$\widehat{A} = \widehat{F} + i\widehat{K} = \left\{\begin{pmatrix} f_1 & -ik_1 \\ ik_1 & f_1 \end{pmatrix}, \ldots, \begin{pmatrix} f_\nu & -ik_\nu \\ ik_\nu & f_\nu \end{pmatrix}, \pm 1, \ldots, \pm 1 \right\}. \tag{37}$$

But it is easily verified that a matrix of the type $\begin{pmatrix} f & -ik \\ ik & f \end{pmatrix}$, where $f^2 - k^2 = 1$, may be represented in the form:

$$\begin{pmatrix} f & -ik \\ ik & f \end{pmatrix} = \pm e^{i\begin{pmatrix} 0 & -\varphi \\ \varphi & 0 \end{pmatrix}}. \tag{38}$$

Here $|f| = \operatorname{ch}\varphi$, $\pm k = \operatorname{sh}\varphi$, and the signs $\pm$ correspond to the signs in

$$f = \pm |f|. \tag{39}$$

Thus

$$\widehat{A} = \widehat{F} + i\widehat{K} =$$

$$= \left\{ \pm e^{i\begin{pmatrix} 0 & -\varphi_1 \\ \varphi_1 & 0 \end{pmatrix}}, \ldots, \pm e^{i\begin{pmatrix} 0 & -\varphi_\nu \\ \varphi_\nu & 0 \end{pmatrix}}, \pm 1, \ldots, \pm 1 \right\} = \widehat{S} e^{i\widehat{\Phi}}, \tag{40}$$

where

$$\widehat{S} = \{\pm 1, \ldots\}, \; \widehat{\Phi} = \left\{\begin{pmatrix} 0 & -\varphi_1 \\ \varphi_1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -\varphi_\nu \\ \varphi_\nu & 0 \end{pmatrix}, 0, \ldots, 0\right\}, \tag{41}$$

and

$$\widehat{S}\widehat{\Phi} = \widehat{\Phi}\widehat{S}. \tag{42}$$

Observing that

$$A = O^{-1}\widehat{A}O \tag{43}$$

and putting

$$S = O^{-1}\hat{S}O, \quad \Phi = O^{-1}\hat{\Phi}O, \tag{44}$$

we come to the equation

$$A = Se^{i\Phi}, \tag{45}$$

where S and Φ possess the properties stated in the theorem.

Let us now return to the equation

$$\overline{P}\overline{P}^{1} = A. \tag{46}$$

Here the automorphism A has the form $A = Se^{i\Phi}$, where S is a real symmetrical orthogonal matrix, Φ a real skew-symmetrical matrix and

$$S\Phi = \Phi S.$$

Then it is easily verified that the formula

$$P \doteq e^{-i\frac{\Phi}{2}}\sqrt{S}R, \tag{47}$$

where R is an arbitrary real matrix and

$$\sqrt{S} = \frac{1-i}{2} S + \frac{1+i}{2} E, \tag{48}$$

gives all solutions of the equation

$$\overline{P}P^{-1} = Se^{i\Phi}. \tag{49}$$

Let us now prove that Φ is an infinitesimal automorphism. The group $\mathfrak{A}$ of all automorphisms of a given semi-simple Lie group consists of a finite number of components: $\mathfrak{A} = \mathfrak{A}_0 + \mathfrak{A}_1 + \ldots + \mathfrak{A}_{k-1}$ ($\mathfrak{A}_0$ is the adjoint group, i. e. the aggregate of all inner automorphisms) [2]. In other words, the factor-group $\mathfrak{A}/\mathfrak{A}_0$ is finite. Therefore there exists such an even number 2ν that

$$A^{2\nu} = e^{2\nu i\Phi} \subset \mathfrak{A}_0.$$

But Φ is a skew-symmetrical real matrix; it has simple elementary divisors. Consequently the inner automorphism $e^{2\nu i\Phi}$ has also simple elementary divisors. In virtue of the canonical representation of inner automorphisms with simple elementary divisors [3] there exists such an inner automorphism U that

$$e^{2\nu i\Phi} = U^{-1}e^{H}U. \tag{50}$$

Here e^{H} is a chief automorphism with respect to a certain maximal Abelian subgroup $\mathfrak{h}$ in $\mathfrak{R}$, containing a regular element, $Hx = [hx]$, $h \subset \mathfrak{h}$. We may assume that this subgroup $\mathfrak{h}$ is obtained by the process of „complexing" from the corresponding subgroup $\mathfrak{h}$ of the original compact form. Then

$$H = H_1 + iH_2, \quad H_1H_2 = H_2H_1, \tag{51}$$

where H_1 and H_2 are two real infinitesimal automorphisms of the given semi-simple Lie group (i. e. H_1 and H_2 are infinitesimal automorphisms of the compact group $\mathfrak{R}_0$).

From (50) and (51) follows

$$e^{2\nu i\Phi} = B_1B_2,$$

[2] Cartan, [4].

[3] Gantmacher, [1], p. 117.

where $B_1 = U^{-1}e^{H_1}U$, $B_2 = U^{-1}e^{iH_2}U$. Since H_1 and H_2 are infinitesimal automorphisms, they are skew-symmetrical [due to the special choice of the basis; see (8)] and consequently have imaginary characteristic roots and simple elementary divisors. Hence B_1 and B_2 have also simple elementary divisors and the modulus of all characteristic numbers of B_1 is equal to one, while all characteristic numbers of B_2 are positive. In virtue of commutability of B_1 and B_2 the characteristic numbers of their product, i. e. of $e^{2\nu i\Phi}$, are the products of the corresponding characteristic numbers of the factors. But the characteristic numbers of $e^{2\nu i\Phi}$ are positive. Hence

$$B_1 = E, \tag{52}$$

and therefore

$$e^{2\nu i\Phi} = B_2 = e^{U^{-1}iH_2U}. \tag{53}$$

But all characteristic numbers of the matrix $2\nu i\Phi$, as well as of the matrix $U^{-1}iH_2U$, are real. Hence from (53) it follows that

$$2\nu i\Phi = U^{-1}iH_2U, \tag{54}$$

i. e. that Φ is an infinitesimal automorphism. Thus we have proved the following

Theorem 5. *In order that the equation*

$$\overline{P}P^{-1} = A, \tag{55}$$

where A is a given automorphism of the complex group $\mathfrak{R}$ and P the required linear transformation in $\mathfrak{R}$ (the complex conjugate is taken with respect to the compact form), should have solutions, it is necessary and sufficient that A should have the form

$$A = Se^{i\Phi}, \tag{56}$$

where S is an involutive automorphism ($S^2 = E$) of the compact group $\mathfrak{R}_0$, Φ an infinitesimal automorphism of the compact group $\mathfrak{R}_0$ and S and Φ are commutable.

If this condition is satisfied, all solutions of (55) *are given by the formula*

$$P = e^{-i\frac{\Phi}{2}}\sqrt{S}\,R, \tag{57}$$

where $\sqrt{S} = \frac{1-i}{2}S + \frac{1+i}{2}E$ and R is an arbitrary real linear transformation in $\mathfrak{R}$ ($\overline{R} = R$).

Let now P realize the transition from the original compact group $\mathfrak{R}_0$ to a certain real group. Then, by Theorem 2, P satisfies the equation (55). For A and P we have thus the expressions (56) and (57).

Since $e^{-i\frac{\Phi}{2}}$ is an automorphism, from (57) it follows (see Theorem 3) that the transformations P and $\sqrt{S}$ realize transitions to isomorphic real groups. We may thus confine ourselves to transitions realizable by transformations of the type $\sqrt{S}$.

So we have proved the following fundamental theorem of Cartan[4]:

Theorem 6 (Cartan). *All different real forms of a given complex semi-simple group may be obtained in the following way:*

First we find all involutive automorphisms of a compact form, i. e. automorphisms S, for which

$$S^2 = E. \tag{58}$$

Then we take the basis composed of the „Eigen"-vectors of the matrix S, multiply those vectors of this basis which correspond to the characteristic number — 1 *by i and leave the remaining vectors of the basis unchanged. To the so obtained basis there corresponds a real form of the given complex semi-simple Lie group.*

§ 4. The connection between a complex semi-simple Lie group and its compact form

Consider an arbitrary automorphism A of a complex semi-simple Lie group $\mathfrak{R}$. Since A and $\overline{A}$ are automorphisms,

$$\overline{A}A^{-1} = Q$$

is also an automorphism. By Theorem 5,

$$Q = Se^{i\Phi}, \tag{59}$$

$$A = e^{-i\frac{\Phi}{2}}\sqrt{S}R, \tag{60}$$

where $S = \overline{S} = S^{-1}$ is an automorphism, $\Phi = \overline{\Phi}$ is an infinitesimal automorphism and

$$S\Phi = \Phi S, \quad R = \overline{R}.$$

Since A and $e^{-i\frac{\Phi}{2}}$ are automorphisms, we conclude from (60) that

$$C = \sqrt{S}R \tag{61}$$

is an automorphism.

