THE INVERSE PROBLEM OF THE CALCULUS OF VARIATIONS FOR SCALAR FOURTH-ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. A simple invariant characterization of the scalar fourth-order ordinary differential equations which admit a variational multiplier is given. The necessary and sufficient conditions for the existence of a multiplier is expressed in terms of the vanishing of two relative invariants which can be associated with any fourth-order equation through the application of Cartan's equivalence method. The solution to the inverse problem for fourth-order scalar equations provides the solution to an equivalence problem for second-order Lagrangians, as well as the precise relationship between the symmetry algebra of a variational equation and the divergence symmetry algebra of the associated Lagrangian.

1. Introduction

Solving the inverse problem of the calculus of variations for scalar differential equations consists of characterizing those equations which may be multiplied by a non-zero function such that the resulting equation arises from a variational principle. Specifically in the case of scalar fourth-order ordinary differential equations we will determine for which equations

$$\frac{d^4u}{dx^4} = f(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}), \tag{1.1}$$

there exist smooth functions $g \neq 0$ (the variational multiplier) and Lagrangian L such that

$$g(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3}) \left[\frac{d^4u}{dx^4} - f(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3})\right] = E(\, L(x, u, \frac{du}{dx}, \frac{d^2u}{dx^2})\,) \tag{1.2}$$

is an identity, where $E(L) = 0$ is the Euler-Lagrange equation for the Lagrangian L.

A complete solution to the inverse problem for the simplest possible case of a scalar second-order ordinary differential equations has been know since Darboux [9]. Darboux determined that every second-order ordinary differential equation admits a multiplier and we will find that this is far from the case for a fourth-order equation. Thus the inverse problem for fourth-order scalar equations is the simplest non-trivial case which admits a complete solution.

The formulation of the variational multiplier problem for scalar equations is easily extended to systems of differential equations where the problem is to determine whether it is possible to multiply a system of equations by a non-singular matrix of functions such that the result is a variation of some Lagrangian. In particular the multiplier problem for a system of two second-order ordinary differential equations has been thoroughly analyzed in the famous work of J. Douglas [10]. Recently Anderson and Thompson [4] have also studied this problem using the variational bicomplex (see section 2 below). Our solution for the scalar fourth-order equations will be based on ideas from these two articles.

The first step in Douglas' solution to the inverse problem involved showing that necessary and sufficient conditions for the existence of a multiplier could be expressed in terms of the existence of solutions to a system of partial differential equations, which arise from the Helmholtz conditions, and where the unknowns are the multipliers (for a discussion of the Helmholtz conditions see [13]). Solving this system of partial differential equation would then provided the multiplier matrix for certain pairs of second-order ordinary

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differential equations. By using Riquier theory to analyze the existence and degree of generality of the solution space of the system of partial differential equations for the multipliers, Douglas discovered that there exist pairs of second-order equations which admit no multipliers, some which admit finitely many different multipliers (with distinct Lagrangians) as well as pairs of equations which admit infinitely many different multipliers (and Lagrangians). Unfortunately due to the overwhelming complexity of the analyses, Douglas sometimes was only able to determine the degree of generality of the solution space to the partial differential equations for the multipliers and was unable to find a pair of ordinary differential equations with the specified degree of generality of the multipliers.

The solution to the inverse problem given by Douglas emphasizes an important aspect in solving the inverse problem for the fourth order equation (1.2). To find a complete solution to the inverse problem we must not only determine which equations admit a multiplier, but we must also determine how unique the multiplier and associated Lagrangian is. Fortunately we will find in the fourth-order inverse problem that if an equation admits a variational multiplier so that (1.2) is satisfied, then the multiplier and the associated Lagrangian are essentially unique. The uniqueness of a variational structure for a variational fourth-order equation will subsequently be used to solve an equivalence problem for second-order Lagrangians as well as provide the relationship between the symmetry group of a variational scalar fourth-order equation and the divergence symmetries of its Lagrangian.

The approach we take in solving the fourth-order inverse problem follows a refined version of Douglas' solution to the multiplier problem given by Anderson and Thompson [4]. Anderson and Thompson derive the system of determining equations for the multiplier in a natural way using the variational bicomplex. They showed that the existence of a multiplier was in direct correspondence with the existence of special cohomology classes arising in the variational bicomplex associated with a differential equation. The advantage in this formulation of the inverse problem is that the invariant nature of the problem is clearly emphasized. Anderson and Thompson proceeded to study the existence problem for the cohomology classes using exterior differential systems which was considerably easier than the intricate analysis of Douglas, and subsequently they were able to determine some of the exceptional examples which had eluded Douglas.

Our solution to the multiplier problem will use in an essential way the formulation of the inverse problem of Anderson and Thompson. In the next section we will recall the theory of the variational bicomplex as it applies to problem (1.2). In particular by using the cohomology formulation of the multiplier problem we determine the exterior differential system which must be integrated in order that a cohomology class and hence a multiplier as in (1.2) exists. The novelty of our solution to the problem relies in writing the exterior differential system in terms of an invariant coframe obtained through Cartan's equivalence method. In section 3 we provide the details of the equivalence method calculations for a fourth-order ordinary differential equation under contact transformations and obtain the associated $\{e\}$-structure and hence the invariant coframe. In section 4 we use this coframe to analyze the exterior differential system for the cohomology class and this provides the solution to the inverse problem which can be described solely in terms of vanishing conditions on two of the relative invariants (torsion) found in section 3. Lastly, in section 5 we consider two applications of the solution to the inverse problem.

2. The Variational Bicomplex

The variational bicomplex was initially introduced in order to formulate and solve the inverse problem in the calculus of variations, and so we recall the basic theory of the bicomplex which allows us to solve the multiplier problem for fourth-order equations.

The infinite jet space (see [1]) $J^\infty(\mathbb{R},\mathbb{R})$ while not a manifold in the standard sense, does admit local coordinates $(x, u_x, u_{xx}, ..., u_r, ...)$ and a contact ideal $\mathcal{C}(J^\infty(\mathbb{R},\mathbb{R}))$ generated by the one-forms

$$\theta^0 = du - u_x dx\,,\ \theta^1 = \ du_x - u_{xx} dx\,,\ \dots,\ \theta^r = du_r - u_{r+1} dx\,,\ \dots\,. \tag{2.1}$$

The one-forms in (2.1) along with the differential form dx form a basis for the exterior algebra of differential forms on $J^\infty(\mathbb{R},\mathbb{R})$. From this basis of forms we define the vector-field $\frac{d}{dx}$ on $J^\infty(\mathbb{R},\mathbb{R})$ by the conditions

$$\frac{d}{dx} \lrcorner dx = 1\,, \quad \frac{d}{dx} \lrcorner \theta^r = 0\,, \quad r = 0, 1, \dots$$

so that the component form of $\frac{d}{dx}$ is

$$\frac{d}{dx} = \frac{\partial}{\partial x} + \sum_{a=0}^{\infty} u_{a+1} \frac{\partial}{\partial u_a} \ . \tag{2.2}$$

$\frac{d}{dx}$ is called the total $x-$derivative.

We now define two subspaces $\Omega^{0,p}(J^\infty(\mathbb{R},\mathbb{R}))$ and $\Omega^{1,p-1}(J^\infty(\mathbb{R},\mathbb{R}))$, for $p \geq 1$, of the set of p-forms on $J^\infty(\mathbb{R},\mathbb{R})$. The first space is defined inductively by

$$\begin{aligned} \Omega^{0,1}(J^\infty(\mathbb{R},\mathbb{R})) &= \{a_i\theta^i,\ a_i \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))\ ,\ 0 \leq i < \infty\} \\ \Omega^{0,p}(J^\infty(\mathbb{R},\mathbb{R})) &= \{\alpha_i \wedge \theta^i,\ \alpha_i \in \Omega^{0,p-1}(J^\infty(\mathbb{R},\mathbb{R}))\ ,\ 0 \leq i < \infty\,, p \geq 1\} \end{aligned}$$

where the summation convention is used here and will be assumed from now on. The second subspace, $\Omega^{1,p-1}(J^\infty(\mathbb{R},\mathbb{R}))$, is then defined as

$$\Omega^{1,p-1}(J^\infty(\mathbb{R},\mathbb{R})) = \{\alpha \wedge dx,\ \alpha \in \Omega^{0,p-1}(J^\infty(\mathbb{R},\mathbb{R})),\, p \geq 1\}\ .$$

These subspaces provide a direct sum decomposition of the p-forms on $J^\infty(\mathbb{R},\mathbb{R})$

$$\Omega^p(J^\infty(\mathbb{R},\mathbb{R})) = \Omega^{0,p}(J^\infty(\mathbb{R},\mathbb{R})) \oplus \Omega^{1,p-1}(J^\infty(\mathbb{R},\mathbb{R})) \tag{2.3}$$

and thus every differential form $\omega \in \Omega^p(J^\infty(\mathbb{R},\mathbb{R}))$ may be written

$$\omega = \omega^{0,p} + \omega^{1,p-1}$$

where

$$\omega^{0,p} \in \Omega^{0,p}(J^\infty(\mathbb{R},\mathbb{R}))\ ,\ \omega^{1,p-1} \in \Omega^{1,p-1}(J^\infty(\mathbb{R},\mathbb{R}))\ .$$

We also define $\Omega^{0,0}(J^\infty(\mathbb{R},\mathbb{R}))$ to be the smooth functions $C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$, and for convenience we take $\Omega^{2,p}(J^\infty(\mathbb{R},\mathbb{R})) = 0$.

