Exterior Differential Systems with Symmetry

Mark Fels, Utah State University

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Definition. *An exterior differential system is a subset of $\mathcal{I} \subset \Omega^*(M)$, which is closed under $\Omega^*(M)\wedge$ and exterior differentiation*

$$\mathcal{I} = <\theta^i, d\theta^i, \beta^a, d\beta^a, \ldots, >$$

where $< \ >$ means algebraically generated by. The EDS that will be of interested are mainly (but not exclusively) **Pfaffian systems** - those generated by one-forms and their derivatives.

$$\mathcal{I} = <\theta^i,\ d\theta^i>$$

Definition. *An integral manifold of $\mathcal{I}$ is an immersion $s : N \to M$ such that $s^*\mathcal{I} = 0$.*

Definition. *A symmetry of an exterior differential system is a diffeomorphism $\phi : M \to M$ such that $\phi^*\mathcal{I} = \mathcal{I}$. A symmetry group of $\mathcal{I}$ will be a Lie group G acting smoothly on M where each diffeomorphism $g : M \to M$ is a symmetry of $\mathcal{I}$.*

If ϕ is a symmetry of the EDS $\mathcal{I}$ and $s : N \to M$ is an integral manifold of $\mathcal{I}$, then

$$(\phi \circ s)^* \mathcal{I} = s^* \phi^* \mathcal{I} = s^* \mathcal{I} = 0.$$

Therefore symmetries map integral manifolds to integral manifolds.

From now on:

1) G is a Lie group acting smoothly on M

2) **g** Lie algebra of right invariant vector-fields

3) $\boldsymbol{\gamma}$ - Lie algebra of infinitesimal generators

4) ρ : **g** $\to \boldsymbol{\gamma}$ the homomorphism

3) $\mathbf{\Gamma} \subset TM$ is the point-wise span of γ

Differential Equations give rise to EDS, and solutions to the differential equations are integral manifolds. Next are some example....

Example 1. The Chazy equation

$$y_{xxx} = 2yy_{xx} - 3y_x^2$$

gives rise to the EDS on $M = (x, y, y_x, y_{xx})$

$$\mathcal{I} = < dy - y_x dx,\ dy_x - y_{xx} dx, dy_{xx} - (2yy_{xx} - 3y_x^2)dx >$$

Solutions to the Chazy equation are integral manifolds.

The EDS is invariant with respect to the (local) action of $SL(2)$ on M:

$$\widehat{x} = \frac{ax+b}{cx+d}, \quad \widehat{y} = (cx+d)^2 y + 6c(cx+d)$$

$$\widehat{y}_{\widehat{x}} = \frac{d\widehat{y}}{dx}\ \frac{dx}{d\widehat{x}} \quad \widehat{y}_{\widehat{x}\widehat{x}} = \frac{d\widehat{y}_{\widehat{x}}}{dx}\ \frac{dx}{d\widehat{x}}$$

where $ad - bc = 1$.

Example 2. The geodesic equation for

$$\eta = e^{-\frac{4}{3}x_4}(dx_1dx_3 - dx_2dx_2) + e^{\frac{2}{3}x_4}dx_3dx_3 + c dx_4dx_4$$

where $c < 0 - Lorentz$ and $c > 0 - split\ sig.$.

The EDS on $M = (t, x_i, \dot{x}_i), 1 \le i \le 4$ is

$$\mathcal{I} = < dx_i - \dot{x}_i dt,$$
$$d\dot{x}_1 - \frac{2}{3}\dot{x}_4(2\dot{x}_1 - 3\dot{x}_3 e^{2x_4})dt$$
$$d\dot{x}_2 - \frac{4}{3}\dot{x}_2\dot{x}_3 dt, \qquad d\dot{x}_3 - \frac{4}{3}\dot{x}_3\dot{x}_4 dt$$
$$d\dot{x}_4 + \frac{1}{3c}\left(4e^{-\frac{4}{3}x_4}(\dot{x}_1\dot{x}_3 + 2\dot{x}_2^2) + e^{\frac{2}{3}x_4}\dot{x}_3^2\right)dt >$$

The geodesics are integral manifolds.