Then from the equation

$$E = C^{-1}\sqrt{S}R$$

we conclude, by Theorem 3, that $P = \sqrt{S}$ realizes the transition from the original compact group to a group isomorphic to it. But then the form φ must be negative definite also for this new group. Hence all characteristic numbers of $\sqrt{S}$ must be equal to one, i. e.

$$\sqrt{S} = E. \tag{62}$$

From (61) and (62) it then follows that

$$R = C,$$

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[4] Cartan, [6], p. 27.

i. e. that R is an automorphism of the compact group. From (60) and (62) follows

$$A = e^{-i\frac{\Phi}{2}} R \tag{63}$$

and

$$A^{-1} = R^{-1} e^{i\frac{\Phi}{2}}. \tag{64}$$

Replacing in (63) $-\frac{\Phi}{2}$ by Φ and in (64) A^{-1} by A and $\frac{\Phi}{2}$ by Φ, we come to the result that any automorphism A of a complex semi-simple Lie group may be represented in the form

$$A = e^{i\Phi} R, \tag{65}$$

as well as in the form

$$A = R e^{i\Phi}, \tag{66}$$

where R is a finite and Φ an infinitesimal automorphism of a compact Lie group. This is the remarkable result of Cartan [1], establishing a close connection between the topological structures of the groups of automorphisms of the complex and the compact semi-simple Lie groups.

§ 5. The canonical representation of automorphisms of a compact semi-simple Lie group

Let us prove the following

Theorem 7. *If two automorphisms A and B of a compact semi-simple group are conjugated with respect to the „complexed" adjoint group, they are conjugated also with respect to the compact adjoint group.*

Proof. Let

$$B = T^{-1} A T, \tag{67}$$

where T is an inner automorphism of the complex group. Then

$$B = \bar{T}^{-1} A \bar{T}. \tag{68}$$

Finding A from (69) and substituting in the so obtained equation for B its expression from (68), we find

$$A = \bar{T} T^{-1} A T \bar{T}^{-1}, \tag{69}$$

i. e. $\bar{T} T^{-1}$ and A are commutable. As it follows from (65)

$$T = e^{i\Phi} R, \tag{70}$$

where R is an inner (finite) automorphism and Φ an infinitesimal automorphism of the compact group.

Observe that since all characteristic numbers of $2i\Phi$ are real and $\bar{T}T^{-1} = e^{-2i\Phi}$, Φ is a function of $\bar{T}T^{-1}$. Consequently Φ is commutable with A.

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[1] Cartan, [5], p.p. 250—251. From this result of Cartan it follows that $\mathfrak{A}$ and $\mathfrak{A}_{\text{compact}}$ have one and the same fundamental group.

Therefore, substituting in (68) instead of T the product $e^{i\Phi}R$, we obtain

$$B = R^{-1}AR, \tag{71}$$

q. e. d.

Now observe that from the compactness of the group of automorphisms it follows that the modulus of all characteristic numbers of every automorphism of a compact group is equal to 1, and all elementary divisors of these automorphisms are simple[1]. Therefore we have for any automorphism A of the compact group[2]

$$A = U^{-1}ZU,$$

where $Z = Z_0 e^{\widehat{H}}$ is a chief automorphism in that component $\mathfrak{A}_i$ which contains A.

Since the modulus of all characteristic numbers of Z must be equal to 1, all parameters λ_i in $\widehat{H}$ must have imaginary values. But then Z will be an automorphism (in fact, chief automorphism) of the compact group. In virtue of the preceding theorem we can also consider U as an inner automorphism of the compact group. Thus we arrive at the following

Theorem 8. *Each automorphism A of the compact group is conjugated, with respect to the compact adjoint group, to a chief automorphism of the compact group:*

$$A = U^{-1}ZU, \tag{72}$$

where U is an inner automorphism of the compact group.

CHAPTER II

THE DETERMINATION OF REAL GROUPS OF THE FIRST CATEGORY WITH SIMPLE COMPLEX STRUCTURE

§ 6. Preliminary remarks

In the preceding chapter we have seen that the problem of determination of all real forms of a simple complex Lie group may be reduced to the determination of all involutive automorphisms of the compact form of this group. By Theorems 3 and 8 we can moreover confine ourselves to consideration of „chief" automorphisms.

We shall refer a real form of a given simple complex group to the first or to the second category according as to whether it is generated by an inner or an outer involutive automorphism.

Consider an involutive chief inner automorphism S. Its characteristic numbers are $+1$ and -1. Consider the decomposition with respect to these characteristic numbers:

$$\mathfrak{R} = \mathfrak{K}_1 + \mathfrak{K}_{-1},$$

---

[1] In fact, from $Ae = \lambda e$ it follows that $A^n e = \lambda^n e$. Further, since from the sequence A^n $(n = 1, 2, \ldots)$ we can choose a subsequence converging to a certain automorphism, $|\lambda|$ must be equal to 1. The same considerations show that we can not have

$$Ae = \lambda e, \quad Ag = \lambda g + e,$$

since then we should have

$$A^n g = \lambda^n g + n\lambda^{n-1} e.$$

[2] Gantmacher, [1], p. 139.

and let $\mathfrak{h} \subset \mathfrak{K}_1$ [1]. Let further

$$\mathfrak{K}_1 = \{\mathfrak{h}, e_\alpha, e_{-\alpha}, \dots\} \quad \text{and} \quad \mathfrak{K}_{-1} = \{e_\rho, e_{-\rho}, \dots\}.$$

Denote by V_1 and V_{-1} the aggregates of roots

$$\alpha, -\alpha, \dots \quad \text{and} \quad \rho, -\rho, \dots$$

Thus to the automorphism S corresponds the decomposition of the root system:

$$V = (V_1, V_{-1}),$$

and from the structure formula

$$[e_\alpha, e_\beta] = N_{\alpha\beta} e_{\alpha+\beta},$$

where $N_{\alpha\beta} \neq 0$, if $\alpha + \beta$ is a root (see also Gantmacher, [1], p. 107), it follows that

$$\left.\begin{array}{l} V_1 + V_1, \\ V_{-1} + V_{-1} \end{array}\right\} \subset V_1, \quad V_1 + V_{-1} \subset V_{-1} \tag{73}$$

(on the left only those roots are added, whose sum is again a root).

Suppose that for another involutive chief automorphism S' [2] we have a similar decomposition of the root system:

$$V = (V_1', V_{-1}').$$

If the systems V_1 and V_1' may be transformed into each other by a certain rotation $\tau \subset \mathfrak{T}$ [3], then the involutive automorphisms S and S' generate isomorphic real groups.

Indeed, the rotation τ transforms the root system into itself. Hence from $V_1' = \tau(V_1)$ follows $V'_{-1} = \tau(V_{-1})$. The rotation τ may be completed to an automorphism A of the compact group [4]. This automorphism transforms $\mathfrak{K}_1$ into $\mathfrak{K}_1'$ and $\mathfrak{K}_{-1}$ into $\mathfrak{K}'_{-1}$. Hence it follows that

$$S' = ASA^{-1}$$

and consequently

$$\sqrt{S'} = A\sqrt{S}A^{-1}. \tag{74}$$

By Theorem 3, S and S' generate isomorphic real groups.

This remark shall be used later. Denote by ν the number of characteristic roots of S equal to 1 and by $\mu = r - \nu$ the number of roots equal to -1. Since for the original compact group $\varphi = -\sum_1^r \tau_i^2$, for the real group corresponding to the automorphism S the form φ will have, by Cartan's theorem, μ positive and ν negative squares. The signature of the form φ

$$\delta = \mu - \nu = 2\mu - r = r - 2\nu \tag{75}$$

Cartan calls the character of the real group under consideration. It is evident that real groups having different characters can not be isomorphic.

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[1] Gantmacher, [1], p. 117.
[2] Here and in the following the dash does not mean the transposed matrix.
[3] Gantmacher, [1], p. 129.
[4] Ibid., p. 130.

Consider the root forms

$$(\alpha\lambda)=\sum_{p=1}^{n} a^p\lambda_p$$

and put $\lambda=\pi i\varphi$ $(i=\sqrt{-1})$. Then

$$(\alpha\lambda)=\pi i(\alpha\varphi)=\pi i\sum_{1}^{n} a^p\varphi_p.$$

Since for S

$$e^{(\alpha\lambda)}=\pm 1, \tag{76}$$

all

$$(\alpha\varphi)=\sum_{1}^{n} a^p\varphi_p$$

must have integral values. In what follows under a root form we shall understand $(\alpha\varphi)$.

We shall write

$$(\varphi_1,\ldots,\varphi_n)\equiv(\varphi'_1,\ldots,\varphi'_n)$$

or, briefly,

$$\varphi\equiv\varphi', \tag{77}$$

if for all roots

$$(\alpha\varphi)\equiv(\alpha\varphi') \pmod 2. \tag{78}$$

In this case the corresponding involutive chief automorphisms S and S' will have the same characteristic numbers: $e^{\pi i(\alpha\varphi)}=e^{\pi i(\alpha\varphi')}$.

We shall say that the systems of numbers $(\varphi_1,\ldots,\varphi_n)$ and $(\varphi'_1,\ldots,\varphi'_n)$ are equivalent and shall write

$$(\varphi_1,\ldots,\varphi_n)\backsim(\varphi'_1,\ldots,\varphi'_n), \tag{79}$$

if there exists a rotation $\tau\subset\mathfrak{T}$ transforming the vector φ into a vector congruent to φ':

$$\tau(\varphi_1,\ldots,\varphi_n)\equiv(\varphi'_1,\ldots,\varphi'_n). \tag{80}$$

In this case, as we have already seen, the corresponding involutive automorphisms S and S' generate isomorphic real groups. Hence, in finding the involutive chief inner automorphisms of the given complex simple structure, we may confine ourselves to consideration of unequivalent systems of numbers $(\varphi_1,\ldots,\varphi_n)$, for which the root forms assume integral values.

After these preliminary remarks we pass now to direct consideration of separate complex simple structures [5].

§ 7. The structure A_n

The root forms are here

$$\varphi_p-\varphi_q \quad (p,q=1,\ldots,n+1) \quad {}^{1} \tag{81}$$

with the additional condition

$$\varphi_1+\ldots+\varphi_{n+1}=0. \tag{82}$$

[5] Gantmacher, [1], p. 126—127.