The direct sum decomposition in (2.3) induces a splitting of the exterior derivative into a direct sum of two derivative operators

$$d = d_H + d_V\ . \tag{2.4}$$

The operator d_H acts on smooth functions $h \in \Omega^{0,0}(J^\infty(\mathbb{R},\mathbb{R}))$ by

$$d_H\, h(x, u_x, ...u_r) = \frac{d}{dx}(h)dx\ \in\ \Omega^{1,0}(J^\infty(\mathbb{R},\mathbb{R}))$$

where

$$\frac{d}{dx}(h) = \frac{\partial h}{\partial x} + \sum_{a=0}^{r} u_{a+1}\frac{\partial h}{\partial u_a}$$

where $\frac{d}{dx}$ is the total derivative operator in (2.2), while the operator d_H on forms is

$$d_H\,\omega = dx \wedge \mathcal{L}_{\frac{d}{dx}}\,\omega \qquad \omega \in \Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R}))\ ,\ r = 0,1\,,\ s \geq 0$$

where $\mathcal{L}_{\frac{d}{dx}}$ is the Lie-derivative along $\frac{d}{dx}$. The operation of $d_V = d - d_H$ on a smooth function h is given in coordinates by

$$d_V\, h(x, u_x, ...u_r) = \sum_{a=0}^{a=r} \frac{\partial h}{\partial u_a}\,\theta^a\ \in\ \Omega^{1,0}(J^\infty(\mathbb{R},\mathbb{R}))\ ,$$

and the action of d_V extends easily to forms. The two derivative operations are then operators

$$\begin{aligned} d_H : \Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R})) &\rightarrow \Omega^{r+1,s}(J^\infty(\mathbb{R},\mathbb{R})) \\ d_V : \Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R})) &\rightarrow \Omega^{r,s+1}(J^\infty(\mathbb{R},\mathbb{R}))\ ,\ r = 0,1; s \geq 0 \end{aligned}$$

where d_H is called the horizontal exterior derivative while d_V is the vertical exterior derivative. These two derivatives satisfy the properties

$$d_H^2 = 0\ ,\quad d_V^2 = 0\ ,\quad d_H d_V + d_V d_H = 0\ . \tag{2.5}$$

Finally the variational bicomplex is defined to be the double complex $\{\Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R})), d_H, d_V\}_{r=0,1,2;s\geq 0}$. The reader should consult [1] for a thorough treatment of the variational bicomplex.

A Lagrangian is represented in the bicomplex by a one-form $\lambda \in \Omega^{1,0}(J^\infty(\mathbb{R},\mathbb{R}))$ whose local coordinate expression is

$$\lambda = L(x, u_x, ..., u_l)dx \ , \tag{2.6}$$

and where the order of the Lagrangian λ is defined to be he highest derivative dependence of L in (2.6), ie. the Lagrangian in (2.6) has order l. The first variational formula in the calculus of variations can be conveniently expressed using the variational bicomplex formalism as

Lemma 2.1. *Let $\lambda \in \Omega^{1,0}(J^\infty(\mathbb{R},\mathbb{R}))$ be an l^{th}-order Lagrangian as in* (2.6)*, then*

$$d_V\lambda = E(\lambda) + d_H\,\eta \tag{2.7}$$

where $E(\lambda) \in \Omega^{1,1}(J^\infty(\mathbb{R},\mathbb{R}))$ is the Euler-Lagrange form

$$E(\lambda) = \left[\frac{\partial L}{\partial u} + \left(-\frac{d}{dx}\right)\frac{\partial L}{\partial u_x} + \left(-\frac{d}{dx}\right)^2\frac{\partial L}{\partial u_{xx}} + ... + \left(-\frac{d}{dx}\right)^l\frac{\partial L}{\partial u_l}\right]\theta^0\wedge dx$$

and $\eta \in \Omega^{0,1}(J^\infty(\mathbb{R},\mathbb{R}))$ is

$$\eta = A_r\theta^r \ , \quad A_r = \sum_{s=0}^{l-r-1}(-1)^{s+1}\left(\frac{d}{dx}\right)^s\left(\frac{\partial L}{\partial u_{r+s+1}}\right) \ , \ 0 \leq r \leq l-1 \ . \tag{2.8}$$

Setting the coefficient of $\theta^0\wedge dx$ in the differential form $E(\lambda)$ to zero generates the Euler-Lagrange equations for λ which we denote by $E(L)$. The Poincare-Cartan form associated with a Lagrangian λ, which is important in the geometry of variational problems (see [14]), is the one-form defined by

$$\Theta = \eta - \lambda \tag{2.9}$$

where η is given in (2.8).

There is a simple procedure, which we will now describe, that allows us to associate a variational bicomplex with a fourth-order ordinary differential equation. A fourth-order ordinary differential equation

$$u_{xxxx} - f(x, u, u_x, u_{xx}, u_{xxx}) = 0 \tag{2.10}$$

defines a 5-dimensional sub-manifold $\mathcal{R}$ of $J^4(\mathbb{R},\mathbb{R})$ by the inclusion

$$i : (\,x, u, u_x, u_{xx}, u_{xxx}) \to (x, u, u_x, u_{xx}, u_{xxx}, f(x, u_x, u_{xx}, u_{xxx})\,) \subset J^4(\mathbb{R},\mathbb{R}) \tag{2.11}$$

where $(x, u, u_x, u_{xx}, u_{xxx})$ are local coordinates for $\mathcal{R}$. We call $\mathcal{R}$ the equation manifold for equation (2.10). The map i in (2.11) extends to a map $i : \mathcal{R} \to J^\infty(\mathbb{R},\mathbb{R})$ by prolongation

$$u_{xxxxx} = \frac{d}{dx}f \ , \ ... \ , u_{4+r} = \frac{d^r}{dx^r}f \ , \ .... \ .$$

The variational bicomplex associated with the differential equation (2.10) whose equation manifold is $\mathcal{R}$ will be denoted by $\{\Omega^{r,s}(\mathcal{R}), d_H, d_V\}_{r=0,1;s\geq 0}$ and this bicomplex is defined to be the pullback of the complex $\{\Omega^{r,s}(J^\infty(\mathbb{R},\mathbb{R})), d_H, d_V\}_{r=0,1;s\geq 0}$ by the inclusion $i : \mathcal{R} \to J^\infty(\mathbb{R},\mathbb{R})$. For example, the contact ideal on $\mathcal{R}$ is

$$C(\mathcal{R}) = \{\theta^0 = du - u_x dx \ , \ \theta^1 = du_x - u_{xx}dx \ , \ \theta^2 = du_x - u_{xxx}dx \ , \ \theta^3 = du_x - f(x, u, u_x, u_{xx}, u_{xxx})dx\} \tag{2.12}$$

while the total derivative of a function $h \in C^\infty(\mathcal{R})$, which we will write as $\frac{dh}{dx}$, has the coordinate expression

$$\frac{dh}{dx} = \frac{\partial h}{\partial x} + u_x\frac{\partial h}{\partial u} + u_{xx}\frac{\partial h}{\partial u_x} + u_{xxx}\frac{\partial h}{\partial u_{xx}} + f(x, u, u_x, u_{xx}, u_{xxx})\frac{\partial h}{\partial u_{xxx}} \ .$$

If we consider those fourth-order ordinary differential equations which admit a multiplier, so that (1.2) is satisfied, then the pullback of the variational formula (2.7) to the equation manifold $\mathcal{R}$ yields

$$d_V\lambda = d_H(i^*\eta) \tag{2.13}$$

($i^*E(\lambda)=0$ by the assumption that the equation admits a multiplier). The essential idea underlying the solution to the inverse problem now lies in defining a differential two-form $\omega\in\Omega^{0,2}(\mathcal{R})$ by

$$\omega = d_V(i^*\eta) \tag{2.14}$$

which from equations (2.13) and (2.4) is found to be closed. The closed differential two-form ω, which may also be written in terms of the Poincare-Cartan form as $\omega = i^* d\Theta$ using (2.9), when written out explicitly is

$$\omega = -\frac{\partial^2 L}{\partial {u_{xx}}^2}(\theta^0\wedge\theta^3-\theta^1\wedge\theta^2)+\frac{d}{dx}\frac{\partial^2 L}{\partial {u_{xx}}^2}\theta^2\wedge\theta^0+\left(\frac{d}{dx}\frac{\partial^2 L}{\partial u_{xx}\partial u_x}+2\frac{\partial^2 L}{\partial u_{xx}\partial u}-\frac{\partial^2 L}{\partial {u_x}^2}\right)\theta^1\wedge\theta^0 \ . \tag{2.15}$$

The differential form ω in (2.14) provides the proof of necessity in the first part of the following key theorem ([4] Theorem 2.6 page 20, as it applies to our case),

Theorem 2.2. *The fourth-order differential equation* (2.10) *admits a multiplier and a non-degenerate Lagrangian of order 2, if and only if there exists a differential form $\omega\in\Omega^{0,2}(\mathcal{R})$ having the algebraic structure*

$$\omega = a_3\theta^0\wedge\theta^3 + a_2\theta^0\wedge\theta^2 + a_1\theta^0\wedge\theta^1 + a_0\theta^1\wedge\theta^2 \tag{2.16}$$

with a_3 non-vanishing, and where ω satisfies the closure condition

$$d\omega = 0 \ .$$

Moreover there is a one-to-one correspondence between these closed two-forms and non-degenerate second-order Lagrangians λ, modulo the addition to λ of a total derivative $d_H h$ where $h\in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$ depends on at most first derivatives.

A non-degenerate Lagrangian λ satisfies by definition

$$\frac{\partial^2 L(x,u,u_x,u_{xx})}{\partial {u_{xx}}^2}\neq 0$$

and comparing this condition with equation (2.15) gives rise to the non-vanishing condition on a_3 in (2.16). We will often identify two non-degenerate Lagrangians λ_1 and λ_2 if

$$\lambda_1 = \lambda_2 + d_H\, h(x,u,u_x) \tag{2.17}$$

where $h\in C^\infty(J^1(\mathbb{R},\mathbb{R}))$. By using this identification the second part of Theorem 2.2 states that the correspondence between λ and ω described in the equations (2.13) and (2.14) is one-to-one.

Theorem 2.2 reduces the multiplier problem to a simple geometric condition on the equation manifold $\mathcal{R}$. In fact this interpretation of the closed form ω in Theorem 2.2 is a special case of a more general phenomena. That is the entire space of closed two-forms

$$\omega\in\Omega^{0,2}(\mathcal{R})\ , \quad d\omega=0 \tag{2.18}$$

can be interpreted in terms of variational operators [2]. An equation is said to admit a variational operator if there exists a total differential operator whose action on the original equation results in an equation which is variational. The existence of a closed form (2.18) implies by a generalization of Theorem 2.2 that the equation determining $\mathcal{R}$ admits a variational operator. A variational multiplier is of course a special case of a variational operator where the total differential operator has no derivative terms.

