SOME APPLICATIONS OF CARTAN'S METHOD OF EQUIVALENCE TO THE GEOMETRIC STUDY OF ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Cartan's method of equivalence is used to prove that there exists two fundamental tensorial invariants which determine the geometry of systems of n (≥ 2) second order ordinary differential equations. These invariants allow us prove that there exist a unique equivalence class of second order equations which admit a Lie point symmetry group of maximal dimension, the dimension being $n^2 + 4n + 3$. For third order systems of ordinary differential equations, we prove that the possible dimension of the point symmetry group is bounded by $n^2 + 3n + 3$. As well we find that there is a unique third order system whose symmetry group has dimension $n^2 + 3n + 3$.

We also characterize invariantly under point transformations some equivalence classes of parabolic quasi-linear second order partial differential equations, and examine their point symmetry groups. We are able to make our characterizations by proving a reduction theorem for principal fibre bundles.

Résumé

La méthode d'équivalence est utilisée afin de prouver qu'il existe deux invariants tensoriels fundamentaux qui determinent la géométrie du systèmes de n (≥ 2) équations differentielles ordinaires du second ordre. Ces invariants nous permettent de démontrer qu'il existe une classe d'équivalence unique d'équations du deuxième ordre admettant un groupe de Lie de symétries ponctuelles de dimension maximale, plus précisément n^2+4n+3. Pour les systèmes d'équations differentielles ordinaires d'ordre trois, nous prouvons que la dimension possible du groupe de symétries ponctuelles est bornée par n^2+3n+3. De même nous trouvons qu'il y a un unique système d'ordre trois dont le groupe de symétries est de dimension n^2+3n+3.

Nous caractérisons des classes d'équivalence d'équations aux dérivées partielles paraboliques quasi-linéaires d'ordre deux de façon invariante sous le groupe de transformations ponctuelles et examinons leurs groupes de symétries ponctuelles. Cette caractérisation est rendue possible grâce à un théorème de réduction que nous avons obtenu pour les fibrés principaux.

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Contents

Notation

V, V^*	- An n-dimensional vector space and its dual
$\{e_i\}_{1\leq i\leq n}, \{e^i\}_{1\leq i\leq n}$	- A basis for V and V^* respectively
$\wedge^* V$	- The exterior algebra of V
$\lrcorner, \odot$	- The left interior multiplication and symmetric tensor product
$(\omega)_{e^i} = e_i \lrcorner \omega$	- The coefficient operator
$(\omega)_{e^i e^j} = e_j \lrcorner e_i \lrcorner \omega$	
$\mathbf{M}$, T($\mathbf{M}$)	- A smooth differentiable manifold and its tangent bundle
$\mathcal{F}(\mathbf{M})$	- The frame bundle of $\mathbf{M}$
$\pi : (\mathbf{F}_f, \mathbf{E}_{m+f}) \rightarrow \mathbf{M}$	- A fibre bundle with fibre $\mathbf{F}$ of dimension f, total space $\mathbf{E}$ of dimension $m + f$, and base $\mathbf{M}$
$\mathcal{V}(\mathbf{E}) = \ker(\pi_*)$	- The vertical bundle of a fibre bundle
$\mathbb{P}^n(\mathbb{R})$	- The n-dimensional real projective space
$GL(n, \mathbb{R})$	- The general linear group of $n \times n$ non-singular real matrices
$Gr^k(l)$	- The Grassman manifold of k-planes in $\mathbb{R}^l$
$Gr^f(T(\mathbf{M}))$	- The Grassman bundle of f-planes in $T(\mathbf{M})$
$M_n(\mathbb{R})$	- The ring of $n \times n$ real matrices
G, $\mathbf{g}$	- A linear Lie group and its Lie algebra
$T_{[ij]}, T_{(ij)}$	- Skew-symmetrization and symmetrization of indices, e.g. $T_{[ij]} = 1/2(T_{ij} - T_{ji})$
${}^tA^i_j$	- The transpose
$< \omega^i >$	- The ideal generated by the one-forms ω^i

Otherwise, we follow the notational conventions in Warner [37], and assume the summation convention.

Introduction

In 1908 Élie Cartan [11] presented a fairly general method for determining whether two given exterior differential systems which are generated by 1-forms are equivalent under a change of local coordinates belonging to a specified group of transformations. This method is now called the Cartan method of equivalence and the problem it solves is known as the Cartan equivalence problem.

With this powerful theory Cartan was able to provide solutions to numerous problems of differential geometry by casting them as equivalence problems. In general Cartan's method provides an invariant coframe whose structure functions can then be used to give necessary and sufficient conditions for the existence of an equivalence or to determine the structure of the symmetry group of the problem. A classic example of where an invariant coframe is produced by the method is the local equivalence problem for Riemannian metrics. In this case the structure functions which appear give rise to the Riemann curvature tensor.

The cases for which the method does not provide an invariant coframe are precisely the problems which admit infinite Lie pseudogroups of symmetries, such as the local conformal equivalence problem for Riemannian metrics in the plane. Infinite pseudogroups arise essentially when the equations governing the equivalence problem are not completely integrable but satisfy the involutivity criterion of the Cartan-Kähler theorem as applied to Pfaffian systems. Cartan applied his equivalence method to produce many impressive results in problems like the equivalence of functionals in the

Calculus of Variations [14], and the equivalence of ordinary and partial differential equations [13], [12]. For example, one famous result from [12] is the (non-linear) representation of the non-compact real form of the exceptional Lie group G_2 as a group of symmetries of a Pfaffian system in five variables (see also [16]).

There is a general consensus (see the introduction in [16]) that Cartan's original papers on the method of equivalence are rather difficult to follow. The reformulation by Chern [10] and by Singer and Sternberg [35] of the equivalence problem using principal fibre bundles put Cartan's work on a more rigorous foundation. Finally today we find a full account of Cartan's method with many explicit examples worked out in the book by Gardner [16]. Gardner has clarified many of the steps in the practical implementation of the equivalence method and has provided a basic algorithm for the solution. His work has generated a lot of new research activity in the subject. We will summarize the principal bundle formulation basically following Sternberg [36] with the intention of providing a proof of an essential result (Theorem 1.3, and Theorems 1.7, 1.8 and 1.9) which we will need for Chapter 3. For the actual calculations in Chapters 2 and 3 we use the formulation provided by Gardner [16], which we summarize in Section 1.6.

Current research using the equivalence method includes Control Theory [18], [38], Calculus of Variations [25], [34] (and references therein), General Relativity [4], [28], and differential equations [27], [24], [23] these references being far from exhaustive. Our concern will be with applications to differential equations, and in particular in Chapter 2 we apply the Cartan method to study the equivalence of second order systems of ordinary differential equations under smooth point transformations. We find there are two fundamental families of tensorial invariants $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ which all other invariants are differential functions of. We use these invariants to prove the property that there is a unique equivalence class of systems of ordinary differential equations

$$\frac{d^2x^i}{dt^2} = f^i\left(t, x^j, \frac{dx^j}{dt}\right) \tag{0.1}$$

which admits a symmetry group of maximal dimension. Every equation in this class is shown to be equivalent by a change of coordinates to the trivial system

$$\frac{d^2x^i}{dt^2} = 0$$

A simple criterion for checking the maximal symmetry property is that the two fundamental invariants must vanish. This is the content of Theorem 2.2, while Lemma 2.3 provides the explicit formula for the invariants. In this chapter we also demonstrate that all systems of n third order ordinary differential equations admit symmetry groups with dimensions less than or equal to $n^2 + 3n + 3$, and that the equivalence class of the trivial equation

$$\frac{d^3x^i}{dt^3} = 0$$

is the unique equation with symmetry group of dimension $n^2 + 3n + 3$.

In Chapter 3 we apply the equivalence method to investigate the Monge-Ampère and quasi-linear parabolic partial differential equations in the plane. In order to give an invariant characterization of Burgers' equation we apply the reduction theorems we have given in Chapter 1. We also provide an invariant classification for the heat equation which is known to admit an infinite Lie pseudogroup of symmetries.

We shall make the blanket assumption that all objects are infinitely differentiable (or real analytic when using the Cartan-Kähler theorem). We will also assume (unless otherwise stated) that we are working in open contractible subsets of real Euclidean space since our considerations are mostly of a local nature.

Chapter 1

Cartan's Method of Equivalence

1.1 Introduction

The goal of this chapter is to give a brief introduction to the equivalence problem of Élie Cartan. Our presentation of the equivalence problem follows a geometric formulation in terms of principal fibre bundles and G-structures. We present this in Sections 1.2 through 1.5 following closely Sternberg [36] while omitting most of the proofs. This summary of the geometric form of the equivalence problem is given in order to have a self-contained proof of Theorem 1.3 and Theorem 1.7. These theorems are described in terms of the local equivalence problem in Section 1.6 and will be used for applications in Chapter 3.

The procedure to be followed for the practical implementation of the solution of the local equivalence problem is know as Elie Cartan's method of equivalence. It is thanks to the fundamental work of R.B. Gardner [16] that the method of equivalence has been greatly clarified and applied to a whole array of equivalence problems which admit a geometric formulation. Although some familiarity with this material will be assumed in this thesis we shall summarize in Section 1.6 the relevant terminology and material we shall need. For a more complete description of the method of equivalence we refer the reader to Gardner [16] and Kamran [24].

1.2 Principal Fibre Bundles

Before giving the preliminary definitions, we would like to warn the reader that in Sections 1.2 through 1.6 we follow the traditional definition for a principal bundle in that we use **right** actions. However, it is more natural when working with differential forms and with the equivalence problem to use left actions, thus we will use principal bundles with **left** actions and in all other sections and chapters besides Sections 1.2 to 1.6.

Let $\mathbf{M}$ be a differentiable manifold of dimension m, and let $\pi : \mathcal{P}_{\mathrm{G}} \to \mathbf{M}$ be a principal fibre bundle with structure group G and right action R. When we fix a point $a \in \mathrm{G}$ the action R is a diffeomorphism of $\mathcal{P}_{\mathrm{G}}$ which we will write as R_a. Let $\mathbf{W}$ be a k-dimensional manifold with a right [1] G action ρ and consider $\mathbf{E} = \mathcal{P}_{\mathrm{G}} \times_{\mathrm{G}} \mathbf{W}$. Any section $\mathbf{s} : \mathbf{M} \to \mathbf{E}$ defines a G-equivariant function $\widehat{\mathbf{s}} : \mathcal{P}_{\mathrm{G}} \to \mathbf{W}$ by requiring the diagram

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{G}} & \xrightarrow{(Id,\widehat{\mathbf{s}})} & \mathcal{P}_{\mathrm{G}} \times \mathbf{W} \\ \downarrow \pi & & \downarrow \widetilde{\pi} \\ \mathbf{M} & \xrightarrow[\mathbf{s}]{} & \mathbf{E} \end{array}$$

be commutative. One of the most important uses of this point of view rests on how to use $\widehat{\mathbf{s}}$ to obtain a reduction of $\mathcal{P}_{\mathrm{G}}$, and to proceed we need

Theorem 1.1: *Let $\sigma : \mathcal{P}_{\mathrm{G}} \to \mathrm{G/H}$ be a smooth map satisfying $\sigma \circ R_a = L_{a^{-1}} \circ \sigma$ then $\mathcal{P}_{\mathrm{G}}$ is reducible to $\mathcal{P}_{\mathrm{H}}$.*

Proof: Let $m \in \mathbf{M}$ and $p \in \mathcal{P}_{\mathrm{G}}$ where $\pi(p) = m$. Then $\phi(m) = (m, p, \sigma(p))$ defines a section of $\mathcal{P}_{\mathrm{G}} \times_{\mathrm{G}} \mathrm{G/H}$. ∎

[1]Given $\mathbf{W}$ with a left action L we use $L_{a^{-1}}$ as the induced right action.

Theorem 1.1 can be applied to our previous considerations as

Lemma 1.1: *Let* $\mathbf{W}$ *be as above and* $\sigma : \mathcal{P}_{\mathrm{G}} \rightarrow \mathbf{W}$ *be a* G*−equivariant map. Let* $\mathcal{O}_w \subset \mathbf{W}$ *be the orbit of a point* $w \in \mathbf{W}_{\mathrm{G}}$ *(* $\mathcal{O}_w$*is a homogeneous space). When* $\widehat{\mathbf{M}} = \pi \circ \sigma^{-1}(\mathcal{O}_w)$ *is a submanifold of* $\mathbf{M}$ *we may apply the above theorem to* $\widehat{\mathcal{P}}_{\mathrm{G}} = \sigma^{-1}(\mathcal{O}_w)$ *to obtain a reduction of* $\widehat{\mathcal{P}}_{\mathrm{G}}$ *to* $\widehat{\mathcal{P}}_{\mathrm{H}_w} = \sigma^{-1}(w)$ *where* H_w *is the stability group of the point* w*.*

In practice we often impose for some $w \in \mathbf{W}$ that $\pi \circ \sigma^{-1}(w) = \mathbf{M}$, thus the above determines a global reduction of $\mathcal{P}_{\mathrm{G}}$. The standard example of this is

Example 1.1: Let $\mathbf{M}$ be such that there exists a non-degenerate $\mathbf{g} : \mathbf{M} \rightarrow T^*(\mathbf{M}) \odot T(\mathbf{M})$ of signature (p,q). We have the following map $\widehat{g} : \mathcal{F}(\mathbf{M}) \rightarrow V^* \odot V$, and $\widehat{g}^{-1}(diag(I_p, -I_q))$ is the $O(p,q)$ reduction of the frame bundle corresponding to g.

We will call any G-equivariant function $\widehat{g} : \mathcal{P}_{\mathrm{G}} \rightarrow V$, where V is a vector space with a left G action, a **tensorial invariant**. For the next example let $\pi : (\mathbf{F}_f, \mathbf{E}_{m+f}) \rightarrow M$ be a fibre bundle and let $\mathcal{V}(\mathbf{E}) \subset T(\mathbf{E}) = ker\, \pi_*$ be the vertical bundle.

Example 1.2: The frame bundle $\mathcal{F}(E)$ admits a reduction to

$$\mathrm{H} = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \qquad A \in GL(f) \quad \text{and} \quad C \in GL(m) \right\} \tag{1.1}$$

where f = fibre dimension of $\mathcal{V}$ (which of course is also the dimension of $\mathbf{F}$). We may view $\mathcal{V}(\mathbf{E})$ as a section of $Gr^f(T(\mathbf{E}))$.

One key point in this example is that $Gr^f(f+m) = GL(f+m)/\mathrm{H}$ (see [37]). We will use this example in the next section.

We now study the geometry of the frame bundle of a principal fibre bundle. We have $T(\mathcal{P}_{\mathrm{G}})$ is the tangent bundle of $\mathcal{P}_{\mathrm{G}}$, $\mathbf{g}$ the Lie algebra of G with $g = \dim \mathbf{g}$ and

$\mathbf{Z}$ the zero-section of $T(\mathcal{P}_{GG})$. For each $A \in \mathbf{g}$ there exists the corresponding vector-field $\tilde{A}$ on $\mathcal{P}_G$ defined by $\tilde{A}(p) = R_*(A, \mathbf{Z}(p))$. $\tilde{A}$ is referred to in Kobayashi and Nomizu [31] as the fundamental vector-field corresponding to A. The fundamental vector-fields satisfy the property,

$$(R_a)_* \circ \tilde{A} \circ R_a^{-1} = (\widetilde{ad_{a^{-1}}A}) \tag{1.2}$$

where ad is the adjoint representation of G. The vertical bundle of $\mathcal{P}_G \subset T(\mathcal{P}_G)$, defined by $\mathcal{V}(\mathcal{P}_G) = ker(\pi_*) : T(\mathcal{F}(\mathbf{M})) \to T(\mathbf{M})$, is isomorphic to $\mathcal{P}_G \times \mathbf{g}$. Choosing a basis A^a of $\mathbf{g}$ defines an isomorphism $\mathcal{V}(\mathcal{P}_G) \leftrightarrow \mathcal{P}_G \times \mathbf{g}$, which is explicitly given by noting, $X_p \in \mathcal{V}(\mathcal{P}_G)$ implies[2] $X = X_a\widetilde{A}^a(p)$ where $X_a \in \mathbb{R}$. The isomorphism is then given by $X \leftrightarrow X_a A^a$.

1.3 G-structures

Perhaps the most useful principal bundle are of reductions of the frame bundle on a manifold, and following Sternberg [36], let $\mathcal{F}(\mathbf{M})$ be the frame bundle of $\mathbf{M}$. A G-structure $\mathcal{B}_G$ is reduction of $\mathcal{F}(\mathbf{M})$ to $G \subset GL(V)$. On the frame bundle there exists a canonical V-valued differential one-form $\omega : T(\mathcal{F}(\mathbf{M})) \longrightarrow V$ defined by $\omega(X) = (u)^{-1} \circ \pi_*(X)$ where $X \in T_u(\mathcal{F}(\mathbf{M}))$ and we may view $u \in \mathcal{F}(\mathbf{M})$ as an isomorphism

$$u : V \longrightarrow T_{\pi(u)}(\mathbf{M}) \ . \tag{1.3}$$

The form ω is defined on any reduction of $\mathcal{F}(\mathbf{M})$ by restriction, and in this situation we will continue to denote the restricted form by ω. Properties of ω which will be needed later are

Lemma 1.2: $(R_a)^*\omega = a^{-1}\omega$ *for* $a \in G$.

[2]The fundamental vector fields $\widetilde{A}^a$ form a global basis of sections for $\mathcal{V}(\mathcal{P}_G)$.

This in infinitesimal form is

Lemma 1.3: $\mathcal{L}_{\tilde{A}}\omega = \tilde{A} \lrcorner d\omega = -A\omega$ *for* $A \in \mathrm{g}$.

Lemma 1.4: *Let* $\phi : \mathbf{M} \to \mathbf{N}$ *be a diffeomorphism. Then there exists a unique lift* $\hat{\phi} : \mathcal{F}(\mathbf{M}) \to \mathcal{F}(\mathbf{N})$ *which is a principal bundle diffeomorphism and satisfies the properties*

$$\hat{\phi}^*\omega^N = \omega^M \qquad \text{and} \qquad \pi^N \circ \hat{\phi} = \phi \circ \pi^M \ .$$

Using the interpretation of $u \in \mathcal{F}(\mathbf{M})$ in equation (1.3), $\hat{\phi}$ may be written explicitly as

$$\hat{\phi}(u) = \phi_* \circ u : V \longrightarrow T_{\phi(u)}(\mathbf{N}) \tag{1.4}$$

This leads us to the definition,

Definition 1.1: (Sternberg [36], pg.313, Def.2.2) *Let* $\mathcal{B}_{\mathrm{G}}^M$ *and* $\mathcal{B}_{\mathrm{G}}^N$ *be G-structures on* $\mathbf{M}$ *and* $\mathbf{N}$ *respectively. We say* $\mathcal{B}_{\mathrm{G}}^M$ *is equivalent to* $\mathcal{B}_{\mathrm{G}}^N$ *if there exist a diffeomorphism* $\phi : \mathbf{M} \to \mathbf{N}$ *such that* $\hat{\phi} : \mathcal{B}_{\mathrm{G}}^M \to \mathcal{B}_{\mathrm{G}}^N$ *is a diffeomorphism.*

We will call the map ϕ between two equivalent G-structures an **equivalence map.** If we now assume G is connected then we have the extension of Lemma 1.4,

Lemma 1.5: *The G-structures* $\mathcal{B}_{\mathrm{G}}^M$ *and* $\mathcal{B}_{\mathrm{G}}^N$ *are equivalent if and only if there exist a diffeomorphism* $\Phi : \mathcal{B}_{\mathrm{G}}^M \to \mathcal{B}_{\mathrm{G}}^N$ *such that*

$$\Phi^*\omega^N = \omega^M \ . \tag{1.5}$$

In other words, if Φ satisfies condition (1.5) then Φ is a lift of a diffeomorphism $\phi : \mathbf{M} \to \mathbf{N}$ and is thus also a bundle map. See Gardner [16] for a proof. If we consider Example 1.2 then two fibre bundles $(\mathbf{E}', \mathbf{F}') \to \mathbf{M}'$ and $(\mathbf{E}, \mathbf{F}) \to \mathbf{M}$

are equivalent (**as fibre bundles**) if there exists an equivalence map between the H$-$structures (contained in $\mathcal{F}(\mathbf{E})$ and $\mathcal{F}(\mathbf{E}')$), where H is given by equation (1.1) of Example 1.2.

We now recall some basic definitions from representation theory. Let G be a linear Lie group. A real representation (μ, V) is a smooth homomorphism $\mu : \mathrm{G} \rightarrow Aut(V)$. Associated with the representation μ are the dual, tensor and exterior algebra representations which will be referred to collectively as the tensor representation of G with respect to μ. We will call a representation space V a G-module.

Example 1.3: (The tensor product representation) Let (μ, V) and (ν, W) be representations of G. The tensor product representation $(\mu \otimes \nu, V \otimes W)$ is defined as

$$\begin{array}{ccc} V \times W & \xrightarrow{\ i\ } & V \otimes W \\ {\scriptstyle (\mu_a, \nu_a)} \downarrow & & \downarrow {\scriptstyle (\mu \otimes \nu)_a} \\ V \times W & \xrightarrow[\ i\ ]{} & V \otimes W \end{array}$$

where $(\mu \otimes \nu)_a$ is the unique map making the above diagram commutative.

Definition 1.2: *Let (μ, V) be a representation of G. A subspace $W \subset V$ is said to be* $\mathbf{G}-$**invariant** *if $\mu(W) \subset W$. W is a G-submodule and we call (μ, W) a* **subrepresentation** *of μ.*

A useful result about representations is

Lemma 1.6: *Let $\mathbf{q} : V^* \otimes V^* \otimes V \rightarrow V^* \wedge V^* \otimes V$ be skew-symmetrization in the first two arguments and, let $W \subset Hom(V, V)$ be a G-invariant subspace. Then $Hom(V, W) \subset Hom(V, V \otimes V^*)$ and $\mathbf{q}(Hom(V, W)) \subset Hom(V \wedge V, V)$ are G-invariant subspaces with respect to the tensor representation of μ. In particular*

$$0 \longrightarrow ker\Pi \longrightarrow Hom(V, W) \xrightarrow{\Pi} \frac{Hom(V \wedge V, V)}{\mathbf{q}(Hom(V, W))} \longrightarrow 0$$

is a short exact sequence of G-modules.

We need to consider a number of special cases of this lemma.

Case 1 : Let (ι, V) be the defining representation of $G \subset GL(V)$, and let $W = \mathbf{g} \subset Hom(V, V)$ be the Lie algebra of G. W is G-invariant, and the subrepresentation of the tensor representation of (ι, V) on $\mathbf{g}$ is ad (the adjoint representation). In this case the kernel from Lemma 1.6 is usually denoted $\mathbf{g}^{(1)}$, and is called the **first prolongation** of $\mathbf{g}$. The subrepresentation on $\mathbf{g}^{(1)}$ we will write as $(\tau, \mathbf{g}^{(1)})$. The subrepresentation of μ on $Hom(V, \mathbf{g})$ is $(ad \otimes \iota^*)$ is given by

$$T \longrightarrow ad_a T \iota_{a^{-1}} \qquad T \in Hom(V, \mathbf{g}) \qquad a \in G \ .$$

The subrepresentation on the quotient in Lemma 1.6 for this case we will denote by σ, and for the representation (ι, V), we will often write $\iota_a(v) = av$.

For the next two cases define j to be the injection $j : Hom(V, \mathbf{g}) \to Hom(V \oplus \mathbf{g}, V \oplus \mathbf{g})$ given in matrix form as

$$j(T) = \begin{pmatrix} I_m & 0 \\ T & I_g \end{pmatrix} \quad , \qquad T \in Hom(V, \mathbf{g}) \ . \tag{1.6}$$

Case 2: Let $(ad \otimes \iota^*, Hom(V, \mathbf{g}))$ and $(\iota \oplus ad, V \oplus \mathbf{g})$ be the representations of G from Case 1, then $W = j(Hom(V, \mathbf{g}))$, is G-invariant. In fact $\iota_a \oplus ad_a j(T) = j(ad_a \otimes \iota_a^*(T))$ for $T \in Hom(V, \mathbf{g})$ follows by the matrix calculation

$$a \oplus ad_a \ j(T) = \begin{pmatrix} a & 0 \\ 0 & ad_a \end{pmatrix} \begin{pmatrix} I_m & 0 \\ T & I_g \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & ad_{a^{-1}} \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ ad_a T a^{-1} & I_g \end{pmatrix}$$

Case 2 will be of use in Section 1.3. We shall denote by μ the tensor representation of $(\iota \oplus ad)$ on $Hom(V \oplus \mathbf{g} \wedge V \oplus \mathbf{g}, V \oplus \mathbf{g})$ and by $\widehat{\mu}$ the representation on the quotient as given in Lemma 1.6.

Case 3: Let $W = j(\mathrm{g}^{(1)})$, (this is just $T \in \mathrm{g}^{(1)}$ in equation (1.6)), is G-invariant. Again as in Case 2 we have $a \oplus ad_a j(T) = j(\tau_a(T))$ for $T \in \mathrm{g}^{(1)}$ and where $(\tau, \mathrm{g}^{(1)})$ was defined in Case 1. This case will be of use in Section 1.4. We shall let λ be the representation on the quotient given in Lemma 1.6 in this case.

We use Sternberg [36], pg.316 in defining the **structure function.** Let H^1 and H^2 be two horizontal subspaces at $u \in \mathcal{B}_{\mathrm{G}}$ and let $v, w \in V$, $X^1, Y^1 \in H^1$ and $X^2, Y^2 \in H^2$ with $\omega(X^1) = \omega(X^2) = v$ and $\omega(Y^1) = \omega(Y^2) = w$ We now define $S_{H^1,H^2} : V \longrightarrow \mathbf{g}$ as

$$S_{H^1,H^2}(v) = X^1 - X^2$$

and $C_{H^i} \in Hom(V \wedge V, V)$, $i = 1, 2$ as

$$C_{H^i}(v \wedge w) = d\omega(X^i \wedge Y^i) \tag{1.7}$$

We then have

$$\begin{aligned} C_{H^2}(v \wedge w) - C_{H^1}(v \wedge w) &= < Y^2, (X^2 - X^1) \lrcorner d\omega > - < X^1, (Y^2 - Y^1) \lrcorner d\omega > \\ &= S_{H^1,H^2}(v)(w) - S_{H^1,H^2}(w)(v) \ . \end{aligned} \tag{1.8}$$

The right hand side of equation (1.8) above is just $\mathbf{q}(S_{H^1,H^2})$, so that

$$C(v \wedge w) = \mathbf{q}(C_H(v \wedge w)) \qquad \text{for any horizontal} \quad H$$

is a well defined function

$$C : \mathcal{B}_{\mathrm{G}} \longrightarrow \frac{Hom(V \wedge V, V)}{\mathbf{q}(Hom(V, \mathbf{g})} \ .$$

C also satisfies

$$C(R_a u) = \sigma_{a^{-1}} \ C(u) \ ,$$

where σ was defined above in Case 1. The role of the structure function in the equivalence problem is illustrated by the following result,

Theorem 1.2: (Sternberg [36], pg.319, Theorem 2.1) *If ϕ is an equivalence map between two G-structures $\mathcal{B}_{\mathrm{G}}^{M}$ and $\mathcal{B}_{\mathrm{G}}^{N}$ on* **M** *and* **N** *then*

$$C_{\mathbf{N}} \circ \hat{\phi} = C_{\mathbf{M}}$$

We will use Lemma 1.1 to simplify questions about the equivalence of G-structures in the following way: Let $L \subset \frac{Hom(V \wedge V, V)}{\mathbf{q}(Hom(V, \mathbf{g}))}$ be a G-invariant subspace with projection π_L, and let $\mathcal{O}_l$ be the orbit of a point $l \in L$. Using $\pi_L \circ C$ in place of σ in Lemma 1.1 we have that if two G-structures $\mathcal{B}_{\mathrm{G}}^{M}$ and $\mathcal{B}_{\mathrm{G}}^{N}$ on **M** and **N** are equivalent then $\widehat{\mathcal{B}}_{\mathrm{H}_l}^{\hat{M}}$ and $\widehat{\mathcal{B}}_{\mathrm{H}_l}^{\hat{N}}$ are equivalent, where $\mathrm{H}_l \subset \mathrm{G}$ is the stability group of $l \in \mathcal{O}_l$ (this assumes the conditions of Lemma 1.1 are satisfied). Often we choose $L = \frac{Hom(V \wedge V, V)}{\mathbf{q}(Hom(V, \mathbf{g}))}$.

1.4 The Frame Bundle of a G-structure

Let $\pi : \mathcal{B}_{\mathrm{G}} \to \mathbf{M}$ be a G-structure, and let $\pi^1 : \mathcal{F}(\mathcal{B}_{\mathrm{G}}) \to \mathcal{B}_{\mathrm{G}}$ be the frame bundle of $\mathcal{B}_{\mathrm{G}}$ with canonical $V \oplus \mathbf{g}$-valued one-form Θ and right action R^1 of $GL(V \oplus \mathbf{g})$. Using the projections

$$\pi_V : V \oplus \mathbf{g} \to V \quad \text{and} \quad \pi_{\mathbf{g}} : V \oplus \mathbf{g} \to \mathbf{g} \ ,$$

we can define a canonical reduction of $\mathcal{F}(\mathcal{B}_{\mathrm{G}})$, which we denote by $\overline{\mathcal{B}}_{\overline{\mathrm{G}}}$, as follows: Let $\overline{u} \in \mathcal{F}(\mathcal{B}_{\mathrm{G}})$, $u = \pi^1(\overline{u})$, and $X \in T_{\overline{u}}(\mathcal{F}(\mathcal{B}_{\mathrm{G}}))$ such that $\pi^1_* X \in \mathcal{V}_u(\mathcal{B}_{\mathrm{G}})$ then $\overline{u} \in \overline{\mathcal{B}}_{\overline{\mathrm{G}}}$ if and only if

$$\textbf{P1)} \qquad \pi_V \circ \Theta_{\overline{u}} = (\pi^1)^* \omega_u \qquad \text{and} \tag{1.9}$$

$$\textbf{P2)} \qquad (\pi_{\mathbf{g}} \circ \widetilde{\Theta}(X))(u) = \pi^1_* X \ . \tag{1.10}$$

These two conditions can be written equivalently as $\overline{u} \in \overline{\mathcal{B}}_{\overline{G}}$ if and only if

$$\omega \circ \overline{u}((v, A)) = v \qquad \text{and} \qquad \overline{u}(0, A) = \tilde{A}(u) \ , \text{ for all } (v, A) \in V \oplus \mathrm{g} \ .$$

Note that the image of V under $\overline{u} \in \overline{\mathcal{B}}_{\overline{G}}$ is a horizontal subspace of $T_u(\mathcal{B}_G)$. The structure group $\overline{G}$ is the **Abelian** group,

$$\overline{G} = \begin{pmatrix} I_m & T \\ 0 & I_g \end{pmatrix} \subset GL(V \oplus \mathrm{g}) \qquad \text{where} \qquad T \in Hom(V, \mathrm{g}) \tag{1.11}$$

We now want to define a right action $\Sigma : \mathcal{F}(\mathcal{B}_G) \times G \to \mathcal{F}(\mathcal{B}_G)$ which has the property of preserving $\overline{\mathcal{B}}_{\overline{G}}$. This will be done in two steps, first by using Lemma 1.4 we find the right action R on $\mathcal{B}_G$ admits a lift $\widehat{R}$ such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(\mathcal{B}_G) \times G & \xrightarrow{\ \widehat{R}\ } & \mathcal{F}(\mathcal{B}_G) \\ {\scriptstyle \pi^1 \times I}\downarrow & & \downarrow{\scriptstyle \pi^1} \\ \mathcal{B}_G \times G & \xrightarrow[\ R\ ]{} & \mathcal{B}_G \end{array} \qquad \text{and} \qquad \widehat{R_a}^{*}\Theta = \Theta \ . \tag{1.12}$$

Now define the map $\rho : \mathcal{F}(\mathcal{B}_G) \times G \to \mathcal{F}(\mathcal{B}_G)$ to be R^1 restricted to the subgroup

$$\begin{pmatrix} a & 0 \\ 0 & ad_a \end{pmatrix} \subset GL(V \oplus \mathrm{g}) \ .$$

That is

$$\rho_a(u^1) = R^1_{a \oplus ad_a}(u^1) \quad , \qquad u^1 \in \mathcal{F}(\mathcal{B}_G) \ . \tag{1.13}$$

ρ_a satisfies

$$\pi^1 \circ \rho_a = \pi^1 \ . \tag{1.14}$$

and because $\widehat{R_a}$ commutes with the action of R^1, we also have

$$\rho_b \circ \widehat{R_a} = \widehat{R_a} \circ \rho_b \qquad b \in G \ . \tag{1.15}$$

We are now able to define Σ by

$$\Sigma_a(u^1) \;=\; \rho_a \circ \widehat{R_a}(u^1) \;=\; \widehat{R_a} \circ \rho_a(u^1)\;, \tag{1.16}$$

and it is straightforward to check that Σ defines a right action of G on $\mathcal{F}(\mathcal{B}_{\mathrm{G}})$. Σ also satisfies the following two lemmas

Lemma 1.7: $\Sigma_a{}^* \,\Theta = a^{-1} \oplus ad_{a^{-1}}\Theta$

Proof: Calculating using equation (1.12) and Lemma 1.2 gives

$$\Sigma_a{}^* \,\Theta = \rho_a{}^* \,\widehat{R_a}^* \,\Theta = \rho_a{}^* \,\Theta = a^{-1} \oplus ad_{a^{-1}}\Theta$$

■

Lemma 1.8: $\pi_* \, \pi^1_* \, X = 0 \qquad \textit{if and only if} \qquad \pi_* \, \pi^1_* \, \Sigma_{a*} \, X = 0$

Proof: The following calculation uses equations (1.14) and (1.12)

$$\pi_* \, \pi^1_* \, \Sigma_{a*} \, X = \pi_* \, \pi^1_* \, \rho_{a*} \, \widehat{R_{a*}} \, X = \pi_* \, \pi^1_* \, \widehat{R_{a*}} \, X = \pi_* \, R_{a*} \, \pi^1_* \, X = \pi_* \, \pi^1_* \, X \;,$$

from which the lemma follows. ■

For the remainder of this section let

$$\overline{u} \in \overline{\mathcal{B}}_{\overline{\mathrm{G}}} \quad , \quad X \in T_{\overline{u}}(\overline{\mathcal{B}}_{\overline{\mathrm{G}}}) \quad , \quad u = \pi^1(\overline{u}) \quad \text{and} \quad a \in \mathrm{G}\;.$$

The importance of Σ and $\overline{\mathcal{B}}_{\overline{\mathrm{G}}}$ lies in the theorem,

Theorem 1.3: *The map Σ_a when restricted to $\overline{\mathcal{B}}_{\overline{\mathrm{G}}}$ defines a diffeomorphism*

$$\Sigma_a : \overline{\mathcal{B}}_{\overline{\mathrm{G}}} \longrightarrow \overline{\mathcal{B}}_{\overline{\mathrm{G}}} \quad \textit{for all } a \in \mathrm{G}$$

Note Σ_a is typically **not** a $\overline{\mathrm{G}}$ bundle map.