[1] Here and in what follows we shall suppose that indices occuring in the denotation of one and the same root form assume different values.

All differences $\varphi_p - \varphi_q$ are integers. Each system of φ_p may be replaced by a congruent system consisting of integers [and, generally speaking, not satisfying the additional condition (82)]. These integral values of φ_p we may reduce to the modulus 2.

Observe that 1) permutations of φ_p are rotations τ, 2) we can add to all φ_p one and the same number without affecting the form $(\alpha\varphi)$.

Using these transformations, we reduce the whole problem to the consideration of the following systems of φ_p:

$$\mathfrak{B}_l = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{l}) \quad (l = 0, 1, \ldots, n+1).$$

But

$$\left.\begin{aligned} \mathfrak{B}_l \backsim (-1, \ldots, -1, \underbrace{0, \ldots, 0}_{l}) = (1, \ldots, 1, \underbrace{0, \ldots, 0}_{l}) \backsim \\ \backsim (0, \ldots, 0, \underbrace{1, \ldots, 1}_{n+1-l}) = \mathfrak{B}_{n+1-l}. \end{aligned}\right\} \qquad (83)$$

Hence we may ascribe to l only the values

$$l = 0, 1, \ldots, \left[\frac{n+1}{2}\right]. \qquad (84)$$

To the system $\mathfrak{B}_l$ corresponds the involutive automorphism S_l, for which

$$\left.\begin{aligned} &\mu = 2l(n+1-l), \ \delta = 2\mu - r = \\ &= 4l(n+1-l) - (n+1)^2 + 1 = 1 - (n+1-2l)^2 = 1 - m^2, \end{aligned}\right\} \qquad (85)$$

where $m = n+1-2l = n+1, n-1, \ldots, \left\{\begin{matrix} 0 \\ 1 \end{matrix}\right.$.

The so obtained $\left[\frac{n+3}{2}\right]$ real simple groups may be realized in groups of linear transformations in $n+1$ complex variables, leaving invariant respectively the Hermitian forms

$$x_1\bar{x}_1 + \ldots + x_l\bar{x}_l - x_{l+1}\bar{x}_{l+1} - \ldots - x_{n+1}\bar{x}_{n+1} \quad [2]$$
$$\left(l = 0, 1, \ldots, \left[\frac{n+1}{2}\right]\right).$$

§ 8. The structure B_n

The root forms:

$$\pm\varphi_p, \ \pm\varphi_p \pm \varphi_q \quad (p, q = 1, \ldots, n). \qquad (86)$$

We may take φ_p to be integers reduced to the modulus 2.

As in the preceding case we may confine ourselves to consideration of only those S, for which

$$\varphi_1 = \ldots = \varphi_{n-l} = 0, \quad \varphi_{n-l+1} = \ldots = \varphi_n = 1,$$

where $l = 0, 1, \ldots, n$. For S_l we have

$$\left.\begin{aligned} &\mu = 2l + 4l(n-l) = 2l(2n-2l+1), \\ &\delta = 2\mu - r = 4l(2n-2l+1) - (2n+1)n = \\ &\qquad = n - 2m(m+1), \\ &m = n - 2l = n, \quad n-2, \ldots, -n, \end{aligned}\right\} \qquad (87)$$

---

[2] Cartan, [2], p. 276.

or, which is in this case equivalent,

$$m = 0, 1, \ldots, n.$$

We have $n+1$ different simple real groups with the complex structure B_n. These groups may be realized in groups of real linear transformations, leaving invariant respectively the quadratic forms

$$x_1^2 + \ldots + x_{2l}^2 - x_{2l+1}^2 - \ldots - x_{2n+1}^2 \quad (l = 0, 1, \ldots, n)$$ [1].

§ 9. The structure C_n

Let us take the root forms in the form

$$\pm \varphi_p, \; \frac{1}{2}(\pm \varphi_p \pm \varphi_q) \quad (p, q = 1, \ldots, n). \tag{88}$$

We may suppose all φ_p to be integers, all odd or all even, reduced to the modulus 4, i. e. either I) all $\varphi_p = 0, 2$, or II) all $\varphi_p = \pm 1$.

Note the following rotations τ:

1) the permutation of φ_p,

2) the change of sign of one of the numbers φ_p.

I. In this case we may confine ourselves to consideration of the systems

$$\mathfrak{B}_l = (0, \ldots, 0, \underbrace{2, \ldots, 2}_{l}) \quad (l = 0, 1, \ldots, n). \tag{89}$$

But the addition of 2 to all φ_p gives a congruent system, and the change of sign of φ_p is a rotation τ. Hence the transformation

$$\varphi_p' = 2 - \varphi_p \quad (p = 1, \ldots, n)$$

realizes a transition to an equivalent system. Therefore

$$\mathfrak{B}_l \backsim \mathfrak{B}_{n-l}. \tag{90}$$

Thus l in (79) may be confined to the values

$$l = 0, 1, \ldots, \left[\frac{n}{2}\right]. \tag{91}$$

For the corresponding S

$$\left.\begin{aligned} \mu &= 4l(n-l), \\ \delta &= 2\mu - r = 8l(n-l) - 2n^2 - 2n = -n - 2m^2, \end{aligned}\right\} \tag{92}$$

where

$$m = n - 2l = n, \; n-2, \ldots, \begin{cases} 0 \\ 1 \end{cases}.$$

The so obtained $\left[\frac{n+2}{2}\right]$ real simple groups may be realized in groups of linear transformations, leaving simultaneously invariant the skew-symmetrical bilinear form

$$x_1 x_2' - x_2 x_1' + \ldots + x_{2n-1} x_{2n}' - x_{2n} x_{2n-1}'$$

---

[1] Cartan, [2], p. 280—281.

and the indefinite Hermitian form

$$x_1\bar{x}_1+\ldots+x_{2l}\bar{x}_{2l}-x_{2l+1}\bar{x}_{2l+1}-\ldots-x_{2n}\bar{x}_{2n}\ {}^{1}$$
$$\left(l=0,1,\ldots,\left[\frac{n}{2}\right]\right).$$

II. The cases, in which $\varphi_p=\pm 1$, may be reduced to the one case: $\varphi_p=1$. It is easily seen that for the corresponding involutive automorphism S we have

$$\delta=n.$$

To this case corresponds the group of real linear transformations, leaving invariant the skew-symmetrical bilinear form

$$x_1x_2'-x_2x_1'+\ldots+x_{2n-1}x_{2n}'-x_{2n}x_{2n-1}'\ {}^{2}.$$

§ 10. The structure D_n

The root forms we take to be

$$\frac{\pm\varphi_p\pm\varphi_q}{2}\quad(p,q=1,\ldots,n). \tag{93}$$

All φ_p must be integers congruent to each other to modulus 2, and these integers may be reduced to modulus 4.

As in the preceding section we may evidently confine ourselves to consideration of

$$\left.\begin{aligned}\mathfrak{B}_l&=(\underbrace{0,\ldots,0}_{l},\ 2,\ldots,2)\quad\left(l=0,1,\ldots,\left[\frac{n}{2}\right]\right),\\ \mathfrak{B}&=(\underbrace{1,1,\ldots,1}_{n}).\end{aligned}\right\} \tag{94}$$

For $\mathfrak{B}_l$ we have

$$\left.\begin{aligned}&\mu=4l(n-l),\quad \delta=2\mu-r=\\ &=8l(n-l)-n(2n-1)=n-2(n-2l)^2=n-2m^2,\end{aligned}\right\} \tag{95}$$

where $m=n,\ n-2,\ldots,\ \left\{\begin{matrix}0\\1\end{matrix}\right.$.

The corresponding real structures may be realized in groups of real linear transformations, leaving invariant the quadratic forms

$$x_1^2+\ldots+x_{2l}^2-x_{2l+1}^2-\ldots-x_{2n}^2$$
$$\left(l=0,1,\ldots,\left[\frac{n}{2}\right]\right)\ {}^{1}.$$

For the system $\mathfrak{B}=(1,\ldots,1)$ we find at once

$$\delta=-n.$$

The corresponding real structure is realized in the group of linear transformations in $2n$ complex variables, leaving simultaneously invariant the quadratic form

$$x_1x_2+\ldots+x_{2n-1}x_{2n}$$

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[1] Cartan, [2], p. 292.
[2] Cartan, [2], p. 291.
[1] Cartan, [2], p. 286.

and the indefinite Hermitian form

$$x_1\bar{x}_1 - x_2\bar{x}_2 + \ldots + x_{2n-1}\bar{x}_{2n-1} - x_{2n}\bar{x}_{2n}$$ [2].

§ 11. The structure G_2

The root forms we may take to be

$$\pm\varphi_p, \quad \varphi_p - \varphi_q \qquad (p, q = 1, 2, 3) \tag{96}$$

with the additional condition

$$\varphi_1 + \varphi_2 + \varphi_3 = 0. \tag{97}$$

Here φ_1 and φ_2 may be confined to the values 0, 1. We consider the following systems:

$$\mathfrak{V}_1 = (0, 0, 0), \ \mathfrak{V}_2 = (0, 1, -1) \equiv (0, 1, 1). \tag{98}$$

1. $\mathfrak{V}_1 = (0, 0, 0), \quad \delta = -14$.
This is a compact real group, which may be realized in the following way[1]:

We define in the seven-dimensional real vector space the operation of „vector multiplication",

$$c = a \times b, \tag{99}$$

where

$$c_i = \begin{vmatrix} a_{i-3} & b_{i-3} \\ a_{i-2} & b_{i-2} \end{vmatrix} + \begin{vmatrix} a_{i+2} & b_{i+2} \\ a_{i-1} & b_{i-1} \end{vmatrix} + \begin{vmatrix} a_{i+1} & b_{i+1} \\ a_{i+3} & b_{i+3} \end{vmatrix} \tag{100}$$

($i = 1, \ldots, 7$; the indices on the right-hand side are to be reduced to modulus 7).

