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Moving Coframes: I. A Practical Algorithm

MARK FELS⋆ and PETER J. OLVER⋆⋆
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A.
e-mail: fels@math.umn.edu;olver@ima.umn.edu

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Abstract. This is the first in a series of papers devoted to the development and applications of a new general theory of moving frames. In this paper, we formulate a practical and easy to implement explicit method to compute moving frames, invariant differential forms, differential invariants and invariant differential operators, and solve general equivalence problems for both finite-dimensional Lie group actions and infinite Lie pseudo-groups. A wide variety of applications, ranging from differential equations to differential geometry to computer vision are presented. The theoretical justifications for the moving coframe algorithm will appear in the next paper in this series.

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1. Introduction

First introduced by Gaston Darboux, and then brought to maturity by Élie Cartan, [6, 8], the theory of moving frames (‘repères mobiles’) is acknowledged to be a powerful tool for studying the geometric properties of submanifolds under the action of a transformation group. While the basic ideas of moving frames for classical group actions are now ubiquitous in differential geometry, the theory and practice of the moving-frame method for more general transformation group actions has remained relatively undeveloped and is as yet not well understood. The famous critical assessment by Weyl in his review, [47], of Cartan’s seminal book, [8], retains its perspicuity to this day:

“I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear.... Nevertheless, I must admit I found the book, like most of Cartan’s papers, hard reading”.

Implementations of the method of moving frames for certain groups having direct geometrical significance – including the Euclidean, affine, and projective groups – can be found in both Cartan’s original treatise, [8], as well as many standard texts in differential geometry; see, for example, the books of Guggenheimer, [19], which gives the method center stage, Sternberg, [44], and Willmore, [50]. The method continues to attract the attention of modern-day

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researchers and has been successfully extended to some additional examples, including, for instance, holomorphic curves in projective spaces and Grassmannians. The papers of Griffiths, [18], Green, [17], Chern, [12], and the lecture notes of Jensen, [23], are particularly noteworthy attempts at placing Cartan's intuitive constructions on a firm theoretical and differential geometric foundation. However, none of the proposed modern geometrical formulations of the theory incorporates the full scope or range of applicability of the method as originally envisioned by Cartan. To this day, both the formulation and construction of moving frames for general Lie group actions has remained obscure, particularly for anyone interested in new applications. Although they strive for generality, the range of examples treated remains rather limited, and Weyl's pointed critique of Cartan's original version still, in our opinion, applies to all of these later efforts.

There are two main goals of this series of papers devoted to a study of Cartan's method of moving frames. The first, of utmost importance for applications and the subject of the present work, is to develop a practical algorithm for constructing moving frames that is easy to implement, and can be systematically applied to concrete problems arising in different applications. Our new algorithm, which we call the method of 'moving coframes', not only reproduces all of the classical moving frame constructions, often in a simpler and more direct fashion, but can be readily applied to a wide variety of new situations, including infinite-dimensional pseudo-groups, intransitive group actions, restricted reparametrization problems, joint group actions, to name a few. Although one can see the germs of our ideas in the above-mentioned references, our approach is different, and, we believe, significantly easier to implement in practical examples. Standard presentations of the method rely on an unusual hybrid of vector fields and differential forms. Our approach is inspired by the powerful Cartan equivalence method [11, 16, 38], which has much of the flavor of moving frame-type computations, but relies solely on the use of differential forms, and the operation of exterior differentiation. The moving coframe method we develop does have a complete analogy with the Cartan equivalence method; indeed, we shall see that the method includes not only all moving frame type equivalence problems, under both finite-dimensional Lie transformation groups and infinite Lie pseudo-groups, but also includes the standard Cartan equivalence problems in a very general framework.

Our second goal is to rigorously justify the moving coframe method by proposing a new theoretical foundation for the method of moving frames. This will form the subject of the second paper in the series [15], and will be based on a second algorithm, known as regularization. The key new idea is to avoid the technically complicated normalization procedure during the initial phases of the computation, leading to a fully regularized moving frame. Once a moving frame and coframe, along with the complete system of invariants, are constructed in the regularized framework, one can easily restrict these invariants to particular classes of submanifolds, producing (in nonsingular cases) the standard moving frame. This approach enables us to successfully bypass branching and singularity complications, and enables one to treat both generic and singular submanifolds on the same general footing. Once the regularized solution to the problem has been properly implemented, the *a posteriori* justification for the usual normalization and reduction procedure can be readily provided. Details and further examples appear in Part II [15].

Beyond the traditional application to the differential geometry of curves and surfaces in certain homogeneous spaces, there are a host of applications of the method that lend great importance to its proper implementation. Foremost are the equivalence and symmetry theorems of Cartan, that characterize submanifolds up to a group transformation by the functional relationships among their fundamental differential invariants. The method provides an effective means of computing complete systems of differential invariants and associated invariant differential operators, which are used to generate all the higher-order invariants. The fundamental differential invariants and their derived invariants, up to an appropriate order, serve to parametrize the 'classifying manifold' associated with a given submanifold; the Cartan solution to the equivalence problem states that two submanifolds are (locally) congruent under a group transformation if and only if their classifying manifolds are identical. Moreover, the dimension of the classifying manifold completely determines the dimension of the symmetry subgroup of the submanifold in question. We note that the differential invariants also form the fundamental building blocks of basic physical theories, enabling one to construct suitably invariant differential equations and variational principles, cf. [38].

Additional motivation for pursuing this program comes from new applications of moving frames to computer vision promoted by Faugeras [13], with applications to invariant curve and surface evolutions, and the use of the classifying (or 'signature') manifolds in the invariant characterization of object boundaries that forms the basis of a fully group-invariant object recognition visual processing system [5]. Although differential invariants have evident direct applications to object recognition in images, the often high order of differentiation makes them difficult to compute in an accurate and stable manner. One alternative approach [35], is to use joint differential invariants, or, as they are known in the computer vision literature, 'semi-differential invariants', which are based on several points on the submanifold of interest. Although a few explicit examples of joint differential invariants are known, there is, as far as we know, no systematic classification of them in the literature. We show how the method of moving coframes can be readily used to compute complete systems of joint differential invariants, and illustrate with some examples of direct interest in image processing. The approximation of higher-order differential invariants by joint differential invariants and, generally, ordinary joint invariants leads to fully invariant finite difference numerical schemes for their computation, which were first proposed in [5]. The moving coframe method should aid in the understanding and extension of such schemes to more complicated situations.

In this paper, we begin with a review of the basic equivalence problems for submanifolds under transformation groups that serve to motivate the method of moving frames. Section 3 provides a brief introduction to one of the basic tools that is used in the moving coframe method – the left-invariant Maurer–Cartan forms on a Lie group. Two practical means of computing the Maurer–Cartan forms, including a novel method based directly on the group transformation rules, are discussed. Section 4 begins the presentation of the moving coframe method for the simplest category of examples – finite-dimensional transitive group actions – and illustrates it with an equivalence problem arising in the calculus of variations and in classical invariant theory. Section 5 extends the basic method to intransitive Lie group actions. The simplest example of an infinite-dimensional pseudo-group, namely the reparametrization pseudo-group for parametrized submanifolds, is discussed in Section 6 and illustrated with a well studied geometrical example – the case of curves in the Euclidean plane. This is followed by a discussion of curves in affine and projective geometry, reproducing classical moving frame computations in a simple direct manner based on the moving coframe approach; in Section 7, the connections between the classical and moving coframe methods are explained in further detail. Section 8 employs the moving coframe method to completely analyze the joint differential invariants in two particular geometrical examples – two-point differential invariants for curves in the Euclidean and affine plane. Section 9 discusses how to analyze more general pseudo-group actions, illustrating the method with two examples arising in classical work of Lie [28], Vessiot [46], and Medolaghi [34]. In addition, we show how to solve the equivalence problem for second-order ordinary differential equations under the pseudo-group of fiber-preserving transformations using the moving coframe method, thereby indicating how all Cartan equivalence problems can be treated by this method. Finally, we discuss some open problems that are under current investigation. In all cases, the paper is designed for a reader who is interested in applications, in that only the basic algorithmic steps are discussed in detail. In order not to cloud the present practically-oriented exposition, precise theoretical justifications for the algorithms proposed here will appear in the second paper in this series [15].

2. The Basic Equivalence Problems

We begin our exposition with a discussion of the basic equivalence problems which can be handled by the method of moving frames; see Jensen [23; p. VI], for additional details. Suppose G is a transformation group acting smoothly on an m-dimensional manifold M. In classical applications, G is a finite-dimensional Lie group, but, as we shall see, the method can be extended to infinite-dimensional Lie pseudo-group actions, e.g., the group of conformal transformations on a Riemannian surface, the group of canonical transformations on a symplectic space, or the group of contact transformations on a jet space. In either situation, a basic equivalence problem is to determine whether two given submanifolds are congruent modulo a group transformation. We shall divide the basic problem into two different versions, depending on whether one allows reparametrizations of the submanifolds in question. Formally, these can be stated as follows.

THE FIXED PARAMETER EQUIVALENCE PROBLEM. Given two embeddings $\iota\colon X \to M$ and $\bar{\iota}\colon X \to M$ of an n-dimensional manifold X into M does there exist a group transformation $g \in G$ such that

$$\bar{\iota}(x) = g \cdot \iota(x) \quad \forall\, x \in X. \tag{2.1}$$

THE UNPARAMETRIZED EQUIVALENCE PROBLEM. Given two submanifolds $N, \bar{N} \subset M$ of the same dimension n, determine whether there exists a group transformation $g \in G$ such that

$$g \cdot N = \bar{N}. \tag{2.2}$$

Submanifolds satisfying (2.2) are said to be *congruent* under the group action.

In both problems we shall only consider the question in the small, meaning that (2.1) only needs to hold on an open subset of X, or that congruence, (2.2), holds in a suitable neighborhood of given points $z_0 \in N$, $\bar{z}_0 \in \bar{N}$. Global issues require global constructions that lie outside the scope of the Cartan approach to equivalence problems.

Note that the problem of determining the symmetries of a submanifold, meaning the set of all group elements that preserve the submanifold, forms a particular case of the equivalence problem. Indeed, a *symmetry* of a submanifold is merely a self-equivalence. For instance, the unparametrized symmetries of a given submanifold $N \subset M$ are those group elements that (locally) satisfy $g \cdot N = N$. Note that the symmetry group of a given submanifold forms a subgroup $H \subset G$ of the full transformation group.

EXAMPLE 2.1. A classical example is inspired by the geometry of curves in the Euclidean plane. A curve $C \subset \mathbb{R}^2$ is parametrized by a smooth map $\mathbf{x}(t) = (x(t), y(t))$ defined on (a subinterval of) $\mathbb{R}$. The underlying group for Euclidean planar geometry is the Euclidean group $\mathrm{E}(2) = \mathrm{O}(2) \ltimes \mathbb{R}^2$ consisting of translations, rotations and (in the nonoriented case) reflections.

In the fixed parametrization problem, we are given two parametrized curves $\mathbf{x}(t)$ and $\bar{\mathbf{x}}(t)$, and want to know when there exists a Euclidean motion such that $\bar{\mathbf{x}}(t) = R \cdot \mathbf{x}(t) + a$ for all t, where the rotation $R \in \mathrm{O}(2)$ and translation $a \in \mathbb{R}^2$ are both independent of t. Physically, we are asking when two moving particles differ by a fixed Euclidean motion at all times, a problem that has significant applications to motion detection and recognition of moving objects.

In the unparametrized problem, we are interested in determining when two curves are congruent under a Euclidean motion, meaning $\bar{C} = R \cdot C + a$ for some fixed Euclidean transformation $(R, a) \in \mathrm{E}(2)$. This occurs if and only if there exists a change of parameter $\bar{t} = \tau(t)$ such that $\bar{\mathbf{x}}(\tau(t)) = R \cdot \mathbf{x}(t) + a$ for some fixed Euclidean transformation (R, a).

A Euclidean symmetry of a curve C is a Euclidean transformation (R, a) that preserves the curve: $R \cdot C + a = C$. For instance, the Euclidean symmetries of a circle consist of the rotations around its center. In the fixed parameter version, the circle must be parametrized by a constant multiple of arc length for this to remain valid.

EXAMPLE 2.2. Consider the action

$$A\colon (x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{u}{\gamma x + \delta} \right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2) \tag{2.3}$$

of the general linear group $\mathrm{GL}(2)$ on $\mathbb{R}^2$. This forms a multiplier representation of $\mathrm{GL}(2)$, cf. [14, 38], which lies at the heart of classical invariant theory. We restrict our attention to curves given by the graphs of functions $u = f(x)$, thereby avoiding issues of reparametrization. Two such curves are equivalent if and only if their defining functions f and $\bar{f}$ are related by the formula

$$f(x) = (\gamma x + \delta)\, \bar{f}\left(\frac{\alpha x + \beta}{\gamma x + \delta} \right) = (\gamma x + \delta) \bar{f}(\bar{x}), \tag{2.4}$$

for some nonsingular matrix A. Equation (2.4) is the fundamental equivalence condition for first-order Lagrangians that depend only on a derivative coordinate in the calculus of variations, cf. [36]. Moreover, if $f(x) = \sqrt[n]{P(x)}$, and $\bar{f}(\bar{x}) = \sqrt[n]{\bar{P}(\bar{x})}$, then (2.4) becomes*

$$P(x) = (\gamma x + \delta)^n\, \bar{P}\left(\frac{\alpha x + \beta}{\gamma x + \delta} \right) = (\gamma x + \delta)^n\, \bar{P}(\bar{x}). \tag{2.5}$$

In the case when P and $\bar{P}$ are polynomials of degree n, (2.5) indicates their equivalence under projective transformations, and so forms the fundamental equivalence problem of classical invariant theory.

In the general unparametrized equivalence problem, typically, the submanifolds N and $\bar{N}$ are formulated via explicit parametrizations $\iota\colon X \to M$, with image $N = \iota(X)$ and $\bar{\iota}\colon X \to M$, with $\bar{N} = \bar{\iota}(X)$, where, for simplicity, the parameter spaces are taken to be the same. (Indeed, since our considerations are always local, we shall not lose any generality by assuming that $X \subset \mathbb{R}^n$ is an open subset of Euclidean space.) In such cases, we can easily reformulate the unparametrized equivalence problem in the following form.

* We are ignoring the branching of the nth root here. See [36, 38] for a more precise version of this construction.

THE REPARAMETRIZATION EQUIVALENCE PROBLEM. Given two embeddings $\iota: X \to M$ and $\bar{\iota}: X \to M$ of an n-dimensional manifold X into M does there exist a local diffeomorphism $\Phi: X \to X$, i.e., a change of parameter, and a group transformation $g \in G$ such that

$$\bar{\iota}(\Phi(x)) = g \cdot \iota(x), \quad \forall\, x \in X. \tag{2.6}$$

We shall see that by solving the fixed parametrization problem, first in the case of G being a finite-dimensional Lie transformation group, then extending this to the case of G being an infinite Lie pseudo-group of transformations, that we will then be able to solve the reparametrization problem. For instance, we can reformulate the unparametrized equivalence problem for curves in the Euclidean plane as a fixed parametrization problem for curves in the extended space $E = \mathbb{R} \times \mathbb{R}^2$, which has coordinates $(t, \mathbf{x}) = (t, x, y)$. The extended curve is given as the graph $\{(t, \mathbf{x}(t))\}$ of the original parametrized curve, and the pseudo-group $\mathcal{G} = \mathcal{D}\mathit{iff}(1) \times \mathrm{E}(2)$ acting on E consists of a finite-dimensional group, the Euclidean group $\mathrm{E}(2)$ acting on $\mathbb{R}^2$, together with the infinite-dimensional pseudo-group $\mathcal{D}\mathit{iff}(1)$ consisting of all smooth (local) diffeomorphisms $\bar{t} = \tau(t)$ of the parameter space $\mathbb{R}$.

The formulation of the reparametrization problem in the form (2.6) indicates an intermediate extension of the two cases, in which one only allows a subclass of all possible reparametrizations.

THE RESTRICTED REPARAMETRIZATION EQUIVALENCE PROBLEM. Given two embeddings $\iota: X \to M$ and $\bar{\iota}: X \to M$ of an n-dimensional manifold X into M and a Lie pseudo-group of transformations $\mathcal{H}$ acting on X, determine whether there exists a group transformation $g \in G$ such that (2.6) holds for some reparametrization $\Phi \in \mathcal{H}$ in the prescribed pseudo-group.

For example, one might consider the problem of equivalence of surfaces in Euclidean space, in which one is only allowed conformal, or area preserving, or Euclidean reparametrizations. The general reparametrization equivalence problem is, of course, a special case when the pseudo-group $\mathcal{H} = \mathcal{D}\mathit{iff}(X)$ is the entire local diffeomorphism group.

In general, the solution to any equivalence problem is governed by a complete system of invariants. In the present context, the invariants are the fundamental differential invariants for the transformation group action in question. Thus, any solution method must, as a consequence, produce the differential invariants in question.

EXAMPLE 2.3. In the case of curves in Euclidean geometry, the ordinary curvature* function $\kappa = |\mathbf{x}_t|^{-3}(\mathbf{x}_t \wedge \mathbf{x}_{tt})$ is the fundamental differential invariant.

* Here $|\mathbf{a}|$ is the usual Euclidean norm and $\mathbf{a} \wedge \mathbf{b}$ is the scalar-valued cross-product between vectors in the plane.

For the fixed parametrization problem, there is a second fundamental differential invariant – the speed $v = |\mathbf{x}_t|$. Furthermore, all higher-order differential invariants are obtained by successively differentiating the curvature (and speed) with respect to arc length $\mathrm{d}s = v\,\mathrm{d}t = |\mathbf{x}_t|\,\mathrm{d}t$, which is the fundamental Euclidean invariant 1-form. (In the fixed problem, one can replace s derivatives by t derivatives since $\mathrm{d}t$ is also invariant if we disallow any changes in parameter.) A similar result holds for general transformation groups – one can obtain all higher-order differential invariants by successively applying certain invariant differential operators to the fundamental differential invariants, cf. [38].

The functional relationships between the fundamental differential invariants will solve the equivalence problem. Roughly speaking, one uses the differential invariants to parametrize a ‘classifying’ or ‘signature’ manifold associated with the given submanifold, and the result is that, under suitable regularity hypotheses, two submanifolds will be congruent under a group transformation if and only if their classifying manifolds are *identical*. For example, in the unparametrized Euclidean curve problem, the classifying curve is parametrized by the two curvature invariants $(\kappa, \kappa_s = \mathrm{d}\kappa/\mathrm{d}s)$, whereas in the fixed problem, one uses all four invariants $(v, \kappa, v_s, \kappa_s)$ to parametrize the classifying curve. See [5] for applications of the classifying curve to the problem of object recognition in computer vision. Of course, this ‘solution’ reduces one to another potentially difficult *identification problem* – when do two parametrized submanifolds coincide? One approach to the latter problem is to use the Implicit Function Theorem to realize the classifying submanifold as the graph of a function, which eliminates the reparametrization ambiguity. Alternatively, in an algebraic context, a solution can be provided by Gröbner basis techniques, cf. [4]. Neither approach completely resolves the general identification problem, but particular cases can often be handled effectively.

Remark. A more standard solution to the equivalence problem depends on the choice of a base point $\mathbf{x}_0 = \mathbf{x}(t_0)$ on the curve. Then the curvature $\kappa(s)$ as a function of arc length $s = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathrm{d}s$ uniquely characterizes the curve up to Euclidean congruence, [19; p. 24]. The classifying curve approach has two distinct advantages: first, there is no choice of base point required, which eliminates the translational ambiguity inherent in the curvature function $\kappa(s)$; second, the classifying curve is completely local, whereas the arc length s is a nonlocal function of the curve. Note that the classifying curve can be computed directly, without appealing to the arc length parametrization.

The differential invariants can also be used to determine the structure of the symmetry group. In the case of an effectively acting Lie group G, the codimension of the symmetry subgroup H of the submanifold N, i.e., $\dim G - \dim H$, is the same as the number of functionally independent differential invariants on the submanifold. In particular, the maximally symmetric submanifolds occur when all differential invariants are constant; if G acts transitively, then these can be identified with the homogeneous submanifolds of M, i.e., the orbits of suitable closed subgroups of G, cf. [23]. For instance, in the Euclidean case, the maximally symmetric curves are where the curvature is constant, which are the circles and straight lines, since these are the orbits of the one-parameter subgroups of $\mathrm{E}(2)$. (Technically, these retain the infinite-dimensional reparametrization group $\mathcal{D}\mathrm{iff}(1)$ as an additional symmetry group.) In the fixed parameter version, the circles and straight lines must be parametrized by a constant multiple of their arc length in order to retain their distinguished symmetry status.

Finally, we remark that differential invariants can be used to construct general invariant differential equations admitting the given transformation group. Specifically, suppose $J_1, \ldots, J_N$ form a complete system of functionally independent kth order differential invariants, defined on an open subset $\mathcal{V}^k \subset \mathrm{J}^k$ of the jet space where the prolonged group action is regular. Then, on $\mathcal{V}^k$, any kth-order system of differential equations admitting G as a symmetry group can be written in terms of the differential invariants: $H_\nu(J_1, \ldots, J_N) = 0$. For example, the most general Euclidean-invariant third order differential equation has the form $\mathrm{d}\kappa/\mathrm{d}s = H(\kappa)$, equating the derivative of curvature with respect to arc length to a function of curvature. Similar comments apply to invariant variational problems, and we refer the reader to [37, 38], for details. These results form the foundations of modern physical field theories, in which one bases the differential equations, or variational principle, on its invariance with respect to the theory's underlying symmetry group. The groups in question range from basic Poincaré and conformal invariance, to the exceptional simple Lie groups lying at the foundations of string theory, as well as infinite-dimensional gauge groups and groups of Kac–Moody type. Remarkably, complete systems of differential invariants are known for only a small handful of transformation groups arising in physical applications – a collection that includes *none* of the above-mentioned groups! Our moving coframe algorithm provides an direct and effective means for providing such classifications.

3. The Maurer–Cartan Forms

In our approach to the theory and practical implementation of the method of moving frames, the left-invariant Maurer–Cartan forms on a finite-dimensional Lie group play an essential role. We therefore begin by reviewing the basic definition, and then present two computationally effective methods for finding the explicit formulae for the Maurer–Cartan forms. The theoretical justification for the second method will appear in Part II [15].

Throughout this section, G will be an r-dimensional Lie group. We let $L_g : h \mapsto g \cdot h$ denote the standard left multiplication map.

DEFINITION 3.1. A 1-form μ on G is called a (left-invariant) *Maurer–Cartan form* if it satisfies

$$(L_g)^* \mu = \mu \quad \text{for all } g \in G. \tag{3.1}$$

Remark. If one uses the right-invariant Maurer–Cartan forms instead, one is led to an alternative theory of right moving frames. Although the left versions appear almost exclusively in the literature, their right counterparts will play an important role in the theoretical justifications and the regularized method introduced in Part II. In this paper, though, we shall exclusively use the left-invariant Maurer–Cartan forms and moving frames; see [15] for details.

The space of Maurer–Cartan forms on G is an r-dimensional vector space, which can naturally be identified with the dual to the Lie algebra $\mathfrak{g}$ of left-invariant vector fields on G. If we choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ of $\mathfrak{g}$, then there is a dual basis $\mu^1, \ldots, \mu^r$ of the space of Maurer–Cartan forms, satisfying $\langle \mu^i; \mathbf{v}_j \rangle = \delta^i_j$, where δ^i_j is the usual Kronecker delta. The basis Maurer–Cartan forms satisfy the fundamental *structure equations*

$$\mathrm{d}\mu^i = -\sum_{j<k} C^i_{jk} \mu^j \wedge \mu^k, \tag{3.2}$$

where the coefficients C^i_{jk} are the structure constants corresponding to our choice of basis of the Lie algebra $\mathfrak{g}$. The Maurer–Cartan forms are a *coframe* on the Lie group G, meaning that they form a pointwise basis for the cotangent space T^*G, or, equivalently, that we can write any 1-form ω on G as a linear combination $\omega = \sum f_i \, \mu^i$ thereof, where the f_i are suitable smooth functions.

The most common method for explicitly determining the Maurer–Cartan forms on a given Lie group is to realize $G \subset \mathrm{GL}(n)$ as a matrix Lie group. The independent entries of the $n \times n$ matrix of 1-forms

$$\boldsymbol{\mu} = A^{-1} \, \mathrm{d}A \tag{3.3}$$

form a basis for the left-invariant Maurer–Cartan forms on G. Here $A = A(g^1, \ldots, g^r) \in G$ represents the general matrix in G, which we have parametrized by local coordinates $(g^1, \ldots, g^r)$ near the identity, and $\mathrm{d}A = \sum (\partial A / \partial g^i) \, \mathrm{d}g^i$ is its differential, which is an $n \times n$ matrix of 1-forms.

For example, in the case $G = \mathrm{GL}(2)$, the four independent Maurer–Cartan forms are the components of the matrix

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 & \mu_2 \\ \mu_3 & \mu_4 \end{pmatrix} = A^{-1} \, \mathrm{d}A = \frac{1}{\alpha\delta - \beta\gamma} \begin{pmatrix} \delta \, \mathrm{d}\alpha - \beta \, \mathrm{d}\gamma & \delta \, \mathrm{d}\beta - \beta \, \mathrm{d}\delta \\ \alpha \, \mathrm{d}\gamma - \gamma \, \mathrm{d}\alpha & \alpha \, \mathrm{d}\delta - \gamma \, \mathrm{d}\beta \end{pmatrix}. \tag{3.4}$$

Similarly, if $G = \mathrm{E}(2) = \mathrm{O}(2) \ltimes \mathbb{R}^2$ is the Euclidean group in the plane, then we can identify $\mathrm{E}(2) \subset \mathrm{GL}(3)$ as a subgroup of $\mathrm{GL}(3)$ by identifying $(R, \mathbf{a}) \in \mathrm{E}(2)$ with the 3×3 matrix

$$\begin{pmatrix} R & \mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & a \\ \sin\phi & \cos\phi & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Substituting into (3.3) leads to

$$\begin{aligned} \boldsymbol{\mu} &= \begin{pmatrix} R^{-1} & -R^{-1}\mathbf{a} \\ \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}R & \mathrm{d}\mathbf{a} \\ \mathbf{0} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\mathrm{d}\phi & \cos\phi\,\mathrm{d}a + \sin\phi\,\mathrm{d}b \\ \mathrm{d}\phi & 0 & -\sin\phi\,\mathrm{d}a + \cos\phi\,\mathrm{d}b \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus, the three independent Euclidean Maurer–Cartan forms are

$$\mu_1 = \mathrm{d}\phi, \qquad \mu_2 = \cos\phi\,\mathrm{d}a + \sin\phi\,\mathrm{d}b, \qquad \mu_3 = -\sin\phi\,\mathrm{d}a + \cos\phi\,\mathrm{d}b. \tag{3.5}$$

In cases when the group is explicitly realized as a local group of transformations on a manifold M, and not necessarily as a matrix Lie group, it is useful to have a direct method for determining the Maurer–Cartan forms. Given $g \in G$ and $z \in M$, we explicitly write the group transformation $\bar{z} = g \cdot z$ in coordinate form

$$\bar{z}^i = H^i(z, g), \qquad i = 1, \ldots, m.$$

We then compute the differentials of the group transformations:

$$\mathrm{d}\bar{z}^i = \sum_{k=1}^{m} \frac{\partial H^i}{\partial z^k}\,\mathrm{d}z^k + \sum_{j=1}^{r} \frac{\partial H^i}{\partial g^j}\,\mathrm{d}g^j, \qquad i = 1, \ldots, m,$$

or, more compactly,

$$\mathrm{d}\bar{z} = H_z\,\mathrm{d}z + H_g\,\mathrm{d}g. \tag{3.6}$$

Next, set $\mathrm{d}\bar{z} = 0$ in (3.6), and solve the resulting system of linear equations for the differentials $\mathrm{d}z^k$. This leads to the formulae

$$-\mathrm{d}z = F\,\mathrm{d}g = (H_z^{-1} \cdot H_g)\,\mathrm{d}g,$$

or, in full detail,

$$-\mathrm{d}z^k = \sum_{j=1}^{r} F_j^k(z, g)\,\mathrm{d}g^j, \qquad k = 1, \ldots, m. \tag{3.7}$$

Then, for each k and each fixed $z_0 \in M$, the 1-form

$$\mu_0 = \sum_{j=1}^{r} F_j^k(z_0, g)\, \mathrm{d}g^j \tag{3.8}$$

is a left-invariant Maurer–Cartan form on the group G. Alternatively, if one expands the right-hand side of (3.7) in a power series (or Fourier series, or ...) in z,

$$\sum_{j=1}^{r} F_j^k(z, g)\, \mathrm{d}g^j = \sum_{I=0}^{\infty} z^I\, \mu_I, \tag{3.9}$$

then each coefficient μ_I also forms a left-invariant Maurer–Cartan form on G. In particular, when G acts locally effectively, the resulting collection of 1-forms spans the space of Maurer–Cartan forms.

