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# Integrating Scalar Ordinary Differential Equations with Symmetry Revisited

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**Abstract.** The process of integrating an $n$th-order scalar ordinary differential equation with symmetry is revisited in terms of Pfaffian systems. This formulation immediately provides a completely algebraic method to determine the initial conditions and the corresponding solutions which are invariant under a one parameter subgroup of a symmetry group. To determine the noninvariant solutions the problem splits into three cases. If the dimension of the symmetry groups is less than the order of the equation, then there exists an open dense set of initial conditions whose corresponding solutions can be found by integrating a quotient Pfaffian system on a quotient space, and integrating an equation of fundamental Lie type associated with the symmetry group. If the dimension of the symmetry group is equal to the order of the equation, then there exists an open dense set of initial conditions whose corresponding solutions are obtained either by solving an equation of fundamental Lie type associated with the symmetry group, or the solutions are invariant under a one-parameter subgroup. If the dimension of the symmetry group is greater than the order of the equation, then there exists an open dense set of initial conditions where the solutions can either be determined by solving an equation of fundamental Lie type for a solvable Lie group, or are invariant. In each case the initial conditions, the quotient Pfaffian system, and the equation of Lie type are all determined algebraically. Examples of scalar ordinary differential equations and a Pfaffian system are given.

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## 1. Introduction

A method to simplify the integration of a scalar ordinary differential equation (ODE) by utilizing its symmetry group was known to Lie and is described in detail in numerous texts [**13**], [**3**]. This classical procedure involves performing a sequence of differential substitutions in which the original differential equation becomes one of lower order. Solutions to the original differential equation are then found by first integrating the lower-order equation and then back integrating the differential substitutions. In this paper we reconsider this integration process geometrically in terms of Pfaffian systems.

Every scalar $n$th-order ODE can be identified with an $(n + 1)$-dimensional differentiable manifold $R$ and a rank $n$ Pfaffian system $I$ on $R$. The manifold $R$ can be considered as the domain of initial conditions for the differential equation, and a solution to the differential equation is an integral manifold of $I$. A symmetry group of the differential equation is then a Lie group $G$ which acts on the manifold $R$ and preserves $I$.

Lie's integration process admits a simple geometric description in terms of $R$, $I$, and $G$. The geometric equivalent to the reduction of order step in Lie's process becomes finding a quotient space $\bar{R}$, and the quotient Pfaffian system $\bar{I}$ on $\bar{R}$. The back integration of the differential substitution becomes the problem of solving the horizontal lift equations for a connection on a principle $G$ bundle. This geometric formulation can be thought of as the dual point of view to the integration process in [**7**] and [**6**].

Approaching the integration problem in this way has a number of theoretical as well as practical advantages. First, the subset of the initial conditions for the solutions which are invariant under a one-parameter subgroup are determined algebraically. Then the subset of initial conditions on which the reduction process is valid is also determined completely algebraically. These domains are determined before any integration is performed. In comparison, the domain of application and the role of the invariant solutions in the algorithm given in [**13**] or [**3**] is difficult to determine. For example, it might not be possible to find the general solution (6.7) in Example 6.3 in Section 6 using the algorithm from these references.

A second advantage to our approach is that the algebraic steps in the reduction process are easily identified. For example, the computation of the quotient Pfaffian system $\bar{I}$ can be done using only algebraic operations.

The last advantage we mention is related to the differential equation which determines a horizontal lift for a connection on a principle fiber bundle. This equation is a special differential equation known as an equation of fundamental Lie type. Equations of fundamental Lie type are geometric differential equations on a Lie group which have many useful properties. For example, if the group is solvable, then the equation can be integrated by quadratures (this may only be true locally if the group is not simply connected). This property leads to a geometric proof of the well-known theorem that "an $n$th-order scalar ordinary differential with an $n$-dimensional solvable symmetry group can be integrated by quadrature."

Our approach allows us to precisely state the hypothesis and domain on which this theorem is valid.

In order to use the symmetry group of a scalar ODE to simplify solving the equation, the first step suggested by the geometry of the problem is to consider the subset $R^{nt} \subset R$ of initial conditions where solutions with initial values in $R^{nt}$ are contained in the orbit of a one-parameter subgroup of the symmetry group. This set of initial conditions and the corresponding maximal solutions, which are the orbits of one-parameter subgroups of the symmetry, are easily found using only algebraic operations.

The geometry of the problem then leads us to consider three distinct cases. The different cases are distinguished by the dimension of the group $G$. The first case is when the dimension of the symmetry group is less than the order of the equation. In this case there exists a dense $G$-invariant open set of initial conditions $R^0 \subset R$ where the reduction process described above is valid. The set $R^0$ is determined algebraically from the symmetry group $G$ and the equation.

The second case is where the dimension of the symmetry is equal to the order of the equation. In this case there exists an algebraically determined $G$-invariant open set $R^0$ of initial condition where the union $R^0 \cup R^{nt}$ is dense in the set $R$ of all initial conditions. Solutions with initial conditions in $R^0$ are found by solving an equation of fundamental Lie type. There are differential equations where $R^0$ is empty and all solutions can be found algebraically. These equations are easily characterized, and an example is given.

The last case consists of differential equations where the dimension of the symmetry group is greater than the order of the equation. In this case there exists an algebraically determined open set of initial conditions $R^0$ such that the union $R^0 \cup R^{nt}$ is dense in $R$. Solutions with initial values in $R^0$ can be found by quadratures.

The contents of the paper are as follows. Section 2 contains the necessary background on Pfaffian systems with symmetry. Included in Section 2 are some basic results on invariant integral manifolds of completely integrable Pfaffian systems. These results are used in Sections 3, 4, 5, and 6 to study the invariant solutions to ODEs.

Section 3 begins with an introduction to group actions on the jet space $J^n(\mathbb{R}, \mathbb{R})$. The second part of Section 3 provides some results on the orbit structure of contact transformation on $J^n(\mathbb{R}, \mathbb{R})$. These results are essential in determining the domain of application of the integration process.

Section 4 begins with the basic definitions which make a scalar ODE into a geometric quantity. External symmetry groups are also defined. We generalize these definition and define a geometric structure on a manifold called an $n$th-order scalar ODE structure. This definition is essential in order to be able to make a precise statement on when the quotient of an ODE by an external symmetry group is an ODE of lower order. The results of Section 3 are used to study the orbit structure of a symmetry group of an $n$th-order ODE structure. The quotient of an ODE structure by a symmetry group is then shown to be an ODE structure of lower order.

Section 5 contains the geometric analogue of the two-step integration method of Lie. At the end of Section 5 the results of the preceding sections are summarized in terms of the practical matter of integrating a scalar ODE.

Lastly, Section 6 contains four examples demonstrating the theory.

## 2. Completely Integrable Pfaffian systems with Symmetry

### 2.1. *Preliminaries*

Let $M$ be an $n$-dimensional differentiable manifold, $T^*M$ the cotangent bundle, and $\Omega^1(M)$ the space of differential one-forms on $M$. A Pfaffian system $\mathcal{I}$ is a submodule $\mathcal{I} \subset \Omega^1(M)$ over $C^\infty(M)$. The Pfaffian system $\mathcal{I}$ is of constant rank $r$ if there exists a rank $r$ subbundle $I \subset T^*M$ such that $\mathcal{I}$ is the space of sections of $I$. We will only consider constant rank Pfaffian systems and simply refer to $I$ as a Pfaffian system.

The algebraic ideal in $\Omega^*(M)$ generated by a Pfaffian system $I$ will be denoted by $\langle I \rangle$. If $\theta \in I$, then by $d\theta$ mod $I$ we mean modulo $\langle I \rangle \cap \Omega^2(M)$. If $U$ is an open subset in $M$, then $I(U) \subset T^*U$ is the restriction of the bundle $I$ to $U$ while $\mathcal{I}(U)$ are the corresponding sections.

Given a set $S = \{\theta^a\}_{a\in A}$ where $\theta^a \in \Omega^1(M)$, we denote by $\mathcal{I} = [\theta^a]_{a\in A}$ the Pfaffian system generated by $S$ over $C^\infty(M)$. If $\mathcal{I}$ is also of constant rank, we also write $I = [\theta^a]_{a\in A}$ for the corresponding bundle.

The derived system $\mathcal{I}' \subset \mathcal{I}$ is defined by

$$\mathcal{I}' = [\theta \in \mathcal{I} \mid d\theta = 0 \text{ mod } I].$$

We will assume that $\mathcal{I}'$ is constant rank so there exists a rank $r'$ subbundle $I' \subset T^*M$ such that $\mathcal{I}'$ is the space of sections of $I'$. The bundle $I'$ is determined pointwise by

$$I'_x = \text{span}\{\theta_x \mid \theta \in \mathcal{I}(U) \text{ and } d\theta = 0 \text{ mod } I(U)\},$$

where $x \in M$ and $U \subset M$ is any open neighborhood of $x$.

Setting $I^{\langle 1\rangle} = I'$, the higher derived systems are defined inductively and, assuming they are of constant rank, we have $I^{\langle k\rangle} = (I^{\langle k-1\rangle})'$. This leads to the derived flag

$$I^{\langle\infty\rangle} \subset \cdots \subset I^{\langle 2\rangle} \subset I^{\langle 1\rangle} \subset I^{\langle 0\rangle} \subset T^*M,$$

where $I^{\langle 0\rangle} = I$.

The annihilator $I^\perp \subset TM$ of the rank $r$ Pfaffian system $I$ is the rank $n - r$ subbundle defined pointwise as

$$I_x^\perp = \{X \in T_xM \mid \theta(X) = 0 \text{ for all } \theta \in I_x\}. \quad (2.1)$$

A Pfaffian system $I$ is *completely integrable* if $I^\perp$ is completely integrable. Equivalently, the Pfaffian system $I$ is completely integrable if and only if $I' = I$. The complete integrability of $I$ can also be checked locally.

**Corollary 2.1.** *A Pfaffian system $I$ is completely integrable if and only if for all $x \in M$ there exists an open set $U$ containing $x$, and a basis of sections $\{\theta^i\}_{1 \le i \le r}$ of $I(U)$ such that*[1]

$$d\theta^i = \rho^i_j \wedge \theta^k \tag{2.2}$$

*for some $\rho^j_k \in \Omega^1(U)$, $1 \le j, k \le n$.*

An *integral manifold* of a Pfaffian system $I$ is an immersion $\iota : N \to M$ satisfying

$$\iota^*\theta = 0 \qquad \text{for all} \quad \theta \in I. \tag{2.3}$$

Condition (2.3) implies $\dim N \le n - r$. If $I$ is of codimension 1 (rank $n - 1$), then the integral manifolds are curves. An integral manifold $N$ is maximal if $N$ is connected and the image of $N$ is not a proper subset of another connected integral manifold. If $I$ is completely integrable, then the existence of integrable manifolds is given by Frobenius' Theorem [**16**].

**Theorem 2.2** (Frobenius' Theorem). *Let $I$ be a completely integrable rank $r$ Pfaffian system on the $n$-dimensional manifold $M$. Then through each point $p \in M$ there exists a unique maximal integral manifold having dimension $n - r$.*

See [**4**] for more information on Pfaffian systems.

### 2.2. *Symmetries of Pfaffian Systems and Quotients*

Let $G$ be a Lie group of dimension $m$ acting on $M$ with multiplication map $\mu : G \times M \to M$. The notation $\mu(g, p) = g\,p$ will be used. Let $\mu_g : M \to M$ be the diffeomorphism $\mu_g(p) = \mu(g, p)$, and let $\mu_p : G \to M$ be the function $\mu_p(g) = \mu(g, p)$. Denote by $\mathbf{g} = T_eG$ the Lie algebra of $G$, with the Lie bracket on $\mathbf{g}$ being induced by the bracket for the *right* invariant vector-fields.

The homomorphism $\rho : \mathbf{g} \to \chi(M)$ to the vector-fields $\chi(M)$ on $M$ is defined by

$$\rho(\mathbf{z}) = Z \qquad \text{where} \quad Z_p = \frac{d}{dt}\mu(\exp(t\mathbf{z}), p)\Big|_{t=0}, \quad \mathbf{z} \in \mathbf{g}, \tag{2.4}$$

and $\exp(t\mathbf{z})$ is the one-parameter subgroup of $G$ generated by $\mathbf{z}$. It follows from this formula that

$$Z_p = (\mu_p)_*\mathbf{z}. \tag{2.5}$$

The image of $\mathbf{g}$ by $\rho$ is the Lie algebra of infinitesimal generators which will be denoted by $\Gamma = \rho(\mathbf{g})$. We also let $\boldsymbol{\Gamma} \subset TM$ be the completely integrable distribution defined pointwise by $\boldsymbol{\Gamma}_p = \text{span}\{\Gamma(p)\}$.

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[1] The summation convention will be used throughout.

**Lemma 2.3.** *Let $\mathbf{z} \in \mathbf{g}$ and let $Z = \rho(\mathbf{z})$. Then, for all $p \in M$ and $g \in G$,*

(1) $g_* Z_p = \rho(Ad_g \mathbf{z})_{gp}$;
(2) $(\mu_p)_*(L_g)_* \mathbf{z} = \rho(Ad_g \mathbf{z})_{gp}$; *and*
(3) $g_* \boldsymbol{\Gamma}_p = \boldsymbol{\Gamma}_{gp}$.

*Proof.* A proof of (1) is given on page 269 in [**1**]. To prove (2), let $L_g$ denote left multiplication in $G$ by $g \in G$. Then, as functions from $G$ to $M$,

$$\mu_p \circ L_g = \mu_g \circ \mu_p. \tag{2.6}$$

The differential of equation (2.6) is

$$(\mu_p)_*(L_g)_* = (\mu_g)_*(\mu_p)_*.$$

Evaluating this on $\mathbf{z} \in \mathbf{g}$ and using part (1) proves part (2).

Part (1) implies that $g_*(\boldsymbol{\Gamma}_p) \subset \boldsymbol{\Gamma}_{gp}$ and $(g^{-1})_*(\boldsymbol{\Gamma}_{gp}) \subset \boldsymbol{\Gamma}_p$. This proves (3). □

A Lie group $G$ acting on $M$ acts locally effectively on subsets if for every nonempty open set $U \subset M$, the subgroup of $G$ which fixes every point in $U$ is discrete.

**Lemma 2.4.** *A Lie group $G$ acting on $M$ acts locally effectively on subsets if and only if for each open set $U \subset M$ the map $\rho : \mathbf{g} \to \Gamma_U$ is an isomorphism.*

*Proof.* Suppose $U$ is an open set of $M$ and there exists $\mathbf{z} \in \mathbf{g}$, with $\rho(\mathbf{z})_U = 0$. Let $H \subset G$ be the one-parameter subgroup generated by $\mathbf{z}$. Then each point in $U$ is invariant under $H$. It easily follows from this argument that the action is locally effective on subsets if and only if $\rho : \mathbf{g} \to \Gamma_U$ is an isomorphism. □

A Lie group $G$ acts *locally freely* on $M$ if, for each $p \in M$, the isotropy subgroup

$$G_p = \{g \in G \mid gp = p\}$$

is discrete. Let $M^\ell \subset M$ be the open subset

$$M^\ell = \{p \in M \mid \operatorname{rank} \boldsymbol{\Gamma}_p = m\}, \tag{2.7}$$

where $m = \dim G$. The next lemma shows that $M^\ell$ is the maximal subset of $M$ on which $G$ acts locally freely.