Proof: We need to show that P1 and P2 in equation (1.9) are satisfied at the point $\Sigma_a(\overline{u}) \in \mathcal{F}(\mathcal{B}_{\mathrm{G}})$. To check property P1 we need to verify

$$(\pi^1)^*\omega(\Sigma_* X) = \pi_V \circ \Theta_{\Sigma_a(\overline{u})}(\Sigma_{a*}X) \ . \tag{1.17}$$

The left hand side of equation (1.17) simplifies to

$$\omega_{ua}(\pi^1_* \rho_{a*}\widehat{R}_{a*}X) = \omega_{ua}(\pi^1_*\widehat{R}_{a*}X) = \omega_{ua}(R_{a*}\pi^1_* X) \tag{1.18}$$

by using equation (1.14) then (1.12). Lemma 1.2 applied to equation (1.18) above then yields

$$R_a{}^* \ \omega_u(\pi^1_* X) = a^{-1}\omega_u(\pi^1_* X) \ . \tag{1.19}$$

Now applying Lemma 1.7 to the right hand side of equation (1.17) we have

$$\pi_V \circ \Theta_{\Sigma_a(\overline{u})}(\Sigma_{a*}X) = \pi_V \circ \Sigma_a{}^*\Theta_{\overline{u}}(X) = a^{-1}\pi_V \circ \Theta_{\overline{u}}(X) \tag{1.20}$$

Since $\overline{u} \in \mathcal{B}^1_{\mathrm{G}}$ we may use property P1 at $\overline{u}$ from which it is clear that equations (1.20) and (1.19) imply equation (1.17). Hence P1 is satisfied at $\Sigma_a(\overline{u})$.
To check property P2, Lemma 1.8 implies we only need to verify that

$$(\pi_{\mathbf{g}} \circ \widetilde{\Theta_{\Sigma_a(\overline{u})}(\Sigma_{a*}X)})(ua) = \pi^1_* \Sigma_{a*} X \tag{1.21}$$

for all $X \in T_{\overline{u}}(\overline{\mathcal{B}}_{\overline{\mathrm{G}}})$ satisfying $\pi^1_* X \in \mathcal{V}_u(\mathcal{B}_{\mathrm{G}})$. Using Lemma 1.7 the left hand side of equation (1.21) above is

$$(\pi_{\mathbf{g}} \circ \widetilde{\Sigma_a{}^*\Theta_{\overline{u}}(X)})(ua) = (\widetilde{ad_{a^{-1}}\pi_{\mathbf{g}} \circ \Theta(X)})(ua) \tag{1.22}$$

while the right hand side of (1.21) by equation (1.12) is

$$R_{a*}\pi^1_* X \ .$$

Now use the fact $\overline{u} \in \overline{\mathcal{B}}_{\overline{G}}$ and equation (1.2), so that equation (1.22) becomes

$$R_{a*}(\widetilde{\pi_{\mathbf{g}} \circ \Theta}(X))(u) = (ad_{a^{-1}}\widetilde{\pi_{\mathbf{g}} \circ \Theta}(X))(ua) \ .$$

Equation 1.21 is now satisfied, completing the proof. ■

We now examine the effect of the action of Σ on the structure function

$$\overline{C} : \overline{\mathcal{B}}_{\overline{G}} \longrightarrow \frac{Hom(V \oplus \mathbf{g} \wedge V \oplus \mathbf{g}, V \oplus \mathbf{g})}{\mathbf{q}(Hom(V \oplus \mathbf{g}, \mathbf{g}^1)}$$

By Case 2 following Lemma 1.6, the image space of $\overline{C}$ admits the left action $\widehat{\mu}$ of G (while the numerator has action μ). We have

Theorem 1.4: $\overline{C}(\Sigma_a(\overline{u})) = \widehat{\mu}_{a^{-1}}\overline{C}(\overline{u})$.

The proof below is analogous to that of Sternberg [36], pg.318.

Proof: Let $\overline{H} \subset T_{\overline{u}}(\overline{\mathcal{B}}_{\overline{G}})$ be a horizontal subspace, (v, A) , $(w, B) \in V \oplus \mathbf{g}$, and X , $Y \in \overline{H}$ such that $\Theta(X) = (v, A)$, and $\Theta(Y) = (w, B)$. By Lemma 1.7 we have

$$\Sigma_a{}^* \, \Theta(X) = (\ a^{-1}v \ , \ ad_{a^{-1}}A \ ) \qquad \text{and} \qquad \Sigma_a{}^* \, \Theta(Y) = (\ a^{-1}w \ , \ ad_{a^{-1}}B \ )$$

as well as

$$\Sigma_a{}^* \, d\Theta(X \wedge Y) = a^{-1} \oplus ad_{a^{-1}} d\Theta(X \wedge Y) \ .$$

Thus

$$\begin{aligned} \overline{C}_{\Sigma_{a*}\overline{H}}((v, A) \wedge (w, B)) &= a^{-1} \oplus ad_{a^{-1}}\overline{C}_{\overline{H}}((av, ad_a A) \wedge (aw, ad_a B)) \\ &= \mu_{a^{-1}}\overline{C}_{\overline{H}}((v, A) \wedge (w, B)) \qquad (1.23) \end{aligned}$$

Since $\mathbf{q}(Hom(V \oplus \mathbf{g}, \mathbf{g}^1) \subset Hom(V \oplus \mathbf{g} \wedge V \oplus \mathbf{g}, V \oplus \mathbf{g})$ is G-invariant with respect to μ_a when we pass to the quotient we have the result. ■

Consider now the situation in which the structure function $\overline{C}$ is independent of the action R^1 of $\overline{G}$ on $\mathcal{B}_{\overline{G}}$. This defines a function

$$C^0 : \mathcal{B}_G \longrightarrow \frac{Hom(V \oplus g \wedge V \oplus g, V \oplus g)}{\cdot\, q(Hom(V \oplus g, g^1)} \qquad (1.24)$$

by $C^0(u) = \overline{C}(\overline{u})$ where $\overline{u}$ is **any** point in $\mathcal{B}_{\overline{G}}$ such that $\pi^1(\overline{u}) = u$. We have

Theorem 1.5: *The function C^0 in equation* (1.24) *satisfies,*

$$C^0(ua) = \hat{\mu}_{a^{-1}} C^0(u)$$

Proof: We have $\pi^1\Sigma_a(\overline{u}) = ua$ along with Theorem 1.4 gives

$$C^0(ua) = \overline{C}(\Sigma_a\overline{u}) = \hat{\mu}_{a^{-1}}\overline{C}(\overline{u}) = \hat{\mu}_{a^{-1}} C^0(u)$$

■

Theorem 1.5 gives rise to necessary conditions for equivalence of two G-structures in the same way the structure function does in Theorem 1.2.

1.5 Prolongation

Let $\mathcal{A} = q(Hom(V, g))$ and let $\mathcal{C}$ be a subspace of $Hom(V \wedge V, V)$ such that $Hom(V \wedge V, V) = \mathcal{C} \oplus \mathcal{A}$. A choice of $\mathcal{C}$ defines a reduction of $\overline{\mathcal{B}}_{\overline{G}}$ by considering $\overline{u} \in \overline{\mathcal{B}}_{\overline{G}}$ satisfying the condition

$$C_{H_u} \in \mathcal{C} \quad , \quad \text{where } u = \pi^1(\overline{u}) \quad \text{and} \quad H_u = \overline{u}(v, 0)$$

(that is H_u is the image of V in $T_u(\mathcal{B}_G)$). The reduced group as a subgroup of $\overline{G}$ we get by considering only $T \in g^{(1)}$ in equation (1.11). This reduction of $\mathcal{F}(\mathcal{B}_G)$ is known as the first prolongation of $\mathcal{B}_G$ which we denote by

$$\pi^1 : \mathcal{B}_{G^{(1)}} \longrightarrow \mathcal{B}_G \ .$$

The importance of this principal bundle is the following,

Theorem 1.6: (Sternberg [36], pg.336, Theorem 1.2) *Two G−spaces* $\mathcal{B}_{\mathrm{G}}^{M}$ *and* $\mathcal{B}_{\mathrm{G}}^{N}$ *on* **M** *and* **N** *are equivalent if and only if the* $\mathrm{G}^{(1)}$*−spaces* $\mathcal{B}_{\mathrm{G}^{(1)}}^{M}$ *and* $\mathcal{B}_{\mathrm{G}^{(1)}}^{N}$ *over* $\mathcal{B}_{\mathrm{G}}^{M}$ *and* $\mathcal{B}_{\mathrm{G}}^{N}$ *are equivalent.*

For the rest of this section we will require that the complement $\mathcal{C} \subset Hom(V \wedge V, V)$ above be G−invariant[3] with respect to the tensor representation of (ι, V) (See Case 1) we then have the extension of Theorem 1.3 to

Theorem 1.7: *The map* Σ_a *when restricted to* $\mathcal{B}_{\mathrm{G}^{(1)}}$ *defines a diffeomorphism*

$$\Sigma_a : \mathcal{B}_{\mathrm{G}^{(1)}} \longrightarrow \mathcal{B}_{\mathrm{G}^{(1)}} \quad \textit{for all } a \in \mathrm{G}$$

Proof: Let $u^1 \in \mathcal{B}_{\mathrm{G}^{(1)}}$, $u = \pi^1(u^1)$ and $H_u = u^1(v, 0)$. We need to check that $H'_{ua} = \Sigma_a(u^1)(v, 0)$ satisfies $C_{H'_{ua}} \in \mathcal{C}$. However, by the definition of $\widehat{R_a}$ and equation (1.4) we have

$$H'_{ua} = R_{a*}\, H_u \ ,$$

and the G−invariance of $\mathcal{C}$ means $C_{R_{a*}H_u} \in \mathcal{C}$, which finishes the proof. ■

Note the G-invariance of $\mathcal{C}$ is crucial since without it Σ_a would map $\mathcal{B}_{\mathrm{G}^{(1)}}$ to $\overline{\mathcal{B}_{\overline{\mathrm{G}}}}$ (see previous section).

Writing C^1 as the structure function on $B_{\mathrm{G}^{(1)}}$, we also have the corresponding extension to Theorems 1.4 and 1.5,

Theorem 1.8: $C^1(\Sigma_a(u^1)) = \lambda_{a^{-1}} C^1(u^1)$.

Where λ_a is defined in Case 3 after Lemma 1.6. If we assume C^1 is independent of $\mathrm{G}^{(1)}$ and let C' denote the function on the $\mathcal{B}_{\mathrm{G}}$ so that,

$$C^1(u^1) = C' \circ \pi^1(u^1) \tag{1.25}$$

then we also have,

[3] For compact G this is always possible

Theorem 1.9: *The function C' satisfies,*

$$C'(ua) = \lambda_{a^{-1}} C'(u)$$

We may then use C' for reduction of G as usual.

As an example consider the case where $\mathbf{g}^{(1)} = 0$, and the existence of a $\mathcal{B}^1_{\mathrm{G}^{(1)}}$ structure is equivalent to the existence of a connection, the horizontal distribution $\mathcal{H} \subset T(\mathcal{B}_{\mathrm{G}})$ is given by

$$\mathcal{H}_u = u^1(v, 0)\,.$$

We have the following,

Lemma 1.9: *Let α be the connection one-form, then the canonical one-form Θ from Section 1.4 is given by*

$$\Theta = (\pi^1)^*\omega \oplus (\pi^1)^*\alpha \tag{1.26}$$

Proof: Let $X \in T_{u^1}(\mathcal{B}^1_{\mathrm{G}^1})$, we have that $u^1(v, A) = \pi_* X = X_H \oplus X_{\mathbf{g}}$ where $X_H = u^1(v, 0) \in H_u$. Thus $X_{\mathbf{g}} = u^1(0, A)$ and by definition we have $\alpha(\pi_* X) = A$ ∎

Now $\mathbf{g}^{(1)} = 0$ means at each point $u^1 \in \mathcal{B}_{\mathrm{G}^{(1)}}$ we have $H_{u^1} = T_{u^1}(\mathcal{B}_{\mathrm{G}^{(1)}})$ and so the structure function C^1 has the property,

$$\begin{aligned} C^1 = C' : \mathcal{B}_{\mathrm{G}} \longrightarrow \;& Hom(V \wedge V, V) \oplus Hom(V \otimes \mathbf{g}, V) \oplus Hom(\mathbf{g} \wedge \mathbf{g}, V) \\ & \oplus Hom(V \wedge V, \mathbf{g}) \oplus Hom(V \otimes \mathbf{g}, \mathbf{g}) \oplus Hom(\mathbf{g} \wedge \mathbf{g}, \mathbf{g})\,. \end{aligned} \tag{1.27}$$

We will write the above as $(C^1_V, C^2_V, C^3_V, C^1_{\mathbf{g}}, C^2_{\mathbf{g}}, C^3_{\mathbf{g}})$ and each term in this case is easy to determine by E. Cartan's structure equations for a connection which are,

$$d\alpha(X, Y) = -\frac{1}{2}[\alpha(X), \alpha(Y)] + \Omega(X, Y)$$

$$d\omega\ (X,Y) = -\frac{1}{2}(\alpha(X)\cdot\omega(Y) - \alpha(Y)\cdot\omega(X)) + \mathbf{T}(X,Y)$$

where Ω is the curvature form and $\mathbf{T}$ the torsion form of the connection α. To determine C' let $v^i \in V, A^i \in \mathbf{g}$, and $X^i \in T_{u^1}(\mathcal{B}^1_{G^1})$, $i = 1..2$ and examine the three cases,

Case 1: Let $u^1(v^i,0) = \pi^1_* X^i$ then using Lemma 1.9

$$C^1(v^1 \wedge v^2) = d\Theta(X^1,X^2) = \mathbf{T}(\pi^1_* X^1, \pi^1_* X^2) \oplus \Omega(\pi^1_* X^1, \pi^1_* X^2)$$

and thus,

$$\begin{aligned} C^1_V(v^1 \wedge v^2) &= \mathbf{T}(\pi^1_* X^1, \pi^1_* X^2) \\ C^1_{\mathbf{g}}(v^1 \wedge v^2) &= \Omega(\pi^1_* X^1, \pi^1_* X^2) \end{aligned} \quad (1.28)$$

Case 2: Let $u^1(0,A^i) = \pi^1_* X^i$, then

$$C^1(A^1 \wedge A^2) = d\Theta(X^1,X^2) = -\frac{1}{2}[\alpha(X^1),\alpha(X^2)] = -\frac{1}{2}[A^1,A^2]\,.$$

From which we have

$$\begin{aligned} C^3_V(A^1 \wedge A^2) &= 0 \\ C^3_{\mathbf{g}}(A^1 \wedge A^2) &= -\frac{1}{2}[A^1,A^2]\,. \end{aligned} \quad (1.29)$$

Case 3: Let $u^1(v^1,0) = \pi^1_* X^1$, $u^1(0,A^2) = \pi^1_* X^2$, , then

$$C^1(v^1 \otimes A^2) = d\Theta(X^1,X^2) - d\Theta(X^2,X^1) = A^2\, v^1 \quad .$$

From this we have

$$\begin{aligned} C^2_V(v^1 \otimes A^2) &= A^2\, v^1 \\ C^2_{\mathbf{g}}(v^1 \otimes A^2) &= 0\,. \end{aligned} \quad (1.30)$$

Particular examples where the G action on the curvature tensor is used to generate further necessary conditions for the equivalence problem can be found in [4] and [28].

1.6 Local Equivalence and Symmetry

As mentioned in the introduction to this chapter, the theory of local equivalence is discussed in detail in both Gardner [16] and Kamran [24] and familiarity with this material will be assumed for the rest of this thesis. We will only extract a few of the important results and mention the standard terminology which we use. As in these references, we use trivial **left** principal bundles which arise more naturally when using differential forms.

The set up for the study of the local equivalence problem is as follows: Let $U, V \subset \mathbb{R}^n$ be open and contractible, and let ω^i_U, ω^i_V be coframes on U and V respectively. We know that **any** diffeomorphism $\phi : U \to V$ satisfies

$$\phi^*\omega^i_V = K^i_j\omega^j_U$$

where $K^i_j : U \to GL(n, \mathbb{R})$. However if we encode in the coframes some geometric structure, we would like to know if there does exist a diffeomorphism preserving this structure. This usually translates into the requirement that K^i_j take values in a given linear group $\hat{\mathrm{H}} \subset GL(n, \mathbb{R})$ so that the problem can be stated as,

Local Equivalence Problem: Does there exist a diffeomorphism $\phi : U \to V$, such that

$$\phi^*\omega^i_V = K^i_j\omega^j_U$$

where $K^i_j : U \to \hat{\mathrm{H}} \subset GL(n, \mathbb{R})$.

A standard example is,

Example 1.4: Let $(\mathbf{M}, g)$ and $(\overline{\mathbf{M}}, \overline{g})$ be Riemannian manifolds, and let (U, ω^i_U), (V, ω^i_V) be local orthonormal coframes. $\mathbf{M}$ and $\overline{\mathbf{M}}$ are locally isometric if and only if there exists $\phi : U \to V$ such that $K^i_j : U \to O(n, \mathbb{R})$.

Let $\mathrm{H} \subset \hat{\mathrm{H}}$ be an open neighbourhood of the identity [4] with a subset of S^i_j the

[4]We will be concerned with local problems from here on, so we will make this assumption

standard coordinate on $GL(n, \mathbb{R})$ being coordinates on H. Define on $U \times$ H the $\mathbb{R}^n$-valued differential form

$$\omega^i = \mathcal{S}^i_j \omega^j_U ,$$

so that $(U \times \mathrm{H}, \omega^i)$ is a (trivial) left principal bundle with ω^i being the canonical form. So we may apply the necessary conditions in Theorem 1.2 obtained by computing the structure function. In order to give the local description of the structure function we first denote by by $(\alpha^b)_{1 \leq b \leq h}$ a maximal linearly independent subset of the one-forms

$$(d\mathcal{S}^i_k)(\mathcal{S}^{-1})^k_j .$$

The right-invariant forms $(\alpha^b)_{1 \leq b \leq h}$. are known as a right-invariant **Maurer-Cartan** forms. On H we have

$$(d\mathcal{S}^i_k)(\mathcal{S}^{-1})^k_j = C^i_{jb}\alpha^b$$

and $(C^i_{jb}\alpha^b)$ is a right-invariant Lie algebra-valued form, known as a Lie algebra-valued **Maurer-Cartan form** .

To compute the structure function first on $U \times$ H take $d\omega^i$

$$d\omega^i = (d\mathcal{S}^i_k)(\mathcal{S}^{-1})^k_j \wedge \omega^j + \mathcal{S}^i_j d\omega^j_U = C^i_{jb}\alpha^b \wedge \omega^j + \Gamma^i_{jk}\omega^j \wedge \omega^k .$$

where we have lifted $C^i_{jb}\alpha^b$ to $U \times$ H. Then perform what often Gardner [16] calls **absorption of torsion**, that is let

$$\widehat{\alpha}^b = \alpha^b + V^b_k \omega^k$$

and solve as many of the linear equations

$$V^b_{[k} C^i_{j]b} = \Gamma^i_{jk} . \tag{1.31}$$

for V^i_b as possible. The purpose of solving these equations is to choosing a splitting $Hom(V \wedge V, V) = \mathcal{C} \oplus \mathcal{A}$ as in section 4. We may write the resulting equations in the form,

$$d\omega^i = C^i_{jb}\alpha^b \wedge \omega^j + \overline{\Gamma}^i_{jk}\omega^j \wedge \omega^k \ . \tag{1.32}$$

where we have dropped the hats on α^b, and where $\overline{\Gamma}^i_{jk}$ takes values in C. The fact that $\overline{\Gamma}^i_{jk}$ takes values in C means that it is a representative for the structure function (which takes values in a quotient). We call the collection of forms ω^i **base** forms, while we call the collection α^b of forms $\mathbf{h}^*$ forms.

When determining the infinitesimal form of the group action on $\overline{\Gamma}^i_{jk}$, of essential importance is

Lemma 1.11: (Cartan's Lemma) Let $\{\omega^i\}$ be an independent set of one-forms, and let $\{\pi^i\}$ be an arbitrary set of one-forms of the same finite cardinality; then $\pi_i \wedge \omega^i = 0$ if and only if,

$$\pi_i = K_{ij}\omega^j \, , \text{where } K_{[ij]} = 0 \ . \tag{1.33}$$

After applying Cartan's Lemma we will often write instead of (1.33) the congruence,

$$\pi^i \equiv 0 \qquad \text{mod}(\omega^j)$$

We will use the Cartan Lemma in the form of congruences without further reference.

The set of solutions of the homogeneous system associated to equations in (1.31) (these equations with right-hand side zero) is just $\mathbf{h}^{(1)}$ the first prolongation of $\mathbf{h}$ as defined in Case 1 of Lemma 1.7. Finding these solutions we call finding the **kernel of the absorption map**, and a parameterization for this set of solutions may be used as local coordinates on $\mathrm{H}^{(1)}$ (or for $\mathbf{h}^{(1)}$), the first prolongation of H (or $\mathbf{h}$). Suppose now that we have a prolonged equivalence problem so we have $U \times H \times H^{(1)} \rightarrow U \times H$, with the structure function $\rho^1 : U \times H \times H^{(1)} \rightarrow W$, where W is the appropriate vector space. If ρ is **independent** of $\mathrm{H}^{(1)}$, then there exists a function $\rho' : U \times \mathrm{H} \rightarrow W$ such that $\rho' \circ \pi^1 = \rho^1$. Theorems 1.7, 1.8, and 1.9 determine the procedure one follows in this situation, to summarize:

Theorem 1.10: *If the complement C used in defining the prolongation* $\mathrm{H}^{(1)}$ *is* H-*equivariant, then ρ' is a* H-*equivariant function, with the* H*−action on W being given by the subrepresentation of* H *on the quotient in equation* (1.24).

A very well understood case in the equivalence problem is when $\mathrm{H} = \{e\}$ and this is known as the equivalence problem for $\{e\}$-structures. Using the (invariant) coframe $\{\omega^i\}$, the covariant derivatives $f_{|i}$ of $f \in C^\infty(U)$ are defined by

$$df = [df]_{\omega^i}\,\omega^i \equiv f_{|i}\omega^i \ .$$

From the structure equations,

$$d\omega^i = \Gamma^i_{jk}\omega^j \wedge \omega^k \tag{1.34}$$

we then define,

$$\mathcal{F}_s = \left\{\Gamma^i_{jk}, \Gamma^i_{jk|i_1}, \Gamma^i_{jk|i_1 i_2}, \ldots, \Gamma^i_{jk|i_1 i_2 \ldots i_s}\right\} \tag{1.35}$$

and

$$k_s(p) = rank(\mathcal{F}_s)_p \ . \tag{1.36}$$

At some finite number for s we have $k_s(p) = k_{s+1}(p) = k(p)$ which is called the **rank** of the $\{e\}$-structure at p. Necessary and sufficient conditions for the existence of an equivalence between two $\{e\}$-structures can be given in terms of the rank of $\mathcal{F}$ and the functional dependencies of its elements, see one of [36], [16], [24].

A diffeomorphism $\phi : U \rightarrow U$ such that

$$\phi^*\omega^i = \omega^i \tag{1.37}$$

is called a **symmetry** (or automorphism). In the case that the one-forms ω^i form a coframe or an $\{e\}$-structure the set of symmetry's form a finite dimensional local Lie transformation group which we call the **symmetry group of the {e}-structure**. The group operation is composition of functions, while the symmetry group has the property,

Theorem 1.11: *The dimension of the symmetry group at $p \in U$ is* dim(M) - k(p).

This of course implies the dimension of the symmetry group is less than or equal to the dimension of the $\{e\}$-structure. The global counterpart to this theorem is classical [30] pg. 13. A corollary of Theorem 1.11 that we will use is,

Corollary 1.1: *An $\{e\}$-structure has a maximal dimension symmetry group if and only if the structure function is constant.*

In this case the $\{e\}$-structure is a local Lie group by the third fundamental theorem of Lie. In the case that we have an $\{e\}$-structure $\{\omega^i\}$ on a principal bundle $U \times H$, Lemma 1.6 allows us to conclude that any symmetry is a prolongation of a diffeomorphism of U.

If the structure function $\overline{\Gamma}^i_{jk}$ in equation (1.32) is independent of H we need to determine whether equations (1.37) admit what is known as an **infinite Lie pseudogroup**. For the precise definition of an infinite pseudogroup we refer the reader to Kamran [24]. In summary, the conditions in (1.37) for the existence of a symmetry are a system of partial differential equations for ϕ. In the case that the system of differential equations admit a family of solutions which can be parameterized by a finite number of arbitrary constants (in an open set) then the collection of symmetries form a finite dimensional Lie transformation group (the constants being the local group coordinates). One case we have already mentioned where this occurs is when the ω^i form a coframe in which case Theorem 1.11 applies. On the other hand it is conceivable that the partial differential equations in (1.37) admit solutions which depend on arbitrary functions. In that case the collection of solutions satisfying (1.37) form what we call an infinite Lie pseudogroup. Cartan devised a criterion based on an existence theorem known as the **Cartan-Kähler theorem** for the existence of integral manifolds of **analytic** exterior differential systems called the involutivity test. This determines whether an analytic exterior differential systems will admit a general solution depending on arbitrary functions. To give this criteria for the equations (1.37), define

Definition 1.3: *The* **Cartan characters** *for* (1.32) *are defined inductively by,*

$$\sigma'_1 + \sigma'_2 + \ldots + \sigma'_L = \begin{matrix} max\ rank \\ v_1, \ldots v_L \in \mathbb{R}^n \end{matrix} \begin{pmatrix} v^j_1 C^i_{jb} \\ . \\ . \\ . \\ v^j_L C^i_{jb} \end{pmatrix} . \tag{1.38}$$

The importance of the Cartan characters are then due to the following,

Theorem 1.12: *(Involutivity test) If*

$$\dim(\mathrm{g}^{(1)}) = \sum_{l=1}^{n} l\sigma'_l \tag{1.39}$$

and $C^i_{jb} \neq 0$*, then the symmetries of equation* (1.32) *form an* **infinite Lie pseudogroup.**

If the terms $\overline{\Gamma}^i_{jk}$ are constant then the infinite Lie pseudogroup is **transitive,** otherwise it is called intransitive.

Chapter 2

Systems of Ordinary Differential Equations

2.1 Introduction

In this chapter we will apply the Cartan method of equivalence to study the equivalence of systems of n (≥ 2) second and third order ordinary differential equations under point transformations. This approach was first utilized by Chern [7] who examined equivalence under the groups of smooth invertible transformations

$$\bar{t} = t \qquad \bar{x}^i = \psi^i(x^j) \qquad \text{and}$$
$$\bar{t} = t \qquad \bar{x}^i = \psi^i(t, x^j) \tag{2.1}$$

for systems of second-order ordinary differential equations. Chern subsequently [9] considered the equivalence of systems of r^{th} order ordinary differential equations under the invertible smooth transformations

$$\bar{t} = t \qquad \bar{x}^i = \psi^i(t, x^j) \ .$$

Chern was able to associate to any system of equations an $\{e\}$-structures or an **invariant coframe**. We shall prove the same result is true under the larger group of point transformations,

$$\bar{t} = \phi(t, x^j) \qquad \bar{x}^i = \psi^i(t, x^j) \ . \tag{2.2}$$

With each second order system of ordinary differential equations the associated $\{e\}$-structure we obtain is of dimension $n^2 + 4n + 3$. This $\{e\}$-structure enjoys the important property that its structure function can be expressed solely in terms of two fundamental families of tensorial invariants $\tilde{P}^i_j, \tilde{S}^i_{jkl}$. When we further consider equations admitting symmetry groups of maximal dimension, an analysis of the integrability conditions yields the rather remarkable fact that there is a **unique** equivalence class of second order systems of ordinary differential equations admitting a symmetry group of maximal dimension. The vanishing of the tensorial invariants $\tilde{P}^i_j, \tilde{S}^i_{jkl}$ characterizes this equivalence class and a representative for this class is

$$\frac{d^2x^i}{dt^2} = 0 \ . \tag{2.3}$$

We may interpret this result another way by saying that given a system of second order equation admitting a symmetry group of dimension $n^2 + 4n + 3$ there exist a set of coordinates such that the equation is of the form (2.3). The upper bound $n^2 + 4n + 3$ was also found in [19], while the uniqueness result for **scalar** equations has been known for a long time (see the discussion in [22]). The structure equations we have in the case of maximal symmetry are those of $\mathbf{sl}(n+2, \mathbb{R})$ (this is true in the scalar case as well [22]).

The fundamental tensorial invariants $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ appear in numerous applications for example the inverse problem of the Calculus of Variations [1] and [32]. However their role and that of any associated invariants one can construct from $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ can still be further explored.

Considerably less is known about systems of differential equations of order greater than 2. In particular it is unknown which r^{th} order systems $r \geq 3$ admit a symmetry group of maximal dimension. In the case of scalar third order equations, Chern has investigated local equivalence under contact transformation [8]. It is interesting to note that here Chern has shown that third order equations admit local contact invariants, while a classic theorem of Lie states that all second order scalar ordinary differential equations are contact equivalent. A proof of this last fact using the equivalence method is given in [16].

The point symmetry properties for systems of ordinary differential equations have been studied in [20] where it is shown that for $r \geq 3$ an r^{th} order system of n equations admits at most an $n^2 + (r+1)n + 3$ dimensional symmetry group. While in [19] it is demonstrated that the trivial equation $x^{(r)} = 0$ admits a symmetry group of dimension $n^2 + r\,n + 3$. Dr. A. González-López the author of these two works, pointed out this discrepancy to me, and felt that perhaps the equivalence method could help to determine whether there are equations whose symmetry groups have higher dimension than $n^2 + r\,n + 3$. In Section 2.4 what we find by applying the equivalence method to third order systems is an associated $\{e\}$-structure of dimension n^2+3n+3. Thus (by Theorem 1.11) the dimension of the symmetry group is less than or equal to n^2+3n+3. We also find, as in the case of second order equations, analysis of the integrability conditions demonstrates that there is a unique equivalence class of third order systems of ordinary differential equations admitting a symmetry group of maximal dimension. Again the trivial equation,

$$\frac{d^3x^i}{dt^3} = 0 \tag{2.4}$$

is a representative for this class.

2.2 Systems of Second Order Ordinary Differential Equations

In order to apply the equivalence method to systems of second ordinary differential equations we must first translate the system of second ordinary differential equations

$$\frac{d^2x^i}{dt^2} = f^i\left(t, x^j, \frac{dx^j}{dt}\right) \quad 1 \leq i \leq n \tag{2.5}$$

into a Pfaffian system. To do this, first let $U \subset J^1(\mathbb{R}, \mathbb{R}^n)$ be an open subset and let (t, x^i, x_1^i) be standard coordinates on $J^1(\mathbb{R}, \mathbb{R}^n)$. We then associate to the equations (2.5) the Pfaffian system generated by

$$\theta^i = dx^i - x_1^i dt \quad , \qquad \pi^i = dx_1^i - f^i dt \tag{2.6}$$

whose importance is,

Lemma 2.1: *The solutions $x^i = x^i(t)$ to equations (2.5) are in one to one correspondence with the one dimensional integral manifolds $\gamma : \mathbb{R} \to U$ of the Pfaffian system (2.6) which satisfy $\gamma^* dt \neq 0$.*

If we now consider another system of second order ordinary differential equations

$$\frac{d^2\bar{x}^i}{d\bar{t}^2} = \bar{f}^i\left(\bar{t}, \bar{x}^j, \frac{d\bar{x}^j}{d\bar{t}}\right) \tag{2.7}$$

and the associated Pfaffian system

$$\bar{\theta}^i = d\bar{x}^i - \bar{x}_1^i d\bar{t} \qquad \bar{\pi}^i = d\bar{x}_1^i - \bar{f}^i d\bar{t} \tag{2.8}$$

on $\bar{U} \subset J^1(\bar{\mathbb{R}}, \bar{\mathbb{R}}^n)$ with coordinates $(\bar{t}, \bar{x}^i, \bar{x}_1^i)$ we may then define equivalence as,

Definition 2.1: *The two systems of differential equations (2.5), (2.7) are equivalent if and only if there exists a point transformation $(\bar{t}, \bar{x}^i) = \Psi(t, x^j)$ whose first prolongation Ψ_1 satisfies*

$$\Psi_1^* < \bar{\theta}^i, \bar{\pi}^i > \; = \; < \theta^i, \pi^i > . \tag{2.9}$$

(See Appendix A for more details on diffeomorphisms of $J^1(\mathbb{R}, \mathbb{R}^n)$). In other words the systems (2.5) and (2.7) are equivalent if and only if there exists a smooth map $\Psi : U \to \overline{U}$ taking integral manifolds of (2.6) with independence condition $\gamma^* dt \neq 0$ to integral manifolds of (2.8) with independence condition $(\Psi \circ \gamma)^* d\bar{t} \neq 0$.

