EXTERIOR DIFFERENTIAL SYSTEMS WITH SYMMETRY

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Abstract. The symmetry group of an exterior differential system is used to simplify finding integral manifolds in the case of Pfaffian systems. This leads to the definition of a moment map for Pfaffian systems.

Key words. Pfaffian systems, symmetry reduction, group invariant solutions.

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1. Introduction. Let $\Delta = 0$ be a system of differential equations, and let $\mathcal{S}$ be its solution space. A symmetry group of the differential equations $\Delta = 0$ is group G which acts on the space of solutions $\mathcal{S}$. Typically a symmetry group of a differential equation is known, even though the solution space is not. A basic idea, which originated with Sophus Lie, is that it might be easier to determine $\mathcal{S}/G$ than it is to determine $\mathcal{S}$. This idea (simplifying finding solutions to differential equations using a group) is one of the principle motivating problems in studying exterior differential systems with symmetry.

To describe the quotient $\mathcal{S}/G$ we first partition $\mathcal{S}$ into orbits having inequivalent stabilizers,

$$\mathcal{S} = \mathcal{S}^G \cup \mathcal{S}^1 \cup \ldots \cup \mathcal{S}^{free}. \tag{1.1}$$

The set $\mathcal{S}^G$ corresponds to fixed points of the action of G. The solutions in $\mathcal{S}^G$ are the *G-invariant solutions*, which are also sometimes called equivariant solutions. The set $\mathcal{S}^{free}$ correspond to points in $\mathcal{S}$ on which G acts freely. Solutions in $\mathcal{S}^{free}$ have no symmetry. The intermediate terms in (1.1) are solutions which are fixed (invariant) respect to some subgroup of G. The quotient can then be partitioned,

$$\mathcal{S}/G = (\mathcal{S}^G/G) \cup (\mathcal{S}^1/G) \cup \ldots \cup (\mathcal{S}^{free}/G). \tag{1.2}$$

where $\mathcal{S}^G/G = \mathcal{S}^G$.

EXAMPLE 1.1. *Laplace's equation on* $\mathbb{R}^2 - (0,0)$,

$$u_{xx} + u_{yy} = 0$$

admits $SO(2)$ *acting on the punctured plane in the usual way*

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

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as a symmetry group. If $u = f(x,y)$ is a solution to Laplace's equation, then it is easy to check that

$$\hat{f}(x,y) = f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$$

is also a solution, and so $SO(2)$ acts on $\mathcal{S}$. The set $\mathcal{S}^G$ are the rotationally invariant solutions to Laplace's equation. These satisfy $f(x,y) = \hat{f}(x,y)$, and so f must have the form

$$f = f(\sqrt{x^2+y^2}).$$

This leads to the fundamental solution. The set $\mathcal{S}^{free}$ are the solutions to Laplace's equation without any rotational symmetry, which would be most solutions.

Lie's original way of thinking was to find a family of *quotient differential equations* $\bar{\Delta} = 0$ whose solutions would be in 1-1 correspondence with the different terms in the quotient $\mathcal{S}/G$ in Equation (1.2). A serious problem with this idea is that it is unclear whether the quotient of a differential equation is another differential equation. For example, what would the quotient of Laplace's equation by $SO(2)$ in Example 1.1 be? The difficulty here lies in the usual coordinate description of differential equations. However exterior differential systems (EDS) are a coordinate invariant way to represent differential equations and in this context the idea of a quotient differential equation can be easily realized.

A second important problem which arises in implementing a quotient is the following. Suppose that $\bar{\Delta}$ was a set of differential equations whose solutions represented one of the quotients in Equation (1.2). We then need to take a solution $\bar{s}$ to this quotient equation $\bar{\Delta}$ and construct a solution s to the original equation $\Delta = 0$. This is sometimes called the *reconstruction problem.*

In this article I will focus on how to find the quotient $\bar{\Delta}$, and what is the "reconstruction problem". I would like to list two other important problems which should be kept in mind in the theory of quotients.

1) The *inverse problem.* Given a differential equation $\delta = 0$, is there "simple" differential equation $\Delta = 0$ such that $\delta = \bar{\Delta}$?

2) How are the geometric properties of $\Delta = 0$ related to those of $\bar{\Delta} = 0$? For example suppose $\Delta = 0$ are the Euler-Lagrange equations for some Lagrangian λ. Are $\bar{\Delta}$ the Euler-Lagrange equations for some quotient Lagrangian $\bar{\lambda}$? For $\mathcal{S}^G$ the question is known as the principle of symmetric criticality [4, 1]. For $\mathcal{S}^{free}$ this is sometimes known as Lagrangian Reduction [9, 10]. Symmetric criticality and Lagrangian reduction lie at the *opposite* ends of the symmetry spectrum!

2. Symmetries and quotients of EDS.

DEFINITION 2.1. *An exterior differential system (EDS) is a differential ideal $\mathcal{I} \subset \Omega^*(M)$.*

Differential equations $\Delta = 0$ give rise to exterior differential systems (EDS). Examples are given below.

DEFINITION 2.2. *An integral manifold of $\mathcal{I}$ is an immersion $s : N \to M$ such that $s^*\mathcal{I} = 0$.*

Solutions to $\Delta = 0$ are integral manifolds to a corresponding EDS $\mathcal{I}$.