**Lemma 2.5.** *Let $G$ be an $m$-dimensional Lie group acting on $M$. Then $G_p$ is discrete if and only if $p \in M^\ell$.*

*Proof.* Let $p \in M$. The isotropy subgroup $G_p$ is discrete if and only if the isotropy subalgebra satisfies

$$\mathbf{g}_p = \ker \rho_p = \{\mathbf{z} \in \mathbf{g} \mid \rho(\mathbf{z})_p = 0\} = 0. \tag{2.8}$$

Therefore $G_p$ is discrete if and only if $\rho_p : \mathbf{g} \to \boldsymbol{\Gamma}_p$ is a vector-space isomorphism which is equivalent to $p \in M^\ell$. □

Clearly if $\dim G > n$, then $G$ can never act locally freely on $M$, and $M^\ell = \varphi$.

Let $M/G$ be the quotient space and let $\mathbf{q} : M \to M/G$ be the projection map. If we use the notation $\overline{M}$ for the set $M/G$, then the action of the Lie group $G$ on $M$ will have been assumed to act on $M$ in a sufficiently regular manner so that $\overline{M}$ is a differentiable manifold with the projection map $\mathbf{q} : M \to \overline{M}$ being a submersion. See [**13**] for more information on this hypothesis. One consequence of this assumption is that the orbits of $G$ on $M$ all have the same dimension. If the orbits are of dimension $q$ then the dimension of $\overline{M}$ is $n - q$. We also note that $\ker \mathbf{q}_* = \boldsymbol{\Gamma}$. We will use the notation $M/G$ when the action of $G$ on $M$ may not have the regularity property above.

A Lie group $G$ acting on $M$ is a *symmetry group of the Pfaffian system $I$* if for all $g \in G$, $g^* I = I$. The elements of a symmetry group of a Pfaffian system map integral manifolds to integral manifolds. That is, if $s : N \to M$ is an integral manifold of $I$ and $g \in G$, then $g \circ s : N \to M$ is an integral manifold of $I$.

We now define the reduction or quotient of a Pfaffian system by a symmetry group. The definition given here is different than the one in [**2**] where the authors work with differential ideals. For the Pfaffian systems we are interested in, the two definitions agree.

**Definition 2.6.** Let $G$ be a symmetry group of the Pfaffian system $I$ on the manifold $M$ with quotient manifold $\overline{M}$. The reduced system $\bar{I} \subset T^*\overline{M}$ is defined by

$$\bar{I}_{\bar{p}} = \{\bar{\theta} \in T^*_{\bar{p}}\overline{M} \mid \mathbf{q}^*\bar{\theta} \in I\}.$$

The subset $\overline{I} \subset T^*\overline{M}$ is not necessarily a bundle without some conditions on the action of $G$. Sufficient conditions that guarantee $\overline{I}$ is a bundle can be described in terms of two canonical intersections. The first intersection is $\boldsymbol{\Gamma} \cap I^\perp \subset TM$ where $I^\perp$ is defined in equation (2.1). This subset of $TM$ plays an important role throughout the paper. The second intersection is $I_{sb} = I \cap \boldsymbol{\Gamma}^\perp \subset T^*M$ where

$$\boldsymbol{\Gamma}_p^\perp = \{\alpha \in T^*_pM \mid \alpha(X) = 0 \text{ for all } X \in \boldsymbol{\Gamma}_p\}.$$

The semibasic forms $I_{sb} = I \cap \boldsymbol{\Gamma}^\perp$ are therefore given pointwise by

$$I_{sb,p} = \{\theta \in I_p \mid \theta(X) = 0 \text{ for all } X \in \boldsymbol{\Gamma}_p\}.$$

**Theorem 2.7.** *Let $I$ be a rank $r$ Pfaffian system on $M$, and $G$ a symmetry group of $I$ with $q$-dimensional orbits. If $\boldsymbol{\Gamma} \cap I^\perp$ is of constant rank $k$, then $I_{sb} \subset T^*M$ and $\overline{I} \subset T^*\overline{M}$ are constant rank $r + k - q$ Pfaffian systems.*

*Proof.* If $\boldsymbol{\Gamma} \cap I^\perp$ has constant rank $k$, then $0 \le k \le \min(n-r, q)$ and $I_{sb} = I \cap \boldsymbol{\Gamma}^\perp$ is of constant rank $r + k - q$. Therefore $I_{sb}$ is a subbundle.

The rank $k$ assumption for $\Gamma \cap I^{\perp}$ implies that the projection $\mathbf{q}_*(I^{\perp})$ is a subbundle of $T^*\overline{M}$ of rank $n-r-k$. We now check that the rank $r+k-q$ subbundle $(\mathbf{q}_*(I^{\perp}))^{\perp}$ is $\bar{I}$. Let $\bar{\alpha} \in T^*_{\bar{p}}M$, then $\bar{\alpha}$ satisfies $\bar{\alpha} \in \bar{I}_{\bar{p}}$ if and only if $\mathbf{q}^*\bar{\alpha}(X_p) = \bar{\alpha}(\mathbf{q}_*(X_p)) = 0$ for all $X \in I^{\perp}_p$. That is, if and only if $\bar{\alpha} \in (\mathbf{q}_*(I^{\perp})_{\bar{p}})^{\perp}$. Therefore $\bar{I} = (\mathbf{q}_* I^{\perp})^{\perp}$ is a subbundle of rank $r+k-q$. □

The next corollary, whose proof can be found in [**2**], shows that local generators for $\bar{I}$ can be computed algebraically from local generators for $I_{sb}$, and conversely.

**Corollary 2.8.** *Let $U \subset M$ be an open set and $\sigma : \overline{U} \to U$ a cross-section, where $\overline{U} = \mathbf{q}(U)$. If $\{\theta^i_{sb}\}_{1\le i\le r+k-q}$ form a local basis of sections for $I_{sb}(U)$, then $\{\sigma^*\theta^i_{sb}\}_{1\le i\le r+k-q}$ form a local basis for $\bar{I}(\overline{U})$. Conversely, if $\{\bar{\theta}^i\}_{i=1,\dots,r+k-q}$ form a local basis for $\bar{I}(\overline{U})$, then $\{\mathbf{q}^*\bar{\theta}^i\}_{1\le i\le r+k-q}$ form a local basis for $I_{sb}(U)$.*

If $I$ is completely integrable, then the next theorem answers whether $\bar{I}$ is also.

**Theorem 2.9.** *Let $G$ be a symmetry group of the rank $r$ completely integrable Pfaffian system $I$ satisfying* $\operatorname{rank}(\Gamma \cap I^{\perp}) = k$. *Then $I_{sb}$ and $\bar{I}$ are completely integrable rank $r+k-q$ Pfaffian systems.*

*Proof.* Let $p \in M$ and chose an open set $U \subset M$ about $p$ such that $T^*U$ and $\bar{I}(\overline{U})$, where $\overline{U} = \mathbf{q}(U)$, are trivial. Let $\{\bar{\theta}^\alpha\}_{1\le\alpha\le r+k-q}$ be a basis of section for $\bar{I}(\overline{U})$ and let $\theta^\alpha = \mathbf{q}^*\bar{\theta}^\alpha$ which form a basis of sections for $I_{sb}(U)$. Extend this to a basis $\{\theta^\alpha, \eta^i\}_{1\le i\le q-k}$ for the sections of $I(U)$, and then further extend this to a basis $\{\theta^\alpha, \eta^i, \omega^a\}_{1\le a\le n-r}$ for $T^*U$.

The forms $\eta^i$ are a basis of sections for $I(U)$ mod $I_{sb}(U)$, and so we may choose $X_j$, $j = 1\dots q-k$, sections of $\Gamma(U)$ satisfying $\eta^i(X_j) = \delta^i_j$ on $U$.

The complete integrability of $I$ implies that

$$d\theta^\alpha = \rho^\alpha_\beta \wedge \theta^\beta + P^\alpha_{ij}\eta^i \wedge \eta^j + Q^\alpha_{aj}\omega^a \wedge \eta^j \tag{2.9}$$

where $\rho^\alpha_\beta \in \Omega^1(U)$ and $P^\alpha_{ij}, Q^\alpha_{aj} \in C^\infty(U)$. Evaluating equation (2.9) on $X_l$ and using the fact that the forms $\theta^\alpha$ are actually $G$-basic, we get

$$0 = \rho^\alpha_\beta(X_l)\theta^\beta + 2P^\alpha_{lj}\eta^j - Q^\alpha_{al}\omega^a, \qquad 1 \le l \le q-k.$$

Therefore, $P^\alpha_{lj} = Q^\alpha_{al} = 0$ and so $I_{sb}$ is completely integrable.

To show that $\bar{I}$ is completely integrable, let $\bar{p} = \mathbf{q}(p)$, and let $\sigma : \overline{U} \to U$ be a cross-section such that $p = \sigma(\bar{p})$. By pulling back equation (2.9) with $\sigma$ and taking into account $P^\alpha_{kj} = Q^\alpha_{ak} = 0$ we get

$$d\bar{\theta}^\alpha = \sigma^*(\rho^\alpha_\beta) \wedge \bar{\theta}^\beta.$$

Therefore $\bar{I}$ is completely integrable. □

2.3. *Transversality*

Let $G$ be a symmetry group of the rank $r$ Pfaffian system $I$ on $M$. Given $p \in M$, the group $G$ *acts transversally* to $I$ at $p$ if

$$\boldsymbol{\Gamma}_p \cap I_p^{\perp} = 0.$$

Let $M^t \subset M$ be the set of points in $M$ where $G$ acts transversally to $I$,

$$M^t = \{p \in M \mid \boldsymbol{\Gamma}_p \cap I_p^{\perp} = 0\} = \{p \in M \mid \operatorname{rank} \theta^i(X_\alpha) = q\}, \tag{2.10}$$

where $\{\theta^i\}_{1 \le i \le r}$ is a basis for $I_p$ and $\{X_\alpha\}_{1 \le \alpha \le q}$ is a basis for $\boldsymbol{\Gamma}_p$. Let $M^{nt} \subset M$ be the complement of $M^{nt}$ in $M$,

$$M^{nt} = M - M^t = \{p \in M \mid \boldsymbol{\Gamma}_p \cap I_p^{\perp} \neq 0\}. \tag{2.11}$$

The set $M^{nt}$ consists of points in $M$ where $G$ is not transverse to $I$.

**Lemma 2.10.** *The subsets $M^t$, $M^{nt} \subset M$ are $G$-invariant.*

*Proof.* Let $p \in M$. The annihilator of the symmetry equation $g^* I_{gp} = I_p$ is

$$I_{gp}^{\perp} = g_*(I_p^{\perp}).$$

By combining this equation with part (3) of Lemma 2.3, we have

$$\boldsymbol{\Gamma}_{gp} \cap I_{gp}^{\perp} = g_*(\boldsymbol{\Gamma}_p \cap I_p^{\perp}).$$

This equation implies that if $p \in M^{nt}$ (or $M^t$), then $gp \in M^{nt}$ (or $M^t$). □

The group $G$ is a *transverse symmetry group of I* if $G$ is transverse to $I$ at every point in $M$, and so $M^t = M$. If $G$ is transverse to $I$, then Theorems 2.7 and 2.9 have the following corollaries.

**Corollary 2.11.** *If $G$ is a transverse symmetry group of the Pfaffian system $I$ with $q$-dimensional orbits on $M$, then $\bar{I}$ is a rank $r - q$ bundle. If $I$ is completely integrable, then so are $I_{sb}$ and $\bar{I}$.*

Other properties of transverse symmetry groups can be found in [**2**]. For example, Proposition 4.1 of [**2**] implies the following.

**Lemma 2.12.** *Let $G$ be a transverse symmetry group of the Pfaffian system $I$. If $s : N \to M$ is an integral manifold of $I$, then $\mathbf{q} \circ s : N \to \overline{M}$ is an integral manifold of $\bar{I}$.*

## 2.4. *Integral Orbits*

In this section we consider the problem of constructing integral manifolds which are orbits of a subgroup of a symmetry group of a completely integral Pfaffian system $I$. Throughout this section $G$ is an $m$-dimensional symmetry group of the completely integrable rank $r$ Pfaffian system $I$.

Let $p \in M$ and define $\delta^p \subset \Gamma$ by

$$\delta^p = \{Z \in \Gamma \mid Z_p \in I_p^\perp\}. \tag{2.12}$$

If $\delta^p \neq 0$, then by equation (2.11) $p \in M^{nt}$.

**Lemma 2.13.** *The subspace $\delta^p \subset \Gamma$ is a Lie subalgebra.*

*Proof.* Let $X, Y \in \delta^p$. We show that $[X, Y]_p \in I_p^\perp$. Let $U$ be an open neighborhood of $p$ and $\{\theta^i\}_{1\le i\le r}$ a local basis of sections of $I$ satisfying (2.2). The vector-fields $X$ and $Y$ are infinitesimal generators of the symmetry group $G$, therefore,

$$\mathcal{L}_X\theta^i = F^i_j\theta^j \qquad \text{and} \qquad \mathcal{L}_Y\theta^i = G^i_j\theta^j, \tag{2.13}$$

for some $F^i_j, G^i_j \in C^\infty(U)$. Taking the identity

$$\theta^i([X, Y]) = (\mathcal{L}_Y\theta^i)(X) - (\mathcal{L}_X\theta^i)(Y) + d\theta^i(X, Y),$$

and substituting from (2.2) and (2.13) we get

$$\theta^i([X, Y]) = G^i_j\theta^j(X) - F^i_j\theta^j(Y) + (\rho^i_j \wedge \theta^j)(X, Y), \qquad \rho^i_j \in \Omega^1(U).$$

By evaluating this equation at $p$ and using the fact that $X_p, Y_p \in I_p^\perp$, proves $\theta^i([X, Y])_p = 0$. Therefore, $[X, Y]_p \in I_p^\perp$ and so $\delta^p$ is a subalgebra. □

The sought after integral manifold through $p$ will be the orbit of a subgroup of $G$ having its Lie algebra isomorphic to $\delta^p$.

Let $\mathbf{g}_p$ be the isotropy subalgebra defined in equation (2.8) and let $\mathbf{h} \subset \mathbf{g}$ be the Lie subalgebra defined by $\mathbf{h} = \rho^{-1}(\delta^p)$, where $\rho$ is the homomorphism in (2.4). Clearly $\mathbf{g}_p \subset \mathbf{h}$, and is a subalgebra. Let $H$ be a Lie subgroup of $G$ with Lie algebra $\mathbf{h}$, and let $H_p \subset H$ be the closed Lie subgroup with Lie algebra $\mathbf{g}_p$,

$$H_p = H \cap G_p = \{h \in H \mid hp = p\}.$$

Let $\pi : H \to H/H_p$ be the standard smooth quotient map.

Let $s : H/H_p \to M$ be the smooth function

$$s([h]) = s(\pi(h)) = \mu(h, p) = \mu_p(h) = h \cdot p, \qquad h \in H. \tag{2.14}$$

The function $s$ identifies the orbit of $H$ through $p$ with the homogeneous space $H/H_p$.

**Theorem 2.14.** *The function $s : H/H_p \to M$ is an integral manifold.*

*Proof.* We begin by checking that $s$ is an immersion, which is standard. Let $h \in H$, $\hat{z} \in T_{[h]}H/H_p$, and $\mathbf{z} \in \mathbf{h}$ such that $h_*\pi_*\mathbf{z} = \hat{\mathbf{z}}$. Consider the following equation:

$$s_*(\hat{\mathbf{z}}) = s_*(h_*\pi_*\mathbf{z}) = s_*(\pi_*h_*\mathbf{z}) = (\mu_p)_*(\mu_h)_*\mathbf{z} = (\mu_h)_*\rho(\mathbf{z})_p, \tag{2.15}$$

which follows from equations (2.14), (2.6), and (2.5). The function $\mu_h$ is a diffeomorphism of $M$, and so if equation (2.15) is 0, then $\rho(\mathbf{z})_p = 0$. Therefore $\mathbf{z} \in \mathbf{g}_p$, which implies $\pi_*\mathbf{z} = 0$, and that $\hat{\mathbf{z}} = 0$. Thus $s$ is an immersion.