EXAMPLE 3.2. Consider the action of $\mathrm{GL}(2)$ given by

$$\bar{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad \bar{u} = \frac{u}{\gamma x + \delta}, \tag{3.10}$$

as discussed above. Differentiating (3.10), we find, as in (3.6),

$$\begin{aligned} \mathrm{d}\bar{x} &= \frac{(\gamma x + \delta)(\alpha\, \mathrm{d}x + x\, \mathrm{d}\alpha + \mathrm{d}\beta) - (\alpha x + \beta)(\gamma\, \mathrm{d}x + x\, \mathrm{d}\gamma + \mathrm{d}\delta)}{(\gamma x + \delta)^2} \\ &= \frac{(\alpha\delta - \beta\gamma)\, \mathrm{d}x + (\gamma x + \delta)(x\, \mathrm{d}\alpha + \mathrm{d}\beta) - (\alpha x + \beta)(x\, \mathrm{d}\gamma + \mathrm{d}\delta)}{(\gamma x + \delta)^2}, \\ \mathrm{d}\bar{u} &= \frac{(\gamma x + \delta)\, \mathrm{d}u + u(\gamma\, \mathrm{d}x + x\, \mathrm{d}\gamma + \mathrm{d}\delta)}{(\gamma x + \delta)^2}. \end{aligned}$$

Setting $\mathrm{d}\bar{x} = 0 = \mathrm{d}\bar{u}$ and solving for $\mathrm{d}x$ and $\mathrm{d}u$, we obtain

$$\begin{aligned} -\mathrm{d}x &= \frac{\delta\, \mathrm{d}\beta - \beta\, \mathrm{d}\delta}{\alpha\delta - \beta\gamma} + \left(\frac{\delta\, \mathrm{d}\alpha + \gamma\, \mathrm{d}\beta - \alpha\, \mathrm{d}\delta - \beta\, \mathrm{d}\gamma}{\alpha\delta - \beta\gamma} \right) x + \\ &\quad + \left(\frac{\gamma\, \mathrm{d}\alpha - \alpha\, \mathrm{d}\gamma}{\alpha\delta - \beta\gamma} \right) x^2, \\ -\mathrm{d}u &= \left(\frac{\alpha\, \mathrm{d}\delta - \gamma\, \mathrm{d}\beta}{\alpha\delta - \beta\gamma} \right) u + \left(\frac{\alpha\, \mathrm{d}\gamma - \gamma\, \mathrm{d}\alpha}{\alpha\delta - \beta\gamma} \right) xu. \end{aligned} \tag{3.11}$$

Note that the coefficients of 1, x and x^2 in the first formula, i.e.,

$$\begin{aligned} \widehat{\mu}_1 &= \frac{\delta\, \mathrm{d}\beta - \beta\, \mathrm{d}\delta}{\alpha\delta - \beta\gamma}, \qquad \widehat{\mu}_2 = \frac{\delta\, \mathrm{d}\alpha + \gamma\, \mathrm{d}\beta - \alpha\, \mathrm{d}\delta - \beta\, \mathrm{d}\gamma}{\alpha\delta - \beta\gamma}, \\ \widehat{\mu}_3 &= \frac{\gamma\, \mathrm{d}\alpha - \alpha\, \mathrm{d}\gamma}{\alpha\delta - \beta\gamma}, \end{aligned} \tag{3.12}$$

recover three of the Maurer–Cartan forms in (3.4), while the coefficient of either u or xu in the second formula provides the remaining one.

Remark. The coefficients in (3.11) are, in fact, immediately found in terms of the coefficients of the infinitesimal generators for the transformation group. See [15] for details.

If G does not act effectively on M, then the forms computed by this method will form a basis for the annihilator

$$(\mathfrak{g}_M)^{\perp} = \{\omega \in \mathfrak{g}^* \mid \langle \omega; \mathbf{v} \rangle = 0 \quad \text{for all } \mathbf{v} \in \mathfrak{g}_M\}$$

of the Lie algebra of the global isotropy subgroup

$$G_M = \{g \in G \mid g \cdot z = z \quad \text{for all } z \in M\},$$

and thus can be identified with the Maurer–Cartan forms for the effectively acting quotient group $\widetilde{G} = G/G_M$. For example, if we only treat the linear fractional transformations in x in (3.10), then the resulting three Maurer–Cartan forms (3.12) all annihilate the generator $\mathbf{v} = \alpha\partial_\alpha + \beta\partial_\beta + \gamma\partial_\gamma + \delta\partial_\delta$ of the isotropy subgroup $\{\lambda 1\!\!1\} \subset \mathrm{GL}(2)$ consisting of scalar multiples of the identity matrix. Hence, the three 1-forms can be identified with a basis for the Maurer–Cartan forms of the effectively acting projective linear group $\mathrm{PSL}(2) = \mathrm{GL}(2)/\{\lambda 1\!\!1\}$.

4. Compatible Lifts and Moving Coframes

In this section, we begin our development of the moving coframe method, starting with the simplest problems and gradually work our way up to more complicated situations. Throughout this section, we assume that G is an r-dimensional Lie group which acts locally effectively and transitively on an m-dimensional manifold M. (As remarked above, we can always assume local effectiveness by quotienting by the global isotropy subgroup.) We begin by choosing a convenient 'base point' $z_0 \in M$.

DEFINITION 4.1. A smooth map $\rho\colon M \to G$ is called a *compatible lift* with base point z_0 if it satisfies

$$\rho(z) \cdot z_0 = z. \tag{4.1}$$

In order to compute the most general compatible lift, we solve the system of m equations (4.1) for m of the group parameters in terms of the coordinates z on M and the remaining $r - m = \dim G - \dim M$ group parameters, which we denote by h. This leads to a general formula $g = \rho_0(z, h)$ for the solution to the compatibility equations (4.1). In other words, by solving the compatibility conditions (4.1), we have effectively 'normalized' m of the original group parameters. Since our considerations are always local, in practice, we only need to solve the compatibility equations (4.1) near z_0. In accordance with Cartan's terminology [6], we will call the general compatible lift $\rho_0(z, h)$ the *moving frame of order zero* for the given transformation group. If $\iota\colon X \to M$ defines a parametrized submanifold $N = \iota(X)$, then one can view the composition

$\rho_0(\iota(x), h)$ as a restriction of the 0th-order moving frame to the submanifold N, where the unnormalized parameters h determine the degree of indeterminacy of the moving frame on N. In geometrical situations, such restrictions can be identified with the classical moving frames; see also Section 7 below.

EXAMPLE 4.2. Consider the planar action (2.3) of the general linear group $\mathrm{GL}(2)$

$$A \cdot (x, u) = \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{u}{\gamma x + \delta} \right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2). \tag{4.2}$$

The action (4.2) is transitive on $M = \mathbb{R}^2 \setminus \{u = 0\}$. Choose the base point to be $z_0 = (0, 1)$. Since $A \cdot z_0 = (\beta/\delta, 1/\delta)$, any compatible lift $A = \rho(x, u)$ must satisfy $\beta/\delta = x$, $1/\delta = u$ and, hence, the solution to (4.1) is

$$\beta = \frac{x}{u}, \qquad \delta = \frac{1}{u}. \tag{4.3}$$

The most general compatible lift thus has the form

$$\rho_0(x, u, \alpha, \gamma) = \begin{pmatrix} \alpha & x/u \\ \gamma & 1/u \end{pmatrix}, \tag{4.4}$$

where $\alpha = \alpha(x, u)$, $\gamma = \gamma(x, u)$ are arbitrary functions, subject only to the condition $\alpha \neq x\gamma$, so that the determinant of (4.4) does not vanish and, hence, ρ_0 does take its values in the group $\mathrm{GL}(2)$.

Note that since G acts transitively, we can locally identify $M \simeq G/H$ with a homogeneous space, where $H = G_{z_0}$ is the isotropy group of the base point. Therefore, a compatible lift is merely a (local) section of the fiber bundle $G \to G/H$.

PROPOSITION 4.3. *Two maps $\rho, \widehat{\rho}\colon M \to G$ are compatible lifts with the same base point if and only if they satisfy*

$$\widehat{\rho}(z) = \rho(z) \cdot \eta(z),$$

where $\eta\colon M \to H$ is an arbitrary map to the isotropy subgroup of the base point z_0.

Thus, in the previous example, the isotropy subgroup H of the point $z_0 = (0, 1)$ consists of all invertible lower triangular matrices of the form $\begin{pmatrix} \alpha' & 0 \\ \gamma' & 1 \end{pmatrix}$. Indeed, we can rewrite (4.4) in the factored form

$$\rho_0(x, u, \alpha, \gamma) = \begin{pmatrix} \alpha & x/u \\ \gamma & 1/u \end{pmatrix} = \begin{pmatrix} 1 & x/u \\ 0 & 1/u \end{pmatrix} \begin{pmatrix} \alpha' & 0 \\ \gamma' & 1 \end{pmatrix}, \tag{4.5}$$

where $\alpha' = \alpha - x\gamma$, $\gamma' = u\gamma$, reconfirming Proposition 4.3 in this particular example.

Although the remaining unspecified group parameters can be identified with the isotropy subgroup coordinates, in any practical implementation of the moving coframe algorithm, it is not necessary to identify the isotropy subgroup explicitly, nor to adopt its particular coordinates to characterize the 0th-order moving frame. Thus, in the present example, the coordinates α, γ, are just as effective as the subgroup coordinates α', γ'. (The interested reader can follow through the ensuing calculations using the subgroup coordinates instead, reproducing the final result.)

The 0th-order moving frame $\rho_0(z, h)$, which is the general solution to the compatible lift equations (4.1), defines a map from the 0th-order moving frame bundle $\mathcal{B}_0 = M \times H \simeq G/H \times H$, coordinatized by (z, h), to the group G, which is, in fact, a local diffeomorphism $\rho_0 \colon \mathcal{B}_0 \xrightarrow{\sim} G$. There is an induced action of G on the moving frame bundle $\mathcal{B}_0$ that makes ρ_0 into a G-equivariant map: $\rho_0(g\cdot(z, h)) = g\cdot\rho_0(z, h)$. Thus, the action on the unnormalized group parameters h can be explicitly determined by multiplying the moving frame on the left by a group transformation. The action of G on $\mathcal{B}_0$ projects to the original action of G on M, so that $g \cdot (z, h) = (g \cdot z, \eta(g, z, h))$ for $g \in G$.

In the present example, the induced action of $\mathrm{GL}(2)$ on the unspecified parameters α, γ, is found by multiplying the moving frame (4.4) on the left by a group element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2)$; explicitly,

$$\begin{pmatrix} \bar{\alpha} & \bar{x}/\bar{u} \\ \bar{\gamma} & 1/\bar{u} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \alpha & x/u \\ \gamma & 1/u \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & (ax+b)/u \\ c\alpha + d\gamma & (cx+d)/u \end{pmatrix}. \tag{4.6}$$

Therefore, the action of $G = \mathrm{GL}(2)$ on the moving frame bundle $\mathcal{B}_0$ is given by

$$\bar{x} = \frac{ax+b}{cx+d}, \qquad \bar{u} = \frac{u}{cx+d}, \qquad \bar{\alpha} = a\alpha + b\gamma, \qquad \bar{\gamma} = c\alpha + d\gamma. \tag{4.7}$$

Note that the (x, u) transformations coincide with the original action (2.3), as they should.

Remark. In practical implementations of the moving coframe algorithm, we do *not* have to explicitly compute this group action. We do this here so as to provide the reader with some justification for our claims.

Remark. The action of G on $\mathcal{B}_0 = M \times H$ does *not* project to an action on the isotropy subgroup H, even if we use the associated subgroup coordinates. In the present example, we find (4.7) implies that the subgroup coordinates α', γ' in (4.5) transform according to

$$\bar{\alpha}' = \frac{ad - bc}{cx+d}\,\alpha', \qquad \bar{\gamma}' = \gamma' + \frac{cu}{cx+d}\,\alpha'.$$

The next step is to characterize the group transformations by a collection of differential forms. In the finite-dimensional situation that we are currently considering, these will be obtained by pulling back the left-invariant Maurer–Cartan forms $\boldsymbol{\mu}$ on G to the 0th-order moving frame bundle $\mathcal{B}_0$ using the compatible lift. The resulting 1-forms $\boldsymbol{\zeta}_0 = (\rho_0)^*\boldsymbol{\mu}$ will provide an invariant coframe on $\mathcal{B}_0$, which we name the *moving coframe* of order zero. The moving coframe forms $\boldsymbol{\zeta}_0$ clearly satisfy the same Maurer–Cartan structure equations (3.2).

THEOREM 4.4. *The* 0th-*order moving coframe forms completely characterize the group transformations on the bundle* $\mathcal{B}_0$. *In other words, a map* $\Psi\colon \mathcal{B}_0 \to \mathcal{B}_0$ *satisfies* $\Psi^*\boldsymbol{\zeta}_0 = \boldsymbol{\zeta}_0$ *if and only if* $\Psi(z,h) = g\cdot(z,h)$ *coincides with the action of a group element* $g \in G$ *on* $\mathcal{B}_0$.

In the present example, we substitute the formulae (4.3) characterizing our compatible lift (4.4) into the Maurer–Cartan forms (3.4). The result is the 0th-order moving coframe

$$\begin{aligned} \zeta_1 &= \frac{\mathrm{d}\alpha - x\,\mathrm{d}\gamma}{\alpha-\gamma x}, \qquad & \zeta_2 &= \frac{\mathrm{d}x}{u(\alpha-\gamma x)},\\ \zeta_3 &= \frac{u(\alpha\,\mathrm{d}\gamma - \gamma\,\mathrm{d}\alpha)}{\alpha-\gamma x}, \qquad & \zeta_4 &= -\frac{\gamma\,\mathrm{d}x}{\alpha-\gamma x} - \frac{\mathrm{d}u}{u}, \end{aligned} \tag{4.8}$$

which forms a basis for the space of 1-forms on $\mathcal{B}_0$. The skeptical reader can explicitly check that these four 1-forms really do completely characterize the group action (4.7), as described in Theorem 4.4.

Let us now consider a curve $N \subset M$. For simplicity, we shall assume that the curve coincides with the graph of a function $u = u(x)$. However, this restriction is not essential for the method to work, and later we show how parametrized curves can also be readily handled by the general method. We restrict the moving coframe forms to the curve, which amounts to replacing the differential du by its 'horizontal' component $u_x\,\mathrm{d}x$. If we interpret the derivative u_x as a coordinate on the first jet space $\mathrm{J}^1 = \mathrm{J}^1M \simeq \mathbb{R}^3$ of curves in M, then the restriction of a differential form to the curve can be reinterpreted as the natural projection of the 1-form $\mathrm{d}u$ on J^1 to its horizontal component, using the canonical decomposition of differential forms on the jet space into horizontal and contact components. Indeed, the vertical component of the form $\mathrm{d}u$ is the contact form $\mathrm{d}u - u_x\,\mathrm{d}x$, which vanishes on all prolonged sections of the first jet bundle J^1M. We refer the reader to [38; Chap. 4] for a comprehensive review of the contact geometry of jet bundles. Therefore, the restricted (or horizontal) moving coframe forms are explicitly given by

$$\begin{aligned} \eta_1 &= \frac{\mathrm{d}\alpha - x\,\mathrm{d}\gamma}{\alpha-\gamma x}, \qquad & \eta_2 &= \frac{\mathrm{d}x}{u(\alpha-\gamma x)},\\ \eta_3 &= \frac{u(\alpha\,\mathrm{d}\gamma - \gamma\,\mathrm{d}\alpha)}{\alpha-\gamma x}, \qquad & \eta_4 &= \frac{\gamma(xu_x - u) - \alpha u_x}{u(\alpha-\gamma x)}\,\mathrm{d}x, \end{aligned} \tag{4.9}$$

which now depend on first-order derivatives.

The next step in the procedure is to look for invariant combinations of coordinates and group parameters. Each such invariant combination will either provide us with a basic differential invariant for the problem, or, in the case that it explicitly depends on the remaining group parameters, a 'lifted invariant' which can be normalized and thereby eliminate one of the remaining group parameters, as discussed below. Specifically, in the present example, a function $J(\alpha, \gamma, x, u, u_x)$ will be a *lifted invariant* provided it is unaffected by the group action on its arguments, meaning that

$$J(\bar{\alpha}, \bar{\gamma}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}) = J(\alpha, \gamma, x, u, u_x), \tag{4.10}$$

wherever $\bar{\alpha}, \bar{\gamma}, \bar{x}, \bar{u}$, are related to α, γ, x, u, according to the induced action (4.7) of the group on the moving frame bundle, and $\bar{u}_{\bar{x}}$ is related to u_x according to the standard prolongation [38], of the action of G on M to the first jet bundle J^1. In the present case, if $\bar{x}, \bar{u}$ are given by (4.7), then a straightforward chain rule computation provides the prolonged action of $\mathrm{GL}(2)$ on the derivative coordinate:

$$\bar{u}_{\bar{x}} = \frac{(cx+d)u_x - cu}{ad - bc}. \tag{4.11}$$

In other words, we interpret $\alpha, \gamma, x, u, u_x$ as coordinates on a bundle $\widetilde{\mathcal{B}}_0 \to \mathrm{J}^1$ over the first jet space, which is merely the pull-back $\widetilde{\mathcal{B}}_0 = (\pi_0^1)^* \mathcal{B}_0$ of the 0th-order moving frame bundle via the standard projection $\pi_0^1 \colon \mathrm{J}^1 \to M$. There is an induced action of G on $\widetilde{\mathcal{B}}_0$ which projects to its prolonged action on J^1. A (first-order) lifted invariant, then, is just a function $J \colon \widetilde{\mathcal{B}}_0 \to \mathbb{R}$ which is invariant under the action of G on $\widetilde{\mathcal{B}}_0$. If the lifted invariant $J = J(x, u, u_x)$ does not, in fact, depend on the group parameters α, γ, then it will be a (first-order) differential invariant. (However, in the present example, there are no nonconstant first-order differential invariants, since $\mathrm{GL}(2)$ acts transitively on J^1.) Alternatively, if J actually depends on either α or γ then it can be used in the normalization procedure.

Fortunately, the lifted invariants can be determined without explicitly computing the prolonged group action, or solving any differential equations. They appear in the linear dependencies among the restricted (horizontal) moving coframe forms! Indeed, because the 1-forms are invariant, each coefficient J_i in a linear relation $\eta_0 = J_1 \eta_1 + \cdots + J_k \eta_k$, in which the forms η_i on the right-hand side are linearly independent, is automatically invariant under the action of the group. In our example, we note that, among the restricted 1-forms (4.9), there is one linear dependency, namely $\eta_4 = J\eta_2$, where

$$J = \gamma(xu_x - u) - \alpha u_x. \tag{4.12}$$

One can explicitly verify that J is indeed a lifted invariant, meaning that it satisfies (4.10) whenever $\bar{\alpha}, \bar{\gamma}, \bar{x}, \bar{u}, \bar{u}_{\bar{x}}$, are related to $\alpha, \gamma, x, u, u_x$, according to (4.7), (4.11).

The ultimate goal of the moving frame method is to eliminate all the ambiguities, i.e., the undetermined group parameters, in the original moving frame, in a suitably invariant manner. Cartan's crucial observation is that, we can, without loss of generality, *normalize* any lifted invariant by setting it equal to any convenient constant value

$$J(\alpha,\gamma,x,u,u_x)=c, \tag{4.13}$$

without affecting the equivalence problem. In (4.13), c can be any constant, subject only to the requirement that the solutions to (4.13) remain in the group, e.g., that the determinant of any resulting matrix (4.4) remains nonzero. Typically, c is taken to be $0, 1$, or -1, although other values can be chosen to simplify the resulting formulae. Assuming that J does actually depend on the parameters α,γ, we can solve the normalization equation (4.13) for one of them; e.g., $\alpha = \alpha(\gamma,x,u,u_x)$. Because J is an invariant, such a normalization will not alter the solution to the equivalence problem, and hence we can use it to eliminate α from the original formulae for the moving frame and moving coframe. The result is a first-order moving frame, depending on one fewer unnormalized group parameter. This produces a corresponding first-order moving coframe, to which one can apply the same procedure, leading to a chain of successive normalizations and reductions, eventually enabling one to completely eliminate all the undetermined parameters and specify a uniquely defined moving frame on some suitable jet bundle $\mathrm{J}^n = \mathrm{J}^n M$.

In accordance with the general procedure, then, we can normalize our particular lifted invariant (4.12) by setting it equal to zero; the solution to the normalization equation $J=0$ is then given by

$$\alpha = \left(\frac{xu_x - u}{u_x}\right)\gamma. \tag{4.14}$$

Substituting (4.14) into (4.4) produces the *first-order moving frame*

$$\rho_1(x,u,u_x,\gamma) = \begin{pmatrix} (xu_x-u)\gamma/u_x & x/u \\ \gamma & 1/u \end{pmatrix}, \tag{4.15}$$

which now depends on first-order derivatives of u, and just one unnormalized group parameter. We can regard the coordinates (x,u,u_x,γ) as parametrizing a bundle $\mathcal{B}_1 \to \mathrm{J}^1$ sitting over the first jet space, which is realized as a G-invariant subbundle of $\widetilde{\mathcal{B}}_0$, namely $\mathcal{B}_1 = J^{-1}\{0\} \subset \widetilde{\mathcal{B}}_0$. As before, one can restrict the first-order moving frame to a curve $u=u(x)$ by restricting the map ρ_1 to the first prolongation or jet of the curve, i.e., we set $u=u(x)$, $u_x=u'(x)$, in (4.15), with γ indicating the remaining ambiguity. There is an induced action of $\mathrm{GL}(2)$ on $\mathcal{B}_1$, which projects to the usual first prolonged action $G^{(1)}$ of the group on J^1, cf. (4.7), (4.11), and makes the first-order moving frame $\rho_1\colon \mathcal{B}_1 \xrightarrow{\sim} G$ into a local

G-equivariant diffeomorphism. In our case, the explicit transformation rules on $\mathcal{B}_1$ are given by

$$\bar{x} = \frac{ax+b}{cx+d}, \qquad \bar{u} = \frac{u}{cx+d}, \qquad \bar{u}_{\bar{x}} = \frac{(cx+d)u_x - cu}{ad-bc},$$
$$\bar{\gamma} = \left(\frac{(cx+d)u_x - cu}{ad-bc}\right)\gamma, \tag{4.16}$$

which coincide with left multiplication of the first-order moving frame (4.15) by the given group element. (Again, these explicit formulae are provided for illustration only, and are not essential for application of the method.) Furthermore, substituting (4.14) into (4.8), we find the first-order moving coframe

$$\zeta_1 = \frac{\mathrm{d}\gamma}{\gamma} - \frac{\mathrm{d}u_x}{u_x} + \frac{\mathrm{d}u - u_x\,\mathrm{d}x}{u}, \qquad \zeta_2 = -\frac{u_x\,\mathrm{d}x}{\gamma u^2},$$
$$\zeta_3 = \frac{\gamma u\,\mathrm{d}u_x}{u_x} - \gamma(\mathrm{d}u - u_x\,\mathrm{d}x), \qquad \zeta_4 = \frac{\mathrm{d}u - u_x\,\mathrm{d}x}{u}. \tag{4.17}$$

As in the 0th-order case, cf. Theorem 4.4, the first-order moving coframe completely characterizes the group transformations on $\mathcal{B}_1$.

As before, we determine new lifted invariants by restricting the first-order moving coframe 1-forms to a curve $u = u(x)$. This amounts to replacing $\mathrm{d}u$ and $\mathrm{d}u_x$ by their horizontal components $u_x\,\mathrm{d}x$ and $u_{xx}\,\mathrm{d}x$, respectively, leading to the restricted forms

$$\eta_1 = \frac{\mathrm{d}\gamma}{\gamma} - \frac{u_{xx}\,\mathrm{d}x}{u_x}, \qquad \eta_2 = -\frac{u_x\,\mathrm{d}x}{\gamma u^2},$$
$$\eta_3 = \frac{\gamma u u_{xx}\,\mathrm{d}x}{u_x}, \qquad \eta_4 = 0, \tag{4.18}$$

that now depend on second-order derivatives. Alternatively, one could deduce these restricted forms by substituting the normalization (4.14) into the previous restricted forms (4.9). Note in particular that the fact that η_4 vanishes is an automatic consequence of our normalization condition $\eta_4 = J\eta_2 = 0$; alternatively, we note that ζ_4 is an invariant contact form, which hence vanishes when restricted to any submanifold. Now there is an additional dependency, namely $\eta_3 = K\eta_2$, where

$$K = -\frac{\gamma^2 u^3 u_{xx}}{u_x^2}$$

is a new lifted invariant. Again, the reader can check that K is invariant under the prolonged action of $\mathrm{GL}(2)$ on the bundle $\widetilde{\mathcal{B}}_1 = (\pi_1^2)^*\mathcal{B}_1 \to \mathrm{J}^2$, where $\pi_1^2 \colon \mathrm{J}^2 \to \mathrm{J}^1$

is the natural projection, and is provided by (4.16) and the second-order prolongation (chain rule) formula

$$\bar{u}_{\bar{x}\bar{x}} = \frac{(cx+d)^3 u_{xx}}{(ad-bc)^2}. \tag{4.19}$$

We can normalize $K = -1$ by setting

$$\gamma = \frac{u_x}{\sqrt{u^3 u_{xx}}}. \tag{4.20}$$

Note that we *cannot* normalize $K = 0$ since this would require $\gamma = 0$, but then the lift (4.15) would have zero determinant, violating the group conditions. The final lift

$$\rho_2(x, u, u_x, u_{xx}) = \begin{pmatrix} \dfrac{x u_x - u}{\sqrt{u^3 u_{xx}}} & \dfrac{x}{u} \\[2ex] \dfrac{u_x}{\sqrt{u^3 u_{xx}}} & \dfrac{1}{u} \end{pmatrix} \tag{4.21}$$

defines the second-order moving frame. The moving frame (4.21) provides an explicit G-equivariant identification $\rho_2 \colon \mathcal{V}^2 \xrightarrow{\sim} G$ of the open subset $\mathcal{V}^2 = \{u u_{xx} \neq 0\} \subset \mathrm{J}^2$ of the second jet bundle with an open subset of the group G, identifying the prolonged action of $G^{(2)}$ on J^2 with the ordinary left multiplication on G; thus

$$\rho_2(g^{(2)} \cdot z^{(2)}) = g \cdot \rho_2(z^{(2)}), \quad g \in \mathrm{GL}(2), \quad z^{(2)} = (x, u, u_x, u_{xx}) \in \mathcal{V}^2.$$

Substituting (4.20) into (4.18) produces the final set of invariant 1-forms

$$\begin{aligned} \zeta_1 &= -\frac{\mathrm{d}u_{xx}}{2u_{xx}} - \frac{\mathrm{d}u}{2u} - \frac{u_x\,\mathrm{d}x}{u}, & \zeta_2 &= -\sqrt{\frac{u_{xx}}{u}}\,\mathrm{d}x, \\ \zeta_3 &= \frac{\mathrm{d}u_x}{\sqrt{u u_{xx}}} - \frac{u_x(\mathrm{d}u - u_x\,\mathrm{d}x)}{\sqrt{u^3 u_{xx}}}, & \zeta_4 &= \frac{\mathrm{d}u - u_x\,\mathrm{d}x}{u}, \end{aligned} \tag{4.22}$$

which form the second-order moving coframe. Note that the second-order moving frame (4.21) provides an equivalence, $\rho_2^* \mu_i = \zeta_i$, mapping the moving coframe forms on the second-order jet space to the Maurer–Cartan forms (3.4) on the group. Consequently, the forms ζ_i uniquely characterize the second-order prolonged action of $\mathrm{GL}(2)$ on $\mathcal{V}^2 \subset \mathrm{J}^2$.

Finally, the restricted (horizontal) moving coframe forms become

$$\eta_1 = -\frac{u u_{xxx} + 3u_x u_{xx}}{2u u_{xx}}\,\mathrm{d}x, \qquad \eta_2 = \sqrt{\frac{u_{xx}}{u}}\,\mathrm{d}x, \qquad \eta_3 = -\eta_2, \qquad \eta_4 = 0.$$

There is one final linear dependency, namely $\eta_1 = -I\,\eta_2$, where

$$I = \frac{uu_{xxx} + 3u_x u_{xx}}{2\sqrt{uu_{xx}^3}} \tag{4.23}$$

is the fundamental differential invariant of the transformation group, also known as the group-invariant curvature. The remaining 1-form $\mathrm{d}s = \eta_2$ is the fundamental invariant 1-form, or group-invariant arc length element. All higher-order differential invariants can be found by differentiating the curvature invariant with respect to the invariant arc length; for instance, the fundamental fourth-order differential invariant is

$$\begin{aligned} J &= \frac{\partial I}{\partial \eta_2} = \frac{\mathrm{d}I}{\mathrm{d}s} = \sqrt{\frac{u}{u_{xx}}}\frac{\mathrm{d}I}{\mathrm{d}x} \\ &= \frac{2u^2 u_{xx} u_{xxxx} - 3u^2 u_{xxx}^2 - 2uu_x u_{xx} u_{xxx} + 6uu_{xx}^3 - 3u_x^2 u_{xx}^2}{4uu_{xx}^3}. \end{aligned} \tag{4.24}$$

From the general theory, we conclude that every differential invariant for the group (2.3) is a function of the curvature and its successive derivatives with respect to the arc length. On the regular part $\mathcal{V}^2$ of the jet space J^2, all $\mathrm{GL}(2)$ invariant ordinary differential equations can be written in terms of these invariants; for instance, the most general invariant third-order ordinary differential equation has the form

$$uu_{xxx} + 3u_x u_{xx} = k\sqrt{uu_{xx}^3}, \tag{4.25}$$

for some constant k.