The compact real group, in which we are interested, consists of all orthogonal transformations T, leaving this operation of vector multiplication invariant:

$$T(a \times b) = Ta \times Tb. \tag{101}$$

2. $\mathfrak{V}_2 \equiv (0, 1, 1), \ \mu = 8, \ \delta = 2\mu - r = 2$.
To this case corresponds the group of linear transformations in the real seven-dimensional space, leaving a certain indefinite quadratic form and the operation of vector multiplication, defined above, invariant[2].

§ 12. The structure F_4

The root forms are

$$\pm\varphi_p, \ \pm\varphi_p \pm \varphi_q, \ \frac{1}{2}(\pm\varphi_1 \pm \varphi_2 \pm \varphi_3 \pm \varphi_4). \tag{102}$$

The φ_p may be confined to the values $0, \pm 1, 2$, but $\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$ must be an even number. Observe also that subtraction of 2 from two of the φ_p gives a system congruent to the original one.

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[2] Loc. cit., p. 286.

[1] Cartan, [2], p. 297—298; Lardy, [1], p. 212—215.

[2] Cartan, [2], p. 297—298.

Note the following rotations τ:

1) the permutation of φ_p,

2) the change of sign of some of the φ_p.

3) the subtraction from each φ_p of the semi-sum of all the φ_p.

Since these operations, as well as the subtraction of 2 from two of the φ_p, transform any system of the φ_p into a system equivalent to it, we may consider the following cases:

$$\left.\begin{array}{ll} \mathfrak{V}_1=(0,\ 0,\ 0,\ 0) & (\delta=-52), \\ \mathfrak{V}_2=(0,\ 0,\ 0,\ 2) & (\delta=-20), \\ \mathfrak{V}_3=(0,\ 0,\ 1,\ 1) & (\delta=4) \end{array}\right\} \tag{103}$$

$$[\mathfrak{V}_4=(1,\ 1,\ 1,\ 1)\underset{2}{\backsim}(1,-1,-1,-1)\underset{3}{\backsim}(2,\ 0,\ 0,\ 0)\underset{1}{\backsim}\mathfrak{V}_2]\ [1].$$

In order to show how these three real structures are realized, let us consider the so called Cartan's normal group [2]. To this end we introduce the following denotations.

If T is an infinitesimal transformation of a linear group defined by the equations

$$z_i'=\delta z_i=\sum_1^n a_{ik}z_k \qquad (i=1,\ldots,n),$$

we shall write this transformation T in the following form:

$$Tf=\delta f=\sum_{i,k=1}^{n} a_{ik}z_k\frac{\partial f}{\partial z_i},$$

where f is an arbitrary differentiable function of the z_i. We take 26 complex variables

$$x_i,\ x_{\alpha\beta\gamma\delta},\ y,\ z$$

$$(i=\pm 1,\ \pm 2,\ \pm 3,\ \pm 4,\quad \alpha=\pm 1,\ \beta=\pm 2,\ \gamma=\pm 3,\ \delta=\pm 4)$$

and put

$$\frac{\partial f}{\partial x_i}=p_i,\quad \frac{\partial f}{\partial x_{\alpha\beta\gamma\delta}}=p_{\alpha\beta\gamma\delta},\quad \frac{\partial f}{\partial y}=q,\quad \frac{\partial f}{\partial z}=r.$$

By the dash we shall denote the change of sign of the index. Then Cartan's normal group is the linear group with the following infinitesimal transformations [3]:

---

[1] We write, for instance, $\mathfrak{V}\underset{1}{\backsim}\mathfrak{V}'$, if the system $\mathfrak{V}'$ can be obtained from the system $\mathfrak{V}$ by the transformation 1), etc.

[2] Cf. Cartan, [1], p. 145.

[3] Loc. cit., p. 145.

$$\left.\begin{aligned}
Y_i f &= x_i p_i - x_{i'} p_{i'} + \frac{1}{2} \sum_{j,k,l} (x_{ijkl} p_{ijkl} - x_{i'jkl} p_{i'jkl}),\\
X_\alpha f &= -x_{\alpha'} q + 2y p_\alpha + \varepsilon_\alpha \sum_{\beta,\gamma,\delta} \varepsilon_\beta \varepsilon_\gamma \varepsilon_\delta x_{\alpha'\beta\gamma\delta} p_{\alpha\beta\gamma\delta},\\
X_\beta f &= -x_{\beta'} q + 2y p_\beta + \varepsilon_\beta \sum_{\alpha,\gamma,\delta} \varepsilon_\delta x_{\alpha\beta'\gamma\delta} p_{\alpha\beta\gamma\delta},\\
X_\gamma f &= -x_{\gamma'} q + 2y p_\gamma + \varepsilon_\gamma \sum_{\alpha,\beta,\delta} \varepsilon_\beta x_{\alpha\beta\gamma'\delta} p_{\alpha\beta\gamma\delta},\\
X_\delta f &= -x_{\delta'} q + 2y p_\delta + \varepsilon_\delta \sum_{\alpha,\beta,\gamma} \varepsilon_\gamma x_{\alpha\beta\gamma\delta'} p_{\alpha\beta\gamma\delta},\\
X_{\alpha\beta} f &= x_{\beta'} p_\alpha - x_{\alpha'} p_\beta - \varepsilon_\alpha \sum_{\gamma,\delta} \varepsilon_\gamma x_{\alpha'\beta'\gamma\delta} p_{\alpha\beta\gamma\delta},\\
X_{\alpha\gamma} f &= x_{\gamma'} p_\alpha - x_{\alpha'} p_\gamma - \varepsilon_\alpha \sum_{\beta,\delta} \varepsilon_\delta x_{\alpha'\beta\gamma'\delta} p_{\alpha\beta\gamma\delta},\\
X_{\alpha\delta} f &= x_{\delta'} p_\alpha - x_{\alpha'} p_\delta - \varepsilon_\alpha \sum_{\beta,\gamma} \varepsilon_\beta x_{\alpha'\beta\gamma\delta'} p_{\alpha\beta\gamma\delta},\\
X_{\gamma\beta} f &= x_{\beta'} p_\gamma - x_{\gamma'} p_\beta - \varepsilon_\gamma \sum_{\alpha,\delta} \varepsilon_\delta x_{\alpha\beta'\gamma'\delta} p_{\alpha\beta\gamma\delta},\\
X_{\beta\delta} f &= x_{\delta'} p_\beta - x_{\beta'} p_\delta - \varepsilon_\beta \sum_{\alpha,\gamma} \varepsilon_\gamma x_{\alpha\beta'\gamma\delta'} p_{\alpha\beta\gamma\delta},\\
X_{\delta\gamma} f &= x_{\gamma'} p_\delta - x_{\delta'} p_\gamma - \varepsilon_\delta \sum_{\alpha,\beta} \varepsilon_\beta x_{\alpha\beta\gamma'\delta'} p_{\alpha\beta\gamma\delta},\\
X_{\alpha\beta\gamma\delta} f &= -(y - 3\varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma\varepsilon_\delta z) p_{\alpha\beta\gamma\delta} + \frac{1}{2} x_{\alpha'\beta'\gamma'\delta'} (q - \varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma\varepsilon_\delta r) +\\
&\quad + \varepsilon_\alpha\varepsilon_\beta\varepsilon_\gamma\varepsilon_\delta (x_{\alpha'} p_{\alpha'\beta\gamma\delta} - x_{\alpha\beta'\gamma'\delta'} p_\alpha) +\\
&\quad + \varepsilon_\beta\varepsilon_\delta (x_{\beta'} p_{\alpha\beta'\gamma\delta} - x_{\alpha'\beta\gamma'\delta'} p_\beta) + \varepsilon_\beta\varepsilon_\gamma (x_{\gamma'} p_{\alpha\beta\gamma'\delta} - x_{\alpha'\beta'\gamma\delta'} p_\gamma) +\\
&\quad + \varepsilon_\gamma\varepsilon_\delta (x_{\delta'} p_{\alpha\beta\gamma\delta'} - x_{\alpha'\beta'\gamma'\delta} p_\delta)
\end{aligned}\right\} \quad (104)$$

$$(i = 1, 2, 3, 4, \quad \alpha = \pm 1, \ \beta = \pm 2, \ \gamma = \pm 3, \ \delta = \pm 4,$$
$$\varepsilon_j > 0, \text{ if } j > 0 \text{ and } \varepsilon_j < 0, \text{ if } j < 0).$$

If we confine the variables in this group to real values, we obtain a real simple group with the character $\delta = 4$.

If we subject the complex variables x_i, $x_{\alpha\beta\gamma\delta}$, y, z to the conditions

$$x_{i'} = \bar{x}_i, \ x_{\alpha'\beta'\gamma'\delta'} = -\bar{x}_{\alpha\beta\gamma\delta}, \ y = \bar{y}, \ z = \bar{z}, \quad (105)$$

we obtain a simple real group with the character $\delta = -20$.

If, finally, we replace the conditions (105) by the conditions

$$x_{i'} = \bar{x}_i, \ x_{\alpha'\beta'\gamma'\delta'} = \bar{x}_{\alpha\beta\gamma\delta}, \ y = \bar{y}, \ z = \bar{z}, \quad (106)$$

we obtain a compact group with the character $\delta = -52$.