DEFINITION 2.3. *A symmetry of an exterior differential system is a diffeomorphism $\phi : M \to M$ such that $\phi^*\mathcal{I} = \mathcal{I}$. A symmetry group of $\mathcal{I}$ will be a Lie group G acting smoothly on M where each diffeomorphism $g : M \to M$ is a symmetry of $\mathcal{I}$.*

Symmetries of differential equations determine symmetries of their corresponding EDS $\mathcal{I}$. Symmetries of EDS behave like symmetries of differential equations. If ϕ is a symmetry of the EDS $\mathcal{I}$ and $s : N \to M$ is an integral manifold of $\mathcal{I}$, then

$$(\phi \circ s)^*\mathcal{I} = s^*\phi^*\mathcal{I} = s^*\mathcal{I} = 0.$$

Therefore symmetries map integral manifolds to integral manifolds. Given a symmetry group G of an EDS $\mathcal{I}$, the *symmetry group of an integral manifold* is

$$G_s = \{ g \in G \mid gs(N) = s(N) \}.$$

If $G_s = G$, then N is a G-invariant integral manifold. The G-invariant solutions to a differential equation correspond to G-invariant integral manifolds.

We use the following notation. If $\{\theta^i\} \in \Omega^*(M)$ is a set of differential forms, then the differential ideal they generate is

$$\mathcal{I} = < \theta^i, d\theta^i >,$$

where $<,\ >$ means the algebraic ideal in $\Omega^*(M)$. If $\theta^i \in \Omega^1(M)$ (one-forms) then $\mathcal{I}$ is a *Pfaffian System.* The EDS which will be of main interest through this article are constant rank Pfaffian systems.

DEFINITION 2.4. *A (constant) rank r Pfaffian system $\mathcal{I}$ is an exterior differential system generated by the sections of a rank r subbundle $I \subset T^*M$.*

We will usually refer to the bundle I as the Pfaffian system, and we will also denote by

$$I = \{\theta^i\}$$

the subbundle of T^*M which is the point-wise span of the differential one-forms $\theta^i \in \Omega^1(M)$.

The annihilator $I^\perp \subset TM$ of a rank r Pfaffian system $I \subset T^*(M)$ is the rank $n - r$ subbundle defined point-wise by

$$I_p^\perp = \{V \in T_pM \mid \theta(V) = 0, \quad \forall\, \theta \in I_p,\ p \in M\ \}.$$

An integral manifold of a constant rank Pfaffian system I is then an immersion $s : N \to M$ such that

$$s_* T_x N \in I^{\perp}_{s(x)}.$$

Symmetries of a constant rank Pfaffian system I are diffeomorphisms $\phi : M \to M$ which preserve the subbundles I and $I^{\perp}$.

REMARK 2.1. Infinitesimal methods will be used when discussing symmetry [13].

EXAMPLE 1.1 (Continued). *Laplace's equation on* $\mathbb{R}^2 - (0,0)$, *gives rise to a rank three Pfaffian system* $I = \{\theta_u, \theta_{u_x}, \theta_{u_y}\}$ *on a seven manifold* $M_7 = (x, y, u, u_x, u_y, u_{xy}, u_{yy})$, *where*

$$\begin{aligned} \theta_u &= du - u_x dx - u_y dy, \\ \theta_{u_x} &= du_x + u_{yy} dy - u_{xy} dy, \\ \theta_{u_y} &= du_y - u_{xy} dx - u_{yy} dy. \end{aligned} \tag{2.1}$$

Solutions $u = f(x,y)$ *to Laplace's equation define integral manifolds* $s : \mathbb{R}^2 - (0,0) \to M_7$,

$$s(x,y) = \Big(x, y, u = f,\ u_x = f_x,\ u_y = f_y,\ u_{xy} = f_{xy},\ u_{yy} = f_{yy}\Big).$$

The infinitesimal generator of the prolonged $SO(2)$ *action on* M_7 *is*

$$X = x\partial_y - y\partial_x + u_x \partial_{u_y} - u_y \partial_{u_x} - 2u_{yy}\partial_{u_{xy}} + 2u_{xy}\partial_{u_{yy}}, \tag{2.2}$$

and is an infinitesimal symmetry of I *(or* $\mathcal{I}$*).*

2.1. The quotient. Let G be a Lie group acting smoothly on M, with infinitesimal generators Γ. Let $\mathbf{\Gamma} \subset TM$, be the corresponding point-wise span of elements of Γ. In the following discussion we will assume that the action of G on M if sufficiently regular so that $\mathbf{q} : M \to M/G = \bar{M}$ is a smooth submersion. With this hypothesis on the action, $\mathbf{\Gamma} \subset TM$ is a rank q subbundle where q is the dimension of the orbits. Furthermore

$$\mathbf{\Gamma} = \ker \mathbf{q}_*$$

which is also the vertical bundle for $\mathbf{q}$.

DEFINITION 2.5. *The quotient of an EDS* $\mathcal{I} \subset \Omega^*(M)$, *is the EDS* $\bar{\mathcal{I}} \subset \Omega^*(\bar{M})$ *defined by*

$$\bar{\mathcal{I}} = \{\ \bar{\theta} \in \Omega^*(\bar{M}) \mid \mathbf{q}^*\bar{\theta} \in \mathcal{I}\ \}.$$

If $\mathcal{I}$ is a constant rank Pfaffian system (with bundle $I \subset T^*M$), the *quotient of the bundle* I is the subset $\bar{I} \subset T^*\bar{M}$ given point-wise by

$$\bar{I}_{\bar{x}} = \{\ \bar{\theta} \in T^*_{\bar{x}}\bar{M} \mid \mathbf{q}^*\bar{\theta}_{\bar{x}} \in I_x \text{ where } \mathbf{q}(x) = \bar{x}\ \}.$$

This can be computed in terms of $I^{\perp} \subset TM$ by

$$\bar{I} = \left(\mathbf{q}_*(I^{\perp})\right)^{\perp}.$$

Necessary and sufficient conditions that the subsets $\bar{I} \subset T^*\bar{M}$, or $\mathbf{q}_*(I^{\perp}) \subset T\bar{M}$ are subbundles are well known [8].