Applications to the equivalence problem for curves (which includes the equivalence problem for first-order Lagrangians as well as that of classical invariant theory) follow directly from the general theorems. Given a function $u = u(x)$, we define its *classifying curve* $\mathcal{C}$ to be the planar curve parametrized by the fundamental differential invariants $I(x), J(x)$. The general result states that two curves are mapped to each other by a group transformation (2.3), so $\bar{C} = g \cdot C$, if and only if their classifying curves are identical, $\bar{\mathcal{C}} = \mathcal{C}$. A curve C is maximally symmetric if and only if its classifying curve reduces to a point; in this case the original curve is, in fact, an orbit of a one-parameter subgroup of $\mathrm{GL}(2)$. Thus, we have, in a very simple and direct manner, recovered the results in [36] on the equivalence and symmetry of binary forms, which were found by a much less direct approach based on the standard Cartan equivalence problem for particle Lagrangians.

There are a few technical points that should have been addressed during the preceding discussion. First, one needs to impose certain conditions on the function $u(x)$ in order to ensure that the computation is valid. For instance, the normalization (4.14) requires $u_x \neq 0$, i.e., the curve does not have a horizontal tangent. (We have already assumed that it does not have a vertical tangent by requiring that it be the graph of a smooth function.) If $u_x = 0$, then we can still normalize $J = 0$ as long as $u \neq x u_x$, in which case we normalize by solving for γ instead of α. Actually, both cases can be simultaneously handled by the normalization $\alpha = \lambda(x u_x - u)$, $\gamma = \lambda u_x$, where $\lambda \neq 0$ is a new parameter whose normalization will be specified at the next stage of the procedure. The reader can check that this alternative procedure leads to the same lift and differential invariants as before. In the second normalization, we have assumed* $u_{xx} > 0$ in order to take the square root. For $u_{xx} < 0$ we would need to normalize $K = +1$, and use $\sqrt{-u_{xx}}$ instead. Thus the problem actually separates into two branches, with the inflection points $u_{xx} = 0$ being interpreted as singular points for the group action. The straight lines, for which $u_{xx} \equiv 0$, form a special class and must be analyzed separately. Finally, the square root itself has a sign ambiguity (or, in the complex case, an ambiguity in its choice of branch). Both signs must, in fact, be allowed in the final expression for the lift and the differential invariants. Such branching and ambiguous sign phenomena will be familiar to practitioners of the Cartan equivalence method; see [38] for a detailed discussion of these issues.

Let us finish this section by summarizing the basic method of moving coframes, in a form which will apply to more general problems. The basic steps are

(a) Determine the general invariant lift, or moving frame of order zero, by choosing a base point and solving (4.1) for the given group action.
(b) Determine the invariant forms. In the finite-dimensional case, they are the Maurer–Cartan forms, which can be computed either by using the matrix approach, or by direct use of the transformation group formulae.
(c) Use the invariant lift to pull-back the invariant forms, leading to the moving coframe of order zero.
(d) Determine lifted invariants by finding linear dependencies among the restricted or horizontal components of the moving coframe forms.
(e) Normalize any group-dependent invariants to convenient constant values by solving for some of the unspecified parameters.
(f) Successively eliminate parameters by substituting the normalization formulae into the moving coframe and recomputing dependencies.
(g) After the parameters have all been normalized, the differential invariants will appear through any remaining dependencies among the final moving coframe elements. The invariant differential operators are found as the dual differential operators to a basis for the invariant coframe forms.

Note that we do not need the explicit isotropy groups for the transformation group actions, nor do we need compute explicit formulae for the prolonged group action in order to successfully apply the method.

* In the complex-valued problem, there is no sign restriction.

Remark. If one is solely interested in the final differential invariants and invariant horizontal 1-forms (i.e., invariant forms on the submanifold itself), then one need only determine the effect of the normalizations on the horizontal components of the moving coframe forms during the computation. The moving coframe itself will also include invariant contact forms, which vanish upon restriction, but which, nevertheless, play an important role in other aspects of the geometry. See [20, 38, 40], for applications of invariant contact forms to the study of invariant evolution equations, with applications to image processing. Applications to the computation of the invariant cohomology of the variational bicomplex (cf. [2]) are also of particular importance in the analysis of symmetries and conservation laws of variational problems.

Remark. The proposed method of moving coframes has the same basic structure as the Cartan equivalence method [11, 16, 38], in that one deals with a system of differential forms depending on arbitrary parameters, and seeks to normalize all the parameters by a suitable collection of lifted invariants. One can, indeed, view the two methods as particular cases of a completely general equivalence procedure. However, it is worth pointing out a few of the differences between the two. First, the Cartan method only deals with lifted coframes, whose constituents are linearly independent differential forms, whereas the differential forms occurring in the moving coframe method are linearly dependent. The invariant combinations (lifted invariants) used to normalize the parameters are found via linear dependencies in the moving coframe method, whereas they arise as unabsorbed torsion coefficients in the differentials of the lifted coframe forms in the Cartan equivalence method. In the moving coframe method, the differentials of the moving coframe 1-forms satisfy the Maurer–Cartan structure equations and, hence, do not provide any nonconstant invariants. Finally, and perhaps most significantly, the group parameters g only occur algebraically in the lifted coframe elements in the Cartan equivalence method, whereas in the moving frame problems their differentials $\mathrm{d}g$ occur as well, since they appear in the Maurer–Cartan forms. One can, of course, imagine solving hybrid equivalence problems, in which aspects of both problems occur during the normalization procedure, although we are not currently aware of any interesting examples where these occur naturally.

5. Intransitive Lie Group Actions

Our next task is to extend the moving coframe method to the case of finite-dimensional Lie groups whose action is no longer transitive. In the intransitive case, we still assume that G is an r-dimensional Lie group acting effectively, and regularly, which implies that its orbits, which we take to have dimension s, form a foliation of M. We choose a local *cross-section* $\mathcal{K} \subset M$ to this foliation, i.e., a submanifold of dimension $m - s$ intersecting the orbits transversally, and introduce a *compatible lift* $\rho\colon M \to G$ by requiring that, for each z near $\mathcal{K}$, the lift ρ satisfies

$$z = \rho(z) \cdot z_0 \quad \text{for some } \; z_0 \in \mathcal{K}. \tag{5.1}$$

The general solution to the compatible lift equations (5.1) will be of the form $\rho(z, h)$ depending on $r - s$ parameters h. Note that unless the isotropy subgroups at each point in the cross-section happen to be identical, we cannot identify the unspecified parameters as local coordinates on any subgroup $H \subset G$, leading us beyond any principal bundle-theoretic interpretation of the method. Nevertheless, the Implicit Function Theorem will allow us to locally write the general compatible lift in this form. In addition, the group admits (locally) $m - s$ functionally independent invariants, $I_1(z), \dots, I_{m-s}(z)$, whose level sets characterize the orbits. The 0th-order moving frame will then be the map

$$\rho_0(z, h) = (\rho(z, h), I(z)), \tag{5.2}$$

whose first components $g = \rho(z, h)$ are those of the general compatible lift (for the given cross-section) and, in addition, has the invariants $w = I(z) = (I_1(z), \dots, I_{m-s}(z))$ as further components. Note that ρ_0 is only locally defined, since z must lie near the cross-section $\mathcal{K}$, and, moreover, the remaining parameters h are determined in accordance with the Implicit Function Theorem.

Note. We can view the range $G \times \mathbb{R}^{m-s}$ of ρ_0 as having the structure of a Cartesian product Lie group, the additive group structure on the second factor formalizing the fact that we can add invariants.

The moving coframe forms in this case are constructed from the Maurer–Cartan forms $\boldsymbol{\mu}$ on the group G, together with the coordinate 1-forms $\mathrm{d}\mathbf{w} = \{\mathrm{d}w_1, \dots, \mathrm{d}w_{m-s}\}$ on $\mathbb{R}^{m-s}$. The group transformations are then characterized by the conditions

$$\Phi^* \bar{\mathbf{w}} = \mathbf{w}, \qquad \Phi^* \,\mathrm{d}\bar{\mathbf{w}} = \mathrm{d}\mathbf{w}, \qquad \Phi^* \bar{\boldsymbol{\mu}} = \boldsymbol{\mu}. \tag{5.3}$$

Using the moving frame lift $g = \rho(z, h)$, $w = I(z)$, to pull back these 1-forms, we are led to the 0th-order moving coframe, consisting of the pulled-back Maurer–Cartan forms $(\rho_0)^* \boldsymbol{\mu}$, along with the differentials $(\rho_0)^* \mathrm{d}\mathbf{w}_\kappa = \mathrm{d}I_\kappa$ of the group invariants. At this stage, the set up of the intransitive problem is complete, and one proceeds, as in the transitive case, to look for dependencies among the restricted coframe forms, and then normalize the resulting lifted invariants.

EXAMPLE 5.1. The intransitive action

$$A\colon (x, u) \longmapsto \left(x, \frac{\alpha u + \beta}{\gamma u + \delta} \right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2), \tag{5.4}$$

of the special linear group $\mathrm{SL}(2)$ on $M = \mathbb{R}^2$ arises in complex function theory [21]. (We restrict to $\mathrm{SL}(2)$ in order to maintain local effectiveness.) The group orbits are vertical lines and so the basic invariant is merely $I(x,u) = x$. We choose the cross-section $\mathcal{K} = \{u = 0\}$. Solving the equation $A(x,u)\cdot z_0 = (x,u)$, where $z_0 = (y, 0)$, leads to the general compatible lift

$$A_0(x,u,\alpha,\delta) = \begin{pmatrix} \alpha & \delta u \\ \dfrac{\alpha\delta - 1}{\delta u} & \delta \end{pmatrix}, \tag{5.5}$$

which forms the group component of the 0th-order moving frame. The other component is just the invariant

$$w = I(x,u) = x. \tag{5.6}$$

Pulling back the Maurer–Cartan forms $\boldsymbol{\mu} = A^{-1}\,\mathrm{d}A$ and $\mathrm{d}w$ via the lift (5.5), (5.6), leads to the 0th-order moving coframe

$$\begin{aligned} \zeta_1 &= (\alpha\delta - 1)\frac{\mathrm{d}u}{u} - \frac{\mathrm{d}\delta}{\delta}, \qquad \zeta_2 = \delta^2\,\mathrm{d}u, \\ \zeta_3 &= \frac{u\,\mathrm{d}(\alpha\delta) + \alpha\delta(1-\alpha\delta)\,\mathrm{d}u}{\delta^2 u^2}, \qquad \zeta_4 = \mathrm{d}x. \end{aligned} \tag{5.7}$$

As before, we restrict ourselves to a curve $u = u(x)$ by replacing $\mathrm{d}u$ by its horizontal component $u_x\,\mathrm{d}x$. Letting η_i denote the horizontal component of ζ_i, we find that there is one resulting linear dependency, namely

$$\eta_2 = \delta^2 u_x\,\mathrm{d}x = J\,\mathrm{d}x = J\eta_4.$$

The leads to the first normalization $\delta = 1/\sqrt{u_x}$ resulting from setting $J = 1$. Substituting this normalized value into (5.5), (5.6), provides the first-order moving frame. Furthermore, substituting into (5.7) produces the second-order moving coframe, with horizontal components

$$\begin{aligned} \eta_1 &= \left(\frac{2\alpha u_x^{3/2} + u u_{xx} - 2u_x^2}{2u u_x}\right)\mathrm{d}x, \qquad \eta_2 = \eta_4 = \mathrm{d}x, \\ \eta_3 &= \frac{\sqrt{u_x}}{u}(\mathrm{d}\alpha - \alpha\,\eta_1). \end{aligned} \tag{5.8}$$

Now we normalize the coefficient of η_1 to 0 by setting $\alpha = u_x^{-3/2}(u_x^2 - \frac{1}{2}u u_{xx})$. The final moving frame (of order 2) is

$$A_2 = \frac{1}{u_x^{3/2}}\begin{pmatrix} u_x^2 - \frac{1}{2}u u_{xx} & u u_x \\ -\frac{1}{2}u_{xx} & u_x \end{pmatrix}, \quad w = x. \tag{5.9}$$

The corresponding restricted moving coframe has reduced to

$$\eta_2 = \eta_4 = \mathrm{d}x, \qquad \eta_3 = -\tfrac{1}{4}S\eta_2, \qquad \eta_1 = 0, \tag{5.10}$$

where

$$S = \frac{2u_x u_{xxx} - 3u_{xx}^2}{u_x^2} \tag{5.11}$$

is the classical Schwarzian derivative of the function $u(x)$, whose invariance under linear fractional transformations is of fundamental importance in complex function theory. Since the 1-form $\mathrm{d}x$ is invariant, all the higher-order differential invariants are found by differentiating S with respect to x.

Actually, the preceding computation can be slightly simplified by extending our general method to noneffective actions. We consider (5.4) as defining a noneffective (and intransitive) action of the general linear group $\mathrm{GL}(2)$ on $\mathbb{R}^2$. We may apply the second algorithm for computing the required Maurer–Cartan forms, leading to the three 1-forms (3.12) that annihilate the global isotropy subalgebra. We substitute the compatible lift formulae $\beta = \delta u$ for the 0th-order moving coframe, which is now $A_0 = \begin{pmatrix} \alpha & \delta u \\ \gamma & \delta \end{pmatrix}$, into (3.12), leading to the restricted moving coframe forms

$$\begin{aligned} \widehat{\eta}_1 &= \frac{\delta u_x \,\mathrm{d}x}{\alpha - u\gamma}, \qquad \widehat{\eta}_2 = \frac{\mathrm{d}\alpha - u\,\mathrm{d}\gamma + \gamma u_x \,\mathrm{d}x}{\alpha - u\gamma} - \frac{\mathrm{d}\delta}{\delta}, \\ \widehat{\eta}_3 &= \frac{\gamma\,\mathrm{d}\alpha - \alpha\,\mathrm{d}\gamma}{\delta(\alpha - u\gamma)}, \qquad \widehat{\eta}_4 = \mathrm{d}x. \end{aligned} \tag{5.12}$$

The first dependency between $\widehat{\eta}_1$ and $\widehat{\eta}_4$ leads to the reduction $\delta = (\alpha - u\gamma)/u_x$. Substituting into $\widehat{\eta}_3$ leads to a second dependency, and the resulting normalization yields $\alpha = \gamma(u - 2u_x^2/u_{xx})$. At this stage, even though we have not normalized the final parameter γ, it no longer appears in the coframe, which coincides with our earlier one, (5.10). It does, of course, occur in the final moving frame lift, which is obtained by multiplying the matrix A_2 in (5.9) by γ. However, γ plays no other role in the problem, and merely reflects a final indeterminacy stemming from the ineffectiveness of the group action. The main point in this solution method is that one does not have to explicitly implement an effective action, as was done in the original lift (5.5), in order to solve the problem. Indeed, in more complicated examples, it may be relatively straightforward to write down the compatible lift for an ineffective group action, whereas doing the same for the effectively acting quotient group G/G_M may be considerably more complicated.

EXAMPLE 5.2. Consider the elementary similarity group $G = \mathbb{R}^+ \ltimes \mathbb{R}^2$ acting transitively on $M = \mathbb{R}^2$ via

$$A\colon (x, u) \longmapsto (\alpha x + a, \alpha u + b). \tag{5.13}$$

For the base point $z_0 = (0, 0)$, the associated moving frame of order 0 is the lift with $a = x$, $b = u$. The Maurer–Cartan forms $\{\mathrm{d}\alpha/\alpha, \mathrm{d}a/\alpha, \mathrm{d}b/\alpha\}$ are pulled back to provide the 0th-order moving coframe, whose horizontal (or, more precisely, non-contact) components are

$$\eta_1 = \frac{\mathrm{d}\alpha}{\alpha}, \qquad \eta_2 = \frac{\mathrm{d}x}{\alpha}, \qquad \eta_3 = \frac{u_x\,\mathrm{d}x}{\alpha}. \tag{5.14}$$

There is a single linear dependency $\eta_3 = I\,\eta_2$, but the resulting invariant $I = u_x$ does *not* depend on the remaining group parameter, and hence *cannot* be used to normalize it. To proceed further in such cases, we work in analogy with the preceding intransitive case. Here the intransitivity is on the first-order jet bundle, and is an indication of the fact that this particular group exhibits the pathology of 'pseudo-stabilization' of its prolonged group orbits [38]. We therefore introduce an additional invariant 1-form $\mathrm{d}u_x$, whose horizontal component is

$$\eta_4 = u_{xx}\,\mathrm{d}x = K\eta_2.$$

The resulting dependency leads to the lifted invariant $K = \alpha u_{xx}$ which yields the desired normalization $\alpha = 1/u_{xx}$ and the second-order moving frame. The associated invariant coframe is

$$\eta_2 = \eta_4 = u_{xx}\,\mathrm{d}x, \qquad \eta_1 = -J\eta_2, \qquad \eta_3 = I\eta_2, \tag{5.15}$$

yielding two fundamental differential invariants

$$I = u_x, \qquad J = u_{xx}^{-2}u_{xxx}. \tag{5.16}$$

The higher-order invariants are found by differentiating J with respect to $\eta_4 = u_{xx}\,\mathrm{d}x \simeq \mathrm{d}u_x$, so that a basic fourth-order invariant is

$$K = \frac{\mathrm{d}J}{\mathrm{d}u_x} = \frac{1}{u_{xx}}\frac{\mathrm{d}J}{\mathrm{d}x} = \frac{u_{xxxx}}{u_{xx}^2} - 2J^2.$$

Note that $\mathrm{d}I/\mathrm{d}u_x = 1$, so that differentiating I produces nothing new. Thus, in this case, we find two fundamental differential invariants, and require three, namely (I, J, K), to parametrize the classifying curve that solves the associated equivalence problem. We conclude that the phenomenon of pseudo-stabilization of group orbits is reflected in the moving coframe procedure by the premature appearance of differential invariants, whose differentials are required to finish the procedure. See [38, 39] for further discussion.

Remark. Interestingly, if the scaling acts differently on x and u, so the group is

$$A\colon (x, u) \longmapsto (\alpha x + a, \alpha^k u + b), \tag{5.17}$$

for $k \neq 1$, then pseudo-stabilization does not occur. Such cases can be readily handled via our basic method without any such intransitive normalizations.

6. Reparametrization Pseudo-Groups

The classical applications of moving frames to curves and surfaces in Euclidean, affine, and projective geometry, cf. [6, 8, 19], can all be readily implemented using the moving coframe algorithm. In each case, we consider the reparametrization equivalence problem for submanifolds, so that the underlying transformation group is the Cartesian product of an infinite Lie pseudo-group, namely the local diffeomorphism group $\mathcal{D}\mathit{iff}(X)$ of the parameter space, and a finite-dimensional Lie group acting on the manifold M. In this case, in addition to the Maurer–Cartan forms for the group, one also includes the 1-forms defining the diffeomorphism pseudo-group. One can then proceed to reduce and normalize as before. For simplicity, we just deal with planar curves, although extensions to surfaces and curves in higher dimensional ambient spaces can also be handled without significant further complications.

EXAMPLE 6.1. *Euclidean geometry of curves.* The most well-known classical example is the reparametrization equivalence problem for curves in the Euclidean plane, introduced in Example 2.1 above. In this case, we are dealing with a finite-dimensional group, the Euclidean group $\mathrm{E}(2)$ on the plane, together with the pseudo-group $\mathcal{D}\mathit{iff}(1)$ consisting of all smooth (local) diffeomorphisms of the line representing the change of parameter. Thus, the entire pseudo-group $\mathcal{G} = \mathcal{D}\mathit{iff}(1) \times \mathrm{E}(2)$ acts on the total space $M = \mathbb{R} \times \mathbb{R}^2$ with coordinates $(t, \mathbf{x}) = (t, x, y)$. For the Euclidean component, we use a compatible lift

$$A_0(x, y, \phi) = \begin{pmatrix} R & \mathbf{x} \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & x \\ \sin\phi & \cos\phi & y \\ 0 & 0 & 1 \end{pmatrix} \tag{6.1}$$

and compute the pull-back of the associated Euclidean Maurer–Cartan forms

$$\begin{aligned} \boldsymbol{\zeta} \;&=\; A_0^{-1}\,\mathrm{d}A_0 = \begin{pmatrix} R^{-1}\,\mathrm{d}R & R^{-1}\,\mathrm{d}\mathbf{x} \\ \mathbf{0} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\mathrm{d}\phi & \cos\phi\,\mathrm{d}x + \sin\phi\,\mathrm{d}y \\ \mathrm{d}\phi & 0 & -\sin\phi\,\mathrm{d}x + \cos\phi\,\mathrm{d}y \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{6.2}$$

On the other hand, the pseudo-group $\mathcal{D}\mathit{iff}(1)$ is characterized by the invariance of the canonical 1-form $\sigma\,\mathrm{d}t$ on the frame bundle $\mathcal{F}(\mathbb{R})$, cf. [26], and, hence, we include this additional 1-form in our moving coframe formulation. Restricting these four 1-forms to a parametrized curve $(x(t), y(t))$ leads to

$$\begin{aligned} &\eta_1 = \mathrm{d}\phi, \qquad \eta_2 = (x_t\cos\phi + y_t\sin\phi)\,\mathrm{d}t, \\ &\eta_3 = (-x_t\sin\phi + y_t\cos\phi)\,\mathrm{d}t, \qquad \eta_4 = \sigma\,\mathrm{d}t. \end{aligned} \tag{6.3}$$

Now $\eta_2 = J_1\eta_4$ and $\eta_3 = J_2\eta_4$ are the linear dependencies, with associated lifted invariants

$$J_1 = \frac{x_t \cos\phi + y_t \sin\phi}{\sigma}, \qquad J_2 = \frac{-x_t \sin\phi + y_t \cos\phi}{\sigma}.$$

We normalize $J_1 = 1$, $J_2 = 0$ by setting

$$\phi = \tan^{-1}(y_t/x_t), \qquad \sigma = \sqrt{x_t^2 + y_t^2}. \tag{6.4}$$

This immediately produces the first-order moving frame

$$R = \frac{1}{\sqrt{x_t^2 + y_t^2}} \begin{pmatrix} x_t & -y_t \\ y_t & x_t \end{pmatrix}, \qquad \mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \sigma = \sqrt{x_t^2 + y_t^2}. \tag{6.5}$$

The canonical 1-form $\sigma\,\mathrm{d}t$ has been reduced to the fundamental arc length form $\mathrm{d}s = \sqrt{x_t^2 + y_t^2}\,\mathrm{d}t$ for the Euclidean group. Substituting into (6.3), we are left with a final set of horizontal 1-forms

$$\eta_1 = \kappa\,\eta_4, \qquad \eta_2 = \eta_4 = \mathrm{d}s = \sqrt{x_t^2 + y_t^2}\,\mathrm{d}t, \qquad \eta_3 = 0. \tag{6.6}$$

Here

$$\kappa = \frac{x_t y_{tt} - x_{tt} y_t}{(x_t^2 + y_t^2)^{3/2}} = \frac{\mathbf{x}_t \wedge \mathbf{x}_{tt}}{|\mathbf{x}_t|^3} = \mathbf{x}_s \wedge \mathbf{x}_{ss} \tag{6.7}$$

is the fundamental differential invariant for the Euclidean group – the curvature of the plane curve. All higher-order differential invariants are obtained by successively differentiating the curvature with respect to arc length.

The classical Frenet equations for curves in the Euclidean plane are reformulations of our final moving frame formulae. (See Section 7 below for more details on the connection with the classical theory.) The rotational component in (6.5) is traditionally written as $R = (\mathbf{e}_1, \mathbf{e}_2)$, where $\mathbf{e}_1$ is the unit tangent and $\mathbf{e}_2$ the unit normal. The translational Maurer–Cartan forms $\eta_2 = \mathrm{d}s$, $\eta_3 = 0$ are computed by the original formula as the entries of $R^{-1}\,\mathrm{d}\mathbf{x} = \begin{pmatrix} \mathrm{d}s \\ 0 \end{pmatrix}$, which reduces to the first Frenet equation $\mathrm{d}\mathbf{x}/\mathrm{d}s = \mathbf{e}_1$. Similarly, the Maurer–Cartan matrix

$$R^{-1}\,\mathrm{d}R = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix} \mathrm{d}s \quad \text{implies that} \quad \frac{\mathrm{d}R}{\mathrm{d}s} = R \cdot \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix}.$$

The columns of the latter matrix differential equation complete the system of Frenet equations:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{e}_1, \qquad \frac{\mathrm{d}\mathbf{e}_1}{\mathrm{d}s} = \kappa\,\mathbf{e}_2, \qquad \frac{\mathrm{d}\mathbf{e}_2}{\mathrm{d}s} = -\kappa\,\mathbf{e}_1. \tag{6.8}$$

Finally, the Maurer–Cartan structure equations (3.2) for the Euclidean group reduce to the classical Frenet–Serret equations for curves. See [13; p. 23], [19; p. 20], for details.

Remark. One can also compute, as in our original example, the full moving coframe forms on the jet bundle, leading to a corresponding set of fundamental Euclidean-invariant contact forms.

Remark. Actually, since we are dealing with the full pseudo-group $\mathcal{D}\mathit{iff}(1)$ consisting of *all* diffeomorphisms of $\mathbb{R}$, the final 1-form $\eta_4 = \sigma\,\mathrm{d}t$ in our moving coframe (6.3) is, in fact, irrelevant – one could perform the same normalization (6.4) of the angle ϕ based on the dependency between η_2 and η_3, the lifted invariant now being J_2/J_1 which is normalized to zero by setting $J_2 = 0$, leading to the same final moving coframe and curvature invariant. Thus, the calculations for parametrized curves and surfaces can, in fact, be done without invoking the diffeomorphism pseudo-group. Nevertheless, in all examples we have treated, the inclusion of the canonical 1-form η_4 on the parameter space leads to an immediate identification of the final invariant arc length element. More generally, the restricted reparametrization equivalence problem does require the introduction of suitable 1-forms that characterize the pseudo-group of allowed reparametrizations.

Remark. The problem of Euclidean equivalence of curves with *fixed* parametrizations, as discussed in Example 2.1, can also be formulated and solved in the moving coframe context. Now we are in the intransitive framework, where the parameter t provides a scalar invariant. Consequently, we retain the first three 1-forms η_1, η_2, η_3 in (6.3), but replace η_4 by $\mathrm{d}t$ to form the moving coframe. We normalize $J_1 = 0$ as before, but now $J_2 = v = \sqrt{x_t^2 + y_t^2}$ forms a first-order differential invariant – the speed of the particle. The final moving frame has $\eta_1 = K\,\mathrm{d}t$, $\eta_2 = 0$, $\eta_3 = v\,\mathrm{d}t$, $\eta_4 = \mathrm{d}t$, where $K = x_t y_{tt} - x_{tt} y_t = v^3 \kappa$ is a second-order differential invariant. The higher-order differential invariants are found by differentiating with respect to t. Note that the arc length $\mathrm{d}s = v\,\mathrm{d}t$ is also an invariant 1-form, being an invariant multiple of $\mathrm{d}t$ and, hence, one can, without loss of generality, apply the arc length derivative $\mathrm{d}/\mathrm{d}s$ to produce the higher-order differential invariants instead. Thus, in this case, a complete list of differential invariants is provided by v, κ, and their derivatives with respect to arc length.

EXAMPLE 6.2. The equi-affine geometry of curves in the plane is governed by the special affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$, acting on $M = \mathbb{R}^2$ according to

$$g\colon \mathbf{x} \longmapsto A\mathbf{x} + \mathbf{a}, \qquad \mathbf{x} \in M, \quad A \in \mathrm{SL}(2), \quad \mathbf{a} \in \mathbb{R}^2. \tag{6.9}$$

We shall adopt a vector notation for the matrix $A = (\boldsymbol{\alpha}\,\boldsymbol{\beta}) \in \mathrm{SL}(2)$, so that the column vectors are subject to the unimodularity constraint

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = 1. \tag{6.10}$$

It will be computationally convenient *not* to explicitly implement the unimodularity constraint (6.10) by solving for one of the parameters, but retain it as an additional constraint that is to be respected during the course of the calculation. This method, i.e., treating a subgroup of a larger Lie group via a collection of algebraic constraints, rather than parametrizing it directly, has general applicability, and can be readily implemented as is done in this particular case.