The EDS is invariant with respect to time translations and the isometry group (5-d solvable) whose Killing vector-fields are

$$X_1 = \partial_{x_1}, X_2 = \partial_{x_2}, \ X_3 = \partial_{x_3},$$
$$X_4 = x_2\partial_{x_1} + x_3\partial_{x_2}$$
$$X_5 = -5x_1\partial_{x_1} - 2x_2\partial_{x_2} + x_3\partial_{x_3} - 3\partial_{x_4}$$

Example 3. The contact system on $M = J^2(I\!R, I\!R^2)$ with coordinates $(t, x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y})$:

$$\mathcal{I} = < dx - \dot{x}dt, dy - \dot{y}dy, d\dot{x} - \ddot{x}dt, d\dot{y} - \ddot{y}dt,$$
$$d\ddot{x} \wedge dt, \;\; d\ddot{y} \wedge dt >$$

Any prolonged graph $(x(t), y(t))$ is an integral manifold.

The $E^+(2)$ action

$$\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\begin{pmatrix} \widehat{\dot{x}} \\ \widehat{\dot{y}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$
$$\begin{pmatrix} \widehat{\ddot{x}} \\ \widehat{\ddot{y}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

is a symmetry group of $\mathcal{I}$.

Example 4. The contact system on $M = J^2(\mathbb{R},\mathbb{R})\times J^2(\mathbb{R},\mathbb{R})$: $(x,u,u_x,u_{xx},y,v,v_y,v_{yy})$

$$\mathcal{I} = < du - u_x dx, du_x - u_{xx}dx, du_{xx}\wedge dx,$$
$$dv - v_y dy, dv_y - v_{yy}dy, dv_{yy}\wedge dy >$$

This is also the system of PDE $u_y = 0, v_x = 0$ prolonged. Any prolonged graph $u = f(x), v = g(y)$ is an integral manifold.

Consider local symmetries of $SL(2,\mathbb{R})$

$$\hat{u} = \frac{au+b}{cu+d}, \qquad \hat{v} = \frac{av+b}{cv+d}$$
$$\hat{u}_x = \frac{u_x}{(cu+d)^2}, \qquad \hat{v} = \frac{v_y}{(cv+d)^2}$$
$$\hat{u}_{xx} = \frac{u_{xx}}{(cu+d)^2} - \frac{2cu_x^2}{(cu+d)^3}$$
$$\hat{v}_{yy} = \frac{v_{yy}}{(cv+d)^2} - \frac{2cv_y^2}{(cv+d)^3}$$

where $ad - bc = 1$.

Suppose $\mathbf{q} : M \to M/G = \bar{M}$ is a submersion.

Then $\mathbf{\Gamma} \subset TM$ is a rank q-subbundle, where q is the dimension of the orbits. Also

$$\ker \mathbf{q}_* = \mathbf{\Gamma}$$

Definition. *The quotient* $\bar{\mathcal{I}} \subset \Omega^*(\bar{M})$ *of* $\mathcal{I}$ *is*

$$\bar{\mathcal{I}} = \{ \ \bar{\theta} \in \Omega^*(\bar{M}) \mid \mathbf{q}^* \bar{\theta} \in \mathcal{I} \ \}$$

This definition is difficult to work with without any assumptions on the EDS.

One invariant condition is constant rank.

If the Pfaffian system $\mathcal{I}$ is constant rank r then $\mathcal{I}^1 = \mathcal{I} \cap \Omega^1(M)$ are sections of a rank r subbundle $I \subset T^*M$.

The symmetries of $\mathcal{I}$ then preserve I: $g^*I = I$.

The quotient $\bar{I} \subset T^*\bar{M}$ defined point-wise is

$$\bar{I}_{\bar{x}} = \{\ \bar{\theta} \in T^*_{\bar{x}}\bar{M} \mid \mathbf{q}^*\bar{\theta}_{\bar{x}} \in I_x \text{ where } \mathbf{q}(x) = \bar{x}\ \}$$

This can be computed using $I^{\mathbf{An}} \subset TM$

$$I_x^{\mathbf{An}} = \{V \in T_xM \mid \theta(V) = 0\ \forall\ \theta \in I_x\}$$

which is a rank $n - r$ sub-bundle. Then

$$\bar{I} = \left(\mathbf{q}_*(I^{\mathbf{An}})\right)^{\mathbf{An}}$$

$\bar{I}$ is a sub-bundle if and only if

$$\dim I_x^{\mathbf{An}} \cap \ker \mathbf{q}_* = \dim I_x^{\mathbf{An}} \cap \mathbf{\Gamma}_x = k, \quad \forall x \in M,$$

in which case rank $\bar{I} = r + k - q$, with rank $\mathbf{\Gamma} = q$.