To show $s$ is an integral manifold we let $\theta \in I_{hp}$ and use equation (2.15) to get

$$\theta(s_*(\hat{\mathbf{z}})) = \theta((\mu_h)_*(\mu_p)_*\mathbf{z}) = (\mu_h^*\theta)(\rho(\mathbf{z})_p). \tag{2.16}$$

By the invariance of $I$, $\mu_h^*\theta \in I^p$, and by the definition of $\mathbf{h}$, $\rho(\mathbf{z})_p \in I_p^\perp$. Therefore (2.16) is zero and $s : H/H_p \to M$ is an integral manifold. □

If $I$ is of codimension one, then Theorem 2.14 has the following corollaries.

**Corollary 2.15.** *Let $I$ be a rank $n$ Pfaffian system on an $(n+1)$-dimensional manifold $M$. If $p \in M^{nt}$, then there exists $\mathbf{z} \in \mathbf{g}$ which is algebraically determined by equation* (2.12) *such that* $\exp(t\mathbf{z})p$ *is the maximal integral curve through $p$.*

Corollary 2.15 is used in the examples to determine the invariant solutions. The next corollary is obvious.

**Corollary 2.16.** *Let $I$ be a rank $n$ Pfaffian system on an $(n+1)$-dimensional manifold $M$. If a symmetry group $G$ acts transitively on $M$, then the maximal integral curve through any point $p \in M$ is the orbit of an algebraically determined one-parameter subgroup.*

## 3. Group Actions on $J^n(\mathbb{R}, \mathbb{R})$

### 3.1. *Preliminaries*

Let $J^n(\mathbb{R}, \mathbb{R})$ be the $(n+2)$-dimensional manifold consisting of the $n$-jets of functions from $\mathbb{R}$ to $\mathbb{R}$, and let $(x, u, u_x, \ldots, u_n)$ be the standard coordinates on $J^n(\mathbb{R}, \mathbb{R})$. The contact one forms

$$\theta^i = du_{i-1} - u_i\, dx, \qquad i = 1, \ldots, n, \tag{3.1}$$

on $J^n(\mathbb{R}, \mathbb{R})$ generate the family of Pfaffian

$$C^a = [\theta^1, \ldots, \theta^a], \qquad a = 1, \ldots, n.$$

The rank of $C^a$ is $a$, and a simple computation shows that $C^{a-1} = (C^a)'$, $a = 1, \ldots, n$, where $C^0 = 0$. The derived flag of $C^n$ is then

$$0 = C^0 \subset C^1 \subset \cdots \subset C^n.$$

The two-dimensional annihilator $(C^n_\sigma)^\perp$ at $\sigma \in J^n(\mathbb{R}, \mathbb{R})$ is computed from equation (3.1) to be

$$(C^n_\sigma)^\perp = \operatorname{span}\{D_x,\ \partial_{u_n}\},$$

where

$$D_x = \partial_x + u_x \partial_u + \cdots + u_n \partial_{u_{n-1}}. \tag{3.2}$$

Let

$$\pi^n_{n-1} : J^n(\mathbb{R}, \mathbb{R}) \to J^{n-1}(\mathbb{R}, \mathbb{R})$$

be the projection, and let $\operatorname{Vert}(\pi^n_{n-1}) = \ker(\pi^n_{n-1})_*$. Then

$$\operatorname{Vert}(\pi^n_{n-1}) = \{\partial_{u_n}\} \subset (C^n)^\perp. \tag{3.3}$$

A Lie group $G$ acting on $J^n(\mathbb{R}, \mathbb{R})$ acts by contact transformations if

$$g^* C^n = C^n. \tag{3.4}$$

That is, $G$ is a symmetry group of the Pfaffian system $C^n$. A symmetry group of a Pfaffian system preserves its derived flag, and so

$$g^* C^a = C^a, \qquad a = 0, \ldots, n.$$

Let $G$ be a Lie group acting on $J^n(\mathbb{R}, \mathbb{R})$ by contact transformations. Bäcklund's Theorem [**13**] states that if $G$ is a Lie group acting on $J^n(\mathbb{R}, \mathbb{R})$ by contact transformations, then the action is the prolongation of the action of $G$ on $J^1(\mathbb{R}, \mathbb{R})$ by contact transformation.

Bäcklund's Theorem has an infinitesimal version. A vector-field $Z$ on $J^n(\mathbb{R}, \mathbb{R})$ is an infinitesimal contact transformation if

$$\mathcal{L}_Z \theta^k = 0 \mod C^n$$

for all $k = 1, \ldots, n$. The infinitesimal form of Bäcklund's Theorem [**12**] states

$$Z = \xi D_x + \sum_{a=0}^{n} Q^a \partial_{u_a}, \tag{3.5}$$

where $Q^0$ is a function on $J^1(\mathbb{R}, \mathbb{R})$, $D_x$ is given in equation (3.2), and

$$\xi = -\partial_{u_x} Q^0, \qquad Q^1 = (\partial_x + u_x \partial_u) Q^0, \qquad Q^a = (D_x)^{a-1} Q^1. \tag{3.6}$$

The functions $Q^a$ can be considered as functions on $J^a(\mathbb{R}, \mathbb{R})$ for each $a = 1, \ldots, n$.

If $\Gamma$ is the Lie algebra of infinitesimal generators of a Lie group of contact transformations and $Z \in \Gamma$, then $Z$ is an infinitesimal contact transformation and has the form of equation (3.5).

The infinitesimal form of Bäcklund's Theorem together with Lemma 2.4 imply the next lemma.

**Lemma 3.1.** *Let $G$ be an $m$-dimensional Lie group acting by contact transformations on $J^n(\mathbb{R}, \mathbb{R})$, $n \geq 2$. The group $G$ acts locally effectively on subsets on $J^n(\mathbb{R}, \mathbb{R})$ if and only if the action of $G$ on $J^k(\mathbb{R}, \mathbb{R})$, $k = 1, \dots, n-1$, is locally effective on subsets.*

### 3.2. *Orbits of Contact Transformations*

In this section we make the following assumptions. The group $G$ is an $m$-dimensional Lie group acting locally effectively on subsets on $J^n(\mathbb{R}, \mathbb{R})$, $n \geq 1$, by contact transformations. The vector-fields $\{Z_\alpha\}_{1 \leq \alpha \leq m}$ are a basis for $\Gamma$ the Lie algebra of infinitesimal generators.

Let $W^\ell \subset J^n(\mathbb{R}, \mathbb{R})$ be the subset defined in equation (2.7), and let $W^t$ and $W^{nt}$ be the subsets defined as in equations (2.10) and (2.11) where $I = C^n$. The set $W^\ell$ is the subset of $J^n(\mathbb{R}, \mathbb{R})$ on which $G$ acts locally freely. The set $W^t$ is the subset on which $G$ is transverse to $C^n$, and $W^{nt}$ is the complement of $W^t$.

The orbit structure of locally effective actions of $G$ on $J^n(\mathbb{R}, \mathbb{R})$ are usually sufficiently complicated so that the quotient $J^n(\mathbb{R}, \mathbb{R})/G$ is not a manifold (as described in Section 2.2). The irregularity of these orbits has important consequences when using symmetry to solve differential equations. The following fundamental theorem will be used to help understand some of the orbit structure.

**Theorem 3.2.** *For any $k \leq \min(m, n)$ the $k \times k$ matrix*

$$P^i_j = \theta^i(Z_j), \qquad 1 \leq i, j \leq k,$$

*is invertible on an open dense subset of $J^n(\mathbb{R}, \mathbb{R})$.*

*Proof.* Let $k \leq \min(m, n)$, and let $U \subset J^n(\mathbb{R}, \mathbb{R})$ be the set where $P^i_j$, $1 \leq i$, $j \leq k$, is invertible,

$$U = \{\sigma \in J^n(\mathbb{R}, \mathbb{R}) \mid \det P^i_j(\sigma) \neq 0, 1 \leq i, j \leq k\}. \tag{3.7}$$

The set $U$ is clearly open.

We will prove $U$ is dense using induction on $k$. We begin by using the infinitesimal form of Bäcklund's Theorem (3.5), and write the infinitesimal generators as

$$Z_\alpha = \xi_\alpha D_x + \sum_{a=0}^{n} Q^a_\alpha \partial_{u_a}, \qquad \alpha = 1, \dots, m.$$

This gives $P^i_j = \theta^i(Z_j) = Q^{i-1}_j$, $1 \leq i, j \leq k$.

Let $k = 1$, and suppose there exists an open set $V \subset J^n(\mathbb{R}, \mathbb{R})$ where at points of $V$,

$$P_1^1 = \theta^1(Z_1) = (du - u_x\, dx)\left(\xi_1 D_x + \sum_{a=0}^{n} Q_1^a \partial_{u_a}\right) = Q_1^0 = 0.$$

Then on $V$, $\xi_1 = 0$, and $Q_1^a = 0$ by (3.6), and so $Z_1 = 0$ on $V$. This contradicts the hypothesis that $G$ acts locally effectively on subsets (Lemma 2.4). Therefore the open set where $P_1^1 \neq 0$ is dense, and the theorem is true for $k = 1$.

Assume the theorem is true for $k - 1$, but that $P_j^i = Q_j^{i-1}$, $1 \leq i, j \leq k$, has rank less than $k$ on an open set $V \subset J^n(\mathbb{R}, \mathbb{R})$. The induction hypothesis implies that there exists an open dense set $V' \subset V$ such that the $k - 1$ by $k - 1$ square matrix $P_b^a = Q_b^{a-1}$, $1 \leq a, b \leq k - 1$, is invertible on $V'$. The submaximality assumption means that the $k \times k$ matrix $P_j^i = Q_j^{i-1}$, $1 \leq i, j \leq k$, is of rank $k - 1$ on $V'$ and, consequently, there exists $r_a \in C^\infty(V')$ such that

$$Q_k^a = \sum_{l=1}^{k-l} r^l Q_l^a, \qquad a = 0 \ldots k - 1, \tag{3.8}$$

at points of $V'$. Now using $Q_k^{a+1} = D_x Q_k^a$, $a = 0 \ldots k - 2$, and differentiating (3.8) we get

$$Q_k^{a+1} - D_x(Q_k^a) = \sum_{l=1}^{k-1} (r^l Q_l^{a+1} - D_x(r^l) Q_l^a - r^l D_x Q_l^a) = 0, \qquad a = 0 \ldots k-2,$$

at points of $V'$. Therefore, on $V'$,

$$\sum_{l=1}^{k-1} D_x(r^l) Q_l^a = 0, \qquad 0 \leq a \leq k - 2.$$

The matrix $Q_l^a$ is invertible on $V'$, and so

$$D_x(r^l) = 0. \tag{3.9}$$

Now differentiating equation (3.8) with respect to $u^n$ and using $\partial_{u^n} Q_j^a = 0$, $j = 1, \ldots, k$, $0 \leq a \leq k-1$, implies that $r^l$ does not depend on $u_n$. This, together with equation (3.9), implies $r^l$ are constants. Now let $Y = Z_k - r^l Z_l \in \Gamma$. On the set $V'$,

$$Y = \left(\xi_k - \sum_{l=1}^{k-1} r^l \xi_l\right) D_x.$$

By applying equations (3.5) and (3.6) to $Y$, we conclude that $Y = 0$ on $V'$. This contradicts the hypothesis that $G$ acts locally effectively on subsets (Lemma 2.4). No such original choice of $V$ exists so $U$ in equation (3.7) is dense. □

When $\dim G = m \le n$, we define the subset $W^0 \subset J^n(\mathbb{R}, \mathbb{R})$ by

$$W^0 = \{\sigma \in J^n(\mathbb{R}, \mathbb{R}) \mid \det \theta^\alpha(Z_\beta)_\sigma \neq 0,\ 1 \le \alpha, \beta \le m\}. \tag{3.10}$$

The set $W^0$ has the following properties.

**Lemma 3.3.** *Let* $\dim G = m \le n$. *Then*:

(1) $W^0$ *is a* $G$*-invariant open dense subset of* $J^n(\mathbb{R}, \mathbb{R})$, *and* $G$ *acts locally freely on* $W^0$.
(2) $G$ *is transverse to* $C^s$, $m \le s \le n$, *on the set* $W^0$.

*Proof.* For part (1), the set $W^0$ is $G$-invariant by the same argument used in the proof of Lemma 2.10. That $W^0$ is dense is just Theorem 3.2 when $m \le n$. The defining condition for $W^0$ implies that at $p \in W^0$, the vectors $\{Z_\alpha(p)\}_{1\le\alpha\le m}$ are linearly independent. By equation (2.7), $W^0 \subset W^\ell$ and so Lemma 2.5 implies $G$ acts locally freely on $W^0$.

For part (2), equation (3.10) implies that equation (2.10) is satisfied at points of $W^0$. Thus $G$ is transverse to $C^m$ on $W^0$. Now $(C^s)^\perp \subset (C^m)^\perp$ for $m \le s \le n$. Therefore, if $\boldsymbol{\Gamma}_p \cap (C^m_p)^\perp = 0$, then $\boldsymbol{\Gamma}_p \cap (C^s_p)^\perp = 0$ for $m \le s \le n$. □

When $\dim G = m \ge n + 1$ the following lemma on the orbit structure of $G$ is known [**12**], but we provide an alternate proof.

**Lemma 3.4.** *If* $\dim G = m \ge n + 1$, *then the set*

$$W^1 = \{\sigma \in J^n(\mathbb{R}, \mathbb{R}) \mid \operatorname{rank} \boldsymbol{\Gamma}_\sigma \ge n + 1\}$$

*is a* $G$*-invariant open dense subset of* $J^n(\mathbb{R}, \mathbb{R})$.

*Proof.* The set $W^1$ is open by definition and $G$-invariant by Lemma 2.3. We only need to show $W^1$ is dense. Theorem 3.2 states that the $n \times n$ matrix, $\theta^i(Z_j)$, $1 \le i$, $j \le n$, is invertible on a dense open set $U$ in $J^n(\mathbb{R}, \mathbb{R})$. Therefore the infinitesimal generators $Z_j$, $j = 1, \ldots, n$, are pointwise linearly independent on $U$.

Assume there exists a nonempty open subset $V \subset J^n(\mathbb{R}, \mathbb{R})$ where the orbits of $G$ have dimension less than $n + 1$. Then on the nonempty open set $V \cap U$ there exists $f^j \in C^\infty(U \cap V)$ such that

$$Z_{n+1} = \sum_{j=1}^{n} f^j Z_j.$$

The Lie derivative of the contact forms with respect to $Z_{n+1}$ on $U \cap V$ is

$$\mathcal{L}_{Z_{n+1}} \theta^i = f^j \mathcal{L}_{Z_j} \theta^i + \theta^i(Z_j)\, df^j.$$

The vector-field $Z_{n+1}$ is a symmetry of the contact structure therefore,

$$\theta^i(Z_j)\, df^j = 0 \mod C^n(U \cap V).$$

Since $\theta^i(Z_j)$ is invertible on $U \cap V$ this equation implies

$$D_x(f^j) = 0 \qquad \text{and} \qquad \frac{\partial f^j}{\partial u_n} = 0.$$

Therefore $f^j$ are constants on $U \cap V$, and the vector-field $Z_{n+1} - f^j Z_j \in \Gamma$ vanishes on $U \cap V$. This contradicts the hypothesis that $G$ acts effectively on subsets (Lemma 2.4). No such $V$ exists and the set $W^1$ is dense. □

The last lemma in this section considers the case where $\dim G = m \geq n + 2$.