The Maurer–Cartan forms are computed directly as in Section 3, leading to

$$\begin{aligned} \mu_1 &= \boldsymbol{\alpha} \wedge \mathrm{d}\boldsymbol{\alpha}, & \mu_2 &= \boldsymbol{\beta} \wedge \mathrm{d}\boldsymbol{\alpha}, & \mu_3 &= \boldsymbol{\beta} \wedge \mathrm{d}\boldsymbol{\beta}, \\ \nu_1 &= \boldsymbol{\alpha} \wedge \mathrm{d}\mathbf{a}, & \nu_2 &= \boldsymbol{\beta} \wedge \mathrm{d}\mathbf{a}. \end{aligned} \tag{6.11}$$

Note that the unimodularity constraint (6.10) implies that

$$\boldsymbol{\alpha} \wedge \mathrm{d}\boldsymbol{\beta} = \boldsymbol{\beta} \wedge \mathrm{d}\boldsymbol{\alpha}, \tag{6.12}$$

which means that the matrix of Maurer–Cartan forms $\boldsymbol{\mu} = A^{-1}\,\mathrm{d}A$ must be trace free.

Choose the base point to be $\mathbf{x}_0 = 0$. Solving the compatible lift equations $\mathbf{x} = g \cdot \mathbf{x}_0 = \mathbf{a}$ yields the 0th-order moving frame, which sets $\mathbf{a} = \mathbf{x}$. Substituting into the Maurer–Cartan forms (6.11), we find that, for a parametrized curve $\mathbf{x}(t)$, the forms ν_1, ν_2 restrict to the following two horizontal forms

$$\eta_1 = (\boldsymbol{\alpha} \wedge \mathbf{x}_t)\,\mathrm{d}t, \qquad \eta_2 = (\boldsymbol{\beta} \wedge \mathbf{x}_t)\,\mathrm{d}t. \tag{6.13}$$

Their ratio produces the lifted invariant $(\boldsymbol{\alpha} \wedge \mathbf{x}_t)/(\boldsymbol{\beta} \wedge \mathbf{x}_t)$, which is normalized to 0 by setting

$$\boldsymbol{\alpha} = \lambda\,\mathbf{x}_t, \tag{6.14}$$

for some scalar parameter λ. Substituting (6.14) into the first Maurer–Cartan form $\mu_1 = \boldsymbol{\alpha} \wedge \mathrm{d}\boldsymbol{\alpha}$, leads to the restricted form $\xi_1 = \lambda^2(\mathbf{x}_t \wedge \mathbf{x}_{tt})\,\mathrm{d}t$. Assuming $\mathbf{x}_t \wedge \mathbf{x}_{tt} \neq 0$, the latter form can be normalized to equal $-\eta_2$ by setting

$$-\boldsymbol{\beta} \wedge \mathbf{x}_t = \lambda^2(\mathbf{x}_t \wedge \mathbf{x}_{tt}), \quad \text{or} \quad \boldsymbol{\beta} = \lambda^2\mathbf{x}_{tt} + \mu\mathbf{x}_t, \tag{6.15}$$

for some scalar μ. However, applying the unimodularity constraint (6.10) to the normalizations (6.14), (6.15), we deduce that $\lambda^3(\mathbf{x}_t \wedge \mathbf{x}_{tt}) = 1$ and, thus,

$$\lambda = \frac{1}{\sqrt[3]{\mathbf{x}_t \wedge \mathbf{x}_{tt}}}. \tag{6.16}$$

Note that (6.15), (6.16) reduce the form η_2 to be minus the equi-affine arc length form

$$\mathrm{d}s = \sqrt[3]{\mathbf{x}_t \wedge \mathbf{x}_{tt}}\,\mathrm{d}t. \tag{6.17}$$

Furthermore, substituting (6.15), (6.16) into the second Maurer–Cartan form, we find it reduces to a multiple of $\xi_1 = \mathrm{d}s$, so

$$\xi_2 = \boldsymbol{\beta} \wedge \mathrm{d}\boldsymbol{\alpha} = J\,\mathrm{d}s,$$

where the lifted invariant

$$J = \mu(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{1/3} + \frac{\mathbf{x}_t \wedge \mathbf{x}_{ttt}}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{4/3}}$$

is normalized to zero in the obvious manner. Therefore, the final moving frame is given by

$$\begin{aligned} \boldsymbol{\alpha} &= \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \frac{\mathbf{x}_t}{\sqrt[3]{\mathbf{x}_t \wedge \mathbf{x}_{tt}}}, \\ \boldsymbol{\beta} &= \frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}s^2} = \frac{\mathbf{x}_{tt}}{(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{2/3}} - \frac{(\mathbf{x}_t \wedge \mathbf{x}_{ttt})\,\mathbf{x}_t}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{5/3}}, \end{aligned} \qquad \mathbf{a} = \mathbf{x}. \tag{6.18}$$

The final Maurer–Cartan form becomes

$$\xi_3 = \boldsymbol{\beta} \wedge \mathrm{d}\boldsymbol{\beta} = \kappa\,\mathrm{d}s,$$

where

$$\kappa = \mathbf{x}_{ss} \wedge \mathbf{x}_{sss} = \frac{(\mathbf{x}_t \wedge \mathbf{x}_{tttt}) + 4(\mathbf{x}_{tt} \wedge \mathbf{x}_{ttt})}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{5/3}} - \frac{5(\mathbf{x}_t \wedge \mathbf{x}_{ttt})^2}{9(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{8/3}} \tag{6.19}$$

defines the equi-affine curvature. As usual, all higher-order differential invariants are obtained by differentiating κ with respect to the equi-affine arc length $\mathrm{d}s$. This reproduces the basic invariants of the equi-affine geometry of curves [19]; see also [5] for applications in computer vision.

As with the Euclidean case, we recover the classical Frenet equations as simple reformulations of the final moving frame formulae. We identify the linear part

$$A = (\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{x}_s, \mathbf{x}_{ss})$$

of the final moving frame with the equi-affine frame at a point $\mathbf{x}(t)$ on the curve, so that $\mathbf{e}_1 = \mathbf{x}_s$ is the unit affine tangent vector, whereas $\mathbf{e}_2 = \mathbf{x}_{ss}$ is the unit equi-affine normal. Combining this with the Maurer–Cartan matrix $A^{-1}\,\mathrm{d}A = \begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix}\mathrm{d}s$ leads to the complete Frenet equations of planar equi-affine geometry [13; p. 27]:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = \mathbf{e}_1, \qquad \frac{\mathrm{d}\mathbf{e}_1}{\mathrm{d}s} = \mathbf{e}_2, \qquad \frac{\mathrm{d}\mathbf{e}_2}{\mathrm{d}s} = \kappa\mathbf{e}_1. \tag{6.20}$$

See [19; Sect. 7–3] for further details.

EXAMPLE 6.3. The most complicated example treated in the literature [7], is the projective geometry of curves in the plane. Here the group is $\mathrm{SL}(3)$, acting on $M = \mathbb{RP}^2$ according to

$$g\colon (x,u) \longmapsto \left(\frac{\alpha x + \beta u + \gamma}{\rho x + \sigma u + \tau}, \ \frac{\lambda x + \mu u + \nu}{\rho x + \sigma u + \tau}\right),$$

$$\det A = \det \begin{vmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{vmatrix} = 1. \tag{6.21}$$

For simplicity, we deal with curves which can be expressed as the graphs of functions, $u = u(x)$, although the general case of parametrized curves can be handled via the same sequence of normalizations. Choose the base point to be $z_0 = (0,0)$. Solving $g \cdot (0,0) = (x,u)$ leads to the 0th-order moving frame in the form*

$$A = \begin{pmatrix} \alpha & \beta & x\tau \\ \lambda & \mu & u\tau \\ \rho & \sigma & \tau \end{pmatrix}, \quad \text{where } \alpha = \frac{1 + \tau[\beta(\lambda - \rho u) + x(\mu\rho - \lambda\sigma)]}{\tau(\mu - \sigma x)}. \tag{6.22}$$

The 1-forms in the first-order moving coframe are the entries of the pull-back of the Maurer–Cartan matrix $A^{-1}\,\mathrm{d}A$, which we label (in row order) as $\eta_1, \ldots, \eta_8$, the final entry being $\eta_9 = -\eta_1 - \eta_5$, reflecting the unimodularity of A. For simplicity, we just indicate the salient features of the computation without dwelling on the details. (These computations were done with the aid of some MATHEMATICA routines written for this purpose.) The first normalization comes from the ratio η_3/η_6, whose vanishing requires

$$\mu = \sigma(u - xu_x) - \beta u_x.$$

Plugging this normalization back into the moving coframe forms and recomputing, we find that we can normalize $\eta_6 = \eta_2$ by requiring

$$\beta = \sigma x - u_{xx}^{-1/3}.$$

In the next stage, we set η_5 to zero by normalizing

$$\sigma = \frac{\tau(\rho u - \lambda)u_{xx}^{1/3}}{u_x} - \frac{u_x u_{xxx} - 3u_{xx}^2}{3u_x u_{xx}^{4/3}}.$$

At the next step, we can no longer just look at 1-forms depending only on $\mathrm{d}x$ – these do not produce any further invariants. However, we discover that $\eta_8 = J\eta_2 + \eta_4$ and, hence, the rather complicated lifted invariant J can be normalized to zero, leading to

$$\lambda = \rho u - \frac{u_x u_{xxx} - 3u_{xx}^2 + u_x\sqrt{18\rho\tau u_{xx}^{8/3} - P_4}}{3\tau u_{xx}^{5/3}},$$

* In this example, we have chosen to implement the unimodularity constraint explicitly.

where

$$P_4 = 3u_{xx}u_{xxxx} - 5u_{xxx}^2.$$

Next, the normalization $\eta_2 = -\eta_7$ requires

$$\tau = \sqrt[3]{\frac{L_5}{54u_{xx}^4}}, \quad \text{where } L_5 = 9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3.$$

The final normalization

$$\rho = \frac{M_6^2 + P_4 L_5^2}{3\sqrt[3]{4u_{xx}^4 L_5^7}}, \quad \text{where } M_6 = (u_{xx}D_x - 4u_{xxx})L_5$$

comes from setting η_1 to zero. The final moving frame is explicitly given by

$$\begin{aligned}
\alpha &= \frac{\lambda + \rho(xu_x - u)}{u_x} - \frac{3}{u_x}\sqrt[3]{\frac{2u_{xx}^5}{L_5}}, \qquad \beta = x\mu - u_{xx}^{-1/3}, \quad \gamma = x\tau, \\
\lambda &= \frac{uM_6^2 + 6u_x u_{xx} L_5 M_6 + K_4 L_5^2}{3\sqrt[3]{4u_{xx}^4 L_5^7}}, \qquad \mu = u\sigma - \frac{u_x}{u_{xx}^{1/3}}, \quad \nu = u\tau, \\
\rho &= \frac{M_6^2 + P_4 L_5^2}{3\sqrt[3]{4\, u_{xx}^4 L_5^7}}, \qquad \sigma = \frac{M_6}{3u_{xx}^{4/3} L_5}, \qquad \tau = \frac{L_5^{1/3}}{3\sqrt[3]{2u_{xx}^4}}.
\end{aligned} \tag{6.23}$$

The corresponding final coframe has

$$\begin{aligned}
\eta_2 &= \eta_6 = -\eta_7 = \mathrm{d}s = \frac{\sqrt[3]{L_5}}{3\sqrt[3]{2}u_{xx}}\,\mathrm{d}x \\
&= \sqrt[3]{\frac{9u_{xx}^2 u_{xxxxx} - 45u_{xx}u_{xxx}u_{xxxx} + 40u_{xxx}^3}{54u_{xx}^3}}\,\mathrm{d}x,
\end{aligned} \tag{6.24}$$

which determines the well-known projective arc length element, while $\eta_4 = \eta_8 = -\kappa\,\mathrm{d}s$ yields the projective curvature invariant

$$\kappa = \frac{6\,u_{xx}L_5 D_x M_6 - 7M_6^2 - 32\,u_{xxx}L_5 M_6 - P_4 L_5^2}{\sqrt[3]{2}\,L_5^{8/3}}. \tag{6.25}$$

Again, all higher-order differential invariants are found by differentiating the projective curvature κ with respect to the projective arc length $\mathrm{d}s$. This relatively straightforward computation reproduces the moving frame and the fundamental invariants for the projective geometry of curves. Cartan, [7], presents a variety of alternative methods to arrive at the same basic result. See also [38] for a Lie-theoretic approach, and Wilczynski [49], for an approach based on differential operators.

In the classical moving-frame method, one identifies the columns of the 3×3 moving frame matrix $A = (\mathbf{P}_2, \mathbf{P}_1, \mathbf{P})$ as homogeneous coordinates for three points in the projective plane $\mathbb{RP}^2$, the last column $\mathbf{P} = \tau \cdot (x, u, 1)^T$ representing the point on the curve. The Maurer–Cartan matrix

$$A^{-1}\,\mathrm{d}A = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa & 0 & 1 \\ -1 & -\kappa & 0 \end{pmatrix} \mathrm{d}s$$

reduces to the full set of projective Frenet equations,

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}s} = \mathbf{P}_1, \qquad \frac{\mathrm{d}\mathbf{P}_1}{\mathrm{d}s} = -\kappa\mathbf{P} + \mathbf{P}_2, \qquad \frac{\mathrm{d}\mathbf{P}_2}{\mathrm{d}s} = -\mathbf{P} - \kappa\mathbf{P}_1. \tag{6.26}$$

See also [13; pp. 33ff.] for applications to projective curvature evolutions and computer vision.

7. Connections with the Classical Moving Frames Method

Our initial identification of a moving frame as an equivariant lift from the underlying space to the Lie group will be familiar to readers of the modern formulations of Griffiths [18], and Jensen [23]. However, since this point of view is not completely standard, it is worth reviewing how it relates to the more usual geometric approaches, e.g., [19, 50]. Traditionally, a moving frame is realized as a collection of vectors (or, in the projective case, points) in the underlying space. The reason that this works in the classical cases, including Euclidean, affine, and projective geometry of submanifolds, is that it is possible to identify the components of the group itself with objects in the underlying transformation space. For example, in the Euclidean case, one identifies a Euclidean group element $(R, \mathbf{a}) \in \mathrm{E}(m) \simeq \mathrm{O}(m) \ltimes \mathbb{R}^m$ with a vector $\mathbf{a} \in \mathbb{R}^m$, together with an orthonormal frame determined by the columns of the orthogonal matrix R. The 0th-order moving frame, then, uses the lift $\mathbf{a} = \mathbf{x}$, where $\mathbf{x}$ is a point on the submanifold $N \subset \mathbb{R}^m$, and the orthogonal matrix is identified with an orthonormal frame in the ambient space based at the point. The remaining ambiguity in the frame is up to orthogonal transformations, which must then be resolved in an invariant manner. Similarly, in the equi-affine case, one identifies a group element $(A, \mathbf{a}) \in \mathrm{SA}(m) \simeq \mathrm{SL}(m) \ltimes \mathbb{R}^m$ with a vector $\mathbf{a} \in \mathbb{R}^m$ together with a unimodular frame determined by the columns of the matrix A. Again, the 0th-order moving frame takes $\mathbf{a} = \mathbf{x}$ to be a point on the submanifold, and the unimodular frame becomes a set of vectors based at the point $\mathbf{x}$. In both cases, the moving coframe method introduces the Maurer–Cartan forms $\boldsymbol{\mu} = (\boldsymbol{\sigma}, \boldsymbol{\nu})$, where $\boldsymbol{\sigma} = A^{-1}\,\mathrm{d}A$, $\boldsymbol{\nu} = A^{-1}\,\mathrm{d}\mathbf{a}$, leading to the initial structure equations

$$\mathrm{d}\mathbf{x} = A \cdot \boldsymbol{\nu}, \qquad \mathrm{d}A = A \cdot \boldsymbol{\sigma}. \tag{7.1}$$

The moving coframe forms satisfy the usual Maurer–Cartan structure equations (3.2), which, in the classical cases, become the fundamental Cartan structure equations for Euclidean or affine geometry.

Usually, one bypasses the 0th-order moving frame entirely, and proceeds directly to the first-order moving frame, in which the frame at the point $\mathbf{x} \in N$ is split into two parts, so that (using column vector notation)

$$A = (E, F) = (\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{f}_1, \dots, \mathbf{f}_{m-n}), \tag{7.2}$$

where the first $n = \dim N$ frame vectors form a basis for the tangent space TN to the submanifold, while the remainder are left arbitrary, subject to the entire frame satisfying the proper orthonormality or unimodularity constraint. Thus, in the Euclidean case, the vectors $\{\mathbf{f}_1, \dots, \mathbf{f}_{m-n}\}$ form an orthonormal basis for the normal space to N, whereas in the equi-affine case they are left arbitrary subject only to the condition that the determinant of the matrix (7.2) be unity. If we parametrize the submanifold by $\mathbf{x}(t_1, \dots, t_n)$, then the most general first-order moving frame (7.2) will have the form

$$E = (\mathbf{e}_1, \dots, \mathbf{e}_n) = V \cdot B, \tag{7.3}$$

where

$$V = (\mathbf{v}_1, \dots, \mathbf{v}_n), \qquad \mathbf{v}_i = \frac{\partial \mathbf{x}}{\partial t_i}, \tag{7.4}$$

is the $m \times n$ Jacobian matrix, whose columns span the tangent space to N, while B is an invertible $n \times n$ matrix. (In the Euclidean case, the matrix B is restricted so that the columns of E are orthonormal, leaving an $\mathrm{O}(n)$ ambiguity.)

Let us show how this preliminary normalization to a first-order moving frame is an immediate consequence of our general normalization procedure. Using the 0th-order moving frame lift, the pull-backs of the subset of Maurer–Cartan forms given by the entries of $\boldsymbol{\nu} = A^{-1}\, \mathrm{d}\mathbf{a}$ can be written in matrix form as

$$\boldsymbol{\nu} = A^{-1}\, \mathrm{d}\mathbf{x} = A^{-1} V\, \mathrm{d}\mathbf{t}.$$

Precisely n of the m 1-forms $\boldsymbol{\nu}$ are linearly independent and, hence, we can normalize so that the last $m - n$ of these forms vanish. This requires that the matrix A satisfy the block matrix equation

$$A^{-1} V = \begin{pmatrix} D \\ 0 \end{pmatrix}, \tag{7.5}$$

where D is a nonsingular $n \times n$ matrix, while 0 denotes the zero matrix of size $(m-n) \times n$. Writing $A = (E, F)$ in block form (7.2), we see that (7.5) requires

$$V = E \cdot D, \quad \text{or} \quad E = V \cdot C, \quad \text{where } C = D^{-1},$$

thereby recovering (7.3). Thus, we can see that in such cases, the first-order frame recovered by a 0th-order normalization coincides with the traditional first-order frame involving tangent and normal directions.

Similar considerations apply to the projective case. According to Cartan [7], the 0th-order frame can be identified with a set of $n+1$ linearly independent points in the projective space which are identified with the columns of the matrix $A \in \mathrm{SL}(n+1)$. The 0th-order lift, as in (6.22), amounts to identifying one of the columns with the point on the curve **x**. More precisely, the column is a vector with $n+1$ components, which are interpreted as the homogeneous coordinates of **x**.

In more sophisticated versions, one realizes the moving frame on the submanifold $N \subset M$ as a section of the frame bundle $\mathcal{F}(M)$ of M, pulled back to N, i.e., a section $\psi\colon N \to \mathcal{F}(M)$. One can also try to handle cases that do not so readily fit into this simple framework by reinterpreting them as sections of a suitable higher-order frame bundle $\mathcal{F}^k(M)$ over N, cf. [26]. Although this is possible for all (regular, transitive) transformation groups, the original geometrical realization has now been obscured, and such a reformulation does not, we think, offer much insight or help in the explicit implementation of the method.

Consequently, the method of moving coframes includes all the classical constructions based on the indicated identification of group elements with geometric objects on the transformation space. However, once one goes beyond the traditional cases, such identifications become much less apparent, and, in our opinion, attempting to mimic the Euclidean, affine, and projective constructions directly on the transformation space has hindered the development of any significant extensions of the method. Furthermore, once one steps outside the realm of 'classical' moving frame geometries, one can no longer use the identification of the first-order frame with tangent and normal directions. Our nontraditional examples all illustrate this – the first-order frames do not include the tangent spaces to the submanifolds in any obvious manner, because their naïve identification with subspaces of Euclidean space is not necessarily invariant with respect to the given transformation group. It is our view that, in order to attain their full range of applicability, the constructions must be viewed in the purely group- or, more generally, bundle-theoretic framework that we have presented here and develop in detail in Part II.

8. Joint Differential Invariants

New applications in image processing and object recognition [35], have demonstrated the need for classification and computation of the joint differential invariants or, as they are known in computer vision, *semi-differential invariants*, for a given transformation group. Specifically, one is given a Lie group (or pseudogroup) G acting on M and considers its diagonal action $g \cdot (z^1, \ldots, z^k) = (g \cdot z^1, \ldots, g \cdot z^k)$ on the k-fold Cartesian product $M^{\times k} = M \times \cdots \times M$.

The invariants $I(z^1, \dots, z^k)$ of such a Cartesian product action are known as the *k-point joint invariants* of the transformation group. Note that for $j < k$, any j-point invariant can be regarded as a k-point invariant, in several different ways. For example, the two-point invariant $I(z_1, z_2)$ produces three invariants on $M^{\times 3}$, namely $\widetilde{I}(z_1, z_2, z_3) = I(z_1, z_2)$ or $I(z_1, z_3)$ or $I(z_2, z_3)$. If I is not symmetric in its arguments, these in turn lead to 3 further invariants by interchanging the points. To avoid this trivial extension, we will reserve the term k-point invariant for a joint invariant which cannot be written as as one depending on fewer than k arguments.

Similarly, the invariants of the prolonged diagonal action of $G^{(n)}$ on a k-fold Cartesian product of jet space $(\mathrm{J}^n)^{\times k}$ are the *joint differential invariants* of k different submanifolds $N_1, \dots, N_k \subset M$, which we view as a single submanifold $N_1 \times \cdots \times N_k$ of the Cartesian product space $M^{\times k}$. In applications, the submanifolds $N_j = N$ are identical, but the joint differential invariants are measured at k different points along the given submanifold.

The method of moving coframes readily adapts to this slightly more general situation, and immediately provides complete classifications of joint differential invariants for all of the standard geometric transformation groups.

EXAMPLE 8.1. *Euclidean joint differential invariants.* Consider the Euclidean group $\mathrm{E}(2)$ acting on the plane $M = \mathbb{R}^2$. We consider two-point differential invariants, corresponding to the Cartesian product action

$$(\mathbf{x}, \mathbf{y}) \longmapsto (R \cdot \mathbf{x} + \mathbf{a}, R \cdot \mathbf{y} + \mathbf{a}), \quad \mathbf{x}, \mathbf{y} \in M, \quad (R, a) \in \mathrm{E}(2), \tag{8.1}$$

on $M^{\times 2} \simeq \mathbb{R}^4$. Note that the action is intransitive on $M^{\times 2}$, with the interpoint distance

$$r = |\mathbf{z}|, \quad \text{where } \mathbf{z} = \mathbf{x} - \mathbf{y}, \tag{8.2}$$

being the fundamental joint Euclidean invariant. (See [48] for a proof that all Euclidean joint invariants can be written in terms of the elementary two-point invariants.) We can choose the cross-section to the orbits given by $\mathbf{x}_0 = 0$, $\mathbf{y}_0 = (r, 0)$, which leads to the compatible lift with

$$\mathbf{a} = \mathbf{x}, \quad (r \cos\phi, r \sin\phi) = \mathbf{z} = \mathbf{x} - \mathbf{y}. \tag{8.3}$$

Therefore, all the group parameters are normalized by the initial compatible lift, and it only remains to substitute (8.3) into the Euclidean Maurer–Cartan forms (3.5). The net result is the following system of invariant forms

$$\zeta_1 = \mathbf{z} \cdot \mathrm{d}\mathbf{x}, \qquad \zeta_2 = \mathbf{z} \cdot \mathrm{d}\mathbf{y}, \qquad \zeta_3 = r^2\, \mathrm{d}\phi = \mathbf{z} \wedge \mathrm{d}\mathbf{z}. \tag{8.4}$$

Note that the forms (8.4) include the differential of the joint invariant (8.2) since $r\,\mathrm{d}r = \zeta_1 + \zeta_2$. Therefore, given two parametrized curves

$$\mathbf{x} = \mathbf{x}(t), \qquad \mathbf{y} = \mathbf{y}(s), \tag{8.5}$$

the first two 1-forms (8.4) restrict to define two invariant 1-forms

$$\eta_1 = (\mathbf{z} \cdot \mathbf{x}_t)\, \mathrm{d}t, \qquad \eta_2 = (\mathbf{z} \cdot \mathbf{y}_s)\, \mathrm{d}s, \tag{8.6}$$

while $\eta_3 = I_1\, \eta_1 + I_2\, \eta_2$, where

$$I_1 = \frac{\mathbf{z} \wedge \mathbf{x}_t}{\mathbf{z} \cdot \mathbf{x}_t}, \qquad I_2 = \frac{\mathbf{z} \wedge \mathbf{y}_s}{\mathbf{z} \cdot \mathbf{y}_s}, \tag{8.7}$$

are the two fundamental first-order differential invariants, which, along with the original joint invariant (8.2), form a complete system of first-order joint differential invariants. The vector identity

$$(\mathbf{a} \cdot \mathbf{b})^2 + (\mathbf{a} \wedge \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 \tag{8.8}$$

demonstrates that

$$J_1 = \frac{|\mathbf{x}_t|}{\mathbf{z} \cdot \mathbf{x}_t} = \frac{\sqrt{1 + (I_1)^2}}{r}$$

is also a joint differential invariant, and hence (in the orientation-preserving case) one can replace the 1-forms (8.6) by the two Euclidean arc-length forms

$$\omega_1 = J_1\, \eta_1 = |\mathbf{x}_t|\, \mathrm{d}t, \qquad \omega_2 = J_2\, \eta_2 = |\mathbf{y}_s|\, \mathrm{d}s. \tag{8.9}$$

THEOREM 8.2. *Every two-point Euclidean joint differential invariant is a function of the interpoint distance* $r = |\mathbf{x} - \mathbf{y}|$ *and its derivatives with respect to the two arc length forms* (8.9).

For example, to recover the Euclidean curvature $\kappa_1 = |\mathbf{x}_t|^{-3}(\mathbf{x}_t \wedge \mathbf{x}_{tt})$ of the first curve, we differentiate

$$\begin{aligned} \frac{\partial I_1}{\partial \eta_1} &= \frac{\mathbf{z} \wedge \mathbf{x}_{tt}}{(\mathbf{z} \cdot \mathbf{x}_t)^2} - \frac{(\mathbf{z} \wedge \mathbf{x}_t)[(\mathbf{z} \cdot \mathbf{x}_{tt}) + |\mathbf{x}_t|^2]}{(\mathbf{z} \cdot \mathbf{x}_t)^3} \\ &= \frac{(\mathbf{z} \wedge \mathbf{x}_{tt})(\mathbf{z} \cdot \mathbf{x}_t) - (\mathbf{z} \wedge \mathbf{x}_t)(\mathbf{z} \cdot \mathbf{x}_{tt})}{(\mathbf{z} \cdot \mathbf{x}_t)^3} - I_1 J_1^2 \\ &= \frac{(\mathbf{x}_t \wedge \mathbf{x}_{tt})|\mathbf{z}|^2}{(\mathbf{z} \cdot \mathbf{x}_t)^3} - I_1 J_1^2 = \kappa_1 r^2 - I_1 J_1^2, \end{aligned}$$

where we have used the first of the following equivalent determinantal identities

$$\begin{aligned} &(\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) + (\mathbf{b} \wedge \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) - (\mathbf{a} \wedge \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) = 0, \\ &(\mathbf{a} \wedge \mathbf{b})(\mathbf{c} \wedge \mathbf{d}) + (\mathbf{b} \wedge \mathbf{c})(\mathbf{a} \wedge \mathbf{d}) - (\mathbf{a} \wedge \mathbf{c})(\mathbf{b} \wedge \mathbf{d}) = 0. \end{aligned} \tag{8.10}$$

EXAMPLE 8.3. *Equi-affine joint differential invariants.* A more substantial example is provided by the two-point differential invariants for the special affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$, acting on $M = \mathbb{R}^2$. The Cartesian product action

$$(\mathbf{x}, \mathbf{y}) \longmapsto (A\mathbf{x} + \mathbf{a}, A\mathbf{y} + \mathbf{a}), \quad \mathbf{x}, \mathbf{y} \in M, \quad A \in \mathrm{SL}(2), \quad \mathbf{a} \in \mathbb{R}^2, \tag{8.11}$$

is transitive on $M^{\times 2}$. As in Example 6.2, we use the vector notation $A = (\boldsymbol{\alpha}\, \boldsymbol{\beta}) \in \mathrm{SL}(2)$, where $\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = 1$.