LEMMA 2.1. *The subsets $\bar{I} \subset T^*\bar{M}$, and $\mathbf{q}_*(I^{\perp}) \subset T\bar{M}$ are subbundles if and only if there exists a non-negative integer k such that*

$$\dim\left(I_x^{\perp} \cap (\ker \mathbf{q}_*)_x\right) = \dim\left(I_x^{\perp} \cap \mathbf{\Gamma}_x\right) = k\,, \quad \forall\, x \in M. \tag{2.3}$$

Then rank $\mathbf{q}_*(I^{\perp}) = n - r - k$ *and* rank $\bar{I} = r + k - q$, *where* rank $\mathbf{\Gamma} = q$.

It is handy to write out the intersection $I_x^{\perp} \cap \mathbf{\Gamma}_x$ in Equation (2.3) in a basis. Let $x \in M$, $\{X_a\}_{1 \le a \le q}$ be a basis for $\mathbf{\Gamma}_x$ and let $\{\theta^i\}_{1 \le i \le r}$ be a basis for I_x. Then

$$I_x^{\perp} \cap \mathbf{\Gamma}_x = \ker \theta^i(X_a), \tag{2.4}$$

where the kernel is computed on the index a.

The first issue that occurs in EDS reduction is that even when condition (2.3) is satisfied, bundle reduction and Pfaffian system reduction do not necessarily commute.

EXAMPLE 2.2. *Consider the three dimensional manifold $M = (x, u, u_x)$, with rank one Pfaffian system*

$$I = \{\ du - u_x dx\ \}, \text{ and } \quad \mathcal{I} = <du - u_x dx,\ du_x \wedge dx>.$$

The group $G = \mathbb{R}$ acting on M by $c\cdot(x, u, u_x) = (x, u+c, u_x)$ is a symmetry of I. The projection map is $\mathbf{q}(x, u, u_x) = (x, u_x)$. The bundle quotient is

$$\mathbf{q}_*(I^{\perp}) = T\mathbb{R}^2, \quad \text{therefore} \quad \bar{I} = 0.$$

While the EDS quotient is,

$$\bar{\mathcal{I}} = <du_x \wedge dx>.$$

If a Pfaffian systems $\mathcal{I}$ is completely integrable then EDS reduction and bundle reduction commute [5]. For Pfaffian systems which are not completely integrable, sufficient conditions are given in [2].

The intersection condition in Equation (2.3) will be the focus for the remainder of the article. We begin with a simple application, see [5].

THEOREM 2.1. *Suppose $\mathcal{I}$ is a constant rank Pfaffian system with symmetry group G. If there exists $X_{x_0} \in I_{x_0}^{\perp} \cap \mathbf{\Gamma}_{x_0}$ with $X_{x_0} \neq 0$, then $e^{tX}x_0$, $X \in \mathbf{g}$ is a one-dimensional integral manifold.*

The integral manifold $e^{tX}x_0$ in this theorem is the integral curve of the infinitesimal generator X through the point x_0. Equivalently, it is the orbit of the one-parameter subgroup e^{tX} through the point x_0.

We now turn to a few examples before continuing with the development of the theory.

3. Examples.

EXAMPLE 3.3. *The Chazy equation*

$$y_{xxx} = 2yy_{xx} - 3y_x^2$$

gives rise to the completely integrable rank three Pfaffian system $I = \{\theta^1, \theta^2, \theta^3\}$ *on a four dimensional manifold* $M_4 = (x, y, y_x, y_{xx})$, *where*

$$\theta^1 = dy - y_x dx, \quad \theta^2 = dy_x - y_{xx} dx, \quad \theta^3 = dy_{xx} - (2yy_{xx} - 3y_x^2)dx \quad (3.1)$$

Solutions to the Chazy equation are integral manifolds of I. *The Pfaffian system (3.1) is invariant with respect to the (infinitesimal) action of* $SL(2)$ *on* M_4 *given by* $\Gamma = \{X_1, X_2, X_3\}$ *where*

$$\begin{aligned} X_1 &= \partial_x, \\ X_2 &= 2x\partial_x - 2y\partial_y - 4y_x\partial_{y_x} - 6y_{xx}\partial_{y_{xx}}, \\ X_3 &= -x^2\partial_x + 2(xy+3)\partial_y + 2(2xy_x + y)\partial_{y_x} + 6(y_x + xy_{xx})\partial_{y_{xx}}. \end{aligned}$$

By Equation (2.4), there exists $X_p \in I_p^\perp \cap \boldsymbol{\Gamma}_p$ *with* $X_p \neq 0$ *if and only if the determinant* $\det(\theta^i(X_j)) = 0$. *This occurs at the points* $p \in M_4$ *satisfying,*

$$y_{xx} = yy_x - \frac{1}{9}y^3 \pm \frac{1}{9}(y^2 - 6y_x)^{\frac{3}{2}}. \quad (3.2)$$

For initial conditions $(x^0, y^0, y_x^0, y_{xx}^0)$ *satisfying this constraint, the (unique) solution obtained from Theorem 2.10 is the one-dimensional orbit,*

$$\begin{aligned} x &= x^0 + 2t\frac{(\delta^0 + 3y_x^0 + y^0\sqrt{\delta^0})}{ty_x^0(y^0 + \sqrt{\delta^0}) - 1}, \\ y &= y^0 + 2ty_x^0\left((3t(y_x^0)^2 - y^0 + ty_x^0\delta^0)\sqrt{\delta^0} + ty_x^0 y^0\delta^0 - 3y_x^0 - \delta^0\right) \end{aligned}$$

where $\delta^0 = (y^0)^2 - 6y_x^0$. *This is a 2 parameter family of invariant solutions to the Chazy equation, which are easily written as a graph.*