In order to simplify the description of the solution to the inverse problem we define a submodule of $V(\mathcal{R})\subset\Omega^{0,2}(\mathcal{R})$, by

$$V(\mathcal{R}) = \{\omega\in\Omega^{0,2}(\mathcal{R})\,|\,\omega = a_3\theta^0\wedge\theta^3 + a_2\theta^0\wedge\theta^2 + a_1\theta^0\wedge\theta^1 + a_0\theta^1\wedge\theta^2\} \tag{2.19}$$

where $(a_i)_{i=0..3}\in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$. We also define a subspace $\overline{V}(\mathcal{R})\subset V(\mathcal{R})$ by

$$\overline{V}(\mathcal{R}) = \{\omega\in V(\mathcal{R})\,|\,d\omega=0\} \ . \tag{2.20}$$

Theorem 2.2 now states that solving the inverse problem corresponds to determining for which equations with corresponding manifolds $\mathcal{R}$ do there exist $\omega\in\overline{V}(\mathcal{R})$ with $a_3\neq 0$. We will find that $\overline{V}(\mathcal{R})$ is a contact invariant subspace of $\Omega^{0,2}(\mathcal{R})$ which demonstrates by Theorem 2.2, that whether or not an an equation admits a variational multiplier is a contact invariant problem. This motivates us to study the contact geometry of $\mathcal{R}$ by using Cartan's equivalence method [7].

3. The $\{e\}$-structure for fourth-order ordinary differential equations

In this section we use the equivalence method of E. Cartan to associate an invariant coframe with any fourth-order scalar ordinary differential equation. In principle the relative invariants arising in the structure equations for this coframe can be used to distinguish between non-equivalent equations. We identify two of the relative invariants which will be used in section 4 to characterize the fourth-order equations which admit a multiplier.

A convenient description of the equivalence method is given in [11], while our calculations in this section are found to be similar to those for the third-order scalar ordinary differential equation case presented in [8]. To begin let (x, u, u_x) and $(\bar{x}, \bar{u}, \bar{u}_{\bar{x}})$ be local coordinates on $J^1(\mathbb{R}, \mathbb{R})$ and $\overline{J}^1(\mathbb{R}, \mathbb{R})$ respectively. Two fourth-order scalar ordinary differential equations

$$u_{xxxx} = f(x, u, u_x, u_{xx}, u_{xxx}) \qquad \bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}} = \bar{f}(\bar{x}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{x}\bar{x}}) \tag{3.1}$$

are defined to be contact equivalent (in the classical sense) if there exists a contact transformation

$$\Psi^1 : J^1(\mathbb{R}, \mathbb{R}) \to \overline{J}^1(\mathbb{R}, \mathbb{R})$$

which is given in local coordinates by

$$\overline{x} = \phi(x, u, u_x)\,,\ \overline{u} = \psi(x, u, u_x)\,,\ \bar{u}_{\bar{x}} = \psi^1(x, u, u_x)\ , \tag{3.2}$$

and a (nowhere vanishing) smooth function $h(x, u, u_x, u_{xx}, u_{xxx})$ such that

$$(\Psi^4)^* \left[\bar{u}_{\bar{x}\bar{x}\bar{x}\bar{x}} - \bar{f}(\bar{x}, \bar{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{x}\bar{x}})\right] = h(x, u, u_x, u_{xx}, u_{xxx})\left[u_{xxxx} - f(x, u, u_x, u_{xx}, u_{xxx})\right] \tag{3.3}$$

where $\Psi^4 : J^4(\mathbb{R}, \mathbb{R}) \to \overline{J}^4(\mathbb{R}, \mathbb{R})$ is the prolongation of Ψ^1. In other words the two differential equations (3.1) are contact equivalent if there exists a contact preserving change of variables of the form (3.2) which transforms one equation into a multiple of the other.

We now express the relation of contact equivalence between the two equations in (3.1) as an equivalence relationship between coframes so that the equivalence method may be applied. This is done by considering each differential equation in (3.1) as defining a smooth submanifold

$$(x, u, u_x, u_{xx}, u_{xxx}) \to J^4(\mathbb{R}, \mathbb{R})\ , \qquad (\overline{x}, \overline{u}, \bar{u}_{\bar{x}}, \bar{u}_{\bar{x}\bar{x}}, \bar{u}_{\bar{x}\bar{x}\bar{x}}) \to \bar{J}^4(\mathbb{R}, \mathbb{R})$$

as in equation (2.11), and then choosing the particular coframes on $\mathcal{R}$ and $\overline{\mathcal{R}}$ by taking the canonical basis for the contact module $\mathcal{C}(\mathcal{R})$ and that for $\mathcal{C}(\overline{\mathcal{R}})$ as given in equation (2.1) together with the differential forms $\sigma = dx$ and $\overline{\sigma} = d\overline{x}$. That is we have the coframes

$$\begin{pmatrix} \sigma = dx \\ \theta^i \end{pmatrix}\ , \qquad \begin{pmatrix} \overline{\sigma} = d\overline{x} \\ \overline{\theta}^i \end{pmatrix} \tag{3.4}$$

on $\mathcal{R}$ and $\overline{\mathcal{R}}$ respectively. By canonically identifying $\mathcal{R}$ with $J^3(\mathbb{R}, \mathbb{R})$, the prolongation of a contact transformation $\Psi^3 : J^3(\mathbb{R}, \mathbb{R}) \to \overline{J}^3(\mathbb{R}, \mathbb{R})$ defines a diffeomorphism $\Psi^3 : \mathcal{R} \to \overline{\mathrm{R}}$. This identification allows us to express the equivalence condition (3.3) as

Lemma 3.1. *Two fourth-order ordinary differential equations* (3.1) *are contact equivalent if and only if there exists a contact transformation contact* $\Psi^1 : J^1(\mathbb{R}, \mathbb{R}) \to \overline{J}^1(\mathbb{R}, \mathbb{R})$ *such that*

$$(\Psi^3)^* \begin{pmatrix} \overline{\sigma} \\ \overline{\theta}^i \end{pmatrix} = \mathcal{S} \begin{pmatrix} \sigma \\ \theta^i \end{pmatrix}\ , \tag{3.5}$$

where $\mathcal{S}:\mathcal{R}\to H$ *is a smooth function on* $\mathcal{R}$ *taking values in the Lie subgroup* $H\subset \mathrm{GL}(5,\mathbb{R})$ *defined by*

$$H=\left\{\begin{pmatrix} a & u & v & 0 & 0\\ 0 & b & 0 & 0 & 0\\ 0 & c_1 & a^{-1}b & 0 & 0\\ 0 & c_2 & c_3 & a^{-2}b & 0\\ 0 & c_4 & c_5 & c_6 & a^{-3}b\end{pmatrix},\ a,b\in\mathbb{R}^*,u,v,c_1,c_2,c_3,c_4,c_5,c_6\in\mathbb{R}\right\}. \tag{3.6}$$

We may now apply the equivalence method by using Lemma 3.1 and lifting the forms in (3.4) to $\mathcal{R}\times H$ and $\overline{\mathcal{R}}\times H$ by defining the one-forms,

$$\begin{pmatrix}\widehat{\sigma}\\ \widehat{\theta}^i\end{pmatrix}=S\begin{pmatrix}\sigma\\ \theta^i\end{pmatrix} \tag{3.7}$$

(with the analogous definition on $\overline{\mathcal{R}}\times H$) where S is the local parameterization of H in equation (3.6). By taking the exterior derivative of equation (3.7) we have the first set of structure equations

$$\begin{pmatrix}d\widehat{\sigma}\\ d\widehat{\theta}^i\end{pmatrix}=(dS)S^{-1}\begin{pmatrix}\widehat{\sigma}\\ \widehat{\theta}^i\end{pmatrix}+S\begin{pmatrix}d\sigma\\ d\theta^i\end{pmatrix} \tag{3.8}$$

where $(dS)S^{-1}$ is a Maurer-Cartan form for H. The Maurer-Cartan form we use is

$$(dS)S^{-1}=\begin{pmatrix}\alpha & \mu & \nu & 0 & 0\\ 0 & \beta & 0 & 0 & 0\\ 0 & \gamma_1 & \beta-\alpha & 0 & 0\\ 0 & \gamma_2 & \gamma_3 & \beta-2\alpha & 0\\ 0 & \gamma_4 & \gamma_5 & \gamma_6 & \beta-3\alpha\end{pmatrix} \tag{3.9}$$

where $\alpha,\mu,\nu,\beta,\gamma_a$ are right invariant one-forms on H, (with the analogous definitions on $\overline{\mathcal{R}}\times H$) and where S is the local parameterization of H in equation (3.6).

With the lifted forms in (3.7) and the Maurer-Cartan form (3.9) the equivalence method can be applied to find,

Theorem 3.2. *Solutions* $\Psi^3:\mathcal{R}\to\overline{\mathcal{R}}$ *to the equivalence problem for fourth-order ordinary differential equations are in one-to-one correspondence with the solutions of an equivalence problem for an 8 dimensional* $\{e\}$*-structure on* $\mathcal{R}\times G$ *where* G *is a three dimensional Lie subgroup of* H*. The essential part of the structure equations of the coframe are given by*

$$\begin{aligned} d\sigma &= \alpha\wedge\sigma+T_1\,\theta^0\wedge\theta^1+T_2\,\theta^0\wedge\theta^2+T_3\,\theta^0\wedge\theta^3+T_4\,\theta^1\wedge\theta^2+T_5\,\theta^1\wedge\theta^3\\ d\theta^0 &= \beta\wedge\theta^0+\sigma\wedge\theta^1\\ d\theta^1 &= (\beta-\alpha)\wedge\theta^1+\gamma\wedge\theta^0+\sigma\wedge\theta^2\\ d\theta^2 &= (\beta-2\alpha)\wedge\theta^2+\frac{4}{3}\gamma\wedge\theta^1+\sigma\wedge\theta^3\\ d\theta^3 &= (\beta-3\alpha)\wedge\theta^3+\gamma\wedge\theta^2+I_0\sigma\wedge\theta^0+I_1\sigma\wedge\theta^1+T_6\,\theta^0\wedge\theta^1+T_7\,\theta^0\wedge\theta^2+T_8\,\theta^1\wedge\theta^2 \end{aligned} \tag{3.10}$$

where $T_a, a=1...8$ *and* $I_s, s=1,2$ *are smooth functions on* $\mathcal{R}\times G$*.*

The reader who is interested in the characterization of the variational fourth order equations (and applications) may skip the derivation of the structure equations and proceed to section 4. In the remainder of this section we derive the structure equations along with some of the parametric forms of some of the invariants.