§ 13. The structure E_6

Here the root forms are

$$\left.\begin{array}{c}
\varphi_p - \varphi_q, \ \pm(\varphi_p + \varphi_q + \varphi_s), \ \pm(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 + \varphi_5 + \varphi_6)\\
(p, q = 1, \ldots, 6).
\end{array}\right\} \quad (107)$$

We have $\varphi_p = \psi_p + \varepsilon$, where ψ_p are integers and ε assumes one of the three values: $0, \frac{1}{3}, -\frac{1}{3}$.

Since we can add to or subtract from all φ_p two thirds, the cases $\varepsilon = \frac{1}{3}$ and $\varepsilon = -\frac{1}{3}$ may be omitted, and we may suppose that the φ_i are integers reduced to the modulus 2.

Note the following rotations τ:

1) the permutation of φ_p,

2) the change of signs of all φ_p,

3) the mapping σ_α effecting mirror images, where $(\alpha\varphi) = \varphi_1 + \varphi_2 + \varphi_3$:

$$\left.\begin{aligned} \varphi'_q &= \varphi_q - \frac{2}{3}(\varphi_1 + \varphi_2 + \varphi_3), \\ \varphi'_{q+3} &= \varphi_{q+3} + \frac{1}{3}(\varphi_1 + \varphi_2 + \varphi_3) \end{aligned}\right\} \quad (q = 1,\ 2,\ 3), \tag{108}$$

4) the mapping σ_α effecting mirror images, where $(\alpha\varphi) = \varphi_1 + \dots + \varphi_6$:

$$\varphi'_p = \varphi_p - \frac{1}{3}\sum_1^6 \varphi_i. \tag{109}$$

In virtue of rotation 1) we may confine ourselves to the systems

$$\mathfrak{B}_l = (0, \dots, 0, \underbrace{1, \dots, 1}_{l}) \qquad (l = 0, 1, \dots, 6).$$

But

$$\mathfrak{B}_5 = (0,\ 1,\ 1,\ 1,\ 1,\ 1) \underset{1}{\backsim} (1,\ 1,\ 1,\ 1, -1,\ 0) \underset{4}{\backsim} (0, 0, 0, 0, -2, -1) \backsim \mathfrak{B}_1$$

and

$$\mathfrak{B}_5 \underset{1}{\backsim} (1,\ 1,\ 1,\ 1,\ 1,\ 0) \underset{3}{\backsim} (-1,\ -1,\ -1,\ 2, 2, 1) \backsim \mathfrak{B}_4.$$

Further,

$$\mathfrak{B}_6 = (1,\ 1,\ 1,\ 1,\ 1,\ 1) \underset{3}{\backsim} (-1,\ -1,\ -1,\ 2, 2,\ 2) \underset{1}{\backsim} \mathfrak{B}_3 \underset{1}{\backsim}$$
$$\underset{1}{\backsim} (3,\ 0,\ 0,\ 1, -1,\ 0) \underset{3}{\backsim} (1,\ -2,\ -2,\ 2,\ 0,\ 1) \backsim \mathfrak{B}_2.$$

It remains to consider the systems $\mathfrak{B}_0$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

1. $\mathfrak{B}_0 = (0,\ 0,\ 0,\ 0,\ 0,\ 0)$, $\delta = -78$.

To this case corresponds the compact real simple group, which may be given as the group of linear transformations in 27 complex variables x_p, y_q, $z_{pq} = -z_{qp}$ $(p, q = 1, 2, \dots, 6)$, leaving invariant the following two forms: the cubic form

$$\sum_{p,q} x_p y_q z_{pq} - \sum_{p,q,s,t,u,v} (p,\ q,\ s,\ t,\ u,\ v)\, z_{pq} z_{st} z_{uv}, \tag{110}$$

where

$$(p,\ q,\ s,\ t,\ u,\ v) = +1, \text{ if the permutation is even,}$$
$$(p,\ q,\ s,\ t,\ u,\ v) = -1, \text{ if the permutation is odd,}$$

and the positive definite Hermitian form

$$\sum_p x_p \bar{x}_p + \sum_q y_q \bar{y}_q + \sum_{p,q} z_{pq} \bar{z}_{pq}\ [1]. \tag{111}$$

2. $\mathfrak{B}_1 = (0,\ 0,\ 0,\ 0,\ 0,\ 1)$, $\mu = 32$, $\delta = -14$.

[1] Cartan, [2], p. 313.

The corresponding real simple group may be defined in the same way as the compact group of the preceding case, with the only difference that instead of the positive definite Hermitian form (111) we must take here the indefinite Hermitian form

$$\sum_1^5 x_p\bar{x}_p - x_6\bar{x}_6 - \sum_1^5 y_q\bar{y}_q + y_6\bar{y}_6 - \sum_1^5 z_{pq}\bar{z}_{pq} - \sum_1^5 z_{p6}\bar{z}_{p6}. \tag{112}$$

3. $\mathfrak{B}_2 = (0, 0, 0, 0, 1, 1)$, $\mu = 40$, $\delta = 2$.

The real simple group corresponding to this case is again determined in the same way as in case 1, with the only difference that we must take here instead of the Hermitian form (111) the form

$$\sum x_p\bar{x}_p + \sum y_q\bar{y}_q - \sum z_{pq}\bar{z}_{pq} \quad ^2. \tag{113}$$

§ 14. The structure E_7

The root forms are

$$\varphi_p - \varphi_q, \quad \varphi_p + \varphi_q + \varphi_s + \varphi_t \tag{114}$$

with the additional condition

$$\varphi_1 + \ldots + \varphi_8 = 0. \tag{115}$$

The φ_p will be here evidently of the form

$$\varphi_p = \psi_p + \varepsilon, \quad \text{where } \varepsilon = 0, \ \frac{1}{4}, \ -\frac{1}{4}, \ \frac{1}{2}. \tag{116}$$

Since we can add $\frac{1}{2}$ to all φ_p without changing the characteristic numbers of the involutive automorphism S, we can omit the cases $\varepsilon = -\frac{1}{4}$ and $\varepsilon = \frac{1}{2}$.

I. $\varepsilon = 0$, φ_p are integers reduced to modulus 2. The equation (115) we replace by a congruence to the modulus 2. We put

$$\mathfrak{B}_l = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{2l}) \qquad (l = 0, 1, 2, 3, 4).$$

Note the following rotations τ:

1) the permutation of φ_p,

2) the mirror image with respect to the origin:

$$\varphi'_p = -\varphi_p \qquad (p = 1, \ldots, 8),$$

3) the mapping σ_α effecting mirror images, where $(\alpha\varphi) = \sum_1^4 \varphi_q$:

$$\varphi'_q = \varphi_q - \frac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4),$$

$$\varphi'_{q+4} = \varphi_{q+4} + \frac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4).$$

Observe, besides, that to all φ_p can be simultaneously added $\frac{1}{2}$.

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[2] Loc. cit., p. 313.

Then

$$\mathfrak{B}_l = (0, \ldots, 0, \underbrace{-1, \ldots, -1}_{2l}) = (1, \ldots, 1, \underbrace{0, \ldots, 0}_{2l}) \underset{1}{\backsim} \mathfrak{B}_{4-l} \qquad (117)$$

and

$$\mathfrak{B}_3 \underset{1}{\backsim} (1, -1, 1, 0, 1, -1, 1, 0) \underset{1}{\backsim} \mathfrak{B}_2. \qquad (118)$$

II. $\varepsilon = \frac{1}{4}$. Let

$$\mathfrak{W}_l = \Big(\frac{1}{4}, \ldots, \frac{1}{4}, \underbrace{1\frac{1}{4}, \ldots, 1\frac{1}{4}}_{2l}\Big) \quad (l = 0, 1, 2, 3, 4).$$

Again

$$\mathfrak{W}_l = \Big(1\frac{1}{4}, \ldots, 1\frac{1}{4}, \underbrace{2\frac{1}{4}, \ldots, 2\frac{1}{4}}_{2l}\Big) \underset{1}{\backsim} \mathfrak{W}_{4-l}.$$

Besides,

$$\mathfrak{W}_2 \underset{3}{\backsim} \Big(-\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 1\frac{3}{4}, 1\frac{3}{4}, 1\frac{3}{4}, 1\frac{3}{4}\Big) =$$
$$= \Big(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 2\frac{1}{4}, 2\frac{1}{4}, 2\frac{1}{4}, 2\frac{1}{4}\Big) = \mathfrak{W}_0.$$

Thus it remains to consider the systems

$$\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{W}_0, \mathfrak{W}_1.$$

1. $\mathfrak{B}_0 = (0, 0, 0, 0, 0, 0, 0, 0)$, $\delta = -133$.

To this case corresponds the linear group, which may be given as the group of linear transformations in 56 complex variables $x_{pq} = -x_{qp}$, $y_{pq} = -y_{qp}$ $(p, q = 1, \ldots, 8)$, leaving invariant the following three forms: the positive definite Hermitian form

$$\sum_{p,q} x_{pq}\bar{x}_{pq} + \sum_{p,q} y_{pq}\bar{y}_{pq}, \qquad (119)$$

the bilinear form

$$\sum_{p,q} (x_{pq}y'_{pq} - y_{pq}x'_{pq}) \qquad (120)$$

and the biquadratic form

$$\sum_{p,\ldots,s} x_{pq}x_{rs}y_{ps}y_{qr} +$$
$$+ \sum_{p,\ldots,w} (p, q, r, s, t, u, v, w)\{x_{pq}x_{rs}x_{tu}x_{vw} + y_{pq}y_{rs}y_{tu}y_{vw}\}. \qquad (121)$$

Here

$(p, q, r, s, t, u, v, w) = +1$, if the permutation is even,
$(p, q, r, s, t, u, v, w) = -1$, if the permutation is odd [1].