EXAMPLE 3.4. *The standard Pfaffian system for the ordinary differential equation*

$$u_{xxxxx} = \frac{5u_{xxx}(9u_{xxxx}u_{xx} - 8u_{xxx}^2)}{9u_{xx}^2},$$

is invariant with respect to the five dimensional special affine group

$$G = SA(2) = \{\ (A, b) \mid A \in SL(2, \mathbb{R}),\ b \in \mathbb{R}^2\ \},$$

with affine action on (x, u) *and then prolonged. Every solution is the orbit of a one-parameter subgroup, from which the general solution,*

$$u = c_0 + c_1 x \pm \sqrt{c_3 x^2 + c_x x + c_2}(4c_3c_2 - c_x^2)$$

where c_0, c_1, c_2, c_3, c_x are constants, can be found. See [5] for the details.

EXAMPLE 3.5. *Consider the pseudo-Riemannian metric on $\mathbb{R}^4$,*

$$\eta = e^{-\frac{4}{3}x_4}(dx_1dx_3 - dx_2dx_2) + e^{\frac{2}{3}x_4}dx_3dx_3 + c\, dx_4dx_4$$

where if $c < 0$ the metric is Lorentzian, and if $c > 0$ the metric has split signature. The geodesic equation define a completely integrable rank eight Pfaffian system I on a nine-dimensional manifold $M_9 = \{(t, x_i, \dot{x}_i),\ 1 \leq i \leq 4\}$ given by

$$I = \left\{ dx_i - \dot{x}_i dt, \quad d\dot{x}_1 - \frac{2}{3}\dot{x}_4(2\dot{x}_1 - 3\dot{x}_3 e^{2x_4})dt, \quad d\dot{x}_2 - \frac{4}{3}\dot{x}_2\dot{x}_4 dt, \right. \\ \left. d\dot{x}_3 - \frac{4}{3}\dot{x}_3\dot{x}_4 dt, \quad d\dot{x}_4 + \frac{1}{3c}\left(e^{-\frac{4}{3}x_4}(4\dot{x}_1\dot{x}_3 - 2\dot{x}_2^2) - e^{\frac{2}{3}x_4}\dot{x}_3^2\right) dt \right\}. \tag{3.3}$$

The geodesics are integral manifolds. The Pfaffian system (3.3) is invariant with respect to time translations and the induced action of the isometry group which has Killing vector-fields

$$X_1 = \partial_{x_1}, X_2 = \partial_{x_2},\ X_3 = \partial_{x_3},\ X_4 = x_2\partial_{x_1} + x_3\partial_{x_2}, \\ X_5 = 5x_1\partial_{x_1} + 2x_2\partial_{x_2} - x_3\partial_{x_3} + 3\partial_{x_4}.$$

We work at the point $p \in M_9$ given by

$$t = 0,\ \mathbf{x} = (0,0,0,0),\ \dot{\mathbf{x}} = (\frac{k}{4}, 0, k, 0), \tag{3.4}$$

where $k \neq 0$. The vector-field $X \in \Gamma$,

$$X = \partial_t + \frac{k}{4}X_1 + kX_3 = \partial_t + \frac{k}{4}\partial_{x_1} + k\partial_{x_3}$$

satisfies $\theta(X) = 0,\ \forall\, \theta \in I_p$ where p is the point (3.4). The integral curve of X in M through the point (3.4) is

$$x_1 = \frac{k}{4}t,\ x_2 = 0,\ x_3 = kt,\ x_4 = 0, \tag{3.5}$$

which by Theorem 2.10 is an integral manifold. The curve (3.5) is a geodesic which is the orbit of a one parameter subgroup corresponding to X.

REMARK 3.1. Geodesics which are orbits of the isometry group are called homogeneous geodesics. Homogeneous geodesics always exist for homogeneous Riemannian manifolds [11], but it is an open question whether every homogeneous pseudo-Riemannian manifold admits a homogeneous geodesic.

EXAMPLE 3.6. *Every geodesic on a Riemannian symmetric space is homogeneous.*

EXAMPLE 3.7. *The contact system on $J^2(\mathbb{R},\mathbb{R}^2)$ is a rank four Pfaffian system $I = \{\theta_x, \theta_y, \theta_{\dot x}, \theta_{\dot y}\}$ on a seven dimensional manifold $M_7 = (t, x, y, \dot x, \dot y, \ddot x, \ddot y)$ where*

$$\theta_x = dx - \dot x dt,\ \theta_y = dy - \dot y dt,\ \theta_{\dot x} = d\dot x - \ddot x dt,\ \theta_{\dot y} = d\dot y - \ddot y dt. \tag{3.6}$$

Any prolonged graph $(x(t), y(t))$ is an integral manifold. The action of the oriented Euclidean group $E(2)^+$,

$$\begin{gathered}\begin{pmatrix}\hat x\\ \hat y\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}a\\ b\end{pmatrix},\\ \begin{pmatrix}\hat{\dot x}\\ \hat{\dot y}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\dot x\\ \dot y\end{pmatrix},\\ \begin{pmatrix}\hat{\ddot x}\\ \hat{\ddot y}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\ddot x\\ \ddot y\end{pmatrix}\end{gathered} \tag{3.7}$$

is a symmetry group of I. The only possible integral curves which are orbits satisfy $\dot x = \dot y = 0$, and are single points. (Re-parameterization of t is not being allowed as part of the symmetries.)