In view of (8.11), we can choose the base point $\mathbf{x}_0 = 0$, $\mathbf{y}_0 = (1, 0)$, noting that the diagonal $\Delta = \{\mathbf{x} = \mathbf{y}\} \subset M^{\times 2}$ is a singular two-dimensional orbit. This leads to the compatible lift with

$$\mathbf{a} = \mathbf{x}, \qquad \boldsymbol{\alpha} = \mathbf{z} = \mathbf{x} - \mathbf{y}. \tag{8.12}$$

Substituting into the Maurer–Cartan forms (6.11), we find that, for a pair of parametrized curves as in (8.5), the following horizontal forms

$$(\mathbf{z} \wedge \mathbf{x}_t)\, \mathrm{d}t, \qquad (\boldsymbol{\beta} \wedge \mathbf{x}_t)\, \mathrm{d}t, \qquad (\mathbf{z} \wedge \mathbf{y}_s)\, \mathrm{d}s, \qquad (\boldsymbol{\beta} \wedge \mathbf{y}_s)\, \mathrm{d}s,$$

the first two being the pull-backs of ν_1, ν_2, and the latter being that of $\nu_1 - \mu_1$, $\nu_2 - \mu_2$. Generically (i.e., provided $\mathbf{x} - \mathbf{y}$ is not parallel to $\mathbf{x}_t$) we can normalize the second form to zero, leading, in view of (8.12) and the unimodularity constraint, to

$$\boldsymbol{\beta} = \frac{\mathbf{x}_t}{\mathbf{z} \wedge \mathbf{x}_t}, \tag{8.13}$$

which, combined with (8.12) provides the complete moving frame. The remaining 1-forms are

$$\eta_1 = \mathbf{z} \wedge \mathrm{d}\mathbf{x} = (\mathbf{z} \wedge \mathbf{x}_t)\, \mathrm{d}t, \qquad \eta_2 = \mathbf{z} \wedge \mathrm{d}\mathbf{y} = (\mathbf{z} \wedge \mathbf{y}_s)\, \mathrm{d}s, \tag{8.14}$$

which provide the two fundamental invariant 1-forms, and

$$\eta_3 = \boldsymbol{\beta} \wedge \mathrm{d}\mathbf{y} = \left[\frac{\mathbf{x}_t \wedge \mathbf{y}_s}{\mathbf{z} \wedge \mathbf{x}_t}\right] \mathrm{d}s, \qquad \eta_4 = \boldsymbol{\beta} \wedge \mathrm{d}\boldsymbol{\beta} = \left[\frac{\mathbf{x}_t \wedge \mathbf{x}_{tt}}{(\mathbf{z} \wedge \mathbf{x}_t)^2}\right] \mathrm{d}t.$$

The resulting linear dependencies provide the two basic differential invariants, consisting of a single first-order invariant

$$I = \frac{\mathbf{x}_t \wedge \mathbf{y}_s}{(\mathbf{z} \wedge \mathbf{x}_t)(\mathbf{z} \wedge \mathbf{y}_s)} \tag{8.15}$$

and the first of the two second-order invariants

$$J_1 = \frac{\mathbf{x}_t \wedge \mathbf{x}_{tt}}{(\mathbf{z} \wedge \mathbf{x}_t)^3}, \qquad J_2 = \frac{\mathbf{y}_s \wedge \mathbf{y}_{ss}}{(\mathbf{z} \wedge \mathbf{y}_s)^3}. \tag{8.16}$$

Clearly J_2 can be obtained from J_1 by the interchange symmetry $\mathbf{x} \leftrightarrow \mathbf{y}$. Alternatively, we use (8.10) to compute

$$\frac{\partial I}{\partial \eta_1} - I^2 = -\frac{(\mathbf{x}_{tt} \wedge \mathbf{y}_s)(\mathbf{z} \wedge \mathbf{x}_t) - (\mathbf{x}_t \wedge \mathbf{y}_s)(\mathbf{z} \wedge \mathbf{x}_{tt})}{(\mathbf{z} \wedge \mathbf{x}_t)^3(\mathbf{z} \wedge \mathbf{y}_s)}$$
$$= \frac{(\mathbf{x}_{tt} \wedge \mathbf{x}_t)(\mathbf{z} \wedge \mathbf{y}_s)}{(\mathbf{z} \wedge \mathbf{x}_t)^3(\mathbf{z} \wedge \mathbf{y}_s)} = J_1,$$

so that (8.16) are equivalent to the invariant first-order derivatives of the single basic joint invariant I.

THEOREM 8.4. *Every two-point equi-affine joint differential invariant is a function of the fundamental first order invariant* (8.15) *and its derivatives with respect to the two 'joint arc length' forms* (8.14).

The reader is invited to try to express the ordinary affine curvature in terms of the derivatives of I. The same method readily extends to multi-point invariants of more general groups, including the projective group, as well as joint invariants for surfaces and higher-dimensional submanifolds. Additional examples and applications will appear elsewhere.

9. Pseudo-Group Actions

The next case is that of infinite Lie pseudo-groups, cf. [10, 28, 30, 42, 43]. See also [29, 45], for classical results on differential invariants of Lie pseudo-groups, and Kumpera [27], for a modern treatment. These are readily fit into the same general framework as follows. Assume, initially, that the pseudo-group $\mathcal{G}$ acts transitively on the space M. By definition, a Lie pseudo-group consists of an infinite-dimensional family of invertible (local) transformations that form the general solution to an involutive system of partial differential equations. We can always characterize the transformations $\psi\colon M \to M$ in $\mathcal{G}$ as the projections of bundle maps $\Psi\colon \mathcal{B} \to \mathcal{B}$, defined on a principal fiber bundle $\mathcal{B} \to M$, that preserve a system of 1-forms $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_k\}$ defined on $\mathcal{B}$:

$$\Psi^*\boldsymbol{\zeta} = \boldsymbol{\zeta}. \tag{9.1}$$

The forms $\boldsymbol{\zeta}$ will play the role of the moving coframe forms for the pseudo-group, and the fiber coordinates of the bundle $\mathcal{B}$ will play the role of the undetermined group parameters. Of course, in this case $\boldsymbol{\zeta}$ does not form a full coframe on $\mathcal{B}$. (It cannot, because the symmetry group of a coframe is necessarily a finite-dimensional Lie group [38].) A compatible lift, or moving frame of 0th-order, is just an arbitrary section $\rho_0\colon M \to \mathcal{B}$. Such a section defines a corresponding moving frame $\rho = \rho_0 \circ \iota\colon X \to \mathcal{B}$ on any parametrized submanifold $\iota\colon X \to M$. With these provisos, the normalization and reduction procedure proceeds as in the finite-dimensional situation.

EXAMPLE 9.1. Consider the pseudo-group $\mathcal{G}$ consisting of (local) diffeomorphisms on $M = \mathbb{R}^2$ of the form

$$\bar{x} = f(x), \qquad \bar{u} = \frac{u}{f'(x)}. \tag{9.2}$$

The Lie algebra of $\mathcal{G}$ is generated by vector fields of the form

$$\mathbf{v}_h = h(x)\partial_x - uh'(x)\partial_u.$$

This pseudo-group was first introduced by Lie [28; p. 353], [32], in his classification of infinite-dimensional pseudo-groups acting on the plane. We are interested in the action of $\mathcal{G}$ on curves which, for simplicity, we assume are graphs of functions $u = u(x)$.

The first step is to construct a bundle $\mathcal{B}$ and 1-forms on the bundle whose invariance characterizes the pseudo-group transformations. In this case, away the axis $u = 0$, the group transformations (9.2) form the general solution to the defining system of partial differential equations

$$z_u = 0, \qquad z_x = \frac{u}{w}, \qquad w_u = \frac{w}{u}, \tag{9.3}$$

for $\bar{x} = z(x,u)$, $\bar{u} = w(x,u)$, cf. [46; p. 325]. The system (9.3) defines a submanifold $\Phi\colon \mathcal{R} \hookrightarrow J^1(\mathbb{R}^2, \mathbb{R}^2)$ of the first jet space, parametrized by the coordinates (x, u, z, w, w_x). The pull-backs of the basic contact forms on $J^1(\mathbb{R}^2, \mathbb{R}^2)$ to the equation submanifold $\mathcal{R}$ are given by

$$\begin{aligned} \theta_z &= \Phi^*(\mathrm{d}z - z_x\,\mathrm{d}x - z_u\,\mathrm{d}u) = \mathrm{d}z - \frac{u}{w}\,\mathrm{d}x, \\ \theta_w &= \Phi^*(\mathrm{d}w - w_x\,\mathrm{d}x - w_u\,\mathrm{d}u) = \mathrm{d}w - w_x\,\mathrm{d}x - \frac{w}{u}\,\mathrm{d}u. \end{aligned} \tag{9.4}$$

The Pfaffian system

$$\theta_z = 0, \qquad \theta_w = 0,$$

with independence condition $\mathrm{d}x \wedge \mathrm{d}u \neq 0$ is involutive on $\mathcal{R}$, cf. [3, 9, 38]. Indeed, the first Cartan character is $s_1 = 1$, as it should be. Following a general procedure* presented by Kamran [24], we set $\mathrm{d}z = \mathrm{d}w = 0$, which amounts to pulling back to a level set of $\mathcal{R}$ where $u = u_0$ and $w = w_0$ are constant. Choosing $w_0 = 1$ we find that the contact forms (9.4) reduce to the invariant 1-forms

$$\zeta_1 = -u\,\mathrm{d}x, \qquad \zeta_2 = -w_x\,\mathrm{d}x - \frac{\mathrm{d}u}{u}.$$

* Interestingly, this method is similar to our construction of the Maurer–Cartan forms in the finite-dimensional case.

Therefore, the desired bundle $\mathcal{B} \simeq M \times \mathbb{R}$ will be coordinatized by x, u, and the remaining jet coordinate, which we rewrite as $\alpha = w_x$ for clarity. In other words, the 0th-order moving coframe forms for the pseudo-group (9.2) will be

$$\zeta_1 = u\,\mathrm{d}x, \qquad \zeta_2 = \alpha\,\mathrm{d}x + \frac{\mathrm{d}u}{u}. \tag{9.5}$$

Restricting to a curve $u = u(x)$, and letting η_i denote the horizontal component of ζ_i, we have the relation

$$\eta_2 = (u\alpha + u_x)\,\mathrm{d}x = (u\alpha + u_x)\eta_1$$

and so we normalize $\alpha = -u_x/u$. Thus, the final invariant moving coframe is

$$\zeta_1 = u\,\mathrm{d}x, \qquad \zeta_2 = \frac{\mathrm{d}u - u_x\,\mathrm{d}x}{u}, \tag{9.6}$$

the first providing a pseudo-group invariant arc length form, and the latter an invariant contact form. Note that there are no dependencies among these 1-forms and, hence, there are *no* differential invariants in this example. Indeed, it is not hard to see that the prolonged actions of $\mathcal{G}$ are transitive on every jet space $\mathrm{J}^n M$, justifying the preceding statement.

EXAMPLE 9.2. We now extend the pseudo-group discussed in the previous example to an intransitive action obtained by augmenting the transformation rules (9.2) by an additional invariant coordinate y, so that the pseudo-group now has the form

$$\bar{x} = f(x), \qquad \bar{y} = y, \qquad \bar{u} = \frac{u}{f'(x)}. \tag{9.7}$$

This pseudo-group was introduced by Lie [31; p. 373], in his study of second-order partial differential equations integrable by the method of Darboux. In his paper on group splitting and automorphic systems, Vessiot [46], used (9.7) as one of two principal examples illustrating his method. More recently, Kumpera [27] again employed this pseudo-group to illustrate his formalization of the Lie theory of differential invariants. Now we are interested in the equivalence problem and differential invariants for surfaces $u = u(x, y)$ under the pseudo-group (9.7). The Maurer–Cartan forms are given by supplementing (9.5) by an additional coframe element $\zeta_0 = \mathrm{d}y$. The linear dependency

$$\eta_2 = -(u\alpha + u_x)\,\eta_1 - \frac{u_y}{u}\,\mathrm{d}y$$

again produces the normalization $\alpha = -u_x/u$, along with the basic first-order differential invariant

$$I = \frac{u_y}{u}.$$

The final invariant moving coframe is

$$\zeta_0 = \mathrm{d}y, \qquad \zeta_1 = u\,\mathrm{d}x, \qquad \zeta_2 = \frac{\mathrm{d}u - u_x\,\mathrm{d}x}{u}. \tag{9.8}$$

The invariant total differential operators associated with the first two horizontal forms are

$$\frac{\partial}{\partial \zeta_0} = D_y, \qquad \frac{\partial}{\partial \zeta_1} = \frac{1}{u}\,D_x. \tag{9.9}$$

Applying them to the fundamental invariant I produce the second-order differential invariants

$$J_1 = \frac{u u_{yy} - u_y^2}{u^2}, \qquad J_2 = \frac{u u_{xy} - u_x u_y}{u^3},$$

agreeing with the classical formulae. All higher-order differential invariants are obtained by successively applying the invariant total derivative operators (9.9) to the invariant I. Similarly, the classifying surface associated with a generic surface $u(x, y)$ is parametrized by the four invariants (y, I, J_1, J_2); two surfaces are congruent under a pseudo-group transformation if and only if their classifying surfaces are identical. Surfaces with higher-order* symmetry occur when I is a function of y only, so that $u(x, y) = f(x)g(y)$ is multiplicatively separable. Finally, the most general second-order partial differential equation admitting (9.7) as a symmetry group can be written in the form

$$H\left(y, \frac{u_y}{u}, \frac{u u_{yy} - u_y^2}{u^2}, \frac{u u_{xy} - u_x u_y}{u^3}\right) = 0. \tag{9.10}$$

These are the class of equations considered by Lie [31; p. 374].

In his classification of planar second-order partial differential equations which admit symmetry pseudo-groups, Medolaghi [34] treats the same example, but rewritten in a slightly different coordinate system. The group transformations take the form

$$\bar{x} = f(x), \qquad \bar{y} = y + f'(x), \qquad \bar{u} = u. \tag{9.11}$$

Applying the same method (or merely changing variables) leads to the invariant moving coframe

$$\zeta_1 = e^{-y}\,\mathrm{d}x, \qquad \zeta_2 = \frac{u_y}{u_x}\,\mathrm{d}x + \mathrm{d}y, \qquad \zeta_3 = \mathrm{d}u.$$

The basic differential invariants are

$$u, \qquad I = u_y, \qquad J_1 = u_{yy}, \qquad J_2 = e^y (u_y u_{xy} - u_x u_{yy}),$$

* See [15] for more details on higher-order submanifolds, including an interpretation as 'nonreducible partially invariant solutions' to partial differential equations, cf. [41].

the latter two being obtained by applying the invariant differential operators

$$D_y, \qquad e^y(D_x - (u_x/u_y)D_y),$$

to I. This recovers Medolaghi's form [34; p. 249],

$$H\left(u, u_y, u_{yy}, e^y(u_y u_{xy} - u_x u_{yy})\right) = 0 \tag{9.12}$$

of Lie's equation (9.10). The pseudo-group (9.11) is the second of nine different pseudo-groups acting on a three-dimensional space that are isomorphic to the diffeomorphism pseudo-group $\mathcal{D}iff(1)$, as classified by Medolaghi [34; p. 242]. The other eight pseudo-groups can be handled by the same method, reproducing the differential invariants and invariant differential equations catalogued there.

EXAMPLE 9.3. Consider the infinite Lie pseudo-group

$$\bar{x} = f(x), \qquad \bar{y} = y f'(x) + g(x), \qquad \bar{u} = u + \frac{f''(x)y + g'(x)}{f'(x)}, \tag{9.13}$$

acting on the space $M \simeq \mathbb{R}^3$ with coordinates (x, y, u). Here $f(x)$ and $g(x)$ are arbitrary smooth functions of a single variable x. The case $g \equiv 0$ corresponds to the third of Medolaghi's pseudo-groups [34]; the present generalization was introduced by J. Pohjanpelto (personal communication). The pseudo-group transformations can be characterized in terms of an involutive system of invariant 1-forms on a rank five bundle $\mathcal{B} \to M$, with coordinates $(x, y, u, \alpha, \beta, \gamma, \delta, \varepsilon)$. These can be found by a similar method to that used in Example 9.1:

$$\begin{aligned} &\zeta_1 = -\alpha\,\mathrm{d}x, \qquad \zeta_4 = \frac{\mathrm{d}\alpha}{\alpha} - \frac{\gamma}{\alpha}\,\mathrm{d}x, \\ &\zeta_2 = -\alpha\,\mathrm{d}y + u\alpha\,\mathrm{d}x, \qquad \zeta_5 = \frac{\mathrm{d}\beta}{\alpha} + \frac{u}{\alpha}\,\mathrm{d}\gamma - \frac{\delta - u\varepsilon}{\alpha}\,\mathrm{d}x - \frac{\varepsilon}{\alpha}\,\mathrm{d}y, \\ &\zeta_3 = -\mathrm{d}u - \beta\,\mathrm{d}x - \gamma\,\mathrm{d}y, \qquad \zeta_6 = \frac{\mathrm{d}\gamma}{\alpha} - \frac{\varepsilon}{\alpha}\,\mathrm{d}x. \end{aligned} \tag{9.14}$$

It is easy to check that a local diffeomorphism $\Psi: \mathcal{B} \to \mathcal{B}$ satisfies $\Psi^*\zeta_i = \zeta_i$, $i = 1, \ldots, 6$, if and only if it is a bundle map whose projection $\psi: M \to M$ has the form (9.13).

We now consider the equivalence problem for surfaces $u = u(x, y)$ under the pseudo-group (9.13). In order to invariantly normalize the bundle parameters, we replace $\mathrm{d}u$ by its horizontal component $u_x\,\mathrm{d}x + u_y\,\mathrm{d}y$, which leads to the linear relation

$$\eta_3 = J_1\,\eta_1 + J_2\,\eta_2,$$

among the horizontal components η_i of ζ_i. The lifted invariants are

$$J_1 = \frac{u_x + \beta + u(u_y + \gamma)}{\alpha}, \qquad J_2 = \frac{u_y + \gamma}{\alpha}.$$

Both J_1 and J_2 can be normalized to zero by choosing $\beta = -u_x$ and $\gamma = -u_y$, which defines the first-order moving frame. Substituting these values in the last two moving coframe forms yields

$$\begin{aligned}\eta_5 &= -\frac{u_{xx}\,\mathrm{d}x + u_{xy}\,\mathrm{d}y}{\alpha} - \frac{u(u_{xy}\,\mathrm{d}x + u_{yy}\,\mathrm{d}y)}{\alpha} - \frac{\delta - u\varepsilon}{\alpha}\,\mathrm{d}x - \frac{\varepsilon}{\alpha}\,\mathrm{d}y \\ &= \frac{u_{xx} + 2uu_{xy} + u^2 u_{yy} + \delta}{\alpha^2}\,\eta_1 - \frac{u_{xy} + uu_{yy} + \varepsilon}{\alpha^2}\,\eta_2, \\ \eta_6 &= -\frac{(u_{xy} + \varepsilon)\,\mathrm{d}x + u_{yy}\,\mathrm{d}y}{\alpha} = -\frac{u_{xy} + uu_{yy} + \varepsilon}{\alpha^2}\eta_1 + \frac{u_{yy}}{\alpha^2}\eta_2.\end{aligned}$$

We can normalize the coefficients of η_1, η_2 in both formulae by choosing

$$\alpha = \sqrt{u_{yy}}, \qquad \varepsilon = -u_{xy} - uu_{yy}, \qquad \delta = -u_{xx} - 2uu_{xy} - u^2 u_{yy},$$

which produces the second-order moving frame, given by

$$\begin{aligned}&\alpha = \sqrt{u_{yy}}, \qquad \beta = -u_x, \qquad \gamma = -u_y, \\ &\delta = -u_{xx} - 2uu_{xy} - u^2 u_{yy}, \qquad \varepsilon = -u_{xy} - uu_{yy}.\end{aligned}$$

Finally, substituting into the last moving coframe form leads to $\eta_4 = -I_1\eta_1 - I_2\eta_2$, where

$$I_1 = \frac{uu_{yyy} + u_{xyy} + 2u_y u_{yy}}{2u_{yy}^{3/2}}, \qquad I_2 = \frac{u_{yyy}}{2u_{yy}^{3/2}}, \tag{9.15}$$

are the principal differential invariants of the pseudo-group. The fundamental invariant horizontal 1-forms are

$$\eta_1 = -\sqrt{u_{yy}}\,\mathrm{d}x, \qquad \eta_2 = -\sqrt{u_{yy}}\,(\mathrm{d}y - u\,\mathrm{d}x),$$

so that the invariant total differential operators are

$$\mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}}\,(D_x + uD_y), \qquad \mathcal{D}_1 = \frac{1}{\sqrt{u_{yy}}}\,D_y.$$

As above, these can be applied to the basic differential invariants (9.15) to generate all higher-order differential invariants.

EXAMPLE 9.4. In this example, we show how the well-known equivalence problem of characterizing second-order ordinary differential equations under the pseudo-group of fiber-preserving transformations, cf. [22, 38], can be recast into the moving frame formulation, and thereby solved by our moving coframe techniques. This example indicates a general procedure for reformulating all Cartan-type equivalence problems, [11, 16, 38], as moving frame equivalence problems under a suitable infinite-dimensional Lie pseudo-group.

We consider the trivial bundle $M \simeq \mathbb{R} \times \mathbb{R}$, with coordinates x, u. Let $\mathcal{G}$ denote the pseudo-group of fiber-preserving transformations, i.e., bundle maps

$$\bar{x} = \varphi(x), \qquad \bar{u} = \psi(x, u). \tag{9.16}$$

We let $\mathcal{G}^{(2)}$ denote the associated second prolongation acting on J^2, cf. [38]. A (regular) second-order differential equation

$$\Delta(x, u, u_x, u_{xx}) = 0 \tag{9.17}$$

can be identified with a hypersurface $\mathcal{S}_\Delta \subset \mathrm{J}^2$. Two such second-order ordinary differential equations are equivalent if and only if their associated surfaces are mapped to each other,

$$g^{(2)}(\mathcal{S}_{\bar{\Delta}}) = \mathcal{S}_\Delta, \tag{9.18}$$

by a prolonged fiber-preserving transformation $g^{(2)} \in \mathcal{G}^{(2)}$.

In order to use the method of moving frames we need the structure equations of the pseudo-group $\mathcal{G}^{(2)}$. These can be found by the Cartan prolongation algorithm, [11, 16, 38], leading to

$$\begin{aligned}
\mathrm{d}\zeta_1 &= \omega_1 \wedge \zeta_1,\\
\mathrm{d}\zeta_2 &= \omega_2 \wedge \zeta_2 - \zeta_3 \wedge \zeta_1,\\
\mathrm{d}\zeta_3 &= (\omega_2 - \omega_1) \wedge \zeta_3 + \omega_3 \wedge \zeta_2 - \zeta_4 \wedge \zeta_1,\\
\mathrm{d}\zeta_4 &= (\omega_2 - 2\omega_1) \wedge \zeta_4 + \omega_4 \wedge \zeta_1 + \omega_5 \wedge \zeta_2 + \omega_6 \wedge \zeta_3,\\
\mathrm{d}\omega_1 &= (\omega_6 - 2\omega_3) \wedge \zeta_1,\\
\mathrm{d}\omega_2 &= -\pi_2 \wedge \zeta_2 - \omega_3 \wedge \zeta_1,\\
\mathrm{d}\omega_3 &= -\pi_1 \wedge \zeta_2 - \pi_2 \wedge \zeta_3 + \omega_3 \wedge \omega_1 - \omega_5 \wedge \zeta_1,\\
\mathrm{d}\omega_4 &= -\pi_3 \wedge \zeta_1 - \pi_4 \wedge \zeta_3 - \pi_5 \wedge \zeta_2 - 3\omega_1 \wedge \omega_4 -\\
&\quad - \omega_4 \wedge \omega_2 + 3(\omega_3 - \omega_6) \wedge \zeta_4,\\
\mathrm{d}\omega_5 &= -2\pi_1 \wedge \zeta_3 - \pi_2 \wedge \zeta_4 - \pi_5 \wedge \zeta_1 - \pi_6 \wedge \zeta_2 + 2\omega_5 \wedge \omega_1 - \omega_3 \wedge \omega_6,\\
\mathrm{d}\omega_6 &= -2\pi_1 \wedge \zeta_2 - 2\pi_2 \wedge \zeta_3 - \pi_4 \wedge \zeta_1 - \omega_1 \wedge \omega_6 + \omega_5 \wedge \zeta_1.
\end{aligned}$$

The Cartan characters are $s_1 = 5$ and $s_2 = 1$, the kernel dimension is 7, hence this differential system is involutive. The parametric values of the 1-forms $\boldsymbol{\zeta}$, $\boldsymbol{\omega}$, are determined by introducing the group transformation matrix

$$S = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_3\alpha_2\alpha_1^{-1} & \alpha_2\alpha_1^{-1} & 0 \\ \alpha_4\alpha_2\alpha_1^{-2} & \alpha_5\alpha_2\alpha_1^{-2} & \alpha_6\alpha_2\alpha_1^{-2} & \alpha_2\alpha_1^{-2} \end{pmatrix}, \tag{9.19}$$

where α_i, β_i are the fiber coordinates on the prolonged bundle. Equation (9.19) parametrizes the structure group corresponding to the action of the fiber-preserving pseudo-group on J^2; see, for instance, [38; p. 398] for the corresponding group on J^1. The first set of lifted forms are

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_3\alpha_2\alpha_1^{-1} & \alpha_2\alpha_1^{-1} & 0 \\ \alpha_4\alpha_2\alpha_1^{-2} & \alpha_5\alpha_2\alpha_1^{-2} & \alpha_6\alpha_2\alpha_1^{-2} & \alpha_2{\alpha_1}^{-2} \end{pmatrix} \begin{pmatrix} dx \\ du - u_x\, dx \\ du_x - u_{xx}\, dx \\ du_{xx} \end{pmatrix}.$$

Furthermore,

$$\begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & \omega_3 & \omega_2 - \omega_1 & 0 \\ \omega_6 & \omega_5 & \omega_4 & \omega_2 - 2\omega_1 \end{pmatrix} = S^{-1}\, dS + \Omega,$$

where Ω represents the absorbed torsion terms. The explicit formulas are

$$\omega_1 = \frac{d\alpha_1}{\alpha_1} + \frac{\alpha_6 - 2\alpha_3}{\alpha_2}\,\zeta_1,$$

$$\omega_2 = \frac{d\alpha_2}{\alpha_2} + \frac{\alpha_3}{\alpha_1}\,\zeta_1 - \beta_2\zeta_2,$$

$$\omega_3 = \frac{d\alpha_3}{\alpha_2} + \frac{\alpha_3\alpha_6 - \alpha_5 - \alpha_3^2}{\alpha_1^2}\zeta_1 - \beta_1\zeta_2 - \beta_2\zeta_3,$$

$$\begin{aligned} \omega_4 \;=\; & \frac{\alpha_2}{\alpha_1^3}\, d\alpha_4 - \beta_3\zeta_1 + \frac{\alpha_6\alpha_5 + \alpha_3\alpha_5 - \alpha_3\alpha_6^2 - \alpha_2^2\beta_5}{\alpha_2^2}\zeta_2 + \\ & + \frac{\alpha_6^2 - \alpha_5 - \beta_4\alpha_1^2}{\alpha_1^2}\zeta_3 + 3\frac{\alpha_3 - \alpha_6}{\alpha_1}\zeta_4, \end{aligned}$$

$$\omega_5 = \frac{d\alpha_5}{\alpha_1^2} - \frac{\alpha_3}{\alpha_1^2}\, d\alpha_4 + \beta_6\zeta_2 - 2\beta_1\zeta_3 - \beta_2\zeta_4 - \beta_5\omega_1,$$

$$\omega_6 = \frac{d\alpha_6}{\alpha_1} - \beta_4\zeta_1 - 2\beta_1\zeta_2 - 2\beta_2\zeta_3.$$

We now assume, for simplicity, that the second-order ordinary differential equation (9.17) is given by the graph of a section $\sigma : J^1 \to J^2$; this is equivalent to assuming that the equation is normal, and solved

$$u_{xx} = Q(x, u, u_x), \qquad (9.20)$$

for its highest-order derivative. (However, the moving frame method could be applied without this assumption; doing the corresponding problem for nonnormal equations using the Cartan equivalence approach would be harder.) Pulling back the Maurer–Cartan forms under the map σ amounts to substituting for u_{xx} according to (9.20) wherever it occurs. We denote the pull-back of ζ_i by η_i and of ω_i by ϖ_i. To apply the moving frame method, we look for dependencies among the resulting 1-forms. The first of these is

$$\eta_4 = J_1\eta_1 + J_2\eta_2 + J_3\eta_3,$$

where

$$\begin{aligned} J_1 &= \frac{\alpha_2}{\alpha_1^3}\left(\alpha_4 + \frac{\mathrm{d}Q}{\mathrm{d}x}\right),\\ J_2 &= \frac{1}{\alpha_1^2}\left(\alpha_5 - \alpha_6\alpha_3 + \frac{\partial Q}{\partial u} - \alpha_3\frac{\partial Q}{\partial u_x}\right),\\ J_3 &= \frac{1}{\alpha_1}\left(\alpha_6 + \frac{\partial Q}{\partial u_x}\right). \end{aligned} \tag{9.21}$$

Here

$$\frac{\mathrm{d}Q}{\mathrm{d}x} = \frac{\partial Q}{\partial x} + u_x\frac{\partial Q}{\partial u} + Q\frac{\partial Q}{\partial u_x}$$

denotes the total derivative of Q, restricted to the equation manifold (9.20). The lifted invariants (9.21) can all be translated to zero by choosing

$$\alpha_4 = -\frac{\mathrm{d}Q}{\mathrm{d}x}, \qquad \alpha_5 = -\frac{\partial Q}{\partial u}, \qquad \alpha_6 = -\frac{\partial Q}{\partial u_x}.$$

We then pull-back the forms ω_5, ω_6, leading to

$$\left.\begin{aligned} \varpi_5 &\equiv -\left(2\beta_1 + \frac{1}{\alpha_1\alpha_2}Q_{uu_x} - \frac{\alpha_3}{\alpha_2^2}Q_{u_xu_x}\right)\eta_3,\\ \varpi_6 &\equiv -\left(2\beta_2 + \frac{Q_{u_xu_x}}{\alpha_2}\right)\eta_3, \end{aligned}\right\} \quad \mathrm{mod}\{\eta_1, \eta_2\}.$$

Translating the coefficients of η_3 to zero in ϖ_5 and ϖ_6 gives

$$\beta_1 = -\frac{1}{2\alpha_1\alpha_2}Q_{uu_x} + \frac{\alpha_3}{2\alpha_2^2}Q_{u_xu_x}, \qquad \beta_2 = -\frac{1}{2\alpha_2}Q_{u_xu_x},$$

which then leads to the pulled-back forms

$$\varpi_1 = \frac{\mathrm{d}\alpha_1}{\alpha_1} - \left(\frac{Q_{u_x} + 2\alpha_3}{\alpha_1}\right)\eta_1,$$

$$
\begin{aligned}
\varpi_2 &= \frac{\mathrm{d}\alpha_2}{\alpha_2} - \frac{\alpha_3}{\alpha_1}\eta_1 + \left(\frac{1}{2\alpha_2}Q_{u_x u_x}\right)\eta_2,\\
\varpi_3 &= \frac{d\alpha_3}{\alpha_1} + \left(\frac{Q_u - \alpha_3^2 - \alpha_3 Q_{u_x}}{\alpha_1^2}\right)\eta_1 + \\
&\quad + \frac{1}{2\alpha_1\alpha_2}\left(Q_{uu_x} - \alpha_3 Q_{u_x u_x}\right)\eta_2 + \left(\frac{1}{2\alpha_2}Q_{u_x u_x}\right)\eta_3.
\end{aligned}
$$

At this stage, we have reproduced the system of 1-forms obtained via the Cartan equivalence method in [22; p. 394]. Further discussion of this example can be found in this reference.