**Lemma 3.5.** *If* $\dim G = m \geq n + 2$, *then either*:

(1) *the set*

$$U = \{\sigma \in J^n(\mathbb{R}, \mathbb{R}) \mid \operatorname{rank} \boldsymbol{\Gamma}_\sigma = n + 2\}$$

*is dense in* $J^n(\mathbb{R}, \mathbb{R})$; *or*

(2) $G$ *is solvable or* $\mathbf{g}$ *is* $sl(2, \mathbb{R})$.

*In case* (2), *there exists an* $(n + 1)$*-dimensional solvable subgroup* $H \subset G$.

*Proof.* The open set $U$ consists of points lying on orbits of maximal dimension. Suppose that $U$ is not dense. The set $W^1$ defined in Lemma 3.4 is open and dense and so there exists $p \in W^1 - U$ and an open set $V \subset W^1$ about $p$ with $V \cap U = \varphi$. On the open set $V$ the rank of $\boldsymbol{\Gamma}$ is $n + 1$. Therefore $\Gamma$ restricted to $U \subset J^n(\mathbb{R}, \mathbb{R})$ is an $(n + 2)$-dimensional Lie algebra of vector-fields of contact transformations whose pointwise span is $(n + 1)$-dimensional. The local classification of contact transformations [**12**], [**9**], [**5**] implies there are only a few possibilities. Either the Lie algebra is initially intransitive on an open set in $J^1(\mathbb{R}, \mathbb{R})$, or the action pseudo-stabilizes (see Theorem 5.24, p. 153 of [**12**]). The Lie algebra $\mathbf{g}$ is then either solvable or $sl(2, \mathbb{R})$. This proves part (2). If $G$ is solvable, then the final comment is trivial. If $\mathbf{g}$ is $sl(2, \mathbb{R})$, then $n = 1$, and again the final comment is trivial. □

## 4. Scalar ODE Structures and Quotients

### 4.1. *Scalar ODEs and External Symmetries*

A scalar $n$th-order ODE is a subset $R \subset J^n(\mathbb{R}, \mathbb{R})$ given by the zero-set of a smooth function $F : J^n(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ which depends explicitly on $u_n$. Specifically,

$$R = \{\sigma \in J^n(\mathbb{R}, \mathbb{R}) \mid F(\sigma) = 0\}$$

where $F$ satisfies

$$\left.\frac{\partial F}{\partial u_n}\right|_R \neq 0.$$

The condition $\partial_{u_n} F|_R \neq 0$ has a number of elementary consequences. First, $R \subset J^n(\mathbb{R}, \mathbb{R})$ is an embedded $(n+1)$-dimensional manifold. Second, taken together with equation (3.3) we have

$$T_\sigma J^n(\mathbb{R}, \mathbb{R}) = T_\sigma R \oplus \mathrm{Vert}(\pi^n_{n-1}) \qquad \text{for all} \quad \sigma \in R. \tag{4.1}$$

This implies that the projection map $\pi^n_{n-1}$ restricted to $\pi^n_{n-1} : R \to J^{n-1}(\mathbb{R}, \mathbb{R})$ is a local diffeomorphism. Lastly, about each point $\sigma \in R$ there exists an open set $U \subset J^n(\mathbb{R}, \mathbb{R})$ and smooth function $f : \pi^n_{n-1}(U) \to \mathbb{R}$ such that

$$\begin{aligned} R \cap U = \{(x, u, u_x, \ldots, u_n = f(x, u, u_x, \ldots, u_{n-1}) \mid \\ (x, u, u_x, \ldots, u_{n-1}) \in \pi^n_{n-1}(U)\}. \end{aligned} \tag{4.2}$$

Let $\iota : R \to J^n(\mathbb{R}, \mathbb{R})$ be the inclusion map, and define the family of Pfaffian systems $\{I^a\}_{0\le a\le n}$ on $R$ by

$$I^a = \iota^* C^a, \qquad a = 0, \ldots, n. \tag{4.3}$$

The Pfaffian systems $I^a$ can be written in terms of the forms $\tilde{\theta}^i = \iota^*\theta^i$, $i = 1, \ldots, n$, as

$$I^0 = 0 \qquad \text{and} \qquad I^a = [\tilde{\theta}^1, \ldots, \tilde{\theta}^a], \qquad a = 1, \ldots, n. \tag{4.4}$$

A solution to the differential equation is then an integral manifold $s : N \to R$ of $I^n$, where $s$ satisfies the condition $s^*\iota^*\, dx \neq 0$.

The Pfaffian system $I^n$ is of codimension 1, and so its rank is its only local invariant. However, the collection $\{I^a\}_{0\le a\le n}$ of Pfaffian systems does have local invariants.

**Lemma 4.1.** *The bundles* $\{I^a\}_{0\le a\le n}$ *satisfy*:

$$\begin{aligned} &(1) \quad \operatorname{rank} I^a = a,\ a = 0, \ldots, n; \\ &(2) \quad I^{k-1} = (I^k)',\ k = 1, \ldots, n-1;\ \textit{and} \\ &(3) \quad dI^1 \bmod \langle I^1, I^n \wedge I^n\rangle,\ \textit{is rank } 1 \textit{ if } n > 1,\ \textit{and rank } 0 \textit{ if } n = 1. \end{aligned} \tag{4.5}$$

*Proof.* Let $\sigma \in R$. Equations (3.3) and (4.1) imply $\dim(T_\sigma R \cap (C^n_\sigma)^\perp) = 1$ and so

$$\dim((T_\sigma R)^\perp \cap C^n_\sigma) = \dim(T_\sigma R)^\perp + \dim C^n_\sigma - \dim(T_\sigma R \cap (C^n_\sigma)^\perp)^\perp = 1.$$

Therefore, $I^n_\sigma = C^n_\sigma \subset T^*_\sigma R$ is rank $n$. For the systems $I^a$ we have $[\theta^1, \ldots, \theta^a]_\sigma \subset C^n_\sigma$ is a subspace of dimension $a$, and so $I^a_\sigma = [\theta^1, \ldots, \theta^a]_\sigma$ is of constant rank $a$.

This proves (1). To show (2) we choose a chart $U$ containing $\sigma$ with coordinates $(x, u, u_x, \dots, u_{n-1})$ as in (4.2). In these coordinates

$$\tilde\theta^n = du_{n-1} - f(x, u, u_x, \dots, u_{n-1})\, dx,$$
$$\tilde\theta^k = du_{k-1} - u_k\, dx, \qquad k = 2, \dots, n-1.$$

Therefore, by equation (4.4), $I^{k-1} = (I^k)'$, $k = 1, \dots, n-1$.

We now check condition (3). By computing in the previously chosen chart, we find

$$d\tilde\theta^1 = \tilde\theta^2 \wedge dx \quad \mathrm{mod}\langle dx \wedge \tilde\theta^1, \tilde\theta^k \wedge \tilde\theta^1, \tilde\theta^i \wedge \tilde\theta^j\rangle, \qquad 1 \le i, j, k \le n.$$

Therefore condition (3) is satisfied for $n > 1$. The case $n = 1$ is trivial. □

Let $\mu : G \times J^n(\mathbb{R}, \mathbb{R}) \to J^n(\mathbb{R}, \mathbb{R})$ be an action of $G$ by contact transformations. Then $G$ is an *external symmetry group* of the equation $R \subset J^n(\mathbb{R}, \mathbb{R})$ if the subset $R \subset J^n(\mathbb{R}, \mathbb{R})$ is invariant with respect to the action of $G$ on $J^n(\mathbb{R}, \mathbb{R})$. In this case the action $\mu$ of $G$ on $J^n(\mathbb{R}, \mathbb{R})$ restricts to an action $\mu_R : G \times R \to R$ which by definition satisfies

$$\iota(\mu_R(g, \sigma)) = \mu(g, \iota(\sigma)) \qquad \text{for all} \quad g \in G,\ \sigma \in R.$$

**Lemma 4.2.** *Let $G$ be an external symmetry group of the nth-order ODE $R$, $n \ge 2$. If $G$ acts locally effectively on subsets of $J^n(\mathbb{R}, \mathbb{R})$, then $G$ acts locally effectively on subsets of $R$.*

*Proof.* We begin by noting that the map $\pi^n_{n-1} : R \to J^{n-1}(\mathbb{R}, \mathbb{R})$ is equivariant with respect to $G$, and that by Lemma 3.1 $G$ acts locally effectively on subsets of $J^{n-1}(\mathbb{R}, \mathbb{R})$. Suppose that $G$ does not act locally effectively on subsets of $R$. By Lemma 2.4, there exists a nonempty open set $U \subset R$ and a continuous subgroup $G_U \subset G$ such that $gp = p$ for all $p \in U$, $g \in G_U$. By the comment following equation (4.1) we may assume that $\pi^n_{n-1} : U \to \pi^n_{n-1}(U)$ is a diffeomorphism. We then find

$$g\pi^n_{n-1}(p) = \pi^n_{n-1}(gp) = \pi^n(p) \qquad \text{for all} \quad g \in G_U.$$

Therefore $G$ does not act locally effectively on subsets of $J^{n-1}(\mathbb{R}, \mathbb{R})$ which contradicts Lemma 3.1. □

## 4.2. *Scalar ODE Structures*

In order to prove that the quotient of a scalar ODE by the action of an external symmetry group is again a scalar ODE, we need a coordinate invariant description of a scalar ODE. The definition we choose is motivated by the properties of the flag of Pfaffian systems $\{I^a\}_{0 \le a \le n}$ on the manifold $R$ for a scalar $n$th-order ODE in Lemma 4.1.

**Definition 4.3.** An *nth-order scalar ODE structure* is a pair $(M, \{I^a\}_{0\leq a\leq n})$ where $M$ is an $(n+1)$-dimensional manifold, and $\{I^a\}_{0\leq a\leq n}$ is a flag of Pfaffian systems satisfying conditions (1), (2), and (3) in equation (4.5).

By the discussion in Section 4.1 above, $(R, \{I^a\}_{0\leq a\leq n})$ is an $n$th-order ODE structure. The next theorem shows that an $n$th-order scalar ODE structure can locally be identified with a scalar $n$th-order ODE.

**Theorem 4.4.** *Let $(M, \{I^a\}_{0\leq a\leq n})$ be an nth-order scalar ODE structure. About each point $p \in M$ there exists an open set $U$ with local coordinates $(x, u_x, \ldots, u_{n-1})$, and a smooth function $f : U \to \mathbb{R}$ such that $I^a(U) = [\theta^1, \ldots, \theta^a]$, $a = 1, \ldots, n$, where*

$$\begin{aligned} \theta^k &= du_{k-1} - u^k\,dx, \quad k = 1, \ldots, n-1, \text{ and} \\ \theta^n &= du_{n-1} - f(x, u_x, \ldots, u_{n-1})\,dx. \end{aligned} \tag{4.6}$$

*Proof.* If $n = 1$ the theorem is trivial, so assume $n \geq 2$. Let $p \in M$ and apply Theorem A.3 using the Pfaffian system $I^{n-1}$. By Theorem A.3 there exists an open neighborhood $U$ of $p$ with local coordinates $(x, u, u_1, u_2, \ldots, u_{n-1})$ such that

$$I^{n-1}(U) = [du - u_1 dx, \ldots, du_{n-2} - u_{n-1} dx].$$

The rank of $I^n$ is $n$, so there exists $\tilde{\theta} \in I^n(U) \bmod I^{n-1}(U)$ of the form $\tilde{\theta} = a\,du_{n-1} + b\,dx$ where $(a, b) \neq (0, 0)$ on $U$. Condition (3) in equation (4.5) implies $a \neq 0$ on $U$. Therefore there exists a generator $\theta^n$ of $I^n(U) \bmod I^{n-1}(U)$ having the form

$$\theta^n = du_{n-1} - f(x, u, u_1, \ldots, u_{n-1})\,dx.$$

This proves the theorem. □

### 4.3. *Symmetries, Orbits, and Quotients*

An $m$-dimensional Lie group $G$ acting on $M$ is a symmetry group of an $n$th-order scalar ODE structure $(M, \{I^a\}_{0\leq a\leq n})$ if

$$g^* I^a = I^a \qquad \text{for all} \quad a = 0, \ldots, n. \tag{4.7}$$

This definition generalizes the definition of an external symmetry group of an $n$th-order scalar ODE.

Throughout this section we will assume $G$ is a symmetry group of the $n$th-order ODE structure $(M, \{I^a\}_{0\leq a\leq n})$ and that $G$ acts locally effectively on subsets of $M$. As usual, $\{Z_\alpha\}_{1\leq\alpha\leq m}$ will be a basis for the infinitesimal generators $\Gamma$. We will use the subset $M^\ell \subset M$ defined in equation (2.7), and the subsets $M^t$ and $M^{nt}$ as defined in equations (2.10) and (2.11) with $I = I^n$.

Theorem 4.4 states that the Pfaffian system $I^{n-1}$ on $M$ is locally equivalent to $C^{n-1}$ on $J^{n-1}(\mathbb{R}, \mathbb{R})$, while equation (4.7) states $g^* I^{n-1} = I^{n-1}$. This suggests that the action of $G$ on $M$ might behave in a similar manner to a group acting on $J^{n-1}(\mathbb{R}, \mathbb{R})$ by contact transformations. This is the case and Lemmas 3.3, 3.4, and 3.5 from Section 3.3 have analogues for scalar ODE structures. We warn the reader to be careful with the value of $n$ when comparing the lemmas below to those in Section 3.3.

We begin by defining a set $M^0 \subset M$ for symmetry groups $G$ with $\dim G = m \leq n$,

$$M^0 = \{p \in M \mid \det \theta^\alpha(Z_\beta) \neq 0, \text{ where } I^m_p = \text{span}\{\theta^\alpha\}, \ 1 \leq \alpha, \beta \leq m\}. \quad (4.8)$$

The analogue of Lemma 3.3 is first.

**Lemma 4.5.** *Let* $\dim G = m < n$. *Then*:

(1) *$M^0$ is $G$-invariant, open, and dense in $M$, and $G$ acts locally freely on $M^0$.*
(2) *$G$ acts transversally to $I^s$, $m \leq s \leq n$, on $M^0$. In particular, $M^0 \subset M^t$.*

*Proof.* The set $M^0$ is $G$-invariant by the argument used in Lemma 2.10. By equation (2.7), $M^0 \subset M^\ell$, so $G$ acts locally freely by Lemma 2.5. We now show $M^0$ is dense. Let $p \in M$ and choose a local chart $U$ about $p$ satisfying the conditions of Theorem 4.4. Let $\{\theta^\alpha\}_{1\leq\alpha\leq m}$ be a local basis for $I^m(U)$ of the form in equations (4.6). The set $V = M^0 \cap U$ is

$$V = M^0 \cap U = \{\sigma \in U \mid \det \theta^\alpha(Z_\beta), 1 \leq \alpha, \beta \leq m\}.$$

By Theorem 3.2, the set $V$ is open and dense in $U$. The set $M^0$ is then the union of these $V$ as $p \in M$ varies. Therefore $M^0$ is open and dense in $M$. This proves (1). Part (2) is proved exactly the same as part (2) in Lemma 3.3. □

The same technique used in the proof of Lemma 4.5 can be used to prove the analogues of Lemmas 3.4 and 3.5. These are the next two lemmas.