Proof. The initial structure equations in (3.8) on $\mathcal{R}\times H$ are determined from the equations

$$\begin{aligned} d(du_i-u_{i+1}dx) &= dx\wedge du_{i+1}\ ,\ i=0,1,2\\ d(du_3-f dx) &= dx\wedge df \end{aligned} \tag{3.11}$$

to be (after absorption of torsion [11])

$$
\begin{aligned}
d\sigma &= \alpha\wedge\sigma + \mu\wedge\theta^0 + \nu\wedge\theta^1 \\
d\theta^0 &= \beta\wedge\theta^0 + \sigma\wedge\theta^1 \\
d\theta^1 &= (\beta-\alpha)\wedge\theta^1 + \gamma_1\wedge\theta^0 + \sigma\wedge\theta^2 \\
d\theta^2 &= (\beta-2\alpha)\wedge\theta^2 + \gamma_2\wedge\theta^0 + \gamma_3\wedge\theta^1 + \sigma\wedge\theta^3 + K_1\,\sigma\wedge\theta^2 \\
d\theta^3 &= (\beta-3\alpha)\wedge\theta^3 + \gamma_4\wedge\theta^0 + \gamma_5\wedge\theta^1 + \gamma_6\wedge\theta^2 + K_2\,\sigma\wedge\theta^3 \ .
\end{aligned} \tag{3.12}
$$

By taking the exterior derivative of the second and third equation in (3.12) above we also have the equations

$$
\begin{aligned}
d\beta - \mu\wedge\theta^1 + \gamma_1\wedge\sigma &\equiv 0 \qquad \mathrm{mod}(\theta^0) \\
d\alpha + 2\gamma_1\wedge\sigma - \gamma_3\wedge\sigma + \nu\wedge\theta^2 &\equiv 0 \qquad \mathrm{mod}(\theta^0,\theta^1) \ .
\end{aligned} \tag{3.13}
$$

We can now determine the group action on the torsion elements K_1 and K_2 in (3.12) by computing $d^2\theta^2\wedge\theta^0\wedge\theta^1\wedge\theta^3$ and $d^2\theta^3\wedge\theta^0\wedge\theta^1\wedge\theta^2$ and using (3.13) to get

$$
\begin{aligned}
dK_1 + \alpha K_1 + \gamma_6 + 3\gamma_1 - 3\gamma_3 &\equiv 0 \qquad \mathrm{mod(base)} \\
dK_2 + \alpha K_2 - \gamma_6 - 3\gamma_3 + 5\gamma_1 &\equiv 0 \qquad \mathrm{mod(base)} \ .
\end{aligned} \tag{3.14}
$$

The action of the structure group on K_1 and K_2 allows us to translate K_1 and K_2 to zero by using the group elements corresponding to γ_6 and γ_3. The reduced group H^1 (which is easily obtained by exponentiation) will have the Maurer-Cartan form of (3.9) subject to the relations

$$
\gamma_6 = \gamma_1 \qquad \gamma_3 = \frac{4}{3}\gamma_1 \ . \tag{3.15}
$$

We will make the substitution $\gamma_1 = \gamma$ from now on.

The new structure equations with group H^1 will have the same first three structure equations of (3.12) (with $\gamma_1 = \gamma$) while the last two are

$$
\begin{aligned}
d\theta^2 &= (\beta-2\alpha)\wedge\theta^2 + \gamma_2\wedge\theta^0 + \frac{4}{3}\gamma\wedge\theta^1 + \sigma\wedge\theta^3 + L_1\,\sigma\wedge\theta^1 + L_2\,\theta^2\wedge\theta^1 + L_3\,\theta^3\wedge\theta^1 \\
d\theta^3 &= (\beta-3\alpha)\wedge\theta^3 + \gamma_4\wedge\theta^0 + \gamma_5\wedge\theta^1 + \gamma\wedge\theta^2 + L_4\,\sigma\wedge\theta^2 + L_5\,\theta^3\wedge\theta^2 \ .
\end{aligned} \tag{3.16}
$$

In the first of these equations we may still absorb L_2 by letting

$$
\alpha = \hat{\alpha} + \frac{1}{2}L_2\,\theta^1 \ , \qquad \nu = \hat{\nu} + \frac{1}{2}L_2\sigma \ , \ \gamma^5 = \hat{\gamma}_5 - \frac{3}{2}L_2\theta^3 \ . \tag{3.17}
$$

As well we have equations (3.13) along with $d^2\theta^1\wedge\theta^1 = 0$ giving (dropping the $\hat{}$)

$$
\begin{aligned}
d\beta - \mu\wedge\theta^1 + \gamma\wedge\sigma &\equiv 0 \qquad \mathrm{mod}(\theta^0) \\
d\alpha + \frac{2}{3}\gamma\wedge\sigma + \nu\wedge\theta^2 + L_3\,\sigma\wedge\theta^3 &\equiv 0 \qquad \mathrm{mod}(\theta^0,\theta^1) \\
d\gamma + \alpha\wedge\gamma - \mu\wedge\theta^2 + \gamma_2\wedge\sigma &\equiv 0 \qquad \mathrm{mod}(\theta^0,\theta^1) \ .
\end{aligned} \tag{3.18}
$$

It now follows from setting $d^2\theta^2\wedge\theta^0\wedge\theta^1 = 0$ and (3.18) that

$$
L_5 = 3L_3 \ . \tag{3.19}
$$

The action of the reduced group on the independent torsion elements L_1, L_3 and L_4 is obtained by taking $d^2\theta^2\wedge\theta^0\wedge\theta^2$ and $d^2\theta^3\wedge\theta^0\wedge\theta^1\wedge\theta^3$ while using (3.18) to find

$$
\begin{aligned}
dL_1 + 2L_1\,\alpha + \gamma_5 - \frac{7}{3}\gamma_2 &\equiv 0 \\
dL_3 + L_3\,(\beta-2\alpha) - \nu &\equiv 0 \qquad \mathrm{mod(base)} \\
dL_4 + 2L_4\,\alpha - \gamma_5 - \gamma_2 &\equiv 0 \ .
\end{aligned} \tag{3.20}
$$

These equations imply that the torsion elements L_1, L_4 and L_3 can be translated to zero using γ_2, γ_5 and ν. The new structure group $H^2 \subset H^1$ resulting from this reduction will have the Maurer-Cartan form of H^1 subject to the conditions $\gamma_2 = 0, \gamma_5 = 0$, and $\nu = 0$.

The new structure equations with group H^2 are (after absorption of torsion)

$$
\begin{aligned}
d\sigma &= \alpha\wedge\sigma + \mu\wedge\theta^0 + M_1\,\theta^1\wedge\theta^2 + M_2\,\theta^1\wedge\theta^3 \\
d\theta^2 &= (\beta-2\alpha)\wedge\theta^2 + \frac{4}{3}\gamma\wedge\theta^1 + \sigma\wedge\theta^3 + M_3\,\sigma\wedge\theta^0 + M_4\,\theta^1\wedge\theta^2 + M_5\,\theta^3\wedge\theta^0 \\
d\theta^3 &= (\beta-3\alpha)\wedge\theta^3 + \gamma_4\wedge\theta^0 + \gamma\wedge\theta^2 + M_6\,\sigma\wedge\theta^1 + M_7\,\theta^1\wedge\theta^3\ .
\end{aligned}
\tag{3.21}
$$

where $d\theta^0$ and $d\theta^1$ are the same as in (3.12) (with $\gamma_1=\gamma$). The M_4 term arises in these equations because the absorption in (3.17) is no longer possible. The action of the structure group on the terms M_3 and M_5 is found by taking $d^2\theta^2\wedge\theta^1\wedge\theta^2$ resulting in

$$
\begin{aligned}
dM_3 + 3M_3\,\alpha + \gamma_4 &\equiv 0 \quad \mathrm{mod(base)}\\
dM_5 + M_5\,(\beta-\alpha) - \mu &\equiv 0 \quad \mathrm{mod(base)}\ .
\end{aligned}
$$

Thus we may translate M_3 and M_5 further reducing the group to the subgroup $G\subset H^2$ which is three dimensional. The Maurer-Cartan form of G is obtained from that of H^2 with the extra conditions $\mu=0, \gamma_4=0$. The structure equations then read

$$
\begin{aligned}
d\sigma &= \alpha\wedge\sigma + N_1\,\theta^1\wedge\theta^2 + N_2\,\theta^1\wedge\theta^3 + N_3\,\theta^1\wedge\theta^0 + N_4\,\theta^2\wedge\theta^0 + N_5\,\theta^3\wedge\theta^0 \\
d\theta^2 &= (\beta-2\alpha)\wedge\theta^2 + \frac{4}{3}\gamma\wedge\theta^1 + \sigma\wedge\theta^3 + N_6\,\theta^1\wedge\theta^2 \\
d\theta^3 &= (\beta-3\alpha)\wedge\theta^3 + \gamma\wedge\theta^2 + I_0\sigma\wedge\theta^0 + I_1\sigma\wedge\theta^1 + N_7\,\theta^1\wedge\theta^3 + N_8\,\theta^1\wedge\theta^0 + N_9\,\theta^2\wedge\theta^0 + N_{10}\,\theta^3\wedge\theta^0 + N_{11}\theta^2\wedge\theta^1\ .
\end{aligned}
$$

where again $d\theta^0$ and $d\theta^1$ are the same as in (3.12) (with $\gamma_1=\gamma$). At this point we have an $\{e\}$ structure on $U\times G$. The torsion coefficients N_6, N_7 and N_{10} in the $\{e\}-$structure vanish as seen by the following calculations

$$
\begin{aligned}
d^2\theta^2\wedge\theta^1\wedge\theta^2 &= N_{10}\,\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3\\
d^2\theta^3\wedge\theta^0\wedge\theta^1 &= N_7\,\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3\\
d^2\theta^2\wedge\theta^0\wedge\theta^2 &= N_6\,\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3\ .
\end{aligned}
$$

By relabeling the non-zero torsion the structure equations in (3.10) are obtained. □

This theorem is well known [5] but the $\{e\}$-structures in Theorem 3.2 and [5] have a different form.