Before proceeding we introduce the following notation for the partial derivatives of a smooth function $f \in C^\infty(U)$,

$$f_t = \frac{\partial f}{\partial t}, \quad f_{,j} = \frac{\partial f}{\partial x^j}, \quad f_{|j} = \frac{\partial f}{\partial x_1^j} .$$

The one-forms given in (2.6) and (2.8) generate the same Pfaffian system as do the one-forms

$$\widehat{\theta}^i = dx^i - x_1^i dt \qquad \widehat{\pi}^i = dx_1^i - f^i dt - \frac{1}{2} f^i{}_{|j} \theta^j \tag{2.10}$$

$$\widehat{\bar{\theta}}^i = d\bar{x}^i - \bar{x}_1^i d\bar{t} \qquad \widehat{\bar{\pi}}^i = d\bar{x}_1^i - \bar{f}^i dt - \frac{1}{2} \bar{f}^i{}_{|j} \bar{\theta}^j \tag{2.11}$$

Thus, we may use (2.10) and (2.11) in Definition 2.1. This new set of Pfaffian forms (2.10) can be obtained from (2.6) by a reduction argument and was used by Chern [7] in his solution to the equivalence problem under the transformation in (2.1). The usefulness of this modified coframe will be apparent in Lemma 2.2.

Extending $(\widehat{\theta}^i, \widehat{\pi}^i)$ in (2.10) to the coframe $(\widehat{\omega} = dt, \widehat{\theta}^i, \widehat{\eta}^i)$ on U we can explicitly compute the covariant derivatives $(dg)_{\widehat{\omega}}$, $(dg)_{\widehat{\theta}^i}$, and $(dg)_{\widehat{\pi}^i}$ of a smooth function $g \in C^\infty(U)$ by

$$dg = (dg)_{\widehat{\omega}} \widehat{\omega} + (dg)_{\widehat{\theta}^i} \widehat{\theta}^i + (dg)_{\widehat{\pi}^i} \widehat{\pi}^i \tag{2.12}$$

where

$$(dg)_{\widehat{\omega}} = \frac{dg}{dt} \quad , \qquad (dg)_{\widehat{\theta}^j} = g_{,j} + \frac{1}{2} g_{|k} f^k_{|j} \quad , \qquad (dg)_{\widehat{\pi}^j} = g_{|j} \, . \tag{2.13}$$

These equations will be used in the parametric calculations.

Making the analogous extension of $(\widehat{\bar{\theta}}^i, \widehat{\bar{\pi}}^i)$ in (2.11) to the local coframe $(\widehat{\bar{\omega}} = d\bar{t}, \widehat{\bar{\theta}}^i, \widehat{\bar{\eta}}^i)$ on $\bar{U}$ we obtain

Lemma 2.2: *The two differential systems* (2.10) *and* (2.11) *are equivalent if and only if there exists a point transformation* $(\bar{t}, \bar{x}^i) = \Psi(t, x^i)$ *with* $\Psi_1 : U \to \bar{U}$ *satisfying*

$$\Psi_1^* \begin{pmatrix} \hat{\bar{\omega}} \\ \hat{\bar{\theta}}^i \\ \hat{\bar{\pi}}^i \end{pmatrix} = \mathcal{S} \begin{pmatrix} \hat{\omega} \\ \hat{\theta}^i \\ \hat{\pi}^i \end{pmatrix}$$

where $\mathcal{S} : U \to \mathrm{H}$ *is a smooth function on* U *taking values in the Lie subgroup* H *of* $GL(2n+1, \mathbb{R})$ *defined by*

$$\mathrm{H} = \left\{ \begin{pmatrix} a & E_j & 0 \\ 0 & A^i_j & 0 \\ 0 & cA^i_j & a^{-1}A^i_j \end{pmatrix}, \; a \in \mathbb{R}^*, \; A^i_j \in GL(n, \mathbb{R}), \; E_j \in \mathbb{R}^n, \; c \in \mathbb{R} \right\} \quad (2.14)$$

Proof: Sufficiency is obvious, and by Appendix A we need to only determine $\Psi_1^*\hat{\bar{\pi}}^i$. We find by Lemma A.2

$$\Psi_1^*(d\bar{x}^i_1 - \bar{f}^i dt - \frac{1}{2}\bar{f}^i{}_{|j}\bar{\theta}^j) = a^{-1}A^i_j(dx^j_1 - f^i dt) + C^i_j\hat{\theta}^j - \frac{1}{2}\Psi_1^*\left(\frac{\partial \bar{f}^i}{\partial \bar{x}^j_1}\right) A^j_k\hat{\theta}^k$$

where

$$A^i_j = \frac{\partial \psi^i}{\partial x^j} - \psi^i_1\frac{\partial \phi}{\partial x^j} \qquad C^i_j = \frac{\partial \psi^i_1}{\partial x^j} - \psi^i_2\frac{\partial \phi}{\partial x^j} \qquad a = \frac{d\phi}{dt}$$

Now Lemma A.1 in Appendix A tells us that if the systems are equivalent then

$$\bar{x}^i_2 = \psi^i_2(t, x) \quad \text{that is} \quad \bar{f}^i \circ \Psi_1 = \psi^i_2 = \frac{1}{a}\frac{d\psi^i_1}{dt}$$

from which we determine that

$$\Psi_1^*\left(\frac{\partial \bar{f}^i}{\partial \bar{x}^j_1}\right) = \frac{\partial \bar{f}^i \circ \Psi_1}{\partial x^k_1}\frac{\partial x^k_1}{\partial \bar{x}^j_1} = \frac{\partial \psi^i_2}{\partial x^k_1}a(A^{-1})^k_j = \left(\frac{\partial}{\partial x^k_1}\frac{d\psi^i_1}{dt} - \psi^i_2\frac{\partial \phi}{\partial x^k}\right)(A^{-1})^k_j$$

Now in this equation we switch the order of the differentiations (as in Lemma A.2) to get

$$\begin{aligned}\Psi_1^*\left(\frac{\partial \bar{f}^i}{\partial \bar{x}_1^j}\right) &= \left[\frac{d}{dt}\frac{\partial}{\partial x_1^k}\left(\frac{1}{a}\frac{d\psi^i}{dt}\right) + \frac{\partial \psi_1^i}{\partial x^k} + \frac{\partial \psi_1^i}{\partial x_1^l}\frac{\partial f^l}{\partial x_1^k} - \psi_2^i\frac{\partial \phi}{\partial x^k}\right](A^{-1})_j^k \\ &= \left[\frac{d}{dt}\frac{\partial}{\partial x_1^k}\left(\frac{1}{a}\frac{d\psi^i}{dt}\right) + \frac{\partial \psi_1^i}{\partial x_1^l}\frac{\partial f^l}{\partial x_1^k} + C_k^i\right](A^{-1})_j^k \qquad (2.15)\end{aligned}$$

which further simplifies to

$$\begin{aligned}\Psi_1^*\left(\frac{\partial \bar{f}^i}{\partial \bar{x}_1^j}\right) &= \left[\frac{d}{dt}\left(\frac{1}{a}A_k^i\right) + a^{-1}A_l^i\frac{\partial f^l}{\partial x_1^k} + C_k^i\right](A^{-1})_j^k \\ &= \left[-\frac{1}{a^2}\frac{da}{dt}A_k^i + \frac{1}{a}\frac{d}{dt}(A_k^i) + a^{-1}A_l^i\frac{\partial f^l}{\partial x_1^k} + C_k^i\right](A^{-1})_j^k.\end{aligned}$$

Now defining

$$c = \frac{1}{2a^2}\frac{da}{dt}$$

and noting that

$$\frac{d}{dt}(A_j^i) = aC_j^i$$

we finally have

$$\Psi_1^*\left(\frac{\partial \bar{f}^i}{\partial \bar{x}_1^j}\right) = 2\left[C_k^i(A^{-1})_j^k - c\delta_j^i\right] + a^{-1}A_l^i\frac{\partial f^l}{\partial x_1^k}(A^{-1})_j^k \quad .$$

This completes the proof of the lemma. ■

The idea of using Chern's adapted coframe in the set up of our equivalence problem so as to reduce the structure group from the outset is an application of the "inductive approach" to equivalence problems presented in [26]. For the Lie group H in this lemma we have the Maurer-Cartan form

$$\begin{pmatrix} \alpha & \kappa_j & 0 \\ 0 & \Omega_j^i & 0 \\ 0 & \sigma\delta_j^i & \Omega_j^i - \alpha\delta_j^i \end{pmatrix} = dS(S^{-1}) \qquad (2.16)$$

which parametrically is given by

$$\begin{pmatrix} da & dE_k & 0 \\ 0 & dA^i_k & 0 \\ 0 & dc\,A^i_k + c\,dA^i_k & a^{-1}\,dA^i_k - a^{-2}\,da\,A^i_k \end{pmatrix} \begin{pmatrix} a^{-1} & -a^{-1}E_k(A^{-1})^k_j & 0 \\ 0 & (A^{-1})^k_j & 0 \\ 0 & -ac(A^{-1})^k_j & a(A^{-1})^k_j \end{pmatrix}$$

$$= \begin{pmatrix} \frac{da}{a} & dE_k(A^{-1})^k_j - \frac{da}{a}EA^{-1}_j & 0 \\ 0 & dA^i_k(A^{-1})^k_j & 0 \\ 0 & (dc + c\frac{da}{a})\delta^i_j & dA^i_k(A^{-1})^k_j - \frac{da}{a}\delta^i_j \end{pmatrix}$$

It will be convenient in the next section to use the following convention

$$EA^{-1}_j \equiv E_k(A^{-1})^k_j$$

2.3 The Associated $\{e\}$-Structure

In this section we will apply the equivalence method of Cartan with the coframe

$$\widehat{\omega} = dt \qquad \widehat{\theta}^i = dx^i - x^i_1 dt \qquad \widehat{\pi}^i = dx^i_1 - f^i dt - \frac{1}{2} f^i{}_{|j}\theta^j \quad . \tag{2.17}$$

and with the structure group given in Lemma 2.2. We first define the lifted coframe

$$\begin{pmatrix} \omega \\ \theta^i \\ \pi^i \end{pmatrix} = \begin{pmatrix} a & E_j & 0 \\ 0 & A^i_j & 0 \\ 0 & cA^i_j & a^{-1}A^i_j \end{pmatrix} \begin{pmatrix} \widehat{\omega} \\ \widehat{\theta}^i \\ \widehat{\pi}^i \end{pmatrix} \tag{2.18}$$

from which we may state,

Theorem 2.1: *Solutions $\Psi_1 : U \to \overline{U}$ to the equivalence problem for systems of n (≥ 2) second order ordinary differential equations are in one-to-one correspondence with the solutions of an equivalence problem for an n^2+4n+3 dimensional $\{e\}$-structure which is obtained by applying the equivalence method to the initial coframe $(\widehat{\omega}, \widehat{\theta}^i, \widehat{\pi}^i)$ with the structure group given in Lemma 2.2.*

Proof: We proceed initially with the parametric calculations in order to be able to have the explicit form of some of the tensorial invariants later. Differentiating the $(\widehat{\omega}, \widehat{\theta}^i, \widehat{\pi}^i)$ forms we find by using equations (2.12) and (2.13) that

$$d\widehat{\omega} = 0 \qquad d\widehat{\theta}^i = \widehat{\omega}\wedge\widehat{\pi}^i + \frac{1}{2}f^i_{|j}\widehat{\omega}\wedge\widehat{\theta}^j$$

and

$$\begin{aligned} d\widehat{\pi}^i &= -df^i\wedge dt - \frac{1}{2}d(f^i_{|j})\wedge\widehat{\theta}^j - \frac{1}{2}f^i_{|j}(\widehat{\omega}\wedge\widehat{\pi}^j + \frac{1}{2}f^j_{|k}\widehat{\omega}\wedge\widehat{\theta}^k) \\ &= \rho^i_j\widehat{\theta}^j\wedge\widehat{\omega} + \frac{1}{2}f^i_{|j}\widehat{\omega}\wedge\widehat{\pi}^j + \frac{1}{2}f^i_{|jk}\widehat{\theta}^j\wedge\widehat{\pi}^k + \tau^i_{jk}\widehat{\theta}^j\wedge\widehat{\theta}^k \end{aligned}$$

where

$$\begin{aligned} \rho^i_j &= -f^i{}_{,j} + \frac{1}{2}\frac{d}{dt}f^i{}_{|j} - \frac{1}{4}f^i{}_{|k}f^k{}_{|j} \\ \tau^i_{jk} &= \frac{1}{2}\left(f^i_{[,k|j]} + \frac{1}{2}f^i_{|l[j}f^l_{|k]}\right) \end{aligned} \tag{2.19}$$

The expressions for $d\widehat{\omega}, d\widehat{\theta}^i$ and $d\widehat{\pi}^i$ back in terms of the lifted frame are then

$$\begin{aligned} d\widehat{\omega} &= 0 \\ d\widehat{\theta}^i &= -\left[(A^{-1})^i_j\pi^j - c(A^{-1})^i_j\theta^j + \tfrac{a^{-1}}{2}(f^i_{|j})(A^{-1})^j_k\theta^k\right]\wedge(\omega - EA^{-1}_j\theta^j) \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} d\widehat{\pi}^i = &\left[\frac{1}{a}(\rho^i_j)(A^{-1})^j_k\theta^k - \frac{1}{2}(f^i_{|j})(A^{-1})^j_k\pi^k + \frac{c}{2}(f^i_{|j})(A^{-1})^j_k\theta^k\right]\wedge(\omega - EA^{-1}_j\theta^j) \\ &+(\tau^i_{jk})(A^{-1})^j_l(A^{-1})^k_m\theta^l\wedge\theta^m - \frac{a}{2}(f^i_{|jk})(A^{-1})^j_l(A^{-1})^k_m\pi^l\wedge\theta^m \end{aligned} \tag{2.21}$$

Thus differentiating (2.18) and using equation (2.16) the structure equations are

$$\begin{pmatrix} d\omega \\ d\theta^i \\ d\pi^i \end{pmatrix} = \begin{pmatrix} \alpha & \kappa_j & 0 \\ 0 & \Omega^i_j & 0 \\ 0 & \sigma\delta^i_j & \Omega^i_j - \alpha\delta^i_j \end{pmatrix} \wedge \begin{pmatrix} \omega \\ \theta^j \\ \pi^j \end{pmatrix} + \begin{pmatrix} a & E_j & 0 \\ 0 & A^i_j & 0 \\ 0 & cA^i_j & a^{-1}A^i_j \end{pmatrix} \begin{pmatrix} d\widehat{\omega} \\ d\widehat{\theta}^j \\ d\widehat{\pi}^j \end{pmatrix}.$$

Now substituting from (2.20) and (2.21) we have the equations

$$
\begin{aligned}
d\omega &= \alpha\wedge\omega + \kappa_j\wedge\theta^j - EA_j^{-1}\left[\pi^j - c\theta^j + \frac{a^{-1}}{2}A_k^j(f_{|l}^k)(A^{-1})_m^l\theta^m\right]\wedge(\omega - EA_j^{-1}\theta^j) \\
d\theta^i &= \Omega_j^i\wedge\theta^j - \left[\pi^i - c\theta^i + \frac{a^{-1}}{2}A_j^i(f_{|k}^j)(A^{-1})_l^k\theta^l\right]\wedge(\omega - EA_j^{-1}\theta^j)
\end{aligned}
$$

and

$$
\begin{aligned}
d\pi^i &= -c\left[\pi^i - c\theta^i + \frac{a^{-1}}{2}A_k^i(f_{|j}^k)(A^{-1})_k^j\theta^k\right]\wedge(\omega - EA_j^{-1}\theta^j) \\
&+a^{-1}A_l^i\left[\frac{1}{a}(\rho_j^l)(A^{-1})_k^j\theta^k - \frac{1}{2}(f_{|j}^l)(A^{-1})_k^j\pi^k + \frac{c}{2}(f_{|j}^l)(A^{-1})_k^j\theta^k\right]\wedge(\omega - EA_j^{-1}\theta^j) \\
&+a^{-1}A_r^i(\tau_{jk}^r)(A^{-1})_l^j(A^{-1})_m^k\theta^l\wedge\theta^m - \frac{1}{2}A_r^i(f_{|jk}^r)(A^{-1})_l^j(A^{-1})_m^k\pi^l\wedge\theta^m
\end{aligned}
$$

and $d\pi^i$ simplifies to

$$
\begin{aligned}
d\pi^i &= \sigma\wedge\theta^i + (\Omega_j^i - \alpha\delta_j^i)\wedge\pi^j + a^{-1}A_r^i(\tau_{jk}^r)(A^{-1})_l^j(A^{-1})_m^k\theta^l\wedge\theta^m \qquad (2.22)\\
&+\left[a^{-2}A_j^i(\rho_j^i)(A^{-1})_k^j\theta^k + c^2\theta^i - c\pi^i - \frac{1}{2a}A_j^i(f_{|k}^j)(A^{-1})_l^k\pi^l\right]\wedge(\omega - EA_j^{-1}\theta^j) \\
&-\frac{1}{2}A_l^i(f_{|jk}^l)(A^{-1})_m^j(A^{-1})_r^k\pi^m\wedge\theta^r
\end{aligned}
$$

In these equations we may now absorb torsion by

$$
\begin{aligned}
\alpha =& \quad \widehat{\alpha} + EA_j^{-1}\pi^j + 2c(\omega - EA_j^{-1}\theta^j) \\
\kappa_j =& \quad \widehat{\kappa}_j - EA_j^{-1}\pi^jEA_j^{-1} - \left[cEA_j^{-1} + \tfrac{a^{-1}}{2}E_l(f_{|m}^l)(A^{-1})_j^m\right](\omega - EA_j^{-1}\theta^j) \\
\Omega_j^i =& \quad \widehat{\Omega}_j^i - \pi^iEA_j^{-1} + \left[c\delta_j^i - \tfrac{a^{-1}}{2}A_l^i(f_{|m}^l)(A^{-1})_j^m\right](\omega - EA_j^{-1}\theta^j) \qquad (2.23)\\
& -\tfrac{1}{2}A_k^i(f_{|lm}^k)(A^{-1})_n^l\theta^n(A^{-1})_j^m \\
\sigma =& \quad \widehat{\sigma} + \left[c^2 + \tfrac{a^{-2}}{n}\rho_i^i\right](\omega - EA_j^{-1}\theta^j) + \tfrac{1}{n-1}\left[EA_k^{-1}\tilde{P}_j^k + \tfrac{2a^{-1}}{1-n}\tau_{ik}^i(A^{-1})_j^k\right]\theta^j
\end{aligned}
$$

which leads to the structure equations (after dropping hats)

$$
\begin{aligned}
d\omega &= \alpha\wedge\omega + \kappa_j\wedge\theta^j \\
d\theta^i &= \Omega_j^i\wedge\theta^j - \pi^i\wedge\omega \qquad (2.24)\\
d\pi^i &= \sigma\wedge\theta^i + (\Omega_j^i - \alpha\delta_j^i)\wedge\pi^j + \tilde{P}_j^i\theta^j\wedge\omega + \tilde{Q}_{jk}^i\theta^j\wedge\theta^k
\end{aligned}
$$

where

$$\tilde{P}^i_j = \frac{1}{a^2} A^i_k \left(\rho^k_l - \frac{1}{n} \rho^m_m \delta^k_l \right) (A^{-1})^l_j \tag{2.25}$$

and

$$\tilde{Q}^i_{jk} = E A^{-1}_{[j} \tilde{P}^i_{k]} + \frac{1}{n-1} E A^{-1}_l \tilde{P}^l_{[j} \delta^i_{k]} + \frac{1}{a} A^i_l \left(\tau^l_{mr} + \frac{2}{n-1} \tau^s_{s[m} \delta^l_{r]} \right) (A^{-1})^m_j (A^{-1})^r_k$$

We point out that the absorption by σ in equation (2.23) has been chosen so that $\tilde{P}^i_j$ and $\tilde{Q}^i_{jk}$ are trace free (we will use this frequently later on). If we now consider the parametric forms for $\tilde{P}^i_j$ and $\tilde{Q}^i_{jk}$ above we see that any further reductions of the structure group will depend on the algebraic structure of $\tilde{P}^i_j$ and $\tilde{Q}^i_{jk}$. For example we see from (2.25) that $\tilde{P}^i_j$ is acted on by conjugation. One could proceed by putting $\tilde{P}^i_j$ into a normal form and possibly further reduce H. This however will not be done, and instead we resort to Theorem 1.6 and prolong. To this effect we compute $\mathrm{H}^{(1)}$ by finding the kernel of the absorption (finding the solution to the homogeneous system of equations (1.31)) as described in Section 1.6. The first two equations in (2.24) tell us that

$$\alpha = \widehat{\alpha} + L\omega + K_i \theta^i \quad \Omega^i_j = \widehat{\Omega}^i_j + M^i_{jk} \theta^k \quad \kappa_j = \widehat{\kappa}_j + K_j \omega + D_{jk} \theta^k \tag{2.26}$$

where $M^i_{[jk]} = D_{[jk]} = 0$, is possibly in the solution space to the homogeneous system. Inserting this along with

$$\sigma = \widehat{\sigma} + q\omega + r_j \theta^j + s_j \pi^j \tag{2.27}$$

into the last equation in (2.24) gives

$$(\widehat{\sigma} + q\omega + r_j \theta^j + s_j \pi^j) \wedge \theta^i + (M^i_{jk} \theta^k - L\omega - K_k \theta^k \delta^i_j) \wedge \pi^j = 0 \tag{2.28}$$

and immediately we have from this $q = r_j = L = 0$. While putting the coefficient of $\theta^k \wedge \pi^j$ to zero gives

$$M^i_{jk} - s_j \delta^i_k - K_j \delta^i_j = 0 \ .$$

Since we assume $n > 1$, skew-symmetrization and symmetrization of this equation gives,

$$s_j = K_j \quad \text{and} \quad M^i_{jk} = 2K_{(k}\delta^i_{j)} \ . \tag{2.29}$$

Thus we have a parameterization of $\mathrm{H}^{(1)}$ by K_j and D_{jk}, and we may lift the coframe $(\omega, \theta^i, \pi^i, \alpha, \kappa_i, \Omega^i_j, \sigma)$ to $U \times \mathrm{H} \times \mathrm{H}^{(1)}$ by

$$\begin{aligned} \overline{\alpha} &= \alpha + K_j\theta^j & \overline{\Omega}^i_j &= \Omega^i_j + 2K_{(k}\delta^i_{j)}\theta^k \\ \overline{\sigma} &= \sigma + K_j\pi^j & \overline{\kappa}_j &= \kappa_j + D_{jk}\theta^k + K_j\omega \end{aligned}$$

We will now drop the overline and we may then write the lifted structure equations in the general form,

$$\begin{aligned} d\alpha &= \beta_j\wedge\theta^j + \mathbf{t}^0 & d\kappa_j &= \Upsilon_{jk}\wedge\theta^k + \beta_j\wedge\omega + \mathbf{T}^2_j \\ d\sigma &= \beta_j\wedge\pi^j + \mathbf{t}^1 & d\Omega^i_j &= 2\beta_{(k}\delta^i_{j)}\wedge\theta^k + \mathbf{T}^i_j \end{aligned} \tag{2.30}$$

where $\mathbf{t}^0$ and $\mathbf{t}^1$ are 2-forms, $\mathbf{T}^2_j$ is a $\mathbb{R}^n$ valued 2-form, and $\mathbf{T}^i_j$ is a $M_n(\mathbb{R})$ valued 2-form all of which are contained in the exterior algebra generated by $(\omega, \theta^i, \pi^i, \mathbf{h}^*)$ and where

$$\beta_j = dK_j \quad \text{and} \quad \Upsilon_{jk} = dD_{jk} \ . \tag{2.31}$$

Now absorb torsion in equation (2.30) by

$$\beta_j = \widehat{\beta}_j - \left(\mathbf{t}^0\right)_{\theta^i} - \left(\mathbf{t}^1\right)_{\theta(^k\pi^j)}\theta^k \tag{2.32}$$

$$\Upsilon_{jk} = \widehat{\Upsilon}_{jk} - \left(\mathbf{T}^2_{(j}\right)_{\theta^{k)}} \tag{2.33}$$

and after dropping the hats, the structure equations (2.30) retain the form with the additional conditions

$$\left(\mathbf{t}^0\right)_{\theta^i} = 0\,, \quad \left(\mathbf{t}^1\right)_{\theta(^k\pi^j)}\theta^k = 0\,, \quad \left(\mathbf{T}^2_{(j}\right)_{\theta^{k)}} = 0 \tag{2.34}$$

Our goal from here on will be to determine the form of the left over torsion (structure function) by applying a sequence of integrability conditions. The condition we use first is $d^2\theta^i = 0$ from equation (2.24)

$$d^2\theta^i = d\Omega^i_j\wedge\theta^j - \Omega^i_j\wedge d\theta^j + d\omega\wedge\pi^i - \omega\wedge d\pi^i$$

and substitute from (2.30) to get

$$\begin{aligned} d^2\theta^i &= (\beta_k\wedge\theta^k\,\delta^i_j + \beta_j\wedge\theta^i + \mathbf{T}^i_j)\wedge\theta^j - \Omega^i_j\wedge(\Omega^j_k\wedge\theta^k - \pi^j\wedge\omega) \\ &\quad +(\alpha\wedge\omega + \kappa_j\wedge\theta^j)\wedge\pi^i - \omega\wedge\left(\sigma\wedge\theta^i + (\Omega^i_j - \alpha\delta^i_j)\wedge\theta^j + \tilde{Q}^i_{jk}\theta^j\wedge\theta^k\right) \\ 0 &= (\mathbf{T}^i_j - \Omega^i_k\wedge\Omega^k_j + \pi^i\wedge\kappa_j + \sigma\wedge\omega\delta^i_j - \tilde{Q}^i_{jk}\theta^k\wedge\omega)\wedge\theta^j \quad . \end{aligned}$$

From which we obtain

$$\mathbf{T}^i_j = \Omega^i_k\wedge\Omega^k_j - \pi^i\wedge\kappa_j - \sigma\wedge\omega\delta^i_j + \tilde{Q}^i_{jk}\theta^k\wedge\omega + \xi^i_{jk}\wedge\theta^k \tag{2.35}$$

where ξ^i_{jk} is a collection of one-forms satisfying

$$\xi^i_{jk}\wedge\theta^j\wedge\theta^k = 0$$

The next integrability condition we use is $d^2\omega = 0$ from equation (2.24) and using equations (2.24) and (2.30) we find

$$\begin{aligned} d^2\omega &= d\alpha\wedge\omega - \alpha\wedge d\omega + d\kappa\wedge\theta - \kappa\wedge d\theta \\ 0 &= (\mathbf{t}^0 + \kappa_j\wedge\pi^j)\wedge\omega + (\mathbf{T}^2_j - \alpha\wedge\kappa_j - \kappa_k\wedge\Omega^k_j)\wedge\theta^j \end{aligned} \tag{2.36}$$

from which we deduce

$$\mathbf{t}^0 = -\kappa_j\wedge\pi^j + \lambda\wedge\omega + \xi^0_i\wedge\theta^i \qquad \mathbf{T}^2_i = \alpha\wedge\kappa_i + \kappa_j\wedge\Omega^j_i + \xi^2_{ij}\wedge\theta^j + \xi^0_i\wedge\omega \tag{2.37}$$

where ξ^2_{ij}, ξ^0_i and λ are one-forms which by equation (2.36) are subject to the conditions,

$$\begin{aligned} &a)\quad \xi^2_{ij}\wedge\theta^j\wedge\theta^i = 0 \\ &b)\quad (\lambda)_\omega = (\lambda)_{\theta^i} = 0 \qquad \text{that is } \lambda \text{ has no } \theta^i \text{ or } \omega \text{ terms.} \end{aligned} \tag{2.38}$$

The condition b) here is really a choice we make in order that the decomposition for $\mathbf{t}^0$ in (2.37) be unique. There is no loss in generality by this. If we now take into account the absorption in equation (2.32) which gives rise to the conditions in (2.34) we find that

$$\left(\mathbf{t}^0\right)_{\theta^i} = \xi^0_i = 0 \tag{2.39}$$

and so the torsion terms in equation (2.37) simplify to

$$\mathbf{t}^0 = -\kappa_j \wedge \pi^j + \lambda \wedge \omega \qquad \mathbf{T}^2_i = \alpha \wedge \kappa_i + \kappa_j \wedge \Omega^j_i + \xi^2_{ij} \wedge \theta^j \ . \tag{2.40}$$

The last integrability condition we have from equation (2.24) is $d^2\pi^i = 0$ and this gives,

$$\begin{aligned} d^2\pi^i \;&=\; d\sigma \wedge \theta^i - \sigma \wedge d\theta^i + d(\Omega^i_j - \alpha\delta^i_j) \wedge \pi^j - (\Omega^i_j - \alpha\delta^i_j) \wedge d\pi^j + d\tilde{P}^i_j \wedge \theta^j \wedge \omega \\ &\quad + \tilde{P}^i_j (d\theta^j \wedge \omega - \theta^j \wedge d\omega) + d\tilde{Q}^i_{jk} \theta^j \wedge \theta^k - 2\tilde{Q}^i_{jk} \theta^j \wedge d\theta^k \end{aligned}$$

where by using equations (2.30) (2.35) and (2.40) this becomes

$$\begin{aligned} d^2\pi^i \;&=\; (\beta_j \wedge \pi^j + \mathbf{t}^1) \wedge \theta^i - \sigma \wedge (\Omega^i_j \wedge \theta^j - \pi^j \wedge \omega) - (\beta_k \wedge \theta^k - b_k \wedge \pi^k + \lambda \wedge \omega) \wedge \pi^i \\ &\quad + (\beta_k \wedge \theta^k \delta^i_j + \beta_j \wedge \theta^i + \Omega^i_k \wedge \Omega^k_j - \pi^i \wedge \kappa_j - \sigma \wedge \omega \delta^i_j + \tilde{Q}^i_{jk} \theta^k \wedge \omega + \xi^i_{jk} \wedge \theta^k) \wedge \pi^j \\ &\quad - (\Omega^i_j - \alpha\delta^i_j) \wedge \left(\sigma \wedge \theta^j + (\Omega^j_k - \alpha\delta^j_k) \wedge \pi^k + \tilde{P}^j_k \theta^k \wedge \omega + \tilde{Q}^j_{kl} \theta^k \wedge \theta^l\right) \\ &\quad + d\tilde{P}^i_j \wedge \theta^j \wedge \omega + d\tilde{Q}^i_{jk} \wedge \theta^j \wedge \theta^k + \tilde{P}^i_j \left(\Omega^j_k \wedge \theta^k \wedge \omega - \theta^j \wedge (\alpha \wedge \omega + \kappa_k \wedge \theta^k)\right) \\ &\quad - 2\tilde{Q}^i_{jk} \theta^j \wedge (\Omega^k_l \wedge \theta^l - \pi^k \wedge \omega) \ . \end{aligned}$$

Further simplification yields

$$\begin{aligned} 0 \;&=\; (d\tilde{P}^i_j - \Omega^i_k \tilde{P}^k_j + \tilde{P}^i_k \Omega^k_j + 2\alpha \tilde{P}^i_j + 3\tilde{Q}^i_{kj} \pi^k) \wedge \theta^j \wedge \omega \\ &\quad + (d\tilde{Q}^i_{jk} - \Omega^i_l \tilde{Q}^l_{jk} - 2\tilde{Q}^i_{l[j} \Omega^l_{k]} + \alpha \tilde{Q}^i_{jk} - \tilde{P}^i_{[j} \kappa_{k]}) \wedge \theta^j \wedge \theta^k \\ &\quad + \xi^i_{jk} \wedge \theta^k \wedge \pi^j - (\lambda + 2\sigma) \wedge \omega \wedge \pi^i + (\mathbf{t}^1 + \alpha \wedge \sigma) \wedge \theta^i \end{aligned} \tag{2.41}$$

By putting the coefficient of $\omega\wedge\pi^i$ in this equation to zero and recalling from equation (2.38) that λ has **no** θ^i or ω terms, we have

$$\lambda = -2\sigma \tag{2.42}$$

and so, by equations (2.40) and (2.30) we have

$$d\alpha = \beta_j\wedge\theta^j - \kappa_j\wedge\pi^j - 2\sigma\wedge\omega \ . \tag{2.43}$$

We would like to continue using equation (2.41) but before this is possible we must know some information about the form of $\mathbf{t^1}$. This can be done by taking $d^2\alpha$ mod(θ^i) and here we find,

$$\begin{aligned} d^2\alpha &= d\beta_j\wedge\theta^j - \beta_j\wedge d\theta^j - db_j\wedge\pi^j + \kappa_j\wedge d\pi^j - 2d\sigma\wedge\omega + 2\sigma\wedge\omega \\ &\equiv 2(\sigma\wedge\alpha - \mathbf{t}^1)\wedge\omega \qquad\qquad \mathrm{mod}(\theta^i) \ . \end{aligned}$$

We then deduce,

$$\mathbf{t}^1 = \sigma\wedge\alpha + \xi^1_i\wedge\theta^i + \chi^1\wedge\omega \tag{2.44}$$

where χ^1 and ξ^1_i are one-forms subject to the conditions,

$a)$ $(\chi^1)_\omega = (\chi^1)_{\theta^i} = 0$

$b)$ $(\mathbf{t}^1)_{\theta(k\pi j)}\,\theta^k = -\left(\xi^1_{(i}\right)_{\pi^{j)}} = 0$ from equation (2.34) .