10. Conclusions

In this paper we have described a systematic procedure for determining moving frames and invariant differential forms for very general Lie group and Lie pseudo-group actions. The moving frame and moving coframe can be used to directly determine a complete system of fundamental differential invariants and invariant differential operators for the given transformation group. These, in turn, have immediate applications, including the solution to equivalence problems, classification of symmetry groups, rigidity theorems, construction of invariant equations and variational principles, and so on. As we have demonstrated, the method not only readily reproduces all of the standard examples of moving frames known in the literature, but is also in a form that can immediately be applied to a host of new and interesting group actions, including intransitive and ineffective actions, infinite-dimensional Lie pseudo-groups, joint actions, and so on. The theoretical foundations of our method will be presented in the second paper in this series, [15]. Additional applications – to differential invariants, to the theory of Lie pseudo-groups, to automorphic systems, and to computer vision – will be the subject of subsequent papers in this series. Some extensions that we intend to investigate include:

(1) The moving coframe method, as described in this paper, parallels the explicit ‘parametric’ approach to the solution of Cartan equivalence problems. Gardner [16], showed how, in such situations, one could perform an ‘intrinsic’ computation, based on the infinitesimal group action on the torsion coefficients, and thereby determine the general structure of the solution. An interesting question is whether one can implement an intrinsic version of the moving coframe algorithm.

(2) In [25], an inductive approach to complicated equivalence problems, based on the solution to a simpler problem based on a subgroup of the full structure group, was proposed; see also [38]. In his thesis, Lisle [33] successfully uses a similar idea in his ‘frame method’ for symmetry classification of

partial differential equations. The inductive approach not only simplifies the computations, but also provides direct correspondences between the invariants of the two problems. Is there a similar inductive version of the moving coframe method? For example, does the computation of the moving frame for curves in the plane under, say, the equi-affine group help simplify the corresponding projective computation, thereby expressing the projective arc length and curvature directly in terms of its equiaffine counterparts?

(3) In [5], a new scheme for generating invariant numerical approximations to differential invariants based on the use of joint invariants was proposed, and illustrated in the planar Euclidean and equi-affine cases. The computation of joint differential invariants using the moving coframe method strongly indicates that it could be applied to the general problem of invariant numerical formulae for more complicated transformation groups. In particular, determining how joint invariants converge to differential invariants as the points coalesce would be of great importance.

(4) An immediate and important application of the moving method would be to the classification of the differential invariants associated many of the transformation groups arising in physics. As remarked above, to date such classifications have not been completed, even for some of the most fundamental groups of physical importance.

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Moving Coframes: II. Regularization and Theoretical Foundations

MARK FELS* and PETER J. OLVER**
School of Mathematics, University of Minnesota, Minneapolis, MN 55455, U.S.A.
e-mail: fels@math.umn.edu olver@ima.umn.edu

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Abstract. The primary goal of this paper is to provide a rigorous theoretical justification of Cartan's method of moving frames for arbitrary finite-dimensional Lie group actions on manifolds. The general theorems are based a new regularized version of the moving frame algorithm, which is of both theoretical and practical use. Applications include a new approach to the construction and classification of differential invariants and invariant differential operators on jet bundles, as well as equivalence, symmetry, and rigidity theorems for submanifolds under general transformation groups. The method also leads to complete classifications of generating systems of differential invariants, explicit commutation formulae for the associated invariant differential operators, and a general classification theorem for syzygies of the higher order differentiated differential invariants. A variety of illustrative examples demonstrate how the method can be directly applied to practical problems arising in geometry, invariant theory, and differential equations.

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Key words: moving frame, Lie group, jet bundle, prolongation, differential invariant, equivalence, symmetry, rigidity, syzygy.

1. Introduction

This paper is the second in a series devoted to the analysis and applications of the method of moving frames and its generalizations. In the first paper [9], we introduced the method of moving coframes, which can be used to practically compute moving frames and differential invariants, and is applicable to finite-dimensional Lie transformation groups as well as infinite-dimensional pseudo-group actions. In this paper, we introduce a second method, called regularization, that not only provides, in a simple manner, the theoretical justification for the method of moving frames in the case of finite-dimensional Lie group actions, but also gives an alternative, practical approach to their construction. The regularized method successfully bypasses many of the complications inherent in traditional approaches by completely avoiding the usual process of normalization during the general computation. In this way, the issues of branching and regularity do not arise. Once a moving

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frame and coframe, along with the complete system of invariants, are constructed in the regularized framework, one can easily restrict these invariants to particular classes of submanifolds, producing (in nonsingular cases) the standard moving frame. Perhaps Griffiths is the closest in spirit to our guiding philosophy; we fully agree with his statement [12, p. 777], that 'The effective use of frames ... goes far beyond the notion that 'frames are essentially the same as studying connections in the principal bundle of the tangent bundle'.' Indeed, by de-emphasizing the group theoretical basis for the method, which, in the past, has hindered the theoretical foundations from covering all the situations to which the practical algorithm could be applied, our formulation of the framework goes beyond what Griffiths envisioned, and successfully realizes Cartan's original vision [5, 7]. The regularized method can be readily used to compute all classical, known examples of moving frames, as well as a vast array of other, nontraditional Lie group actions. Indeed, the method is not restricted to transitive group actions on homogeneous spaces, although these form an important subclass of transformation groups that can be handled by our general procedure.

In general, given a finite-dimensional Lie group G acting on a manifold M, a moving frame (of order zero) is defined as a G-equivariant map $\rho\colon M \to G$. Moving frames on submanifolds $N \subset M$ are then obtained by restriction. This general definition appears in Griffiths [12], Green [11], and Jensen [14], and can be readily reconciled with classical geometrical constructions [9]. It is not hard to see that an order zero moving frame can only exist when the group action is free and regular. Consequently, the first part of this paper will be devoted to developing the theory of moving frames in the simple context of free group actions on manifolds. We show how a moving frame and a complete system of invariants can be constructed via the process of normalization. Normalization amounts to choosing a cross-section $K \subset M$ to the group orbits, and computing the group element $g = \rho(z)$ which maps a point $z \in M$ in the manifold to the chosen cross-section, so $g \cdot z \in K$. The resulting map $\rho\colon z \mapsto g$ from the manifold to the group is the moving frame. With this data in hand, the group action can be characterized as the local diffeomorphisms which preserve a system of invariant functions and one-forms that are prescribed by the choice of cross-section and the pull-back of the Maurer–Cartan forms on the group via the moving frame. By restricting the invariant functions and one-forms to a submanifold, the solution to the basic congruence and symmetry problems follow directly from Cartan's solution to the general equivalence problem for coframes [8, 18]. That is, the invariants and the derived invariants of a submanifold serve to parameterize a classifying manifold that uniquely characterizes the equivalence class and symmetries of the submanifold under the action of the group.

If the prescribed group action is not free on M, then an order zero moving frame cannot be determined. The strategy then is to prolong the group action to the jet bundles $\mathrm{J}^n = \mathrm{J}^n(M, p)$ of n-jets of p-dimensional submanifolds of the underlying manifold M. Assuming that the group G acts effectively on subsets* then the prolonged transformation group will act locally freely on an open subset of J^n for n sufficiently large and, hence, one can use the moving frame construction described in the previous paragraph to determine a moving frame of order n for regular submanifolds. In general, the invariants and derived invariants associated with such a moving frame can be identified with a complete system of nth order differential invariants for the transformation group. Thus, the congruence and symmetry theorems for regular submanifolds are easily restated in terms of differential invariants and their associated classifying manifold. Moreover, our methods have the widest range of generality possible; by using sufficiently high order jets, we are able to establish moving frames for all submanifolds except those which are 'totally singular'. The latter can be geometrically characterized as submanifolds whose isotropy subgroup does not act freely thereon, and hence cannot be endowed with fully determined moving frames. For example, in equi-affine geometry, the straight lines are totally singular, and do not possess equi-affine moving frames. In this manner, the regularized procedure also sheds light on a comment of Weyl [27, p. 600], on the desirability of investigating 'special classes of manifolds by imposing conditions on the invariants', using the example of minimal curves in Euclidean geometry where the usual normalization procedure breaks down. A related idea of I. Anderson (personal communication) involves the regularization of differential invariants for transformation groups by introducing additional parametric coordinates in order to avoid 'phantom' singularities in jet space. The regularized moving frame method provides a general construction that allows one to rigorously implement the ideas of Weyl and Anderson in practical situations.

A key idea that underlies our theory of regularization is to replace any complicated group action on a manifold by a 'lifted action' of the group on the trivial principal bundles $\mathcal{B}^{(n)} = G \times \mathrm{J}^n$ over the original manifold and its associated jet spaces. Once the action of the group is free on a particular jet space, the moving frame map is nothing but an equivariant section of the principal bundle $\mathcal{B}^{(n)}$ under the lifted action. The equivariant section so obtained allows one to pull back invariant objects on the principal bundle to the base. Fortunately, all the invariant objects on the principal bundle are trivial to construct, and so the particularities of the construction are all embodied in the chosen moving frame section, and can thereby be systematically analyzed.

The regularization approach to moving frames provides new, effective tools for understanding the geometry of submanifolds and their jets under a transformation group. Applications include a new and more general proof of the fundamental theorem on classification of differential invariants, a general classification theorem for syzygies of differential invariants, as well as new explicit commutation formulae for the associated invariant differential operators. We demonstrate a simple but striking generalization of a 'replacement theorem' due to T. Y. Thomas [24]. Two

* This condition is very mild. See Section 2 for the precise definition, and a demonstration that it always holds, without loss of generality, in the analytic category.

types of general rigidity theorems, uniquely characterizing congruent submanifolds by finite order jets, are proved, thereby extending known results for submanifolds of homogeneous spaces. We also give a new basis for Ovsiannikov's theory of partially invariant solutions of partial differential equations [22]. All of our theoretical results are provided in a form that can be applied to practical examples, which we illustrate with several explicit examples of independent interest in Section 5. This wide range of both theoretical and practical applications clearly demonstrates the power of our approach to the general theory of moving frames.

2. Lie Transformation Groups

Let us begin by collecting some basic terminology associated with finite-dimensional transformation groups. See [18] for details. Throughout this paper, G will denote an r-dimensional Lie group acting smoothly on an m-dimensional manifold M.

DEFINITION 2.1. The *isotropy subgroup* of a subset $S \subset M$ is

$$G_S = \{g \in G \mid g \cdot S = S\}. \tag{2.1}$$

The *global isotropy subgroup* is the subgroup

$$G_S^* = \bigcap_{x \in S} G_x = \{g \in G \mid g \cdot s = s \text{ for all } s \in S\}$$

consisting of those group elements which fix *all* points in S.

DEFINITION 2.2. The group G acts

(i) *freely* if $G_z = \{e\}$ for all $z \in M$,
(ii) *locally freely* if G_z is a discrete subgroup of G for all $z \in M$,
(iii) *effectively* if $G_M^* = \{e\}$,
(iv) *effectively on subsets* if $G_U^* = \{e\}$ for every open $U \subset M$,
(v) *locally effectively* if G_M^* is a discrete subgroup of G,
(vi) *locally effectively on subsets* if G_U^* is a discrete subgroup of G for every open $U \subset M$.

If the group G does not act effectively, one can, without any loss of generality, replace G by the effectively acting quotient group G/G_M^*, which acts in essentially the same manner as G does, cf. [18]. Clearly, if G acts effectively on subsets, then G acts effectively. Analytic continuation demonstrates that the converse is true in the analytic category. However, it does not hold for more general smooth actions as the following elementary example shows.

EXAMPLE 2.3. Let $h(x)$ be any C^∞ function such that $h(x) > 0$ for $x > 0$, but $h(x) = 0$ for $x \leqslant 0$. Let $G \simeq \mathbb{R}^2$ be the two-parameter Abelian transformation group acting on $M = \mathbb{R}^2$ via $(x, u) \mapsto (x, u + ah(x) + bh(-x))$, where $(a, b) \in G$ and $(x, u) \in M$. Then G acts effectively on M, but not effectively on any open subset that is contained in either the right or left half plane.

Since they do not arise in usual applications, we will not attempt to analyze pathological smooth actions which are effective but not subset effective. Thus we shall, without significant loss of generality, only consider transformation groups that act effectively on subsets.

DEFINITION 2.4. A group G acts *semi-regularly* on M if all its orbits have the same dimension. A semi-regular group action is *regular* if, in addition, each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof.

PROPOSITION 2.5. *An r-dimensional Lie group G acts locally freely on M if and only if its orbits all have dimension r.*

DEFINITION 2.6. Suppose G acts semi-regularly on the m-dimensional manifold M with s-dimensional orbits. A (local) *cross-section* is a $(m - s)$-dimensional submanifold $K \subset M$ such that K intersects each orbit transversally. The cross-section is *regular* if K intersects each orbit at most once.

If G acts semi-regularly, then the Implicit Function Theorem guarantees the existence of local cross-sections at any point of M. Regular actions admit regular local cross-sections.

EXAMPLE 2.7. The following simple construction, based on the Frobenius Theorem, cf. [18], is of fundamental importance for the theoretical justification of the method of moving frames. Suppose G acts freely and regularly on M. Then we can introduce *flat local coordinates*

$$z = (x, y) = (x^1, \ldots, x^r, y^1, \ldots, y^{m-r}), \quad x \in G, \ y \in Y, \tag{2.2}$$

that locally identify M with a subset of the Cartesian product $G \times Y$, with $Y \simeq \mathbb{R}^{m-r}$, and such that the action of G reduces to the trivial left action $g \cdot z = (g \cdot x, y)$. The y coordinates provide a complete system of functionally independent invariants for the group action. In these coordinates, a general cross-section is given by the graph $K = \{(a(y), y)\}$ of a smooth map $a\colon Y \to G$. When we use flat coordinates, we shall always assume, without loss of generality, that the identity cross-section $\{e\} \times Y$, i.e., when $a(y) \equiv e$, belongs to the flat coordinate chart.

Remark. In practice, of course, the determination of the flat coordinates for a given transformation group action may be extremely difficult. A significant achievement of the method of moving frames is that it allows one to compute invariants without having to find the flat coordinates, or integrate any differential equations.

Throughout this paper, we shall let $\mathfrak{g}$ denote the *right* Lie algebra of G consisting of right-invariant vector fields on G. The map $\psi\colon \mathbf{v} \mapsto \widehat{\mathbf{v}}$ that associates a Lie algebra element $\mathbf{v} \in \mathfrak{g}$ to the corresponding infinitesimal generator $\widehat{\mathbf{v}} = \psi(\mathbf{v})$ of the associated one-parameter subgroup forms a Lie algebra homeomorphism from $\mathfrak{g}$ to the space of vector fields on M. The kernel of ψ coincides with the Lie algebra of the global isotropy subgroup G_M^*, thereby identifying the Lie algebra of infinitesimal generators $\widehat{\mathfrak{g}} = \psi(\mathfrak{g})$ with the quotient Lie algebra of the effectively acting quotient group G/G_M^*. In particular, G acts locally effectively if and only if $\ker \psi = \{0\}$.

3. Regularization

Our approach to the theory of moving frames is based on the following simple but remarkably powerful device. In general, any complicated transformation group action can be 'regularized' by lifting it to a suitable bundle sitting over the original manifold. The construction is reminiscent of the regularization procedure based on universal bundles used to compute equivariant cohomology, cf. [3, 13, §4.11], although our method is considerably simpler in that we only require finite-dimensional bundles.

Let G be a smooth transformation group acting on a manifold M. Let $\mathcal{B} = G \times M$ denote the trivial left* principal G bundle over M.

DEFINITION 3.1. The *left regularization* of the action of G on M is the diagonal action of G on $\mathcal{B} = G \times M$ provided by the maps

$$\widehat{L}_g(h, z) = \widehat{L}(g, (h, z)) = (g \cdot h, g \cdot z), \quad g \in G,\ (h, z) \in \mathcal{B}. \tag{3.1}$$

The *right regularization* of G is given by

$$\widehat{R}_g(h, z) = \widehat{R}(g, (h, z)) = (h \cdot g^{-1}, g \cdot z), \quad g \in G,\ (h, z) \in \mathcal{B}. \tag{3.2}$$

We will also refer to the regularized actions (3.1), (3.2), as the left or right *lifted action* of G since either projects back to the given action on M via the G equivariant projection $\pi_M\colon \mathcal{B} \to M$. In the sequel, the left (respectively right) regularization of a group action will lead to left (right) moving frames associated with submanifolds of M. The key, elementary result is that regularizing any group action immediately eliminates all singularities and irregularities, e.g., lower dimensional orbits, nonembedded orbits, etc. Moreover, the orbits of G in M are the projections of their lifted counterparts in $\mathcal{B}$; all of the lifted orbits have the same dimension as G itself.

THEOREM 3.2. *The right and left regularizations of any transformation group G define regular, free actions on the bundle $\mathcal{B} = G \times M$.*

* Modern treatments of principal bundles, e.g., [13, 23], tend to concentrate on right principal bundles. However, we find the left version more convenient for our purposes.

Thus, lifting the action of G on M to the bundle $\mathcal{B}$ has the effect of completely eliminating any irregularities appearing in the original action.

DEFINITION 3.3. A *lifted invariant* is a (locally defined) smooth function L: $\mathcal{B} \to N$ which is invariant with respect to the (either left or right) lifted action of G on $\mathcal{B}$.

Both regularized actions admit a complete system of globally defined, functionally independent lifted invariants.

DEFINITION 3.4. The *fundamental right lifted invariant* is the multiplication function w: $\mathcal{B} \to M$ given by

$$w = g \cdot z. \tag{3.3}$$

The *fundamental left lifted invariant* is the function $\widetilde{w}$: $\mathcal{B} \to M$ given by

$$\widetilde{w} = g^{-1} \cdot z. \tag{3.4}$$

From the point of view of invariants and moving frames, right regularization is the simpler of the two because its fundamental invariant does not require the computation of the inverse transformation g^{-1}. On the other hand, in the literature, most examples are constructed using the left regularization. Moreover, the final formulae for the moving frame are typically simpler if the left regularization is used. However, the theoretical and practical aspects of our regularized moving frame method underline the primacy of the right version. Therefore, from now on, the terms ‘regularization’ or ‘lift’ without qualification will always mean the *right* versions of these objects. All results will automatically have a left counterpart, typically found by applying the group inversion $g \mapsto g^{-1}$.

PROPOSITION 3.5. *The fundamental lifted invariant $w = g \cdot z$ is invariant with respect to the regularized action* (3.2) *of G on $\mathcal{B}$. Moreover, given $z \in M$, the corresponding level set $w^{-1}\{z\}$ coincides with the orbit of G through the point $(z, e) \in \mathcal{B}$.*

If we introduce local coordinates on M, then the components of w form a complete system of $m = \dim M$ functionally independent invariants on $\mathcal{B}$.

PROPOSITION 3.6. *Any lifted invariant L: $\mathcal{B} \to N$ can be locally written as a function of the fundamental lifted invariants, $L(g, z) = F[w(g, z)]$, so that $L = F \circ w$ for some F: $M \to N$.*

In particular, if $F(z)$ is any function on M, then we can produce a lifted invariant $F \circ w$ on $\mathcal{B}$ by replacing z by $w = g \cdot z$ in the formula for F. The ordinary invariants I: $M \to N$ of the group action are particular cases of lifted invariants, where we identify I with its composition $I \circ \pi_M$ with the standard projection. Therefore, Proposition 3.6 indicates that ordinary invariants are particular functional combinations of lifted invariants that happen to be independent of the group parameters. For such functions, a simple but striking 'replacement theorem' provides an explicit formula expressing an ordinary invariant in terms of the lifted invariants.

THEOREM 3.7. *If $I(z) = F(w(g,z)) = F(g \cdot z)$ is an ordinary invariant, then $F(z) = I(z)$.*

Proof. In other words, replacing z by w in the formula for the invariant does not change its value, i.e., $I(z) = I(w)$. To prove this result, we use the invariance of I and the fact that at the identity $g = e$, the lifted invariant reduces to $w = z$. □

EXAMPLE 3.8. Let $G = \mathrm{SO}(2)$ be the rotation group acting on $M = \mathbb{R}^2$ via

$$(x, u) \longmapsto (x \cos\theta - u \sin\theta, x \sin\theta + u \cos\theta). \tag{3.5}$$

The (right) regularized action on the cylinder $\mathcal{B} = \mathrm{SO}(2) \times \mathbb{R}^2$ is given by supplementing the planar transformation rules (3.5) with the group law $\phi \mapsto (\phi - \theta) \bmod 2\pi$. Note that the action on $\mathcal{B}$ is regular, so we have effectively replaced the singular orbit at the origin by a regular orbit $\{(0,0)\} \times \mathrm{SO}(2) \subset \mathcal{B}$. There are two fundamental right lifted invariants:

$$y = x \cos\phi - u \sin\phi, \qquad v = x \sin\phi + u \cos\phi. \tag{3.6}$$

Note that

$$r^2 = y^2 + v^2 = x^2 + u^2$$

is an invariant for the lifted action which reduces to the ordinary radial invariant for the action back on M. The fact that r has the same formula in terms of x, u as it does in y, v is a simple manifestation of the general Replacement Theorem 3.7.

A differential form ω on the principal bundle $\mathcal{B} = G \times M$ is (right) G-invariant if it satisfies $(\widehat{R}_g)^* \omega = \omega$ for every $g \in G$. Of particular importance are the (pulled-back) Maurer–Cartan forms associated with the Lie group G. We introduce a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ for the (right) Lie algebra $\mathfrak{g}$ of G. The corresponding dual basis $\boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}$ for the right-invariant differential forms on G are known as the *Maurer–Cartan forms*. We shall also use $\boldsymbol{\mu}$ to denote the corresponding Maurer–Cartan one-forms on $\mathcal{B}$, namely the pull-backs $(\pi_G)^* \boldsymbol{\mu}$ of the forms on G under the standard projection $\pi_G \colon \mathcal{B} \to G$. The Maurer–Cartan forms $\boldsymbol{\mu}$ on $\mathcal{B}$ are invariant under the right regularized action of G.

Since $\mathcal{B} = G \times M$ is a Cartesian product, its differential d naturally splits into a group and manifold components: $\mathrm{d} = \mathrm{d}_G + \mathrm{d}_M$. Moreover, since the regularized action (3.2) is a Cartesian product action, the splitting is G-invariant.

PROPOSITION 3.9. *If ω is any G-invariant differential form on $\mathcal{B}$, then both $\mathrm{d}_M \omega$ and $\mathrm{d}_G \omega$ are invariant forms. In particular, if L is any lifted invariant, then $\mathrm{d}_M L$ and $\mathrm{d}_G L$ are invariant one-forms on $\mathcal{B}$.*

In particular, the differential $\mathrm{d}w$ of the fundamental lifted invariant $w = g \cdot z$ will split into two sets of invariant one-forms on $\mathcal{B}$, namely $\mathrm{d}_M w = g \cdot \mathrm{d}z$ and the group component $\mathrm{d}_G w$. The notation $g \cdot \mathrm{d}z$ is meant suggestively; in terms of local coordinates $(z^1, \ldots, z^m)$ on M, the components of $g \cdot \mathrm{d}z$ are the pull-backs $g^* \mathrm{d}z^i$ of the coordinate differentials via the group transformation g. There is a beautiful explicit formula that expresses group components $\mathrm{d}_G w$ as invariant linear combinations of the Maurer–Cartan forms $\boldsymbol{\mu}$ on $\mathcal{B}$.

THEOREM 3.10. *Let G act on M. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ be a basis for the Lie algebra $\mathfrak{g}$, and let*

$$\widehat{\mathbf{v}}_\kappa = \sum_{i=1}^{m} f_\kappa^i(z) \frac{\partial}{\partial z^i}, \quad \kappa = 1, \ldots, r, \tag{3.7}$$

be the corresponding infinitesimal generators on M, written in local coordinates $z = (z^1, \ldots, z^m)$. Let $\boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}$ be the dual Maurer–Cartan forms, pulled back to $\mathcal{B}$. Let $w = (w^1, \ldots, w^m)$ be the components of the the fundamental lifted invariant $w = g \cdot z$, expressed in the same local coordinates. Then the group differential of the components of w are given by

$$\mathrm{d}_G w^i = \sum_{\kappa=1}^{r} f_\kappa^i(w)\, \mu^\kappa, \quad i = 1, \ldots, m. \tag{3.8}$$

In other words, the coefficients of the Maurer–Cartan forms in (3.8) *are the lifted invariant counterparts of the coefficients of the infinitesimal generators* (3.7)*, obtained by replacing z by the lifted invariant w.*

Proof. Let $\mathbf{v} \in \mathfrak{g}$ correspond to the infinitesimal generator $\widehat{\mathbf{v}}$ on M. For simplicity, we use the same notation for the corresponding vertical and horizontal vector fields on $\mathcal{B}$, which generate the actions* $(h, z) \mapsto (\exp(t\mathbf{v}) \cdot h, z)$ and $(h, z) \mapsto (h, \exp(t\widehat{\mathbf{v}}) \cdot z)$ respectively. (The infinitesimal generators of the left regularization (3.1), then, are the sums $\mathbf{v} + \widehat{\mathbf{v}}$ of these vector fields.) We then notice that

$$\mathbf{v}(w) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\big(\exp(t\mathbf{v}) \cdot g\big) \cdot z\right]\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(t\widehat{\mathbf{v}}) \cdot w\right]\Big|_{t=0}.$$

The latter expression is equal to the value of the vector field $\widehat{\mathbf{v}}$ at the point $w = g \cdot z$; therefore, in local coordinates,

$$\mathbf{v}_\kappa(w^i) = f_\kappa^i(w), \quad i = 1, \ldots, m, \ \kappa = 1, \ldots, r.$$

On the other hand, duality** of the Maurer–Cartan forms implies that

$$\mathrm{d}_G w^i = \sum_{\kappa=1}^{r} \mathbf{v}_\kappa(w^i)\, \mu^\kappa = \sum_{\kappa=1}^{r} f_\kappa^i(w)\, \mu^\kappa,$$

completing the proof. □

* Recall that the right-invariant vector fields generate the left action of G on itself.

