THE EQUIVALENCE PROBLEM FOR SYSTEMS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

The equivalence problem for systems of second-order differential equations under point transformations is found to give rise to an $\{e\}$-structure of dimension n^2+4n+3. It is then shown that the structure function for this $\{e\}$-structure is a differential function of two fundamental tensor invariants. The parametric forms of the fundamental invariants are given and their vanishing characterizes the trivial equation $\ddot{x}^i = 0$. We also show that the vanishing of the fundamental invariants characterizes the unique system of second-order ordinary differential equations admitting a maximal-dimension Lie symmetry group. Thus, equations not equivalent to $\ddot{x}^i = 0$ admit symmetry groups of dimension strictly less than n^2+4n+3.

1. *Introduction*

In this article we consider the following problem: given two systems of second-order differential equations

$$\frac{d^2x^i}{dt^2} = f^i\left(t, x^j, \frac{dx^j}{dt}\right) \quad (1 \leqslant i, j \leqslant n), \tag{1}$$

$$\frac{d^2\bar{x}^i}{d\bar{t}^2} = \bar{f}^i\left(\bar{t}, \bar{x}^j, \frac{d\bar{x}^j}{d\bar{t}}\right) \quad (1 \leqslant i, j \leqslant n), \tag{2}$$

are they equivalent under the pseudo-group of smooth invertible local point transformations? In other words, can one make an invertible change of coordinates

$$(\bar{t}, \bar{x}^i) = \Psi(t, x^j) = (\phi(t, x^j), \psi^i(t, x^j)) \tag{3}$$

such that when the system (1) is expressed in the coordinates $(\bar{t}, \bar{x}^i)$ it is identical to the system (2). This notion of two systems of second-order differential equations being equivalent defines an equivalence relation on the set of differential equations of the form (1). We analyse this problem by applying Élie Cartan's method of equivalence [**2**] which, in theory, provides a way to distinguish between equivalence classes of the systems (1).

The problem of equivalence for scalar equations ($n = 1$) was originally solved by Cartan [**3**]. Subsequently Chern [**5**] investigated the two equivalence problems for systems under the restricted pseudo-groups of smooth invertible local transformations which preserve the independent variable as given by

$$(\bar{t}, \bar{x}^i) = (t, \psi^i(x^j)) \quad \text{and} \quad (\bar{t}, \bar{x}^i) = (t, \psi^i(t, x^j)). \tag{4}$$

Chern was able to cast the question of equivalence between equations (1) and (2)

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into a question about the equivalence of exterior differential systems associated to the corresponding equations, at which point the equivalence method of Élie Cartan was immediately applicable. By applying Cartan's method, Chern was able to associate to any system (1) an $\{e\}$-structure or an *invariant coframe* which may be used to determine whether the systems (1) and (2) are equivalent by either of the transformations in (4). Similarly we obtain an invariant coframe (of different dimension to that of Chern) by applying the equivalence method with the larger group of point transformations (3).

The invariant coframe or $\{e\}$-structure in our problem is of dimension n^2+4n+3, and enjoys the important property that its structure function can be expressed solely in terms of two fundamental families of tensor invariants (which we call $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$), and their successive covariant derivatives. From the general theory of $\{e\}$-structures we then know that $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ (along with their corresponding covariant derivatives) can in principle be used to characterize the different equivalence classes of systems of the form (1). Thus, tensors $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ are essential in determining the geometric properties of the system (1). One example where this can be seen is the inverse problem in the calculus of variations [**1**]. Here the structure of the tensor invariant $\tilde{P}^i_j$ plays a significant role in determining whether a system of the form (1) is a multiple of a set of Euler–Lagrange equations for some Lagrangian (see the memoir by Anderson and Thompson for details [**1**]). One importance of the tensor $\tilde{S}^i_{jkl}$ is that its vanishing characterizes systems of second-order equations which may be associated with a projective connection. This is demonstrated in Appendix B.

It is classical (see Cartan [**4**]) that the dimension of the symmetry group of an $\{e\}$-structure is bounded above by its dimension which in our case is n^2+4n+3. The fact that the symmetry group of the system (1) is identical with the symmetry group of the $\{e\}$-structure immediately re-establishes the fact proved by F. Gonzáles-Gascón and A. González-López [**8**] that the symmetry group admitted by a system of second-order differential equations is bounded above by n^2+4n+3. For comparison, we point out that the proof of González-Gascón and González-López uses Lie-theoretic techniques.

González-Gascón and González-López [**8**] have demonstrated that the simple system

$$\frac{d^2x^i}{dt^2}=0 \tag{5}$$

is an example of a system which admits a Lie symmetry group of the maximal dimension n^2+4n+3. Our contribution here consists of sharpening this result by analysing the integrability conditions of our $\{e\}$-structure to show the remarkable fact that (5) is the *unique* system of second-order differential equations (up to change of coordinates) which admits a point symmetry group of maximal dimension. We may state this in another way by saying that given a system (1) admitting a symmetry group of dimension n^2+4n+3, there exists a set of coordinates such that the system may be put in the form (5). Furthermore, we also show that the vanishing of the fundamental tensor invariants provides a simple coordinate-invariant characterization of the systems equivalent to (5).

It is interesting to note that this uniqueness result does *not* hold for either of the pseudo-groups of transformations in (4) considered by Chern. That is, there exists more than one equivalence class of equations which are maximally symmetric under transformations of either form given in (4). In the scalar case ($n = 1$), the characterization of the equation $\ddot{x} = 0$ by the property of having a point symmetry group of maximal dimension (which is 8) was known to Lie [**11**]. Furthermore, Lie was interested in maximally symmetric systems of equations due to their relationship with the projective groups. Finally, we also note that a complete classification of scalar equations which admit fibre-preserving symmetry groups has been given by L. Hsu and N. Kamran [**10**].

2. *Equivalence of systems of second-order ordinary differential equations*

By following Chern [**5**] we cast the equivalence problem for equations (1) and (2) into a question of equivalence for exterior differential systems on $J^1(\mathbb{R}, \mathbb{R}^n)$. Let (t, x^i, p^i) be coordinates on $J^1(\mathbb{R}, \mathbb{R}^n)$ and $\hat{\theta}^i = dx^i - p^i\,dt$ the contact forms. We assume that $f^i(t, x^j, p^j)$ are smooth functions on a contractible open subset U of $J^1(\mathbb{R}, \mathbb{R}^n)$ and associate to equation (1) the Pfaffian system on U generated by

$$\hat{\theta}^i = dx^i - p^i\,dt, \quad \hat{\pi}^i = dp^i - f^i(t, x^j, p^j)\,dt. \tag{6}$$

The significance of this Pfaffian system is provided by the following lemma.

LEMMA 2.1. *The solutions $x^i = x^i(t)$ to equations* (1) *are in one-to-one correspondence with the one-dimensional integral manifolds γ: $\mathbb{R} \to U$ of the Pfaffian system* (6) *which satisfy the independence condition $\gamma^* dt \neq 0$.*

Similarly if we let $\bar{U} \subset J^1(\bar{\mathbb{R}}, \bar{\mathbb{R}}^n)$ be a contractible open subset with coordinates $(\bar{t}, \bar{x}^i, \bar{p}^i)$ and contact forms $\hat{\bar{\theta}}^i = d\bar{x}^i - \bar{p}^i\,dt$, we then associate with equations (2) the Pfaffian system generated by

$$\hat{\bar{\theta}}^i = d\bar{x}^i - \bar{p}^i\,d\bar{t}, \quad \bar{\pi}^i = d\bar{p}^i - \bar{f}^i(\bar{t}, \bar{x}^i, \bar{p}^i)\,d\bar{t}. \tag{7}$$

To express the condition for equivalence of equations (1) and (2) in terms of the Pfaffian systems (6) and (7) we need the explicit form of the prolongation of the point transformation $(\bar{t}, \bar{x}^i) = \Psi(t, x^i)$ in (3). We write the prolongation

$$\Psi_1\colon J^1(\mathbb{R}, \mathbb{R}^n) \to \bar{J}^1(\bar{\mathbb{R}}, \bar{\mathbb{R}}^n)$$

in coordinates as

$$(\bar{t}, \bar{x}^i, \bar{p}^i) = \Psi_1(t, x^j, p^j) = (\phi(t, x^j), \psi^i(t, x^j), \psi_1^i(t, x^j, p^j))$$

where

$$\bar{p}^i = \psi_1^i(t, x^j, p^j) = \left(\frac{d\phi}{dt}\right)^{-1} \frac{d\psi^i}{dt}.$$

Let $\omega = dt$ and $\bar{\omega} = d\bar{t}$ so that with the collection of 1-forms in equations (6) and (7) we have $(\omega, \theta^i, \pi^i)$ and $(\bar{\omega}, \bar{\theta}^i, \bar{\pi}^i)$ being local coframes on U and $\bar{U}$ respectively. It is now possible to express the equivalence condition as an equivalence problem for the Pfaffian systems (6) and (7).