where condition a) gives the unique decomposition in (2.44). We may now place $\mathbf{t}^1$ in equation (2.41) to further investigate ξ^i_{jk}. In particular the $\theta^k\wedge\pi^j$ term from equation (2.41) being zero is

$$\left[\xi^i_{jk} - \left(d\tilde{P}^i_k\right)_{\pi^j}\omega - 3\tilde{Q}^i_{jk}\omega + \left(d\tilde{Q}^i_{lk}\right)_{\pi^j}\theta^l + \delta^i_{[k}\left(\xi^1_{l]}\right)_{\pi^j}\theta^l\right]\wedge\theta^k\wedge\pi^j = 0 \tag{2.45}$$

so that we may write in general

$$\xi^i_{jk} = T^i_{jk}\omega + R^i_{jlk}\theta^l + S^i_{jlk}\pi^l \ . \tag{2.46}$$

where T^i_{jk}, S^i_{jlk} and $R^i_{j[lk]} = R^i_{jlk}$ are smooth functions on $U \times \mathrm{H} \times \mathrm{H}^{(1)}$. Taking this expression for ξ^i_{jk} and imposing condition a) in (2.38) we also have

$$T^i_{[jk]} = S^i_{[j|l|k]} = R^i_{[jkl]} = 0 \quad .$$

As well using the expression for ξ^i_{jk} in equation (2.45) and setting the coefficient of $\pi^l \wedge \theta^k \wedge \pi^j$ to zero we have

$$S^i_{[jl]k} = 0 \qquad \text{or} \qquad S^i_{(jkl)} = S^i_{jkl} \; .$$

In any case using (2.35) and (2.46) we have

$$\begin{aligned} d\Omega^i_j &= 2\beta_{(k}\delta^i_{j)} \wedge \theta^k + \Omega^i_k \wedge \Omega^k_j - \pi^i \wedge \kappa_j - \sigma \wedge \omega \delta^i_j \\ &\quad + (\tilde{Q}^i_{jk} - T^i_{jk})\theta^k \wedge \omega + R^i_{jkl}\theta^k \wedge \theta^l + S^i_{jkl}\pi^k \wedge \theta^l \end{aligned} \tag{2.47}$$

Using the trace of $d\Omega^i_j$ and $d\alpha$ from (2.43) we then have

$$d\Omega^i_i - (n+1)d\alpha = (n+2)(\kappa_i \wedge \pi^i + \sigma \wedge \omega) + (T^i_{ik}\omega + R^i_{ikl}\theta^k + S^i_{ikl}\pi^k) \wedge \theta^l \tag{2.48}$$

and thus setting

$$d^2\Omega^i_i - (n+1)d^2\alpha \equiv 0 \qquad \mathrm{mod}(\,\mathrm{base},\, \mathbf{h}^*\,) \tag{2.49}$$

we determine that,

$$dS^i_{ikl} - (n+2)\Upsilon_{kl} \equiv 0 \qquad \mathrm{mod}(\,\mathrm{base},\, \mathbf{h}^*\,) \tag{2.50}$$

which allows us to translate the trace S^i_{ijk} to zero. We emphasize here that this also implies that the $\mathrm{H}^{(1)}$ action on T^i_{jk} and R^i_{jkl} is trivial. The translation of the trace of S^i_{jkl} to zero gives the reduction of $D_{ij} = 0$ in the prolonged group $\mathrm{H}^{(1)}$ and that

$$\Upsilon_{jk} \equiv 0 \qquad \mathrm{mod(base)} \; .$$

in equation (2.30). Explicitly we will denote

$$\tilde{S}^i_{jkl} = S^i_{jkl} - \frac{3}{n+2} S^m_{m(jk}\delta^i_{l)} \quad . \tag{2.51}$$

and the reduction of $\mathrm{H}^{(1)}$ by $\mathrm{H}^{(1)}_1$. To continue now we may use all the previous equations except the absorption in (2.33) and thus the conditions on $\mathbf{T}^2_i$ in (2.34) must be dropped. It should be pointed out that in the first round of computation with $\mathrm{H}^{(1)}$ we never actually needed to impose the condition in (2.34) on $\mathbf{T}^2_i$ in order to determine the group action in (2.50). Thus we may summarize the structure equations on $U \times \mathrm{H} \times \mathrm{H}^{(1)}$ as

$$\begin{aligned}
d\alpha &= \beta_j \wedge \theta^j - \kappa_j \wedge \pi^j - 2\sigma\wedge\omega \\
d\sigma &= \beta_j \wedge \pi^j + \sigma\wedge\alpha + \xi^1_j \wedge \theta^j + \chi^1 \wedge \omega \\
d\Omega^i_j &= 2\beta_{(k}\delta^k_{j)} \wedge \theta^i + \Omega^i_k \wedge \Omega^k_j - \pi^i \wedge \kappa_j - \sigma\wedge\omega\delta^i_j \\
&\quad + (\tilde{Q}^i_{jk} - T^i_{jk})\theta^k \wedge \omega + R^i_{jkl}\theta^k \wedge \theta^l + \tilde{S}^i_{jkl}\pi^k \wedge \theta^l \\
d\kappa_i &= \beta_i \wedge \omega + \alpha\wedge\kappa_i + \kappa_j \wedge \Omega^j_i + \xi^2_{ij} \wedge \theta^j
\end{aligned} \tag{2.52}$$

with the conditions on the functions $\tilde{Q}, R, \tilde{S}, T$,

$$\tilde{Q}^i_{(jk)} = 0 \ , \quad R^i_{[jkl]} = R^i_{j(kl)} = 0 \ , \quad \tilde{S}^i_{(jkl)} = \tilde{S}^i_{jkl} \ , \quad T^i_{[jk]} = 0 \tag{2.53}$$

(the trace of $\tilde{Q}$ and $\tilde{S}$ are zero) and conditions on the one-forms ξ^1_j, χ^1 and ξ^2_{ij}

$$\left(\xi^1_{(j}\right)_{\theta^{k)}} = \left(\xi^1_{(j}\right)_{\pi^{k)}} = 0 \, , \ \left(\chi^1\right)_\omega = \left(\chi^1\right)_{\theta^i} = 0 \, , \ \left(\xi^2_{i(j}\right)_{\theta^{k)}} = 0 \, , \ \xi^2_{ij} \wedge \theta^i \wedge \theta^j = 0 \, . \tag{2.54}$$

If we now try to prolong the structure equations we see by the equations in (2.52) that the kernel of the absorption by β_i is zero. In other words the β_i forms are invariant. This finally allows us to conclude that we have an $\{e\}$-structure on $U \times \mathrm{H} \times \mathrm{H}^{(1)}_1$ of dimension $n^2 + 4n + 3$, the final invariant coframe being $(\omega, \theta^i, \pi^i, \alpha, \kappa_j, \Omega^i_j, \sigma, \beta_j)$. ■

The two tensorial invariants $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ which are components of the structure function play a fundamental role in this $\{e\}$-structure and so we will determine their parametric form at the identity of $\mathrm{H} \times \mathrm{H}^{(1)}$. We have

Lemma 2.3: *The parametric forms for $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ at the identity of the structure group $\mathrm{H} \times \mathrm{H}^{(1)}_1$ are*

$$(\tilde{P}^i_j)|_e = \frac{1}{2}\frac{d}{dt}f^i{}_{|j} - f^i{}_{,j} - \frac{1}{4}f^i{}_{|k}f^k{}_{|j} - \frac{1}{n}\delta^i_j\left(\frac{1}{2}\frac{d}{dt}f^k{}_{|k} - f^k{}_{,k} - \frac{1}{4}f^l{}_{|k}f^k{}_{|l}\right)$$

$$\tilde{S}^i_{jkl}|_e = f^i_{|jkl} - \frac{3}{n+2}f^m_{|m(jk}\delta^i_{l)}$$

Proof: The form of $(\tilde{P}^i_j)|_e$ is immediate from equations (2.19) and (2.25), while to find $(\tilde{S}^i_{jkl})|_e$ we need to first determine

$$S^i_{jkl} = \left(d\Omega^i_j|_e\right)_{\widehat{\pi}^k\widehat{\theta}^l} \tag{2.55}$$

in equation (2.47) before the reduction of $\mathrm{H}^{(1)}$ to $\mathrm{H}^{(1)}_1$. To compute this we take $d\Omega^i_j$ in equation (2.23) and evaluate at the identity. To do this first notice

$$\widehat{\omega} = \omega - EA^{-1}_j\theta^j \qquad \text{thus} \quad d(\omega - EA^{-1}_j\theta^j) = 0$$

from which we find

$$\begin{aligned}(d\Omega^i_j)|_e &= (\beta_k)|_e\wedge\widehat{\theta}^k\delta^i_j + (\beta_j)|_e\wedge\widehat{\theta}^i + (dA^i_k\wedge dA^k_j)|_e - \widehat{\pi}^i\wedge(dE_j)|_e \\ &-d(c\delta^i_j - \frac{a^{-1}}{2}A^i_l(f^l_{|m})(A^{-1})^m_j)|_e\wedge\widehat{\omega} - \frac{1}{2}d(A^i_k(f^k_{|lm})(A^{-1})^l_n(A^{-1})^m_j)|_e\wedge\widehat{\theta}^n\,.\end{aligned} \tag{2.56}$$

Then use equation (2.23)

$$da|_e \;=\; \alpha|_e$$

to find

$$(\beta_j|_e)_{\widehat{\pi}^k} = 0 \tag{2.57}$$

Now use equation (2.23) again, giving

$$dE_j|_e = \kappa_j|_e \qquad dA^i_j|_e = \Omega^i_j|_e + \frac{1}{2}(f^i_{|j})\widehat{\omega} + \frac{1}{2}(f^i_{|jk})\widehat{\theta}^k$$

which determines the $\widehat{\pi}^k\widehat{\theta}^l$ term of $(d\Omega^i_j|_e)$ from (2.56) as

$$S^i_{jkl} = \left(d\Omega^i_j|_e\right)_{\widehat{\pi}^k\widehat{\theta}^l} = \left(df^i_{|jk}\right)_{\widehat{\pi}^l} = \frac{\partial^3 f^i}{\partial x^j_1 \partial x^k_1 \partial x^l_1}\,.$$

Finally we note that by equation (2.51) we have $\tilde{S}^i_{jkl}$ as the trace free part of (S^i_{jkl}) completing the proof. ■

The form given for $\tilde{S}^i_{jkl}$ in this theorem can also be checked explicitly by carrying out the calculation with the frame change

$$(\widehat{\kappa}_i)|_e = (\kappa)|_e - \frac{1}{n+2} f^m_{|mij}\widehat{\theta}^j$$

which corresponds to the final reduction. We see from equation (2.25) the actual parametric form for $\tilde{P}^i_j$ at an arbitrary point in the structure group. We will say more about $\tilde{S}^i_{jkl}$ later.

Now we would like to prove the main result of this section,

Theorem 2.2: *There exists a* **unique** *$\{e\}$-structure with a maximal dimensional symmetry (automorphism) group. For this $\{e\}$-structure the structure function vanishes, and a representative for the system of equations giving rise to this $\{e\}$-structure is the "free particle" equation*

$$\frac{d^2x^i}{dt^2} = 0 \tag{2.58}$$

The equivalence class of this equation is invariantly characterized by the two vanishing conditions

$$(\tilde{P}^i_j)|_e = \tfrac{1}{2}\tfrac{d}{dt}f^i{}_{|j} - f^i{}_{,j} - \tfrac{1}{4}f^i{}_{|k}f^k{}_{|j} - \tfrac{1}{n}\delta^i_j\left(\tfrac{1}{2}\tfrac{d}{dt}f^k{}_{|k} - f^k{}_{,k} - \tfrac{1}{4}f^l{}_{|k}f^k{}_{|l}\right) = 0$$

$$(\tilde{S}^i_{jkl})|_e = f^i_{|jkl} - \tfrac{3}{n+2}f^m_{|m(jk}\delta^i_{l)} = 0$$

Before we proceed with the proof of this theorem we should point out that this result is a special case of Theorem 2.3 in the next section. The reader could skip this proof altogether and proceed to Theorem 2.3 of which this is a simple corollary.

Proof: What we intend to show is that by making the assumption that the two components of the torsion (or structure function) $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ in Theorem 2.1 are constant, then **they must be zero, as well this implies that all other torsion elements must be zero**. This will follow from the integrability conditions for the $\{e\}$-structure in Theorem 2.1. Our initial assumptions first imply

$$d\tilde{P}^i_j = 0 \qquad \text{and} \quad d\tilde{S}^i_{jkl} = 0 \; . \tag{2.59}$$

We now use $d^2\pi^i = 0$ which is easily taken from equations (2.41) (2.43) (2.44) (or use the structure equations in (2.52)), to get

$$\begin{aligned} 0 \;&= \left(-\Omega^i_k\tilde{P}^k_j + \tilde{P}^i_k\Omega^k_j + 2\alpha\tilde{P}^i_j + 3\tilde{Q}^i_{kj}\pi^k - T^i_{kj}\pi^k - \chi^1\delta^i_j\right)\wedge\theta^j\wedge\omega \\ + &\left(d\tilde{Q}^i_{jk} - \Omega^i_l\tilde{Q}^l_{jk} - 2\tilde{Q}^i_{l[j}\Omega^l_{k]} + \alpha\tilde{Q}^i_{jk} - \tilde{P}^i_{[j}\kappa_{k]} + \xi^1_{[j}\delta^i_{k]} + R^i_{ljk}\pi^l\right)\wedge\theta^j\wedge\theta^k \, . \end{aligned} \tag{2.60}$$

The requirement that the coefficient of $\alpha\wedge\theta^j\wedge\omega$ be zero is

$$\tilde{P}^i_j = 0 \; . \tag{2.61}$$

From (2.60) we now have the following equation

$$3\tilde{Q}^i_{kj}\pi^k - T^i_{kj}\pi^k - \chi^1\delta^i_j \;\; \equiv 0 \qquad\qquad \mathrm{mod}(\omega, \theta^i)$$

where by taking the trace of this and noting by equation (2.54) that χ^1 has no θ^i or ω terms we arrive at

$$\chi^1 = -\frac{1}{n}T^l_{kl}\pi^k$$

Then putting the coefficient of $\pi^k\wedge\theta^j\wedge\omega$ in equation (2.60) to zero we have

$$3\tilde{Q}^i_{kj} - T^i_{kj} + \frac{1}{n}T^l_{kl}\delta^i_j = 0$$

However skew-symmetrizing this on j,k and taking the trace gives

$$T^i_{ki} = 0 \tag{2.62}$$

while just skew-symmetrization and symmetrization using (2.53) leads to

$$\tilde{Q}^i_{kj} = 0 \qquad \text{and} \qquad T^i_{jk} = 0 \ . \tag{2.63}$$

What is left of equation (2.60) is

$$(\xi^1_{[j}\delta^i_{k]} + R^i_{ljk}\pi^l)\wedge\theta^j\wedge\theta^k = 0 \tag{2.64}$$

So we may write

$$\xi^1_j = W_{kj}\pi^k + X_{kj}\theta^k \tag{2.65}$$

where W_{jk}, and X_{jk} are functions satisfying

$$W_{(jk)} = X_{(jk)} = 0 \tag{2.66}$$

where the skew-symmetry comes from the conditions on ξ^l_i in equation (2.54). We actually have from (2.64) and (2.65) that

$$R^i_{ljk} = -W_{l[j}\delta^i_{k]} \qquad \text{or} \qquad W_{jk} = R^i_{ijk} \tag{2.67}$$

The next step will be to compute as in equation (2.48) but use $\tilde{Q}^i_{jk} = T^i_{jk} = 0$ and equations (2.65), (2.67) to find

$$\begin{aligned} d^2\Omega^i_i - (n+1)d^2\alpha &= (n+2)(d\kappa_i\wedge\pi^i - \kappa_i\wedge\pi^i + d\sigma\wedge\omega - \sigma\wedge d\omega) + d(R^i_{ijk}\theta^j\wedge\theta^k) \\ 0 &= (n+2)(\xi^2_{ij}\wedge\theta^j\wedge\pi^i + W_{ij}\pi^i\wedge\theta^j\wedge\omega + X_{ij}\theta^i\wedge\theta^j\wedge\omega) \\ &\quad + dW_{ij}\wedge\theta^i\wedge\theta^j + 2W_{ij}(\Omega^i_k\wedge\theta^k - \pi^i\wedge\omega)\wedge\theta^j \end{aligned} \tag{2.68}$$

where here equation (2.67) is used. From this we may conclude that ξ^2_{ij} is of the form

$$\xi^2_{ij} = \widehat{W}_{ilj}\pi^l + \widehat{X}_{ilj}\theta^l + \omega\widehat{Y}_{ij} \tag{2.69}$$

where $\widehat{W}_{ilj}, \widehat{X}_{ilj}$ and $\widehat{Y}_{ij}$ are functions which by equation (2.54) are subject to the conditions,

$$\widehat{W}_{[i|l|j]} = 0 \quad , \quad \widehat{X}_{i(lj)} = \widehat{X}_{[ilj]} = 0 \quad , \quad \widehat{Y}_{[ij]} = 0 \, . \tag{2.70}$$

Inserting this expression for ξ^2_{ij} into equation (2.68) and using the conditions (2.66) and (2.70) readily gives,

$$W_{ij} = 0 \qquad X_{ij} = 0 \qquad \xi^2_{ij} \wedge \theta^j \wedge \pi^i = 0 \, . \tag{2.71}$$

(This implies $\xi^1_j = 0$) The last of these conditions gives,

$$\widehat{X}_{ilj} = 0 \qquad \widehat{Y}_{ij} = 0 \qquad \widehat{W}_{[il]j} = 0 \quad (\text{thus } \widehat{W}_{(ilj)} = W_{ilj}\,) \tag{2.72}$$

while W_{jk} being zero implies by equation (2.67) that

$$R^i_{jkl} = 0 \quad .$$

At this point the only possibly non-zero torsion coefficients are $\tilde{S}^i_{jkl}$, $\widehat{W}_{jlk}$ in (2.69) and the torsion in $d\beta_j$.

We continue to apply the integrability conditions, the next one being

$$\begin{aligned}
d^2\alpha &= d\beta_j \wedge \theta^j - \beta_j \wedge d\theta^j - d\kappa_j \wedge \pi^j + \kappa_j \wedge d\pi^j - 2d\sigma \wedge \omega + 2\sigma \wedge d\omega \\
0 &= d\beta_j \wedge \theta^j - \beta_j \wedge (\Omega^j_k \wedge \theta^k - \pi^j \wedge \omega) - (\beta_j \wedge \omega + \alpha \wedge \kappa_j + \kappa_k \wedge \Omega^k_j + \xi^2_{jk} \wedge \theta^k) \wedge \pi^j \\
&+ \kappa_j \wedge (\sigma \wedge \theta^j + (\Omega^j_k - \alpha\delta^j_k) \wedge \pi^k) - 2(\beta_j \wedge \pi^j + \sigma \wedge \alpha) \wedge \omega + 2\sigma \wedge (\alpha \wedge \omega + \kappa_j \wedge \theta^j) \\
0 &= \left(d\beta_j - \beta_k \wedge \Omega^k_j - \kappa_j \wedge \sigma + \xi^2_{kj} \wedge \pi^k\right) \wedge \theta^j \, .
\end{aligned}$$

If we use (2.71) then we may write

$$d\beta_j = \beta_k \wedge \Omega^k_j + \kappa_j \wedge \sigma + \lambda_{jk} \wedge \theta^k$$

where

$$\lambda_{ij} \wedge \theta^i \wedge \theta^j = 0 \tag{2.73}$$

Now we compute $d^2\Omega^i_j$ from (2.52)

$$\begin{aligned} d^2\Omega^i_j &= 2(d\beta_k\delta^{(k}_j\wedge\theta^{i)} - \beta_{(k}\delta^i_{j)}\wedge d\theta^k) + d\Omega^i_k\wedge\Omega^k_j - \Omega^i_k\wedge d\Omega^k_j + d\kappa_j\wedge\pi^k - \kappa_j\wedge d\pi^k \\ &\quad +(d\omega\wedge\sigma - \omega\wedge d\sigma)\delta^i_j + d\tilde{S}^i_{jkl}\wedge\pi^k\wedge\theta^l + \tilde{S}^i_{jkl}(d\pi^k\wedge\theta^l - \pi^k\wedge d\theta^l) \end{aligned}$$

and using equations (2.52) with the assumption that $d\tilde{S}^i_{jkl} = 0$ we get

$$\begin{aligned} 0 &= \left[-\Omega^i_m\tilde{S}^m_{jkl} + \tilde{S}^i_{jml}(\Omega^m_k - \alpha\delta^m_k) + \tilde{S}^i_{jkm}\Omega^m_l + \tilde{S}^i_{mkl}\Omega^m_j\right]\wedge\pi^k\wedge\theta^l \\ &\quad +\lambda_{jk}\wedge\theta^k\wedge\theta^i + \widehat{W}_{jlk}\pi^l\wedge\theta^k\wedge\pi^i\ . \end{aligned} \tag{2.74}$$

Two immediate consequences we have by putting the coefficient of $\alpha\wedge\pi^k\wedge\theta^l$ and of $\pi^l\wedge\theta^k\wedge\pi^i$ to zero are that

$$\tilde{S}^i_{jkl} = 0 \qquad \text{and} \qquad \widehat{W}_{ijk} = 0\ . \tag{2.75}$$

This now implies that [1]

$$\lambda_{jk} = X'_{jlk}\theta^l \tag{2.76}$$

Now put the coefficient of $\pi^j\wedge\theta^k\wedge\theta^l$ in $d^2\sigma$ to zero and finally we have,

$$X'_{jkl} = 0\ . \tag{2.77}$$

Thus the only possible constant values for the torsion is zero, proving the theorem. ■

To summarize this theorem we note that equation (2.60) implies that $\tilde{P}^i_j$ is acted on by scaling by the one dimensional subgroup of H generated by a. This dependency is also seen in the parametric form of $\tilde{P}^i_j$ in equation (2.25). While for $\tilde{S}^i_{jkl}$ we also find by equation (2.74) that $\tilde{S}^i_{jkl}$ is scaled by the action of subgroup generated by a. Thus the only way that these tensorial objects can thus be absolute invariants is if they

[1] For dimension $n > 2$, we actually have $\lambda_{jk} = 0$

vanish. The condition in Theorem 2.2 agree with those obtained for linear equations to be equivalent to (2.58) in [21].

For completeness we have

Corollary 2.1: *The $\{e\}$-structure admitting a maximal symmetry group has structure equations*

$$\begin{aligned}
d\omega &= \alpha\wedge\omega + \kappa_j\wedge\theta^j \\
d\theta^i &= \Omega^i_j\wedge\theta^j - \pi^i\wedge\omega \\
d\pi^i &= \sigma\wedge\theta^i + (\Omega^i_j - \alpha\delta^i_j)\wedge\pi^j \\
d\alpha &= \beta_j\wedge\theta^j - \kappa_j\wedge\pi^j - 2\sigma\wedge\omega \\
d\Omega^i_j &= 2\beta_{(k}\delta^k_{j)}\wedge\theta^i + \Omega^i_k\wedge\Omega^k_j - \pi^i\wedge\kappa_j - \sigma\wedge\omega\delta^i_j \\
d\sigma &= \beta_j\wedge\pi^j + \sigma\wedge\alpha \\
d\kappa_i &= \beta_i\wedge\omega + \alpha\wedge\kappa_i + \kappa_j\wedge\Omega^j_i\ .
\end{aligned}$$

These are the Maurer-Cartan equations of $sl(n+2,\mathbb{R})$.

It is possible to realize the symmetry group of $\ddot{x}^i = 0$ as $PGL(n+2,\mathbb{R})$ in the following way [19]: Let $(x^i, x^{n+1} = t, x^{n+2} = 1)$ be standard affine coordinates on an open set U of $\mathbb{P}_{n+1}(\mathbb{R})$, and let

$$L = (l^a_b) \in PGL(n+2,\mathbb{R}),\ 1 \le a, b \le n+2\,.$$

Acting with L on $\mathbb{P}^{n+1}(\mathbb{R})$ takes a point p with coordinates (x^i, t) to a point $\overline{p}$ with coordinates,

$$\overline{x}^i = \frac{l^i_j x^j + l^i_{n+1}\,t + l^i_{n+2}}{l^{n+2}_j x^j + l^{n+2}_{n+1}\,t + l^{n+2}_{n+2}}\quad,\qquad \overline{t} = \frac{l^{n+1}_j x^j + l^{n+1}_{n+2}}{l^{n+2}_j x^j + l^{n+2}_{n+2}}\quad 1 \le j \le n$$

where we have assumed L is sufficiently close to the identity so that the denominator does not vanish. If we now consider the set points in U given implicitly by $x^i = B^i t + C^i$ where B^i and C^i are constants then under L,

$$\overline{x}^i = \overline{B}^i\ \overline{t} + \overline{C}^i$$

where

$$\overline{B}^i = \frac{(l^i_j C^j + l^i_{n+2})(l^{n+2}_j B^j + l^{n+2}_{n+1}) - (l^i_j B^j + l^i_{n+1})(l^{n+2}_j C^j + l^{n+2}_{n+2})}{(l^{n+2}_j B^j + l^{n+2}_{n+1})(l^{n+1}_j C^j + l^{n+1}_{n+2}) - (l^{n+1}_j B^j + l^{n+1}_{n+1})(l^{n+2}_j C^j + l^{n+2}_{n+2})}$$

$$\overline{C}^i = \frac{(l^i_j B^j + l^i_{n+1})(l^{n+1}_j C^j + l^{n+1}_{n+2}) - (l^i_j C^j + l^i_{n+2})(l^{n+1}_j B^j + l^{n+1}_{n+1})}{(l^{n+2}_j B^j + l^{n+2}_{n+1})(l^{n+1}_j C^j + l^{n+1}_{n+2}) - (l^{n+1}_j B^j + l^{n+1}_{n+1})(l^{n+2}_j C^j + l^{n+2}_{n+2})}$$

Thus we see for L sufficiently close to the identity, L maps solutions of $\dddot{x}^i = 0$ to solutions of $\dddot{\overline{x}}^i = 0$. The structure equations in Corollary 2.1 are related to this example by the fact there is an **injection** of $PGL(n+2, \mathbb{R})$ into the second order frame bundle of $\mathbb{P}^{n+1}(\mathbb{R})$, see Kobayashi [29].

2.4 The Fundamental Invariants

In this section we provide the proof that in the $\{e\}$-structure of Theorem 2.1 all the tensorial invariants are differential functions of $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$. The proof of Theorem 2.2 is a special case.

Theorem 2.3: *The $\{e\}$-structure with invariant coframe $\left(\omega, \theta^i, \pi^i, \alpha, \Omega^i_j, \kappa_j, \sigma, \beta_j\right)$ in Theorem 2.1 has the structure equations,*

$$\begin{aligned}
d\omega &= \alpha\wedge\omega + \kappa_j\wedge\theta^j \\
d\theta^i &= \Omega^i_j\wedge\theta^j - \pi^i\wedge\omega \\
d\pi^i &= \sigma\wedge\theta^i + (\Omega^i_j - \alpha\delta^i_j)\wedge\pi^j + \tilde{P}^i_j\theta^j\wedge\omega + \tilde{Q}^i_{jk}\theta^j\wedge\theta^k \\
d\Omega^i_j &= 2\beta_{(k}\delta^k_{j)}\wedge\theta^i + \Omega^i_k\wedge\Omega^k_j - \pi^i\wedge\kappa_j - \sigma\wedge\omega\delta^i_j \\
&\quad +(\tilde{Q}^i_{jk} - T^i_{jk})\theta^k\wedge\omega + R^i_{jkl}\theta^k\wedge\theta^l + \tilde{S}^i_{jkl}\pi^k\wedge\theta^l \\
d\sigma &= \beta_i\wedge\pi^i + \sigma\wedge\alpha + \tfrac{1}{n-1}\kappa_i\tilde{P}^i_j\wedge\theta^j + \omega V_j\wedge\pi^j + W_{ij}\pi^i\wedge\theta^j + X_{ij}\theta^i\wedge\theta^j + \omega Y_j\wedge\theta^j \\
d\kappa_j &= \beta_j\wedge\omega + \alpha\wedge\kappa_j + \kappa_k\wedge\Omega^k_j + \widehat{W}_{jkl}\pi^k\wedge\theta^l + \widehat{X}_{jkl}\theta^k\wedge\theta^l + \omega\widehat{Y}_{jk}\wedge\theta^k \\
d\beta_j &= \beta_i\wedge\Omega^i_j + \kappa_j\wedge\sigma + \kappa_i\tilde{P}^i_j\wedge\omega + \kappa_i\wedge(\tilde{Q}^i_{jk}\theta^k - T^i_{jk})\theta^k + \tfrac{1}{n-1}T^i_{i(j}\kappa_{k)}\wedge\theta^k \\
&\quad +W'_{jkl}\pi^k\wedge\theta^l + X'_{jkl}\theta^k\wedge\theta^l + \omega Y'_{jk}\wedge\theta^k
\end{aligned}$$

where the torsion elements as differential function of $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ are,

$$
\begin{array}{lll}
V_j = \frac{1}{n-1}\left(d\tilde{P}^i_j\right)_{\pi^i} & \tilde{Q}^i_{jk} = \frac{1}{3}\left[\left(d\tilde{P}^i_{[j}\right)_{\pi^{k]}} - V_{[j}\delta^i_{k]}\right] & T^i_{jk} = \left(d\tilde{P}^i_{(j}\right)_{\pi^{k)}} + V_{(j}\delta^i_{k)} \\
Y_j = \frac{1}{n-1}\left(d\tilde{P}^i_j\right)_{\theta^i} & R^i_{jkl} = \left(d\tilde{Q}^i_{lk}\right)_{\pi^j} + W_{j[l}\delta^i_{k]} & W_{jk} = \frac{1}{3n}\left(dT^i_{i[j}\right)_{\pi^{k]}} \\
\widehat{Y}_{jk} = -\frac{1}{n+2}\left(dT^i_{i(j}\right)_{\pi^{k)}} & X_{kl} = \frac{1}{n+2}\left[(dR^i_{ikl})_\omega - \left(dT^i_{i[k}\right)_{\theta^{l]}}\right] & \widehat{X}_{jlk} = -\frac{1}{n+2}(dR^i_{ilk})_{\pi^j} \\
\widehat{W}_{jkl} = \frac{1}{1-n}\left(d\tilde{S}^i_{jkl}\right)_{\pi^i} & W'_{jlk} = \frac{1}{n-1}\left[\left(d\tilde{S}^i_{ljk}\right)_{\theta^i} - \widehat{X}_{(jk)l}\right] &
\end{array}
$$

$$
X'_{jkl} = \left(d\widehat{Y}_{j[l}\right)_{\theta^{k]}} - \widehat{W}_{jm[l}\tilde{P}^m_{k]} - \left(d\widehat{X}_{jkl}\right)_\omega
$$

$$
Y'_{jk} = \frac{1}{n-1}\left[\left(dT^i_{jk}\right)_{\theta^i} - \left(dT^i_{i(j}\right)_{\theta^{k)}} - \tilde{S}^i_{jmk}\tilde{P}^m_i\right]
$$

This proof is similar to that of Theorem 2.2 only more care is needed for handling the indices. Again, we apply integrability conditions to obtain the results.