2. $\mathfrak{B}_1 = (0, 0, 0, 0, 0, 0, 1, 1)$, $\delta = -5$.

The corresponding real group may be realized in the group of linear transformations in 56 complex variables, leaving invariant beside the bilinear form

[1] Cartan, [2], p. 323.

(120) and the biquadratic form (121) the following indefinite Hermitian form [2]:

$$\left.\begin{array}{l}\sum\limits_{p,q} \lambda_p\lambda_q (x_{pq}\bar{x}_{pq}+y_{pq}\bar{y}_{pq}),\\ \lambda_1=\lambda_2=-1, \quad \lambda_\rho=+1 \quad (\rho\neq 1,2).\end{array}\right\} \tag{122}$$

3. $\mathfrak{W}_0=\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right), \ \delta=7.$

To this case corresponds the group of linear transformations in 56 real parameters, leaving invariant the bilinear and the biquadratic forms (120) and (121) [2].

4. $\mathfrak{W}_1=\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 1\frac{1}{4}, 1\frac{1}{4}\right), \quad \delta=-25.$

To this case corresponds the group of linear transformations in 56 variables x_{pq}, y_{pq}, connected by the relations

$$\left.\begin{array}{l}y_{pq}=\lambda_p\lambda_q\bar{x}_{pq},\\ \lambda_1=\lambda_2=-1, \quad \lambda_3=\ldots=\lambda_8=1,\end{array}\right\} \tag{123}$$

where

which leave the forms (120) and (121) invariant [2].

§ 15. The structure E_8

The root forms are

$$\varphi_p-\varphi_q, \quad \pm(\varphi_p+\varphi_q+\varphi_s) \quad (p, q, s=1, \ldots, 9) \tag{124}$$

with the additional condition

$$\varphi_1+\varphi_2+\ldots+\varphi_9=0. \tag{125}$$

In this case $\varphi_p=\psi_p+\varepsilon$, where ψ_p are integers and $\varepsilon=0, \frac{1}{3}, -\frac{1}{3}$. The numbers ψ_p may be reduced to modulus 2, if we replace in (125) the sign $=$ by the sign $\equiv$ (mod 2). Since we can simultaneously add to or subtract from all φ_p two thirds, we can omit the cases $\varepsilon=\frac{1}{3}$ and $\varepsilon=-\frac{1}{3}$ and assume that all φ_p are integers reduced to the modulus 2.

Note the following rotations τ:

1) the permutation of φ_p,

2) the mapping σ_α effecting mirror images, where $(\alpha\varphi)=\varphi_1+\varphi_2+\varphi_3$:

$$\varphi'_q=\varphi_q-\frac{2}{3}(\varphi_1+\varphi_2+\varphi_3) \quad (q=1, 2, 3),$$

$$\varphi'_s=\varphi_s+\frac{1}{3}(\varphi_1+\varphi_2+\varphi_3) \quad (s=4, 5, 6, 7, 8, 9).$$

Putting $\mathfrak{B}_l=(0, \ldots, 0, \underbrace{1, \ldots, 1}_{2l})$ $(l=0, 1, 2, 3, 4)$ and using the rotations 1) and 2), we find

$$\mathfrak{B}_1 \underset{1}{\backsim} (3, 0, \ldots, 1) \underset{2}{\backsim} (1, -2, -2, 1, 1, 1, 1, 1, 2) \underset{1}{\backsim} \mathfrak{B}_3, \tag{126}$$

$$\left.\begin{array}{l}\mathfrak{B}_2 \underset{1}{\backsim} (1, 1, 1, 1, 0, 0, 0, 0, 0) \underset{2}{\backsim}\\ \underset{2}{\backsim} (-1, -1, -1, 2, 1, 1, 1, 1, 1) \underset{1}{\backsim} \mathfrak{B}_4.\end{array}\right\} \tag{127}$$

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[2] Ibid., p. 323.

It remains to consider $\mathfrak{B}_0$, $\mathfrak{B}_1$ and $\mathfrak{B}_2$:

1) $\mathfrak{B}_0 = (0, 0, 0, 0, 0, 0, 0, 0, 0)$, $\delta = -248$,
2) $\mathfrak{B}_1 = (0, 0, 0, 0, 0, 0, 0, 1, 1)$, $\delta = -24$,
3) $\mathfrak{B}_2 = (0, 0, 0, 0, 0, 1, 1, 1, 1)$, $\delta = 8$.

For realizations we take the corresponding adjoint groups [1].

CHAPTER III

THE DETERMINATION OF SIMPLE REAL GROUPS OF THE SECOND CATEGORY

§ 16. Preliminary remarks

In the preceding sections we determined the simple real groups of the first category more or less on the lines of Cartan [1] and Lardy [2]. Passing now to groups of the second category we shall however base our deductions on the canonical representation of outer automorphisms and reduce the whole problem to the determination of all outer chief involutive automorphisms [3]. If the component $\mathfrak{A}_i$, in which we are interested, is given by the particular rotation τ_i in the given subgroup $\mathfrak{h}$, then the chief involutive automorphisms in $\mathfrak{A}_i$ may be taken to be of the form

$$\left.\begin{aligned}(Z-\tau_i)\,\mathfrak{h}=0, \quad Ze_\alpha = \varkappa_\alpha e^{(\alpha\widehat{\lambda})} e_\alpha, \\ \varkappa_\alpha^2 = 1, \quad \alpha^* = \tau_i(\alpha), \quad \widehat{\mathfrak{h}} = \mathfrak{h}^{+} \text{ [4]}.\end{aligned}\right\} \tag{128}$$

The signs of the $\varkappa_\alpha$ are the same for all values of the parameters λ_p occuring in $\widehat{\lambda}$.

Put $Z_0 = \{Z\}_{\lambda=0}$. Then from (128) we obtain

$$Z = Z_0 e^{\widehat{H}}. \tag{129}$$

We introduce now parameters φ_p connected with the λ_p by the relation $\lambda = \pi i \varphi$. Then each involutive Z is characterized by a certain system of real values $(\varphi_1, \ldots, \varphi_n)$.

Let us prove the following proposition:

Two involutive automorphisms $Z = Z_0 e^{H_\varphi}$ and $Z' = Z_0 e^{H_{\varphi'}}$ are equivalent (i. e. generate isomorphic real groups), if $h_{\varphi'-\varphi} \subset \mathfrak{h}^-$.

In fact, let $A = e^{H_\chi}$. Then it is easily verified by means of (128) that

$$AZA^{-1} = Z_0 e^{H_{\varphi + \tau_i(\chi) - \chi}}. \tag{130}$$

Since $h_{\varphi'-\varphi} \subset \mathfrak{h}^-$, by an appropriate choice of the real system χ we can achieve that $\tau_i(\chi) - \chi = \varphi' - \varphi$. Then

$$Z' = AZA^{-1}, \tag{131}$$

where A is an automorphism of the compact group.

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[1] Cartan, [2], p. 338.

[1] Cf. Cartan, [6].

[2] Cf. Lardy, [1], p. 209 and f.

[3] Gantmacher, [1], Chapter III.

[4] Gantmacher, [1], § 14. The rotation τ_i, being involutive, has characteristic numbers $+1$ and -1. Accordingly we have the decomposition $\mathfrak{h} = \mathfrak{h}^+ + \mathfrak{h}^-$, where $(\tau_i - 1)\,\mathfrak{h}^+ = 0$ and $(\tau_i + 1)\,\mathfrak{h}^- = 0$.

§ 17. The structure A_n

Chief automorphisms from the component $\mathfrak{A}_1$ may be defined as follows [1]:

$$\left.\begin{aligned} &(Z-\tau_1)\mathfrak{h}=0,\\ Ze_{pq}=(-1)^{p-q-1}e^{\pi i(\varphi_p-\varphi_q)}e_{q_1p_1} \quad &(p, q=1, \ldots, n+1),\\ &(\varphi_p+\varphi_{p_1}=0). \end{aligned}\right\} \tag{132}$$

Here, for the root form $(\alpha\varphi)=\varphi_p-\varphi_q$ we have $e_{pq}=e_\alpha$, and by the subindex $_1$ is denoted the transition to the conjugated index, i. e. to the index, which together with the original one forms one of the pairs

$$\left.\begin{aligned} &(1, 2),\quad (3, 4), \ldots, (2f-1, 2f),\quad (n+1, n+1)\ ^{2},\\ \text{where}\quad & f=\left[\frac{n+1}{2}\right]. \end{aligned}\right\} \tag{133}$$

τ_1 is a particular rotation in $\mathfrak{h}$, defined by the equations

$$\varphi_p^*=-\varphi_{p_1} \qquad (p=1, 2, \ldots, n+1). \tag{134}$$

The equations

$$\varphi_p+\varphi_{p_1}=0 \qquad (p=1, \ldots, n+1) \tag{135}$$

determine the subspace $\mathfrak{h}^+$, and the equations

$$\varphi_p=\varphi_{p_1} \qquad (p=1, \ldots, n+1) \tag{136}$$

the subspace $\mathfrak{h}^-$.

Let now Z be an involutive automorphism, i. e. let $Z^2=E$. Then from (132) it follows that

$$(-1)^{p+p_1+q+q_1}e^{2\pi i(\varphi_p-\varphi_q)}=1. \tag{137}$$

Consider now the two possible cases:

I. $n+1=2f$ is an even number. In this case for any p

$$(-1)^{p+p_1}=1, \tag{138}$$

and hence the equation (127) is equivalent to the condition: all $\varphi_p-\varphi_q$ are integers.