LEMMA 2.2. *The two systems of differential equations* (1) *and* (2) *are equivalent if and only if there exists a point transformation $(\bar{t}, \bar{x}^i) = \Psi(t, x^i)$ with first* *prolongation* Ψ_1: $U \to \bar{U}$ *satisfying*

$$\Psi_1^* \begin{pmatrix} \hat{\bar{\omega}} \\ \hat{\bar{\theta}}^i \\ \hat{\bar{\pi}}^i \end{pmatrix} = \mathcal{S} \begin{pmatrix} \hat{\omega} \\ \hat{\theta}^i \\ \hat{\pi}^i \end{pmatrix}, \tag{8}$$

where $\mathcal{S}$: $U \to \mathrm{H}$ *is a smooth function on* U *taking values in the Lie subgroup* $\mathrm{H} \subset \mathrm{GL}(2n+1, \mathbb{R})$ *defined by*

$$\mathrm{H} = \left\{ \begin{pmatrix} a & E_j & 0 \\ 0 & A_j^i & 0 \\ 0 & C_j^i & a^{-1}A_j^i \end{pmatrix}, a \in \mathbb{R}^*, A_j^i \in \mathrm{GL}(n, \mathbb{R}), E_j \in \mathbb{R}^n, C_j^i \in M_n(\mathbb{R}) \right\}. \tag{9}$$

Proof. In order to verify this lemma, we need to compute $\Psi_1^* \hat{\bar{\pi}}^i$. This is found to be

$$\begin{aligned} \Psi_1^*(d\bar{p}^i - \bar{f}^i\,dt) &= \frac{\partial \psi_1^i}{\partial t}dt + \frac{\partial \psi_1^i}{\partial x^j}dx^j + \frac{\partial \psi_1^i}{\partial p^j}dp^j - \bar{f}^i \circ \Psi_1 \left(\frac{\partial \phi}{\partial t}dt + \frac{\partial \phi}{\partial x^j}dx^j \right) \\ &= a^{-1}A_j^i(dp_1^j - f^i\,dt) + C_j^i\theta^j + \left(\bar{f}^i \circ \Psi_1 - a^{-1}\frac{d\psi_1^i}{dt} \right)\omega, \end{aligned} \tag{10}$$

where

$$A_j^i = \frac{\partial \psi^i}{\partial x^j} - \psi_1^i \frac{\partial \phi}{\partial x^j}, \quad C_j^i = \frac{\partial \psi_1^i}{\partial x^j} - \bar{f}^i \circ \Psi_1 \frac{\partial \phi}{\partial x^j}, \quad a = \frac{d\phi}{dt}.$$

From equation (10) it is then clear that a necessary and sufficient condition for equivalence is

$$\bar{f}^i \circ \Psi_1 = a^{-1}\frac{d\psi_1^i}{dt},$$

which proves the lemma.

By this lemma the problem of the equivalence of (1) and (2) is set in a form to which the Cartan method may be readily applied. We now describe the equivalence method.

Given two coframes (ω_U^a), (ω_V^a) with $1 \leqslant a, b \leqslant N$ on open contractible subsets $U, V \subset \mathbb{R}^N$, the Cartan equivalence method provides a solution to the problem of whether there exists a diffeomorphism ϕ: $U \to V$ such that

$$\phi^* \omega_V^a = K_b^a \omega_U^b \quad \text{for } 1 \leqslant a, b \leqslant N, \tag{11}$$

where K_b^a maps U to H and $\mathrm{H} \subset \mathrm{GL}(N, \mathbb{R})$ is a linear Lie subgroup of dimension h. In Gardner's book [**7**], we find a detailed presentation of an algorithm which implements the equivalence method and we present the essentials of this algorithm. First define the lift of the coframe ω_U^a to the product space $U \times \mathrm{H}$ by

$$\omega^a = \mathcal{S}_b^a \omega_U^b. \tag{12}$$

Subsequently, we complete the collection of forms ω^a to a coframe on $U \times \mathrm{H}$ by lifting a maximal set of right-invariant Maurer–Cartan forms on H to $U \times \mathrm{H}$. We label the lifted forms by α^s where $1 \leq s \leq h$, and note that any other such collection of forms $\hat{\alpha}^s$ which complete ω^a to a coframe on $U \times \mathrm{H}$ are of the form

$$\hat{\alpha}^s = \alpha^s + V^s_a \omega^a, \tag{13}$$

with $V^s_a \in C^\infty(U \times \mathrm{H})$. Differentiating (12) we obtain the structure equations

$$d\omega^a = (d\mathcal{S}^a_b)(\mathcal{S}^{-1})^b_c \wedge \omega^c + \mathcal{S}^a_b\, d\omega^b_U = C^a_{bs}\alpha^s \wedge \omega^b + \Gamma^a_{bc}\omega^b \wedge \omega^c. \tag{14}$$

At this point we perform what Gardner [7] calls *absorption of torsion*. This amounts to using the freedom in the forms α^s in equation (13) to eliminate as many of the $\Gamma^a_{bc}\omega^a \wedge \omega^b$ terms in equation (14) as possible. This requires solving for as many of the functions V^s_a as possible in the equation

$$V^s_{[c}C^a_{b]s} = \Gamma^a_{bc}. \tag{15}$$

The resulting structure equations then take the form

$$d\omega^a = C^a_{bs}\hat{\alpha}^s \wedge \omega^b + \bar{\Gamma}^a_{bc}\omega^b \wedge \omega^c, \tag{16}$$

where the $\bar{\Gamma}^a_{bc}$ are linear combinations of the original Γ^a_{bc}. The importance of this procedure is that $\bar{\Gamma}^a_{bc}$ is now a tensor function on $U \times \mathrm{H}$ (H-equivariant). The absolute invariants of $\bar{\Gamma}^a_{bc}$ will provide necessary conditions for the existence of an equivalence. The invariants may then be used to reduce the structure group H to a linear subgroup $\mathrm{H}_1 \subset \mathrm{H}$. After a reduction of the group H to H_1 we return to the start of the algorithm with H_1 in place of H.

If the tensor $\bar{\Gamma}^a_{bc}$ vanishes or if its structure is difficult to normalize, we *prolong* the problem. The first step in prolongation is to compute the degree of freedom we have in the coframe (ω^a, α^i) which leaves $\bar{\Gamma}^a_{bc}$ unchanged. The freedom here is given by the solution to the homogeneous system of equations for V^s_a in (15) which are

$$V^s_{[c}C^a_{b]s} = 0. \tag{17}$$

The abelian group

$$\mathrm{H}^{(1)} = \begin{pmatrix} I_{N\times N} & 0 \\ V^s_a & I_{h\times h} \end{pmatrix}, \tag{18}$$

with V^s_a satisfying (17), is called the (first) prolongation of H. We then start the process again by lifting the coframe $(\omega^a, \hat{\alpha}^s)$ on $U \times \mathrm{H}$ to $U \times \mathrm{H} \times \mathrm{H}^{(1)}$. The algorithm terminates when we obtain the trivial group by either reduction or prolongation. In this case the equivalence problem is known as the equivalence problem for $\{e\}$-structures.

The entire equivalence method may be given by a geometric description using principal fibre bundles. This approach is given by Sternberg in [12].

3. *The associated $\{e\}$-structure*

In this section we apply the Cartan equivalence method with the coframes and structure group given in Lemma 2.2. For convenience we will introduce the following notation: let $V \subset \mathbb{R}^k$ and η^a be a coframe on V with X_a being the dual frame to η^a, and let ω be a 1-form and Θ a 2-form. We then define the coefficient operator by

$$(\omega)_{\eta^a} = X_a \lrcorner\, \omega, \quad (\Theta)_{\eta^a\eta^b} = X_b \lrcorner\, X_a \lrcorner\, \Theta,$$

with the obvious extension to forms of higher degree. As well, we use the summation convention and (jk), $[jk]$ for symmetrization and skew-symmetrization on indices defined by

$$V_{(ij)} = \tfrac{1}{2}(V_{ij} + V_{ji}), \quad V_{[ij]} = \tfrac{1}{2}(V_{ij} - V_{ji}).$$

By using the coframe in equation (6) and the structure group of Lemma 2.2, we can see that the lift of the coframe to $U \times \mathrm{H}$ is

$$\begin{pmatrix} \omega \\ \theta^i \\ \pi^i \end{pmatrix} = \begin{pmatrix} a & E_j & 0 \\ 0 & A^i_j & 0 \\ 0 & C^i_j & a^{-1}A^i_j \end{pmatrix} \begin{pmatrix} \hat{\omega} \\ \hat{\theta}^j \\ \hat{\pi}^j \end{pmatrix}. \tag{19}$$

The structure equations for the lifted forms are then

$$\begin{pmatrix} d\omega \\ d\theta^i \\ d\pi^i \end{pmatrix} = (d\mathscr{S})\mathscr{S}^{-1} \wedge \begin{pmatrix} \omega \\ \theta^j \\ \pi^j \end{pmatrix} + \mathscr{S} \begin{pmatrix} d\hat{\omega} \\ d\hat{\theta}^j \\ d\hat{\pi}^j \end{pmatrix}, \tag{20}$$

where $(d\mathscr{S})\mathscr{S}^{-1}$ is the lift to $U \times \mathrm{H}$ of the Maurer–Cartan form for H given by

$$d\mathscr{S}(\mathscr{S}^{-1}) = \begin{pmatrix} \alpha & \kappa_j & 0 \\ 0 & \Omega^i_j & 0 \\ 0 & \Sigma^i_j & \Omega^i_j - \alpha\delta^i_j \end{pmatrix}. \tag{21}$$

We now apply the Cartan equivalence method to prove the following theorem.

THEOREM 3.1. *Solutions Ψ_1: $U \to \bar{U}$ to the equivalence problem for systems of n ($\geqslant 2$) second-order ordinary differential equations are in one-to-one correspondence with the solutions of an equivalence problem for an (n^2+4n+3)-dimensional $\{e\}$-structure which is obtained by applying the equivalence method to the initial coframe $(\hat{\omega}, \hat{\theta}^i, \hat{\pi}^i)$ with the structure group* H *given in Lemma* 2.2.