Proof: We begin by putting $d^2\pi^i = 0$, which is found in equations (2.41), (2.43) and (2.44) (or use the structure equations in (2.52)), to get

$$
\begin{aligned}
0 \;=\; & \left(d\tilde{P}^i_j - \Omega^i_k\tilde{P}^k_j + \tilde{P}^i_k\Omega^k_j + 2\alpha\tilde{P}^i_j + 3\tilde{Q}^i_{kj}\pi^k - T^i_{kj}\pi^k - \chi^1\delta^i_j\right)\wedge\theta^j\wedge\omega \\
+ & \left(d\tilde{Q}^i_{jk} - \Omega^i_l\tilde{Q}^l_{jk} - 2\tilde{Q}^i_{l[j}\Omega^l_{k]} + \alpha\tilde{Q}^i_{jk} - \tilde{P}^i_{[j}\kappa_{k]} + \xi^1_{[j}\delta^i_{k]} + R^i_{ljk}\pi^l\right)\wedge\theta^j\wedge\theta^k
\end{aligned} \tag{2.78}
$$

which gives

$$
\left.\begin{aligned}
d\tilde{P}^i_j - \Omega^i_k\tilde{P}^k_j + \tilde{P}^i_k\Omega^k_j + 2\alpha\tilde{P}^i_j + 3\tilde{Q}^i_{kj}\pi^k - T^i_{kj}\pi^k - \chi^1\delta^i_j \equiv 0 \\
d\tilde{Q}^i_{jk} - \Omega^i_l\tilde{Q}^l_{jk} - 2\tilde{Q}^i_{l[j}\Omega^l_{k]} + \alpha\tilde{Q}^i_{jk} - \tilde{P}^i_{[j}\kappa_{k]} + \xi^1_{[j}\delta^i_{k]} + R^i_{ljk}\pi^l \equiv 0
\end{aligned}\right. \quad \mathrm{mod}(\omega,\theta^i) \tag{2.79}
$$

Taking the trace in the first of these equations and noting from (2.54) that χ^1 has no ω or θ^i terms we find

$$
\chi^1 = -V_k\pi^k = -\frac{1}{n}T^i_{ki}\pi^k
$$

with V_j being functions. Then taking the coefficients of $\pi^k\wedge\theta^j\wedge\omega$ in equation (2.78) we obtain

$$
\left(dP^i_j\right)_{\pi^k} + 3\tilde{Q}^i_{kj} - T^i_{kj} + V_k\delta^i_j = 0 \tag{2.80}
$$

Skew-symmetrizing on j,k and taking the trace gives

$$V_j = \frac{1}{n-1}\left(d\tilde{P}^i_j\right)_{\pi^i} \tag{2.81}$$

where we have used the conditions on $\tilde{Q}^i_{jk}$ and T^i_{jk} in (2.53). Skew-symmetrization and symmetrization in equation (2.80) leads to

$$\begin{aligned} \tilde{Q}^i_{jk} &= \frac{1}{3}\left(\left(d\tilde{P}^i_{[j}\right)_{\pi^{k]}} - V_{[j}\delta^i_{k]}\right) \\ T^i_{jk} &= \left(d\tilde{P}^i_{(j}\right)_{\pi^{k)}} + V_{(j}\delta^i_{k)} \ . \end{aligned}$$

We now take the trace in the second equation in (2.79) and find

$$(n-1)\xi^1_j - \kappa_k\tilde{P}^k_j - 2R^i_{lji}\pi^l \equiv 0 \qquad \mathrm{mod}(\omega, \theta^i) \tag{2.82}$$

In other words we may write

$$\xi^1_j = \frac{1}{n-1}\kappa_k\tilde{P}^k_j + W_{kj}\pi^k + X_{kj}\theta^k + Y_j\omega \tag{2.83}$$

where W_{jk}, X_{jk} and Y_j are functions satisfying,

$$W_{(jk)} = X_{(jk)} = 0 \tag{2.84}$$

by the conditions on ξ^1_i in equation (2.54). Thus substituting (2.83) into (2.82) gives

$$W_{kj} = \frac{2}{1-n}R^i_{[kj]i} \qquad \text{and} \qquad R^i_{(jk)i} = 0 \ . \tag{2.85}$$

Now inserting this expression for ξ^1_i into 2.78 and putting the coefficients of $\pi^l\wedge\theta^j\wedge\theta^k$ and $\theta^j\wedge\theta^k\wedge\omega$ to zero gives

$$\begin{aligned} \left(d\tilde{Q}^i_{jk}\right)_{\pi^l} + W_{l[j}\delta^i_{k]} + R^i_{ljk} &= 0 \\ -\left(d\tilde{P}^i_{[j}\right)_{\theta^{k]}} + \left(d\tilde{Q}^i_{jk}\right)_{\omega} + Y_{[j}\delta^i_{k]} &= 0 \ . \end{aligned} \tag{2.86}$$

Taking the trace of the second equation above we have

$$Y_j = \frac{1}{n-1}\left(d\tilde{P}^i_j\right)_{\theta^i} \tag{2.87}$$

At this point all information from the first integrability condition in equation (2.78) has been obtained. We have W_{jk} and X_{jk} in (2.83) still undetermined. We now continue by setting $d^2\alpha = 0$ and use equations (2.24) and (2.30)

$$\begin{aligned}
d^2\alpha &= d\beta_j \wedge \theta^j - \beta_j \wedge d\theta^j - d\kappa_j \wedge \pi^j + \kappa_j \wedge d\pi^j - 2d\sigma \wedge \omega + 2\sigma \wedge d\omega \\
0 &= d\beta_j \wedge \theta^j - \beta_j \wedge (\Omega^j_k \wedge \theta^k - \pi^j \wedge \omega) - (\beta_j \wedge \omega + \alpha \wedge \kappa_j + \kappa_k \wedge \Omega^k_j + \xi^2_{kj}\theta^k) \wedge \pi^j \\
&\quad + \kappa_j \wedge (\sigma \wedge \theta^j + (\Omega^j_k - \alpha\delta^j_k) \wedge \pi^k + \tilde{P}^j_k \theta^k \wedge \omega + \tilde{Q}^j_{kl}\theta^k \wedge \theta^l) \\
&\quad - 2(\beta_j \wedge \pi^j + \sigma \wedge \alpha + \xi^1_j \wedge \theta^j) \wedge \omega + 2\sigma \wedge (\alpha \wedge \omega + \kappa_j \wedge \theta^j) \\
0 &= \left[d\beta_j - \beta_k \wedge \Omega^k_j - \kappa_j \wedge \sigma + \xi^2_{kj} \wedge \pi^k - \kappa_k \wedge \tilde{P}^k_j \wedge \omega + 2\xi^1_j \wedge \omega - \kappa_l \tilde{\wedge} Q^l_{jk}\theta^k \wedge \right] \wedge \theta^j
\end{aligned}$$

thus

$$d\beta_j = \beta_k \wedge \Omega^k_j + \kappa_j \wedge \sigma - \xi^2_{kj} \wedge \pi^k + \kappa_k \wedge \tilde{P}^k_j \omega - 2\xi^1_j \wedge \omega + \kappa_l \wedge \tilde{Q}^l_{jk}\theta^k + \lambda_{jk} \wedge \theta^k \tag{2.88}$$

where the one-forms λ_{ij} satisfy,

$$\lambda_{ij} \wedge \theta^i \wedge \theta^j = 0 \ . \tag{2.89}$$

Now compute $d^2\Omega^i_j$ from (2.52)

$$\begin{aligned}
d^2\Omega^i_j &= 2(d\beta_{(k}\delta^k_{j)} \wedge \theta^i - \beta_{(k}\delta^i_{j)} \wedge d\theta^k) + d\Omega^i_k \wedge \Omega^k_j - \Omega^i_k \wedge d\Omega^k_j + d\kappa_j \wedge \pi^k - \kappa_j \wedge d\pi^k \\
&+ (d\omega \wedge \sigma - \omega \wedge d\sigma)\delta^i_j + d(\tilde{Q}^i_{jk} - T^i_{jk})\theta^k \wedge \omega + (\tilde{Q}^i_{jk} - T^i_{jk})(d\theta^k \wedge \omega - \theta^k \wedge d\omega) \\
&+ dR^i_{jkl}\theta^k \wedge \theta^l + 2R^i_{jkl} d\theta^k \wedge \theta^l + d\tilde{S}^i_{jkl} \wedge \pi^k \wedge \theta^l + \tilde{S}^i_{jkl}(d\pi^k \wedge \theta^l - \pi^k \wedge d\theta^l)
\end{aligned}$$

and using equations (2.52)

$$
\begin{aligned}
0 = {} & 2\left(\beta_l\wedge\Omega^l_k + \kappa_k\wedge\sigma - \xi^2_{lk}\wedge\pi^l + \kappa_l\wedge\tilde{P}^l_k\omega - 2\xi^1_k\wedge\omega + \kappa_m\wedge\tilde{Q}^m_{kl}\theta^l\right)\delta^{(k}_j\wedge\theta^{i)} \\
& +\lambda_{jk}\wedge\theta^k\wedge\theta^i - 2\beta_{(k}\delta^i_{j)}\wedge\left(\Omega^k_l\wedge\theta^l - \pi^k\wedge\omega\right) \\
& +\left(\beta_k\wedge\theta^i - \pi^i\wedge\kappa_k - \sigma\wedge\omega\delta^i_k + (\tilde{Q}^i_{km} - T^i_{km})\theta^m\wedge\omega + R^i_{kml}\theta^m\wedge\theta^l + \tilde{S}^i_{kml}\pi^m\wedge\theta^l\right)\wedge\Omega^k_j \\
& -\Omega^i_k\wedge\left(\beta_j\wedge\theta^k - \pi^k\wedge\kappa_j - \sigma\wedge\omega\delta^k_j + (\tilde{Q}^k_{jm} - T^k_{jm})\theta^m\wedge\omega + R^k_{jml}\theta^m\wedge\theta^l + \tilde{S}^k_{jml}\pi^m\wedge\theta^l\right) \\
& +(\beta_j\wedge\omega + \alpha\wedge\kappa_j + \kappa_k\wedge\Omega^k_j + \xi^2_{jk}\wedge\theta^k)\wedge\pi^i \\
& -\kappa_j\wedge\left(\sigma\wedge\theta^i + (\Omega^i_k - \alpha\delta^i_k)\wedge\pi^k + \tilde{P}^i_k\theta^k\wedge\omega + \tilde{Q}^i_{kl}\theta^k\wedge\theta^l\right) \\
& +(\alpha\wedge\omega + \kappa_j\wedge\theta^j)\wedge\sigma - \omega\wedge(\beta_j\wedge\pi^j + \sigma\wedge\alpha + \xi^1_j\wedge\theta^j) + d(\tilde{Q}^i_{jk} - T^i_{jk})\wedge\theta^k\wedge\omega \\
& +(\tilde{Q}^i_{jk} - T^i_{jk})(\Omega^k_l\wedge\theta^l\wedge\omega - \theta^k\wedge\alpha\wedge\omega - \theta^k\wedge\kappa_l\wedge\theta^l) \\
& +dR^i_{jkl}\wedge\theta^k\wedge\theta^l + 2R^i_{jkl}(\Omega^k_m\wedge\theta^m - \pi^k\wedge\omega)\wedge\theta^l + d\tilde{S}^i_{jkl}\wedge\pi^k\wedge\theta^l \\
& +\tilde{S}^i_{jkl}\left((\sigma\wedge\theta^k + \Omega^k_m\wedge\pi^m - \alpha\wedge\pi^k + \tilde{P}^k_m\theta^m\wedge\omega + \tilde{Q}^k_{mr}\theta^m\wedge\theta^r)\wedge\theta^l - \pi^k\wedge(\Omega^l_m\wedge\theta^m - \pi^m\wedge\omega)\right)
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
& \left[d(\tilde{Q}^i_{jk} - T^i_{jk}) - \Omega^i_l(\tilde{Q}^l_{jk} - T^l_{jk}) + (\tilde{Q}^i_{jl} - T^i_{jl})(\Omega^l_k + \alpha\delta^l_k) + (\tilde{Q}^i_{lk} - T^i_{lk})\Omega^l_j\right]\wedge\theta^k\wedge\omega \\
& +(\tilde{Q}^i_{jk} - T^i_{jk})\theta^k\wedge\kappa_l\wedge\theta^l + \left[dR^i_{jkl} - \Omega^i_m R^m_{jkl} + 2R^i_{jml}\Omega^m_k + R^i_{mkl}\Omega^m_j\right]\wedge\theta^k\wedge\theta^l \\
& +2R^i_{jkl}\pi^k\wedge\theta^l\wedge\omega + \left[d\tilde{S}^i_{jkl} - \Omega^i_m\tilde{S}^m_{jkl} + \tilde{S}^i_{jml}(\Omega^m_k - \alpha\delta^m_k) + \tilde{S}^i_{jkm}\Omega^m_l + \tilde{S}^i_{mkl}\Omega^m_j\right]\wedge\pi^k\wedge\theta^l \\
& +\lambda_{jk}\wedge\theta^k\wedge\theta^i + \tilde{S}^i_{jkl}(\tilde{P}^k_m\theta^m\wedge\omega + \tilde{Q}^k_{mr}\theta^m\wedge\theta^r)\wedge\theta^l \\
& +2(\kappa_l\wedge\tilde{P}^l_k\omega + \kappa_m\wedge\tilde{Q}^m_{kl}\theta^l - \xi^2_{lk}\wedge\pi^l)\wedge\delta^{(k}_j\theta^{i)} - 2\xi^1_j\wedge\omega\wedge\theta^i + \delta^i_j\xi^1_k\wedge\theta^k\wedge\omega \\
& +(\kappa_j\wedge\tilde{P}^i_k\omega + \kappa_j\wedge\tilde{Q}^i_{kl}\theta^l)\wedge\theta^k + \xi^2_{jk}\wedge\theta^k\wedge\pi^i = 0\ .
\end{aligned}
\tag{2.90}
$$

Taking the trace of this we have,

$$
\begin{aligned}
0 = {} & [dR^i_{ikl} + 2R^i_{iml}\Omega^m_k]\wedge\theta^k\wedge\theta^l - \left[dT^i_{ik} + T^i_{il}(\Omega^l_k + \alpha^l_k)\right]\wedge\theta^k\wedge\omega + 2R^i_{ikl}\pi^k\wedge\theta^l\wedge\omega \\
& -T^i_{ik}\theta^k\wedge\kappa_l\wedge\theta^l - (n+2)\xi^2_{jk}\wedge\pi^j\wedge\theta^k + (n+2)\xi^1_k\wedge\theta^k\wedge\omega - (n+2)\kappa_m\wedge\tilde{Q}^m_{jk}\theta^j\wedge\theta^k \\
& -(n+2)\kappa_m\wedge\tilde{P}^m_j\theta^j\wedge\omega
\end{aligned}
\tag{2.91}
$$

We find X_{kl} by using (2.83) then putting the coefficient of $\theta^l\wedge\theta^k\wedge\omega$ in the above equation to zero, that is

$$
\left(dR^i_{ikl}\right)_\omega - \left(dT^i_{i[k}\right)_{\theta^{l]}} + (n+2)X_{lk} = 0 \tag{2.92}
$$

Now put the term with $\pi^j \wedge \theta^k$ in (2.91) to zero so

$$\left[2R^i_{ijk}\omega + (n+2)W_{jk}\omega - (n+2)\xi^2_{jk} - \left(dT^i_{ik}\right)_{\pi^j}\omega + \left(dR^i_{ilk}\right)_{\pi^j}\theta^l\right]\wedge\pi^j\wedge\theta^k = 0 \quad (2.93)$$

From which we may conclude that ξ^2_{ij} is of the form

$$\xi^2_{jk} = \widehat{W}_{jlk}\pi^l + \widehat{X}_{jlk}\theta^l + \omega\widehat{Y}_{jk} \quad (2.94)$$

where $\widehat{W}_{ijk}, \widehat{X}_{ijk}$ and $\widehat{Y}_{ij}$ are functions which by equation (2.54) are subject to the conditions,

$$\widehat{W}_{[i|j|k]} = 0 \quad , \quad \widehat{X}_{[ijk]} = 0 \quad , \quad \widehat{Y}_{[ij]} = 0 \quad (2.95)$$

Inserting the expansion for ξ^2_{jk} from equation (2.94) into (2.93) and putting the coefficient of $\pi^i \wedge \pi^j \wedge \theta^k$ to zero gives,

$$\widehat{W}_{[ij]k} = 0 \qquad \text{or} \qquad \widehat{W}_{ijk} = \widehat{W}_{(ijk)} \quad (2.96)$$

The coefficient of $\omega \wedge \pi^j \wedge \theta^k$ in equation (2.93) being zero gives after skew-symmetrization

$$W_{jk} = \frac{1}{n+2}\left(2R^i_{ijk} - \left(dT^i_{i[j}\right)_{\pi^k]}\right) \quad (2.97)$$

which simplifies by equation (2.85) and the symmetry properties of R^i_{jkl} in equation (2.53) to

$$W_{jk} = \frac{1}{3n}\left(dT^i_{i[j}\right)_{\pi^k]} \quad (2.98)$$

The symmetric part of the coefficient of $\omega \wedge \pi^j \wedge \theta^k$ in (2.93) gives,

$$\widehat{Y}_{jk} = -\frac{1}{n+2}\left(dT^i_{i(j}\right)_{\pi^k)} \quad (2.99)$$

The coefficient of $\theta^l \wedge \pi^j \wedge \theta^k$ in equation (2.93) being zero gives,

$$\widehat{X}_{jlk} = \frac{1}{n+2}\left(dR^i_{ilk}\right)_{\pi^j} \quad (2.100)$$

We now need to find $\widehat{W}_{ijk}$, which can be done by putting the coefficient of $\pi^m \wedge \pi^l \wedge \theta^k$ in equation (2.90) to zero, which gives

$$\left(d\tilde{S}^i_{jl[k}\right)_{\pi^{m]}} - \widehat{W}_{jl[m}\delta^i_{k]} = 0 \qquad \text{or} \qquad \widehat{W}_{jkl} = \frac{1}{1-n}\left(d\tilde{S}^i_{jkl}\right)_{\pi^i} . \qquad (2.101)$$

Finally we are left with the determination of λ_{ij}, so put the coefficient of $\theta^k \wedge \theta^l$ in equation 2.90 to zero

$$\begin{aligned}
&\left(d\tilde{Q}^i_{j[k} - dT^i_{j[k}\right)_{\theta^{l]}}\omega - (\tilde{Q}^i_{j[k} - T^i_{j[k})\kappa_{l]} + dR^i_{jkl} - \Omega^i_m R^m_{jkl} + 2R^i_{jm[k}\Omega^m_{l]} + R^i_{mkl}\Omega^m_j \\
&+\lambda_{j[k}\delta^i_{l]} - \left(d\tilde{S}^i_{mj[k}\right)_{\theta^{l]}}\pi^m - \tilde{S}^i_{jm[l}\tilde{P}^m_{k]}\omega + \tilde{S}^i_{jn[m}\tilde{Q}^n_{kl]}\theta^m - \widehat{X}_{mkl}\delta^i_j\pi^m + \widehat{X}_{mj[k}\delta^i_{l]}\pi^m \\
&+\widehat{X}_{jkl}\pi^i - (X_{kl}\delta^i_j + 2X_{j[k}\delta^i_{l]})\omega - \kappa_m\tilde{Q}^m_{kl}\delta^i_j + \kappa_m\tilde{Q}^m_{j[k}\delta^i_{l]} - \kappa_j\tilde{Q}^i_{kl} \;=\; 0
\end{aligned}$$

If in this equation we now substitute

$$\lambda_{ij} = \lambda'_{ij} + X'_{ikj}\theta^k \qquad \text{where by 2.89} \quad \lambda'_{[jk]} = 0 \quad X'_{[ikj]} = 0$$

and take the trace on i, l and symmetrize on jk we have

$$\begin{aligned}
(n-1)\lambda'_{jk} \;=\; & \left(\left(dT^i_{jk}\right)_{\theta^i} - \left(dT^i_{i(j}\right)_{\theta^{k)}} - \tilde{S}^i_{jmk}\tilde{P}^m_i\right)\omega + \left(\left(d\tilde{S}^i_{ljk}\right)_{\theta^i} - \widehat{X}_{(jk)l}\right)\pi^l \\
& +T^i_{i(j}\kappa_{k)} - \kappa_l T^l_{jk}
\end{aligned}$$

note that here we have used (2.85). This still leaves X'_{ijk} undetermined. It can be found by taking $d^2\kappa_j$ and setting to zero the coefficient of $\theta^k \wedge \theta^l \wedge \omega$, which is

$$0 = X'_{jkl} + \widehat{W}_{jm[l}\tilde{P}^m_{k]} + \left(d\widehat{X}_{jkl}\right)_{\omega} - \left(d\widehat{Y}_{j[l}\right)_{\theta^{k]}} \qquad (2.102)$$

This completes the determination of the coframe as given in Theorem 2.3 ∎

We can determine the infinitesimal group action on $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ from equations (2.78) and (2.90) as,

$$\begin{aligned}
d\tilde{P}^i_j - \Omega^i_k\tilde{P}^k_j + \tilde{P}^i_k\Omega^k_j + 2\alpha\tilde{P}^i \;&\equiv\; 0 \\
d\tilde{S}^i_{jkl} - \Omega^i_m\tilde{S}^m_{jkl} + \tilde{S}^i_{jml}(\Omega^m_k - \alpha\delta^m_k) + \tilde{S}^i_{jkm}\Omega^m_l + \tilde{S}^i_{mkl}\Omega^m_j \;&\equiv\; 0
\end{aligned} \qquad \text{mod(base)}.$$

From this Theorem 2.2 is an immediate corollary.

2.5 Systems of Third Order Ordinary Differential Equations

In this section we will study the question of equivalence of systems of third order differential equations,

$$\frac{d^3x^i}{dt^3} = f^i\left(t, x^j, \frac{dx^j}{dt}, \frac{d^2x^j}{dt^2}\right) \tag{2.103}$$

$$\frac{d^3\bar{x}^i}{d\bar{t}^3} = \bar{f}^i\left(\bar{t}, \bar{x}^j, \frac{d\bar{x}^j}{d\bar{t}}, \frac{d^2\bar{x}^j}{d\bar{t}^2}\right) \tag{2.104}$$

We proceed in a manner similar to section 2. Let $U \subset J^2(\mathbb{R}, \mathbb{R}^n)$ and $\bar{U} \subset J^2(\bar{\mathbb{R}}, \bar{\mathbb{R}}^n)$ with standard coordinates, (t, x^i, x^i_1, x^i_2) and $(\bar{t}, \bar{x}^i, \bar{x}^i_1, \bar{x}^i_2)$ and associate to the systems of differential equations in (2.103), and (2.104) the Pfaffian systems,

$$\begin{array}{ll} \widehat{\theta}^i_b = dx^i_{b-1} - x^i_b dt \quad \widehat{\theta}^i_3 = dx^i_2 - f^i(t, x^j, x^j_1, x^j_2)dt \\ \widehat{\bar{\theta}}^i_b = d\bar{x}^i_{b-1} - \bar{x}^i_b d\bar{t} \quad \widehat{\bar{\theta}}^i_3 = d\bar{x}^i_2 - \bar{f}^i(\bar{t}, \bar{x}^j, \bar{x}^j_1, \bar{x}^j_2)dt \end{array} \quad b = 1, 2 \tag{2.105}$$

As in the second order case finding a one-dimensional integral manifold γ of the first Pfaffian system in (2.105) satisfying

$$\gamma^* dt \neq 0$$

is identical to finding a solution to the system of equations in (2.103).

The two differential systems in (2.105) are equivalent if and only if there exists a diffeomorphism $(\bar{t}, \bar{x}^i) = (\phi(t, x^j), \psi^i(t, x^j)) = \Psi(t, x^j)$, with second prolongation Ψ_2 satisfying

$$\Psi_2^* < \bar{\theta}_c >_{c=1\ldots3} \; = \; < \theta_c >_{c=1\ldots3} \quad . \tag{2.106}$$

Extending the one-forms in (2.105) to the coframes

$$(\widehat{\omega} = dt, \widehat{\theta}^i_c) \quad \text{and} \quad (\widehat{\bar{\omega}} = d\bar{t}, \widehat{\bar{\theta}}^i_c) \quad c = 1, 2, 3 \; , \tag{2.107}$$

Appendix A allows us to simplify condition (2.106) to

Lemma 2.4: *The two differential systems in* (2.105) *are equivalent if and only if there exists a point transformation,*

$$(\bar{t}, \bar{x}^i) = (\phi(t, x^j), \psi^i(t, x^j)) = \Psi(t, x^j) \tag{2.108}$$

whose second prolongation Ψ_2 *satisfies*

$$\Psi_2^* \begin{pmatrix} \bar{\omega} \\ \bar{\theta}_1^i \\ \bar{\theta}_2^i \\ \bar{\theta}_3^i \end{pmatrix} = \mathcal{S} \begin{pmatrix} \omega \\ \theta_1^i \\ \theta_2^i \\ \theta_3^i \end{pmatrix}$$

where $\mathcal{S} : U \to \mathrm{H}$ *is a smooth function on* U *taking values in the Lie subgroup* H *of* $GL(3n+1, \mathbb{R})$ *defined by*

$$\mathrm{H} = \left\{ \begin{pmatrix} a & E_j & 0 & 0 \\ 0 & A_j^i & 0 & 0 \\ 0 & B_j^i & a^{-1}A_j^i & 0 \\ 0 & C_j^i & D_j^i & a^{-2}A_j^i \end{pmatrix} \quad \begin{matrix} a \in \mathbb{R}^*,\ A_j^i \in GL(n, \mathbb{R}),\ E_j \in \mathbb{R}^n \\ (B_j^i), (C_j^i), (D_j^i) \in M_n(\mathbb{R}) \end{matrix} \right\}.$$

The proof of this comes directly from Appendix A. For the Lie group H we have the Maurer-Cartan form

$$\begin{pmatrix} \alpha & \kappa_j & 0 & 0 \\ 0 & \Omega_j^i & 0 & 0 \\ 0 & \beta_j^i & \Omega_j^i - \alpha\delta_j^i & 0 \\ 0 & \Sigma_j^i & \lambda_j^i & \Omega_j^i - 2\alpha\delta_j^i \end{pmatrix} = d\mathcal{S}(\mathcal{S}^{-1}) \tag{2.109}$$

where $(\mathcal{S}^{-1})$ is

$$\begin{pmatrix} a^{-1} & -a^{-1}EA_j^{-1} & 0 & 0 \\ 0 & (A^{-1})_j^i & 0 & 0 \\ 0 & -a(A^{-1})_k^i B_l^k (A^{-1})_j^k & a(A^{-1})_j^i & 0 \\ 0 & a^2(A^{-1})_k^i (C_j^k - a^{-1}D_l^k (A^{-1})_m^l B_r^m)(A^{-1})_j^r & -a^3(A^{-1})_k^i D_l^k (A^{-1})_j^l & a^2(A^{-1})_j^i \end{pmatrix}$$

We will use the conventions of the last section.

Theorem 2.4: *Solutions $\Psi : U \to \overline{U}$ to the equivalence problem for systems of n (≥ 2) third order ordinary differential equations are in one-to-one correspondence with the solutions of an equivalence problem for an n^2+3n+3 dimensional $\{e\}$-structures which is obtained by applying the equivalence method to the initial coframe $(\widehat{\omega}, \widehat{\theta}^i_1, \widehat{\theta}^i_2, \widehat{\theta}^i_3)$ with the structure group given in Lemma 2.4.*

In this proof we will have the range of indices $b = 1, 2$ and $c = 1, 2, 3$.

Proof: To proceed we first write $d\widehat{\theta}^i_1 = -dx^i_2 \wedge dt$ and $d\widehat{\theta}^i_2 = -dx^i_3 \wedge dt$ and $d\widehat{\theta}^i_3 = -df^i \wedge dt$ in the lifted frame

$$
\begin{aligned}
d\widehat{\theta}^i_1 &= -(A^{-1})^i_j \left(\theta^j_2 - B^j_k (A^{-1})^k_l \theta^l_1\right) \wedge (\omega - EA^{-1}_j \theta^j_1) \\
d\widehat{\theta}^i_2 &= -a(A^{-1})^i_j \left(\theta^j_3 + (C^j_k - a^{-1} D^j_k (A^{-1})^k_l B^l_m)(A^{-1})^m_r \theta^r_1 - a D^j_k (A^{-1})^k_l \theta^l_2\right) \\
&\qquad \wedge (\omega - EA^{-1}_j \theta^j_1) \\
d\widehat{\theta}^i_3 &= -df^i \wedge a^{-1} (\omega - EA^{-1}_j \theta^j_1)
\end{aligned}
\tag{2.110}
$$

From these equations and $d\widehat{\omega} = 0$ the first two structure equations are

$$
\begin{aligned}
d\omega &= \alpha \wedge \omega + \kappa_j \wedge \theta^j_1 - EA^{-1}_j (\theta^j_2 - B^j_k (A^{-1})^k_l \theta^l_1) \wedge (\omega - EA^{-1}_j \theta^j_1) \\
d\theta^i_1 &= \Omega^i_j \wedge \theta^j_1 - \theta^i_2 \wedge \omega + \theta^i_2 \wedge EA^{-1}_j \theta^j_1 + B^i_k (A^{-1})^k_l \theta^l_1 \wedge (\omega - EA^{-1}_j \theta^j_1) .
\end{aligned}
\tag{2.111}
$$

By making the absorptions,

$$
\begin{aligned}
\alpha &= \widehat{\alpha} + EA^{-1}_j \theta^j_2 \\
\kappa_j &= \widehat{\kappa}_j + EA^{-1}_k \theta^k_2 EA^{-1}_j + (\omega - EA^{-1}_j \theta^j_1) EA^{-1}_k B^k_l (A^{-1})^l_j \\
\Omega^i_j &= \widehat{\Omega}^i_j - \theta^i_2 EA^{-1}_j + (\omega - EA^{-1}_k \theta^k_1) B^i_l (A^{-1})^l_j
\end{aligned}
\tag{2.112}
$$

we see that equations (2.111) become

$$
\begin{aligned}
d\omega &= \widehat{\alpha} \wedge \omega + \widehat{\kappa}_j \wedge \theta^j_1 \\
d\theta^i_1 &= \widehat{\Omega}^i_j \wedge \theta^j_1 - \theta^i_2 \wedge \omega .
\end{aligned}
\tag{2.113}
$$

We still have the following freedom to absorb using $\widehat{\alpha}$ and $\widehat{\Omega}^i_j$,

$$
\begin{aligned}
\widehat{\alpha} &= \bar{\alpha} + V\omega + W_j \theta^j_1 \\
\widehat{\Omega}^i_j &= \bar{\Omega}^i_j + X^i_{(jk)} \theta^k_1
\end{aligned}
\tag{2.114}
$$

Considering the last two equations in (2.110) and the absorption in (2.112) we may write in general (and dropping hats)

$$\begin{aligned} d\theta_2^i &= \beta_j^i\wedge\theta_1^j + (\Omega_j^i - \alpha\delta_j^i)\wedge\theta_2^j - \theta_3^i\wedge\omega + T_j^i\theta_2^j\wedge\omega + P_j^i\theta_1^j\wedge\omega + Q^{bi}{}_{jk}\theta_b^j\wedge\theta_1^k \\ d\theta_3^i &= \Sigma_j^i\wedge\theta_1^j + \lambda_j^i\wedge\theta_2^j + (\Omega_j^i - 2\alpha\delta_j^i)\wedge\theta_3^j + Y^{ci}_j\theta_c^j\wedge\omega + Z^{cbi}{}_{jk}\theta_c^j\wedge\theta_b^k \end{aligned} \quad (2.115)$$

where $b = 1, 2$, $c = 1...3$ and P_j^i, $Q_{jk}^i, T_j^i, Y^{bi}_j, Z^{bci}{}_{jk}$ are function whose particular form is not important. We may now further absorb torsion by

$$\begin{aligned} \alpha &= \hat{\alpha} - \frac{1}{n}T_l^l\omega \\ \beta_j^i &= \hat{\beta}_j^i + P_j^i\omega - Q^{ci}_{kj}\theta_c^k \\ \Sigma_j^i &= \hat{\Sigma}_j^i + Y^{1i}_j\omega - Z^{b1i}{}_{kj}\theta_b^k \\ \lambda_j^i &= \hat{\lambda}_j^i + Y^{2i}_j\omega - Z^{22i}{}_{kj}\theta_2^k - Z^{32i}{}_{kj}\theta_3^k \end{aligned}$$

which leads to the last two structure equations,

$$\begin{aligned} d\theta_2^i &= \beta_j^i\wedge\theta_1^i + (\Omega_j^i - \alpha\delta_j^i)\wedge\theta_2^j - \theta_3^i\wedge\omega + \tilde{T}_j^i\theta_2^j\wedge\omega \\ d\theta_3^i &= \Sigma_j^i\wedge\theta_1^i + \lambda_j^i\wedge\theta_2^j + (\Omega_j^i - 2\alpha\delta_j^i)\wedge\theta_3^j + S_j^i\theta_3^j\wedge\omega \end{aligned}$$

where $\tilde{T}_j^i$ and S_j^i are functions, and $\tilde{T}_i^i = 0$. To determine the group action on the torsion we first compute $d^2\theta_1^i$ and find

$$d^2\theta_1^i = (d\Omega_j^i - \Omega_k^i\wedge\Omega_j^k - \omega\wedge\beta_j^i - \kappa_j\wedge\theta_2^i)\wedge\theta_1^j$$

giving

$$d\Omega_j^i \equiv \Omega_k^i\wedge\Omega_j^k + \omega\wedge\beta_j^i + \kappa_j\wedge\theta_2^i \qquad \mathrm{mod}(\theta_1^i\ )$$

Now take $d^2\theta_2^i$ mod(θ_1^i) ,

$$\begin{aligned} d^2\theta_2^i \equiv\ & \beta_k^i\wedge\theta_2^k\wedge\omega + (\Omega_k^i\wedge\Omega_j^k + \omega\wedge\beta_j^i + \kappa_j\wedge\theta_2^i)\wedge\theta_2^j - d\alpha\wedge\theta_2^i \\ -\ & (\Omega_k^i - \alpha\delta_k^i)\wedge(\Omega_j^k\wedge\theta_2^j - \alpha\wedge\theta_2^k - \theta_3^k\wedge\omega + \tilde{T}_j^k\theta_2^j\wedge\omega) \\ -\ & (\lambda_j^i\wedge\theta_2^j + \Omega_j^i\wedge\theta_3^j - 2\alpha\wedge\theta_3^i)\wedge\omega + \theta_3^i\wedge\alpha\wedge\omega \\ +\ & d\tilde{T}_j^i\wedge\theta_2^j\wedge\omega + \tilde{T}_k^i\Omega_j^k\wedge\theta_2^j\wedge\omega \qquad \mathrm{mod}(\theta_1^i) \end{aligned}$$

To continue we substitute into this equation $\beta^i_j = \tilde{\beta}^i_j + \beta\delta^i_j$ and $\lambda^i_j = \tilde{\lambda}^i_j + \lambda\delta^i_j$ where $\tilde{\beta}^i_j$ and $\tilde{\lambda}^i_j$ are trace-free which results in

$$\begin{aligned} &-[(d\tilde{T}^i_j - \Omega^i_k\tilde{T}^k_j + \tilde{T}^i_k\Omega^k_j + 2\tilde{\beta}^i_j - \tilde{\lambda}^i_j)\wedge\omega \\ &+(d\alpha + \kappa_k\wedge\theta^k_2 + 2\beta\wedge\omega - \lambda\wedge\omega)\delta^i_j]\wedge\theta^j_2 \qquad \equiv 0 \qquad \mathrm{mod}(\theta^l_1)\,. \end{aligned}$$

Taking the trace of this gives

$$d\alpha \equiv -\kappa_j\wedge\theta^j_2 - 2\beta\wedge\omega + \lambda\wedge\omega \qquad \mathrm{mod}(\theta^k_1) \tag{2.116}$$

while from the trace-free part we have,

$$d\tilde{T}^i_j - \Omega^i_k\tilde{T}^k_j + \tilde{T}^i_k\Omega^k_j + 2\tilde{\beta}^i_j - \tilde{\lambda}^i_j \equiv 0 \qquad \mathrm{mod(base)} \tag{2.117}$$

We now compute $d^2\theta^i_3$ $\mathrm{mod}(\theta^i_1, \theta^i_2)$ and use (2.116)

$$\begin{aligned} d^2\theta^i_3 \;\; &\equiv \;\; \tilde{\lambda}^i_j\wedge\theta^j_3\wedge\omega + \lambda\wedge\theta^i_3\wedge\omega + (\omega\wedge\tilde{\beta}^i_j + \omega\wedge\beta\delta^i_j + \Omega^i_k\wedge\Omega^k_j)\wedge\theta^j_3 \\ &+ \;\; 2(-2\beta + \lambda)\wedge\theta^i_3\wedge\omega + (\Omega^i_k - 2\alpha\delta^i_k)\wedge\theta^k_3\wedge\omega \\ &+ \;\; dS^i_j\wedge\theta^j_3\wedge\omega + S^i_j(\Omega^j_k\wedge\theta^k_3\wedge\omega - \alpha\wedge\theta^j_3\wedge\omega) \qquad \mathrm{mod}(\theta^l_1, \theta^m_2) \end{aligned}$$

where equation (2.116) has been used. From this we find

$$dS^i_j - \Omega^i_k S^k_j + S^i_k\Omega^k_j - \alpha S^i_j + \tilde{\lambda}^i_j + \tilde{\beta}^i_j + 3(\lambda - \beta)\delta^i_j \equiv 0 \qquad \mathrm{mod(base)} \tag{2.118}$$