Since we can add to all φ_p one and the same number without affecting the corresponding involutive automorphism Z, we may assume φ_p to be integers reduced to the modulus 2, if we replace the equations

$$\varphi_p+\varphi_{p_1}=0 \qquad (p=1, \ldots, n+1) \tag{139}$$

by the congruences

$$\varphi_p+\varphi_{p_1}\equiv\varphi_q+\varphi_{q_1}\ (\text{mod } 2) \qquad (p, q=1, \ldots, n+1). \tag{140}$$

Observe that if $\psi_p=\psi_{p_1}=1$ and all other $\psi_p=0$, $h_\psi\subset\mathfrak{h}^-$. Hence, in virtue of the proposition formulated in the preceding section, we can add 1 to any pair φ_p, φ_{p_1}; consequently we may confine ourselves to consideration of the following systems φ_p:

$$(0, \ldots, 0) \quad \text{and} \quad (0, 1, 0, 1, \ldots, 0, 1).$$

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[1] Gantmacher, [1], p. 139—140.

[2] This last pair occurs only when $n+1$ is an odd number.

1. $(0, \ldots, 0)$. Let us compute the corresponding δ. The rotation τ_1 itself contributes $n_1 = f = \frac{n+1}{2}$ roots equal to 1 and $\frac{n-1}{2}$ roots equal to -1. Further, all roots fall with respect to τ_1 into $n+1$ monomial cycles (α) and binomial cycles $(\beta\gamma)$. In corresponding invariant subspaces $\mathfrak{R}_\alpha$, $\mathfrak{R}_{\beta\gamma}$

$$Ze_\alpha = e_\alpha \qquad (\mathfrak{R}_\alpha),$$
$$Ze_\beta = \varkappa e_\gamma, \quad Ze_\gamma = \varkappa e_\beta \quad (\varkappa^2 = 1) \quad (\mathfrak{R}_{\beta\gamma}).$$

The binomial cycles contribute thus an equal number of roots $+1$ and -1, while the monomial cycles contribute $n+1$ roots equal to $+1$. Consequently

$$\delta = \mu - \nu = -n - 1.$$

To this case corresponds the linear quaternion group with $\frac{n+1}{2}$ quaternion variables [3].

2. $(0, 1, 0, 1, \ldots, 0, 1)$. In the same way as in case 1 we find here $\delta = n$. The corresponding real structure is realized in the group of linear real unimodular transformations in $n+1$ variables.

II. $n+1 = 2f+1$ is an odd number. In this case in the equation (135) $(n+1)_1 = n+1$, $\varphi_{n+1} = 0$, and so we obtain for any p, q

$$e^{2\pi i \varphi_p} = -1, \quad e^{2\pi i (\varphi_p - \varphi_q)} = 1 \qquad (p, q = 1, \ldots, n),$$

i. e. $\varphi_p = \frac{k_p}{2}$, where k_p are odd numbers, $\varphi_p - \varphi_q$ are integers $(p, q = 1, \ldots, n)$,

$$\varphi_{n+1} = 0.$$

As in the preceding case we can replace for $p \neq n+1$ any pair φ_p, φ_{p_1} by $\varphi_p + 1$, $\varphi_{p_1} + 1$. Then we have to consider only one system, namely

$$\left(\frac{1}{2}, -\frac{1}{2}, \ldots, \frac{1}{2}, -\frac{1}{2}, 0\right).$$

In this case the binomial cycles contribute again an equal number of roots $+1$ and -1, while the monomial cycles give the relations

$$Ze_\alpha = e^{2\pi i \varphi_p} = -e_\alpha.$$

Therefore we have in this case $\delta = n$.

This real structure is realized, like that of the preceding case, in the group of real linear unimodular transformations in $n+1$ variables [4].

§ 18. The structure D_n

Consider first the component $\mathfrak{A}_1$ (for $n = 4$ we have beside this component the components $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$, $\mathfrak{A}_5$). The chief automorphisms Z from $\mathfrak{A}_1$ have the form [1]:

$$\left.\begin{aligned} &(Z - \tau_1)\mathfrak{h} = 0, \\ &Ze_{pq} = e^{\pi i (\varphi_p + \varphi_q)} e_{pq} \quad (p, q = \pm 2, \ldots, \pm n), \\ &Ze_{pq} = e^{\pi i (\varphi_p + \varphi_q)} e_{p_1 q} \quad (p = -p_1 = \pm 1,\ q = \pm 2, \ldots, \pm n), \\ &\varphi_1 = 0,\ \varphi_{-p} = -\varphi_p \quad (p = 1, \ldots, n). \end{aligned}\right\} \tag{141}$$

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[3] Cartan, [2], p. 273—274.

[4] Ibid, p. 276.

[1] Gantmacher, [1], p. 140—141.

Here the rotation τ_1 in $\mathfrak{h}$ is defined by

$$\varphi_1^* = -\varphi_1, \quad \varphi_q^* = \varphi_q \qquad (q = 2, \ldots, n). \tag{142}$$

The subspace $\mathfrak{h}^+$ consists of all vectors h, for which $\varphi_1 = 0$, and the subspace $\mathfrak{h}^-$ of all vectors h, for which $\varphi_2 = \ldots = \varphi_n = 0$.

Since Z must be involutive, from (140) it follows that all φ_p have integral values. The involutive automorphism will not be affected, if we add one and the same number to all φ_p (we omit the condition $\varphi_1 = 0$) or change the sign of some of the φ_p. Therefore we may confine the values of φ_p to 0 and 1. Any permutation of the numbers $\varphi_2, \ldots, \varphi_n$ gives an equivalent chief automorphism Z'.

In fact, such substitution τ may be completed to a certain automorphism A of the compact group [2]:

$$\left. \begin{array}{l} (A - \tau)\,\mathfrak{h} = 0, \\ Ae_\alpha = \mu_\alpha e_\alpha \quad (\mu_\alpha^2 = 1). \end{array} \right\} \tag{143}$$

Then $Z' = AZA^{-1}$. Besides, in virtue of the remark in § 16, φ_1 may be replaced by any number. Therefore we may confine ourselves to the systems

$$\mathfrak{B}_l = (0, \ldots, 0, \underbrace{1, \ldots, 1}_{l}) \qquad (l = 0, 1, \ldots, n).$$

Since the transformation $\varphi' = 1 - \varphi$ does not affect the involutive automorphism, we may confine l to the values

$$l = 0, 1, \ldots, \left[\frac{n}{2}\right]. \tag{144}$$

Computing δ for $\mathfrak{B}_l$, we obtain

$$\delta = n - 2m^2,$$

where

$$m = n - 2l + 1 = n - 1,\ n - 3, \ldots, \begin{cases} 0 \\ 1 \end{cases}.$$

These structures are realized in groups of linear real transformations in $2n$ variables, leaving invariant the indefinite quadratic forms

$$x_1^2 + \ldots + x_{2l+1}^2 - \ldots - x_n^2 \qquad \left(l = 0, 1, \ldots, \left[\frac{n}{2}\right]\right)\ ^{3}.$$

Passing now to the case $n = 4$, we observe that here we shall have, beside the components $\mathfrak{A}_0$ and $\mathfrak{A}_1$, the components $\mathfrak{A}_2, \mathfrak{A}_3, \mathfrak{A}_4, \mathfrak{A}_5$. The particular rotations $\tau_2, \tau_3, \tau_4, \tau_5$, corresponding to these components, may be chosen in such a way [4] that

$$\tau_3 = \tau_2^2, \quad \tau_4 = \tau_2\tau_1 = \tau_1\tau_2^2, \quad \tau_5 = \tau_1\tau_2 = \tau_2^2\tau_1. \tag{145}$$

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[2] Loc. cit., p. 130.

[3] Cartan, [2], p. 285 and f.

[4] Loc. cit., p. 285 and f.

Observe that $\tau_2^2 \neq 1$ and $\tau_3^2 \neq 1$. Hence involutive automorphisms will exist only in $\mathfrak{A}_4$ and $\mathfrak{A}_5$. But from (145) follows

$$\left.\begin{aligned} &\tau_5 = \tau_2 \cdot \tau_2 \tau_1 = \tau_2 \tau_1 \tau_2^2 = \tau_2 \tau_1 \tau_2^{-1} \\ \text{and} \quad & \\ &\tau_4 = \tau_1 \tau_2 \tau_2 = \tau_2^{-1} \tau_1 \tau_2 . \end{aligned}\right\} \tag{146}$$

Completing τ_2 to an automorphism A_2 of the compact group, we shall have

$$\mathfrak{A}_5 = A_2 \mathfrak{A}_1 A_2^{-1} \quad \text{and} \quad \mathfrak{A}_4 = A_2^{-1} \mathfrak{A}_1 A_2, \tag{147}$$

i. e. $\mathfrak{A}_4$ and $\mathfrak{A}_5$ do not yield new real structures.