Proof. By using the Maurer–Cartan form in (21) and the coframe in (6) we find that the structure equations (20) can be written as (see Lemma 4.1 for more details)

$$\begin{aligned} d\omega &= \alpha \wedge \omega + \kappa_j \wedge \theta^j + g_j \pi^j \wedge \omega + h_j \theta^j \wedge \omega + k_{jk} \theta^j \wedge \theta^k, \\ d\theta^i &= \Omega^i_j \wedge \theta^j - \pi^i \wedge \omega + l^i_j \theta^j \wedge \omega + m^i_{jk} \theta^j \wedge \theta^k + \pi^i \wedge n_j \theta^j, \\ d\pi^i &= \Sigma^i_j \wedge \theta^j + (\Omega^i_j - \alpha\delta^i_j) \wedge \pi^j + P^i_j \theta^j \wedge \omega + Q^i_{jk} \theta^j \wedge \theta^k \\ &\quad + K^i_j \pi^j \wedge \omega + L^i_{jk} \pi^j \wedge \theta^k, \end{aligned}$$

where g_j, h_j, k_{jk}, l^i_j, m^i_j, n_j, P^i_j, Q^i_{jk}, K^i_j, L^i_{jk} are functions on $U \times \mathrm{H}$. If we absorb torsion by setting

$$\begin{aligned} \alpha &= \hat{\alpha} + g_j \pi^j - n^{-1}(K^j_j - l^j_j)\omega, & \Omega^i_j &= \hat{\Omega}^i_j + l^i_j \omega + m^i_{jk} \theta^k - \pi^i n_j, \\ \kappa_j &= \hat{\kappa}_j + h_j \omega + k_{jk} \theta^k, & \Sigma^i_j &= \hat{\Sigma}^i_j + P^i_j \omega + Q^i_{jk} \theta^k + (m^i_{kj} - L^i_{kj})\pi^k, \end{aligned} \tag{22}$$

the structure equations become (after dropping the hats)

$$\begin{aligned} d\omega &= \alpha \wedge \omega + \kappa_j \wedge \theta^j, \\ d\theta^i &= \Omega^i_j \wedge \theta^j - \pi^i \wedge \omega, \\ d\pi^i &= \Sigma^i_j \wedge \theta^j + (\Omega^i_j - \alpha\delta^i_j) \wedge \pi^j + \tilde{K}^i_j \pi^j \wedge \omega + (n_{[j}\delta^i_{k]} - g_{[j}\delta^i_{k]})\pi^j \wedge \pi^k, \end{aligned} \tag{23}$$

where $\tilde{K}^i_j$ is trace free. Setting $d^2\theta^i = 0$ determines that $n_j = g_j$, and

$$d\Omega^i_j \equiv \Omega^i_k \wedge \Omega^k_j - \pi^i \wedge \kappa_j - \Sigma^i_j \wedge \omega \mod (\theta^j), \tag{24}$$

which allows us to compute $d^2\pi^i \equiv 0 \mod (\theta^i)$ to find

$$d\tilde{K}^i_j + 2\Sigma^i_j - 2n^{-1}\Sigma^k_k\delta^i_j - \Omega^i_k\tilde{K}^k_j + \tilde{K}^i_k\Omega^k_j + \alpha\tilde{K}^i_j \equiv 0 \mod (\omega, \theta^i, \pi^i).$$

In this equation the appearance of the forms Σ^i_j allows us to use the group action of C^i_j to translate $\tilde{K}^i_j$ to zero and resulting in a reduced structure group $\mathrm{H}_1 \subset \mathrm{H}$ and reduced Maurer–Cartan form given by

$$C^i_j = cA^i_j, \text{ with } c \in \mathbb{R}, \quad \text{and} \quad \Sigma^i_j = \sigma\delta^i_j$$

where σ is a right-invariant form on H_1. In the new structure equations with the reduced group H_1 we may use the absorption for α, κ^i and Ω^i_j in equation (22) and the only change for the equations in (23) is

$$d\pi^i = \sigma \wedge \theta^i + (\hat{\Omega}^i_j - \hat{\alpha}\delta^i_j) \wedge \pi^j + P^i_j\theta^j \wedge \omega + Q^i_{jk}\theta^j \wedge \theta^k + (L^i_{jk} - m^i_{jk})\pi^j \wedge \theta^k.$$

We may further absorb torsion in this equation by setting (assuming that $n \geq 2$)

$$\begin{aligned} \hat{\Omega}^i_j &= \bar{\Omega}^i_j + \bar{L}^i_{(jk)}\theta^k, & \sigma &= \bar{\sigma} + \frac{1}{n}P^k_k\omega + \frac{2}{n-1}Q^j_{jk}\theta^k - \frac{1}{n-1}\bar{L}^l_{[jl]}\pi^j, \\ \hat{\alpha} &= \bar{\alpha} - \frac{1}{n-1}\bar{L}^l_{[lk]}\theta^k, & \hat{\kappa}_j &= \bar{\kappa}_j - \frac{1}{n-1}\bar{L}^l_{[lj]}\omega, \end{aligned} \tag{25}$$

where $\bar{L}^i_{jk} = L^i_{jk} - m^i_{jk}$, so that the resulting equation for $d\pi^i$ is (dropping the overlines on σ, α, κ_j, and Ω^i_j)

$$d\pi^i = \sigma \wedge \theta^i + (\Omega^i_j - \alpha\delta^i_j) \wedge \pi^j + \tilde{P}^i_j\theta^j \wedge \omega + \tilde{Q}^i_{jk}\theta^j \wedge \theta^k + \tilde{L}^i_{jk}\pi^j \wedge \theta^k,$$

where $\tilde{P}^i_j$, $\tilde{Q}^i_{jk}$ and $\tilde{L}^i_{jk}$ are the trace-free parts of P^i_j, Q^i_{jk} and $\bar{L}^i_{[jk]}$ respectively. By using (24) we now find that

$$(d^2\pi^i)_{\pi^j\pi^k\omega} = \tilde{L}^i_{jk} = 0.$$

At this point we prolong the structure equations, and a parametrization for the prolonged group $\mathrm{H}^{(1)}_1$ (the solution to equations (17)) is given by (K_j, D_{jk}) in the following:

$$\begin{aligned} \bar{\alpha} &= \alpha + K_j\theta^j, & \bar{\Omega}^i_j &= \Omega^i_j + 2K_{(k}\delta^i_{j)}\theta^k, \\ \bar{\sigma} &= \sigma + K_j\pi^j, & \bar{\kappa}_j &= \kappa_j + D_{jk}\theta^k + K_j\omega, \end{aligned}$$

where for these formulas we require $n \geq 2$. The overlined forms provide the canonical form (or lifted forms) for the principal bundle $U \times \mathrm{H}_1 \times \mathrm{H}^{(1)}_1 \to U \times \mathrm{H}_1$

and we apply the equivalence method in this case. The Maurer–Cartan form (β_j, Υ_{jk}) for $H_1^{(1)}$ is given parametrically by

$$\beta_j = dK_j \quad \text{and} \quad \Upsilon_{jk} = dD_{jk}, \tag{26}$$

allowing the structure equations to be written in the general form (by dropping the overline)

$$\begin{aligned} d\omega &= \alpha \wedge \omega + \kappa_j \wedge \theta^j, \\ d\theta^i &= \Omega^i_j \wedge \theta^j - \pi^i \wedge \omega, \\ d\pi^i &= \sigma\theta^i + (\Omega^i_j - \alpha\delta^i_j) \wedge \pi^j + \tilde{P}^i_j \theta^j \wedge \omega + \tilde{Q}^i_{jk}\theta^j \wedge \theta^k, \\ d\alpha &= \beta_j \wedge \theta^j + \mathbf{t}^0, \\ d\kappa_j &= \Upsilon_{jk} \wedge \theta^k + \beta_j \wedge \omega + \mathbf{T}^2_j, \\ d\sigma &= \beta_j \wedge \pi^j + \mathbf{t}^1, \\ d\Omega^i_j &= 2\beta_{(k}\delta^i_{j)} \wedge \theta^k + \mathbf{T}^i_j, \end{aligned} \tag{27}$$

where $\mathbf{t}^0$ and $\mathbf{t}^1$ are 2-forms, $\mathbf{T}^2_j$ is an $\mathbb{R}^n$-valued 2-form, and $\mathbf{T}^i_j$ is an $M_n(\mathbb{R})$-valued 2-form all of which are contained in the exterior algebra generated by $(\omega, \theta^i, \pi^i, \alpha, \sigma, \Omega^i_j, \kappa_j)$. Now we absorb torsion in equation (27) by setting

$$\beta_j = \hat{\beta}_j - (\mathbf{t}^0)_{\theta^j} - (\mathbf{t}^1)_{\theta^{(k}\pi^{j)}}\theta^k, \tag{28}$$

$$\Upsilon_{jk} = \hat{\Upsilon}_{jk} - (\mathbf{T}^2_{(j})_{\theta^{k)}}, \tag{29}$$

so that (after dropping the hats) the structure equations (27) retain the same form but with the additional conditions

$$(\mathbf{t}^0)_{\theta^i} = 0, \quad (\mathbf{t}^1)_{\theta^{(k}\pi^{j)}} = 0, \quad (\mathbf{T}^2_{(j})_{\theta^{k)}} = 0. \tag{30}$$