4. Quotients.

4.1. Invariant integral manifolds. In order to compute the quotient of a constant rank Pfaffian system I, we need to partition M into G-invariant subsets on which we can control the behavior of the quotient. One key is the intersection condition

$$I_x^\perp \cap \mathbf{\Gamma}_x$$

from Equation (2.3) of Lemma 2.1, or (2.4). Let $K \subset M$ be the G-invariant subset

$$K = \{\ x \in M \mid \mathbf{\Gamma}_x \subset I_x^\perp\ \}. \tag{4.1}$$

This is the set of points $x \in M$ where every form $\theta \in I_x$ vanishes on every $X_x \in \mathbf{\Gamma}_x$. It is also the set of points where the matrix in (2.4) has the smallest possible rank (zero).

Assume $\iota : K \to M$ is an embedded submanifold and that $T_xK \cap I_x^\perp$ has constant dimension independent of x. Then $\mathcal{I}_K = \iota^*\mathcal{I}$ is a constant rank Pfaffian system. Further assume that the action of G on K is sufficiently regular so that $\mathbf{q} : K \to K/G$ is a smooth submersion. At points $x \in K$, $\mathbf{\Gamma}_x \subset (I_{K,x})^\perp$, and so by Lemma 2.1 the quotient $\bar I_K$ has the same rank as I_K. It is also easy to show that bundle reduction and Pfaffian system reduction commute in this case, and therefore $\bar{\mathcal{I}}_K$ is constant rank Pfaffian system.

The Pfaffian system $\bar I_K$ determines the "G-invariant solutions".

THEOREM 4.1. *Let $\bar{N} \subset M$ be an embedded integral manifold of $\bar{I}_K$. Then $N = \mathbf{q}^{-1}(\bar{N}) \subset M$ is a G-invariant integral manifold of I.*

Proof. The manifold N is clearly G-invariant, so we only need to show it is an integral manifold. Let $x \in N$, $X \in T_x N$, and $\bar{x} = \mathbf{q}(x)$, $\bar{X} = \mathbf{q}_* X$. Note $\bar{x} \in \bar{N}$ and $\bar{X} \in T_{\bar{x}} \bar{N}$.

Choose an open set $\bar{U} \subset \bar{K}$ containing $\bar{x}$, and a cross-section $\sigma : \bar{M} \to M$ with $\sigma(\bar{x}) = x$. Then

$$X = \sigma_* \bar{X} + V$$

for some $V \in \mathbf{\Gamma}_x$ (the vertical bundle at x). Evaluating on $\theta \in \mathcal{I}$,

$$\begin{aligned} \theta(X) &= \theta(\sigma_* \bar{X}_{\bar{x}} + V) \\ &= \sigma^* \theta(\bar{X}) + \theta(V) \\ &= 0. \end{aligned}$$

The first term vanishes because $\sigma^* \theta \in \bar{\mathcal{I}}_K$, and $\bar{N}$ is an integral manifold. The second term vanishes because we are at point of K (4.1). □

A few important remarks about this theorem are appropriate.

REMARK 4.1. The reconstruction problem is algebraic. The inverse image process in the theorem provides the reconstruction.

REMARK 4.2. The converse of this theorem is also true. Every invariant integral manifold factors though the set K, and projects to an integral manifold of $\bar{I}_K$.

REMARK 4.3. Integral manifolds of $\mathcal{I}_K$ can always be enlarged (locally) to be invariant.

REMARK 4.4. The set K is the subset of M on which $\mathbf{\Gamma}$ are Cauchy-characteristics for $\mathcal{I}_K$. Perhaps this makes Theorem 4.1 not so surprising.

EXAMPLE 1.1 (Continued). *For Laplace's equation with Pfaffian system (2.1), we determine the set K. With the forms in (2.1) and X in (2.2) we get*

$$\theta^\alpha(X) = \left(y u_x - x u_y, -u_y - x u_{xy} - u u_{yy}, u_x - x u_{yy} + y u_{xy}\right).$$

The set K is then given by $\theta^\alpha(X) = 0$. Solving these equations we find K is four dimensional and choosing coordinates $K = (x, y, u, a)$, the inclusion $\iota : K \to M_7$ is

$$\left(x = x, y = y, u = u, \ u_x = ax, \ u_y = ay, \ u_{xy} = -\frac{2xya}{x^2+y^2}, u_{yy} = \frac{(x^2-y^2)a}{x^2+y^2} \right).$$

The one-forms in $\mathcal{I}$ pullback by ι to give,

$$\begin{aligned} \iota^* \theta_u &= du - ax dx - ay dy, \\ \iota^* \theta_{u_x} &= x da + \frac{2x^2 a}{x^2+y^2} dx + \frac{2xya}{x^2+y^2} dy, \\ \iota^* \theta_{u_y} &= y da + \frac{2xya}{x^2+y^2} dx + \frac{2y^2 a}{x^2+y^2} dy. \end{aligned}$$

Therefore I_K is rank two. The quotient $K/G = \{\, (r,u,a)\,|\, r > 0 \,\}$ is three dimensional and the rank two quotient Pfaffian system is,

$$\bar{I}_K = \{\; du - ardr, \quad rda + 2adr \;\}.$$

The submanifolds

$$s(r) = (\, r,\, c_1 + c_2 \log r,\, c_2 r^{-2}\,) \tag{4.2}$$

are integral manifolds of $\bar{I}_K$. By Theorem 4.1, the inverse image of (4.2) leads to the invariant solutions

$$\left(x,\, y,\, u = c_1 + \frac{c_2}{2}\log(x^2+y^2),\, a = c_2(x^2+y^2)^{-1} \right).$$

4.2. The transverse set. A second G-invariant subset of M is the transverse set

$$M^t = \{\; x \in M \;|\; I_x^{\perp} \cap \mathbf{\Gamma}_x = 0 \;\}. \tag{4.3}$$

This is the set of points in M where the rank of the matrix in (2.4) is as large as possible.