** See Example 5.13 below for details.

Remark. Theorem 3.10 justifies the method for computing Maurer–Cartan forms directly from the group transformations introduced in Part I [9].

EXAMPLE 3.11. Return to the rotation group acting on $M = \mathbb{R}^2$ as in (3.5). Applying d_M and d_G to the lifted invariants (3.6) will produce four lifted invariant one-forms on $\mathcal{B}$. The manifold components are

$$\mathrm{d}_M y = (\cos\phi)\,\mathrm{d}x - (\sin\phi)\,\mathrm{d}u, \qquad \mathrm{d}_M v = (\sin\phi)\,\mathrm{d}x + (\cos\phi)\,\mathrm{d}u.$$

On the other hand, the group components can be written as invariant multiples of the Maurer–Cartan form $\mu = \mathrm{d}\phi$, namely

$$\begin{aligned} \mathrm{d}_G y &= -(x\sin\phi + u\cos\phi)\,\mathrm{d}\phi = -v\,\mathrm{d}\phi, \\ \mathrm{d}_G v &= (x\cos\phi - u\sin\phi)\,\mathrm{d}\phi = y\,\mathrm{d}\phi. \end{aligned}$$

Equation (3.8) implies that the coefficients $(-v, y)$ can be computed directly as the invariant counterparts of the coefficients $(-u, x)$ of the infinitesimal generator $\widehat{\mathbf{v}} = -u\partial_x + x\partial_u$.

Remark. A lifted invariant $L(g, z) = F(w)$ is independent of all group parameters and, hence, reduces to an ordinary invariant as in Theorem 3.7 if and only if $\mathrm{d}_G L = 0$. In view of (3.8), the equation $\mathrm{d}_G L(g, z) = \mathrm{d}_G F(w) = 0$ is equivalent to the usual Lie infinitesimal invariance conditions $\mathbf{v}_\kappa(F(z)) = 0$, $\kappa = 1, \ldots, r$, rewritten in terms of w instead of z.

4. Moving Frames

Let us now define moving frames in the context of a Lie group acting on a manifold. The justification for this definition appears in Part I [9], and is based on the earlier work of Green [11], Griffiths [12], and Jensen [14].

DEFINITION 4.1. Given a transformation group G acting on a manifold M, a *moving frame* is a smooth G-equivariant map

$$\rho\colon\ M \longrightarrow G. \tag{4.1}$$

In (4.1), we can use either the right or the left action of G on itself, and thus speak of right and left moving frames. As in the usual method of moving frames, we shall only be interested in their local existence and construction. Thus, we can relax our condition and only require local G-equivariance of the moving frame map, i.e., for group elements near the identity. There is an elementary correspondence between right and left moving frames.

LEMMA 4.2. *If $\widetilde{\rho}(z)$ is a left moving frame on M, then $\rho(z) = \widetilde{\rho}(z)^{-1}$ is a right moving frame.*

EXAMPLE 4.3. An important example is when G is a Lie group acting on itself, so $M = G$, by left multiplication $h \mapsto g \cdot h$. If $a \in G$ is any fixed element, then the map $\widetilde{\rho}(g) = g \cdot a$ clearly defines a (left) moving frame. Moreover, every (left) moving frame necessarily has this form, with $a = \widetilde{\rho}(e)$. Similarly, every right moving frame is provided by a map $\rho(g) = a \cdot g^{-1}$ for some fixed $a \in G$.

Not every group action admits a moving frame. The key condition is that the action be both free and regular.

THEOREM 4.4. *If G acts on M, then a moving frame exists in a neighborhood of a point $z \in M$ if and only if G acts freely and regularly near z.*

Proof. To see the necessity of freeness, suppose $z \in M$, and let $g \in G_z$ belong to its isotropy subgroup. Let $\widetilde{\rho}\colon M \to G$ be a left moving frame. Then, by left equivariance of $\widetilde{\rho}$,

$$\widetilde{\rho}(z) = \widetilde{\rho}(g \cdot z) = g \cdot \widetilde{\rho}(z).$$

Therefore $g = e$ and, hence, $G_z = \{e\}$ for all $z \in M$. To prove regularity, suppose that $z \in M$ and that there exist points $z_\kappa = g_\kappa \cdot z$ belonging to the orbit of z such that $z_\kappa \to z$ as $\kappa \to \infty$. Thus, by continuity,

$$\widetilde{\rho}(z_\kappa) = \widetilde{\rho}(g_\kappa \cdot z) = g_\kappa \cdot \widetilde{\rho}(z) \longrightarrow \widetilde{\rho}(z) \quad \text{as } \kappa \to \infty,$$

which implies that $g_\kappa \to e$ in G. This suffices to ensure regularity of the orbit through z.

To prove sufficiency, we use the flat local coordinates $z = (x, y) \in G \times Y$ introduced in Example 2.2. A general local cross-section $K \subset M$ is given by a graph $x = a(y)$. Then the map

$$\widetilde{\rho}(x, y) = x \cdot a(y) \tag{4.2}$$

is clearly G-equivariant under left multiplication on G and, hence, defines a left moving frame. Moreover, every left moving frame has this form, provided we define the cross-section via $a(y) = \widetilde{\rho}(e, y)$. □

Remark. If G acts only semi-regularly and/or locally freely*, then the preceding proof can be easily adapted to find a locally G-equivariant moving frame.

THEOREM 4.5. *If $\rho(z)$ is a right moving frame, then the components of the map $I\colon M \to M$ defined by $I(z) = \rho(z) \cdot z$ provide a complete system of invariants for the group.*

Proof. Using our flat local coordinates, Lemma 4.2 implies that the right moving frame corresponding to (4.2) is

$$\rho(z) = a(y)^{-1} \cdot x^{-1}, \quad z = (x, y). \tag{4.3}$$

* A (locally) free action is automatically semi-regular.

Therefore

$$\rho(z) \cdot z = (a(y)^{-1}, y) \in K. \tag{4.4}$$

In particular, the last $m - r$ components of (4.4) provide the invariants y, while the first r components are functions of the invariants. □

The proof of Theorem 4.4 shows that the determination of a moving frame is intimately connected to the process of choosing a cross-section to the group orbits. Example 4.3 is a particular case of this construction since a cross-section to a transitive group action is just a single point. Equation (4.4) shows that the group element $g = \rho(z)$ given by the right moving frame map can be geometrically characterized as the unique group transformation that moves the point z onto the cross-section K. Moreover, $I(z) = \rho(z) \cdot z$ is the point on the cross-section K that lies on the G orbit passing through z.

Remark. In fact, any map $\rho\colon M \to G$ that satisfies $I(z) = \rho(z) \cdot z \in K$ will produce invariants by a choice of local coordinates on K. The action of G need not be free and the map ρ need not be equivariant; moreover, the group can equally well be a pseudo-group.

Theorem 4.5 implies that if $J(z)$ is any other invariant function, then, locally, we can write $J(z) = H(I(z))$ in terms of the moving frame invariants I. As noted in the proof, the components of I are not necessarily functionally independent, but one can always locally choose a set of $m - r$ components which do provide a complete system of functionally independent invariants, or, equivalently, a system of local coordinates on the quotient manifold M/G.

An alternative way of understanding the moving frame construction presented above is to view the regularization of a group action as giving rise to the double fibration

$$\begin{array}{ccc} & G \times M & \\ {}^{\pi_M}\swarrow & & \searrow^{w} \\ M & & M \end{array} \tag{4.5}$$

of the regularized bundle $\mathcal{B}$ over M. Given a cross-section K to the G orbits, the set

$$\mathcal{L} = w^{-1}(K) \subset \mathcal{B} = G \times M$$

forms an m-dimensional submanifold of $\mathcal{B}$ that is invariant with respect to the lifted action of G on $\mathcal{B}$. Projection onto M defines a locally equivariant diffeomorphism $\pi_M\colon \mathcal{L} \xrightarrow{\sim} M$ and hence $\mathcal{L}$ is the graph of a local section $\sigma = (\pi_M \mid \mathcal{L})^{-1}$, called the *moving frame section*. It is not hard to see that σ defines the graph of the moving frame, so $\sigma(z) = (\rho(z), z)$ for $z \in M$, i.e.,

$$\rho = \pi_G \circ \sigma \colon M \longrightarrow G.$$

Since $\sigma\colon M \to \mathcal{L}$ is G-equivariant, any invariant object on $\mathcal{B}$ pulls-back, via σ, to an invariant object on M. In particular, the invariant $I(z) = \rho(z) \cdot z$ constructed in Theorem 4.5 is given by

$$I = \sigma^*(w) = w \circ \sigma \colon M \longrightarrow K.$$

As noted above, given any function $F\colon M \to \mathbb{R}$, the composition $F \circ w\colon \mathcal{B} \to \mathbb{R}$ defines a lifted invariant, $L(g, z) = F(g \cdot z)$. Moreover, pulling back L via the moving frame section $\sigma\colon M \to \mathcal{B}$, defines an ordinary invariant $J(z) = F(w(\sigma(z))) = F(\rho(z) \cdot z)$. Thus, a moving frame provides a natural way to construct invariants from arbitrary functions!

DEFINITION 4.6. The *invariantization* of a function $F\colon M \to N$ with respect to a moving frame $\rho\colon M \to G$ is the composition $J = F \circ w \circ \sigma = F \circ I$.

Invariantization *does* depend on the choice of moving frame. Geometrically, $J(z)$ equals the value of F at the point on the cross-section that lies on the G orbit through z. Theorem 3.7 says that if F itself is an invariant, then $F \circ w$ is independent of the group parameters, and hence $J = F$, i.e., the invariantization process leaves invariants unchanged. Thus, one can view invariantization as a projection operator from the space of functions to the space of invariants.

EXAMPLE 4.7. Consider the usual action (3.5) of SO(2), which is regular on $M = \mathbb{R}^2 \setminus \{0\}$. The positive u axis defines a cross-section $K = \{(0, v) \mid v > 0\}$ to the orbits. The map $g = \rho(x, u)\colon M \to K$ which rotates the point (x, u) to the point $(0, r) \in K$, where $r = \sqrt{x^2 + u^2}$, is clearly SO(2)-equivariant. The moving frame $\rho\colon M \to \mathrm{SO}(2)$ induced by this choice of cross-section is therefore given by the equivariant map $\phi = \tan^{-1}(x/u)$ that determines the the rotation angle needed to map (x, u) to K. The corresponding moving frame section $\sigma\colon M \to \mathcal{B} = \mathrm{SO}(2) \times M$ is given by $\sigma(x, u) = (\tan^{-1}(x/u), x, u)$. Pulling back the lifted invariants (3.6) produces the invariants $\sigma^* y = 0$, $\sigma^* v = r$. If $F(x, u)$ is any function, then $L(\phi, x, u) = F(y, v)$ is its lifted counterpart, and so its invariantization is the radial invariant $J = F(0, r)$. The reader should try computing other moving frames and the corresponding invariants by choosing other cross sections, e.g., $\{(v, v) \mid v > 0\}$, or $\{(v, v^2) \mid v > 0\}$.

Our construction is intimately tied to the Cartan procedure of normalization of group parameters, which is, traditionally, the basic process used in the practical construction of moving frames [5, 7]. Normalization can be interpreted as the restriction of the regularized group action to an invariant submanifold of the regularized bundle $\mathcal{B}$. In particular, when G acts freely on M, we can restrict to a local section of $\mathcal{B}$ and thereby uniquely specify all of the group parameters.

DEFINITION 4.8. A lifted invariant $L\colon \mathcal{B} \to N$ is *regular* provided its group differential $\mathrm{d}_G L$ has maximal rank $n = \dim N$ at every point in its domain of definition.

The essence of the normalization procedure that appears both in the method of moving frames, as well as the Cartan equivalence method, is captured by the following simple definition.

DEFINITION 4.9. A *normalization* of the regularized group action consists of its restriction to a nonempty level set $\mathcal{L}_c = L^{-1}\{c\}$ of a regular lifted invariant $L\colon \mathcal{B} \to N$.

Every level set of a lifted invariant forms a G-invariant submanifold of the regularized action. Note that the regularity assumption on the lifted invariant implies that the projection $\pi_M\colon \mathcal{L}_c \to M$ maps $\mathcal{L}_c$ onto an open subset of M. Thus, regularity ensures that the normalization does not introduce any dependencies among the z coordinates, since that would introduce unacceptable constraints on the original manifold M.

In local coordinates, if $L(g,z) = (L_1(g,z), \ldots, L_n(g,z))$ is a regular lifted invariant and $c = (c_1, \ldots, c_n) \in N$ belongs to the image of L, then the implicit function theorem says that we can (locally) solve the system of n equations

$$L_1(g,z) = c_1, \quad \ldots, \quad L_n(g,z) = c_n, \tag{4.6}$$

for n of the group parameters, say $\hat{g} = (g^1, \ldots, g^n)$, in terms of the remaining $r-n$ group parameters, which we denote by $h = (h^1, \ldots, h^{r-n}) = (g^{n+1}, \ldots, g^r)$, and the z coordinates:

$$g^1 = \gamma^1(h,z), \quad \ldots, \quad g^n = \gamma^n(h,z), \tag{4.7}$$

or, simply, $\hat{g} = \gamma(h,z)$. The coordinates h and z serve to parametrize the G-invariant level set $\mathcal{L}_c = L^{-1}\{c\}$. The remaining group parameters h can be interpreted as parametrizing the isotropy subgroup of the submanifold $\{z \mid L(e,z) = c\}$.

PROPOSITION 4.10. *If G acts freely and regularly on M, then we can completely normalize all of the group parameters by choosing a regular lifted invariant $L\colon \mathcal{B} \to N$ having maximal* rank $\mathrm{d}_G L = r = \dim N = \dim G$ *everywhere.*

DEFINITION 4.11. Let $K \subset M$ be a local cross-section to the G orbits. The *normalization equations* associated with K are the system of equations

$$w = g \cdot z = k, \qquad \text{where } k \in K. \tag{4.8}$$

Remark. The normalization Equations (4.8) are the same as the compatible lift equations discussed in Part I [9].

If we assume that G acts freely and K is a regular cross-section, then there is a unique solution $g = \rho(z)$ to the normalization equations, determining the right moving frame associated with K. More explicitly, if we choose the flat local coordinates $z = (x, y) \in G \times Y$ from Example 2.2, then the fundamental lifted invariant has the form $w = g \cdot z = (g \cdot x, y)$. Choosing a cross-section $x = a(y)$ reduces the normalization equations (4.8) to $g \cdot x = a(y)$, with G-equivariant solution $g = \rho(x, y) = a(y) \cdot x^{-1}$, which agrees with the right moving frame (4.3) after applying the group inversion to the cross-section map $a(y)$.

In practice, one constructs a 'standard' cross-section by solving the normalization equations in the following manner. Locally we choose r components of the fundamental lifted invariant $w = g \cdot z$, say $w^1, \ldots, w^r$, which satisfy the regularity condition

$$\frac{\partial(w^1, \ldots, w^r)}{\partial(g^1, \ldots, g^r)} \neq 0. \tag{4.9}$$

Solving the equations

$$w^1(g, z) = c_1, \quad \ldots, \quad w^r(g, z) = c_r, \tag{4.10}$$

where the constants $c_1, \ldots, c_r$ are chosen to lie in the range of the w's, leads to a complete system of normalizations $g = \rho(z)$ for the group parameters. The resulting map determines a moving frame, and corresponds to the local cross-section $K = \{z^1 = c_1, \ldots, z^r = c_r\}$. Furthermore, Theorem 4.5 implies that if we substitute the normalization formulae $g = \rho(z)$ into the remaining lifted invariants $\widetilde{w} = (w^{r+1}, \ldots, w^m)$, we obtain a complete system of $m-r$ functionally independent invariants for the group action on M:

$$I^{r+1}(z) = w^{r+1}(\rho(z), z), \quad \ldots, \quad I^m(z) = w^m(\rho(z), z). \tag{4.11}$$

Thus, barring algebraic complications, the normalization procedure provides a simple direct method for determining the invariants of free group actions. Note particularly that, unlike Lie's infinitesimal method, cf. [17], we do *not* have to integrate⋆ any differential equations in order to compute invariants.

Remark. If $L(g, z)$ is any other regular lifted invariant of rank r, then we can introduce local coordinates on M to make L agree with the first r components of $w = g \cdot z$ when written in the new coordinates. Thus, changing the normalized invariants is equivalent to changing coordinates on M.

⋆ In a sense, though, we have integrated the differential equations by being able to explicitly write down the group transformation formulae for $w = g \cdot z$. However, it is rare that one can integrate the ordinary differential equations for the invariants without being able to find the group transformations!

EXAMPLE 4.12. Let $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ be the planar Euclidean group, parametrized by (ϕ, a, b). Consider the free, local action of SE(2) on $M = \mathbb{R}^4$ that maps a point $(x, u, p, q) \in M$ to

$$\left(x\cos\phi - u\sin\phi + a,\; x\sin\phi + u\cos\phi + b,\right.$$
$$\left.\frac{\sin\phi + p\cos\phi}{\cos\phi - p\sin\phi},\; \frac{q}{(\cos\phi - p\sin\phi)^3}\right). \tag{4.12}$$

The fundamental lifted invariants are the individual components (y, w, r, s) of (4.12). Let us normalize the first three lifted invariants to all be zero, leading to the normalization equations

$$\begin{aligned} y &= x\cos\phi - u\sin\phi + a = 0, \qquad v = x\sin\phi + u\cos\phi + b = 0,\\ r &= \frac{\sin\phi + p\cos\phi}{\cos\phi - p\sin\phi} = 0. \end{aligned} \tag{4.13}$$

This corresponds to choosing the cross-section $K = \{(0, 0, 0, \kappa) \mid \kappa \in \mathbb{R}\}$ to the three-dimensional orbits of SE(2). The solution to (4.13) is

$$\phi = -\tan^{-1} p, \qquad a = -\frac{x + up}{\sqrt{1+p^2}}, \qquad b = \frac{xp - u}{\sqrt{1+p^2}}, \tag{4.14}$$

which defines the right moving frame $\rho\colon M \to \mathrm{SE}(2)$. The left moving frame is obtained by inverting the group element parametrized by (4.14), whereby

$$\tilde\phi = \tan^{-1} p, \qquad \tilde a = x, \qquad \tilde b = u. \tag{4.15}$$

Finally, if we substitute (4.14) into the final lifted invariant $s = (\cos\phi - p\sin\phi)^{-3} q$, we recover the fundamental invariant

$$\kappa = \frac{s}{(1+r^2)^{3/2}} = \frac{q}{(1+p^2)^{3/2}}. \tag{4.16}$$

Note again the common functional dependency on the coordinates on M and the associated lifted invariants, in accordance with Theorem 3.7. If we identify $p = u_x$, $q = u_{xx}$, then (4.12) coincides with the second prolongation of the standard action of SE(2) on curves in the plane, (4.15) agrees with the classical left moving frame for Euclidean curves, cf. [9], and the invariant (4.16) is, of course, the Euclidean curvature. See Example 10.10 below.

EXAMPLE 4.13. Consider the joint action $(x, y) \mapsto (Rx + a, Ry + a)$ of the Euclidean group $(R, a) \in \mathrm{SE}(2)$ on $(x, y) \in M = \mathbb{R}^2 \times \mathbb{R}^2$. The action is free on $M \setminus D$, where $D = \{x = y\}$ is the diagonal. The lifted invariants are the components of $z = Rx + a$, $w = Ry + a$. We normalize $z = 0$ by setting $a = -Rx$. The remaining normalized invariant now reduces to $w = R(y - x)$. Away from the diagonal, we can further normalize the second component of w to be zero by specifying the rotation matrix R to have angle $\phi = -\arg(y - x)$; this amounts to picking the cross-section $K = \{(0, 0, d, 0)\}$. The resulting normalizations specify the right moving frame for the joint Euclidean action. The first component of w then reduces to the distance $|\, y - x \,| = d$, which forms the fundamental joint invariant for the Euclidean group. A similar construction for the n-dimensional Euclidean group $\mathrm{E}(n)$ provides a simple proof of an analytical version of a theorem in Weyl [28], that the only joint Euclidean invariants are functions of the distances between points. Extensions to joint invariants for other transformation groups are straightforward. See [9] for recent results on joint differential invariants.

In applications to equivalence problems, one restricts the moving frame to a submanifold of the underlying space M. The resulting maps from the submanifold to the group agree with the traditional definition of a moving frame in classical geometrical situations. Assume that $S = \iota(X)$ is an immersed submanifold parametrized by a smooth map $\iota\colon X \to M$ of maximal rank equal to the dimension of X.

DEFINITION 4.14. A *moving frame* on a submanifold $S = \iota(X)$ is a map $\lambda\colon X \to G$ that factors through a G-equivariant map $\rho\colon M \to G$, so that $\lambda = \rho \circ \iota$.

In other words, the moving frame λ on S can be realized by the following commutative diagram

$$\begin{array}{ccc} & M & \\ {}^{\iota}\nearrow & & \searrow^{\rho} \\ X & \xrightarrow{\ \lambda\ } & G \end{array} \tag{4.17}$$

The moving frame ρ must, of course, be defined in a neighborhood of S. As before, we can consider either left or right moving frames on the submanifold S. Lemma 4.2 still applies and shows that they are merely inverses of each other.

5. Equivalence Problems for Coframes

We now turn to the applications of moving frames to equivalence problems for submanifolds. In preparation, we first review a very particular equivalence problem, that of coframes on a manifold. The goal of both the Cartan equivalence method and the moving coframe method is to produce, via the normalization and reduction process, an invariant coframe, and thereby reduce the original equivalence problem to an equivalence problem for coframes. Thus it is essential that we understand the known solution to this particular equivalence problem before proceeding further. We refer the reader to [8, 10, 18] for more details on the basic theory as well as numerous examples.

Let M and $\overline{M}$ be m-dimensional manifolds, and let $\boldsymbol{\omega} = \{\omega^1, \ldots, \omega^m\}$ and $\overline{\boldsymbol{\omega}} = \{\overline{\omega}^1, \ldots, \overline{\omega}^m\}$ be respective coframes thereon. The basic *coframe equivalence* *problem* is to determine when there exists a (local) diffeomorphism $\psi\colon M \to \overline{M}$ such that

$$\psi^* \overline{\omega}^i = \omega^i, \quad i = 1, \dots, m. \tag{5.1}$$

More generally, one might also include a collection of smooth scalar-valued functions $I_\nu\colon M \to \mathbb{R}$ and $\overline{I}_\nu\colon \overline{M} \to \mathbb{R}$, where $\nu = 1, \dots, l$, that are required to be mapped to each other, meaning that $\overline{I}_\nu(\bar{x}) = I_\nu(x)$ whenever $\bar{x} = \psi(x)$, or, equivalently,

$$\psi^* \overline{I}_\nu = I_\nu, \quad \nu = 1, \dots, l. \tag{5.2}$$

We formalize this as follows.

DEFINITION 5.1. An *extended coframe* on a manifold M is a collection $\Omega = \{\boldsymbol{\omega}, I\}$ consisting of a coframe $\boldsymbol{\omega}$ along with a collection $I = (I_1, \dots, I_l)$ of smooth scalar functions.

DEFINITION 5.2. A local diffeomorphism $\psi\colon M \to \overline{M}$ is an *equivalence* between extended coframes $\Omega = \{\boldsymbol{\omega}, I\}$ on M, and $\overline{\Omega} = \{\overline{\boldsymbol{\omega}}, \overline{I}\}$, on $\overline{M}$ if and only if ψ satisfies (5.1), (5.2), which we abbreviate as $\psi^* \overline{\Omega} = \Omega$. In particular, a *symmetry* of an extended coframe Ω is a self-equivalence, i.e., a local diffeomorphism $\psi\colon M \to M$ such that $\psi^*\Omega = \Omega$.

The *symmetry group* G of an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ is the local transformation group consisting of all symmetries. The functions I_ν in Ω are then invariants for the group G, hence their common level sets are G-invariant subsets of M. In view of this remark, we shall often refer to the functions I in an extended coframe Ω as its *invariants*.

Two equivalent extended coframes *must* have the same number of invariants. Moreover, if there is a functional dependency $I_l = H(I_1, \dots, I_{l-1})$ among the invariants of Ω, then the corresponding invariants of any equivalent coframe $\overline{\Omega}$ must satisfy an *identical* functional relation: $\overline{I}_l = H(\overline{I}_1, \dots, \overline{I}_{l-1})$. The function $H(y_1, \dots, y_{l-1})$ in such a functional relation is known as a *classifying function* for the extended coframe. As argued in [18], the most natural way to keep track of such functional dependencies between the structure invariants is to introduce the associated classifying manifold.

DEFINITION 5.3. The *classifying manifold* $\mathcal{C}(\Omega)$ of an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ is the subset $I(M) \subset Z = Z(\Omega)$ of the *classifying space* $Z \simeq \mathbb{R}^l$ that is parametrized by the invariant functions $I = (I_1, \dots, I_l)\colon M \to Z$.

DEFINITION 5.4. An extended coframe Ω is called *semi-regular* of *rank* t if its invariants have constant rank $t = \operatorname{rank} \mathrm{d}I$. Note that t equals the number of functionally independent invariants near any point. An extended coframe Ω is called *regular* if its classifying manifold is an embedded submanifold of its ambient classifying space. In this case, the rank of the coframe equals the dimension of $\mathcal{C}(\Omega)$.

LEMMA 5.5. *If* $\Omega = \psi^* \overline{\Omega}$ *are equivalent extended coframes, then their classifying manifolds are identical,* $\mathcal{C}(\overline{\Omega}) = \mathcal{C}(\Omega)$.

Remark. If the equivalence map ψ is only locally defined, then one must restrict the classifying manifolds to the open subsets $U = \operatorname{dom} \psi \subset M$ and $\overline{U} = \psi(U) \subset \overline{M}$.

The converse to Lemma 5.5 is not true in general – one must impose an additional 'involutivity condition' on the extended coframes in order to prove sufficiency of the classifying manifold condition. In preparation, we note that one can (simultaneously) perform two elementary operations on extended coframes that preserve their symmetry and equivalence constraints.

DEFINITION 5.6. Two regular extended coframes $\Omega = \{\boldsymbol{\omega}, I\}$ and $\Theta = \{\boldsymbol{\theta}, J\}$ on M are said to be *invariantly related* if

(a) There exists a local diffeomorphism $\varphi\colon \mathcal{C}(\Omega) \to \mathcal{C}(\Theta)$ such that $J = \varphi \circ I$, and
(b) There is a smooth map $A\colon \mathcal{C}(\Omega) \to \mathrm{GL}(m, \mathbb{R})$ such that $\boldsymbol{\theta} = (A \circ I)\boldsymbol{\omega}$.

We shall write $\Theta = \Phi(\Omega)$, where $\Phi = (\varphi, A)$, for such an invariant relation.

Note that the first condition means that the two classifying manifolds have the same dimension, and so both coframes contain the same number of functionally independent invariants. Moreover, each invariant in Θ is functionally dependent on the invariants in Ω, i.e., $J_\nu = \varphi_\nu(I_1, \ldots, I_l)$, and conversely. The second condition means that the one-forms in the two coframes are invariant linear combinations of each other, so

$$\theta^i = \sum_{j=1}^{m} (A^i_j \circ I)\omega^j, \quad i = 1, \ldots, m. \tag{5.3}$$

PROPOSITION 5.7. *If* $\psi^* \overline{\Omega} = \Omega$ *are equivalent extended coframes, and* $\Theta = \Phi(\Omega)$ *and* $\overline{\Theta} = \Phi(\overline{\Omega})$ *are invariantly related coframes for the same functions* $\Phi = (\varphi, A)$, *then* $\psi^* \overline{\Theta} = \Theta$ *are also equivalent.*

The proof is immediate. Note that this allows us to always assume (at least locally) that the functions occurring in our extended coframe are functionally independent; for example, we can use those corresponding to a consistent choice of local coordinates on the classifying manifold.

COROLLARY 5.8. *Two invariantly related extended coframes* $\Theta = \Phi(\Omega)$ *have the same symmetry group.*

The complete solution to the extended coframe equivalence problem (5.1), (10.9), is based on the Frobenius Theorem for closed differential ideals [4, 18]. One effectively determines a complete system of functionally independent invariants by successive differentiation, and adjoins them to the original invariance conditions (10.9). There are two ways in which additional scalar invariants can arise. First of all, since the one-forms $\boldsymbol{\omega}$ form a coframe, we can re-express their differentials in terms of wedge products thereof, leading to the structure equations

$$\mathrm{d}\omega^k = -\sum_{i<j} I^k_{ij}\omega^i \wedge \omega^j, \quad k = 1, \dots, m. \tag{5.4}$$

The structure coefficients I^k_{ij} are readily seen to be invariants of the problem, i.e., satisfy (10.9), and hence should be included in our list of invariants. Thus, even if we began with no additional invariants, the structure equations automatically produce some for us, whose invariance must be taken into account in the resolution of the problem. Secondly, the coefficients $I_{\nu,k} = \partial I_\nu/\partial\omega^k$ of the differential

$$\mathrm{d}I_\nu = \sum_{k=1}^{m} I_{\nu,k}\omega^k = \sum_{k=1}^{m} \frac{\partial I_\nu}{\partial\omega^k}\,\omega^k, \tag{5.5}$$

of any invariant are also automatically invariant, and are known as the (first-order) *derived invariants* corresponding to the original invariant I_ν. The invariant differential operators $\partial/\partial\omega^k$ are known as *coframe* (or *covariant*) *derivatives*; these coincide with the dual frame vector fields to $\boldsymbol{\omega}$.