To determine the left-over torsion (the structure function) we apply a sequence of integrability conditions. The first condition we use is $d^2\theta^i = 0$ which, on account of equation (27), gives

$$\mathbf{T}^i_j = \Omega^i_k \wedge \Omega^k_j - \pi^i \wedge \kappa_j - \sigma \wedge \omega\delta^i_j + \tilde{Q}^i_{jk}\theta^k \wedge \omega + \xi^i_{jk} \wedge \theta^k, \tag{31}$$

where ξ^i_{jk} is a collection of one-forms satisfying

$$\xi^i_{jk} \wedge \theta^j \wedge \theta^k = 0. \tag{32}$$

The next integrability condition, $d^2\omega = 0$, gives

$$\mathbf{t}^0 = -\kappa_j \wedge \pi^j + \lambda \wedge \omega + \xi^0_i \wedge \theta^i, \tag{33}$$

$$\mathbf{T}^2_i = \alpha \wedge \kappa_i + \kappa_j \wedge \Omega^j_i + \xi^2_{ij} \wedge \theta^j + \xi^0_i \wedge \omega, \tag{34}$$

where ξ^2_{ij}, ξ^0_i and λ are one-forms subject to the conditions

$$\xi^2_{ij} \wedge \theta^j \wedge \theta^i = 0, \tag{35}$$

$$(\lambda)_\omega = (\lambda)_{\theta^i} = 0. \tag{36}$$

The condition (36) means that λ has no ω or θ^i terms, and arises in order that the decomposition for $\mathbf{t}^0$ in (33) be unique. The absorption in equation (28) (or see (30)) implies that

$$(\mathbf{t}^0)_{\theta^i} = \xi^0_i = 0$$

in equation (33). The next integrability condition, $d^2\pi^i = 0$, becomes, on account of (27),

$$\begin{aligned} 0 = {} & (d\tilde{P}^i_j - \Omega^i_k \tilde{P}^k_j + \tilde{P}^i_k \Omega^k_j + 2\alpha \tilde{P}^i_j + 3\tilde{Q}^i_{kj}\pi^k) \wedge \theta^j \wedge \omega \\ & + (d\tilde{Q}^i_{jk} - \Omega^i_l \tilde{Q}^l_{jk} - 2\tilde{Q}^i_{l[j}\Omega^l_{k]} + \alpha \tilde{Q}^i_{jk} + \tilde{P}^i_{[j}\kappa_{k]}) \wedge \theta^j \wedge \theta^k \\ & + \xi^i_{jk} \wedge \theta^k \wedge \pi^j - (\lambda + 2\sigma) \wedge \omega \wedge \pi^i + (\mathbf{t}^1 + \alpha \wedge \sigma) \wedge \theta^i. \end{aligned} \tag{37}$$

Putting the coefficient of $\omega \wedge \pi^i$ in this equation to zero and recalling from equation (36) that λ has *no* θ^i or ω terms, we have

$$\lambda = -2\sigma. \tag{38}$$

Before proceeding with equation (37) we need information about the form of $\mathbf{t}^1$. This we get by setting $d^2\alpha = 0$ to find

$$\begin{aligned} d^2\alpha &= d\beta_j \wedge \theta^j - \beta_j \wedge d\theta^j - d\kappa_j \wedge \pi^j + \kappa_j \wedge d\pi^j \\ &\quad - 2\, d\sigma \wedge \omega + 2\sigma \wedge d\omega \\ &\equiv 2(\sigma \wedge \alpha - \mathbf{t}^1) \wedge \omega \quad \mathrm{mod}\,(\theta^i). \end{aligned} \tag{39}$$

Thus we have

$$\mathbf{t}^1 = \sigma \wedge \alpha + \xi^1_i \wedge \theta^i + \chi^1 \wedge \omega, \tag{40}$$

where χ^1 and ξ^1_i are one-forms subject to the conditions

$$(\chi^1)_\omega = (\chi^1)_{\theta^i} = 0, \tag{41}$$

$$(\mathbf{t}^1)_{\theta^{(i}\pi^{j)}} = -(\xi^1_{(i})_{\pi^{j)}} = 0, \tag{42}$$

where condition (41) gives a unique decomposition in (40) and condition (42) comes from equation (30). Now substitute $\mathbf{t}^1$ from equation (40) into (37) and set the $\theta^k \wedge \pi^j$ term to zero to obtain

$$[\xi^i_{jk} - (d\tilde{P}^i_k)_{\pi^j}\omega - 3\tilde{Q}^i_{jk}\omega + (d\tilde{Q}^i_{lk})_{\pi^j}\theta^l + \delta^i_{[k}(\xi^1_{l]})_{\pi^j}\theta^l] \wedge \theta^k \wedge \pi^j = 0. \tag{43}$$

From this we may conclude that ξ^i_{jk} may be written as

$$\xi^i_{jk} = T^i_{jk}\omega + R^i_{jlk}\theta^l + S^i_{jlk}\pi^l, \tag{44}$$

with T^i_{jk}, S^i_{jlk} and $R^i_{j[lk]} = R^i_{jlk}$ being functions on $U \times \mathrm{H} \times \mathrm{H}^{(1)}$. By considering the condition in equation (32) and equation (43) we have the following constraints on these functions

$$T^i_{[jk]} = R^i_{[jkl]} = 0, \quad S^i_{jkl} = S^i_{(jkl)}. \tag{45}$$

At this point it is possible to determine the action of $\mathrm{H}^{(1)}_1$ on S^i_{jkl} by taking the exterior derivative of

$$\begin{aligned} d\Omega^i_i - (n+1)\, d\alpha = {} & (n+2)(\kappa_i \wedge \pi^i + \sigma \wedge \omega) \\ & + (T^i_{il}\omega + R^i_{ikl}\theta^k + S^i_{ikl}\pi^k) \wedge \theta^l, \end{aligned} \tag{46}$$

to get

$$d^2\Omega^i_i - (n+1)\, d^2\alpha \equiv dS^i_{ikl} - (n+2)\Upsilon_{kl} \equiv 0 \quad \mathrm{mod}\,(\omega, \theta^i, \pi^i, \alpha, \kappa_j, \sigma, \Omega^i_j).$$

Thus we may translate the trace S^i_{ijk} to zero by using the action of D_{jk}. The corresponding reduction of $\mathrm{H}^{(1)}$ has

$$D_{ij} = 0 \quad \text{and} \quad \Upsilon_{jk} = 0.$$

Explicitly we will write

$$\tilde{S}^i_{jkl} = S^i_{jkl} - \frac{3}{n+2} S^m_{m(jk}\delta^i_{l)}, \tag{47}$$

and the reduction of $H^{(1)}_1$ by $(H^{(1)}_1)_1$. To continue now we may use all the previous equations except the absorption in (29) and thus the conditions on $\mathbf{T}^2_i$ in (30) must be dropped. If we try to prolong $(H^{(1)}_1)_1$, it is easily seen from equations (27) that the prolongation will be trivial, and thus the theorem is proved.

We now summarize what is known at this point about the structure equations for the $\{e\}$-structure $(\omega, \theta^i, \pi^i, \alpha, \sigma, \kappa_i, \Omega^i_j, \beta_i)$ on $U \times H_1 \times (H^{(1)}_1)_1$. By using the equations in (27) and the information in equations (31), (33), (34), (38), (40), and (44), we have

$$\begin{aligned}
d\omega &= \alpha \wedge \omega + \kappa_j \wedge \theta^j,\\
d\theta^i &= \Omega^i_j \wedge \theta^j - \pi^i \wedge \omega,\\
d\pi^i &= \sigma \wedge \theta^i + (\Omega^i_j - \alpha\delta^i_j) \wedge \pi^j + \tilde{P}^i_j\theta^j \wedge \omega + \tilde{Q}^i_{jk}\theta^j \wedge \theta^k,\\
d\alpha &= \beta_j \wedge \theta^j - \kappa_j \wedge \pi^j - 2\sigma \wedge \omega,\\
d\sigma &= \beta_j \wedge \pi^j + \sigma \wedge \alpha + \xi^1_j \wedge \theta^j + \chi^1 \wedge \omega,\\
d\Omega^i_j &= 2\beta_{(k}\delta^i_{j)} \wedge \theta^k + \Omega^i_k \wedge \Omega^k_j - \pi^i \wedge \kappa_j - \sigma \wedge \omega\delta^i_j + (\tilde{Q}^i_{jk} - T^i_{jk})\theta^k \wedge \omega\\
&\quad + R^i_{jkl}\theta^k \wedge \theta^l + \tilde{S}^i_{jkl}\pi^k \wedge \theta^l,\\
d\kappa_i &= \beta_i \wedge \omega + \alpha \wedge \kappa_i + \kappa_j \wedge \Omega^j_i + \xi^2_{ij} \wedge \theta^j,
\end{aligned} \tag{48}$$

with the conditions on $\tilde{Q}^i_{jk}$, R^i_{jkl}, $\tilde{S}^i_{jkl}$, T^i_{jk} from (45) being

$$\tilde{Q}^i_{(jk)} = 0, \quad R^i_{[jkl]} = R^i_{j(kl)} = 0, \quad \tilde{S}^i_{(jkl)} = \tilde{S}^i_{jkl}, \quad T^i_{[jk]} = 0, \tag{49}$$

and the traces of $\tilde{P}^i_j$, $\tilde{Q}^i_{jk}$ and $\tilde{S}^i_{jkl}$ are zero. The conditions on the 1-forms ξ^1_j, χ^1 and ξ^2_{ij} from equations (35), (36), (41) and (42) are

$$(\xi^1_{(j})_{\theta^{k)}} = (\xi^1_{(j})_{\pi^{k)}} = 0, \quad (\chi^1)_\omega = (\chi^1)_{\theta^i} = 0, \quad (\xi^2_{i(j})_{\theta^{k)}} = 0, \quad \xi^2_{ij} \wedge \theta^i \wedge \theta^j = 0. \tag{50}$$

We now proceed to examine further the torsion in this $\{e\}$-structure.