Suppose M^t is an embedded submanifold and that G acts regularly on M^t. Let

$$\mathcal{I}_{M^t} = \mathcal{I}|_{M^t}$$

be the restriction of $\mathcal{I}$ to M^t. The transversality condition $I_x^{\perp} \cap \mathbf{\Gamma}_x = 0$ implies that $\mathcal{I}_{M^t}$ is a constant rank Pfaffian system with the same rank as $\mathcal{I}$. Unlike the case for the invariant integral manifolds in section 4.1, the quotient $\bar{\mathcal{I}}_{M^t}$ is not necessarily a Pfaffian system.

The set M^t was studied in detail in [2]. We recall a few things from that reference. First, the integral manifolds of $\mathcal{I}_{M^t}$ have essentially no continuous symmetry. Second, if $s : N \to M$, an integral manifold of $\mathcal{I}_{M^t}$, then $\mathbf{q} \circ s : N \to \bar{M}^t$ is an integral manifold of $\bar{\mathcal{I}}_{M^t}$. (The immersion property still holds).

A generalization of Proposition 6.1 from [2], solves the reconstruction problem.

THEOREM 4.2. *Let $\bar{N} \to \bar{M}^t$ be an embedded integral manifold of $\bar{\mathcal{I}}_{M^t}$. Then $\mathcal{I}|_{\mathbf{q}^{-1}(\bar{N})}$ is completely integrable, and the leaves are integrable manifolds of $\mathcal{I}$.*

An immediate consequence of this theorem is that the integral manifolds of $\mathcal{I}$ are surjective (locally) by $\mathbf{q}$ onto the integral manifolds of $\bar{\mathcal{I}}_{M^t}$.

There is a particularly nice geometric way to think about the reconstruction problem when the action of G on M^t is free. Starting with an integral manifold $s : \bar{N} \to \bar{M}^t$ of $\bar{\mathcal{I}}_{M^t}$, let $\hat{s} : \bar{N} \to M$ be any cover of $\bar{N}$. Any other cover $s : \bar{N} \to M$ of $\bar{s}$ is of the form

$$s(t) = \mu(A(t), \hat{s}(t))$$

where $A : \bar{N} \to G$ is unique. If we require $s(t)$ to be an integral manifold of $\mathcal{I}$, then $A(t)$ satisfies a generalized equation of Lie type. Equations of Lie type are differential equations on Lie groups which have many applications and interesting properties. For example, the equations are integrable by quadratures for (simply connected) solvable Lie groups.

EXAMPLE 3.7 (Continued). *The infinitesimal generators of the action of the oriented Euclidean group $E(2)^+$ in (3.7) on the 7 dimensional manifold $J^2(\mathbb{R}, \mathbb{R}^2)$ are*

$$\Gamma = \text{ span } \{ \ \partial_x, \ \partial_y, \ x\partial_y - y\partial_x + \dot{x}\partial_{\dot{y}} - \dot{y}\partial_{\dot{x}} + \ddot{x}\partial_{\ddot{y}} - \ddot{y}\partial_{\ddot{x}} \ \}.$$

Using the forms in Equation (3.6) for I, the transverse subset M^t in Equation (4.3) for the $E^+(2)$ action is

$$M^t = \{p \in J^2(\mathbb{R}, \mathbb{R}^2) \mid (\dot{x}, \dot{y}) \neq (0,0)\}.$$

The group $E(2)^+$ acts freely on M^t and the quotient is $\bar{M}^t = M^t/G$ is 4 dimensional, $\bar{M}^t = \{(t, v, k_1, k_2), v \neq 0\}$. The quotient EDS is

$$\bar{\mathcal{I}}_{M^t} = < dv - k_2 dt, dk_2 \wedge dt, k_1 dk_1 \wedge dt >,$$

which is not a Pfaffian system. A typical integral manifold for $\bar{\mathcal{I}}_{M^t}$ is

$$\bar{s}(t) = (t, v = v(t), k_2 = \frac{dv}{dt}, k_1 = k(t)), \ \ v(t) \neq 0.$$

An integral manifold in M^t which projects to $\bar{s}$ is of the form

$$s(t) = \mu(A(t), \sigma \circ \bar{s}(t))$$

where $\sigma(t, v, k_1, k_2) = (t, 0, 0, 0, v, k_1, k_2)$, and $A : \mathbb{R} \to E(2)^+$ satisfies

$$\frac{da}{dt} = -v(t) \sin\theta(t) = 0, \quad \frac{db}{dt} = v(t)\cos\theta(t), \quad \frac{d\theta}{dt} = -\frac{k_1(t)}{v(t)}.$$

This is an equation of Lie type for the curve $\alpha : \mathbb{R} \to \mathbf{g}$,

$$\alpha(t) = \left(0, -v(t), \frac{k_1(t)}{v(t)}\right).$$

REMARK 4.5. The example above can be generalized to $J^k(\mathbb{R}, G/H)$ with the following interpretation. Compute the quotient of the contact structure $\mathcal{I}$ on $J^k(\mathbb{R}, G/H)/G$ on the transverse set for k sufficiently large. Let $\bar{s}$ be an integral manifold to the quotient system. By finding an integral manifold to $\mathcal{I}$ projecting to $\bar{s}$, we have solved the prescribed "curvature" problem for curves in a homogeneous space. The curve $\bar{s}$ is the prescribed curvature. The reconstruction is done (in general) by solving an equation of Lie type on the group G.