In section 4 we will the following information about the structure equations for $d\alpha$ and $d\beta$.

Corollary 3.3. *The forms $d\alpha$ and $d\beta$ satisfy,*

$$
\begin{aligned}
d\beta + \gamma\wedge\sigma + T_2\,\theta^2\wedge\theta^1 + T_3\,\theta^3\wedge\theta^1 &\equiv 0 \qquad \mathrm{mod}(\theta^0)\\
d\alpha + \frac{2}{3}\gamma\wedge\sigma - T_5\,\theta^3\wedge\theta^2 &\equiv 0 \qquad \mathrm{mod}(\theta^0,\theta^1)\ .
\end{aligned}
\tag{3.22}
$$

This calculation easily follows from the equations in Theorem 3.2. The two torsion coefficients T_5 and I_1 also play an important role in the next section and by a simple consequence of the structure equations in Theorem 3.2 we find

Corollary 3.4. *The torsion coefficient T_5 and I_1 are relative invariants. They satisfy*

$$
\begin{aligned}
dT_5 + T_5(2\beta-5\alpha) &\equiv 0 \qquad \mathrm{mod(base)}\\
dI_1 + 3I_1\alpha &\equiv 0 \qquad \mathrm{mod(base)}\ .
\end{aligned}
$$

This corollary implies that the vanishing of T_5 and I_1 is an invariant property of a fourth-order differential equation. In section 4 we will show that the invariant subclass of fourth-order equations defined by the vanishing of T_5 and I_1 are precisely those equations which admit a variational multiplier. We compute the parametric values of these relative invariants to be

Lemma 3.5. *The parametric value of relative invariants T_5 and I_1 (at the identity of G) are*

$$
\begin{aligned}
T_5 &= \frac{1}{6}\frac{\partial^3 f}{\partial u_{xxx}^3} \\
I_1 &= \frac{\partial f}{\partial u_x} + \frac{1}{2}\frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xxx}} - \frac{d}{dx}\frac{\partial f}{\partial u_{xx}} - \frac{3}{4}\frac{\partial f}{\partial u_{xxx}}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}} + \frac{1}{2}\frac{\partial f}{\partial u_{xx}}\frac{\partial f}{\partial u_{xxx}} + \frac{1}{8}\left(\frac{\partial f}{\partial u_{xxx}}\right)^3 .
\end{aligned}
$$

Proof. In order to determine the parametric values of the torsion elements we use some of the intermediate calculations from Theorem 3.2 and restrict to the identity of the group in question. By using equations (3.11) we have

$$K_1 = 0\,,\ K_2 = \frac{\partial f}{\partial u_{xxx}} . \tag{3.23}$$

The modification in the coframe obtained by the translating K_1 and K_2 to zero in (3.14) is by setting

$$c_6 = -\frac{1}{2}\frac{\partial f}{\partial u_{xxx}}\ ,\ c_3 = \frac{1}{3}c_6\ .$$

The resulting coframe on $\mathcal{R}$ is given by

$$dx\ ,\ \theta_0^0 = du - u_x dx\ ,\ \theta_0^1 = du_x - u_{xx}dx$$

together with the twisted forms

$$
\begin{aligned}
\theta_0^2 &= du_{xx} - u_{xxx}dx - \frac{1}{6}\frac{\partial f}{\partial u_{xxx}}\theta_0^1\ , \\
\theta_0^3 &= du_{xxx} - fdx - \frac{1}{2}\frac{\partial f}{\partial u_{xxx}}(du_{xx} - u_{xxx}dx) \;=\; du_{xxx} - fdx - \frac{1}{2}\frac{\partial f}{\partial u_{xxx}}(\theta_0^2 + \frac{1}{6}\frac{\partial f}{\partial u_{xxx}}\theta_0^1)\ .
\end{aligned}
\tag{3.24}
$$

To find the torsion in equations (3.16) we take the exterior derivative of the forms in (3.24) to get

$$
\begin{aligned}
d\theta_0^2\wedge\theta_0^0\wedge\theta_0^2\wedge\theta_0^3 &= \left[-du_{xxx}\wedge dx - \frac{1}{6}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}}dx\wedge\theta_0^1 + \frac{1}{6}\frac{\partial f}{\partial u_{xxx}}du_{xx}\wedge dx\right]\wedge\theta_0^0\wedge\theta_0^2\wedge\theta_0^3 \\
d\theta_0^3\wedge\theta_0^0\wedge\theta_0^1\wedge\theta_0^3 &= \left[-\frac{\partial f}{\partial u_{xx}}du_{xx}\wedge dx - \frac{1}{2}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}}dx\wedge\theta_0^2 + \frac{1}{2}\frac{\partial f}{\partial u_{xxx}}du_{xxx}\wedge dx\right]\wedge\theta_0^0\wedge\theta_0^1\wedge\theta_0^3 \\
d\theta_0^3\wedge\theta_0^0\wedge\theta_0^1\wedge dx &= -\frac{1}{2}\frac{\partial^2 f}{\partial u_{xxx}^2}du_{xxx}\wedge\theta_0^2\wedge\theta_0^0\wedge\theta_0^1\wedge dx
\end{aligned}
$$

which after substituting from (3.24) easily gives

$$
\begin{aligned}
L_1 &= \frac{1}{18}\left(\frac{\partial f}{\partial u_{xxx}}\right)^2 - \frac{1}{6}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}} \\
L_4 &= \frac{\partial f}{\partial u_{xx}} + \frac{1}{4}\left(\frac{\partial f}{\partial u_{xxx}}\right)^2 - \frac{1}{2}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}} \\
L_5 &= -\frac{1}{2}\frac{\partial^2 f}{\partial u_{xxx}^2}\ .
\end{aligned}
$$

In order to determine the twist in the coframe on $\mathcal{R}$ corresponding to translating L_1 and L_4 to zero in (3.14) we solve the equations

$$c_5 - \frac{7}{3}c_2 = L_1\ ,\ c_2 + c_5 = -L_4$$

which along with the translation of $L_5 = 3L_3$ to zero in (3.20) gives the coframe

$$
\begin{aligned}
\sigma_1 &= dx + \frac{1}{6}\frac{\partial^2 f}{\partial u_{xxx}^2}\theta_0^1\ \ ,\ \ \theta_1^0 \;=\; \theta_0^0\ ,\ \theta_1^1 \;=\; \theta_0^1 \\
\theta_1^2 &= \theta_0^2 - \frac{3}{10}(L_4 + L_1)\theta_0^0\ ,\ \theta_1^3 \;=\; \theta_0^3 + (\frac{3}{10}L_1 - \frac{7}{10}L_4)\theta_0^1\ .
\end{aligned}
\tag{3.25}
$$

We are now in a position to compute T_5 from

$$d\sigma_1\wedge\sigma_1\wedge\theta_1^0\wedge\theta_1^2 = \frac{1}{6}\frac{\partial^3 f}{\partial u_{xxx}^3}\sigma_1\wedge\theta_1^3\wedge\theta_1^1\wedge\theta_1^0\wedge\theta_1^2$$

which gives the expression for T_5 as stated in the lemma. Next we compute M_3 in equation (3.21) by

$$d\theta_1^2\wedge\theta_1^1\wedge\theta_1^2\wedge\theta_1^3 = \left[-du_{xxx}\wedge dx + \frac{1}{6}\frac{\partial f}{\partial u_{xx}}du_{xx}\wedge dx - \frac{3}{10}\frac{d}{dx}(L_4+L_1)dx\wedge\theta_0^0\right]\wedge\theta_1^1\wedge\theta_1^2\wedge\theta_1^3$$

which upon substitution from (3.24) and (3.25) gives

$$M_3 = \frac{1}{5}\frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xxx}} - \frac{3}{10}\frac{d}{dx}\frac{\partial f}{\partial u_{xx}} - \frac{1}{4}\frac{\partial f}{\partial u_{xxx}}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}} + \frac{1}{10}\frac{\partial f}{\partial u_{xx}}\frac{\partial f}{\partial u_{xxx}} + \frac{11}{360}\left(\frac{\partial f}{\partial u_{xxx}}\right)^3 .$$

In a similar manner we compute M_6 to be

$$M_6 = \frac{\partial f}{\partial u_x} + \frac{3}{10}\frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xxx}} - \frac{7}{10}\frac{d}{dx}\frac{\partial f}{\partial u_{xx}} - \frac{1}{2}\frac{\partial f}{\partial u_{xxx}}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}} + \frac{2}{5}\frac{\partial f}{\partial u_{xx}}\frac{\partial f}{\partial u_{xxx}} + \frac{17}{180}\left(\frac{\partial f}{\partial u_{xxx}}\right)^3$$

The relative invariant I_1 is then computed by noting from equation (3.21) that

$$dM_6 - \gamma_4 \equiv 0 \qquad \mathrm{mod(base)}$$

so

$$I_1 = M_6 + M_3$$

which is the expression for I_1 in the statement of the lemma. □

4. The Variational Multiplier Problem

At the end of section 2 we described necessary and sufficient conditions for a fourth-order scalar ordinary differential equation

$$u_{xxxx} - f(x,u,u_x,u_{xx},u_{xxx}) = 0 \ , \tag{4.1}$$

with equation manifold $\mathcal{R}$, to admit a multiplier as being equivalent to the existence of a closed differential two-form $\tilde{\omega}\in\overline{V}(\mathcal{R})$. Specifically $\tilde{\omega}$ had to be of the form

$$\tilde{\omega} = a_3\,\tilde{\theta}^0\wedge\tilde{\theta}^3 + a_2\,\tilde{\theta}^0\wedge\tilde{\theta}^2 + a_1\,\tilde{\theta}^0\wedge\tilde{\theta}^1 + a_0\,\tilde{\theta}^1\wedge\tilde{\theta}^2 \qquad a_3\neq 0 \ , \tag{4.2}$$

where $\{a_i\}_{i=0..3}\in C^\infty(\mathcal{R})$ and $\{\tilde{\theta}^i\}_{i=0,\dots,3}$ are the contact forms on the manifold $\mathcal{R}$ defined in (2.12). If we subject the differential form $\tilde{\omega}$ in (4.2) to a contact transformation of $\mathcal{R}$, then by using Lemma 3.1 to transform the differential forms $(\tilde{\theta}^i)_{i=0,\dots,3}$, it is clear that the algebraic form of ω in (4.2) is invariant. Thus the module $V(\mathcal{R})$ in (2.19) and the subspace $\overline{V}(\mathcal{R})$ in (2.20) are invariant with respect to contact transformations. The fact that the space $\overline{V}(\mathcal{R})$ is invariant implies through Theorem 2.2 that determining whether an equation admits a multiplier is a contact invariant problem. The invariant nature of determining which scalar fourth-order ordinary differential equations admit a multiplier will allow us us to demonstrate that the vanishing of the relative invariants in Corollary 3.4 (or Lemma 3.5) associated with a fourth-order equation (4.1) characterize variational equations.