4. *The fundamental invariants*

In this section we examine the structure equations for the $\{e\}$-structure of Theorem 3.1 to prove that all the tensor invariants which arise in the structure equations are differential functions of $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$.

THEOREM 4.1. *All tensor invariants which arise in the structure equations for the $\{e\}$-structure $(\omega, \theta^i, \pi^i, \alpha, \kappa_j, \sigma, \Omega^i_j, \beta_j)$ are homogeneous linear differential functions of $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$. That is, the tensor invariants may be expressed as homogeneous linear combinations of $\tilde{P}^i_{jkl}$, $\tilde{S}^i_{jkl}$ and their successive covariant derivatives.*

The method of proof involves taking the integrability conditions for the

$\{e\}$-structure in (48) and showing that all the coefficients in the structure equations are determined by $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$. We give the explicit dependencies for only a few of the invariants.

Proof. We go back to equation (37) and take into account equations (38), (40) and (44) (or use equation (48)) to get

$$\begin{aligned} 0 = (d\tilde{P}^i_j - \Omega^i_k\tilde{P}^k_j + \tilde{P}^i_k\Omega^k_j + 2\alpha\tilde{P}^i_j + 3\tilde{Q}^i_{kj}\pi^k - T^i_{kj}\pi^k - \chi^1\delta^i_j) \wedge \theta^j \wedge \omega \\ + (d\tilde{Q}^i_{jk} - \Omega^i_l\tilde{Q}^l_{jk} - 2\tilde{Q}^i_{l[j}\Omega^l_{k]} + \alpha\tilde{Q}^i_{jk} + \tilde{P}^i_{[j}\kappa_{k]} + \xi^1_{[j}\delta^i_{k]} + R^i_{ljk}\pi^l) \wedge \theta^j \wedge \theta^k. \end{aligned} \tag{51}$$

From this equation we obtain the following congruences:

$$\begin{aligned} d\tilde{P}^i_j - \Omega^i_k\tilde{P}^k_j + \tilde{P}^i_k\Omega^k_j + 2\alpha\tilde{P}^i_j + 3\tilde{Q}^i_{kj}\pi^k - T^i_{kj}\pi^k - \chi^1\delta^i_j \equiv 0 \quad \mathrm{mod}\,(\theta^i), \\ d\tilde{Q}^i_{jk} - \Omega^i_l\tilde{Q}^l_{jk} - 2\tilde{Q}^i_{l[j}\Omega^l_{k]} - \alpha\tilde{Q}^i_{jk} + \tilde{P}^i_{[j}\kappa_{k]} + \xi^1_{[j}\delta^i_{k]} + R^i_{ljk}\pi^l \equiv 0 \quad \mathrm{mod}\,(\omega). \end{aligned} \tag{52}$$

If we take the trace in the first of these equations and take into account equation (50) which shows that χ^1 has no ω or θ^i terms, we find that

$$\chi^1 = -n^{-1}T^i_{ki}\pi^k.$$

Now taking the coefficients of $\pi^k \wedge \theta^j \wedge \omega$ in equation (51) we obtain

$$(d\tilde{P}^i_j)_{\pi^k} + 3\tilde{Q}^i_{kj} - T^i_{kj} + n^{-1}T^l_{kl}\delta^i_j = 0. \tag{53}$$

From this equation we determine T^l_{kl} by skew-symmetrizing on j, k and taking the trace to get

$$T^l_{kl} = (n-1)^{-1}(d\tilde{P}^i_k)_{\pi^i},$$

where we have used the conditions on $\tilde{Q}^i_{jk}$ and T^i_{jk} in (49). Now skew-symmetrization and symmetrization in equation (53) leads to

$$\begin{aligned} \tilde{Q}^i_{jk} &= \tfrac{1}{3}((d\tilde{P}^i_{[j})_{\pi^{k]}} - V_{[j}\delta^i_{k]}), \\ T^i_{jk} &= (d\tilde{P}^i_{(j})_{\pi^k} + (n-1)^{-1}\delta^i_{(k}(d\tilde{P}^l_{j)})_{\pi^l}. \end{aligned} \tag{54}$$

If we now take the trace in the second equation in (52), we have

$$\xi^1_j = (1-n)^{-1}\kappa_k\tilde{P}^k_j + W_{kj}\pi^k + X_{kj}\theta^k + Y_j\omega, \tag{55}$$

where W_{jk}, X_{jk} and Y_j are functions yet to be determined. By the conditions on ξ^1_i in equation (50) these functions satisfy

$$W_{(jk)} = X_{(jk)} = 0. \tag{56}$$

Upon substituting (55) into (51) and applying symmetry arguments similar to those above we find that

$$(d\tilde{Q}^i_{jk})_{\pi^l} + W_{l[j}\delta^i_{k]} + R^i_{ljk} = 0, \quad Y_j = (n-1)^{-1}(d\tilde{P}^i_j)_{\theta^i}, \tag{57}$$

with the first of these equations giving

$$W_{kj} = 2(1-n)^{-1}R^i_{[kj]i}, \quad R^i_{(jk)i} = 0. \tag{58}$$

At this point all information from equation (51) has been obtained while W_{jk} and

X_{jk} in (55) are still undetermined. The next integrability condition we use is by taking the exterior derivative of equation (46) to get

$$0 = (n+2)(\xi^2_{jk} \wedge \theta^k \wedge \pi^j - \kappa_i \wedge \tilde{P}^i_j \theta^j \wedge \omega - \kappa_i \wedge \tilde{Q}^i_{jk} \theta^j \wedge \theta^k + \xi^1_j \wedge \theta^j \wedge \omega)$$
$$+ d(T^i_{ij}\theta^j \wedge \omega + R^i_{ijk}\theta^j \wedge \theta^k), \tag{59}$$

which implies that ξ^2_{jk} has the form

$$\xi^2_{jk} = \hat{W}_{jlk}\pi^l + \hat{X}_{jlk}\theta^l + \omega \hat{Y}_{jk}, \tag{60}$$

where $\hat{W}_{ijk}$, $\hat{X}_{ijk}$ and $\hat{Y}_{ij}$ are smooth functions. By equations (50) and (59) these functions are subject to the conditions

$$\hat{W}_{ijk} = \hat{W}_{(ijk)}, \quad \hat{X}_{[ijk]} = 0, \quad \hat{Y}_{[ij]} = 0. \tag{61}$$

The coefficient of $\omega \wedge \theta^k \wedge \pi^j$ in equation (59) being zero gives

$$(n+2)(\hat{Y}_{jk} - W_{jk}) + 2R^i_{ijk} + (dT^i_{ik})_{\pi^j} = 0. \tag{62}$$

By observing the symmetry properties of $\hat{Y}_{jk}$ and W_{jk} from equations (61) and (56) we find that symmetrization on j, k in equation (62) determines $\hat{Y}_{jk}$ while skew-symmetrization gives

$$W_{jk} = \frac{1}{3n}(dT^i_{i[j})_{\pi^{k]}} \tag{63}$$

where we have used equation (58) and the symmetry properties of R^i_{jkl} in (49) to arrive at this. Thus we have $\hat{Y}_{jk}$ and W_{jk} in terms of covariant derivatives of T^i_{jk} which is given in (54). As well we may use W_{jk} to determine R^i_{jkl} by equation (57). Lastly, the coefficient of $\pi^l \wedge \theta^j \wedge \pi^k$ in equation (59) determines $\hat{W}_{jlk}$, and we only have to determine $\hat{X}_{jlk}$ in (60).

To determine $\hat{X}_{jlk}$ we first use equation (39) to get

$$d\beta_j = \beta_k \wedge \Omega^k_j + \kappa_j \wedge \sigma - \xi^2_{kj} \wedge \pi^k + \kappa_k \wedge \tilde{P}^k_j \omega - 2\xi^1_j \wedge \omega$$
$$+ \kappa_l \wedge \tilde{Q}^l_{jk}\theta^k + \lambda_{jk} \wedge \theta^k, \tag{64}$$

where the 1-forms λ_{ij} satisfy $\lambda_{ij} \wedge \theta^i \wedge \theta^j = 0$. This allows us to compute the following:

$$(d^2\Omega^i_j)_{\pi^m\pi^k\theta^l} = (d\tilde{S}^i_{jl[k})_{\pi^{m]}} - \hat{W}_{jl[m}\delta^i_{k]} = 0.$$

This determines $\hat{W}_{jlk}$ in terms of $\tilde{S}^i_{jkl}$. To complete the theorem it is clear that the integrability condition $d^2\kappa_j = 0$ along with the expression in (64) will completely determine $d\beta_j$. This allows us to conclude that all the tensor invariants in the structure equations for the $\{e\}$-structure are homogeneous linear functions of the tensor invariants $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ and their successive covariant derivatives, thus proving the theorem.

(For a complete list of the explicit dependencies of the tensor invariants in terms of $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ see [**6**].)

Now we will perform some of the parametric calculations to find the explicit form of the tensor invariants $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$. In order to simplify the formulas we use the following notation for differentiation:

$$f^i_{,j}=\frac{\partial f^i}{\partial x^j},\quad f^i_{|j}=\frac{\partial f^i}{\partial p^j},\quad \frac{df^i}{dt}=\frac{\partial f^i}{\partial t}+\frac{\partial f^i}{\partial x^j}p^j+\frac{\partial f^i}{\partial p^j}f^j.$$

We now give the parametric forms for the tensor invariants $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$.