Computing

$$I_x^{\mathbf{An}} \cap \mathbf{\Gamma}_x = \{Z \in \mathbf{\Gamma}_x \mid \theta(Z) = 0, \ \forall\ \theta \in I_x\}$$

in the basis θ^α for I_x, and Z_i a basis $\mathbf{\Gamma}_x$ is

$$I_x^{\mathbf{An}} \cap \mathbf{\Gamma}_x = \ker \theta^\alpha(Z_i). \qquad (1)$$

How to compute $\bar{\mathcal{I}}$ (or $\bar{I}$)?

1) Choose $\sigma : \bar{M} \to M$ a cross-section

2) Compute the semi-basic forms in $\mathcal{I}$ (or I):

$$I_{x,sb} = \{\theta \in I_x \mid \theta(Z) = 0 \ \forall\ Z \in \mathbf{\Gamma}_x\}$$

(the kernel in (1) on the form side)

3) The pullback $\sigma^*\theta_{sb} \in \bar{\mathcal{I}}$, and generate $\bar{\mathcal{I}}$.

$\bar{I}$ being constant rank is also equivalent to I_{sb} being a constant rank sub-bundle'.

Observations:

Theorem. *If $\mathcal{I}$ is a rank r completely integrable Pfaffian system, and* rank $I^{\mathbf{An}} \cap \mathbf{\Gamma} = k$, *then $\bar{\mathcal{I}}$ is a rank $r+k-q$ completely integrable Pfaffian system.*

If rank $I^{\mathbf{An}} \cap \mathbf{\Gamma} = k$ but $\mathcal{I}$ is not completely integrable, then $\bar{\mathcal{I}}$ is not necessarily a Pfaffian system. It is possible to give sufficient conditions so that $\bar{\mathcal{I}}$ is Pfaffian system.

Theorem. *Suppose $\mathcal{I}$ is a constant rank Pfaffian system invariant with respect to G and $Z_{x_0} \in I^{\mathbf{An}}_{x_0} \cap \mathbf{\Gamma}_{x_0}$ with $Z_{x_0} \neq 0$. Then $e^{t\mathbf{z}}x_0$, $\mathbf{z} \in \mathbf{g}$ is a one-dimensional integral manifold, where $Z_{x_0} = \rho(\mathbf{z})_{x_0}$.*

Example 1. $SL(2,\mathbb{R})$, Chazy. The EDS is

$$\theta^1 = dy - y_x dx,$$
$$\theta^2 = dy_x - y_{xx} dx,$$
$$\theta^3 = dy_{xx} - (2yy_{xx} - 3y_x^2)dx.$$

A set of generators for γ are,

$$X_1 = \partial_x,$$
$$X_2 = 2x\partial_x - 2y\partial_y - 4y_x\partial_{y_x} - 6y_{xx}D_{y_{xx}} - 8y_{xxx}\partial_{y_{xxx}}$$
$$X_3 = -x^2\partial_x + 2(xy+3)\partial_y + 2(2xy_x + y)\partial_{y_x}$$
$$+ 6(y_x + xy_{xx})D_{y_{xx}} + (12y_{xx} + 8xy_{xxx})\partial_{y_{xxx}}$$

The determinant $\det(\theta^i(X_j)) = 0$ if and only if

$$y_{xx} = yy_x - \frac{1}{9}y^3 \pm (y^2 - 6y_x)^3.$$

For initial conditions $(x^0, y^0, y_x^0, y_{xx}^0)$ satisfying this constraint , the (unique) solution is the one-dimensional orbit:

$$x = x^0 + 2t\frac{(\delta^0 + 3y^0_x + y^0\sqrt{\delta^0})}{ty^0_x(y^0 + \sqrt{\delta^0} - 1)},$$
$$y = y^0 + 2ty^0_x\left((3t(y^0_x)^2 - y^0 + ty^0_x\delta)\sqrt{\delta} + ty^0_x y^0\delta\right)$$

where $\delta_0 = (y^0)^2 - 6y^0_x$.