**Lemma 4.6.** *If* $\dim G = m \geq n$, *then the set*

$$M^1 = \{p \in M \mid \text{rank}\, \boldsymbol{\Gamma}_p \geq n\} \quad (4.9)$$

*is $G$-invariant, open, and dense in $M$.*

**Lemma 4.7.** *If* $\dim G = m > n$, *then either*:

(1) *the set $M^{nt}$ is dense in $M$; or*
(2) *$G$ is solvable, or* $\mathbf{g} = sl(2, \mathbb{R})$.

*In case* (2), *there exists an* $n$*-dimensional solvable subgroup* $H \subset G$.

*Proof.* Let $W \subset M^{nt}$ be

$$W = \{p \in M \mid \operatorname{rank} \boldsymbol{\Gamma}_p = n + 1\},$$

and suppose $M^{nt}$ is not dense. Then $W$ is not dense. Choose an open set $U$ such that $U \cap W = \varphi$, and which forms a local chart about $p$ satisfying the conditions of Theorem 4.4. Applying Lemma 3.5 to the open set $U$ implies (2). □

The next lemma has no analogy on jet space.

**Lemma 4.8.** *Let* $\dim G = n$. *The* $G$*-invariant subset* $M^0 \cup M^{nt} \subset M$ *is dense.*

*Proof.* We start by using the identity $M = M^t \cup M^{nt}$ and write the subset $M^1$ defined in equation (4.9) above as

$$M^1 = (M^1 \cap M^t) \cup (M^1 \cap M^{nt}). \tag{4.10}$$

It follows from equations (4.8), (4.9), and (2.10) that $(M^1 \cap M^t) = M^0$. Therefore $M^1 \subset M^0 \cup M^{nt}$ by equation (4.10). By Lemma 4.6, $M^1$ is dense, and so $M^0 \cup M^{nt}$ is also. □

Lemmas 4.5, 4.7, and 4.8 will be used to determine the domains on which the reduction procedure is valid. The next lemma also simplifies determining the domains in the reduction.

**Lemma 4.9.** *Let* $\dim G = m \leq n$. *If* $G$ *acts locally freely on* $M$, *then* $M^t = M^0$, *and* $M = M^0 \cup M^{nt}$.

*Proof.* Let $p \in M$. The action is locally free so $\{Z_\beta(p)\}_{1 \leq \beta \leq m}$ form a basis for $\boldsymbol{\Gamma}_p$. Let $\{\theta^\alpha\}_{1 \leq \alpha \leq m}$ be a basis for $I_p^m$. By definition, $p \in M^t$ if $\boldsymbol{\Gamma}_p \cap I_p^\perp = 0$, which by equation (2.10) is true if and only if $\det \theta^\alpha(Z_\beta) \neq 0$. Therefore, $M^t = M^0$ and $M = M^0 \cup M^{nt}$. □

We now come to the principle theorem of this section. This theorem states that the quotient of an $n$th-order scalar ODE structure by a locally free action of an $m$-dimensional symmetry group with $m < n$ is an $(n-m)$th-order scalar ODE structure.

**Theorem 4.10.** *If* $\dim G = m < n$, $G$ *is transverse to* $I^m$, *and* $G$ *acts locally freely on* $M$, *then the quotient* $(\overline{M}, \{J^v\}_{0 \leq v \leq n-m})$, *where* $J^v = \bar{I}^{v+m}$, *is an* $(n-m)^{th}$*-order ODE structure.*

*Proof.* To show condition (1) in Definition 4.3, we begin by using the argument in part (2) of Lemma 3.3. This shows $G$ acts transversally to the bundles $I^{v+m}$, $v = 0, \ldots, n-m$. Corollary 2.10 then implies that the quotients $= \bar{I}^{v+m}$, $v = 0, \ldots, n-m$, are rank $v$ subbundles of $T^*\overline{M}^0$, and condition (1) is satisfied.

Applying Theorem 5.1 on page 14 in [**2**] we have

$$\bar{I}^{v+m-1} = \overline{(I^{v+m})'} = (\bar{I}^{v+m})', \qquad v = 1, \ldots, n-m-1.$$

This shows that the flag $\{J^v\}_{0\leq v\leq n-m}$ on $\overline{M}$ satisfies condition (2) in Definition 4.3.

We check condition (3) for $m+1 < n$. The case $m+1 = n$ is left as an exercise. Let $\bar{p} \in \overline{M}$, $\overline{U}$ be an open neighborhood of $\bar{p}$ in $\overline{M}$, and let $U \subset \mathbf{q}^{-1}(\overline{U})$ be an open set where the following properties hold. First, $\overline{U} = \mathbf{q}(U)$, and on $\overline{U}$ there is a coframe $\{\bar{\sigma}, \bar{\eta}^v\}$, $v = 1, \ldots, n-m$, adapted to the flag $J^v$, $v = 1, \ldots, n-m$. Second, on $U$ the forms $\{dx, \theta^i\}_{1\leq i\leq n}$ form a coframe, where $\theta^i$ are given by (4.6). The last property we require is that there exists a cross-section $\gamma : \overline{U} \to U$.

Now let $\eta^{m+v} = \mathbf{q}^*\bar{\eta}^v$, $v = 1, \ldots, n-m$ and $\sigma = \mathbf{q}^*\bar{\sigma}$. A coframe on $U$ is given by the forms $\{\sigma, \theta^\alpha, \eta^{m+v}\}_{1\leq\alpha\leq m, 1\leq v\leq n-m}$, and for $a > m$ we have $I^a(U) = [\theta^\alpha, \eta^v]_{1\leq\alpha\leq m, m+1\leq v\leq a}$. The structure equation for $\eta^{m+1}$ is then

$$d\eta^{m+1} = f\sigma \wedge \eta^{m+2} + \tau \wedge \eta^{m+1} + \sum_{\alpha=1}^{m} \tau_\alpha \wedge \theta^\alpha,$$

where $f \neq 0$ on $U$ and $\tau, \tau_\alpha \in \Omega^1(U)$. Since $\eta^{m+1}$ is basic, then $d\eta^{m+1}$ is also basic. Therefore $\iota_Z\, d\eta^{m+1} = 0$ for all $Z \in \Gamma$. The fact that the action of $G$ is locally free and transverse to $I^m$ together with $\iota_Z\, d\eta^{m+1} = 0$ implies $\tau_\alpha = 0$ and $\tau$ is semibasic. This results in the equation

$$d\eta^{m+1} = f\sigma \wedge \eta^{m+2} + \tau \wedge \eta^{m+1} \tag{4.11}$$

where $\tau \in \Omega^1_{sb}(U)$ (the semibasic one forms on $U$). Taking the pullback of equation (4.11) by the cross-section $\gamma : \overline{U} \to U$ we get

$$d\bar{\eta}^1 = \bar{f}\bar{\sigma} \wedge \bar{\eta}^2 + (\gamma^*\tau) \wedge \bar{\eta}^1,$$

where $\bar{f} = f \circ \gamma$ doesn't vanish on $\overline{U}$. This equation proves that condition (3) in Definition 4.3 is satisfied for the flag $\{J^v\}_{0\leq v\leq n-m}$. Therefore $(\overline{M}, \{J^v\}_{0\leq v\leq n-m})$ is an $(n-m)^{\text{th}}$-order scalar ODE structure. □

**Remark 4.11.** Lemma 4.5 implies that the hypothesis of Theorem 4.10 are valid on the subset $M^0$ in equation (4.8). See Case 1 in Section 5.3 for the practical implications of this.

## 5. Integration

### 5.1. *Connections and Equations of Lie Type*

Let $G$ be an $m$-dimensional Lie group acting freely on the $n$-dimensional manifold $M$ so that $\mathbf{q} : M \to \overline{M}$ is a left principle fiber bundle. Assume $G$ is also a symmetry group of the rank $m$ Pfaffian system $I$ on $M$ and that $G$ acts transversally to $I$.

**Theorem 5.1.** *Let $Q = I^{\perp} \subset TM$, then $Q$ is a principle $G$ bundle connection for $\mathbf{q} : M \to \overline{M}$.*

*Proof.* We check conditions (a), (b), and (c) on page 63 of [**10**]. To check condition (a), we need to show that $Q$ is a horizontal distribution. The dimension of $Q$ is $n-m$, and $\boldsymbol{\Gamma}$ is of dimension $m$. The transversality condition (2.10) implies $Q \cap \boldsymbol{\Gamma} = 0$. Therefore, $TM = Q \oplus \boldsymbol{\Gamma}$. The vertical bundle is $\boldsymbol{\Gamma}$, so $Q$ is horizontal.

Condition (b) is that $Q$ is $G$-invariant. The group $G$ is a symmetry group of $I$ and therefore if $g \in G$, then $g^* I = I$. By taking complements, then $g_* Q = Q$.

Condition (c) requires $Q$ to be smooth, which is trivially true. □

Let $\omega : TM \to \mathbf{g}$ be the Lie algebra valued connection form corresponding to $Q$. The form $\omega$ satisfies (using our conventions for $\mathbf{g}$)

$$\omega(\rho(\mathbf{z})) = \mathbf{z} \qquad \text{and} \qquad (\mu_g)^* \omega = Ad_g(\omega). \tag{5.1}$$

These are the formulas on page 64 in [**10**] expressed in terms of left principle bundles.

We now give a local description of $\omega$. Let $\{\mathbf{e}_\alpha\}_{1 \le \alpha \le m}$ be a basis for $\mathbf{g}$ and let $\{Z_\alpha = \rho(\mathbf{e}_\alpha)\}_{1 \le \alpha \le m}$ be the corresponding basis for the Lie algebra of infinitesimal generators. Let $U \subset M$ be an open set where $\{\theta^\alpha\}_{1 \le \alpha \le m}$ are a basis for $I(U)$. Let $P^\alpha_\beta = \theta^\alpha(Z_\beta)$, $1 \le \alpha, \beta \le m$. The matrix $P^\alpha_\beta$ is invertible on account of transversality.

**Lemma 5.2.** *On the open set $U$, the Lie algebra valued one form $\omega$ is*

$$\omega = (P^{-1})^\alpha_\beta \theta^\beta \otimes \mathbf{e}_\alpha.$$

*Proof.* It is sufficient to check the first condition in equation (5.1), which is the defining equation for $\omega$. This is checked on a basis by computing

$$\omega(Z_\gamma) = (P^{-1})^\alpha_\beta \theta^\beta(Z_\gamma) \otimes \mathbf{e}_\alpha = (P^{-1})^\alpha_\beta P^\beta_\gamma \mathbf{e}_\alpha = \mathbf{e}_\gamma.$$

This proves the lemma. □

Finding the horizontal lift of a curve in $\overline{M}$ with respect to a connection on $M$ requires solving a special type of differential equation called an equation of fundamental Lie type, which we now define. Let $\mathbf{g}$ be the Lie algebra of $G$, and recall that we use the right invariant vector-fields to define the brackets on $\mathbf{g}$. Let $\tau : \mathbf{g} \to \chi(G)$ be the homomorphism $\tau(\mathbf{z}) = X_{\mathbf{z}}$ where $X_{\mathbf{z}}$ is the left invariant vector-field,

$$X_z(g) = -(L_g)_* \mathbf{z}.$$

Given a curve $\alpha : \mathbb{R} \to \mathbf{g}$, the homomorphism above leads to the ODE,

$$\dot{\gamma}(t) = \tau(\alpha(t))(\gamma(t)) = -(L_\gamma)_*(\alpha(t))$$

for a curve $\gamma : \mathbb{R} \to G$. This equation is called an equation of fundamental Lie type. If $G$ is simply connected and solvable, then an equation of fundamental Lie type can be solved by quadratures. See page 55 in [**8**] for more information on equations of Lie type.

For the rest of this section let $G$ be an $m$-dimensional Lie symmetry group of the $n$th-order ODE structure $(M, \{I^a\}_{0 \le a \le n})$ with $m \le n$. Assume that $G$ acts transversally to $I^m$ and freely on $M$. By Theorem 5.1, $(I^m)^\perp$ is a connection, and so let $\omega$ be the corresponding connection form for $I^m$ defined by (5.1).

**Lemma 5.3.** *Let $\bar{I}^n$ be the quotient Pfaffian system on $\overline{M}$ of $I^n$. Then*

$$I^n = I^m \oplus (\mathbf{q}^* \bar{I}^n). \tag{5.2}$$

*Proof.* As usual $G$ acts transversally to $I^n$, and so the subbundle $\mathbf{q}^* \bar{I}^n \subset I^n$ is of rank $n - m$ (Corollary 2.11). Now $I^m \cap \boldsymbol{\Gamma}^\perp \neq 0$ and $\mathbf{q}^* \bar{I}^n \cap \boldsymbol{\Gamma}^\perp = 0$. Therefore $\mathbf{q}^* \bar{I}^n \cap I^m = 0$. The subbundle $I^m \subset I^n$ is rank $m$, and thus equation (5.2) holds. $\square$

The fundamental theorem of Section 5 which is given next is a decomposition property of integral manifolds (see also Proposition 6.1 in [**2**]).

**Theorem 5.4.** *Let $\bar{I}^n$ be the rank $n - m$ quotient Pfaffian system on $\overline{M}$ of the rank $n$ Pfaffian system $I^n$ on $M$. Let $\bar{s} : N \to \overline{M}$ be an integral curve of $\bar{I}^n$, $\hat{s} : N \to M$ a lift of $\bar{s}$, and $\gamma : N \to G$ a smooth curve, where $N \subset \mathbb{R}$ is an open interval. The curve $s : N \to M$ given by $s(t) = \mu(\gamma(t), \hat{s}(t))$ is an integral manifold of $I^n$ if and only if $\gamma$ satisfies the equation of fundamental Lie type,*

$$\dot{\gamma}(t) = -(L_\gamma)_* \omega(\dot{\hat{s}}). \tag{5.3}$$

*Proof.* Let $\bar{s}$, $\hat{s}$, $\gamma$, and $s$ be as stated in the theorem (the curve $\hat{s}$ exists because $N$ is an interval). To find conditions on $\gamma$ so that $s$ is an integral manifold of $I^n$ we use Lemma 5.3. We first show that $\dot{s}$ vanishes on $\mathbf{q}^*(\bar{I}^n)$. Let $p \in s(N)$ and let $\theta \in (\mathbf{q}^* \bar{I}^n)_p$. Then $\theta = \mathbf{q}^* \bar{\theta}$, $\bar{\theta} \in \bar{I}^n_{\mathbf{q}(p)}$. The curve $\bar{s}$ is an integral manifold of $\bar{I}^n$, therefore,

$$\theta(\dot{s}) = \mathbf{q}^* \bar{\theta}(\dot{s}) = \bar{\theta}(\mathbf{q}_* \dot{s}) = \bar{\theta}(\dot{\bar{s}}) = 0.$$

By this computation and Lemma 5.3, the curve $s : N \to M$ is an integral manifold of $I^n$ if and only if $\omega(\dot{s}) = 0$.