LEMMA 4.1. *The parametric forms for* $\tilde{P}^i_j$ *and* $\tilde{S}^i_{jkl}$ *at the identity of the structure group* $\mathrm{H}\times(\mathrm{H}^{(1)}_1)_1$ *are*

$$(\tilde{P}^i_j)\big|_{\mathrm{Id}}=\frac{1}{2}\frac{d}{dt}f^i_{|j}-f^i_{,j}-\tfrac{1}{4}f^i_{|k}f^k_{|j}-\frac{1}{n}\,\delta^i_j\Big(\frac{1}{2}\frac{d}{dt}f^k_{|k}-f^k_{,k}-\tfrac{1}{4}f^l_{|k}f^k_{|l}\Big),$$

$$(\tilde{S}^i_{jkl})\big|_{\mathrm{Id}}=f^i_{|jkl}-3(n+2)^{-1}f^m_{|m(jk}\delta^i_{l)}.$$

Proof. We first need to determine the frame corresponding to the reduction of H to H_1. From the initial coframe in (6) we have

$$d\hat{\omega}=0,\quad d\hat{\theta}^i=-dp^i\wedge dt,\quad d\hat{\pi}^i=-df^i\wedge dt, \tag{65}$$

and by using

$$\begin{pmatrix}\hat{\omega}\\ \hat{\theta}^i\\ \hat{\pi}^i\end{pmatrix}=\begin{pmatrix}a^{-1} & -a^{-1}E_k(A^{-1})^k_j & 0\\ 0 & (A^{-1})^i_j & 0\\ 0 & -a(A^{-1})^i_kC^k_l(A^{-1})^l_j & a(A^{-1})^i_j\end{pmatrix}\begin{pmatrix}\omega\\ \theta^j\\ \pi^j\end{pmatrix}$$

we find that the structure equations are

$$\begin{aligned}
d\omega &= \alpha\wedge\omega+\kappa_j\wedge\theta^j-E_l(A^{-1})^l_i(\pi^i-C^i_k(A^{-1})^k_j\theta^j)\wedge(\omega-E_k(A^{-1})^k_j\theta^j),\\
d\theta^i &= \Omega^i_j\wedge\theta^j-(\pi^i-C^i_k(A^{-1})^k_j\theta^j)\wedge(\omega-E_k(A^{-1})^k_j\theta^j),\\
d\pi^i &= \Sigma^i_j\wedge\theta^j+(\Omega^i_j-\alpha\delta^i_j)\wedge\pi^j-C^i_l(A^{-1})^l_j(\pi^j-C^j_k(A^{-1})^k_m\theta^m)\wedge(\omega-E_k(A^{-1})^k_j\theta^j)\\
&\quad-[a^{-1}A^i_kf^k_{|l}(A^{-1})^l_j(\pi^j-C^j_m(A^{-1})^m_r\theta^r)+A^i_kf^k_{,l}(A^{-1})^l_j\theta^j]\wedge(\omega-E_j(A^{-1})^j_k\theta^k).
\end{aligned}$$

From these equations it is easy to determine $\tilde{K}^i_j$ in equation (23) to be

$$\tilde{K}^i_j=-a^{-1}A^i_k(f^k_{|l})(A^{-1})^l_j-2C^i_k(A^{-1})^k_j+n^{-1}\delta^i_j(A^m_k(f^k_{|l})(A^{-1})^l_m+2C^m_k(A^{-1})^k_m).$$

Thus changing to the frame

$$\hat{\omega}=dt,\quad \hat{\theta}^i=dx^i-p^i\,dt,\quad \hat{\pi}^i=dp^i-f^i\,dt-\tfrac{1}{2}f^i_{|j}\hat{\theta}^j, \tag{66}$$

will give rise to an equivalence problem with structure group H_1. We now compute $d\hat{\pi}^i$, by using the frame (66), to be

$$\begin{aligned}
d\hat{\pi}^i &= -df^i\wedge dt-\tfrac{1}{2}d(f^i_{|j})\wedge\hat{\theta}^j-\tfrac{1}{2}f^i_{|j}(\hat{\omega}\wedge\hat{\pi}^j+\tfrac{1}{2}f^j_{|k}\hat{\omega}\wedge\hat{\theta}^k)\\
&= \rho^i_j\hat{\theta}^j\wedge\hat{\omega}+\tfrac{1}{2}f^i_{|j}\hat{\omega}\wedge\hat{\pi}^j+\tfrac{1}{2}f^i_{|jk}\hat{\theta}^j\wedge\hat{\pi}^k+\tau^i_{jk}\hat{\theta}^j\wedge\hat{\theta}^k,
\end{aligned} \tag{67}$$

where

$$\rho^i_j=-f^i_{,j}+\frac{1}{2}\frac{d}{dt}f^i_{|j}-\tfrac{1}{4}f^i_{|k}f^k_{|j}\quad\text{and}\quad \tau^i_{jk}=\tfrac{1}{2}(f^i_{[,k|j]}+\tfrac{1}{2}f^i_{|l[j}f^l_{|k]}). \tag{68}$$

From this it is immediate that

$$(\tilde{P}^i_j)|_{\mathrm{Id}} = \rho^i_j - n^{-1}\delta^i_j\rho^k_k,$$

which proves part of the lemma. In order to find $(\tilde{S}^i_{jkl})|_{\mathrm{Id}}$ we need first to determine

$$S^i_{jkl} = (d\Omega^i_j|_{\mathrm{Id}})_{\hat{\pi}^k\hat{\theta}^l}$$

in equation (44) which is before the reduction of $\mathrm{H}^{(1)}_1$ to $(\mathrm{H}^{(1)}_1)_1$. To determine this we need to compute $(d\Omega^i_j)|_{\mathrm{Id}}$ parametrically and this will require the parametric form of the structure equations for the lift of the coframe (66). These equations are given by equations (78) and (79) in Appendix A. Thus we are in a position which allows us to write the parametric form of the absorption occurring in (22) and (25) as

$$\begin{aligned}
\alpha &= \hat{\alpha} + E_k(A^{-1})^k_j\pi^j + 2c(\omega - E_k(A^{-1})^k_j\theta^j),\\
\kappa_j &= \hat{\kappa}_j - E_k(A^{-1})^k_l\pi^l E_i(A^{-1})^i_j\\
&\quad - [cE_k(A^{-1})^k_j + \tfrac{1}{2}a^{-1}E_l(f^l_{|m})(A^{-1})^m_j](\omega - E_k(A^{-1})^k_r\theta^r),\\
\sigma &= \hat{\sigma} + \left[c^2 + \frac{a^{-2}}{n}\rho^i_i\right](\omega - E_k(A^{-1})^k_j\theta^j)\\
&\quad + \frac{1}{n-1}\left[E_k(A^{-1})^k_l\tilde{P}^l_j + \frac{2a^{-1}}{1-n}\tau^i_{ik}(A^{-1})^k_j\right]\theta^j,\\
\Omega^i_j &= \hat{\Omega}^i_j - \pi^i E_k(A^{-1})^k_j + [c\delta^i_j - \tfrac{1}{2}a^{-1}A^i_l(f^l_{|m})(A^{-1})^m_j](\omega - E_k(A^{-1})^k_r\theta^r)\\
&\quad - \tfrac{1}{2}A^i_k(f^k_{|lm})(A^{-1})^l_n\theta^n(A^{-1})^m_j.
\end{aligned} \tag{69}$$

To begin by computing $d\Omega^i_j$, we first notice that

$$\hat{\omega} = a^{-1}(\omega - E_k(A^{-1})^k_j\theta^j),$$

so that

$$d[a^{-1}(\omega - E_k(A^{-1})^k_j\theta^j)] = 0.$$

By using this we find that

$$\begin{aligned}
(d\Omega^i_j)|_{\mathrm{Id}} &= (\beta_k)|_{\mathrm{Id}} \wedge \hat{\theta}^k\delta^i_j + (\beta_j)|_{\mathrm{Id}} \wedge \hat{\theta}^i + (dA^i_k \wedge dA^k_j)|_{\mathrm{Id}}\\
&\quad - \hat{\pi}^i \wedge (dE_j)|_{\mathrm{Id}} - d(c\delta^i_j - \tfrac{1}{2}a^{-1}A^i_l(f^l_{|m})(A^{-1})^m_j)|_{\mathrm{Id}} \wedge \hat{\omega}\\
&\quad - \tfrac{1}{2}d(A^i_k(f^k_{|lm})(A^{-1})^l_n(A^{-1})^m_j)|_{\mathrm{Id}} \wedge \hat{\theta}^n.
\end{aligned} \tag{70}$$

Now use equation (69) in the form $da|_{\mathrm{Id}} = \alpha|_{\mathrm{Id}}$, to find by the absorption in (28) that

$$(\beta_j|_{\mathrm{Id}})_{\hat{\pi}^k} = 0.$$

Finally using equation (69) in the form

$$dE_j|_{\mathrm{Id}} = \kappa_j|_{\mathrm{Id}}, \quad dA^i_j|_{\mathrm{Id}} = \Omega^i_j|_{\mathrm{Id}} + \tfrac{1}{2}(f^i_{|j})\hat{\omega} + \tfrac{1}{2}(f^i_{|jk})\hat{\theta}^k,$$

we determine the $\hat{\pi}^k \wedge \hat{\theta}^l$ term of $(d\Omega^i_j|_{\mathrm{Id}})$ from (70) to be

$$S^i_{jkl} = (d\Omega^i_j|_{\mathrm{Id}})_{\hat{\pi}^k\hat{\theta}^l} = (df^i_{|jk})_{\hat{\pi}^l} = \frac{\partial^3 f^i}{\partial p^j\,\partial p^k\,\partial p^l}.$$

To complete the proof we note that, from equation (47), $\tilde{S}^i_{jkl}$ is the trace-free part of S^i_{jkl} in the above equation.