There is a 2 parameter family of invariant solutions. (The solutions as a graph are easily found).

Example 2. Geodesics: The EDS is

$$\mathcal{I} = < dx_i - \dot{x}_i dt, d\dot{x}_1 - \frac{2}{3}\dot{x}_4(2\dot{x}_1 - 3\dot{x}_3 e^{2x_4})dt$$

$$d\dot{x}_2 - \frac{4}{3}\dot{x}_2\dot{x}_3 dt, \qquad d\dot{x}_3 - \frac{4}{3}\dot{x}_3\dot{x}_4 dt$$

$$d\dot{x}_4 + \frac{1}{3c}\left(4e^{-\frac{4}{3}x_4}(\dot{x}_1\dot{x}_3 + 2\dot{x}_2^2) + e^{\frac{2}{3}x_4}\dot{x}_3^2\right)dt >$$

At the point

$$t = 0,\ \mathbf{x} = (0,0,0,0),\ \dot{\mathbf{x}} = (\frac{c_0}{4}, 0, c_0, 0), \quad (2)$$

where $c_0 \neq 0$ the vector-field $X \in \gamma$,

$$X = \partial_t + \frac{c_0}{4}X_1 + c_0 X_3 = \partial_t + \frac{c_0}{4}\partial_{x_1} + c_0\partial_{x_3}$$

satisfies $\theta(X) = 0,\ \forall\, \theta \in \mathcal{I}$ at (2). The integral curve of X in M through the point (2)

$$x_1 = \frac{c_0}{4}t,\ x_2 = 0,\ x_3 = c_0 t,\ x_4 = 0$$

and is a geodesic which is the orbit of a one parameter sub-group corresponding to X.

Homogeneous geodesics always exist for homogeneous Riemannian manifolds.

Example 5. The standard EDS for the differential equation

$$u_{xxxxx} = \frac{5u_{xxx}(9u_{xxxx}u_{xx} - 8u_{xxx}^2)}{9u_{xx}^2}$$

Is invariant with respect to the five dimensional symmetry group

$$G = SA(2) = \{\ (A, b) \mid A \in SL(2, \mathbb{R}),\ b \in \mathbb{R}^2\ \},$$

acting on (x, u) and then prolonged. Every solution is the orbit of a one-parameter subgroup, giving the general solution,

$$u = c_0 + c_1 x \pm \sqrt{c_3 x^2 + c_x x + c_2}(4c_3 c_2 - c_x^2).$$

Example 6. Every geodesic on a Riemannian symmetric space is homogeneous.

Remark: It is sometimes possible to generate integral manifolds with dimension > 1 when rank $I_x^{\mathbf{An}} \cap \mathbf{\Gamma}_x > 1$.

Why the quotient?

1) We want to split the problem of finding integral manifolds for $\mathcal{I}$ into finding integral manifolds for $\bar{\mathcal{I}}$, and then build integral manifolds utilizing G. (This second part is sometimes called the reconstruction problem).

2) Classify the integral manifolds that are inequivalent with respect to G.

3) The inverse problem: Use $\mathcal{I}$ to simplify finding integral manifolds to $\bar{\mathcal{I}}$.

Let's look at problem 1. In order to implement the decomposition we need to partition M into G-invariant subsets on which we can control the behavior of the quotient. The key is:

$$I_x^{\textsf{\textbf{An}}} \cap \mathbf{\Gamma}_x$$

Consider the two G-invariant subsets

$$K = \{ \ x \in M \mid \mathbf{\Gamma}_x \subset I_x^{\textsf{\textbf{An}}} \ \}$$

and the transverse subset

$$M^t = \{ \ x \in M \mid \mathbf{\Gamma}_x \cap I_x^{\textsf{\textbf{An}}} = 0 \ \}.$$

What can be said about $\mathcal{I}$ and $\bar{\mathcal{I}}$ and the reconstruction problem on these subsets? (which can be empty)

We start with K. Assume

1) $\iota : K \to M$ is an embedded submanifold,

2) the action of G on K is regular: $\mathbf{q} : K \to \bar{K}$

Let $\mathcal{I}_K = \iota^* \mathcal{I}$, which under obvious conditions is a constant rank Pfaffian system (rank depends on the embedding). In this case $\bar{\mathcal{I}}$ is a constant rank Pfaffian system with the same rank as $\mathcal{I}_K$.