Using equations (2.117) and (2.118) we may translate $\tilde{T}^i_j$ and S^i_j to zero. With this reduction we have

$$\lambda \equiv \beta \qquad \text{and} \qquad \beta^i_j \equiv \lambda^i_j \equiv \sigma\delta^i_j \quad \mathrm{mod(base)} \tag{2.119}$$

where σ is a right-invariant one-form on the reduced group. To continue with this reduction note that everything up to equations (2.115) stays the same (in the modified frame) while equations (2.115) become

$$\begin{aligned} d\theta^i_2 &= \sigma\wedge\theta^i_1 + (\Omega^i_j - \alpha\delta^i_j)\wedge\theta^j_2 - \theta^i_3\wedge\omega + S^i_j\theta^j_1\wedge\omega + T^{bi}{}_{jk}\theta^j_b\wedge\theta^k_1 + U^i_{jk}\theta^j_3\wedge\theta^k_1 \\ d\theta^i_3 &= \Sigma^i_j\theta^j_1 + \sigma\wedge\theta^i_2 + (\Omega^i_j - 2\alpha\delta^i_j)\wedge\theta^j_3 + R^i_j\theta^j_1\wedge\omega + V^i_j\theta^j_2\wedge\omega + U^{ci}{}_{jk}\theta^j_c\wedge\theta^k_2 \end{aligned}$$

In the first equation we may now absorb

$$\begin{aligned}
\alpha &= \widehat{\alpha} + \frac{1}{2(n-1)}(T^{2l}_{\ jl} - T^{2l}_{\ lj})\theta^j_1 \\
\Omega^i_j &= \widehat{\Omega}^i_j + T^{2i}_{\ (jk)}\theta^k_1 \\
\sigma &= \widehat{\sigma} + \frac{1}{n}S^l_l\omega - \frac{1}{2(n-1)}(T^{2l}_{\ jl} - T^{2l}_{\ lj})\theta^j_2 + \frac{2}{(n-1)}T^{1l}_{\ jl}\theta^j_1 - U^l_{lk}\theta^k_3
\end{aligned}$$

so that we have

$$d\theta^i_2 = \sigma\wedge\theta^i_1 + (\Omega^i_j - \alpha\delta^i_j)\wedge\theta^j_2 - \theta^i_3\wedge\omega + \tilde{S}^i_j\theta^j_1\wedge\omega + \hat{T}^{bi}_{\ jk}\theta^j_b\wedge\theta^k_1 + \widehat{U}^i_{jk}\theta^j_3\wedge\theta^k_1$$

where the functions satisfy,

$$\tilde{S}^l_l = \tilde{T}^i_{(jk)} = \tilde{T}^l_{lj} = \widehat{U}^l_{lj} = 0 \qquad (2.120)$$

By using Σ^i_j any terms with θ^i_1 in equation (2.120) for $d\theta^i_3$ can be absorbed so we have,

$$d\theta^i_3 = \Sigma^i_j\wedge\theta^j_1 + \sigma\wedge\theta^i_2 + (\Omega^i_j - 2\alpha\delta^i_j)\wedge\theta^j_3 + V^i_j\theta^j_2\wedge\omega + U^{2i}_{\ jk}\theta^j_2\wedge\theta^k_2 + U^{3i}_{\ jk}\theta^j_3\wedge\theta^k_2$$

We now compute the group action on part of the structure function by taking $d^2\theta^i_2 \equiv 0 \bmod(\theta^j_2)$,

$$\begin{aligned}
d^2\theta^i_2 \equiv\; & d\sigma\wedge\theta^i_1 - \sigma\wedge\Omega^i_j\wedge\theta^j - (\Sigma^i_j\wedge\theta^j_1 + (\Omega^i_j - 2\alpha\delta^i_j)\wedge\theta^j_3)\wedge\omega + \theta^i_3\wedge(\alpha\wedge\omega + \kappa_j\wedge\theta^j_1) \\
& -(\Omega^i_j - \alpha\delta^i_j + \tilde{T}^{2i}_{\ jk}\theta^k_1)\wedge(\sigma\wedge\theta^j_1 - \theta^j_3\wedge\omega + \tilde{S}^j_k\theta^k_1\wedge\omega + \tilde{T}^{cj}_{\ kl}\theta^k_c\wedge\theta^l_1 + \widehat{U}^j_{kl}\theta^k_3\wedge\theta^l_1) \\
& +(d\tilde{S}^i_j + \tilde{S}^i_k\Omega^k_j + \alpha\tilde{S}^i_j)\wedge\theta^j_1\wedge\omega - \tilde{S}^i_j\theta^j_1\wedge\kappa_k\wedge\theta^k_1 + (d\tilde{T}^{1i}_{\ jk} + 2\tilde{T}^{1i}_{\ lk}\Omega^l_j)\wedge\theta^j_1\wedge\theta^k_1 \\
& +d\widehat{U}^i_{jk}\wedge\theta^j_3\wedge\theta^k_1 + \widehat{U}^i_{lk}\left((\Sigma^l_j\wedge\theta^j_1 + (\Omega^l_j - 2\alpha\delta^l_j)\wedge\theta^j_3)\wedge\theta^k_1 - \theta^l_3\wedge\Omega^k_m\wedge\theta^m_1\right) \quad \bmod(\theta^j_2)
\end{aligned}$$

From this we may extract the following two properties,

$$\begin{aligned}
d\tilde{S}^i_j - \Omega^i_k\tilde{S}^k_j + \tilde{S}^i_k\Omega^k_j + 2\alpha\tilde{S}^i_j - (d\sigma)_\omega\delta^i_j - \Sigma^i_j &\equiv 0 \\
d\widehat{U}^i_{jk} - \Omega^i_l\tilde{U}^l_{jk} + \tilde{U}^i_{lk}\Omega^l_j + \tilde{U}^i_{jl}\Omega^l_k - \alpha\tilde{U}^i_{jk} + (d\sigma)_{\theta^j_3}\delta^i_k - \kappa_k\delta^i_j &\equiv 0\,.
\end{aligned} \qquad \bmod(\text{base})$$

By writing $\Sigma^i_j = \tilde{\Sigma}^i_j + \Sigma\delta^i_j$ and setting the trace and trace-free part of the first equation the trace-free on i,j part of the second equation leads to

$$\begin{aligned} -(d\sigma)_\omega &\equiv \Sigma \\ d\tilde{S}^i_j - \Omega^i_k\tilde{S}^k_j + \tilde{S}^i_k\Omega^k_j + 2\alpha\tilde{S}^i_j - \tilde{\Sigma}^i_j &\equiv 0 \quad \text{mod(base)} \\ d\hat{U}^i_{jk} - \Omega^i_l\hat{U}^l_{jk} + \hat{U}^i_{lk}\Omega^l_j + \hat{U}^i_{jl}\Omega^l_k - \alpha\hat{U}^i_{jk} + n\kappa_j\delta^i_k - \kappa_k\delta^i_j &\equiv 0 \end{aligned} \tag{2.121}$$

We may then use the group action to translate

$$\tilde{S}^i_j = 0 \qquad \text{and} \qquad \hat{U}^l_{jl} = 0 \tag{2.122}$$

where we note the translation on $\hat{U}^i_{jk}$ is the trace on the second index. Next we compute $d^2\theta^i_3$ mod(θ^j_1, θ^j_3)

$$\begin{aligned} d^2\theta^i_3 &\equiv (\tilde{\Sigma}^i_j + \Sigma\delta^i_j)\wedge\theta^j_2\wedge\omega + d\sigma\wedge\theta^i_2 - \sigma\wedge(\Omega^i_j - \alpha\delta^i_j)\wedge\theta^j_2 \\ &- (\Omega^i_j - 2\alpha\delta^i_j)\wedge(\sigma\wedge\theta^i_2 + V^i_j\theta^j_2\wedge\omega + U^{2i}_{jk}\theta^j_2\wedge\theta^k_2) + (dV^i_j + V^i_k\Omega^k_j)\wedge\theta^j_2\wedge\omega \\ &+ (dU^{2i}_{jk} + 2U^{2i}_{lk}(\Omega^l_j - \alpha\delta^l_j))\wedge\theta^j_2\wedge\theta^k_2 \qquad \text{mod}(\theta^j_1, \theta^j_3)\ . \end{aligned}$$

By using equation (2.121) in this we have

$$dV^i_j - \Omega^i_k V^k_j + V^i_k\Omega^k_j + 2\alpha V^i_j + \tilde{\Sigma}^i_j + 2\Sigma\delta^i_j \equiv 0 \qquad \text{mod(base)} \tag{2.123}$$

so we may translate the trace of V^i_j to zero. With the corresponding reduction of the structure group we have thus eliminated Σ^i_j and κ_j. With the same absorptions (using the new frame) we may then write the structure equations as

$$\begin{aligned} d\omega &= \alpha\wedge\omega + W^c{}_{jk}\theta^j_c\wedge\theta^k_1 \\ d\theta^i_1 &= \Omega^i_j\wedge\theta^j - \theta^i_2\wedge\omega \\ d\theta^i_2 &= \sigma\wedge\theta^i_1 + (\Omega^i_j - \alpha\delta^i_j)\wedge\theta^j_2 - \theta^i_3\wedge\omega + \tilde{T}^{ci}_{jk}\theta^j_c\wedge\theta^k_1 \\ d\theta^i_3 &= \sigma\wedge\theta^i_2 + (\Omega^i_j - 2\alpha\delta^i_j)\wedge\theta^j_3 + R^i_j\theta^j_1\wedge\omega + \tilde{V}^i_j\theta^j_2\wedge\omega + U^{bci}{}_{jk}\theta^j_b\wedge\theta^k_c \end{aligned} \tag{2.124}$$

where $W^c{}_{jk}$, $\tilde{T}^{ci}_{jk}$, $\tilde{V}^i_j$, R^i_j, and $U^{bci}{}_{jk}$ are functions with,

$$\tilde{T}^{ci}_{ik} = \tilde{V}^i_i = 0$$

If at this point we try to prolong these equations we obtain an $\{e\}$-structure. ■

Next we look at the case of maximal symmetry,

Theorem 2.5: *There exists a* **unique** *$\{e\}$-structure with a maximal dimensional symmetry (automorphism) group. For this $\{e\}$-structure the structure function vanishes, and a representative for the system of equations giving rise to this $\{e\}$ -structure is*

$$\frac{d^3x^i}{dt^3} = 0 \tag{2.125}$$

Proof: The assumption of maximal symmetry implies that the structure function (the torsion above) is constant. We will then demonstrate that the only possible constant values for the torsion are zero. The first implication for the structure function to be constant is

$$dW^{ci}_{\ \ jk} = d\tilde{T}^{ci}_{\ \ jk} = d\tilde{V}^i_j = dR^i_j = dU^{bci}_{\ \ \ jk} = 0$$

We set $d^2\omega = 0$ mod(ω) to find

$$d^2\omega = -\alpha\wedge(W^c_{\ jk}\theta^j_c\wedge\theta^k_1) + W^c_{\ jk}d\theta^j_c\wedge\theta^k_1 - W^c_{\ jk}\theta^j_c\wedge\Omega^k_l\wedge\theta^l_1 \quad \text{mod}(\ \omega\ )$$

where by equations (2.124) we then have

$$\begin{aligned} -c\,W^c_{\ jk}\alpha + W^c_{\ lk}\Omega^l_j + W^c_{\ jl}\Omega^l_k + W^{c+1}_{\ \ jk}\sigma &= 0 \qquad c = 1,2 \\ -3\,W^3_{\ jk}\alpha + W^3_{\ lk}\Omega^l_j + W^3_{\ jl}\Omega^l_k &= 0 \end{aligned} \qquad \text{mod(base)}$$

It now easily concluded by choosing the different values for c that

$$W^c_{\ jk} = 0 \qquad \text{and so} \qquad d\omega = \alpha\wedge\omega \tag{2.126}$$

$d^2\omega = 0$ then gives

$$d\alpha = \rho\wedge\omega \tag{2.127}$$

where ρ is a one-form (and $(\rho)_\omega = 0$ w.l.o.g). Now put $d^2\theta_1^i = 0$

$$\begin{aligned} d^2\theta_1^i &= d\Omega^i_j\wedge\theta_1^j - \Omega^i_k\wedge(\Omega^k_j\wedge\theta_1^j - \theta_2^k\wedge\omega) + \alpha\wedge\omega\wedge\theta_2^i \\ & -\omega\wedge\left(\sigma\wedge\theta_1^i + (\Omega^i_j - \alpha\delta^i_j)\wedge\theta_2^j - \theta_3^i\wedge\omega + \tilde{T}^{ci}{}_{jk}\theta_c^j\wedge\theta_1^k\right) \end{aligned}$$

so that

$$d\Omega^i_j = \Omega^i_k\wedge\Omega^k_j + \omega\wedge\sigma\delta^i_j + \tilde{T}^{ci}{}_{kj}\omega\wedge\theta_c^j + \tau^i_{jk}\wedge\theta_1^k + S^i_{jkl}\theta_1^k\wedge\theta_1^l \tag{2.128}$$

where τ^i_{jk} are one forms and S^i_{jkl} are functions subject to

$$\tau^i_{[jk]} = 0 \qquad S^i_{[jkl]} = S^i_{j(kl)} = 0 \tag{2.129}$$

Next compute $d^2\theta_2^i$

$$\begin{aligned} & d^2\theta_2^i = d\sigma\wedge\theta_1^i - \sigma\wedge(\Omega^i_j\wedge\theta_1^j - \theta_2^i\wedge\omega) - \rho\wedge\omega\wedge\theta_2^i \\ & +(\Omega^i_k\wedge\Omega^k_j + \omega\wedge\sigma\delta^i_j + \tilde{T}^{ci}{}_{kj}\omega\wedge\theta_c^k + \tau^i_{jk}\wedge\theta_1^k + S^i_{jkl}\theta_1^k\wedge\theta_1^l)\wedge\theta_2^j \\ & -(\Omega^i_j - \alpha\delta^i_j)\wedge\left(\sigma\wedge\theta_1^j + (\Omega^j_k - \alpha\delta^j_k)\wedge\theta_2^k - \theta_3^j\wedge\omega + \tilde{T}^{cj}{}_{kl}\theta_c^k\wedge\theta_1^l\right) \\ & -(\sigma\wedge\theta_2^i + (\Omega^i_j - 2\alpha\delta^i_j)\wedge\theta_3^j + U^{bci}{}_{jk}\theta_b^j\wedge\theta_c^k)\wedge\omega + \theta_3^i\wedge\alpha\wedge\omega \\ & +\tilde{T}^{ci}{}_{lm}\left((1-c)\alpha\delta^l_j\delta^m_k + \Omega^l_j\delta^m_k + \delta^l_j\Omega^m_k\right)\wedge\theta_c^j\wedge\theta_1^k + (1-\delta^c_3)\tilde{T}^{c+1i}{}_{jk}\sigma\wedge\theta_b^j\wedge\theta_1^k \\ & +\tilde{T}^{1i}{}_{jk}\theta_1^k\wedge\theta_2^j\wedge\omega + \tilde{T}^{2i}_{lk}(\tilde{T}^{cl}{}_{jm}\theta_c^j\wedge\theta_1^m - \theta_3^l\wedge\omega)\wedge\theta_1^k \\ & +\tilde{T}^{3i}_{lk}(\tilde{V}^l_j\theta_2^j\wedge\omega + R^l_j\theta_1^j\wedge\omega + U^{bc}{}_{jm}\theta_b^j\wedge\theta_c^m)\wedge\theta_1^k + \theta_3^i\wedge\alpha\wedge\omega - \tilde{T}^{ci}{}_{jk}\theta_c^j\wedge\theta_2^k\wedge\omega\,. \end{aligned} \tag{2.130}$$

If we now let

$$\xi = d\sigma - \sigma\wedge\alpha \qquad \eta = \rho + \sigma \tag{2.131}$$

we then have

$$\begin{aligned} \tilde{T}^{1i}{}_{lm}(\alpha\delta^l_j\delta^m_k + 4\Omega^l_{[j}\delta^m_{k]}) - \Omega^i_l\tilde{T}^{1l}{}_{jk} + \tilde{T}^{2i}{}_{jk}\sigma + (\xi)_{\theta_1^{[j}}\delta^i_{k]} &\equiv 0 \\ \tilde{T}^{2i}_{lm}(\delta^m_k\Omega^l_j + \delta^l_j\Omega^m_k) - \Omega^i_l\tilde{T}^{1l}_{jk} + \tilde{T}^{3i}_{jk}\sigma + (\xi)_{\theta_2^j}\delta^i_k + \tau^i_{jk} &\equiv 0 \qquad \text{mod(base)} \\ \tilde{T}^{3i}_{lm}(-\alpha\delta^l_j\delta^m_k + \delta^m_k\Omega^l_j + \delta^l_j\Omega^m_k) - \Omega^i_l\tilde{T}^{1l}_{jk} + (\xi)_{\theta_3^j}\delta^i_k &\equiv 0 \end{aligned} \tag{2.132}$$

where in the first equation we have used that $T^{2i}{}_{(jk)} = 0$. Taking the trace on the first and third of these equations we get

$$(\xi)_{\theta^i_1} = (\xi)_{\theta^i_3} = 0$$

from which we conclude

$$\tilde{T}^{ci}{}_{jk} = 0 \ .$$

By the condition $\tau^i_{[jk]} = 0$ the second equation in (2.132) gives

$$(\xi)_{\theta^i_1} \equiv \tau^i_{jk} \equiv 0 \qquad \mathrm{mod(base)} \tag{2.133}$$

We have that equation (2.130) at this point is

$$\xi \wedge \theta^i_1 - \eta \wedge \omega \wedge \theta^i_2 + (\tau^i_{jk} \wedge \theta^k_1 + S^i_{jkl}\theta^k_1 \wedge \theta^l_1) \wedge \theta^j_2 - U^{bci}{}_{jk}\theta^j_b \wedge \theta^k_c \wedge \omega = 0 \tag{2.134}$$

from which we find

$$(\xi)_\omega \equiv \eta \equiv 0 \qquad \mathrm{mod(base)} \tag{2.135}$$

and this with (2.133) gives

$$\xi \equiv 0 \qquad \mathrm{mod(base)} \tag{2.136}$$

We now take $d^2\theta^i_3$

$$\begin{aligned}
d^2\theta^i_3 \ &= (\sigma \wedge \alpha + \xi) \wedge \theta^i_2 - \sigma \wedge ((\Omega^i_j - \alpha\delta^i_j) \wedge \theta^j_1 - \theta^i_3 \wedge \omega) \\
&+ \ (\Omega^i_k \wedge \Omega^k_j + \omega \wedge \sigma\delta^i_j + \tau^i_{jk} \wedge \theta^k_1 + S^i_{jkl}\theta^k_1 \wedge \theta^l_1) \wedge \theta^j_3 - 2(-\sigma + \eta) \wedge \omega \wedge \theta^i_3 \\
&- \ (\Omega^i_k - 2\alpha\delta^i_k) \wedge \left(\sigma \wedge \theta^k_2 + (\Omega^k_j - 2\alpha\delta^k_j) \wedge \theta^j_3 + R^k_j\theta^j_1 \wedge \omega + \tilde{V}^k_j\theta^j_2 \wedge \omega + U^{bck}{}_{jl}\theta^j_b \wedge \theta^l_c\right) \\
&+ \ R^i_k(\Omega^k_j \wedge \theta^j_1 - \theta^k_1 \wedge \alpha) \wedge \omega + \tilde{V}^i_j(\sigma \wedge \theta^j_1 + (\Omega^j_k - \alpha\delta^j_k) \wedge \theta^k_2 - \theta^j_2 \wedge \alpha) \wedge \omega \\
&+ \ U^{bci}{}_{lm}(\delta^l_j\delta^m_k(2 - b - c)\alpha + \delta^m_k\Omega^l_j + \delta^l_j\Omega^m_k) \wedge \theta^j_b \wedge \theta^k_c \\
&+ \ (V^{b+1bi}{}_{jk} + U^{bc+1i}{}_{jk})\sigma \wedge \theta^j_b \wedge \theta^k_c
\end{aligned}$$

We then find

$$
\begin{aligned}
\left(d^2\theta^i_3\right)_{\alpha\theta^j_1\omega} &= 3R^i_j = 0 \\
\left(d^2\theta^i_3\right)_{\alpha\theta^j_2\omega} &= 2\tilde{U}^i_j = 0 \\
\left(d^2\theta^i_3\right)_{\alpha\theta^j_b\theta^k_c} &= (4 - b - c)U^{bci}{}_{jk} = 0
\end{aligned}
$$

and so

$$
d\theta^i_3 = \sigma\wedge\theta^i_2 + (\Omega^i_j - 2\alpha\delta^i_j)\wedge\theta^j_3 + U^{13i}{}_{jk}\theta^j_1\wedge\theta^k_3 + U^{22i}{}_{jk}\theta^j_2\wedge\theta^k_2.
$$

Now use (2.133) and write

$$
\tau^i_{jk} = W^i_{jk}\omega + X^i_{jlk}\theta^l_2 + Y^i_{jlk}\theta^l_3 \quad .
$$

where we are assuming that $W^i_{jk}, X^i_{jlk}, Y^i_{jlk}$ are constant by maximal symmetry. We then compute $d^2\Omega^i_j = 0$

$$
\begin{aligned}
d^2\Omega^i_j &= d\Omega^i_k\wedge\Omega^k_j - \Omega^i_k\wedge d\Omega^k_j + d\omega\wedge\sigma\delta^i_j - \omega\wedge(\sigma\wedge\alpha + \xi\delta^i_j) \\
&+ W^i_{jk}\alpha\wedge\omega\wedge\theta^k_1 + X^i_{jlk}\left(\sigma\wedge\theta^l_1 + \Omega^l_m\wedge\theta^m_2 - \alpha\wedge\theta^l_2\right)\wedge\theta^k_1 \\
&+ Y^i_{jlk}\left(\sigma\wedge\theta^l_2 + \Omega^l_m\wedge\theta^m_3 - 2\alpha\wedge\theta^l_3 + U^{13l}{}_{mr}\theta^m_1\wedge\theta^r_3 + U^{22l}{}_{mr}\theta^m_2\wedge\theta^r_2\right)\wedge\theta^k_1 \\
&- (W^i_{jk}\omega + 2S^i_{jlk}\theta^l_1 + X^i_{jlk}\theta^l_2 + Y^i_{jlk}\theta^l_3)\wedge d\theta^k_1
\end{aligned}
$$

so that

$$
\begin{aligned}
\left(d^2\Omega^i_j\right)_{\alpha\omega\theta^k_1} &= W^i_{jk} = 0 \\
\left(d^2\Omega^i_j\right)_{\alpha\theta^k_1\theta^l_2} &= X^i_{jlk} = 0 \\
\left(d^2\Omega^i_j\right)_{\alpha\theta^k_1\theta^l_3} &= 2Y^i_{jlk} = 0
\end{aligned}
$$

Thus $\tau^i_{jk} = 0$. Similarly we may write

$$
\eta = W_j\theta^j_1 + X_j\theta^j_2
$$

where W_j, X_j are constants by the assumption of maximal symmetry, and by using equation (2.134) there are no θ_3^j terms. Now take

$$d^2\alpha = \sigma\wedge\alpha\wedge\omega - (\sigma\wedge\alpha + \xi)\wedge\omega + W_j(d\theta_1^j - \theta_1^j\wedge\alpha)\wedge\omega + X_j(\sigma\wedge\theta_1^j + \Omega_k^j\wedge\theta_2^k)\wedge\omega \qquad (2.137)$$

and putting the following coefficients to zero we have

$$\left(d^2\alpha\right)_{\alpha\theta_1^j\omega} = W_j = 0$$
$$\left(d^2\alpha\right)_{\sigma\theta_1^j\omega} = X_j = 0 \qquad (2.138)$$

and so $\eta = 0$. Equation (2.137) now tells us that

$$\xi\wedge\omega = 0 \qquad (2.139)$$

so that when we wedge equation (2.134) with ω we have $S^i_{jk} = 0$. At this point equation (2.134) is

$$\xi\wedge\theta_1^i - U^{13i}_{jk}\theta_1^j\wedge\theta_3^k\wedge\omega = 0 - U^{22i}_{jk}\theta_2^j\wedge\theta_2^k\wedge\omega = 0$$

and by wedging with θ_1^i gives $U^{13i}_{jk} = U^{22i}_{jk} = 0$. This of course implies $\xi = 0$. We have thus determined that the structure function must be zero. It is shown in [19] that the trivial system of third order equations

$$\frac{d^3x^i}{dt^3} = 0$$

admits a symmetry group of dimension $n^2 + 3n + 3$ and so these equations generate the $\{e\}$-structure with maximal symmetry. ■

From this theorem we obtain,

Corollary 2.2: *All systems of third order differential not equivalent by a point transformation to* $\dddot{x}^i = 0$ *admit a symmetry group of dimension strictly less than* $n^2 + 3n + 3$.

For completeness we give,

Corollary 2.3: *The {e}-structure with maximal symmetry has the structure equations,*

$$\begin{aligned}
d\omega &= \alpha\wedge\omega \\
d\theta^i_1 &= \Omega^i_j\wedge\theta^j - \theta^i_2\wedge\omega \\
d\theta^i_2 &= \sigma\wedge\theta^i_1 + (\Omega^i_j - \alpha\delta^i_j)\wedge\theta^j_2 - \theta^i_3\wedge\omega \\
d\theta^i_3 &= \sigma\wedge\theta^i_2 + (\Omega^i_j - 2\alpha\delta^i_j)\wedge\theta^j_3 \\
d\alpha &= -\sigma\wedge\omega \\
d\Omega^i_j &= \Omega^i_k\wedge\Omega^k_j + \omega\wedge\sigma\delta^i_j \\
d\sigma &= \sigma\wedge\alpha
\end{aligned}$$

Chapter 3

Parabolic Equations

3.1 Introduction

In this Chapter we would like to apply the Cartan equivalence method to study second order quasi-linear parabolic partial differential equations in the plane under the pseudogroup of point transformations. This study is to be contrasted with the one undertaken by Kamran and Shadwick [27], and Kamran [24] for quasi-linear hyperbolic and elliptic equations in the plane, where necessary and sufficient conditions were given for a quasi-linear **non-parabolic** second order partial differential equation to be equivalent to certain types of f-Gordon equations, with emphasis on equations admitting infinite Lie pseudogroups of symmetries.

Following [17], a partial differential equations of second order in one dependent and two independent variables

$$\mathbf{F}(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}) = 0 \tag{3.1}$$

defines a locus in the space $J^2(\mathbb{R}^2, \mathbb{R})$ given by

$$\mathcal{L} = \Big\{(x, y, z, p, q, r, s, t) \in J^2(\mathbb{R}^2, \mathbb{R}) \mid \mathbf{F}(x, y, z, p, q, r, s, t) = 0\Big\}$$

where (x, y, z, p, q, r, s, t) are standard coordinates for $J^2(\mathbb{R}^2, \mathbb{R})$. We assume that $\mathcal{L}$

can be identified locally with an imbedded 7-dimensional manifold $\Sigma \subset \mathbb{R}^7$ and that the function satisfies the non-degeneracy condition

$$(\frac{\partial \mathbf{F}}{\partial r}, \frac{\partial \mathbf{F}}{\partial s}, \frac{\partial \mathbf{F}}{\partial t}) \neq (0,0,0) \ . \tag{3.2}$$

so the equation is truy second order. A solution to (3.1) is then a function

$$l : U \to \mathbb{R} \quad \text{satisfying} \quad j^2 l \in \Sigma \ .$$

For a recent analysis of the role of characteristics in geometry of second order hyperbolic equations see [17].

We will be interested in the parabolic Monge-Ampère equations that is equations of the form,

$$\mathbf{F}(x,y,z,p,q,r,s,t) = e\,(rt - s^2) + g\,r + 2b\,s + k\,t - f = 0 \tag{3.3}$$

where b, e, f, g, k are smooth functions of (x,y,z,p,q) satisfying,

$$e\,f + g\,k - b^2 = 0 \ . \tag{3.4}$$

We shall see in the next section that these equations enjoy the property that they can be cast into an exterior differential system on $J^1(\mathbb{R}^2, \mathbb{R})$. The original investigations on the equivalence of a restricted class of parabolic Monge-Ampère equations under contact transformations were made in a famous paper of Cartan [12]. More recently these equations have been considered from the point of view of conservation laws by Bryant and Griffiths [6]. In contrast to these works we consider the problem of equivalence under smooth invertible point transformations,

$$(\overline{x}, \overline{y}, \overline{z}) = \Psi(x,y,z) \ .$$

We proceed first in the next section by giving the differential geometric framework for the Monge-Ampère equations which will lead to the equivalence problem. We then proceed to study some particular invariant classes of (3.3) which culminates in determining the invariant classification of the heat equation and by using Theorem 1.10, an invariant coframe for Burgers' equation,

$$r - q - z\,p = 0\ . \tag{3.5}$$

The symmetry properties of this equation are well known and discussed in Olver [33] and Bluman and Kumei [3]. By using the equivalence method we determine an invariant frame associated with (3.5) which determines the Lie algebra of the symmetry group all without solving any differential equations. This is in contrast with what is required in the standard infinitesimal approach in [3] and [33].

3.2 Parabolic Monge-Ampère Equations

We begin by briefly discussing some of the geometry of what are known as Monge-Ampère structures. The flavour of this discussion follows the exposition of Bryant and Griffiths [6]. To define Monge-Ampère structure we first use Bryant et. al.[5] or Kobayashi [30] in defining

Definition 3.1: *A contact manifold* $(\mathbf{M}^{2k+1}, \mathcal{I}_0)$ *is an odd-dimensional manifold with a Pfaffian system* $\mathcal{I}_0$ *which is locally generated by a one-form* θ, *with* $d\theta$ *being of rank* $2k$.

A locally generated differential system means in this case for $\mathbf{M}^{2k+1}$ there exist a cover $\{U_\alpha, \phi_\alpha\}$ of $\mathbf{M}$, a collection of one-forms θ_α, and a collection of smooth functions $\lambda_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{R}^*$ such that

$$(\phi_{\alpha\beta,\alpha})^*\theta_\alpha = \lambda_{\alpha\beta}\,(\phi_{\alpha\beta,\beta})^*\theta_\alpha$$

where $(\phi_{\alpha\beta,\alpha}) : U_{\alpha\beta} \to U_\alpha$ and $(\phi_{\alpha\beta,\alpha}) : U_{\alpha\beta} \to U_\beta$ are the canonical injections. This notion extends to arbitrary differential systems in an obvious way. If the manifold is orientable the contact structure is generated by a global one-form [30].

For the contact manifold M^{2k+1} the two-form $d\theta$ defines a conformal symplectic structure on $\mathcal{E} = ker(\theta) \subset T(\mathrm{M})$ (the structure group of the frame bundle of $\mathcal{E}$ is reducible to $csp(k)$) and $\mathcal{E}^* = T^*(\mathrm{M})\ mod\ \theta = T^*(\mathrm{M})/\mathcal{I}_0$ so

$$d\theta : \mathcal{E} \to \mathcal{E}^*$$

is an isomorphism. The forms θ and $d\theta$ define a reduction $\mathcal{P}_G \subset \mathcal{F}^*(\mathrm{M})$, as in Chapter 1 Section 1.1, to

$$H = \left\{ \begin{pmatrix} c & 0 \\ v & T \end{pmatrix} \quad , \ c \in \mathbb{R} - \{0\} \, , \ T \in \mathrm{csp}(k) \, , \ v \in \mathbb{R}^{2k} \right\}$$

where T correspond to the conformal symplectic structure on $\mathcal{E}$ (and $\mathcal{E}^*$). What this reduction corresponds locally is that a local coframe $(\theta, \omega^i, \eta^i)$ lies in $\mathcal{P}_G$ if and only if

$$d\theta = \lambda \sum_{i=1}^{n} \omega^i \wedge \eta^i \quad \mathrm{mod}(\theta) \qquad \lambda \in C^\infty(\mathrm{M}) \, . \tag{3.6}$$

A diffeomorphism $\phi : (\mathrm{M}, \mathcal{I}_0) \to (\overline{\mathrm{M}}, \overline{\mathcal{I}}_0)$ satisfying the $\phi^* \overline{\mathcal{I}}_0 = \mathcal{I}_0$, is called a **contact transformation.** Locally ϕ satisfies $\phi^* \overline{\theta} = \lambda \, \theta$ where $\lambda \in C^\infty(\mathrm{M})$.

The basic result about the local structure of contact manifolds is the following [30]

Lemma 3.1: *Let $(\mathrm{M}^{2k+1}, \theta)$ be a contact manifold. About each point $p \in \mathbf{M}$ there exists an open set U with local coordinates (z, x^i, p^i) such that*

$$\theta = dz - \sum_{i=1}^{n} p^i \, dx^i \, . \tag{3.7}$$

We call a coframe $(\theta, \omega^i, \eta^i)$ on U admissible if it satisfies (3.6) and coordinates (z, x^i, p^i) as in Lemma 3.1 **standard coordinates** on U. The frame (θ, dx^i, dp^i) is admissible.