In the next section we refer to the following simple corollary of Theorem 4.1.

COROLLARY 4.1. *The structure function for the* {*e*}*-structure of Theorem* 3.1 *is constant if and only if the tensor invariants* $\tilde{P}^i_j$ *and* $\tilde{S}^i_{jkl}$ *are constant.*

5. *Systems with maximal symmetry*

In this final section we examine the {*e*}-structures which are maximally symmetric. Maximally symmetric {*e*}-structures are characterized by having a constant structure function and, in the light of Corollary 4.1, the {*e*}-structure we have obtained in Theorem 3.1 will be maximally symmetric if and only if the tensor invariants $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ are constant. To examine the possibility that $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ are constant we determine their dependence on the group parameters.

THEOREM 5.1. *The infinitesimal form of the group action of* $\mathrm{H}^{(1)}_1 \times (\mathrm{H}^{(1)}_1)_1$ *on* $\tilde{P}^i_j$ *and* $\tilde{S}^i_{jkl}$ *is given by*

$$\begin{aligned} d\tilde{P}^i_j - \Omega^i_k\tilde{P}^k_j + \tilde{P}^i_k\Omega^k_j + 2\alpha\tilde{P}^i &\equiv 0 \mod (\omega, \theta^i, \pi^i), \\ d\tilde{S}^i_{jkl} - \Omega^i_m\tilde{S}^m_{jkl} + \tilde{S}^i_{jml}(\Omega^m_k - \alpha\delta^m_k) + \tilde{S}^i_{jkm}\Omega^m_l + \tilde{S}^i_{mkl}\Omega^m_j &\equiv 0 \mod (\omega, \theta^i, \pi^i). \end{aligned} \tag{71}$$

Proof. We begin the proof by using equation (52) from which the expression giving the infinitesimal action of the structure group on $\tilde{P}^i_j$ above follows immediately. In order to find the infinitesimal action on $\tilde{S}^i_{jkl}$ set $d^2\Omega^i_j \equiv 0 \mod (\omega)$, which will lead to the second equation in (71) and prove the theorem.

The presence of the α term in the equations (71) implies that the one-parameter subgroup of $\mathrm{H}^{(1)}_1$ generated by the element a acts on both $\tilde{P}^i_j$ and $\tilde{S}^i_{jkl}$ by scaling. This allows us to make the following observation.

COROLLARY 5.1. *The tensor invariants* $\tilde{P}^i_j$ *and* $\tilde{S}^i_{jkl}$ *are constant if and only if they are zero. In other words there exists a unique* {*e*}*-structure which is maximally symmetric, and hence a unique* (*up to equivalence*) *system of second-order ordinary differential equations which admit a Lie symmetry group of dimension* $n^2 + 4n + 3$. *This equivalence class is invariantly characterized by the conditions*

$$\tilde{P}^i_j = 0, \quad \tilde{S}^i_{jkl} = 0, \tag{72}$$

and a representative for this class is given by

$$\frac{d^2x^i}{dt^2} = 0. \tag{73}$$

For completeness we also state the following corollary.

COROLLARY 5.2. *All systems of second-order equations not equivalent to* (73) *admit Lie symmetry groups only of dimension strictly less than* n^2+4n+3.

It is interesting to note that in the paper by González-López [9] the most general *linear* system which can be transformed to (73) by a point transformation is given. The expression given in [9] for this system of linear equations can easily be checked to satisfy the conditions of Corollary 5.1. Lastly we remark that putting $\tilde{P}^i_j=0$ and $\tilde{S}^i_{jkl}=0$ in the structure equations for the $\{e\}$-structure in Theorem 3.1 gives the symmetry algebra of the equation (73). That is, the structure equations are the Maurer–Cartan equations of $\mathrm{sl}(n+2,\mathbb{R})$ (see [8]).

6. *Appendix A*

Here we give the parametric form of the structure equations for the lifted frame

$$\begin{pmatrix}\omega\\ \theta^j\\ \pi^j\end{pmatrix}=\begin{pmatrix}a & E_j & 0\\ 0 & A^i_j & 0\\ 0 & cA^i_j & a^{-1}A^i_j\end{pmatrix}\begin{pmatrix}\hat{\omega}\\ \hat{\theta}^j\\ \hat{\pi}^j\end{pmatrix},$$

with $(\hat{\omega},\hat{\theta}^i,\hat{\pi}^i)$ being defined in (66). The structure equations for the lifted forms are

$$\begin{pmatrix}d\omega\\ d\theta^i\\ d\pi^i\end{pmatrix}=\begin{pmatrix}\alpha & \kappa_j & 0\\ 0 & \Omega^i_j & 0\\ 0 & \sigma\delta^i_j & \Omega^i_j-\alpha\delta^i_j\end{pmatrix}\wedge\begin{pmatrix}\omega\\ \theta^j\\ \pi^j\end{pmatrix}$$
$$+\begin{pmatrix}a & E_j & 0\\ 0 & A^i_j & 0\\ 0 & cA^i_j & a^{-1}A^i_j\end{pmatrix}\begin{pmatrix}d\hat{\omega}\\ d\hat{\theta}^j\\ d\hat{\pi}^j\end{pmatrix}, \tag{74}$$

where

$$\alpha=a^{-1}\,da,\qquad \kappa_j=dE_k(A^{-1})^k_j-a^{-1}\,daE_k(A^{-1})^k_j,$$
$$\Omega^i_j=dA^i_k(A^{-1})^k_j,\qquad \sigma=(dc+ca^{-1}\,da)\delta^i_j.$$

Now we write the expressions for $d\hat{\omega}$, $d\hat{\theta}^i$ and $d\hat{\pi}^i$ back in terms of the lifted frame as

$$d\hat{\omega}=0, \tag{75}$$
$$d\hat{\theta}^i=-[(A^{-1})^i_j\pi^j-c(A^{-1})^i_j\theta^j+\tfrac{1}{2}a^{-1}(f^i_{|j})(A^{-1})^j_k\theta^k]\wedge(\omega-E_k(A^{-1})^k_j\theta^j), \tag{76}$$

and (from (67))

$$d\hat{\pi}^i=\left[\frac{1}{a}(\rho^i_j)(A^{-1})^j_k\theta^k-\tfrac{1}{2}(f^i_{|j})(A^{-1})^j_k\pi^k+\tfrac{1}{2}c(f^i_{|j})(A^{-1})^j_k\theta^k\right]$$
$$\wedge(\omega-E_l(A^{-1})^l_m\theta^m)$$
$$+(\tau^i_{jk})(A^{-1})^j_l(A^{-1})^k_m\theta^l\wedge\theta^m-\tfrac{1}{2}a(f^i_{|jk})(A^{-1})^j_l(A^{-1})^k_m\pi^l\wedge\theta^m. \tag{77}$$

Now we substitute from (75), (76) and (77) into (74) to get the equations

$$\begin{aligned} d\omega &= \alpha\wedge\omega + \kappa_j\wedge\theta^j - E_r(A^{-1})^r_j[\pi^j - c\theta^j + \tfrac{1}{2}a^{-1}A^j_k(f^k_{|l})(A^{-1})^l_m\theta^m] \\ &\quad \wedge(\omega - E_k(A^{-1})^k_j\theta^j), \qquad (78)\\ d\theta^i &= \Omega^i_j\wedge\theta^j - [\pi^i - c\theta^i + \tfrac{1}{2}a^{-1}A^i_j(f^j_{|k})(A^{-1})^k_l\theta^l]\wedge(\omega - E_k(A^{-1})^k_j\theta^j), \end{aligned}$$

and

$$\begin{aligned} d\pi^i &= \sigma\wedge\theta^i + (\Omega^i_j - \alpha\delta^i_j)\wedge\pi^j - c[\pi^i - c\theta^i + \tfrac{1}{2}a^{-1}A^i_k(f^k_{|j})(A^{-1})^j_k\theta^k] \\ &\quad \wedge(\omega - E_k(A^{-1})^k_j\theta^j) \\ &\quad + a^{-1}A^i_l[a^{-1}(\rho^l_j)(A^{-1})^j_k\theta^k - \tfrac{1}{2}(f^l_{|j})(A^{-1})^j_k\pi^k + \tfrac{1}{2}c(f^l_{|j})(A^{-1})^j_k\theta^k] \\ &\quad \wedge(\omega - E_k(A^{-1})^k_j\theta^j) \\ &\quad + a^{-1}A^i_r(\tau^r_{jk})(A^{-1})^j_l(A^{-1})^k_m\theta^l\wedge\theta^m - \tfrac{1}{2}A^i_r(f^r_{|jk})(A^{-1})^j_l(A^{-1})^k_m\pi^l\wedge\theta^m. \end{aligned}$$

We then simplify $d\pi^i$ to

$$\begin{aligned} d\pi^i &= \sigma\wedge\theta^i + (\Omega^i_j - \alpha\delta^i_j)\wedge\pi^j + a^{-1}A^i_r(\tau^r_{jk})(A^{-1})^j_l(A^{-1})^k_m\theta^l\wedge\theta^m \\ &\quad + [a^{-2}A^i_l(\rho^l_j)(A^{-1})^j_k\theta^k + c^2\theta^i - c\pi^i - \tfrac{1}{2}a^{-1}A^i_j(f^j_{|k})(A^{-1})^k_l\pi^l] \\ &\quad \wedge(\omega - E_k(A^{-1})^k_j\theta^j) - \tfrac{1}{2}A^i_l(f^l_{|jk})(A^{-1})^j_m(A^{-1})^k_r\pi^m\wedge\theta^r. \qquad (79) \end{aligned}$$

7. *Appendix B*

The purpose of this appendix is to demonstrate that systems of second-order ordinary differential equations with n dependent variables for which the tensor $\tilde{S}^i_{jkl}$ vanishes are in one-to-one correspondence with projective connections on an $(n+1)$-dimensional space. Before proceeding I would like to thank the referee for clarifying this point.