Note that at each point in K, all forms in $\mathcal{I}_K$ are semi-basic!

An integral manifold $s : N \to M$ is G invariant if $gs(N) = N$. These can be found as integral manifolds to $\mathcal{I}_K$. Here's how:

Theorem. *Let $\bar{N} \subset M$ be an embedded integral manifold of $\bar{\mathcal{I}}_K$. Then $N = \mathbf{q}^{-1}(\bar{N}) \subset M$ is a G-invariant integral manifold of $\mathcal{I}$.*

Proof: N is clearly G-invariant. So need to show it is an integral manifold.

Let $x \in N$, $X \in T_x N$, and $\bar{x} = \mathbf{q}(x)$, $\bar{X} = \mathbf{q}_* X$. Note $\bar{x} \in \bar{N}$ and $\bar{X} \in T_{\bar{x}} \bar{N}$.

Choose and open set $\bar{U} \subset \bar{K}$ containing $\bar{x}$, and a cross-section $\sigma : \bar{M} \to M$ with $\sigma(\bar{x}) = x$. Then

$$X = \sigma_* \bar{X} + V$$

for some $V \in \mathbf{\Gamma}_x$. Evaluating on $\theta \in \mathcal{I}$,

$$\begin{aligned} \theta(X) &= \theta(\sigma_* \bar{X}_{\bar{x}} + V) \\ &= \sigma^* \theta(\bar{X}) + \theta(V) \\ &= 0. \end{aligned}$$

The first term vanishes because, all one-forms in $\mathcal{I}_K$ are semi-basic so pullback to $\bar{\mathcal{I}}_K$, and $\bar{N}$ is an integral manifold. The second term vanishes because we are at point of K (so that $V \in I_x^{\textbf{An}}$). QED.

Remarks:

1) The reconstruction problem is algebraic.

2) Integral manifolds of $\mathcal{I}_K$ can always be enlarged (locally) to be invariant.

3) K is the subset of M on which $\mathbf{\Gamma}$ are Cauchy-characteristics for $\mathcal{I}_K$. This theorem is not so surprising.

The transverse subset M^t.

1) Integral manifolds in M^t don't have continuous symmetry.

2) For $s : N \to M$, an integral manifold $\mathbf{q} \circ s : N \to \bar{M}^t$ is an integral manifold. (Ie. immersion property still holds).

If M^t is non-empty then the restriction

$$\mathcal{I}_{M^t} = \mathcal{I}|_{M^t}$$

has the same rank as $\mathcal{I}$, but the quotient $\bar{\mathcal{I}}_{M^t}$ is not necessarily a Pfaffian system. (It is similar though.)

The reconstruction problem is:

Theorem. *Let $\bar{N} \to \bar{M}^t$ be an embedded integral manifold of $\bar{\mathcal{I}}_{M^t}$. Then $\mathcal{I}|_{\mathbf{q}^{-1}(\bar{N})}$ is completely integrable, and the leaves are integrable manifolds of $\mathcal{I}$.*

As a consequence of this theorem, the integral manifolds of $\mathcal{I}$ are surjective via. **q** onto the integral manifolds of $\bar{\mathcal{I}}_{M^t}$.

If the action of G on M^t is free, there is a nice geometric way to think about the reconstruction problem.

Let $s : \bar{N} \subset \bar{M}^t$ be an integral manifold of $\bar{\mathcal{I}}_{M^t}$, and let $\widehat{s} : \bar{N} \to M$ be any cover of $\bar{N}$. The integral manifold N of $\mathcal{I}$ which projects to $\bar{N}$ is of the form

$$s(t) = \mu(A(t), \widehat{s}(t))$$

where $A : \bar{N} \to G$.

In order for $s(t)$ to be an integral manifold of $\mathcal{I}$, $A(t)$ satisfies a generalized equation of Lie type. These are integrable by quadratures for (s.c) solvable Lie groups.