Now we restrict our attention to 5-dimensional contact manifolds. We first point out the following interesting geometric property of Monge-Ampère equations which distinguishes them among all second order partial differential equations,

Lemma 3.2: *Solutions to equation* (3.3) *are in one-to-one correspondence with two-dimensional integral manifolds* $L : U \longrightarrow J^1(\mathbb{R}^2, \mathbb{R})$ *of the differential system on* $(J^1(\mathbb{R}^2, \mathbb{R}), \theta)$ *generated by the two-form*

$$\kappa = \begin{pmatrix} dx, dy, dp, dq \end{pmatrix} \wedge K \, {}^t\begin{pmatrix} dx, dy, dp, dq \end{pmatrix}$$

$$\text{where} \quad K = \begin{pmatrix} F & H \\ -{}^tH^i_j & E^i_j \end{pmatrix}$$

$$E^i_j = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix}, \; F^i_j = \begin{pmatrix} 0 & -f \\ f & 0 \end{pmatrix}, \; H^i_j = \begin{pmatrix} b & k \\ -g & -b \end{pmatrix}$$

and the contact one-form $\theta = dz - pdx - qdy$ *which satisfy the independence condition* $L^*(dx \wedge dy) \neq 0$.

This property property leads to the general geometric definition,

Definition 3.2: *A Monge-Ampère structure on* $\mathbf{M}^5$ *is an exterior differential system* $\mathcal{I}$ *which is locally generated by* θ *and a two-form* κ, *where* κ *and* $d\theta$ *are linearly independent.*

The rank of a Monge-Ampère structure is defined to be the rank of κ (as a skew-symmetric form) and this invariant can be either 2 or 4. We assume the rank of κ to be constant on $\mathbf{M}$. (These definitions could of course be easily extended to manifolds of higher dimension.)

Definition 3.3: *If the rank of the Monge-Ampère structure is 2 then it is said to be* **parabolic** .

Locally the rank two condition is equivalent to the condition

$$K \neq 0 \qquad det(K) = (e\,f + g\,k - b^2)^2 = 0 \ . \tag{3.8}$$

A Monge-Ampère equation satisfying these conditions would be parabolic by the standard definitions for second order equations, see [17]. It is now easy to define the equivalence of Monge-Ampère equations by using the differential system $\mathcal{I}$ generated by κ and θ,

Definition 3.4: *Two Monge-Ampère structures* $(\mathbf{M}^5, \mathcal{I})$ *and* $(\overline{\mathbf{M}}^5, \overline{\mathcal{I}})$ *are equivalent if and only if there exists a contact transformation* $\Psi : \mathbf{M} \to \overline{\mathbf{M}}$ *such that* $\phi^* \overline{\mathcal{I}} = \mathcal{I}$. *Locally this is the condition,*

$$\Psi^* \overline{\kappa} = \lambda\,\kappa \qquad \mathrm{mod}(\theta) \qquad \text{where } \lambda \in C^\infty(\mathbf{M}) \ . \tag{3.9}$$

Our discussion up until now has been given rather generally in terms of contact transformations, however out main goal is to study local equivalence of Monge-Ampère equations under point transformations. Thus we assume $\mathbf{M}^5$ is now an open set $U \subset J^1(\mathbb{R}^2, \mathbb{R})$ with the standard coordinates on $J^1(\mathbb{R}^2, \mathbb{R})$. We then have,

Lemma 3.3: *Let* (θ, dx, dy, dp, dq) *and* $(\overline{\theta}, d\overline{x}, d\overline{y}, d\overline{p}, d\overline{q})$ *be as in Lemma* 3.1, *then a contact transformation* $(\overline{x}, \overline{y}, \overline{z}, \overline{p}, \overline{q}) = \Psi(x, y, z, p, q)$ *is the first prolongation of a point transformation* $(\overline{x}, \overline{y}, \overline{z}) = \Psi(x, y, z)$ *if and only if*

$$\Psi_1^* \begin{pmatrix} \overline{\theta} \\ d\overline{x} \\ d\overline{y} \\ d\overline{p} \\ d\overline{q} \end{pmatrix} = S \begin{pmatrix} \theta \\ dx \\ dy \\ dp \\ dq \end{pmatrix}$$

where $S : U \to \mathrm{G}$ where S is a smooth function taking values in the Lie subgroup G *of* $GL(5, \mathbb{R})$ *defined by*

$$G = \left\{ \begin{pmatrix} c & 0 & 0 \\ B^i & A^i_j & 0 \\ ({}^tA^{-1})^i_k \mathrm{E}^k & ({}^tA^{-1})^i_k D^k_j & c({}^tA^{-1})^i_j \end{pmatrix} \begin{array}{c} c \in \mathbb{R}\,,\ A^i_j \in GL(n, \mathbb{R}) \\ \mathrm{E}^i, B^i \in \mathbb{R}^2 \\ D^i_j \in M_{2\times 2} \text{ (symmetric)} \end{array} \right\} . \quad (3.10)$$

This can be verified by a calculation similar to appendix A or see [24]. Now given a contact structure κ we have the local invariant under point transformations:

Lemma 3.4: *The condition $e = 0$ is invariant under point transformations.*

Proof : After a point transformation the coefficient matrix K of κ becomes

$$\widehat{K} = \begin{pmatrix} {}^tA^i_j & (A^{-1})^i_k D^k_j \\ 0 & a(A^{-1})^i_j \end{pmatrix} \begin{pmatrix} F & H \\ -({}^tH) & E \end{pmatrix} \begin{pmatrix} A^i_j & 0 \\ ({}^tA^{-1})^i_k D^k_j & c({}^tA^{-1})^i_j \end{pmatrix}$$

so that

$$\widehat{\mathrm{E}}^i_j = c^2 (A^{-1})^i_k \mathrm{E}^k_l ({}^tA^{-1})^l_j$$

From this, the invariance property $e = 0$ follows immediately. ∎

Equations that satisfy this invariant condition are just the classical **quasi-linear** second order equations. This Lemma simply expresses the fact that the property of being quasi-linear is invariant under point transformation. It is important to note the condition $e = 0$ is **not** an invariant under arbitrary contact transformations see [15] pg. 295.

We continue now studying equation (3.3) with $e = 0$ and $b^2 - g\,k = 0$. By the non-degeneracy $\kappa \neq 0$ (which is the same as (3.2)) we may assume without loss of generality that $g \neq 0$ and dividing (3.3) by g we have that any parabolic quasi-linear equation can be put in the following form

$$\mathbf{F} = r + 2h\,s + h^2\,t - f = 0 \quad (3.11)$$

where $h(x,y,z,p,q) = k\,g^{-1}$ from (3.3). Suppose now that $(U,\widehat{\theta},\widehat{\kappa})$ and $(V,\widehat{\bar{\theta}},\widehat{\bar{\kappa}})$ are local Monge-Ampère structures with

$$K = \begin{pmatrix} 0 & -f & 2h & h^2 \\ f & 0 & -1 & -2h \\ -2h & 1 & 0 & 0 \\ -h^2 & 2h & 0 & 0 \end{pmatrix}$$

and corresponding $\bar{K}$ giving rise to quasi-linear parabolic equations of the form (3.11) above. By changing to the (admissible) coframe

$$\begin{pmatrix} \widehat{\omega}^1 \\ \widehat{\omega}^2 \\ \widehat{\eta}^1 \\ \widehat{\eta}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -h & 1 & 0 & 0 \\ -f & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dp \\ dq \end{pmatrix} \tag{3.12}$$

we find $\widehat{\kappa} = \widehat{\eta}^1 \wedge \widehat{\omega}^2$. Performing now the same change of coframe in the V system we have

Lemma 3.5: *Two parabolic quasi-linear equations are equivalent by a point transformation $(\bar{x},\bar{y},\bar{z}) = \Psi(x,y,z)$ if and only if*

$$\Psi^* \begin{pmatrix} \widehat{\bar{\theta}} \\ \widehat{\bar{\omega}}^i \\ \widehat{\bar{\eta}}^i \end{pmatrix} = S \begin{pmatrix} \widehat{\theta} \\ \widehat{\omega}^j \\ \widehat{\eta}^j \end{pmatrix}$$

where $S : U \to \mathrm{H}'$, with H' being the subgroup of the Lie group in Lemma 3.3 with

$$A^i_j = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \quad \text{and} \quad D^i_j = \begin{pmatrix} 0 & d_1 \\ d_1 & d_2 \end{pmatrix} \tag{3.13}$$

Proof The frames $(\widehat{\theta},\widehat{\omega}^i,\widehat{\eta}^i)$ and $(\widehat{\widehat{\theta}},\widehat{\widehat{\omega}}^i,\widehat{\widehat{\eta}}^i)$ being admissible means we need to only impose the condition (3.9) in Definition 3.4. Writing

$$\widehat{\widehat{\kappa}} = \widehat{\widehat{\omega}}_i \wedge Q^i_j \widehat{\widehat{\eta}}^j \quad \text{where} \quad Q^i_j = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

we have

$$\Psi_1^* \widehat{\widehat{\kappa}} = \widehat{\omega}_i \wedge {}^tA^i_k Q^k_j {}^t(A^{-1})^j_l \left(a\,\widehat{\eta}^l + D^l_m \widehat{\omega}^m\right) \qquad \text{mod}(\widehat{\theta})\ .$$

and the equivalence condition then requires,

$${}^tA^i_k Q^k_j {}^t(A^{-1})^j_l = \lambda\, Q^i_j \quad {}^tA^i_k Q^k_j {}^t(A^{-1})^j_l D^l_m = 0 \quad \lambda \in C^\infty(U \times \mathrm{G}) \qquad (3.14)$$

with λ nowhere vanishing on $U \times \mathrm{G}$. Explicitly we have

$${}^tA^i_k Q^k_j {}^t(A^{-1})^j_l = (det A)^{-1} \begin{pmatrix} -A^2_1 A^2_2 & (A^2_1)^2 \\ -(A^2_2)^2 & A^2_1 A^2_2 \end{pmatrix}$$

which by first condition in (3.14) we have $A^2_1 = 0$ giving the form for A^i_j in (3.13). The structure of D^i_j follows by a similar computation. ■

We now proceed to study the equivalence problem with the structure group in this last lemma.

3.3 Parabolic Quasi-Linear Equations

In order to apply the equivalence method we first need to compute the Lie algebra valued Maurer-Cartan form for H'. From Lemma 3.5, we find

$$(dS)(S^{-1}) = \begin{pmatrix} \sigma & 0 & 0 \\ \beta^i & \Omega^i_j & 0 \\ \Upsilon^i & \Sigma^i_j & \sigma\,\delta^i_j - {}^t\Omega^i_j \end{pmatrix}$$

where,

$$\mathcal{S}^{-1} = \begin{pmatrix} c^{-1} & 0 & 0 \\ -c^{-1}(A^{-1})^i_j B^j & (A^{-1})^i_j & 0 \\ -c^{-2}(D^i_k (A^{-1})^k_j B^j - E^i) & -c^{-1} D^i_k (A^{-1})^k_j & c^{-1}\, {}^t A^i_j \end{pmatrix}$$

and

$$\Omega^i_j = dA^i_k\,(A^{-1})^k_j \quad \beta^j = \tfrac{1}{c}(dB^j - \Omega^j_k B^k) \quad \sigma = \tfrac{1}{c}dc$$
$$\Sigma^i_j = ({}^t A^{-1})^i_k (dD^k_l - \sigma D^k_l)(A^{-1})^l_j \quad \Upsilon^i = -\tfrac{1}{c}\left(\Sigma^i_k B^k + (A^{-1})^i_k (dE^k - \sigma E^k)\right) \tag{3.15}$$

The form of A^i_j and D^i_j in Lemma 3.5 tell us that Ω^i_j and Σ^i_j may be written

$$\Omega^i_j = \begin{pmatrix} \Omega^1_1 & \Omega^1_2 \\ 0 & \Omega^2_2 \end{pmatrix} \quad , \quad \Sigma^i_j = \begin{pmatrix} 0 & \Sigma_1 \\ \Sigma_1 & \Sigma_2 \end{pmatrix}$$

Next we would like to apply the equivalence method to the lifted coframe

$$\mathcal{S}\,{}^t(\theta, \widehat{\omega}^1, \widehat{\omega}^2, \widehat{\eta}^1, \widehat{\eta}^2)) \tag{3.16}$$

where the hatted forms given in (3.12) and the structure group of Lemma 3.5. The task of examining all possible invariantly defined branches which may arise would be extremely lengthy and not particularly useful in view of the large number of subcases which would need to be considered. While we will start by applying the equivalence method in as much generality as possible we will inevitably be lead to making choices for the branches we pursue. Thus we will chose our branches so as to obtain characterizations of of Burgers' equation the heat equation and some others, which satisfy our invariant assumptions. It will be seen that even with our assumptions the computations are rather extensive. If we start with the coframe in (3.12) which is associated to the quasi-linear equation (3.11) we may write the structure equations in the general form

$$\begin{aligned} d\theta &= \sigma\wedge\theta + \omega_j\wedge\eta^j + g_j\,\omega^j\wedge\theta + h_j\,\eta^j\wedge\theta \\ d\omega^i &= \beta^i\wedge\theta + \Omega^i_j\wedge\omega^j + R^i_j\omega^j\wedge\theta + S^i_j\eta^j\wedge\theta + T^i_{jk}\omega^j\wedge\eta^k + s^i\,\omega^1\wedge\omega^2 \\ d\eta^i &= \Upsilon^i\wedge\theta + \Sigma^i_j\wedge\omega^j + (\sigma\delta^i_j - {}^t\Omega^i_j)\wedge\eta^j + V^i_j\omega^j\wedge\theta + W^i_j\eta^j\wedge\theta \\ &\quad + Y^i_{jk}\omega^j\wedge\eta^k + k^i\,\omega^1\wedge\omega^2 + l^i\,\eta^1\wedge\eta^2 \end{aligned} \tag{3.17}$$

where g, h, k, l, s, R, S, T are functions on $U \times \mathrm{H}$. We now apply Cartan's method of equivalence to this problem. First we absorb torsion by

$$\begin{aligned} &\sigma = \widehat{\sigma} - g_j\omega^j - h_j\eta^j & &\beta^i = \widehat{\beta}^i - R^i_j\omega^j - S^i_j\eta^j \\ &\Sigma_i = \widehat{\Sigma}_i - k_i\omega^1 + (Y^i_{2k} - g_2\delta^i_j)\eta^k & &\Upsilon^i = \widehat{\Upsilon}^i - V^i_j\omega^j - W^i_j\eta^j \end{aligned} \tag{3.18}$$

$$\Omega^i_j = \widehat{\Omega}^i_j + T^i_{jk}\eta^k - \begin{pmatrix} g_1 - Y^1_{11} & s^1 \\ 0 & s^2 \end{pmatrix}\omega^1 \qquad 1 \le i \le j \le 2$$

so the resulting equations can be written dropping hats,

$$\begin{aligned} d\theta &= \sigma\wedge\theta + \omega_j\wedge\eta^j \\ d\omega^1 &= \beta^1\wedge\theta + \Omega^1_1\wedge\omega^1 + \Omega^1_2\wedge\omega^2 \\ d\omega^2 &= \beta^2\wedge\theta + \Omega^2_2\wedge\omega^2 + \overline{T}^2_{1j}\omega^1\wedge\eta^j \\ d\eta^1 &= \Upsilon^1\wedge\theta + \Sigma_1\wedge\omega^2 + (\sigma - \Omega^1_1)\wedge\eta^1 + \overline{Y}^1_{12}\omega^1\wedge\eta^2 + \overline{l}^1\,\eta^1\wedge\eta^2 \\ d\eta^2 &= \Upsilon^2\wedge\theta + \Sigma_1\wedge\omega^1 + \Sigma_2\wedge\omega^2 + (\sigma - \Omega^2_2)\wedge\eta^2 - \Omega^1_2\wedge\eta^1 \\ &\quad + \overline{Y}^2_{1k}\omega^1\wedge\eta^k + \overline{l}^2\,\eta^1\wedge\eta^2 \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} &\overline{T}^2_{1j} = T^2_{1j} & &\overline{Y}^2_{11} = Y^2_{11} + s_1 - Y^1_{21} + g_2 \\ &\overline{Y}^1_{12} = Y^1_{12} & &\overline{Y}^2_{12} = Y^2_{12} - g_1 + s_2 - Y^1_{22} \\ &\overline{l}^1 = l^1 + h_2 + T^1_{12} & &\overline{l}^2 = l^2 - h_1 + T^1_{22} - T^2_{21} \end{aligned} \tag{3.20}$$

By expressing the condition $d^2\theta \wedge \theta = 0$ we get

$$\overline{l}^1 = \overline{T}^2_{11}, \quad \overline{Y}^2_{1j} = 0 \quad \overline{l}^2 = 0 \tag{3.21}$$

so that $\overline{T}^2_{11}, \overline{T}^2_{12}$ and $\overline{Y}^1_{12}$ are the only possible non-zero torsion terms in (3.19). The infinitesimal group action on $\overline{T}^2_{11}$ and $\overline{T}^2_{12}$ are determined by putting $(d^2\omega^2)\wedge\omega^2\wedge\theta = 0$, which gives

$$\begin{aligned} d\overline{T}^2_{11} + \overline{T}^2_{11}(\sigma - \Omega^2_2) - \beta^2 - \overline{T}^2_{12}\Omega^1_2 &\equiv 0 \\ d\overline{T}^2_{12} + \overline{T}^2_{12}(\sigma + \Omega^1_1 - 2\Omega^2_2) &\equiv 0 \end{aligned} \qquad \text{mod(base)} . \tag{3.22}$$

While $(d^2\eta^1)\wedge\omega^2\wedge\theta\wedge\eta^1$ gives,

$$d\overline{Y}^1_{12} + \overline{Y}^1_{12}(2\Omega^1_1 - \Omega^2_2) - \overline{T}^2_{12}\Sigma_1 \ \equiv \ 0 \qquad \text{mod(base)} . \tag{3.23}$$

We will make the invariant assumptions $\overline{T}^2_{12} = 0$ and $\overline{Y}^1_{12} \neq 0$, and then use the group action translational group action which we see in (3.22) to translate $\overline{T}^2_{11}$ to 0 (thus by equation (3.21) $\overline{l}^1$ translates to 0), and scale $\overline{Y}^1_{12}$ to 1. Thus in a modified coframe which gives rise to this choice of invariants the new structure group satisfies,

$$\Omega^2_2 \equiv 2\Omega^1_1 \qquad \beta_2 \equiv 0 \qquad \text{mod(base)} \tag{3.24}$$

The structure equations in the next round of computations will be such that the invariant conditions

$$\overline{l}^1 = \overline{T}^2_{11} = \overline{T}^2_{12} = 0 \qquad \text{and} \quad \overline{Y}^1_{12} = 1 \tag{3.25}$$

are satisfied (after the absorption has been performed). In this next round of computations we return to equations (3.17) but append the conditions (3.24), and thus with this first reduced group and modified coframe we perform the absorptions,

$$\begin{aligned} \sigma &= \hat{\sigma} - g_j\omega^j - h_j\eta^j \ , \quad \beta^1 = \hat{\beta}^1 - R^1_j\,\omega^j - S^1_j\,\eta^j + \frac{1}{2}R^2_2\omega^1 \\ \Omega^1_j &= \hat{\Omega}^1_j + T^1_{jk}\eta^k - (g_1 - Y^1_{11}, s^1)\omega^1 + (\frac{1}{2}R^2_2\theta,\, 0) \\ \Sigma_i &= \hat{\Sigma}_i - k_i\omega^1 + (Y^i_{2k} - g_2\delta^i_k)\eta^k \\ \Upsilon^i &= \hat{\Upsilon}^i - V^i_j\omega^j - W^i_j\eta^j - \begin{pmatrix} \frac{1}{2}R^2_2\eta^1 \\ R^2_2\eta^2 \end{pmatrix} . \end{aligned} \tag{3.26}$$

With this absorption and taking into account the invariant conditions from (3.25) above the structure equations are then,

$$\begin{aligned}
d\theta &= \sigma\wedge\theta + \omega_j\wedge\eta^j \\
d\omega^1 &= \beta^1\wedge\theta + \Omega^1_1\wedge\omega^1 + \Omega^1_2\wedge\omega^2 \\
d\omega^2 &= 2\Omega^1_1\wedge\omega^2 + \overline{R}^2_1\omega^1\wedge\theta + \overline{S}^2_j\eta^j\wedge\theta + \overline{T}^2_{2j}\omega^2\wedge\eta^j + \overline{s}^2\omega^1\wedge\omega^2 \\
d\eta^1 &= \Upsilon^1\wedge\theta + \Sigma_1\wedge\omega^2 + (\sigma - \Omega^1_1)\wedge\eta^1 + \omega^1\wedge\eta^2 \\
d\eta^2 &= \Upsilon^2\wedge\theta + \Sigma_1\wedge\omega^1 + \Sigma_2\wedge\omega^2 + (\sigma - 2\Omega^1_1)\wedge\eta^2 - \Omega^1_2\wedge\eta^1 \\
&\quad + \overline{Y}^2_{1j}\omega^1\wedge\eta^j + \overline{l}^2\eta^1\wedge\eta^2.
\end{aligned} \tag{3.27}$$

where

$$\begin{array}{lll}
\overline{R}^2_1 = R^2_1 & \overline{s}^2 = s^2 + 2Y^1_{11} - 2g_1 & \overline{Y}^2_{11} = Y^2_{11} + g_2 - Y^1_{21} + s^1 \\
\overline{S}^2_j = S^2_j & \overline{l}^2 = l^2 - h_1 - 2T^1_{11} & \overline{Y}^2_{12} = Y^2_{12} + g_1 - 2Y^1_{11} - Y^1_{22} \\
 & \overline{T}^2_{2j} = T^2_{2j} - 2T^1_{1j} &
\end{array} \tag{3.28}$$

Expressing $d^2\theta \wedge \theta = 0$ gives

$$\overline{Y}^2_{11} = 0\,, \quad \overline{Y}^2_{12} = -\overline{s}^2\,, \quad \overline{l}^2 = \overline{T}^2_{21}\,.$$

To determine the infinitesimal group action on the left over torsion terms we first take $(d^2\omega^2)\wedge\omega^2 = 0$, to get,

$$\begin{aligned}
d\overline{R}^2_1 + \overline{R}^2_1(\sigma - \Omega_1) + \overline{S}^2_2\Sigma_1 &\equiv 0 \\
d\overline{S}^2_1 + \overline{S}^2_1(2\sigma - 3\Omega_1) - \overline{S}^2_2\Omega_2 &\equiv 0 \qquad \text{mod(base)} \\
d\overline{S}^2_2 + \overline{S}^2_2(2\sigma - 4\Omega_1) &\equiv 0
\end{aligned} \tag{3.29}$$

The form of the infinitesimal group action in these equations demonstrate that the assumptions

$$\overline{R}^2_1 = 0 \quad , \qquad \overline{S}^2_j = 0\,. \tag{3.30}$$

are invariant. We now assume this to be true. Then use

$$\left(2(d^2\omega^1)\wedge\omega^2 + (d^2\omega^2)\wedge\omega^1\right)\wedge\theta = 0$$

to find,

$$\begin{aligned} d\overline{T}^2_{21} + \overline{T}^2_{21}(\sigma - \Omega_1) - \overline{T}^2_{22}\Omega_2 + 2\beta^1 &\equiv 0 \\ d\overline{T}^2_{22} + \overline{T}^2_{22}(\sigma - 2\Omega_1) &\equiv 0 \end{aligned} \qquad \text{mod(base)} \tag{3.31}$$

which we assume $\overline{T}^2_{22} = 0$ and translate $\overline{T}^2_{21}$ to 0. Lastly setting

$$d^2\omega^2 \wedge \eta^1 \wedge \theta - 2d^2\eta^1 \wedge \omega^2 \wedge \theta - 2d^2\theta \wedge \omega^2 \wedge \eta^1 = 0$$

results in,

$$d\overline{s}^2 + \overline{s}^2\Omega^1_1 - 4\Upsilon^1 - 2\Omega^1_2 \equiv 0 \qquad \text{mod(base)} \tag{3.32}$$

and so we may translate $\overline{s}^2$ to 0. The translation of $\overline{T}^2_{21}$ to 0 and $\overline{s}^2$ to 0 give rise to a new coframe and structure group so that

$$\beta^1 \equiv 0 \qquad \Omega^1_2 \equiv -2\Upsilon^1 \quad \text{mod(base)} \tag{3.33}$$

In the next round of the computations we will have the invariant conditions,

$$\overline{R}^2_1 = \overline{S}^2_j = \overline{T}^2_{2j} = \overline{s}^2 = 0\ . \tag{3.34}$$

after absorption. With our reduction (3.33) we then perform the absorption,

$$\begin{aligned} \sigma &= \hat{\sigma} - g_j\omega - h_j\eta^j + (W^1_1 + \frac{1}{2}R^2_2)\theta \\ \Omega^1_1 &= \hat{\Omega}^1_1 + T^1_{1k}\eta^k - (g^1 - Y^1_{11})\omega^1 + \frac{1}{2}R^2_2\theta + (s^1 + 2V^1_1)\omega^2 \\ \Upsilon^1 &= \hat{\Upsilon}^1 - \frac{1}{2}(R^1_2\theta + T^1_{21}\eta^1) - V^1_j\omega^j - W^1_2\eta^2 \\ \Sigma_1 &= \hat{\Sigma}_1 - k_1\omega^1 + (Y^1_{2k} - g_2)\delta^1_k)\eta^k - (s^1 + 2V^1_1)\eta^1 \\ \Sigma_2 &= \hat{\Sigma}_2 - k^2\omega^1 + (Y^2_{2k} - g_2\delta^2_k)\eta^k - 2(s^1 + 2V^1_1)\eta^2 \\ \Upsilon^2 &= \hat{\Upsilon}^2 - V^2_j\omega^j - W^2_j\eta^j - R^1_2\eta^1 + (W^1_1 - \frac{1}{2}R^2_2)\eta^2\ . \end{aligned} \tag{3.35}$$

Taking into consideration the invariant conditions (3.25), (3.34) the structure equations are then

$$
\begin{aligned}
d\theta &= \sigma\wedge\theta + \omega_j\wedge\eta^j \\
d\omega^1 &= \Omega_1^1\wedge\omega^1 - 2\Upsilon^1\wedge\omega^2 + \overline{T}_{22}^1\omega^2\wedge\eta^2 + \overline{R}_1^1\omega^1\wedge\theta + \overline{S}_j^1\eta^j\wedge\theta \\
d\omega^2 &= 2\Omega_1^1\wedge\omega^2 \\
d\eta^1 &= \Upsilon^1\wedge\theta + \Sigma_1\wedge\omega^2 + (\sigma - \Omega_1^1)\wedge\eta^1 + \omega^1\wedge\eta^2 \\
d\eta^2 &= \Upsilon^2\wedge\theta + \Sigma_1\wedge\omega^1 + \Sigma_2\wedge\omega^2 + (\sigma - 2\Omega_1^1)\wedge\eta^2 + 2\Upsilon^1\wedge\eta^1 \\
&\quad + \overline{Y}_{1k}^2\omega^1\wedge\eta^k + \overline{l}^2\eta^1\wedge\eta^2
\end{aligned}
\tag{3.36}
$$

where

$$
\begin{aligned}
&\overline{R}_1^1 = R_1^1 - \tfrac{1}{2}R_2^2 && \overline{Y}_{11}^2 = Y_{11}^2 - Y_{21}^2 + g_2 + s^1 \\
&\overline{T}_{22}^1 = T_{22}^1 - 2W_2^1 && \overline{Y}_{12}^2 = Y_{12}^2 - Y_{22}^1 + g_1 - 2Y_{11}^1 \\
&\overline{S}_j^1 = S_j^1 && \overline{l}^2 = l^2 - h_1 - 2T_{11}^1 + 2W^{12}
\end{aligned}
\tag{3.37}
$$

As well $d^2\theta\wedge\theta = 0$ gives

$$
\overline{Y}_{1j}^2 = 0 \qquad \overline{l}^2 = -\overline{T}_{22}^1 \tag{3.38}
$$

Computing

$$
2(d^2\omega^1)\wedge\omega^2 + (d^2\omega^2)\wedge\omega^1 = 0
$$

determines,

$$
\begin{aligned}
d\overline{R}_1^1 + \overline{R}_1^1\sigma + \overline{S}_2^1\Sigma_1 &\equiv 0 \\
d\overline{S}_1^1 + 2\overline{S}_1^1(\sigma - \Omega_1^1) + 2\overline{S}_2^1\Upsilon^1 &\equiv 0 \qquad \mathrm{mod(base)} \\
d\overline{S}_2^1 + \overline{S}_2^1(2\sigma - 3\Omega_1^1) &\equiv 0
\end{aligned}
$$

from which we will make the invariant assumption $\overline{R}_1^1 = \overline{S}_j^1 = 0$. While

$$
\left((d^2\omega^1)\wedge\theta\wedge\eta^1 - 2(d^2\eta^1)\wedge\omega^2\wedge\eta^1\right)\wedge\omega^1 = 0 \tag{3.39}
$$

gives,

$$d\overline{T}^1_{22} + \overline{T}^1_{22}\sigma \equiv 0 \qquad \mathrm{mod(base)}$$

which we also assume to be zero. We have at this point assumed,

$$\overline{R}^1_1 = \overline{S}^1_j = 0 = T^1_{22} = 0 \ . \tag{3.40}$$

Thus all the torsion in equation (3.36) is zero, and the structure equations are of the form

$$\begin{aligned} d\theta &= \sigma\wedge\theta + \omega_j\wedge\eta^j \\ d\omega^1 &= \Omega^1_1\wedge\omega^1 - 2\Upsilon^1\wedge\omega^2 \\ d\omega^2 &= 2\Omega^1_1\wedge\omega^2 \\ d\eta^1 &= \Upsilon^1\wedge\theta + \Sigma_1\wedge\omega^2 + (\sigma - \Omega^1_1)\wedge\eta^1 + \omega^1\wedge\eta^2 \\ d\eta^2 &= \Upsilon^2\wedge\theta + \Sigma_1\wedge\omega^1 + \Sigma_2\wedge\omega^2 + (\sigma - 2\Omega^1_1)\wedge\eta^2 + 2\Upsilon^1\wedge\eta^1 \end{aligned} \tag{3.41}$$

We will now impose the invariant conditions on an arbitrary parabolic equation and find a canonical form for the equations which satisfy the conditions.