The geodesic equations for two affine connections Γ, $\bar{\Gamma}$, on U, $\bar{U}\subset\mathbb{R}^{n+1}$ with coordinates x^a, $\bar{x}^a$ and affine parameters s, $\bar{s}$ are given by

$$\frac{d^2x^a}{ds^2} = -\Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} \quad \text{with } 0\leq a, b, c\leq n, \qquad (80)$$

$$\frac{d^2\bar{x}^a}{d\bar{s}^2} = -\bar{\Gamma}^a_{bc}\frac{d\bar{x}^b}{d\bar{s}}\frac{d\bar{x}^c}{d\bar{s}} \quad \text{with } 0\leq a, b, c\leq n. \qquad (81)$$

The two affine connections are projectively equivalent if there exists a diffeomorphism Φ: $U\to\bar{U}$ such that the *paths,* or solutions to (80), are in one-to-one correspondence with the paths (or solutions) of (81). It is important to note that the existence of Φ is independent of the affine parameters s, $\bar{s}$. The following lemma, which can be found in **[13]**, gives necessary and sufficient conditions for Γ to be projectively equivalent to $\bar{\Gamma}$.

LEMMA B.1. *Two affine connections* Γ *and* $\bar{\Gamma}$ *are projectively equivalent if and only if there exists a collection of functions* ϕ_b *such that*

$$\Gamma^a_{bc} = \bar{\Gamma}^a_{bc} + \phi_{(b}\delta^a_{c)}. \qquad (82)$$

An equivalence class of affine connections is called a *projective connection.*

In order to determine the role of the tensor $\tilde{S}^i_{jkl}$ we first associate with (80) a system of second-order ordinary differential equations in n dependent variables in the following way. Given a solution $x^a = \phi^a(s)$ to (80) we rewrite the system (80) by eliminating the parameter s by solving for one of the dependent variables as $x^0 = \phi^0(s)$. With the notation $t = x^0 = \phi(s)$, we then find that the system (80) becomes

$$\frac{d^2\phi}{ds^2} = -\left(\frac{d\phi}{ds}\right)^2\left(\Gamma^0_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + 2\Gamma^0_{0j}\frac{dx^j}{dt} + \Gamma^0_{00}\right),$$

$$\frac{d^2\phi}{ds^2}\frac{dx^i}{dt} + \left(\frac{d\phi}{ds}\right)^2\frac{d^2x^i}{dt^2} = -\left(\frac{d\phi}{ds}\right)^2\left(\Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} + 2\Gamma^i_{0j}\frac{dx^j}{dt} + \Gamma^i_{00}\right) \quad \text{for } 1 \leq i, j, k \leq n.$$

From these two equations we immediately obtain the second-order system in n dependent variables

$$\frac{d^2x^i}{dt^2} = \Gamma^0_{(jk}\delta^i_{l)}\frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt} + (2\Gamma^0_{0j}\delta^i_k - \Gamma^i_{jk})\frac{dx^j}{dt}\frac{dx^k}{dt}$$
$$+ (\Gamma^0_{00}\delta^i_j - 2\Gamma^i_{0j})\frac{dx^j}{dt} - \Gamma^i_{00}, \tag{83}$$

where Γ^a_{bc} are functions of (t, x^i) in our notation. Thus the connections Γ, $\bar{\Gamma}$ are projectively equivalent if and only if the system (83) is equivalent to the corresponding system for $\bar{\Gamma}$ by a point transformation.

Among all second-order systems of ordinary differential equations in n dependent variables the ones which satisfy the invariant condition $\tilde{S}^i_{jkl} = 0$ are related to projectively equivalent affine connections by the following lemma.

LEMMA B.2. *The systems of second-order ordinary differential equations*

$$\frac{d^2x^i}{dt^2} = f^i\left(t, x^j, \frac{dx^j}{dt}\right) \quad \textit{with } i \leq i, j \leq n \tag{84}$$

which satisfy the invariant condition $\tilde{S}^i_{jkl} = 0$ are in one-to-one correspondence with projectively equivalent affine connections on an $(n+1)$-dimensional space.

Proof. First, given an affine connection Γ we have the associated system of second-order equations in (83) which is easily seen to satisfy the condition $\tilde{S}^i_{jkl} = 0$.

We now proceed to show that for any second-order system in n dependent variables with $\tilde{S}^i_{jkl} = 0$ we may associate an $(n+1)$-dimensional affine connection. By differentiating the equation $\tilde{S}^i_{jkl} = 0$ with respect to p^r we obtain

$$f^i_{|jklr} - \frac{3}{n+2}f^m_{|mr(jk}\delta^i_{l)} = 0. \tag{85}$$

Now taking the trace on i, r in this equation we find

$$\frac{n-1}{n+2}f^m_{|jklm} = 0,$$

which, by (85), gives

$$f^i_{|jklm} = 0.$$

In other words, $f^i(t, x^j, \dot{x}^j)$ is at most cubic in $\dot{x}^j$. If we substitute the most general cubic polynomial in $\dot{x}^j$ into the equation $\tilde{S}^i_{jkl} = 0$, we find the form of $f^i(t, x^j, \dot{x}^j)$ and the corresponding system of equations to be

$$\frac{d^2x^i}{dt^2} = \delta_{(j}A_{kl)} \frac{dx^j}{dt}\frac{dx^k}{dt}\frac{dx^l}{dt} + B^i_{jk}\frac{dx^k}{dt}\frac{dx^k}{dt} + C^i_j \frac{dx^k}{dt} + D^i, \tag{86}$$

where A_{jk}, B^i_{jk}, C^i_j, and D^i are functions of (t, x^i). Given the system (86) we may then define the affine connection

$$\hat{\Gamma}^i_{jk} = B^i_{jk}, \quad \hat{\Gamma}^0_{jk} = A_{jk}, \quad \hat{\Gamma}^i_{0j} = C^i_j, \quad \hat{\Gamma}^i_{00} = D^i. \tag{87}$$

Finally, to complete the lemma we note that by constructing the affine connection $\hat{\Gamma}$ in (87) using the system of equations (83), the connection $\hat{\Gamma}$ is projectively equivalent to the original connection Γ (Lemma B.1) which gave rise to the system (83).

The notion of projectively equivalent affine connections originated with Weyl [**14**] and Cartan [**3**].

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References

1. I. Anderson and G. Thompson, 'The inverse problem of the calculus of variations for ordinary differential equations', *Mem. Amer. Math. Soc.* 98 (1992).

2. E. Cartan, 'Les sous-groupes des groupes continus de transformations', *Ann. École Norm. Sup.* 25 (1908) 57–194; *Oeuvres complètes*, vol. 2 (Gauthier-Villers, Paris, 1955), pp. 719–856.

3. E. Cartan, 'Sur les variétés à connexion projective', *Bull. Soc. Math.* 52 (1924) 205–241.

4. E. Cartan, 'Les problèmes d'équivalence', *Séminaire de Mathématiques, exposé du 11 janvier, 1937*, pp. 113–136; *Oeuvres complètes*, vol. 2 (Gauthier-Villars, Paris, 1955), pp. 1311–1334.

5. S. S. Chern, 'Sur la géométrie d'un système d'équations différentielles du second ordre', *Bull. Sci. Math.* 63 (1939) 206–212.

6. M. E. Fels, 'Some applications of Cartan's method of equivalence to the geometric study of ordinary and partial differential equations', Ph.D. thesis, McGill University, Canada, 1993.

7. R. B. Gardner, 'The method of equivalence and its applications', CBMS-NSF Regional Conference Series in Applied Mathematics 58 (SIAM, Philadelphia, 1989), pp. 1–127.

8. F. González-Gascón and A. González-López, 'Symmetries of differential equations IV', *J. Math. Phys.* 24 (1983) 2006–2021.

9. A. González-López, 'Symmetries of linear systems of second-order differential equations', *J. Math. Phys.* 29 (1988) 1097–1105.

10. L. Hsu and N. Kamran, 'Classification of second-order ordinary differential equations admitting Lie groups of fibre-preserving point symmetries', *Proc. London Math. Soc.* (3) 58 (1989) 387–416.

11. S. Lie and G. Scheffers, *Vorlesung über continuierliche Gruppen* (Chelsea, New York, 1971).

12. S. Sternberg, *Differential geometry* (Chelsea, New York, 1982).

13. T. Y. THOMAS, *The differential invariants of generalized spaces* (Cambridge University Press, 1934).
14. H. WEYL, 'Zur Infinitesimalgeometrie: Einordnung der projektiven etc. Auffassung', *Göttinger Nachrichten* (1921) 99–112.

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