Example 3. $E^+(2)$ symmetry on $J^2(\mathbb{R},\mathbb{R}^2)$ with the standard contact structure:

$$\mathcal{I} = < dx - \dot{x}dt, dy - \dot{y}dy, d\dot{x} - \ddot{x}dt, d\dot{y} - \ddot{y}dt,$$
$$d\ddot{x} \wedge dt, \ \ddot{y} \wedge dt >$$

(Curves $(x(t), y(t))$ are integral manifolds)

The infinitesimal generators are

$\gamma =$ span $\{ \partial_x, \ \partial_y, \ x\partial_y - y\partial_x + \dot{x}\partial_{\dot{y}} - \dot{y}\partial_{\dot{x}} + \ddot{x}\partial_{\ddot{y}} - \ddot{y}\partial_{\ddot{x}} \}$

and $E^+(2)$ action is transverse at

$$M^t = \{\sigma \in J^2(\mathbb{R},\mathbb{R}^2) \mid (\dot{x}, \dot{y}) \neq (0,0)\}$$

The quotient is 4 dimensional $\bar{M}^t = (t, v, k_1, k_2)$.

Let $\sigma : \bar{M}^t \to M$ be the cross-section,

$$(t, x = 0, y = 0, \dot{x} = 0, \dot{y} = v, \ddot{x} = k_1, \ddot{y} = k_2).$$

The quotient EDS is (pullback semi-basic forms)

$$\bar{\mathcal{I}}_{M^t} = < dv - k_2 dt, dk_2 \wedge dt, (k_1 dk_1 + k_2 dk_2) \wedge dt >$$

A typical integral manifold for $\bar{\mathcal{I}}_{M^t}$ is

$$\bar{s}(t) = (t, v = v(t), k_2 = \frac{dv}{dt}, k_1 = k(t)),\ v(t) \neq 0.$$

An integral manifold in M^t which projects to $\bar{s}$ is of the form *

$$s(t) = \mu(A(t), \sigma \circ \bar{s}(t))$$

where $A : I\!R \to E^+(2)$ satisfies

$$\frac{da}{dt} = -v(t) \sin\theta(t) = 0,\ \frac{db}{dt} = v(t) \cos\theta(t),$$
$$\frac{d\theta}{dt} = -\frac{k_1(t)}{v(t)}.$$

An equation of Lie type for $\alpha : I\!R \to \mathbf{g}$,

$$\alpha(t) = \left(0, -v(t), \frac{k_1(t)}{v(t)}\right).$$

*Finding an integral manifold to $\mathcal{I}$ projecting to $\bar{s}$ is the prescribed "curvature" problem

Example 4. $SL(2)$ on two copies of jet-space. On the set $u_x v \neq 0$, the action satisfies

$$I^{\mathbf{An}} \cap \mathbf{\Gamma} = 0.$$

Choosing the cross-section on $\bar{M}^t = (x, y, w, w_x, w_y)$

$$x = x, y = x, u = 0, v = 1, u_x = w, v_y = 1,$$
$$u_{xx} = w_x - 2w^2, v_{yy} = w_y/w + 2$$

The quotient EDS is

$$\bar{\mathcal{I}}_{M^t} =< dw - w_x dx - w_y dy, dw_x dx + dw_y dy,$$
$$\left(dw_x + (\frac{w_x w_y}{w} - w^2)dy)\right) \wedge dx \ >.$$

Project integral manifold $f(x), g(y)$ of $\mathcal{I}$ to

$$w = \frac{u_x v_y}{(u - v)^2} = \frac{f'(x) g'(y)}{(f(x) - g(y))^2}$$

integral manifolds of $\bar{\mathcal{I}}$.

If $w = e^u$ - Monge Ampere form of Liouville.

Exercise: Compute the equation of Lie type for the Chazy equation on the set of transverse initial condtions.

Is there a way to do this all at once?

Yes: In fact it might look familiar:

Suppose $\mathcal{I}$ is constant rank r Pfaffian and θ^i are a basis of sections.

Then $\mu : M \to \mathbb{R}^r \otimes \mathbf{g}^*$ given by

$$\theta^i(X)$$

is the moment map.

K is the zero-set of the moment map.

M^t is the full rank set for the moment map.

Reduction is the same as symplectic or contact reduction - restriction then quotient.