Theorem 3.1: *Any parabolic partial differential equation of the form,*

$$z_{xx} = \pm(g^0(x,y,z) + g^1(x,y,z)z_x)^2 z_y + f^0(x,y,z,z_x) \quad \textit{with} \quad g^0_z = g^1_x \tag{3.42}$$

and g^1, g^0, f^0 *otherwise arbitrary, admits by application of the equivalence method the structure equations in* (3.41) *on* $U \times \mathrm{H}$ *where* H *is the Lie subgroup of* H' *given parametrically by*

$$a_2 = -2\frac{E^1 a_1}{c} \ , \quad a_3 = (a_1)^2 \ , \quad B^i = 0 \ . \tag{3.43}$$

Proof : We need to verify that any parabolic equation satisfying the invariant criteria up to this point must be of the form given in (3.42). If we begin with the initial coframe as in (3.12) and note that the assumptions that $R^2_1 = 0$ and

$S^2_j = 0$ imply that $d\omega^2 \wedge \omega^2 = 0$ which in turn implies there exists coordinates such that $\omega^2 = \lambda\, dy'$. In turn any partial differential equation satisfying these conditions may be written as, (dropping primes)

$$z_{xx} = f(x, y, u, z, z_x, z_y) \ . \tag{3.44}$$

We will assume that these coordinates have been chosen. We then continue our parametric calculations with the coframe $\widehat{\theta} = dz - pdx - qdy$,

$$\begin{aligned} \widehat{\omega}^1 &= dx & \qquad \widehat{\eta}^1 &= dp - f dx \\ \widehat{\omega}^2 &= dy & \qquad \widehat{\eta}^2 &= dq \ . \end{aligned} \tag{3.45}$$

The first step is to find the torsion terms $\overline{T}^2_{1j}$ and $\overline{Y}^1_{12} = Y^1_{12}$ in equation (3.19). At the identity of the group they are found to be,

$$\begin{aligned} (\overline{T}^2_{1j})|_e &= (T^2_{1j})_e = 0 \\ (\overline{Y}^1_{12})|_e &= (Y^1_{11})|_e = \left(d\widehat{\eta}^1\right)_{\widehat{\omega}^1\widehat{\eta}^2} = f_q \end{aligned}$$

From equation (3.23) we use a_3 to scale $\overline{Y}^1_{12}$ to 1 leading to the reduction of the structure group by,

$$a_3 = (a_1)^2 \qquad B^2 = 0$$

where $B^2 = 0$ comes from the translation in (3.22). We then change the coframe (3.45) by

$$\widehat{\omega}^2 = \frac{1}{f_q} dy \qquad\qquad \widehat{\eta}^2 = f_q dq \tag{3.46}$$

which gives rise to these reductions. Now using this modified coframe we determine the next set of invariant conditions (3.34) by computing,

$$d\widehat{\omega}^2 = -\frac{1}{f_q}\left(\frac{df_q}{dt}\widehat{\omega}^1 + f_{zq}\widehat{\theta} + f_{pq}\widehat{\eta}^1 + f_{qq}\widehat{\eta}^2\right)\wedge\widehat{\omega}^2 \tag{3.47}$$

from which we determine,

$$(\overline{T}^2_{22})|_c = \frac{f_{qq}}{f_q} \quad (\overline{T}^2_{21})|_c = \frac{f_{pq}}{f_q} \quad (T^2_{1j})|_c = 0 \ . \tag{3.48}$$

The last torsion term we need at this point is

$$(\overline{s}^2)|_c = (s^2)_e + 2Y^1_{11} + \left(d\widehat{\omega}^2\right)_{\widehat{\omega}^1\widehat{\omega}^2} + 2\left(d\widehat{\eta}^1\right)_{\widehat{\omega}^1\widehat{\eta}^1} = 2f_p - \frac{1}{f_q}\frac{df_q}{dx}$$

where the second term arises from the absorption in (3.28). The new coframe corresponding to the translations of $\overline{T}^2_{21}$ and $\overline{Y}^1_{11}$ to 0 is then

$$\begin{aligned} \widehat{\omega}^1 &= dx + \left(-f_p + \tfrac{1}{2f_q}\tfrac{df_q}{dx}\right)\widehat{\omega}^2 + \tfrac{f_{pq}}{2f_q}\,\widehat{\theta} \\ \widehat{\eta}^2 &= f_q dq + \left(f_p - \tfrac{1}{2f_q}\tfrac{df_q}{dx}\right)\,\widehat{\eta}^1 \\ \widehat{\omega}^2 &= \tfrac{1}{f_q}dy \\ \widehat{\eta}^1 &= dp - f dx \end{aligned} \tag{3.49}$$

where ω^1 and η^1 haven't changed. The parametric reduction of the structure group is given by

$$B^1 = 0 \qquad \text{and} \qquad a_2 = -2E^1\frac{a_1}{c} \ .$$

In order to determine the restrictions on f by the assumption $\overline{T}^2_{22}$ being zero which from (3.48) gives,

$$f_{qq} = 0 \quad \text{thus} \quad f = f^1(x,y,z,p)q + f^0(x,y,z,p) \ . \tag{3.50}$$

From the frame (3.49) we may determine the final invariants by taking $d\omega^1$. We first find,

$$\begin{aligned} (\overline{S}^1_1)|_c &= (S^1_1)|_c = \frac{\partial}{\partial p}\left(\frac{f_{pq}}{2f_q}\right) + \frac{f_{pq}}{2f_q}\left(d\widehat{\theta}\right)_{\widehat{\eta}^1\widehat{\theta}} \\ &= \frac{f_{ppq}f_q - (f_{pq})^2}{2{f_q}^2} + \left(\frac{f_{pq}}{2f_q}\right)^2 \\ &= \frac{2f_{ppq}f_q - (f_{pq})^2}{4{f_q}^2} \end{aligned}$$

while $(\overline{S}^1_2) = 0$ is easily seen by using (3.50). Next we compute,

$$\begin{aligned}(\overline{T}^1_{22})|_e &= (T^1_{22})|_e - 2(W^1_2)|_e \\ &= \frac{f_{pq}}{f_q} - \frac{1}{2f_q}\frac{\partial}{\partial q}\left(\frac{df_q}{dx}\right) + \frac{f_{pq}}{2f_q}\left(d\hat{0}\right)_{\hat{\omega}^2\hat{\eta}^2} - 2\left(\frac{1}{2}\frac{f_{pq}}{f_q}\right) = 0\end{aligned}$$

Finally we compute,

$$\begin{aligned}(\overline{R}^1_1)|_e &= (R^1_1)|_e - \frac{1}{2}(R^2_2)|_e = \frac{1}{2}\frac{d}{dx}\left(\frac{f_{pq}}{f_q}\right) - \frac{1}{2(f_q)}\left(f_{zq} - \frac{f_{pq}}{2f_q}\frac{df_q}{dx}\right) \\ &= \frac{1}{4(f_q)^2}\left(2f_q\frac{df_{pq}}{dx} - 2f_{zq}f_q - f_{pq}\frac{df_q}{fx}\right) \\ &= \frac{2f_q(f_{xpq} - f_{zq}) + p(2f_{zpq}f_q - f_{pq}f_{zq}) + f(2f_{ppq}f_q - (f_{pq})^2) - f_{pq}f_{xq}}{4c{f_q}^2}\end{aligned}$$

The assumption that $\overline{S}^1_1 = 0$ is zero is

$${f_{pq}}^2 - 2f_{ppq}f_q = 0$$

Substituting the form of f in equation (3.50) in here gives

$$(f^1_p)^2 - 2f^1_{pp}f^1 = 0$$

whose general solution is,

$$f^1 = \pm\left(g^0(x,y,z) + g^1(x,y,z)p\right)^2 . \tag{3.51}$$

While requiring $\overline{R}^1_1 = 0$ and taking into account $\overline{S}^1_1 = 0$ gives,

$$2f^1(f^1_{xp} + pf^1_{zp} - f^1_z) - f^1_p(f^1_x + pf^1_z) = 0$$

and then inserting the form of f^1 given in (3.51), into this equation we find

$$\frac{\partial}{\partial z}g^0(x,y,z) = \frac{\partial}{\partial x}g^1(x,y,z)$$

and finally we have finished the proof. ∎

We may summarize this result by saying that any parabolic partial differential equation which satisfies the invariant conditions in equation (3.25), (3.34) and (3.40), can be put into the canonical form (3.41). We now check the conditions of Theorem 1.11 to see whether the structure equations satisfy the conditions for an infinite Lie pseudogroup. The dimension of the solution space of the homogeneous system (the kernel of the absorption) is of dimension 5 and can be parameterized by K_1, K_2, K_3, K_4, K_5. This results in the freedom

$$\begin{array}{llll} \Omega^1_1 = \widehat{\Omega}^1_1, & \Upsilon^1 = \widehat{\Upsilon}^1 + K_1\omega^2, & & \Sigma_2 = \widehat{\Sigma}_2 + K_2\omega^1 + K_3\omega^2 + K_4\theta + 2K_1\eta^1 \\ \sigma = \widehat{\sigma}, & \Sigma_1 = \widehat{\Sigma}_1 + K_2\omega^2 + K_1\theta, & \Upsilon^2 = \widehat{\Upsilon}^2 + K_1\omega^1 + K_4\omega^2 + K_5\theta \ . & \end{array} \tag{3.52}$$

while the Cartan characters are found to be

$$\sigma'_1 = 3 \qquad \sigma'_2 = 2 \tag{3.53}$$

We thus have $\dim(\mathrm{h}^{(1)}) = 5 \neq 3+2(2)$ equations (3.41) are not the structure equations of an infinite Lie pseudogroup.

From equation (3.52) we may use K_i to parameterize the first prolonged group $\mathrm{H}^{(1)}$ and we now determine,

Proposition 3.1: *The prolongation of the structure equations*

$$\begin{aligned} d\theta &= \sigma\wedge\theta + \omega_j\wedge\eta^j \\ d\omega^1 &= \Omega^1_1\wedge\omega^1 - 2\Upsilon^1\wedge\omega^2 \\ d\omega^2 &= 2\Omega^1_1\wedge\omega^2 \\ d\eta^1 &= \Upsilon^1\wedge\theta + \Sigma_1\wedge\omega^2 + (\sigma - \Omega^1_1)\wedge\eta^1 + \omega^1\wedge\eta^2 \\ d\eta^2 &= \Upsilon^2\wedge\theta + \Sigma_1\wedge\omega^1 + \Sigma_2\wedge\omega^2 + (\sigma - 2\Omega^1_1)\wedge\eta^2 + 2\Upsilon^1\wedge\eta^1 \end{aligned} \tag{3.54}$$

gives rise to a G-structure on $U \times \mathrm{H} \times \mathrm{H}^{(1)}_1$, where $\mathrm{H}^{(1)}_1 \subset \mathrm{H}^{(1)}$ is a two dimensional subgroup, with 10 tensorial invariants $T_a : U \times \mathrm{H} \to \mathbb{R}$ which are acted on trivially by $\mathrm{H}^{(1)}_1$.

Proof : In order to find the structure equations for the lifted coframe first denote by $\beta_i = dK_i$ the Maurer-Cartan forms for $H^{(1)}$, and then apply $d^2 = 0$ to the equations (3.54) of Theorem 3.1. Using $d^2\omega^2 = 2d\Omega_1^1 \wedge \omega^2 = 0$ we conclude,

$$d\Omega_1^1 = \xi_1 \wedge \omega^2 \tag{3.55}$$

where ξ_1 is a one-form such that $(\xi_1)_{\omega^2} = 0$. $d^2\omega^1 = 0$ gives,

$$d\Upsilon^1 = \beta_1 \wedge \omega^2 + \Upsilon^1 \wedge \Omega_1^1 - \frac{1}{2}\xi_1 \wedge \omega^1 + \xi_2 \wedge \omega^2 \tag{3.56}$$

where ξ_2 is a one-form and $(\xi_2)_{\omega^2} = 0$. Next we take,

$$\begin{aligned} d^2\theta &= d\sigma \wedge \theta - \sigma \wedge (\omega_j \wedge \eta^j) + (\Omega_1^1 \wedge \omega^1 - 2\Upsilon^1 \wedge \omega^2) \wedge \eta^1 \\ &\quad -\omega^1 \wedge (\Sigma_1 \wedge \omega^2 + \Upsilon^1 \wedge \theta + (\sigma - \Omega_1^1) \wedge \eta^1) + 2\Omega_1^1 \wedge \omega^2 \wedge \eta^2 \\ &\quad -\omega^2 \wedge (\Sigma_1 \wedge \omega^1 + \Upsilon^2 \wedge \theta + (\sigma - 2\Omega_1^1) \wedge \eta^2 + 2\Upsilon^1 \wedge \eta^1) \\ &= (d\sigma - \omega^1 \wedge \Upsilon^1 - \omega^2 \wedge \Upsilon^2) \wedge \theta \end{aligned}$$

so that

$$d\sigma = \omega^1 \wedge \Upsilon^1 + \omega^2 \wedge \Upsilon^2 + \xi_3 \wedge \theta \tag{3.57}$$

where ξ_3 is a one-form such that $(\xi_3)_\theta = 0$. Continuing we have,

$$\begin{aligned} d^2\eta^1 &= d\Upsilon^1 \wedge \theta - \Upsilon^1 \wedge (\sigma \wedge \theta + \omega_j \wedge \eta^j) + d\Sigma_1 \wedge \omega^1 - 2\Sigma_1 \wedge \Omega_1^1 \wedge \omega^2 \\ &\quad + (d\sigma - d\Omega_1^1) \wedge \eta^1 - (\sigma - \Omega_1^1) \wedge (\Sigma_1 \wedge \omega^2 + \Upsilon^1 \wedge \theta + \omega^1 \wedge \eta^2) + 2d\Upsilon^1 \wedge \eta^1 \\ &\quad + (\Omega_1^1 \wedge \omega^1 - 2\Upsilon^1 \wedge \omega^2) \wedge \eta^2 - \omega^1 \wedge (\Sigma_1 \wedge \omega^2 + \Upsilon^1 \wedge \theta + (\sigma - \Omega_1^1) \wedge \eta^1 + \omega^1 \wedge \eta^2) \\ &= (d\Sigma_1 - 3\Sigma_1 \wedge \Omega_1^1 + 3\Upsilon^1 \wedge \eta^2 + \Upsilon^2 \wedge \eta^1 - \omega^1 \wedge \Sigma_2 - \sigma \wedge \Sigma^1 + \xi_1 \wedge \eta^1 - \xi_2 \wedge \theta) \wedge \omega^2 \\ &\quad -(\xi_3 \wedge \eta^1 + \omega^1 \wedge \Upsilon^2 + \frac{1}{2}\xi_1 \wedge \omega^1) \wedge \theta \end{aligned} \tag{3.58}$$

where we have used (3.55), (3.57) and (3.56). Now wedging this with ω^2 we conclude,

$$\begin{aligned} \xi_1 &= 2\Upsilon^2 + T_{11}\theta + T_{12}\omega^1 + T_{14}\eta^1 \\ \xi_3 &= \tfrac{1}{2}T_{14}\omega^1 + T_{33}\omega^2 + T_{34}\eta^1 . \end{aligned}$$

Using these expressions in equation (3.58) that we just computed, we find

$$\begin{aligned} d\Sigma_1 &= \beta_1\wedge\theta + \beta_2\wedge\omega^2 + 3\Sigma_1\wedge\Omega_1^1 + 3\eta^2\wedge\Upsilon^1 + \eta^1\wedge\Upsilon^2 + \sigma\wedge\Sigma_1 \\ &\quad +\omega^1\wedge\Sigma_2 - \xi_1\wedge\eta^1 + T_{33}\eta^1\wedge\theta + \xi_2\wedge\theta + \xi_4\wedge\omega^2 \end{aligned} \tag{3.59}$$

where ξ_4 is a one-form with $(\xi_4)_{\omega^2} = 0$. Last take $d^2\eta^2$ to find,

$$\begin{aligned} d^2\eta^2 &= d\Upsilon^2\wedge\theta - \Upsilon^2\wedge(\sigma\wedge\theta + \omega_j\wedge\eta^j) - \Sigma_1\wedge(\Omega_1^1\wedge\omega^1 - 2\Upsilon^1\wedge\omega^2) \\ &+(3\Sigma_1\wedge\Omega_1^1 + 3\eta_2\wedge\Upsilon^1 + \eta^1\wedge\Upsilon^2 + \sigma\wedge\Sigma_1 - \xi_1\wedge\eta^1 + \xi_2\wedge\theta + \xi_4\wedge\omega^2 + T_{33}\eta^1\wedge\theta)\wedge\omega^1 \\ &+d\Sigma_2\wedge\omega^2 - 2\Sigma_2\wedge\Omega_1^1\wedge\omega^2 + (\omega^1\wedge\Upsilon^1 + \omega^2\wedge\Upsilon^2 + \xi_3\wedge\theta - 2\xi_1\wedge\omega^2)\wedge\eta^2 \\ &-(\sigma - 2\Omega_1^1)\wedge(\Sigma_1\wedge\omega^1 + \Sigma_2\wedge\omega^2 + \Upsilon^2\wedge\theta + 2\Upsilon^1\wedge\eta^1) \\ &+(2\Upsilon^1\wedge\Omega_1^1 - \xi_1\wedge\omega^1 + 2\xi_2\wedge\omega^2)\wedge\eta^1 - 2\Upsilon^1\wedge(\Sigma_1\wedge\omega^2 + (\sigma - \Omega_1^1)\wedge\eta^1 + \omega^1\wedge\eta^2) \end{aligned}$$

which gives,

$$\begin{aligned} d\Upsilon^2 &= \beta_5\wedge\theta + \beta_4\wedge\omega^2 + \beta_1\wedge\omega^1 + 2\Upsilon^2\wedge\Omega_1^1 + \xi_2\wedge\omega^1 + \xi_3\wedge\eta^2 \\ &\quad +T_{33}\eta^1\wedge\omega^1 + \xi_6\wedge\omega^2 + \xi_7\wedge\theta \end{aligned} \tag{3.60}$$

$$\begin{aligned} d\Sigma_2 &= \beta_2\wedge\omega^1 + \beta_3\wedge\omega^2 + 2\beta_1\wedge\eta^1 + \beta_4\wedge\theta + 4(\Upsilon^1\wedge\Sigma_1 + \Upsilon^2\wedge\Sigma_2) + 2\eta^2\wedge\Upsilon^2 \\ &\quad +\sigma\wedge\Sigma_2 - 2\xi_1\wedge\eta^2 + 2\xi_2\wedge\eta^1 + \xi_4\wedge\omega^1 + \xi_5\wedge\omega^2 + \xi_6\wedge\theta \end{aligned} \tag{3.61}$$

where ξ_5, ξ_6, ξ_7 are one-forms with $(\xi_5)_{\omega^2} = (\xi_6)_{\omega^2} = (\xi_7)_{\omega^2} = 0$. The fact that no β_i terms enter into $d\Omega_1^1$, allows us to use T_{1j} in equation (3.59) to reduce the structure group with out worrying about the absorption. By taking $d^2\Omega_1^1$ from (3.55) and (3.59) we get

$$d^2\Omega_1^1 \equiv 2d\Upsilon^2 + dT_{11}\wedge\theta + dT_{12}\wedge\omega^1 + dT_{14}\wedge\eta^1 \qquad \mathrm{mod}(\omega^2) \tag{3.62}$$

Thus,

$$dT_{11} + 2\beta_5 \equiv dT_{12} + 2\beta_1 \equiv 0 \qquad \mathrm{mod}(\text{base}, \mathbf{h}^*)$$

and so we may translate T_{11} and T_{12} to 0 resulting in

$$\xi_1 = 2\Upsilon^2 + T_{14}\eta^1 \tag{3.63}$$

from (3.59). We have thus eliminated the group parameters K_1, K_5 in equation (3.52) (and hence β_1 and β_5 vanish in all equations that follow). In order to further simplify the ξ_2 and ξ_7 terms we put $d^2\Omega_1^1 = 0$

$$\begin{aligned} d^2\Omega_1^1 &= 2(\xi_2\wedge\omega^1 + \xi_3\wedge\eta^2 + T_{33}\eta^1\wedge\omega^1 + \xi_7\wedge\theta)\wedge\omega^2 + dT_{14}\wedge\eta^1\wedge\omega^2 \\ &\quad + T_{14}(\Upsilon^1\wedge\theta + (\sigma+\Omega_1^1)\wedge\eta^1 + \omega^1\wedge\eta^2)\wedge\omega^2 \end{aligned}$$

which gives,

$$\begin{aligned} \xi_2 &= T_{21}\theta + T_{22}\omega^1 + T_{24}\eta^1 + T_{14}\eta^2 \\ \xi_7 &= -\tfrac{1}{2}T_{14}\Upsilon^1 + T_{21}\omega^1 + T_{74}\eta^1 \end{aligned} \tag{3.64}$$

Now we proceed as usual and perform the absorption,

$$\beta_2 = \widehat{\beta}_2 - \xi_4\ , \quad \beta_4 = \widehat{\beta}_4 - \xi_6\ , \quad \beta_3 = \widehat{\beta}_3 - \xi_5$$

in equations (3.59), (3.60) and (3.61). We may now compute the action of $\mathrm{H}^{(1)}$ on ξ_2 (3.64)) by taking

$$\begin{aligned} d^2\Upsilon^1 &= d(\Upsilon^1\wedge\Omega_1^1 - \Upsilon^2\wedge\omega^1 - \tfrac{1}{2}T_{14}\eta^1\wedge\omega^1 + (T_{21}\theta + T_{22}\omega^1 + T_{24}\eta^1 + T_{14}\eta^2)\wedge\omega^2) \\ &\equiv dT_{22} + \beta_4 \equiv 0 \qquad \mathrm{mod}(\mathrm{base}, \mathbf{h}^*) \end{aligned}$$

where we have used (3.64). So we may also eliminate K_4 (and β_4) in the prolonged group by translating T_{22} to 0. We may now use $d^2\Upsilon^1$ to find,

$$\xi_6 = -\frac{3}{2}T_{14}\Sigma_1 + T_{61}\theta + T_{62}\omega^1 + T_{64}\eta^1 + T_{65}\eta^2$$

To make the last round of computation we absorb torsion by,

$$\beta_2 = \widehat{\beta}_2 - \xi_4 \qquad \beta_3 = \widehat{\beta}_3 - \xi_5 \tag{3.65}$$

and find the final structure equations are,

$$
\begin{aligned}
d\Omega_1^1 &= 2\Upsilon^2\wedge\omega^2 + T_{14}\eta^1\wedge\omega^2 \\
d\sigma &= \omega^1\wedge\Upsilon^1 + \omega^2\wedge\Upsilon^2 + (\tfrac{1}{2}T_{14}\omega^1 + T_{33}\omega^2 + T_{34}\eta^1)\wedge\theta \\
d\Upsilon^1 &= \Upsilon^1\wedge\Omega_1^1 - \Upsilon^2\wedge\omega^1 - \tfrac{1}{2}T_{14}\eta^1\wedge\omega^1 + (T_{21}\theta + T_{24}\eta^1 + T_{14}\eta^2)\wedge\omega^2 \\
d\Upsilon^2 &= 2\Upsilon^2\wedge\Omega_1^1 - \tfrac{3}{2}T_{14}\Sigma_1\wedge\omega^2 - \tfrac{1}{2}T_{14}\Upsilon^1\wedge\theta - (T_{24}+T_{33})\omega^1\wedge\eta^1 - \tfrac{1}{2}T_{14}\omega^1\wedge\eta^2 \\
&\quad + T_{34}\eta^1\wedge\eta^2 + (T_{61}\theta^1 + T_{62}\omega^1 + T_{64}\eta^1 + (T_{65} - T_{33})\eta^2)\wedge\omega^2 + T_{74}\eta^1\wedge\theta \\
d\Sigma_1 &= \beta_2\wedge\omega^2 + 3(\Sigma_1\wedge\Omega_1^1 + \eta^2\wedge\Upsilon^1 + \eta^1\wedge\Upsilon^2) + \sigma\wedge\Sigma_1 + (T_{33}\eta^1 + T_{24}\eta^1 + T_{14}\eta^2)\wedge\theta \\
d\Sigma_2 &= \beta_2\wedge\omega^1 + \beta_3\wedge\omega^2 + 4(\Upsilon^1\wedge\Sigma_1 + \Upsilon^2\wedge\Sigma_2) + 6\eta^2\wedge\Upsilon^2 + \sigma\wedge\Sigma_2 \\
&\quad -\tfrac{3}{2}T_{14}\Sigma_1\wedge\theta - 4T_{14}\eta^1\wedge\eta^2 + (2T_{21} - T_{64})\theta\wedge\eta^1 + T_{62}\omega^1\wedge\theta + T_{65}\eta^2\wedge\theta\ .
\end{aligned}
\tag{3.66}
$$

The fact that the forms β_2 or β_3 will not appear in taking d of the first 4 of these equations implies that the 10 different torsion terms

$$T_{14},\ T_{21},\ T_{24},\ T_{33},\ T_{34},\ T_{61},\ T_{62}\ ,\ T_{64},\ T_{65},\ T_{74} \tag{3.67}$$

do not depend on the group parameters K_2, K_3, and thus the conditions of the theorem are satisfied. ∎

Since the tensorial invariants in (3.67) are acted on trivially by $\mathrm{H}_1^{(1)}$ this G-structure is of the type considered in Theorem 1.10 and we may apply our reduction theorem. This will be done below in the case of Burgers' equation. While the general parametric form for the tensorial invariants in (3.67) are too lengthy to write down we can make one simple claim in the case all they all vanish,

Theorem 3.2: *If the invariant conditions,*

$$T_{14} = T_{21} = T_{24} = T_{33} = T_{34} = T_{61} = T_{62} = T_{64} = T_{65} = T_{74} = 0 \tag{3.68}$$

in equations (3.66) *are satisfied then the resulting equations are the equations of a transitive infinite Lie pseudogroup. As well, any parabolic equation satisfying the conditions of Theorem* 3.1 *and the condition* (3.68) *is equivalent to the heat equation* $z_{xx} = z_t$

Proof : By taking d of the equations in (3.66) the condition that all the torsion be zero is easily found to be invariant. For example we have from equation (3.62)

$$dT_{14} + T_{14}(\Omega^1_1 + \sigma) \equiv 0 \qquad \text{mod(base)} \; . \tag{3.69}$$

Thus we only need to check the condition of Theorem 1.11 to see if we have the equations of a transitive infinite Lie pseudogroup. The first Cartan character is

$$\sigma'_1 = 2 \; . \tag{3.70}$$

while the kernel of the absorption by Σ_1 and Σ_2 in (3.66) is seen to be 2 dimensional, that is

$$\dim\left((\mathrm{h}^1_1)^{(1)}\right) = 2$$

and thus the conditions of Theorem 1.11 are satisfied. It is a straight forward calculation to check that the heat equation gives rise to the condition in (3.68) by using the frame from Theorem 3.1 given in equation (3.49) which in this case is,

$$(\theta, dx, dy, dp, dq) \; .$$

The last part of the theorem follows from the standard results on infinite transitive Lie pseudogroups [16]. ∎

One interesting observation we should also make in the example of the heat equation is,

Lemma 3.6: *The structure equations for the differential forms* $(\theta, \omega^1, \omega^2, \Omega^1_1, \sigma, \Upsilon^1, \Upsilon^2)$ *with the conditions* (3.68) *are the Maurer-Cartan equations the for the finite dimensional subgroup of the symmetry group of the heat equation.*

Proof : We write the structure equations,

$$\begin{aligned}
d\theta &= \sigma\wedge\theta + \omega_j\wedge\eta \\
d\omega^1 &= \Omega^1_1\wedge\omega^1 - 2\Upsilon^1\wedge\omega^2 \\
d\omega^2 &= 2\Omega^1_1\wedge\omega^2 \\
d\Omega^1_1 &= 2\Upsilon^2\wedge\omega^2 \\
d\sigma &= \omega^1\wedge\Upsilon^1 + \omega^2\wedge\Upsilon^2 \\
d\Upsilon^1 &= \Upsilon^1\wedge\Omega^1_1 - \Upsilon^2\wedge\omega^1 \\
d\Upsilon^2 &= 2\Upsilon^2\wedge\Omega^1_1
\end{aligned}$$

and a comparison with the reference [33] pg. 122. gives the result. ■

We now turn to the case of Burgers' equation (3.5) and apply Theorem 1.10 to reduce H. The calculations are too extensive to be done by hand but we have written a MAPLE program and summarize the computations.

Theorem 3.3 *The equivalence class of Burgers' equation $z_{xx} = z_y + z\, z_x$ is invariantly characterized by the invariant coframe,*

$$\begin{aligned}
\omega^1 &= \lambda(dx - zdy) \\
\omega^2 &= \lambda^2 dy \\
\theta^1 &= \lambda^{-1}(dz - pdx - qdy) \qquad\qquad \text{where} \quad \lambda = (q + zp)^{\frac{1}{3}} \\
\eta^1 &= \lambda^{-2}dp - \lambda dx + \lambda^{-2}(p^2 + zq + z^2 q)dy \\
\eta^2 &= \lambda^{-3}(dq + pdz + zdp) + 3pdy
\end{aligned}$$

with structure equations

$$\begin{aligned}
d\theta &= \tfrac{1}{3}\theta\wedge\eta^2 + \omega^1\wedge\eta^1 + \omega^2\wedge\eta^2 \\
d\omega^1 &= \omega^2\wedge\theta - \tfrac{1}{3}\omega^1\wedge\eta^2 \\
d\omega^2 &= -\tfrac{2}{3}\omega^2\wedge\eta^2 \\
d\eta^1 &= \omega^1\wedge\eta^2 - \omega^2\wedge\theta + \tfrac{2}{3}\eta^1\wedge\eta^2 \\
d\eta^2 &= 3\omega^1\wedge\omega^2 - 3\omega^2\wedge\eta^1
\end{aligned} \tag{3.71}$$

Proof : To determine the invariant frame we have a number of options, either compute all of the torsion in (3.66) or compute part of the torsion perform a reduction and proceed with a standard equivalence problem on the base. We will give the expressions for the torsion in equations (3.66) at the identity of H,

$$T_{14} = -1\,,\ T_{21} = -p\,,\ T_{24} = 0\,,\ T_{33} = 0\,,\ T_{34} = 0\,,\ T_{61} = -\tfrac{1}{2}(q + zp)\,,$$
$$T_{62} = (4p + z^2)(q + zp)\,,\ T_{64} = -\tfrac{5}{2}p\,,\ T_{65} = 0\,,\ T_{74} = 0$$

The following normalization for the torsion above

$$T_{14} = -1\,,\ T_{21} = 0\,,\ T_{33} = 0\,,\ T_{61} = 0\,,\ T_{62} = -\frac{1}{2}$$

give rise to the corresponding congruences,

$$\Omega^1_1 \equiv -\sigma\,,\ \Upsilon^2 \equiv 0\,,\ \Upsilon^1 \equiv 0\,,\ \Sigma_2 \equiv 0\,,\ \sigma \equiv 0 \quad \text{mod(base)} \tag{3.72}$$

This leaves only Σ_1 in the structure equations (3.54) after group reduction. The frame change corresponding to this reduction is

$$\begin{pmatrix} \overline{\theta} \\ \overline{\varpi}^1 \\ \overline{\varpi}^2 \\ \overline{\eta}^1 \\ \overline{\eta}^2 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda^2 & 0 & 0 \\ 0 & 0 & 0 & \lambda^{-2} & 0 \\ p\lambda^{-3} & 0 & 4p + z^2 & 0 & \lambda^{-3} \end{pmatrix} \begin{pmatrix} \overline{\theta} \\ \overline{\varpi}^1 \\ \overline{\varpi}^2 \\ \overline{\eta}^1 \\ \overline{\eta}^2 \end{pmatrix}$$

One last group reduction allows us to make $\Sigma_1 \equiv 0$ mod(base), and the corresponding the frame change is

$$\eta^1 = \overline{\eta}^1 + (p^2 + zq + z^2 p)\lambda^{-4}\,\overline{\varpi}^2$$
$$\eta^2 = \overline{\eta}^2 + (p^2 + zq + z^2 p)\lambda^{-4}\,\overline{\varpi}^1\ .$$

which leads to the frame of the theorem.

The structure equations in (3.71) are the Maurer-Cartan equations for the symmetry group of Burgers' equation (compare with [3] pg. 266). In the case of linear equations we find all the invariants except T_{62} vanish.

Appendix A

Contact Transformations

Let $U \subset J_1^\alpha(\mathbf{M}_{n+1})$ be open, and let $(t, x_0^i, x_1^i, ..., x_\alpha^i)$ be a system of local coordinates on U. The contact system in these coordinates is generated by

$$\theta_\rho^i = dx_{\rho-1}^i - x_\rho^i dt \qquad \text{where} \quad i = 1..n \quad , \quad \rho = 1..\alpha \quad .$$

Any contact transformation $\Psi_\alpha : J^\alpha \to \bar{J}^\alpha$ preserves the contact structure on J^ρ for $\rho \leq \alpha$, and from this we determine the explicit form of Ψ_α in terms of the point transformation [1] Ψ_0 on $\mathbf{M}$.

Lemma A.1: *Let $(\bar{t}, \bar{x}_0^i) = (\phi(t, x_0), \psi_0^i(t, x_0)) = \Psi_0(t, x)$ be a point transformation. The prolongation $\Psi_\alpha : J^\alpha \to \bar{J}^\alpha$ is given in local coordinates by*

$$\bar{x}_{\rho+1}^i = \psi_{\rho+1}^i(t, x_0^i, x_1^i, ..., x_{\rho+1}^i) = \left(\frac{d\phi}{dt}\right)^{-1} \frac{d\psi_\rho^i}{dt} \qquad \textit{where} \quad \rho = 1...\alpha - 1$$

Proof This will follow by induction. First Ψ_1 is obtained by requiring

$$\Psi_0^* \bar{\theta}_1 = 0 \qquad mod(\theta_1) \tag{A.1}$$

By computing in coordinates

$$\Psi_0^*(d\bar{x}_0^i - \bar{x}_1^i d\bar{t}) = \left(\frac{\partial \psi_0^i}{\partial x_0^j} - \psi_1^i \frac{\partial \phi}{\partial x_0^j}\right) \theta_1^j + \left(\frac{d\psi_0^i}{dt} - \left(\frac{d\phi}{dt}\right) \bar{x}_1^i \circ \Psi_0\right) dt$$

[1]For $n > 1$ any contact transformation is the prolongation of a point transformation by Bäcklund's theorem [2].

so that condition (A.1) gives

$$\bar{x}_1^i = \left(\frac{d\phi}{dt}\right)^{-1} \frac{d\psi_0^i}{dt}$$

Continuing by induction we find

$$\begin{aligned}
\Psi_{\rho+1}^* \theta_{\rho+1} &= \Psi_\rho^*(d\bar{x}_\rho^i - \bar{x}_{\rho+1}^i d\bar{t}) \\
&= d(\psi_\rho^i) - \psi_{\rho+1}^i \left(\frac{d\phi}{dt}dt + \frac{\partial \phi}{\partial x_0^j}\theta_0^j\right) \\
&= \frac{\partial \psi_\rho^i}{\partial x_s^j}\theta_s^j + \left(\frac{\partial \psi_\rho^i}{\partial x_0^j} - \psi_{\rho+1}^i \frac{\partial \phi}{\partial x_0^j}\right)\theta_0^j + \left(\frac{d\phi_\rho^i}{dt} - \psi_{\rho+1}^i \frac{d\phi}{dt}\right) dt
\end{aligned}$$

which by requiring

$$\Psi_\rho^* \bar{\theta}_\rho = 0 \qquad mod(\theta_1, \theta_2, ..., \theta_\rho) \tag{A.2}$$

gives the result. ■

To simplify the next lemma let $a = \frac{d\phi}{dt}$ and $E_i = \frac{\partial \phi}{\partial x_0^i}$. We may now write the contact transformations in more detail as follows,

Lemma A.2:

$$\Psi_\alpha^* \begin{pmatrix} \bar{\theta}_1 \\ . \\ . \\ . \\ \bar{\theta}_\alpha \end{pmatrix} = \begin{pmatrix} A_0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1 & a^{-1}A_0 & 0 & 0 & 0 & 0 & 0 \\ A_2 & B_1^2 & a^{-2}A_0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & 0 & 0 \\ . & . & B_r^s & . & a^{-s}A_0 & 0 & 0 \\ . & . & . & . & . & . & 0 \\ A_n & B_1^n & . & . & . & . & a^{-n}A_0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ . \\ . \\ . \\ \theta_\alpha \end{pmatrix}$$

where $A_r = \frac{\partial \psi_r^i}{\partial x_0^j} - \psi_{r+1}^i E_j$ *and* $B_r^s = \frac{\partial \psi_s^i}{\partial x_r^j}$.

Proof Lemma A.1 gives everything here except the diagonal terms, which we denote by D_ρ. Lemma A.1 gives

$$D_{\rho+1} = \frac{\partial \psi^i_{\rho+1}}{\partial x^j_{\rho+1}} = \frac{\partial}{\partial x^j_{\rho+1}} \left(\frac{1}{a} \frac{d\psi^i_\rho}{dt} \right) \qquad \rho = 1 \ldots n-1 \tag{A.3}$$

Now using the commutator,

$$\left[\frac{\partial}{\partial x^j_{\rho+1}}, \frac{d}{dt} \right] = \frac{\partial}{\partial x^j_\rho}$$

and that

$$\frac{\partial a}{\partial x^j_{\rho+1}} = 0 \qquad \frac{\partial \psi^i_\rho}{\partial x^j_{\rho+1}} = 0$$

we find that equation A.3 gives

$$D_{\rho+1} = \frac{1}{a} \frac{\partial \psi^i_\rho}{\partial x^j_\rho} = \frac{1}{a} D_\rho$$

A simple induction now completes the proof. ∎

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