EXAMPLE 1.1 (Continued). *Laplace's equation with the basis for I in Equation (2.1), and infinitesimal generator (2.2), the matrix $\theta^\alpha(X)$ is*

$$(yu_x - xu_y, -u_y - xu_{xy} - uu_{yy}, u_x - xu_{yy} + yu_{xy}).$$

On the $SO(2)$-invariant set $M_0^t = \{yu_x - xu_y \neq 0\}$, which is a subset of M^t, the matrix $\theta^\alpha(X)$ is full rank. The quotient is six dimensional $M_0^t/SO(2) = \{(r, v, p, q, s, t) \mid r > 0\}$ with projection map

$$r = \frac{1}{2}\log(x^2 + y^2), \; v = u, \; p = xu_x + yu_y, \; q = xu_y - yu_x,$$
$$s = \frac{xu_y - yu_x + 2yxu_{yy} + (x^2 - y^2)u_{xy}}{xu_y - yu_x},$$
$$t = \frac{u_{yy}(x^2 - y^2) - 2xyu_{xy} - xu_x - yu_y}{xu_y - yu_x}.$$

The quotient $\bar{I}_{M_0^t}$ is the rank two Pfaffian system

$$\bar{I}_{M_0^t} = \{ \, dp - sdv + (tq + ps)dr, \; dq - tdu + (tp - sq)dr \, \}.$$

The submanifold

$$(\, r = r, \; v = v, \; p = \frac{v}{r}, \; q = r, \; s = r^{-1}, \; t = 0 \,)$$

is an integral manifold of $\bar{I}_{M_0^t}$.

The reconstruction problem leads to the completely integrable system of partial differential equations for $\theta(r, v)$,

$$r^2\partial_r\theta = -v, \quad r\partial_v\theta = 1.$$

The solution is $\theta = vr^{-1} + c_0$. This leads to the integral manifolds (as a graph)

$$u = \frac{1}{2}\log(x^2 + y^2)(\arctan\frac{y}{x} - c),$$

which are not $SO(2)$ invariant.

REMARK 4.6. It is a good exercise to compute the equation of Lie type on $SL(2)$ for the Chazy equation on the set of transverse initial conditions. The transverse initial conditions will be the complement to those in Equation (3.2).

EXAMPLE 4.8. *In this last example, we demonstrate an inverse problem. Consider the "Cartan-Hilbert equation"*[1] *in the form of the rank three Pfaffian system $I = \{\theta^1, \theta^2, \theta^3\}$ on the five dimensional manifold $M_5 = (z_1, z_2, z_3, t_1, t_2)$, where*

$$\theta^1 = \frac{1}{2}dz_1 - t_2 dt_1, \quad \theta^2 = dz_2 - \frac{1}{2}t_1^2 dt_2, \quad \theta^3 = dz_3 + \frac{1}{2}t_2^2 dt_1. \qquad (4.4)$$

[1]The usual form of Cartan Hilbert equation is $z' = (y'')^2$.

The derived flag for this Pfaffian system is $(3,2,0)$, *and the Lie algebra of the symmetry group is the split form of* G_2. *The integral manifolds are easily determined with* $t = t_1$ *to be,*

$$z_1 = f(t),\ t_2 = \frac{1}{2}\frac{df}{dt},\ z_2 = F_1(t),\ z_3 = F_2(t)$$

where F_1, F_2 *satisfy,*

$$\frac{dF_1}{dt} = \frac{1}{4}t^2\frac{d^2 f}{dt^2},\quad \frac{dF_2}{dt} = -\frac{1}{8}\left(\frac{df}{dt}\right)^2. \tag{4.5}$$

The group $G = \mathbb{R}^3$ *acting on* M_5,

$$(a,b,c)\cdot(z_1,z_2,z_3,t_1,t_2) = (z_1+a, z_2+b, z_3+c, t_1, t_2) \tag{4.6}$$

is a symmetry group of I_1.

Now let $I \coloneqq I_1 \oplus I_2$ *be the direct sum two copies of the "Cartan-Hilbert" Pfaffian system (4.4), on the ten-dimensional manifold* $M_5 \times M_5$ *where,*

$$I_2 = \left\{\frac{1}{2}dw_1 - s_2 ds_1,\ ds_2 - \frac{1}{2}s_1^2 ds_2,\ ds_3 + \frac{1}{2}s_2^2 ds_1\right\}$$

is on the second five-dimensional manifold $M_5 = (w_1, w_2, w_3, s_1, s_2)$. *Let* $G = \mathbb{R}^3$ *from (4.6) act diagonally on* $M_5 \times M_5$ *by*

$$\mathbf{a}(\mathbf{z},\mathbf{t}) \times (\mathbf{w},\mathbf{s}) = (\mathbf{z}+\mathbf{a},\mathbf{t}) \times (\mathbf{w}+\mathbf{a},\mathbf{s})$$

where $\mathbf{a} \in \mathbb{R}^3$. *The quotient* $(M_5 \times M_5)/G$ *is a seven dimensional manifold* $M_7 = (\bar z_1, \bar z_2, \bar z_3, t_1, t_2, s_1, s_2)$, *and the quotient map* $\mathbf{q} : M_5 \times M_5 \to M_7$ *is*