In analogy with definitions (2.19) and (2.20) at that end of section 2, we define the spaces

$$\begin{aligned} V(\mathcal{R}\times G) &= \{\omega\in\Omega^2(\mathcal{R}\times G)\,|\,\omega = a_3\theta^0\wedge\theta^3 + a_2\theta^0\wedge\theta^2 + a_1\theta^0\wedge\theta^1 + a_0\theta^1\wedge\theta^2\}\ , \\ \overline{V}(\mathcal{R}\times G) &= \{\omega\in V(\mathcal{R}\times G)\,|\,d\omega = 0\} \end{aligned} \tag{4.3}$$

where $\{a_i\}_{i=0..3}\in C^\infty(\mathcal{R}\times G)$, and $\{\theta^i\}_{i=0,\dots,3}$ are the components of the invariant coframe determined in Theorem 3.2. We may express Theorem 2.2 in terms of conditions on the geometry of the $\{e\}$-structure $\mathcal{R}\times G$, by using definitions (4.3), as

Lemma 4.1. *A fourth-order equation admits a variational multiplier if and only if there exists a (closed) two-form $\omega\in\overline{V}(\mathcal{R}\times G)$ with a_3 non-vanishing.*

The proof of this lemma is a consequence of the geometric relationship between the spaces $\overline{V}(\mathcal{R})$ and $\overline{V}(\mathcal{R}\times G)$. In fact if we let $\omega \in \overline{V}(\mathcal{R}\times G)$ and X be any infinitesimal generator of the left action of G on $\mathcal{R}\times G$, we have $X \lrcorner \omega = 0$ and so

$$\mathcal{L}_X \omega = 0 \ .$$

Thus every closed differential form $\omega \in \overline{V}(\mathcal{R}\times G)$ is invariant with respect to the left action of G on $\mathcal{R}\times G$, and consequently there exists a unique closed differential $\tilde{\omega} \in \overline{V}(\mathcal{R})$ such that

$$\omega = \pi^* \tilde{\omega} \ , \tag{4.4}$$

where π is the projection $\pi : \mathcal{R}\times G \to \mathcal{R}$. The correspondence in (4.4) is one-to-one and the spaces $\overline{V}(\mathcal{R})$ and $\overline{V}(\mathcal{R}\times G)$ are canonically isomorphic. This being so we let

$$\nu = dimension\, \overline{V}(\mathcal{R}\times G) = dimension\, \overline{V}(\mathcal{R}) \tag{4.5}$$

and we may re-express Lemma 4.1 as

Corollary 4.2. *Equation (4.1) admits a variational multiplier if and only if* $\nu \neq 0$.

The partial differential equations $d\omega = 0$ for the unknowns $\{a_i\}_{i=0,\dots,3}$ which arise in Lemma 4.1 can be written in terms of the invariant coframe in Theorem 3.2. The existence of a solutions to $d\omega = 0$ will then be expressed in terms of the relative invariants. Fortunately we the partial differential equations $d\omega = 0$ for the unknowns $\{a_i\}_{i=0,\dots,3}$ dramatically simplify due to several algebraic relationships which must hold amongst the terms $\{a_i\}_{i=0,\dots,3}$ in order for the dimension ν in (4.5) to be non-zero. We find

Lemma 4.3. *If there exists a non-zero differential form* $\omega \in \overline{V}(\mathcal{R}\times G)$ *then* ω *must have the algebraic structure*

$$\omega = a(\theta^0\wedge\theta^3 - \theta^1\wedge\theta^2) \ . \tag{4.6}$$

Proof. We work with the structure equations (3.10) modulo (α,β,γ). By taking the exterior derivative of an arbitrary $\omega \in V(\mathcal{R}\times G)$ (using the structure equations (3.10)) and concentrating on terms which contain σ, we find

$$d\omega\wedge\theta^0\wedge\theta^2 = -(a_3+a_0)\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3 \ .$$

which demonstrates that ω can be closed only if it has the algebraic form

$$\omega = a_3(\theta^0\wedge\theta^3 - \theta^1\wedge\theta^2) + a_1\theta^0\wedge\theta^1 + a_2\theta^0\wedge\theta^2 \ .$$

In a similar manner we compute

$$\begin{aligned} d\omega\wedge\theta^1\wedge\theta^2 &= (\dot{a}_3 + a_2)\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3 \\ d\omega\wedge\theta^0\wedge\theta^3 &= (-\dot{a}_3 + a_2)\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3 \end{aligned} \tag{4.7}$$

where

$$\dot{a}_3\sigma \equiv da_3 \qquad \mathrm{mod}(\theta^0,\theta^1,\theta^2,\theta^3) \ .$$

The equations in (4.7) clearly imply that ω could be closed only if $a_2 = 0$. Lastly we find

$$d\omega\wedge\theta^1\wedge\theta^3 = a_1\,\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3$$

and so a_1 must be zero in order for ω to be closed, which proves the lemma. □

The lemma implies that the partial differential equations $d\omega = 0$ contain only the single unknown function a. The differential equations for a obtained by setting the exterior derivative of ω in (4.6) to zero giving

$$da\,\wedge\,(\theta^0\wedge\theta^3 - \theta^1\wedge\theta^2) + a\; d(\theta^0\wedge\theta^3 - \theta^1\wedge\theta^2) = 0 \tag{4.8}$$

Dividing this equation by a leads immediately to the lemma

Lemma 4.4. *If there exists a non-zero solution* a *to* (4.8) *then there exists a one-form* $\lambda \in \Omega^1(\mathcal{R}\times G)$ *such that*

$$d(\theta^0\wedge\theta^3 - \theta^1\wedge\theta^2) = \lambda\wedge(\theta^0\wedge\theta^3 - \theta^1\wedge\theta^2) \ . \tag{4.9}$$

The contact invariant condition (4.9) is not necessarily satisfied by an arbitrary $\{e\}$-structure on $\mathcal{R}\times G$ from section 3. In fact the next lemma shows that being able to solve equation (4.9) provides the first non-trivial condition on the geometry of $\mathcal{R}\times G$ which must be satisfied in order to be able to find a closed two-form ω as in Lemma 4.1. The result is

Lemma 4.5. *There exists a one-form $\lambda \in \Omega^1(U\times G)$ such that equation (4.9) holds if and only if the relative invariant I_1 in the structure equations (3.10) vanishes. If $I_1 = 0$ then*

$$\lambda = 2\beta - 3\alpha \ . \tag{4.10}$$

Proof. Using the structure equations in (3.10) we find

$$d(\theta^0{\wedge}\theta^3 - \theta^1{\wedge}\theta^2) = (2\beta-3\alpha){\wedge}(\theta^0{\wedge}\theta^3 - \theta^1{\wedge}\theta^2) + I_1\sigma{\wedge}\theta^0{\wedge}\theta^1 - T_8\theta^0{\wedge}\theta^1{\wedge}\theta^2 \tag{4.11}$$

and we may immediately conclude that the vanishing of I_1 and T_8 are necessary conditions for (4.9) to hold. If we now assume that $I_1 = 0$ the computation

$$d^2\theta^3{\wedge}\theta^0{\wedge}\theta^2 + \frac{3}{2}d^2\theta^2{\wedge}\theta^0{\wedge}\theta^3 + \frac{1}{2}d^2\theta^0{\wedge}\theta^2{\wedge}\theta^3 = -\frac{5}{2}T_8\sigma{\wedge}\theta^0{\wedge}\theta^1{\wedge}\theta^2{\wedge}\theta^3$$

implies that T_8 vanishes as a consequence of the assumption $I_1 = 0$. This proves that the vanishing of I_1 is both necessary and sufficient for (4.9) to be satisfied which proves the first part of the lemma. Substituting the hypothesis $I_1 = 0$, which implies $T_8 = 0$, into the computation in (4.11) we obtain equation (4.10) which finishes the proof of the lemma. □

This lemma in conjunction with Lemma 4.1 implies that any fourth-order ordinary differential equation satisfying $I_1 \neq 0$ will not admit a multiplier.

From now on we consider only those geometries $\mathcal{R}\times G$ in Theorem 3.2 which satisfy the invariant condition $I_1 = 0$ so that equation (4.10) holds. The differential equations for a in (4.8) are then

$$(da + a(2\beta - 3\alpha)){\wedge}(\theta^0{\wedge}\theta^3 - \theta^1{\wedge}\theta^2) = 0 \ . \tag{4.12}$$

It is now easy to check that given any $\rho \in \Omega^1(\mathcal{R}\times G)$ with $\rho{\wedge}\omega = 0$ then $\rho = 0$. We then conclude that the differential equations (4.12) for a are

$$da + a(2\beta - 3\alpha) = 0 \ . \tag{4.13}$$

The degree of generality of the possible space of solutions a to this equation is

Lemma 4.6. *If there exists a non-zero solution a to the partial differential equations (4.13), then the solution a is unique up to multiplication by a non-zero real scalar.*

A simple but important consequence of this lemma is

Corollary 4.7. *$\nu = 0$ or 1. Equation (4.1) is variational if and only if $\nu = 1$*

At this point we may conclude that if a fourth order ordinary differential equation admit a variational multiplier then the Lagrangian (and multiplier) are unique up to scaling.