The tangent vector $\dot{s}$ is

$$\dot{s}(t) = (\mu_\gamma)_* \dot{\hat{s}} + (\mu_{\hat{s}})_* \dot{\gamma}. \tag{5.4}$$

Evaluating the connection form $\omega$ on $\dot{s}$ in equation (5.4) gives

$$\omega(\dot{s}) = \omega((\mu_\gamma)_* \dot{\hat{s}}) + \omega((\mu_{\hat{s}})_* \dot{\gamma}). \tag{5.5}$$

By the equivariance property of $\omega$ in equation (5.1) we have

$$\omega((\mu_\gamma)_* \dot{\hat{s}}) = Ad_\gamma \omega(\dot{\hat{s}}). \tag{5.6}$$

By part (2) in Lemma 2.3, the properties of $\omega$, and that $(L_{\gamma^{-1}})_* \dot{\gamma}(t) \in \mathbf{g}$ for all $t \in N$,

$$\begin{aligned} \omega((\mu_{\hat{s}})_* \dot{\gamma}) &= \omega((\mu_{\hat{s}})_* (L_\gamma)_* (L_{\gamma^{-1}})_* \dot{\gamma}) = Ad_\gamma \omega((\mu_{\hat{s}})_* (L_{\gamma^{-1}})_* \dot{\gamma}) \\ &= Ad_\gamma ((L_{\gamma^{-1}})_* \dot{\gamma}). \end{aligned}$$

Combining this equation with (5.6), equation (5.5) is then

$$\omega(\dot{s}) = Ad_\gamma ((L_{\gamma^{-1}})_* \dot{\gamma} + \omega(\dot{\hat{s}})).$$

This is zero if and only if $\gamma$ satisfies equation (5.3). □

The curve $s$ in this Theorem 5.4 is a horizontal lift of $\bar{s}$ with respect to the connection $(I^m)^\perp$. Also note that every integral manifold of $I^n$ can be determined by Theorem 5.4.

**Corollary 5.5.** *Let $s : N \to M$ be an integral manifold to $I^n$, $\bar{s} = \mathbf{q} \circ s$, and let $\hat{s} : N \to M$ be any lift of $\bar{s}$. Then there exists a smooth curve $\gamma : N \to G$ such that $s(t) = \mu(\gamma(t), \hat{s}(t))$, $t \in N$, and $\gamma$ satisfies the equation of fundamental Lie type* (5.3).

*Proof.* By Lemma 2.12, $\bar{s} : N \to \overline{M}$ is an integral manifold to $\bar{I}^n$. Let $\hat{s} : N \to M$ be a lift of $\bar{s}$. The projections of $\hat{s}$ and $s$ agree, so there exists a function $\gamma : N \to G$ such that $s(t) = \mu(\gamma(t), \hat{s}(t))$, $t \in N$. The action of $G$ is free and so $\gamma$ is unique and smooth. By the proof in Theorem 5.4, $\gamma$ satisfies equation (5.3). □

### 5.2. *Remarks on Theorem* 5.4

Theorem 5.4 and Corollary 5.5 can be interpreted as decomposing the problem of finding integral manifolds for $I^n$ into finding integral manifolds for the quotient system $\bar{I}^n$, and then constructing integral manifolds for $I^n$ from those for $\bar{I}^n$ by solving an equation of Lie type. The process of taking a solution to the quotient and constructing a solution to the original problem is known as the reconstruction problem [**2**]. Solving the equation of Lie type is the reconstruction problem in the case at hand.

We now summarize the process of finding integral manifolds using Theorem 5.4 and highlight the steps which are algebraic. Suppose we are interested in finding an integral manifold of $I^n$ through $p \in M$. In the first step we compute the quotients $\overline{M}$ and $\bar{I}^n$. Only algebraic operations are required to compute these quotients [**2**]. Now let $\bar{p} = \mathbf{q}(p)$, and let $\bar{s} : N \to \overline{M}$ be an integral manifold to $\bar{I}^n$ satisfying $\bar{s}(t_0) = \bar{p}$, $t_0 \in N$. If $m < n$, then integration is required at this step to determine $\bar{s}$. If $\dim G = n$, then $\bar{I}^n = 0$ and we may choose $\bar{s} : N \to \overline{M}$ to be any immersion.

Next choose a lift $\hat{s} : N \to M$ of $\bar{s}$ satisfying $\hat{s}(t_0) = p$ for some $t_0 \in N$. This is an algebraic problem. The equation of Lie type in (5.3) of Theorem 5.4 is then determined algebraically from $\Gamma$, $I^n$, and $\hat{s}$. Solving the equation of Lie type, which requires integration, with the initial condition $\gamma(t_0) = e$ gives the integral manifold $s : N \to M$, $s = \mu(\gamma(t), \hat{s}(t))$. The function $s$ satisfies $s(t_0) = p$.

The proof of Theorem 5.4 does not use the ODE structure in an essential way. This leads to a number of simple corollaries for quotients of Pfaffian systems with symmetry. We give one example.

**Corollary 5.6.** *Let $G$ be a freely acting $n$-dimensional symmetry group of the rank $n$ Pfaffian system $I$ on the $(n + 1)$-dimensional manifold $M$, with $\overline{M}$ the quotient. If $G$ is transverse to $I$, then every integral curve $s : N \to M$, $N \subset \mathbb{R}$ an open interval, can be found by solving an equation of fundamental Lie type. If $G$ simply connected and solvable, then $s$ can be found by quadratures.*

Example 4 in Section 6 is an application of this corollary to a system of second-order ODEs.

The results of Section 5.1 can be extended to the case where the action of the Lie group $G$ on $M$ is locally free, with only some minor changes. First the bundle $M \to \overline{M}$ is not a principle $G$ bundle, but the set $(I^m)^\perp$ in Theorem 5.1 is still (a) horizontal, (b) invariant, and (c) smooth. The connection form $\omega$ can still be constructed locally by Lemma 5.2, Lemma 5.3 and Theorem 5.4 also hold as stated. However, Corollary 5.5 may only hold locally.

### 5.3. *Integration of Scalar ODEs with Symmetry*

Let $G$ be an $m$-dimensional external symmetry group of the $n$th-order ODE $R \subset J^n(\mathbb{R}, \mathbb{R})$, where $G$ acts locally effectively on subsets of $J^n(\mathbb{R}, \mathbb{R})$. Let $I^a$, $a = 0, \ldots, n$, be the Pfaffian systems on $R$ from equation (4.3). By Lemma 4.2, $G$ acts locally effectively on subsets of $R$ and the results of Section 4 hold for the ODE structure $(R, \{I^a\}_{0 \le a \le n})$.

We begin by noting that on the domain $R^{nt}$ which is determined algebraically from $I^n$ and $\boldsymbol{\Gamma}$ in equation (2.11), Corollary 2.15 applies. Any connected solution to the ODE $R$ with initial condition in $R^{nt}$ is contained in the orbit of some one-parameter subgroups of $G$. The subgroups and their orbits (and so these solutions) are determined algebraically. This is demonstrated in the examples.

There are now three cases to consider depending on the dimension of $G$.

**Case 1.** If $m < n$, then by Lemma 4.5 the $G$-invariant subset $R^0 \subset R$, defined in equation (4.8), is dense. The group $G$ acts locally freely on $R^0$ and transversally to $I^m$. The set $R^0$ is determined algebraically. Theorems 4.10 and 5.4 and Corollary 5.5 apply on $R^0$. Every solution with initial condition in $R^0$ is found by solving an $(n-m)$th-order ODE, together with an equation of fundamental Lie type. To solve the differential equation for initial conditions in $R^0$, we first integrate the $(n-m)$th-order quotient ODE $(\bar{R}, \{\bar{I}^v\}_{n-m \leq v \leq n})$. The solutions for the quotient are then used to construct, using only algebraic operations, the equation (5.3) of fundamental Lie type. Solutions to the original equation are then found by multiplying the lift $\hat{s}$ of a solution to the quotient and the solution $\gamma$ to the equation of Lie type. See Example 1 in Section 6.

**Case 2.** If $m = n$, then Lemma 4.8 implies there exists a $G$-invariant open dense set $R^0 \cup R^{nt} \subset R$ where $R^0$ is defined in equation (4.8) and $R^{nt}$ is defined in (2.11). The subsets $R^0$ and $R^{nt}$ are determined algebraically from $I^n$ and $\Gamma$. By equation (2.7) $G$ acts locally freely on $R^0$ and transversally to $I^n$. Theorem 5.4 and Corollaries 5.5 and 5.6 apply on the open set $R^0$. Solutions with initial values in $R^0$ can be found by choosing a smooth curve $\hat{s}$ transverse to the orbits of $G$ on $R^0$ and solving the corresponding equation of fundamental Lie type. The Lie equation (5.3) is determined algebraically from $\hat{s}$. If $G$ is simply connected and solvable, then Corollary 5.6 states that every solution with initial value in $R^0$ can be found by quadratures. See Example 2. Unlike Case 1 above, the set $R^0$ can be empty, see Example 3.

**Case 3.** If $m > n$, Lemma 4.7 applies. If $R^{nt}$ is dense, then every integral curve with initial condition in $R^{nt}$ is contained in the orbit of some one-parameter subgroup of $G$. If $R^{nt}$ is not dense, then part (2) of Lemma 4.8 implies there exists a subgroup $H \subset G$ which is solvable and of dimension $n$. Case 2 applies with $G$ being replaced by $H$. Let $R^0_H$ be the subset of $R$ determined by the subgroup $H$ in equation (4.8), and let

$$R^D = R^0_H \cup R^{nt}.$$

The subset $R^D \subset R$ is dense (by Lemma 4.8), and any solution to $R$ with initial condition in $R^0_H$ can be determined by quadratures (Coroallary 5.6). Therefore, if $\dim G > n$, then there exists a dense subset of initial conditions where the solutions can be found algebraically or by quadrature. This domain is determined algebraically.

**Remark 5.7.** In practice one has only the Lie algebra infinitesimal external symmetries of $R$. These are the infinitesimal contact transformations tangent of $J^n(\mathbb{R}, \mathbb{R})$ which are tangent to $R$. This Lie algebra often occurs as the Lie algebra of a *local* group action on $J^n(\mathbb{R}, \mathbb{R})$ by contact transformations. Fortunately, the definitions and most results from Sections 3, 4, and 5, hold exactly as written for local group actions. The reason is that most of the constructions depend only on the infinitesimal generators $\Gamma$ and the distribution $\boldsymbol{\Gamma}$.

Theorem 5.4 and Corollary 5.5 do need to be modified when only a local action of $G$ is given. For example, Theorem 5.4 holds with the following modification. Let $\bar{s} : N \to \overline{M}$ be an integral manifold of $\bar{I}$, and let $\hat{s}$ be a lift of $\bar{s}$. About each point $t_0 \in N$ there exists a connected open set $N_0$ containing $t_0$, and open neighborhood of $G_0 \subset G$ containing the identity $e$, such that $s(t) = \mu(\gamma(t), \hat{s}(t)),\ t \in N_0$, is an integral manifold if and only if $\gamma$ satisfies equation (5.3). A similar local statement for Corollary 5.5 is easily made.

If $G$ acts only locally and locally effectively on $R$, the summary in this section is valid taking these comments into consideration.

## 6. Examples

**Example 6.1.** This example is of type 1 in Section 5.3. The fourth-order differential equation on page 156 in [**3**]

$$u_{xxxx} = \frac{1}{(uu_x)^2}(5u^2 u_{xx} u_{xxx} u_x + 4uu_x^2 u_{xx}^2 - u_x^4 u_{xx} - 3uu_x^3 u_{xxx} - 4u^2 u_{xx}^3), \quad (6.1)$$

defines the equation manifold $R = \{(x, u, u_x, u_{xx}, u_{xxx}) \in \mathbb{R}^5 \mid uu_x \neq 0\}$, and by equation (4.3) the Pfaffian system

$$\begin{aligned} I = [&du - u_x\, dx,\ du_x - u_{xx}\, dx,\ du_{xx} - u_{xxx}\, dx, \\ &du_{xxx} - (uu_x)^{-2}(5u^2 u_{xx} u_{xxx} u_x + 4uu_x^2 u_{xx}^2 - u_x^4 u_{xx} \\ &\qquad - 3uu_x^3 u_{xxx} - 24u^2 u_{xx}^3)\, dx]. \end{aligned}$$

The three-dimensional solvable external symmetry group

$$G = \{(a, b, c) \mid a \in \mathbb{R},\ b, c \in \mathbb{R}^*\}$$

with multiplication law

$$(a', b', c')(a, b, c) = (a'a, b' + a'b, c'c),$$

and action on $R$ given by

$$x' = ax + b, \quad u' = cu, \quad u'_x = a^{-1}cu_x, \quad u'_{xx} = a^{-2}cu_{xx}, \quad u'_{xxx} = a^{-3}cu_{xxx},$$

is an external symmetry group of equation (6.1). The action is free on $R$.

Let $\mathbf{e}_\alpha \in T_eG$, $\alpha = 1, 2, 3$, be the basis

$$\mathbf{e}_1 = \partial_a, \qquad \mathbf{e}_2 = \partial_b, \qquad \mathbf{e}_3 = \partial_c. \tag{6.2}$$

The infinitesimal generators $Z_\alpha = \rho(\mathbf{e}_\alpha)$ of the action of $G$ on $R$ are then

$$\begin{aligned} Z_1 &= x\partial_x - u_x\partial_{u_x} - 2u_{xx}\partial_{u_{xx}} - 3u_{xxx}\partial_{u_{xxx}}, \qquad Z_2 = \partial_x, \\ Z_3 &= u\partial_u + u_x\partial_{u_x} + u_{xx}\partial_{u_{xx}} + u_{xxx}\partial_{u_{xxx}}. \end{aligned}$$

The dense open subset $R^0 \subset R$ from equation (4.8), is given by $\det\theta^\alpha(Z_\beta) \neq 0$, $\alpha, \beta = 1, 2, 3$, and is

$$R^0 = \{(x, u, u_x, u_{xx}, u_{xxx}) \in R \mid u_{xxx} \neq 2u_{xx}^2 u_x^{-1} - u_{xx}u_x u^{-1} \text{ and } uu_x \neq 0\}.$$

The action is free and so, by Lemma 4.9, $R^t = R^0$, and $R^{nt} = R - R^0$ is then given by

$$R^{nt} = \{(x, u, u_x, u_{xx}, u_{xxx}) \mid u_{xxx} = 2u_{xx}^2 u_x^{-1} - u_{xx}u_x u^{-1} \text{ and } uu_x \neq 0\}.$$

The union $R^0 \cup R^{nt} = R$ consists of all initial conditions.

We start with $R^0$. The first part of the decomposition for finding solutions with initial values in $R^0$ requires computing the quotient Pfaffian system $\bar{I}$. This is done using Corollary 2.8. Let $(t, y)$ be coordinates on the quotient $\overline{R}^0 = R^0/G$ and let $\delta(t, y) = (0, 1, 1, t, y)$ be a cross-section of $\mathbf{q} : R^0 \to \overline{R}^0$. The semibasic form is

$$\begin{aligned} \theta_{sb} = {} & (2u_{xxx}uu_x - 2u_{xx}^2 u + u_{xx}u_x^2)u_x\, du - (3u_{xxx}uu_x - 4u_{xx}^2 u + 2u_{xx}u_x^2)u du_x \\ & - (2u_{xx}u - u_x^2)uu_x\, du_{xx} + (uu_x)^2\, du_{xxx} \end{aligned}$$

and so

$$\bar{I} = [\delta^*\theta_{sb}] = [dy + (1 - 2t)\, dt].$$

The most general integral manifold $\bar{s}$ of $\bar{I}$ satisfying the independence condition $dt \neq 0$ can be written as a graph as $y = t^2 - t + c_0$.