$$\mathbf{q} : (\mathbf{z},\mathbf{t}) \times (\mathbf{w},\mathbf{s}) = (\bar{\mathbf{z}} = \mathbf{z} - \mathbf{w}, \mathbf{t}, \mathbf{s}),$$

where $\bar z_i = z_i - w_i, 1 \le i \le 3$. *The quotient Pfaffian system* $(I_1 \oplus I_2)/G = \bar I$ *is easy to compute and is the rank three Pfaffian system,*

$$\bar I = \left\{\frac{1}{2}d\bar z_1 - t_2 dt_1 + s_2 ds_1,\ d\bar z_2 - \frac{1}{2}t_1^2 dt_2 + \frac{1}{2}s_1^2 ds_2, d\bar z_3 + \frac{1}{2}t_2^2 dt_1 - \frac{1}{2}s_2^2 ds_1\right\}.$$

By making a change of coordinates

$$\bar z_1 = \frac{4(u_y - y u_{yy} - x u_{xy})}{u_{yy}},\quad \bar z_2 = \frac{6u_{yy}^3 u_x - x - 3x u_{xy}^2 u_{yy}^2}{u_{yy}^3},$$

$$\bar z_3 = \frac{x^2(1 + 3u_{yy}^2 u_{xy}^2)}{3u_{yy}^3} + 2(x u_x - u) + \frac{u_y(u_y - 2x u_{xy})}{u_{yy}},$$

$$t_1 = \frac{u_{xy}u_{yy} - 1}{u_{yy}},\quad t_2 = \frac{x u_{xy} u_{yy} - u_y u_{yy} - x}{u_{yy}},$$

$$s_1 = \frac{u_{xy}u_{yy} + 1}{u_{yy}},\quad s_2 = \frac{x u_{xy} u_{yy} - u_y u_{yy} + x}{u_{yy}},$$

we get

$$\bar{I} = \left\{ du - u_x dx - u_y dy,\ du_x + \frac{1}{3u_{yy}^3} dx - u_{xy} dy,\ du_y - u_{xy} dx - u_{yy} dy \right\}. \quad (4.7)$$

This is the standard Pfaffian system for the non-Monge-Ampere partial differential equation,

$$3u_{xx}u_{yy}^3 + 1 = 0. \quad (4.8)$$

By taking integral manifolds to I_1 *and* I_2*, the map* $\mathbf{q}$ *produces the general solution to this non-Monge-Ampere equation by a non-linear superposition of solutions to the Cartan-Hilbert system. The solution is given implicity by*

$$\begin{aligned} x &= \frac{(g' - \dot{f})}{2(t-s)}, \quad y = \frac{1}{8}(\dot{f} + g')(t-s) + \frac{1}{4}(g - f) \\ u &= \frac{1}{2}(G_2 - F_2) - \frac{(tg' - s\dot{f})^2}{24(t-s)} - \frac{(sg' + t\dot{f})((2s-t)g' + (2t-s)\dot{f})}{48(t-s)} \\ &\quad + \frac{ts\dot{f}g'}{12(t-s)} - \frac{(F_1 - G_1)(g' - \dot{f})}{4(t-s)} \end{aligned}$$

where

$$(f(t_1), F_1(t_1), F_2(t_1)), \quad \text{and} \quad (g(s_1), G_1(s_1), G_2(s_1))$$

are integral manifolds of the corresponding system (4.5).

REMARK 4.7. The Pfaffian system in Equation (4.7) for the non Monge-Ampere partial differential equation in the plane (4.8) is the quotient of two fairly simple Pfaffian systems. The quotient allows us to find the general solution to the partial differential equation (4.8). The Pfaffian system (4.7) is called *Darboux Integrable.* It can be shown that a Darboux integrable EDS can be given explicitly by a non-linear superposition (or G-quotient) of two "simple" EDS. Darboux integrability also occurs for systems of equations, such as the harmonic map and the Toda Molecule equation. See [3] for more details.

5. The moment map. The procedure in Sections 4.1 and 4.2 for finding integral manifolds involves two steps. The first step is to restrict the Pfaffian system to a G-invariant set (K or M^t). The second step is to compute the quotient Pfaffian system on the invariant set.

This two step process is very similar to symplectic [13] or contact reduction [12] where a moment map is used. In the reduction process we have outlined, there is a moment map which is a direct generalization of the moment map in contact geometry (a contact manifold is a particular rank one Pfaffian system).

We begin with a local description of this map. Suppose I is constant rank r Pfaffian and $\{\theta^i\}_{1\leq i\leq r}$ is a basis of local sections. The moment $\mu : M \to \mathbb{R}^r \otimes \mathbf{g}^*$ given by

$$\theta^i(X)\,, \quad X \in \mathbf{g}.$$

In this equation we have identified $\mathbf{g}$ with the infinitesimal generators Γ. The set K in Section 4.1 is then the zero-set of the moment map, while M^t is the full rank set for the moment map. The reduction process we have described is then similar to that for symplectic or contact reduction - restriction then quotient.

A global description of μ can be given [12]. Let $L = TM/I^\perp$ be the quotient vector-bundle, and let $\Sigma : TM \to TM/I^\perp$ be the vector-bundle projection map. The moment map μ is a section of $Hom(\mathbf{g}, L) = \mathbf{g}^* \otimes L$ given by

$$\mu_x(X) = \Sigma(X_x) \quad X \in \mathbf{g}.$$

The argument in [12] for contact manifolds proves the following theorem.

THEOREM 5.1. *The moment map $\mu : M \to Hom(\mathbf{g}, L)$, is equivariant.*

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