We continue studying the integrability conditions for the first-order partial differential equations (4.13) for a by taking the exterior derivative of (4.13). An application of the Poincare lemma proves

Lemma 4.8. *There exists a (non-zero) solution a to the partial differential equation (4.13) if and only if*

$$2d\beta - 3d\alpha = 0 \ . \tag{4.14}$$

Thus equation (4.14) together with the hypothesis that $I_1 = 0$ finally provide necessary and sufficient conditions in terms of the geometry of $\{e\}$-structures of Theorem 3.2 which would guarantee the existence of the closed form ω as in Lemma 4.1.

Continuing with the assumption $I_1 = 0$ and using the equations (3.22) in Corollary 3.3, the integrability conditions in (4.14) could in general have the form

$$2d\beta - 3d\alpha = B_{ij}\,\theta^i{\wedge}\theta^j + C_i\,\theta^i{\wedge}\omega = 0 \tag{4.15}$$

where the functions B_{ij}, $C_i \in C^\infty(\mathcal{R}\times G)$ can be written in terms of the structure functions of the $\{e\}$-structure and their covariant derivatives. However the integrability conditions (4.15) simplify because the functions B_{ij} and C_i are not all independent. In fact under the assumption $I_1 = 0$ if we take the exterior derivative of (4.9) to get

$$(2d\beta - 3d\alpha)\wedge(\theta^0\wedge\theta^3 - \theta^1\wedge\theta^2) = 0$$

and then substitute from (4.15) it follows that we can write

$$2d\beta - 3d\alpha = b_1\theta^0\wedge\theta^1 + b_2\theta^0\wedge\theta^2 + b_3(\theta^1\wedge\theta^2 + \theta^0\wedge\theta^3) + b_4\theta^1\wedge\theta^3 + b_5\theta^2\wedge\theta^3 \tag{4.16}$$

where $(b_r)_{r=1,\dots,5} \in C^\infty(\mathcal{R}\times G)$. Thus there are at most five independent integrability conditions for the partial differential equations (4.13). This simplification of the integrability conditions in (4.14) allows us to prove the theorem

Theorem 4.9. *The condition $\nu = 1$ is satisfied if and only if the two relative invariants T_5 and I_1 vanish.*

Proof. We have already established from the arguments above that the vanishing of I_1 and $(b_r)_{r=1..5}$ are necessary and sufficient conditions for the existence of a closed form (4.2). The coefficient b_5 in (4.16) can be expressed in terms of the torsion in the structure equations (3.22) using Corollary 3.3 to give

$$b_5 = 3T_5\ . \tag{4.17}$$

This implies that $T_5 = 0$ along with $I_1 = 0$ are necessary conditions for the existence of a closed form ω. In order to establish that the vanishing of T_5 and I_1 guarantees that the form ω is closed , we need to show these two conditions imply $(b_r)_{r=1,\dots,5} = 0$. First the assumption $T_5 = 0$ and equation (4.17) trivially imply that $b_5 = 0$ so that equation (4.16) becomes

$$2d\beta - 3d\alpha = b_1\theta^0\wedge\theta^1 + b_2\theta^0\wedge\theta^2 + b_3(\theta^1\wedge\theta^2 + \theta^0\wedge\theta^3) + b_4\theta^1\wedge\theta^3\ = 0.$$

Taking the exterior derivative of this equation and concentrating on terms which contain σ we find

$$(2d^2\beta - 3d^2\alpha)\wedge\theta^0\wedge\theta^1\wedge\alpha\wedge\beta\wedge\gamma = b_4\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\alpha\wedge\beta\wedge\gamma\,,$$

which implies $b_4 = 0$. In a similar manner we also have

$$(2d^2\beta - 3d^2\alpha)\wedge\theta^0\wedge\theta^2\alpha\wedge\beta\wedge\gamma = 2b_3\sigma\wedge\theta^0\wedge\theta^1\wedge\theta^2\wedge\theta^3\wedge\alpha\wedge\beta\wedge\gamma$$

which implies $b_3 = 0$. Again by similar arguments we have $b_1 = 0$ and $b_2 = 0$ which proves that the assumptions $T_5 = 0$ and $I_1 = 0$ imply $(b_r)_{r=1,\dots,5} = 0$. □

By using Lemma 3.5 the characterization of fourth-order scalar ordinary differential equations which admit a multiplier can then be expressed parametrically as:

Corollary 4.10. *A fourth-order ordinary differential equation admits a variational multiplier and non-degenerate second-order Lagrangian such that (1.2) is satisfied if and only if*

$$\begin{aligned}
T_5 &= \frac{1}{6}\frac{\partial^3 f}{\partial u_{xxx}^3} = 0\\
I_1 &= \frac{\partial f}{\partial u_x} + \frac{1}{2}\frac{d^2}{dx^2}\frac{\partial f}{\partial u_{xxx}} - \frac{d}{dx}\frac{\partial f}{\partial u_{xx}} - \frac{3}{4}\frac{\partial f}{\partial u_{xxx}}\frac{d}{dx}\frac{\partial f}{\partial u_{xxx}} + \frac{1}{2}\frac{\partial f}{\partial u_{xx}}\frac{\partial f}{\partial u_{xxx}} + \frac{1}{8}\left(\frac{\partial f}{\partial u_{xxx}}\right)^3 = 0.
\end{aligned}$$

We also have as a corollary of Lemma 4.6 and Theorem 2.2

Corollary 4.11. *If a fourth-order ordinary differential equation admits a variational multiplier and non-degenerate second-order Lagrangian such that (1.2) is satisfied then the multiplier, Lagrangian, and associated closed two-form ω are unique up to multiplication by a non-zero real scalar.*

The uniqueness of the Lagrangian in Corollary 4.11 is of course subject to the identification in (2.17). Corollary 4.10 and 4.4 provide a complete solution to the multiplier problem for fourth-order scalar ordinary differential equations.

As a final remark to conclude this section we would like to point out that the proof of Theorem 4.9 could be shortened by simultaneously trying to solve the equivalence method and determining the necessary and sufficient conditions for the existence of the form ω. This would require one less step in the equivalence method in section 3. For our particular problem it was easy enough to obtain the final $\{e\}$-structure in Theorem 3.2, but for more complicated problems the shorter solution would be preferred. The technique of running the equivalence method while imposing the conditions which control the existence of ω is used in [6] to characterize the second-order parabolic partial differential equations in the plane which admit multiple conservation laws.

5. Applications

In this section we provide two simple applications of the characterization of variational fourth-order ordinary differential equations given in section 4. Our first application is based on the fact that the solution to the problem of determining whether two $\{e\}$-structures are equivalent is well known [11]. With this in mind, we define an equivalence relation on the set of non-degenerate second-order Lagrangians such that we can associate a unique $\{e\}$-structure with each Lagrangian equivalence class thus solving the equivalence problem. The $\{e\}$-structure we associate with a given Lagrangian λ is of course the $\{e\}$-structure in Theorem 5.1 defined by the Euler-Lagrange equations of λ.

Let

$$\lambda = L(x,u,u_x,u_{xx})\,dx \in \Omega^{1,0}(J^\infty(\mathbb{R},\mathbb{R}))\ , \quad \overline{\lambda} = \overline{L}(\bar{x},\bar{u},\bar{u}_{\bar{x}},\bar{u}_{\bar{x}\bar{x}})\,d\bar{x} \in \Omega^{1,0}(\overline{J}^\infty(\mathbb{R},\mathbb{R})) \tag{5.1}$$

be two non-degenerate second-order Lagrangians with corresponding Euler-Lagrange equations $E(L)=0$ and $E(\overline{L})=0$. We define the two Lagrangians in (5.1) to be equivalent if there exist a classical contact transformation $\Psi^1 : J^1(\mathbb{R},\mathbb{R}) \to \overline{J}^1(\mathbb{R},\mathbb{R})$ with prolongation $\Psi : J^\infty(\mathbb{R},\mathbb{R}) \to \overline{J}^\infty(\mathbb{R},\mathbb{R})$, such that

$$\Psi^\#(\overline{\lambda}) = c\,\lambda + d_H\,\mu(x,u,u_x) \tag{5.2}$$

where $c \in \mathbb{R}^*$, $\mu \in C^\infty(J^1(\mathbb{R},\mathbb{R}))$, and $\Psi^\#$ is the projected pullback [3]. The projected pullback in equation (5.2) can be written as

$$\Psi^\#(\overline{\lambda}) = \Psi^*(\overline{\lambda}) \mod(\theta^0,\theta^1)\,,$$

and this formula for $\Psi^\#(\overline{\lambda})$ is the standard definition for the transformation law of a Lagrangian.