We now find the equation of Lie type in (5.3). First we compute the connection form $\omega = \omega^1 \otimes \mathbf{e}_1 + \omega^2 \otimes \mathbf{e}_2 + \omega^3 \otimes \mathbf{e}_3$ on $R^0$ which is determined from Lemma 5.2 to be

$$\begin{aligned} \omega^1 &= \Delta((u_x u_{xxx} - u_{xx}^2)\, du - (uu_{xxx} - u_x u_{xx})\, du_x + (uu_{xx} - u_x^2)\, du_{xx}), \\ \omega^2 &= dx - \Delta((u_x u_{xx} - xu_{xx}^2 + xu_x u_{xxx})\, du + (2uu_{xx} + xuu_{xxx} - xu_x u_{xx})\, du_x \\ &\quad - (uu_x + xuu_{xx} - xu_x^2)\, du_{xx}), \\ \omega^3 &= \Delta((u_x u_{xxx} - 2u_{xx}^2)\, du + 2u_x u_{xx}\, du_x - u_x^2\, du_{xx}), \end{aligned}$$

where $\Delta = (u_x^2 u_{xx} + u_x uu_{xxx} - 2uu_{xx}^2)^{-1}$. Following Theorem 5.4, let $\hat{s}(t) = \delta \circ \bar{s} = (0, 1, 1, t, t^2 - t + c_0)$, and we find the curve $\omega(\dot{\hat{s}}) : \mathbb{R} \to \mathbf{g}$ to be

$$\omega(\dot{\hat{s}}) = (\omega^1(\dot{\hat{s}}), \omega^2(\dot{\hat{s}}), \omega^3(\dot{\hat{s}})) = (t^2 - c_0)^{-1}(1 - t, 1, 1).$$

By using the basis for $T_eG$ in equation (6.2), the equation (5.3) of Lie type is

$$\dot{a}(t) = a(t)(1-t)(c_0-t^2)^{-1}, \quad \dot{b}(t) = a(t)(c_0-t^2)^{-1}, \quad \dot{c}(t) = c(t)(c_0-t^2)^{-1}. \tag{6.3}$$

The group $G$ is solvable, and the coordinates on $G$ have been chosen so that equations (6.3) can be integrated by quadratures.

We now consider the invariant solutions for equation (6.1) by applying Corollary 2.15. At a point $\sigma^0 \in R^{nt}$, we have

$$\sigma^0 = (x^0, u_x^0, u_{xx}^0, u_{xxx}^0 = 2(u_{xx}^0)^2(u_x^0)^{-1} - u_{xx}^0 u_x^0 (u^0)^{-1}).$$

Then

$$\Gamma_\sigma \cap (I_\sigma)^\perp = \operatorname{span}\{((u_x^0)^2 - u^0 u_{xx}^0) Z_1 + (u^0 u_x^0 - (u_x^0)^2 x^0 + u^0 u_{xx}^0 x^0) Z_2 + (u_x^0)^2 Z_3\}.$$

Corollary 2.15 implies that the orbit through $\sigma^0$ of the one-parameter subgroup determined the vector in the equation above is the maximal solution. By computing this subgroup, the solution (or orbit) with initial value $\sigma^0$ is given in parametric form by

$$x = x^0 + \lambda^{-1} u_x^0 u^0 (1 - e^{\lambda t}), \qquad u = u^0 e^{-(u_x^0)^2 t},$$

where $\lambda = u^0 u_{xx}^0 - (u_x^0)^2$. As a graph these solutions are

$$u = u^0 \left( \frac{(x_0 - x)\lambda + u_x^0 u^0}{u_x^0 u^0} \right)^{-\lambda^{-1}(u_x^0)^2}.$$

The solutions $u = u^0 e^{(x-x^0)u_x^0(u^0)^{-1}}$ when $\lambda = 0$ are found in a similar way. The entire set of initial conditions for this example have now been accounted for.

**Example 6.2.** This example is of type 2 in Section 5.3. The differential equation

$$u_{xxx} = \frac{3u_{xx}^2}{u_x} + \frac{u_{xx}^3}{u_x^5} \tag{6.4}$$

from page 152 in [**13**], defines the equation manifold $R = \{(x, u, u_x, u_{xx}) \in \mathbb{R}^4 \mid u_x \neq 0\}$, and from equation (4.3) the Pfaffian system

$$I = [du - u_x\, dx,\ du_x - u_{xx}\, dx,\ du_{xxx} - (3u_{xx}^2 u_x^{-1} + u_{xx}^3 u_x^{-5})\, dx].$$

Equation (6.4) admits the three-dimensional solvable external symmetry group

$$G = \{(a, b, c) \mid a, b, c \in \mathbb{R}\},$$

with multiplication law

$$(a', b', c')(a, b, c) = (a' + a + c'b, b' + b, c' + c),$$

and local action on $R$ given by

$$x' = x + a + cu, \quad u' = u + b, \quad u'_x = (1 + cu_x)^{-1} u_x, \quad u'_{xx} = (1 + cu_x)^{-3} u_{xx}.$$

Let $\mathbf{e}_\alpha \in T_eG$, $\alpha = 1, 2, 3$, be

$$\mathbf{e}_1 = \partial_a, \qquad \mathbf{e}_2 = \partial_b, \qquad \mathbf{e}_3 = \partial_c. \tag{6.5}$$

The corresponding infinitesimal generators $Z_\alpha = \rho(\mathbf{e}_\alpha)$ on $R$ are

$$Z_1 = \partial_x, \qquad Z_2 = \partial_u, \qquad Z_3 = u\partial_x - u_x^2 \partial_{u_x} - 3u_x u_{xx} \partial_{u_{xx}}.$$

The local action of $G$ is free everywhere on $R$, and the sets $R^0$ and $R^{nt}$ in Lemma 4.9 are

$$R^0 = \{\sigma \in R \mid u_{xx} \neq 0\}, \qquad R^{nt} = \{\sigma \in R \mid u_{xx} = 0\}.$$

Theorem 5.4 applies on the set $R^0$, and the connection form $\omega = \omega^1 \otimes \mathbf{e}_1 + \omega^2 \otimes \mathbf{e}_2 + \omega^3 \otimes \mathbf{e}_3$ on $R^0$ is computed using Lemma 5.2 to be

$$\omega^1 = dx + \frac{1}{(u_x u_{xx})^2}(3u_x^6 + 3u_{xx}uu_x^4 + u_{xx}^2 u)\, du_x - \frac{u_x^3}{u_{xx}^3}(u_x^2 + u_{xx}u)\, du_{xx},$$

$$\omega^2 = du + \frac{3u_x^5}{u_{xx}^2}\, du_x - \frac{u_x^6}{u_{xx}^3}\, du_{xx}, \qquad \omega^3 = -\frac{(3u_x^4 + u_{xx})}{u_{xx}u_x^2}\, du_x + \frac{u_x^3}{u_{xx}^2}\, du_{xx}.$$

The curve $\hat{s}(t) = (0, 0, 1, t)$ is transverse to $\boldsymbol{\Gamma}$ and $\omega(\dot{\hat{s}}) : \mathbb{R} \to \mathbf{g}$ in Theorem 5.4 is

$$\omega(\dot{\hat{s}}(t)) = (-t^{-3}, -t^{-3}, t^{-2}).$$

With the basis in equation (6.5) for $T_eG$, the corresponding equations of Lie type (5.3) are

$$\dot{a} = (1 + c)t^{-3}, \qquad \dot{b} = t^{-3}, \qquad \dot{c} = -t^{-2}.$$

The coordinates on $G$ have been chosen so that these equations can be integrated by quadratures.

For initial conditions $\sigma = (x^0, u^0, u_x^0, 0) \in R^{nt}$ the solutions are contained in the orbit of the one-parameter subgroup with infinitesimal generator $Z_1 + u_x^0 Z_2$ (Corollary 2.15). In parametric form these orbits are

$$x = x_0 + t, \qquad u = u_0 + u_x^0 t.$$

All initial conditions have been accounted for.

**Example 6.3.** This example is of type 2 in Section 5.3. The fifth-order differential equation

$$u_{xxxxx} = \frac{5u_{xxx}(9u_{xxxx}u_{xx} - 8u_{xxx}^2)}{9u_{xx}^2} \tag{6.6}$$

defines the manifold $R = \{(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}) \in \mathbb{R}^6 \mid u_{xx} \neq 0\}$, and the Pfaffian system

$$I = [du - u_x\,dx,\ du_x - u_{xx}\,dx,\ du_{xx} - u_{xxx}\,dx,\ du_{xxx} - u_{xxxx}\,dx,\\ du_{xxxx} - 5u_{xxx}(9u_{xxxx}u_{xx} - 8u_{xxx}^2)/(9u_{xx}^2)\,dx].$$

Equation (6.6) admits the five-dimensional external symmetry group $G = SA(2) = \{(A, b) \mid A \in SL(2,\mathbb{R}),\ b \in \mathbb{R}^2\}$, with local action on $R$,

$$x' = ax + bu + b_1, \qquad u' = cx + du + b_2, \qquad u'_x = \delta^{-1}(c + du_x),\\ u'_{xx} = \delta^{-3}u_{xx}, \qquad u'_{xxx} = \delta^{-4}u_{xxx} - 3b\delta^{-5}(u_{xx})^2,\\ u'_{xxxx} = \delta^{-5}u_{xxxx} - 10b\delta^{-6}u_{xx}u_{xxx} + 15b^2\delta^{-7}u_{xx}^3,$$

where $\delta = (a + bu_x)$ and $ad - bc = 1$.

Let $\mathbf{e}_\alpha$, $\alpha = 1, \ldots, 5$, of $T_eG = \mathbf{sl}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ be the basis given by

$$\mathbf{e}_1 = \left(\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right),\quad \mathbf{e}_2 = \left(\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right),\quad \mathbf{e}_3 = \left(\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right),\\ \mathbf{e}_4 = \left(\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right),\quad \mathbf{e}_5 = \left(\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right),$$

the infinitesimal generators $Z_\alpha = \rho(\mathbf{e}_\alpha)$ on $R$ are

$$Z_1 = \partial_x, \quad Z_2 = \partial_u, \quad Z_3 = \mathrm{pr}(x\partial_x - u\partial_u), \quad Z_4 = \mathrm{pr}(u\partial_x), \quad Z_5 = \mathrm{pr}(x\partial_u)$$

where pr means prolongation. The local action of $G$ on $R$ is free. A simple calculation shows $\det\theta^\alpha(Z_\beta) = 0$, $1 \leq \alpha, \beta \leq 5$. Therefore $R^0$ from equation (4.8) is empty and so, by Lemma 4.9, $R = R^{nt}$. Consequently, from the discussion in Case 2 of Section 5.3, every integral curve (or solution to (6.6)) is an orbit.

The solutions can be found algebraically and so we compute them. Let $\sigma^0 = (x^0, u^0, u_x^0, u_{xx}^0, u_{xxx}^0, u_{xxxx}^0) \in R$ and let

$$\delta = 3u_{xx}^0u_{xxxx}^0 - 4(u_{xxx}^0)^2, \quad \tau_1 = u_x^0\delta - 3(u_{xx}^0)^2u_{xxx}^0, \quad \tau_0 = u^0\delta + 9(u_{xx}^0)^3 - x^0\tau_1.$$

If $\delta \neq 0$, then an infinitesimal generator lying in $\ker\theta^\alpha(Z_\beta)_{\sigma^0}$ is

$$Z_{\sigma^0} = -\tau_0 Z_1 + \frac{1}{2\delta}(9(u_{xx}^0)^4\tau_x - 2\tau_1\tau_0)Z_2 - \tau_1 Z_3 + \delta Z_4 + \frac{1}{\delta}(9(u_{xx}^0)^4\tau_3 - \tau_1^2)Z_5.$$

By Corollary 2.15, the solution through $\sigma^0$ is the integral curve given in $\mathbb{R}^2$ by

$$x = -\frac{\tau_x}{2\tau_3} - \frac{3u_{xx}^0}{\sqrt{\tau_3}}\sinh(\alpha_0 t) - \frac{3u_{xxx}^0u_{xx}^0}{\tau_3}\cosh(\alpha_0 t),$$

$$u = \frac{2\tau_0\tau_3 - \tau_1\tau_x}{2\delta\tau_3} - \frac{3u_{xx}^0(3(u_{xx}^0)^2u_{xxx}^0 + \tau_1)}{\sqrt{\tau_3}\delta}\sinh(\alpha_0 t)\\ - \frac{3u_{xx}^0(3\tau_3(u_{xx}^0)^2 + \tau_1u_{xxx}^0)}{\tau_3\delta}\cosh(\alpha_0 t)$$

where

$$\alpha_0 = 3\sqrt{\tau_3}(u^0_{xx})^2, \qquad \tau_3 = (u^0_{xxx})^2 + \delta.$$

Writing the curve as a graph yields the general solution

$$u = c_0 + c_1 x \pm \sqrt{c_3 x^2 + c_x x + c_2(4c_3 c_2 - c_x^2)}. \tag{6.7}$$

In terms of the initial conditions $\sigma^0$, the constants are

$$\begin{aligned} &c_0 = \delta^{-1}\tau_0, \quad c_1 = \delta^{-1}\tau_1, \quad c_2 = \tfrac{1}{4}12^{-2/5}\delta^{-4/5}\tau_3^{-1}(\tau_x^2 - 36(u^0_{xx})^2\delta), \\ &c_3 = 12^{-2/5}\delta^{-4/5}\tau_3, \qquad\quad c_x = -\tfrac{2}{5}\delta^{-4/5}\tau_x. \end{aligned}$$

The solution for the initial conditions where $\delta = 0$ or $\tau_3 = 0$ are computed in a similar manner.

**Example 6.4.** In this example we apply Corollaries 5.6 and 2.15 to the problem of solving a system of ODEs. The two second-order differential equations on page 14 in [**14**],

$$\ddot{x} = \dot{x}^2 f\left(\frac{\dot{x}}{\dot{y}}\right), \qquad \ddot{y} = \dot{x}^2 g\left(\frac{\dot{x}}{\dot{y}}\right), \tag{6.8}$$

give rise to the Pfaffian system

$$I = \left[dx - \dot{x}\,dt,\ dy - \dot{y}\,dt,\ d\dot{x} - \dot{x}^2 f\left(\frac{\dot{x}}{\dot{y}}\right)dt,\ d\dot{y} - \dot{x}^2 f\left(\frac{\dot{x}}{\dot{y}}\right)\right]$$

on the manifold $M = \{(t, x, y, \dot{x}, \dot{y}) \in \mathbb{R}^5 \mid \dot{y} \neq 0\}$. Equations (6.8) admit the four-dimensional solvable external symmetry group

$$G = \{(a, b, c_1, c_2) \mid a \in \mathbb{R}^*,\ b, c_1, c_2 \in \mathbb{R}\}$$

with multiplication law

$$(a', b', c_1', c_2')(a, b, c_1, c_2) = (a'a, b' + a'b, c_1' + c_1, c_2 + c_2')$$

and action on $M$ given by

$$t' = at + b, \quad x' = x + c_1, y' = y + c_2, \quad \dot{x}' = \frac{\dot{x}}{a}, \quad \dot{y}' = \frac{\dot{y}}{a}. \tag{6.9}$$

The action of $G$ is free on $M$.