§ 19. The structure E_6

The chief outer automorphisms Z for E_6 may be defined in the following way [1]:

$$\left.\begin{aligned} &(Z - \tau_1)\,\mathfrak{h} = 0, \\ Ze_{pq} &= (-1)^{p-q+1} e^{\pi i (\varphi_p - \varphi_q)} e_{q_1 p_1}, \\ Ze_{pqs} &= e^{\pi i (\varphi_p + \varphi_q + \varphi_s)} e_{pqs}, \text{ if } p < 3 \leqslant q < 5 \leqslant s, \\ Ze'_{pqs} &= e^{-\pi i (\varphi_p + \varphi_q + \varphi_s)} e'_{pqs}, \text{ if } p < 3 \leqslant q < 5 \leqslant s, \\ Ze_{pp_1 s} &= -e^{\pi i (\varphi_p + \varphi_{p_1} + \varphi_s)} e_{qq_1 s} \quad (s \neq p, p_1), \\ Ze'_{pp_1 s} &= -e^{\pi i (\varphi_p + \varphi_{p_1} + \varphi_s)} e_{qq_1 s} \quad (s \neq p, p_1), \\ Ze_0 &= e^{\pi i \varphi_0} e_0,\ Ze'_0 = e^{-\pi i \varphi_0} e_0,\ \varphi_0 = \sum_1^6 \varphi_p, \\ &\varphi_1 + \varphi_2 = \varphi_3 + \varphi_4 = \varphi_5 + \varphi_6 \\ &(p, q = 1, 2, \ldots, 6). \end{aligned}\right\} \tag{148}$$

By p_1 we denote the index conjugated with p, i. e. the index, which together with p forms one of the pairs

$$(1, 2),\ (3, 4),\ (5, 6).$$

The root forms are here

$$\left.\begin{aligned} &\varphi_p - \varphi_q, \quad \pm(\varphi_p + \varphi_q + \varphi_s), \quad \pm\sum_1^6 \varphi_p \\ &(p, q, s = 1, \ldots, 6). \end{aligned}\right\} \tag{149}$$

The corresponding vectors e_α we denote here by

$$e_{pq},\ e_{pqs} = -e_{qps} = \ldots,\ e'_{pqs} = -e'_{qps} = \ldots,\ e_0,\ e'_0.$$

The particular rotation τ_1 is defined by the equation

$$\varphi_p^* = -\varphi_{p_1} + \frac{1}{3}\sum_1^6 \varphi_q. \tag{150}$$

[1] Gantmacher, [1], p. 143.

Here $\mathfrak{h}^+$ is determined by the equations

$$\varphi_1 + \varphi_2 = \varphi_3 + \varphi_4 = \varphi_5 + \varphi_6 \tag{151}$$

and $\mathfrak{h}^-$ by the equations

$$\varphi_1 = \varphi_2, \ \varphi_3 = \varphi_4, \ \varphi_5 = \varphi_6, \ \sum_1^6 \varphi_p = 0. \tag{152}$$

Suppose now that Z is an involutive automorphism, i. e. that $Z^2 = E$. Then all root forms (149) must have integral values. Hence

$$\varphi_p = \psi_p + \varepsilon \qquad (p = 1, \ldots, 6),$$

where $\varepsilon = 0, \ -\frac{1}{3}, \frac{1}{3}$.

Since we can add to all φ_p any of the numbers $\frac{2}{3}, \ -\frac{2}{3}$ without affecting Z, we may assume that the φ_p are integers. We may further reduce the φ_p to the modulus 2, replacing at the same time the equations (151) by the congruences

$$\varphi_1 + \varphi_2 \equiv \varphi_3 + \varphi_4 \equiv \varphi_5 + \varphi_6 \pmod 2. \tag{153}$$

It is easily seen that for any two pairs of conjugated indices p, p_1 and q, q_1 we may replace φ_p, φ_{p_1}, φ_q, φ_{q_1} by $\varphi_p + 1$, $\varphi_{p_1} + 1$, $\varphi_q - 1$, $\varphi_{q_1} - 1$ (without changing the two remaining φ_s, φ_{s_1}), since the system $(\psi_1, \ldots, \psi_6)$, where $\psi_p = +1$, $\psi_{p_1} = -1$, $\psi_q = +1$, $\psi_{q_1} = -1$, all other $\psi = 0$, satisfies the equations (152) and the addition of this system to the system φ does not disturb the validity of the congruences (153). Therefore we have to consider only the following four systems:

$$\mathfrak{V}_1 = (0, \ldots, 0), \quad \mathfrak{V}_2 = (1, \ldots, 1),$$
$$\mathfrak{V}_3 = (0, 1, 0, 1, 0, 1), \quad \mathfrak{V}_4 = (0, 1, 0, 1, 1, 0).$$

We shall show that the systems $\mathfrak{V}_3$ and $\mathfrak{V}_4$ may be omitted. Indeed, take, for instance, the system $\mathfrak{V}_3$. Denote by Z_3 the corresponding automorphism. Take two root forms

$$(\rho\varphi) = \varphi_1 + \varphi_3 + \varphi_5 \quad \text{and} \quad (\omega\varphi) = \varphi_1 + \varphi_2 + \cdots + \varphi_6$$

and observe that

$$Z_3 e_\rho = e_\rho, \ Z_3 e_\omega = -e_\omega. \tag{154}$$

Consider now the mapping σ_α, with $(\alpha\varphi) = \varphi_2 + \varphi_4 + \varphi_6$, effecting mirror images

$$\left.\begin{aligned} \varphi^*_{2p} &= \varphi_{2p} - \frac{2}{3}(\varphi_2 + \varphi_4 + \varphi_6), \\ \varphi^*_{2p-1} &= \varphi_{2p-1} + \frac{1}{3}(\varphi_2 + \varphi_4 + \varphi_6) \end{aligned}\right\} \tag{155}$$

and complete σ_α to an inner automorphism U of the compact group, for which [2]

$$(U - \tau_1)\mathfrak{h} = 0, \quad Ue_\alpha = \pm e_{\alpha^*}. \tag{156}$$

[2] Ibid., p. 130.

Consider the automorphism

$$A_3 = UZ_3U^{-1}. \tag{157}$$

Since the rotation σ_α interchanges the roots ρ and ω, from (154) we find

$$A_3 e_\omega = e_\omega. \tag{158}$$

On the other hand, the rotations σ_α and τ_1 are commutable [3]. The permutation σ_α transforms therefore a cycle of the permutation τ_1 again into a cycle and so permutes the cycles of τ_1 among themselves. From (157) it then follows that the automorphism A_3 realizes in $\mathfrak{h}$ the rotation τ_1 and consequently has the same invariant subspaces $\mathfrak{R}_\alpha$, $\mathfrak{R}_{\beta\gamma}$ as Z_3. Moreover, in each of the $\mathfrak{R}_{\beta\gamma}$,

$$A_3 e_\beta = \mu_\beta e_\gamma, \; A_3 e_\gamma = \mu_\gamma e_\beta, \; \mu_\beta = \mu_\gamma, \tag{159}$$

i. e. A_3 is commutable with Z and consequently is itself one of the Z. Thus A_3 can be obtained from Z for a certain system of the φ_p. But the equation (158) shows that the sum of these φ_p is even, so that this system may be reduced to one of the systems

$$(0, \dots, 0) \quad \text{and} \quad (1, \dots, 1).$$

Consider the system $(0, \dots, 0)$. It can be easily calculated that for it $\delta = -26$. The corresponding simple real structure may be realized in the group of linear transformations in 27 complex variables x_p, y_q, $z_{pq} = -z_{qp}$ $(p, q = 1, \dots, 6)$, leaving invariant the cubic form

$$\sum_{p,q} x_p y_q z_{pq} + \sum_{p,\dots,v} (p, q, s, t, u, v)\, z_{pq} z_{st} z_{uv}, \tag{160}$$

where the variables are subject to the following conditions:

$$y_{2p-1} = \bar{x}_{2p}, \quad y_{2p} = -\bar{x}_{2p-1},$$
$$z_{2p-1,2q-1} = \bar{z}_{2p,2q},$$
$$z_{2p-1,2q} = \bar{z}_{2q-1,2p}.$$

For the system $(1, \dots, 1)$ we find $\delta = 6$. To this system corresponds the group of linear real transformations in 27 variables x_p, y_q, z_{pq} $(z_{pq} = -z_{qp}$, $p, q = 1, \dots, 6)$, leaving invariant the cubic form (160) [4].

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[3] This follows from the fact that $\tau_1 \sigma_\alpha \tau_1^{-1} = \sigma_\beta$, where $\beta = \tau_1(\alpha)$; in fact, since $h_\alpha \subset \mathfrak{h}^+$, we have $\beta = \alpha$ and $\tau_1 \sigma_\alpha = \sigma_\alpha \tau_1$.

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О классификации простых вещественных групп Ли

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(Резюме)

Киллинг [1] и Картан [1] дали классификацию всех простых комплексных групп Ли. После этого определение всех неизоморфных простых вещественных групп Ли свелось к нахождению различных вещественных форм данной простой комплексной группы. Эта проблема была решена Картаном [2] в 1914 г., но весьма громоздким и, в известной степени, кустарным методом. Картан перебирает различные простые комплексные структуры и в пределах каждой структуры оперирует специфическими для этой структуры приемами. В 1929 г.

Картан [6] установил изящную теорему, дающую общий подход к нахождению простых вещественных групп Ли. Хотя сама теорема имеет непосредственный алгебраический характер, доказательство ее у Картана тесно связано с развитой им теорией специальных римановых пространств. В этой же работе Картан показывает, каким образом каноническое представление внутренних автоморфизмов простой компактной группы Ли может быть использовано для нахождения простых вещественных групп. Но отсутствие аналогичного представления для внешних автоморфизмов не дает ему возможности применить свой метод к некоторым комплексным структурам, например, к E_6. Этот пробел был восполнен в работе Ларди [1] несколько обходным и сложным путем.

В главе I настоящей работы дается алгебраическое доказательство основной теоремы Картана. При этом существенно используется установленное автором в предыдущей работе [1] каноническое представление автоморфизмов простой комплексной группы Ли, имеющих простые элементарные делители.

Попутно получается доказательство замечательного предложения Картана о связи между топологической структурой комплексной простой группы Ли и структурой ее вещественной компактной формы. Здесь же устанавливается каноническое представление внешних автоморфизмов компактной простой группы Ли. В дальнейших главах (II и III) все это используется для непосредственного и сравнительно несложного проведения классификации простых вещественных групп Ли.