The equivalence relation (5.2) on the space of Lagrangians along with the definition of equivalence for fourth order ordinary differential equations given in (3.3) allows us to prove

Theorem 5.1. *There exists a one-to-one correspondence between the equivalence classes of non-degenerate second-order Lagrangians and the equivalence classes of the associated Euler-Lagrange equations.*

Proof. Two Lagrangians as in (5.1) satisfying (5.2) clearly have equivalent Euler-Lagrange equations (in the sense of (3.3)), so that we need to only prove sufficiency. Suppose $\lambda = Ldx$ and $\overline{\lambda} = \overline{L}d\overline{x}$ are two non-degenerate second-order Lagrangians with equivalent Euler-Lagrange equations. Let $\Psi : J^\infty(\mathbb{R},\mathbb{R}) \to \overline{J}^\infty(\mathbb{R},\mathbb{R})$ be the map that provides the equivalence between $E(L)=0$ and $E(\overline{L})=0$ and define the Lagrangian

$$\widehat{\lambda} = \Psi^\#(\overline{\lambda})\ .$$

The condition that the two Euler-Lagrange equations for λ and $\overline{\lambda}$ are equivalent simply implies

$$E(\widehat{\lambda}) = \psi\,E(\lambda)$$

for some $\psi \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$. Using this in the first variational formula (2.7) which in terms of $\widehat{\lambda}$ is

$$d_V\widehat{\lambda} = E(\widehat{\lambda}) + d_H\widehat{\eta}$$

where $\widehat{\eta} \in \Omega^{0,1}(J^\infty(\mathbb{R},\mathbb{R}))$, leads to

$$d_V\,\widehat{\lambda} = \psi\,E(\lambda) + d_H\widehat{\eta}\ .$$

Pulling this equation back to the equation manifold $i : \mathcal{R} \to J^\infty(\mathbb{R},\mathbb{R})$ defined by $E(L)=0$, as described in equations (2.13) and (2.14) we associate with $\widehat{\lambda}$ the two-form

$$\widehat{\omega} = d_V\ i^*\widehat{\eta}\ . \tag{5.3}$$

The form $\widehat{\omega}$ is closed on the equation manifold $\mathcal{R}$ and hence by Corollary 4.11 we may conclude

$$\widehat{\omega} = c\,\omega$$

where $c \in \mathbb{R}^*$ and ω is the closed two-form on $\mathcal{R}$ associated with λ. If we now define the Lagrangian $\lambda' = c^{-1}\widehat{\lambda}$, the above procedure produces the two-form $\omega' = c^{-1}\widehat{\omega}$ associated with λ' and clearly $\omega' = \omega$. The second part of Theorem 2.2 then implies that (5.2) holds. □

Thus in summary Theorem 5.1 allows us to answer the question of whether two Lagrangians are equivalent by determining whether the corresponding Euler-Lagrange equations are equivalent. As a direct application of Theorem 5.1 combined with Theorem 4.9 we state

Corollary 5.2. *The equivalence classes of non-degenerate second-order Lagrangians are in one-to-one correspondence with the equivalence classes of $\{e\}$-structures given in Theorem 3.2 which satisfy $I_1 = 0$ and $T_5 = 0$.*

This provides a complete solution to the equivalence problem for non-degenerate second-order Lagrangians with respect to the equivalence relationship (5.2), and completes our first application.

In our second application we determine the relationship between the symmetry algebra of a fourth-order Euler-Lagrange equation and the divergence symmetry algebra of the corresponding Lagrangian.

An infinitesimal symmetry of a fourth-order scalar ordinary differential equation

$$u_{xxxx} - f(x, u, u_x, u_{xx}, u_{xxx}) = 0 \tag{5.4}$$

with corresponding equation manifold $\mathcal{R}$, is a vector-field $\tilde{X}$ on $J^1(\mathbb{R},\mathbb{R})$ which preserves the contact structure on $J^1(\mathbb{R},\mathbb{R})$ and whose evolutionary representative (see [13])

$$X = \phi\frac{\partial}{\partial u} \tag{5.5}$$

satisfies

$$X^\infty\left(u_{xxxx} - f(x, u, u_x, u_{xx}, u_{xxx})\right) = \psi\left(u_{xxxx} - f(x, u, u_x, u_{xx}, u_{xxx})\right) \tag{5.6}$$

for some $\psi \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$, and where X^∞ is the prolongation of X to $J^\infty(\mathbb{R},\mathbb{R})$. On the other hand, an infinitesimal divergence symmetry of a Lagrangian λ is a vector-field $\tilde{X}$ on $J^1(\mathbb{R},\mathbb{R})$ which preserves the contact structure on $J^1(\mathbb{R},\mathbb{R})$ and whose evolutionary representative as in (5.5) satisfies

$$\mathcal{L}_{X^\infty}\lambda = d_H\mu$$

for some $\mu \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$. It is a classical result [13] that every infinitesimal divergence symmetry of a Lagrangian λ is an infinitesimal symmetry of the associated Euler-Lagrange equations, however the converse of this theorem is often not true. That is, an infinitesimal symmetry of an Euler-Lagrange equation need not define a divergence symmetry of the associated Lagrangian. This discrepancy can be precisely described in the case of fourth-order scalar Euler-Lagrange equations.

Let $\lambda = L dx$ be a non-degenerate second-order Lagrangian with Euler-Lagrange equation $E(L) = 0$ which defines the equation manifold $i : \mathcal{R} \to J^\infty(\mathbb{R},\mathbb{R})$. Furthermore let $\mathbf{g}$ be the Lie algebra of infinitesimal symmetries of the fourth order ordinary differential equation $E(L) = 0$. We then find

Theorem 5.3. *Let X be an evolutionary representative of an infinitesimal symmetry $\tilde{X} \in \mathbf{g}$. There exists a constant $c \in \mathbb{R}$ such that*

$$\mathcal{L}_{X^\infty}\lambda - c\,\lambda = d_H\mu \tag{5.7}$$

for some $\mu \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$.

Note that if $c = 0$ in this theorem then the symmetry $\tilde{X}$ is, by definition, a variational symmetry.

Proof. Let $\tilde{X} \in \mathbf{g}$ and define the Lagrangian

$$\widehat{\lambda} = \mathcal{L}_{X^\infty}\lambda\ .$$

The symmetry condition (5.6) implies

$$\mathcal{L}_{X^\infty} E(\lambda) = \kappa\, E(\lambda)$$

for some $\kappa \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$. Substituting these two equations into the Lie derivative with respect to X^∞ of the first variation equation in (2.7) and using the standard properties of evolutionary vector-fields [1] we get

$$d_V\, \widehat{\lambda} = \kappa\, E(\lambda) + d_H \widehat{\eta}$$

where $\widehat{\eta} = \mathcal{L}_{X^\infty}\eta$. We may use this equation to define the two-form on $\mathcal{R}$

$$\widehat{\omega} = d_V\ i^*\widehat{\eta}$$

and as usual, $\widehat{\omega} \in \overline{V}(\mathcal{R})$ (ω is closed and has the appropriate algebraic form). We conclude from Corollary 4.11 that

$$\widehat{\omega} = c\,\omega \tag{5.8}$$

where $c \in \mathbb{R}$ and $\omega \in \overline{V}(\mathcal{R})$ is the closed form associated with λ through Theorem 2.2. There are now two possibilities to consider depending on whether the constant c in (5.8) is zero or not. First assume that $c \neq 0$ in equation (5.8), and define $\lambda' = c^{-1}\widehat{\lambda}$. The differential two-form ω' associated with λ' satisfies

$$\omega' = \omega$$

where again ω is the closed form associated with λ. This equation along with the second part of Theorem 2.2 allows us to conclude that (5.7) holds. The case $c = 0$ is similar. ☐

Let $\mathbf{h}$ be the Lie algebra of infinitesimal divergence symmetries of λ. The Lie algebra $\mathbf{h}$ is a subalgebra of $\mathbf{g}$ and the precise relationship between these two symmetry algebras is

Theorem 5.4. *The Lie algebra* $\mathbf{g}$ *splits as a direct sum of vector-spaces*

$$\mathbf{g} = \mathbf{h} \oplus \mathbf{s}$$

where $\mathbf{s}$ *is a vector-space of dimension 0 or 1. Thus* $g = h + s$ *where* $g = \dim \mathbf{g}$, $h = \dim \mathbf{h}$, *and* $s = \dim \mathbf{s}$ *. The dimension of* $\mathbf{s}$ *is 1 if and only if there exists a symmetry* $\tilde{X} \in \mathbf{g}$ *and a constant* $c \in \mathbb{R}^*$ *such that*

$$\mathcal{L}_{X^\infty}\lambda - c\,\lambda = d_H\mu\ .$$

Proof. Let $\{X_a\}_{a=1..g}$ be a set of evolutionary vector-fields corresponding to a basis for the Lie algebra $\mathbf{g}$. According to Theorem 5.3 we have a collection of constants $c_a \in \mathbb{R}$ and functions $\mu_a \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$ such that

$$\mathcal{L}_{X_a^\infty}\lambda - c_a\lambda = d_H\mu_a\ ,\quad a = 1...g\ .$$

If $c_a = 0$ for $a = 1...g$, then the theorem is true and $\mathbf{g} = \mathbf{h}$. Thus we assume, with out loss of generality, $c_1 \neq 0$ and define a new basis of $\mathbf{g}$ so that the corresponding evolutionary representatives are

$$\widehat{X}_a = X_a - \frac{c_a}{c_1}X_1 \quad a = 2...g\ .$$

By taking the Lie derivatives of λ with respect to these vector-fields we find

$$\mathcal{L}_{\widehat{X}_a^\infty}\lambda = d_H\widehat{\mu}_a\ ,\quad a = 2...g$$

where $\widehat{\mu}_a \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$. Thus $\widehat{X}_a, a = 2...g$ are the evolutionary form of divergence symmetries of λ, and the theorem is proved. ☐

This theorem shows that in the case of a scalar fourth-order variational problem that the symmetry algebra of a variational equation and the corresponding divergence symmetry algebra of the Lagrangian will be isomorphic if and only if there is no symmetry of the Euler-Lagrange equation which scales the Lagrangian. In light of Theorem 5.4, if we chose to define a vector-field $\tilde{X}$ to be a divergence symmetry of a Lagrangian λ if there exist $c \in \mathbb{R}$ such that

$$\mathcal{L}_{X^\infty}\lambda = c\lambda + d_H\widehat{\mu} \tag{5.9}$$

for some $\mu \in C^\infty(J^\infty(\mathbb{R},\mathbb{R}))$, then these divergence symmetries are in one-to-one correspondence with symmetries of the Euler-Lagrange equations for the Lagrangian.

To conclude this article, we would like to point out that Theorem 5.3 and Theorem 5.4 are true for scalar ordinary differential equations of arbitrary (even) order [2]. Finally, by using the fact that bounds on the maximal dimension of the point and contact symmetry groups of scalar ordinary differential equations are known [14], Theorem 5.3 and 5.4 provide bounds on the maximal dimension of the divergence symmetry algebra of a Lagrangian which are in agreement with [12].

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