Let $\mathbf{e}_1 = \partial_a, \mathbf{e}_2 = \partial_b, \mathbf{e}_3 = \partial_{c_1}, \mathbf{e}_4 = \partial_{c_2}$ be a basis for $T_eG$. The corresponding infinitesimal generators $Z_\alpha = \rho(\mathbf{e}_\alpha)$ are

$$Z_1 = t\partial_t - \dot{x}\partial_{\dot{x}} - \dot{y}\partial_{\dot{y}}, \quad Z_2 = \partial_t, \quad Z_3 = \partial_x, \quad Z_4 = \partial_y.$$

$G$ satisfies the transversality condition (2.10) on the set $M^t \subset M$ computed from equation (2.10) to be

$$M^t = \{\sigma \in M \mid \dot{x}(\dot{x}g - \dot{y}f) \neq 0\}. \tag{6.10}$$

The invariant solutions will have initial condition in

$$M^{nt} = \{\sigma \in M \mid \dot{x}(\dot{x}g - \dot{y}f) = 0\}.$$

We first look at solutions to equation (6.8) with initial conditions in $M^t$. Corollary 5.6 applies on the set $M^t$, and solutions can be found by solving an equation of fundamental Lie type. The components of the connection form on $M^t$ are computed using Lemma 5.2 to be

$$\omega^1 = \Delta(f\,d\dot{y} - g\,d\dot{x}), \qquad \omega^2 = dt + \Delta((\dot{y}\dot{x}^{-2} + tg)\,d\dot{x} - (\dot{x}^{-1} + tf)\,d\dot{y}),$$
$$\omega^3 = dx + \Delta(\dot{y}\dot{x}^{-1}\,d\dot{x} - d\dot{y}), \qquad \omega^4 = dy + \Delta(\dot{y}^2\dot{x}^{-2}\,d\dot{x} - \dot{y}\dot{x}^{-1}\,d\dot{y}),$$

where $\Delta = (\dot{x}g - \dot{y}f)^{-1}$. The curve $\hat{s}(\tau) = (0, 0, \tau, 1)$, $\tau \in \mathbb{R}$, is transverse to the orbits of $G$, and $\omega(\dot{\hat{s}}) : \mathbb{R} \to \mathbf{g}$ in Theorem 5.4 is

$$\omega(\dot{\hat{s}}(\tau)) = \left( \frac{g(\tau)}{f(\tau) - \tau g(\tau)}, \frac{1}{\tau^2(\tau g(\tau) - f(\tau))}, \frac{1}{\tau(\tau g(\tau) - f(\tau))}, \frac{1}{\tau^2(\tau g(\tau) - f(\tau))} \right).$$

The equations of Lie type (5.3) corresponding to our choice of basis for $\mathbf{g}$ are

$$a_\tau = \frac{a(\tau)g(\tau)}{\tau g(\tau) - f(\tau)}, \qquad b_\tau = \frac{a(\tau)}{\tau^2(f(\tau) - \tau g(\tau))},$$
$$c_{1,\tau} = \frac{1}{\tau(f(\tau) - \tau g(\tau))}, \qquad c_{2,\tau} = \frac{1}{\tau^2(f(\tau) - \tau g(\tau))}.$$

The coordinates have been chosen so that these can be integrated by quadratures.

We now find the invariant solutions with initial conditions $\sigma \in M^{nt}$. By Corollary 2.15, the solutions are orbits of $Z \in \Gamma$ such that $Z_\sigma \in \Gamma_\sigma \cap I_\sigma$, $Z_\sigma \neq 0$. For initial conditions where $\sigma = (t_0, x_0, y_0, \dot{x}_0, \dot{y}_0) \in M^{nt}$ which satisfy $(\dot{x}_0 g(\dot{x}_0/\dot{y}_0) - \dot{y}_0 f(\dot{x}_0/\dot{y}_0)) = 0$, $\dot{x}^0 \neq 0$, the solution is the orbit of the one-parameter subgroup with infinitesimal generator

$$Z = -\alpha_0 X_1 + (1 + \alpha_0 t)X_2 + \dot{x}^0 X_3 + \dot{y}^0 X_4, \qquad \text{where} \quad \alpha_0 = \dot{x}_0 f(\dot{x}_0/\dot{y}_0).$$

The explicit parametrized solutions (or orbits) are

$$t = t_0 + \frac{1}{\alpha_0}(1 - e^{-\alpha_0 \tau}), \qquad x = x_0 + \dot{x}_0 \tau, \qquad y = y_0 + \dot{y}_0 \tau, \qquad \tau \in \mathbb{R}. \quad (6.11)$$

As a graph these solutions are

$$x = x_0 - \frac{\dot{x}_0}{\alpha_0}\ln(1 + \alpha_0(t_0 - t)), \qquad y = y_0 - \frac{\dot{y}_0}{\alpha_0}\ln(1 + \alpha_0(t_0 - t)), \qquad t \in \mathbb{R}.$$

Solutions with $\alpha_0 = 0$ or $\dot{x}_0 = 0$ are written down similarly.

**Remark 6.5.** If the functions $f$ and $g$ in equation (6.8) satisfy $\dot{x}g - \dot{y}f = 0$, then every solution is the orbit of a one-parameter subgroup of the symmetry group $G$.

## Appendix A

The theorems in this appendix give conditions that a codimension two Pfaffian system is locally equivalent to $J^n(\mathbb{R}, \mathbb{R})$ in terms of invariants of the Pfaffian system. These theorems are a modification of Proposition 6.3, p. 162 in [**15**].

**Theorem A.1.** *Let $M$ be an $(n+2)$-dimensional manifold, and $I^{\langle 0\rangle}$ a rank $n$ Pfaffian system on $M$ with the derived flag $I^{\langle k\rangle}$ satisfying* $\operatorname{rank}(I^{\langle k\rangle}) = n-k$, *$k = 0, \ldots, n$. There exists an open dense subset $M^0 \subset M$ where, about each point $p \in M^0$, there is an open set $U$ containing $p$ and a local coframe $\{\rho, \sigma, \theta^i\}_{i=1,\ldots,n}$ on $U$ such that $I^{\langle n-k\rangle}(U) = [\theta^1, \ldots, \theta^k]$, $k = 1, \ldots, n$, and*

$$\begin{aligned} &(1)\quad d\theta^k = \rho \wedge \theta^{k+1} \mod I^{\langle k\rangle}(U), \qquad k = 1, \ldots, n-1,\\ &(2)\quad d\theta^n = \rho \wedge \sigma \mod I^{\langle 0\rangle}(U). \end{aligned} \tag{A.1}$$

*Proof.* Let $U$ be an open set such that $T^*U$ is trivial. Let $\{\omega, \tau, \eta^i\}_{1\le i\le n}$ be a coframe on $U$ such that $I^{\langle k\rangle} = [\eta^1, \ldots, \eta^{n-k}]$. By the definition of the derived series and the choice of coframe we have the structure equations,

$$\begin{aligned} d\eta^k &= \omega^k \wedge \eta^{k+1} \mod I^{\langle n-k\rangle}(U), \qquad k = 1, \ldots, n-1,\\ d\eta^n &= f^n \omega \wedge \tau \mod I^{\langle 0\rangle}(U), \end{aligned} \tag{A.2}$$

where

$$\omega^k = a^k\omega + b^k\tau + \sum_{\alpha=k+2}^{n} c^k_\alpha \eta^\alpha, \tag{A.3}$$

and $f^n$, $\omega^k$ are nonzero at every point in $U$. This last condition implies, for $k = n-1$, that

$$(a^{n-1}, b^{n-1}) \neq (0, 0) \tag{A.4}$$

at every point in $U$.

We now show there exists an open dense subset $V \subset U$ where $(a^k, b^k) \neq (0, 0)$ at any point of $V$ for each $k = 1, \ldots, n-1$. If $k = n-1$, then by (A.4) $V = U$. We proceed by (reverse) induction. Suppose for some $k \in \{1, \ldots, n-2\}$ that $(a^k, b^k) = (0, 0)$ on an open set $\tilde{W} \subset U$. By the induction hypothesis choose $W \subset \tilde{W}$ a nonempty open set where $(a^{k+t}, b^{k+t}) \neq (0, 0)$ at any point of $W$ for $t = 1, \ldots, n-k-1$. Taking the exterior derivative of (A.2) and using (A.3), we have

$$0 = d^2\eta^k \wedge \eta^1 \wedge \cdots \wedge \eta^{k+1} = -\left(\sum_{\alpha=k+2}^{n} c^k_\alpha \eta^\alpha\right) \wedge \omega^{k+1} \wedge \eta^{k+2} \wedge \eta^1 \wedge \cdots \wedge \eta^{k+1}.$$

Substituting $\omega^{k+1}$ from equation (A.3) into this equation, and using $(a^{k+1}, b^{k+1}) \neq (0, 0)$ at points of $W$, implies $\omega^k = c^k_{k+2}\eta^{k+2}$ and $c^k_{k+2} \neq 0$ on $W$. Assuming $k < n-2$, and computing

$$0 = d^2\eta^k \wedge \eta^1, \ldots, \eta^k \wedge \eta^{k+2} = c^k_{k+2}\omega^{k+2} \wedge \eta^{k+3} \wedge \eta^{k+1} \wedge \eta^1 \wedge \cdots \wedge \eta^k \wedge \eta^{k+2}$$

implies $c^k_{k+2} = 0$, which is a contradiction. For $k = n-2$, the argument is similar. Thus $(a^k, b^k)$ can't vanish on an open subset of $U$ for any $k = 1, \dots, n-1$. This finishes the induction.

Let $V_k \subset U$ be the open dense subset of $U$ where $(a^k, b^k) \neq (0,0)$, $k = 1 \dots n-1$, and let

$$V = \bigcap_{k=1}^{n-1} V_k$$

which is open and dense in $U$. By making the change of coframe

$$\tilde{\omega} = a^1\omega + b^1\tau + \sum_{\alpha=3}^{n} c^1_\alpha \eta^\alpha, \qquad \tilde{\tau} = -b^1\omega + a^1\tau, \tag{A.5}$$

on $V$, we have

$$d\eta^1 = \tilde{\omega} \wedge \eta^2 \mod I^{\langle n-1\rangle}(V). \tag{A.6}$$

Now equations (A.2) and (A.3) hold with $\omega$ and $\tau$ replaced by $\tilde{\omega}$ and $\tilde{\tau}$ and a redefining $\omega^k$, $a^k$ and $b^k$, $c^k_\alpha$ appropriately.

We now show by induction that

$$\omega^k = f^k\tilde{\omega}, \qquad k = 1 \dots n-1, \tag{A.7}$$

where $f^k \neq 0$ on $V$. Equation (A.7) holds for $k = 1$ with $f^1 = 1$. Assume (A.7) is true for $k$ and compute

$$0 = d^2\eta^k \wedge \eta^1 \wedge \cdots \wedge \eta^{k+1} = -f^k\tilde{\omega} \wedge \omega^{k+1} \wedge \eta^{k+2} \wedge \eta^1 \wedge \cdots \wedge \eta^{k+1},$$

and then substitute for $\omega^{k+1}$ from (A.3). This implies $\tilde{\omega} \wedge \omega^{k+1} = 0$. Therefore (A.7) holds for $k+1$, and hence for all $k$, $k = 1 \dots n-1$. The structure equations at this point are (A.6) together with

$$\begin{aligned} d\eta^k &= f^k\tilde{\omega} \wedge \eta^{k+1} && \mod I^{\langle n-k\rangle}(U), \qquad k = 2, \dots, n-1, \\ d\eta^n &= \frac{f^n}{(a^1)^2 + (b^1)^2}\tilde{\omega} \wedge \tilde{\tau} && \mod I^{\langle n-1\rangle}(U). \end{aligned}$$

Now make a final change of coframe

$$\rho = \tilde{\omega}, \qquad \sigma = \frac{f_2 f_3, \dots f_n}{(a^1)^2 + (b^1)^2}\tilde{\tau}, \qquad \theta^k = (f_2 f_3 \dots f_{k-1})\eta^k, \qquad k = 3, \dots, n.$$

The coframe $\{\rho, \sigma, \theta^1, \dots, \theta^n\}$ satisfies the structure equations (A.1) on $V$. To finish the proof of the theorem, we simply let $M^0$ be the union of all $V$ constructed in this manner. □

**Remark A.2.** If $M$ is a four-dimensional manifold, then Theorem A.1 holds everywhere on $M$, not just on an open dense subset.

**Theorem A.3.** *Let $M$ be an $(n+2)$-dimensional manifold, and $I^{\langle 0\rangle}$ a rank $n$ Pfaffian system on $M$ with the derived flag $I^{\langle k\rangle}$ satisfying* $\operatorname{rank}(I^{\langle k\rangle}) = n-k$, $k=0,\dots,n$. *Suppose that either $n=1,2$, or $n>2$ and*

$$dI^{\langle n-1\rangle} \mod \langle I^{\langle n-1\rangle}, I^{\langle 0\rangle}\wedge I^{\langle 0\rangle}\rangle \tag{A.8}$$

*is constant rank* 1. *Then about each point $p\in M$, there exists an open set $U$ and local coordinates $(x,u,u_1,u_2,\dots,u_n)$ such that*

$$I^{\langle n-k\rangle}(U) = [du-u_1\,dx,\dots,du_{k-1}-u_k\,dx], \qquad k=1,\dots,n. \tag{A.9}$$

*In other words the Pfaffian system $I^{\langle 0\rangle}$ on $M$ is locally equivalent to the contact structure $C^n$ on $J^n(\mathbb{R},\mathbb{R})$.*

*Proof.* Suppose that for each $p\in M$, there exists an open set $V$ and coframe $\{\omega,\tau,\theta^i\}_{1\le i\le n}$ on $V$ such that $I^{\langle n-k\rangle}(V)=[\theta^1,\dots,\theta^k]$, and the structure equations (A.1) are satisfied. The forms $\{\theta^i\}_{1\le i\le n}$ generate $I^{\langle 0\rangle}(V)$ and satisfy the conditions in the Goursat Normal Form Theorem [**4**]. Hence about any point $p\in M$, there exists an open set $U$ with local coordinates $(x,u,u_1,u_2,\dots,u_{n-1})$ such that equation (A.9) holds. If $n=1,2$, the existence of the coframe $\{\omega,\tau,\theta^i\}_{1\le i\le n}$ is proved in Theorem A.1.

To prove the existence of such a coframe for $n>2$ we start off exactly as in Theorem A.1. Let $p\in M$, and $U$ an open neighborhood of $p$ with coframe $\{\omega,\tau,\eta^i\}_{1\le i\le n}$ on $U$ such that $I^{\langle n-k\rangle}=[\eta^1,\dots,\eta^k]$. The structure equations are

$$\begin{aligned} d\eta^k &= \omega^k\wedge\eta^{k+1} \mod I^{\langle n-k\rangle}(U), \qquad k=1,\dots,n-1,\\ d\eta^n &= f^n\omega\wedge\tau \mod I^{\langle 0\rangle}(U), \end{aligned}$$

where

$$\omega^k = a^k\omega + b^k\tau + \sum_{\alpha=k+2}^{n} c^k_\alpha\eta^\alpha,$$

and $f^n$, $\omega^k$, and $(a^{n-1},b^{n-1})\ne(0,0)$ are nonzero at every point in $U$. The hypothesis in equation (A.8) implies that $(a^1,b^1)\ne(0,0)$ at every point in $U$. We now make a change of coframe as in equation (A.5). By continuing to follow the proof of Theorem A.1 word for word from (A.6) onward, proves the existence of the required coframe $\{\omega,\tau,\theta^i\}_{1\le i\le n}$. □

**Remark A.4.** The Goursat Normal Form Theorem [**4**] implies that the Pfaffian system in Theorem A.1 is locally equivalent to $J^n(\mathbb{R},\mathbb{R})$ on a dense subset of $M$.

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