Remark. The coframe derivative operators do not necessarily commute. Applying d to (5.5) and comparing with (5.4) produces the basic commutation formulae:

$$\left[\frac{\partial}{\partial\omega^i}, \frac{\partial}{\partial\omega^j}\right] = \sum_{k=1}^{m} I^k_{ij}\,\frac{\partial}{\partial\omega^k}\,. \tag{5.6}$$

If the one-forms $\omega^k = \mathrm{d}f^k$ are all (locally) exact, then all the structure coefficients vanish, and so the coframe derivatives do commute in this particular case.

DEFINITION 5.9. The *derived invariants* of an extended coframe $\{\boldsymbol{\omega}, I\}$ with l invariants $I = (I_1, \dots, I_l)$ are the $l(m+1) + \frac{1}{2}m^2(m-1)$ functions

$$I^{(1)} = (\dots, I_\nu, \dots, I_{\nu,k}, \dots, I^k_{ij}, \dots)$$

consisting of

(a) the original invariants $I_1, \dots, I_l$,
(b) their first order coframe derivatives $I_{\nu,k} = \partial I_\nu/\partial\omega^k$, $\nu = 1, \dots, l$, $k = 1, \dots, m$, and

(c) the coefficients I_{ij}^k, $k = 1, \ldots, m$, $1 \leqslant i < j \leqslant m$, in the structure equations (5.4) for each $d\omega^k$.

DEFINITION 5.10. The *derived coframe* associated with an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ is the extended coframe $\Omega^{(1)} = \{\boldsymbol{\omega}, I^{(1)}\}$ consisting of the original coframe along with all its derived invariants.

LEMMA 5.11. *A map $\psi\colon M \to \overline{M}$ determines an equivalence between extended coframes Ω and $\overline{\Omega}$ if and only if it determines an equivalence between their corresponding derived coframes $\Omega^{(1)}$ and $\overline{\Omega}^{(1)}$.*

In this manner, one recursively defines the higher order derived coframes $\Omega^{(k)} = (\Omega^{(k-1)})^{(1)}$ by computing the higher-order derived invariants. Lemma 5.11 shows that all such higher order derived coframes are also equivalent under the given map. The process will terminate whenever the set of first order derived invariants arising from the current list of invariants fails to produce any new, meaning functionally independent, invariants.

DEFINITION 5.12. An extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ is called *involutive* if it is regular and is invariantly related to its derived coframe $\Omega^{(1)}$.

Thus, a regular extended coframe is involutive if and only if $\operatorname{rank} \Omega = \operatorname{rank} \Omega^{(1)}$, which occurs if and only if its derived invariants are functionally dependent on the original invariants: $I^{(1)} = H \circ I$.

EXAMPLE 5.13. The most familiar example of an involutive coframe is the Maurer–Cartan coframe $\boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}$ on an r-dimensional Lie group G. The symmetry group of the Maurer–Cartan coframe coincides with the right action of G on itself. Involutivity follows from the basic Maurer–Cartan structure equations

$$d\mu^k = -\sum_{i<j} C_{ij}^k \, \mu^i \wedge \mu^j, \quad k = 1, \ldots, r, \tag{5.7}$$

where C_{ij}^k are the structure constants for the dual basis $\mathbf{v}_i = \partial/\partial\mu^i$ of the Lie algebra $\mathfrak{g}$. Since all the derived invariants are constant, the Maurer–Cartan coframe has rank 0. In fact, any rank 0 coframe is locally equivalent to a Maurer–Cartan coframe; see [19] for global versions of this result, based on the theory of 'non-associative local Lie groups'.

LEMMA 5.14. *Let Ω be an extended coframe. If the derived coframe $\Omega^{(s)}$ is involutive, then so are all higher order derived coframes $\Omega^{(k)}$, $k \geqslant s$. Moreover,* $\operatorname{rank} \Omega^{(k)} = \operatorname{rank} \Omega^{(s)}$ *for all $k \geqslant s$.*

Proof. Any functional dependency among the invariants $I = H(I_1, \ldots, I_l)$ automatically induces a functional dependency among the corresponding derived invariants:

$$\frac{\partial I}{\partial \omega^i} = \sum_{\nu=1}^{l} \frac{\partial H}{\partial I_\nu} \, \frac{\partial I_\nu}{\partial \omega^i}, \quad i = 1, \ldots, p. \tag{5.8}$$

This observation suffices to prove the result. □

Remark. Equation (5.8) implies that if an invariant in $\Omega^{(k)}$ is functionally dependent on the others, then one does not need to include its derived invariants in the higher-order derived coframes $\Omega^{(k+1)}$ since their associated functional dependencies are automatic. In other words, we can reduce the number of invariants in $\Omega^{(k+1)}$ by a well-determined invariant relation, as in Definition 5.6. Therefore, at each step, one really only needs compute the coframe derivatives of the independent invariants.

DEFINITION 5.15. The *order* of an extended coframe Ω is the minimal integer s such that $\Omega^{(s)}$ is regular and involutive. We call $t = \operatorname{rank} \Omega^{(s)}$ the *involutivity rank* of Ω.

Remark. Our definition of order is slightly different than that in [18]. If we start with an ordinary coframe $\boldsymbol{\omega}$, under the present construction the structure invariants I^i_{jk} will appear at order 1 and, hence, unless the coframe has rank 0, involutivity will not occur until at least order 2.

Let us call an extended coframe Ω *fully regular* if it and its derived coframes $\Omega^{(k)}$, $k = 0, 1, 2, \ldots$, are regular. In the fully regular case, the ranks $t_k = \operatorname{rank} \Omega^{(k)}$ are nondecreasing, $t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant m$ and bounded by the dimension of M. Moreover, if $t_s = t_{s+1}$, then $\Omega^{(s)}$ is involutive and, hence, Ω has order s and involutivity rank $t = t_s = t_{s+1} = \cdots$. In particular, a fully regular coframe has order $s \leqslant m$. Coframes of order greater than m can occur if singularities are present, but can be resolved at some higher order.

The fundamental equivalence and symmetry theorems for coframes can now be stated. Both are direct consequences of Frobenius' Theorem; details can be found in [18].

THEOREM 5.16. *Let M and $\overline{M}$ be m-dimensional manifolds. Two finite order extended coframes Ω on M and $\overline{\Omega}$ on $\overline{M}$ are locally equivalent if and only if they have the same order s, and their $(s+1)$st order classifying manifolds are identical: $\mathcal{C}(\overline{\Omega}^{(s+1)}) = \mathcal{C}(\Omega^{(s+1)})$. In this case, if $z_0 \in M$ and $\bar{z}_0 \in \overline{M}$ map to the same point $\overline{I}^{(s+1)}(\bar{z}_0) = I^{(s+1)}(z_0)$ in the common classifying manifold, then there is a unique local diffeomorphism $\Phi \colon M \to \overline{M}$ with $\Phi(z_0) = \bar{z}_0$ and $\Phi^* \overline{\Omega} = \Omega$.*

Remark. One can replace the order s by any higher-order $k \geqslant s$ in the theorem. Thus, in fully regular cases, one can always determine the equivalence of two extended coframes on an m-dimensional manifold by comparing the $(m + 1)$st order classifying manifolds.

Remark. Regularity relies on two conditions: first, the invariants have constant rank, and, second, they parametrize an embedded submanifold of the classifying space. The latter can clearly be weakened to include immersed classifying manifolds, since the result is local anyway, and so one can restrict to a subdomain where the classifying manifold is embedded. In fact, one can resolve singularities and self-intersections of the classifying manifold by going to a yet higher order coframe. Indeed, if $\Omega^{(s)}$ is 'semi-involutive', meaning that it is semi-regular and of the same rank as $\mathcal{C}(\Omega^{(s+1)})$, then formula (5.8) implies that one can identify the classifying manifold $\mathcal{C}(\Omega^{(k)})$ for any $k > s$ with the $k-s-1$ jet of $\mathcal{C}(\Omega^{(s+1)})$. Thus, if $\mathcal{C}(\Omega^{(s+1)})$ intersects itself transversally, then $\mathcal{C}(\Omega^{(s+2)})$ will not intersect itself at all, and can be used instead. Thus, in the analytic category, one can eliminate all such singularities and self-intersections by going to a classifying manifold of sufficiently high order.

THEOREM 5.17. *The symmetry group of an extended coframe Ω of order s is a freely acting local Lie group of transformations of dimension $r = m - t$, where $t = \dim \mathcal{C}(\Omega^{(s)})$ is the involutivity rank of Ω. The orbits of G are the common level sets of the $(s+1)$st order invariants $I^{(s+1)}$.*

This completes our survey of the basic equivalence problem for (extended) coframes. One can also investigate the equivalence of more general 'extended one-form systems' $\Omega = \{\boldsymbol{\omega}, I\}$ containing of a collection of one-forms that do not necessarily form a coframe. The *overdetermined* case, where the one-forms $\boldsymbol{\omega}$ span the cotangent space, is easily reduced to the case of an extended coframe. One can locally choose a coframe, say $\{\omega^1, \ldots, \omega^m\}$ from among the one-forms in Ω. Any additional one-forms in Ω can be written as linear combinations of the given coframe,

$$\omega^k = \sum_{i=1}^{m} J_i^k \omega^i, \qquad k > m. \tag{5.9}$$

The coefficients J_i^k will be invariant functions for the problem, and should be included among the functions in an invariantly related extended coframe. Thus, the overdetermined equivalence problem reduces to an extended coframe equivalence problem (5.1), (5.2), where the invariant functions include all the original invariants I_ν as well as the coefficients J_i^k stemming from the linear dependencies (5.9). The *underdetermined* case, when the one-forms fail to span the relevant cotangent spaces, can be treated by the Cartan equivalence method until it is reduced to either a coframe equivalence problem, or to an involutive differential system defining a Lie pseudo-group via the Cartan–Kähler Theorem, cf. [4, 18]. For brevity, we will not discuss the latter more complicated theory here.

6. Moving Coframes

The method of moving coframes was introduced in [9] as a practical means of determining moving frames for general transformation groups, and will now be incorporated into our regularized approach. The following definition is inspired by Cartan's approach to equivalence problems, which always begins by characterizing the (pseudo-)group of allowable transformations by a suitable collection of differential forms.

DEFINITION 6.1. Let G be a finite-dimensional Lie group acting on a manifold M. A *G-coframe* is, by definition, a regular, involutive extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ on M, whose symmetry group coincides with the transformation group G.

In other words, if $\Omega = \{\boldsymbol{\omega}, I\}$ is a G-coframe, then a local diffeomorphism $\psi\colon M \to M$ satisfies the symmetry conditions

$$\psi^*\boldsymbol{\omega} = \boldsymbol{\omega}, \qquad \psi^* I = I, \tag{6.1}$$

if and only if $\psi(z) = g \cdot z$ coincides with the action of a group element $g \in G$. For example, the right Maurer–Cartan coframe on a Lie group forms a G-coframe for the right action of G on itself. Since G-coframes are always assumed to be involutive, the solution to the equivalence problem for coframes implies that they are essentially unique.

PROPOSITION 6.2. *Let Ω be a G-coframe on M. An extended coframe Θ is also a G-coframe if and only if Ω and Θ are invariantly related.*

Proof. Corollary 5.8 implies that if the two extended coframes are invariantly related, then their symmetry groups are the same. Conversely, according to Theorem 5.17, the orbits of the symmetry group of an involutive extended coframe are the level sets of its invariants. Since the two collections of invariants have the same level sets, they are necessarily functionally related, as in part (a) of Definition 5.6. Moreover, since the symmetry groups coincide, the coefficients A^i_j relating the coframes, as in (5.3), must also be invariants, proving the result. □

Theorem 5.17 implies that the symmetry group of an involutive extended coframe acts locally freely. This condition also turns out to be sufficient; see Theorem 6.5 below. The moving frame method provides a simple mechanism for constructing G-coframes. Suppose G acts freely and regularly on the m-dimensional manifold M. Let $\rho\colon M \to G$ be a (right) moving frame. We let $\boldsymbol{\zeta} = \rho^*\boldsymbol{\mu}$ denote the pull-back of the Maurer–Cartan forms to M. If G acts transitively on M, whence $m = r$, then $\boldsymbol{\zeta}$ forms a coframe on M, called the *moving coframe* associated with the given moving frame. The coframe $\boldsymbol{\zeta}$ has the same structure equations (5.7) as the Maurer–Cartan coframe on G and, hence, forms an involutive coframe of rank zero on M.

Remark. The pull-back of the left Maurer–Cartan coframe $\widetilde{\boldsymbol{\mu}}$ on G under the left moving frame map $\widetilde{\rho}$ leads, up to sign, to the same collection of moving coframe forms: $\widetilde{\rho}^*\widetilde{\boldsymbol{\mu}} = -\rho^*\boldsymbol{\mu} = \boldsymbol{\zeta}$. This is because the inversion $g \mapsto g^{-1}$ maps the right Maurer–Cartan forms on G to minus their left-invariant counterparts.

If G does not act transitively, then the one-forms $\boldsymbol{\zeta} = \rho^*\boldsymbol{\mu}$ only form a coframe when restricted to the orbits, and we need to supplement them by an additional $m - r$ one-forms to construct a full coframe. Locally, if we choose a complete system of functionally independent invariants $y = (y^1, \ldots, y^{m-r})$, then the m one-forms

$$\{\boldsymbol{\zeta},\ \mathrm{d}y\} = \{\zeta^1, \ldots, \zeta^r,\ \mathrm{d}y^1, \ldots,\ \mathrm{d}y^{m-r}\} \tag{6.2}$$

form a coframe on M.

DEFINITION 6.3. The *moving coframe* associated with a given moving frame map $\rho\colon\ M \to G$ is the extended coframe $\Sigma = \{\boldsymbol{\zeta},\ \mathrm{d}y, y\}$ consisting of the pulled-back Maurer–Cartan forms $\boldsymbol{\zeta} = \rho^*\boldsymbol{\mu}$, along with the invariant functions y and their differentials.

LEMMA 6.4. *The moving coframe Σ forms an involutive G-coframe on M.*

Proof. Involutivity is immediate, since the Maurer–Cartan structure equations (5.7) along with the equations $\mathrm{d}(\mathrm{d}y^i) = 0$ imply that all the derived invariants for the moving coframe are constant. To prove that the only symmetries are the group transformations $z \mapsto g \cdot z$, we note that, in the flat local coordinates of Example 2.7, the associated moving coframe consists of the Maurer–Cartan forms $\boldsymbol{\mu}$ pulled back to the orbits $G \times \{y_0\}$, along with the invariants and their differentials. Invariance of the y's implies that any symmetry of the moving coframe must have the form $\psi(x, y) = (\varphi(x), y)$, where $\varphi\colon\ G \to G$ is a symmetry of the Maurer–Cartan coframe, and hence agrees with right multiplication by a group element. □

We have thus proved the following basic existence theorem.

THEOREM 6.5. *Let G be a Lie group acting on a manifold M. Then the following are equivalent*:

(i) *G acts freely and regularly on M.*
(ii) *G admits a moving frame in a neighborhood of each point $z \in M$.*
(iii) *There exists a G-coframe in a neighborhood of each point $z \in M$.*

There is a second important method that can be used to construct an alternative G-coframe for a free group action without appealing to the Maurer–Cartan forms. First, the invariants $I(z) = \rho(z) \cdot z$ were earlier interpreted as the pull-back, via $\sigma\colon\ M \to \mathcal{B}$, of the fundamental lifted invariants $w = g \cdot z$. Second, by applying Proposition 3.9, the differential dw of the fundamental lifted invariant will split into two sets of invariant forms on $\mathcal{B}$, namely $\mathrm{d}_M w = g \cdot \mathrm{d}z$ and the group component $\mathrm{d}_G w$. Theorem 3.10 implies that the latter are invariant linear combination of the Maurer–Cartan forms $\boldsymbol{\mu}$ on $\mathcal{B}$. Therefore

$$\mathrm{d}w = \mathrm{d}_M w + \mathrm{d}_G w = g \cdot \mathrm{d}z + F(w)\boldsymbol{\mu}, \tag{6.3}$$

where the coefficients $F(w)$ are explicitly determined by (3.8). We now pull back $\mathrm{d}_M w$ via our moving frame section σ to construct a system of G-invariant one-forms on M.

THEOREM 6.6. *Let G act freely on M. Let $\rho\colon M \to G$ be a right-moving frame. Then the extended coframe $\Gamma = \{\boldsymbol{\gamma}, I\}$ consisting of the invariant functions $I(z) = \rho(z)\cdot z$ along with the one-forms $\boldsymbol{\gamma} = \rho(z)\cdot \mathrm{d}z$ forms a G-coframe on M.*

Proof. The fact that $I = \sigma^* w$ include a complete system of functionally independent invariants was given in Theorem 4.5. Applying σ^* to (6.3), we find

$$\mathrm{d}I = \sigma^*(\mathrm{d}_M w) + \sigma^*(F)\boldsymbol{\zeta} = \boldsymbol{\gamma} + (F \circ I)\boldsymbol{\zeta}.$$

Therefore, the one-forms $\boldsymbol{\gamma} = \sigma^*(\mathrm{d}_M w)$ are invariantly related to the moving coframe forms $\{\boldsymbol{\zeta}, \mathrm{d}I\}$, as in (5.3). It is not hard to see that the $\boldsymbol{\gamma}$ define a coframe on M, and so the result follows from Corollary 5.8. □

Formula (3.8) provides an explicit local coordinate formula relating the normalized coframe forms $\boldsymbol{\gamma}$ with the Maurer–Cartan forms:

$$\gamma^k = \mathrm{d}I^k - \sum_{\kappa=1}^{r} (f_\kappa^k \circ I)\zeta^\kappa, \quad k = 1, \ldots, m, \tag{6.4}$$

where $I^k(z)$ is the kth component of $I(z) = \rho(z)\cdot z$. Note that the coefficients in (6.4) are obtained by invariantization, as in Definition 4.6, of the coefficients $f_\kappa^k(z)$ of the infinitesimal generators (3.7) with respect to the moving frame ρ. In particular, if we normalize $w^k = c^k$ to be constant, then $I^k = c^k$ is constant also, and the $\mathrm{d}I^k$ term in (6.4) disappears.

EXAMPLE 6.7. Consider the action (4.12) of the planar Euclidean group on $\mathbb{R}^4$. The right Maurer–Cartan forms on SE(2) are

$$\mu^1 = \mathrm{d}\phi, \qquad \mu^2 = \mathrm{d}a + b\,\mathrm{d}\phi, \qquad \mu^3 = \mathrm{d}b - a\,\mathrm{d}\phi. \tag{6.5}$$

The corresponding components of the right-moving coframe $\boldsymbol{\zeta} = \rho^*\boldsymbol{\mu}$ are obtained by pulling back the Maurer–Cartan forms (6.5) using the right-moving frame (4.14), so

$$\zeta^1 = -\frac{\mathrm{d}p}{1+p^2}, \qquad \zeta^2 = -\frac{\mathrm{d}x + p\,\mathrm{d}u}{\sqrt{1+p^2}}, \qquad \zeta^3 = \frac{p\,\mathrm{d}x - \mathrm{d}u}{\sqrt{1+p^2}}. \tag{6.6}$$

In order to complete (6.6) to a G-coframe on $M = \mathbb{R}^4$, we must supplement the forms (6.5) by the fundamental invariant (4.16) and its differential

$$\zeta^4 = \mathrm{d}\kappa = \frac{(1+p^2)\,\mathrm{d}q - 3pq\,\mathrm{d}p}{(1+p^2)^{5/2}}, \qquad \kappa = \frac{q}{(1+p^2)^{3/2}}. \tag{6.7}$$

The complete extended coframe (6.6), (6.7) forms a SE(2) coframe on M – its symmetries coincide with the group transformations (4.12).

On the other hand, computing the G-coframe $\{\boldsymbol{\gamma}, I\}$ as in Theorem 6.6, we only need compute the differentials of the fundamental lifted invariants, and then pull-back via the moving frame map, thereby avoiding explicit determination of the Maurer–Cartan forms. Differentiating (4.12) with respect to the coordinates (x, u, p, q) on M leads to the one-forms

$$\begin{aligned} \mathrm{d}_M y &= \cos\phi\,\mathrm{d}x - \sin\phi\,\mathrm{d}u, \qquad \mathrm{d}_M v = \sin\phi\,\mathrm{d}x + \cos\phi\,\mathrm{d}u,\\ \mathrm{d}_M r &= \frac{\mathrm{d}p}{(\cos\phi - p\sin\phi)^2},\\ \mathrm{d}_M s &= \frac{(\cos\phi - p\sin\phi)\,\mathrm{d}q + 3q\sin\phi\,\mathrm{d}p}{(\cos\phi - p\sin\phi)^4}, \end{aligned} \tag{6.8}$$

which, along with the Maurer–Cartan forms (6.5) form a SE(2) coframe for the lifted action on $\mathcal{B} = \mathrm{SE}(2) \times M$. The corresponding SE(2) coframe on M is found by pulling back (6.8) via the right moving frame (4.14); the result is

$$\begin{aligned} \gamma^1 &= \sigma^*(\mathrm{d}_M y) = -\zeta^2, \qquad \gamma^3 = \sigma^*(\mathrm{d}_M r) = -\zeta^1,\\ \gamma^2 &= \sigma^*(\mathrm{d}_M v) = -\zeta^3, \qquad \gamma^4 = \sigma^*(\mathrm{d}_M s) = \zeta^4. \end{aligned} \tag{6.9}$$

The formulae (6.9) relating the two coframes can be deduced from (6.4), as we now explicitly show. The group components of the differentials are

$$\begin{aligned} \mathrm{d}_G y &= \mathrm{d}_G(x\cos\phi - u\sin\phi + a) = -v\mu^1 + \mu^2,\\ \mathrm{d}_G v &= \mathrm{d}_G(x\sin\phi + u\cos\phi + b) = y\,\mu^1 - \mu^3,\\ \mathrm{d}_G r &= \mathrm{d}_G\left(\frac{\sin\phi + p\cos\phi}{\cos\phi - p\sin\phi}\right) = (1 + r^2)\,\mu^1,\\ \mathrm{d}_G s &= \mathrm{d}_G\left(\frac{q}{(\cos\phi - p\sin\phi)^3}\right) = 3rs\mu^1. \end{aligned} \tag{6.10}$$

The lifted invariant coefficients in (6.10) follow directly from (3.8) and the formulae

$$\begin{aligned} \mathbf{v}_1 &= -u\partial_x + x\partial_u + (1 + p^2)\partial_p + 3pq\partial_q,\\ \mathbf{v}_2 &= \partial_x, \qquad \mathbf{v}_3 = \partial_u, \end{aligned} \tag{6.11}$$

for the infinitesimal generators of the Euclidean action (4.12) that are dual to the chosen Maurer–Cartan form basis (6.5). Indeed, if we write down the coefficient matrix

$$\begin{pmatrix} -u & 1 & 0\\ x & 0 & 1\\ 1 + p^2 & 0 & 0\\ 3pq & 0 & 0 \end{pmatrix} \tag{6.12}$$

for the vector fields (6.11), then (6.10) can be written in matrix form

$$\begin{pmatrix} d_G y \\ d_G v \\ d_G r \\ d_G s \end{pmatrix} = \begin{pmatrix} -v & 1 & 0 \\ y & 0 & 1 \\ 1+r^2 & 0 & 0 \\ 3rs & 0 & 0 \end{pmatrix} \begin{pmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{pmatrix}, \tag{6.13}$$

and the coefficient matrix is the lifted version of (6.12), obtained by replacing the coordinates on M by their corresponding lifted counterparts. Formula (6.9) then follows from (6.4):

$$\begin{pmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \\ \gamma^4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d\kappa \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \\ \zeta^3 \end{pmatrix}. \tag{6.14}$$

In (6.14), the first term is the pull-back of the lifted coordinate differentials $(dy,\ dv,\ dr,\ ds)$ via the normalization map (4.13), while the coefficient matrix in the second is the invariantization of the infinitesimal generator coefficient matrix (6.12) with respect to the given moving frame.

7. Equivalence of Submanifolds

We now apply our general results to the equivalence problem for submanifolds under a freely acting transformation group. Given submanifolds $S, \overline{S} \subset M$, we want to know whether or not they are *congruent* under a group transformation, meaning that $g \cdot S = \overline{S}$ for some $g \in G$. In this section, we review the solution to this problem in the case when G acts regularly and freely on M. Actually, we shall only consider the local problem here, so that the congruence condition is only required to hold in a neighborhood of a point. Global questions can be handled by continuation processes (e.g., analytic continuation).

DEFINITION 7.1. Let $S = \iota(X)$ and $\overline{S} = \bar{\iota}(\overline{X})$ be two embedded p-dimensional submanifolds parametrized by maps $\iota\colon X \to M$, and $\bar{\iota}\colon \overline{X} \to M$. The submanifolds are said to be (locally) *congruent* under a transformation group G provided there exists a group element $g \in G$ and a (local) diffeomorphism $\psi\colon X \to \overline{X}$ such that

$$\bar{\iota}(\psi(x)) = g \cdot \iota(x) \tag{7.1}$$

for all x in the domain of ψ.

In the case when G acts freely, the solution to the congruence problem follows directly from the theorem for submanifolds embedded in Lie groups [12].

THEOREM 7.2. *Let G be a free, regular Lie transformation group acting on M. Let $\Omega = \{\boldsymbol{\omega}, I\}$ be a G-coframe on M. Then two embedded p-dimensional* *submanifolds* $S = \iota(X)$ *and* $\overline{S} = \overline{\iota}(\overline{X})$ *are locally congruent under* G *if and only if the pulled-back extended coframes* $\Xi = \{\boldsymbol{\xi}, J\} = \iota^*\Omega$ *on* X *and* $\overline{\Xi} = \{\overline{\boldsymbol{\xi}}, \overline{J}\} = \overline{\iota}^*\overline{\Omega}$ *on* $\overline{X}$ *are locally equivalent.*

The diffeomorphism $\psi\colon X \to \overline{X}$ determining the reparametrization part in the correspondence (7.1) must satisfy $\psi^*\,\overline{\Xi} = \Xi$. In other words, in terms of the defining coframe and invariants,

$$(\overline{\iota} \circ \psi)^*\boldsymbol{\omega} = \iota^*\boldsymbol{\omega}, \qquad (\overline{\iota} \circ \psi)^* I = \iota^* I. \tag{7.2}$$

Remark. For the fixed parameter equivalence problem, we do not allow the reparametrization map ψ in the equivalence condition (7.1), which thus reduces to $\overline{\iota}(x) = g \cdot \iota(x)$, $x \in X$. The solution to this problem follows from Theorem 7.2: the pulled-back G-coframes must now be identical: $\overline{\Xi} = \Xi$.

The original G-coframe Ω is involutive; in particular, if $\Omega = \Sigma$ is the moving coframe, it will have constant derived invariants. The pull-back $\Xi = \iota^*\Omega$ will, of course, have the same structure equations as Ω. However, if the submanifold S has strictly smaller dimension than M, i.e., $\dim S = p < m$, the one-forms $\boldsymbol{\xi} = \iota^*\boldsymbol{\omega}$ are *not* a coframe on X, because there are too many of them. Thus, Ξ will constitute an overdetermined one-form system, as discussed at the end of Section 5. In order to apply our general equivalence theorems, we need to reduce Ξ to an extended coframe by eliminating the linear dependencies among the pulled-back one-forms. Near each point $x \in X$ we can choose* p linearly independent one-forms $\boldsymbol{\varpi} = \{\varpi^1, \ldots, \varpi^p\}$ from among the pulled-back forms $\boldsymbol{\xi}$. The choice of $\boldsymbol{\varpi}$ is governed by a transversality condition on the submanifold S.

DEFINITION 7.3. Let $\widetilde{\boldsymbol{\omega}} = \{\omega^1, \ldots, \omega^p\}$ be a collection of p pointwise linearly independent one-forms on an m-dimensional manifold M. A p-dimensional submanifold $S = \iota(X)$ is *transverse* with respect to $\widetilde{\boldsymbol{\omega}}$ if and only if the one-forms $\boldsymbol{\varpi} = \iota^*\widetilde{\boldsymbol{\omega}}$ forms a coframe on the parameter space X, and so S satisfies the *independence condition*

$$\varpi^1 \wedge \cdots \wedge \varpi^p = \iota^*\big(\omega^1 \wedge \cdots \wedge \omega^p\big) \neq 0. \tag{7.3}$$

We refer the reader to [4] for a detailed discussion of the role of independence conditions and transversality in the context of exterior differential systems. Thus, given an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$, we shall impose an independence condition on p-dimensional submanifolds $S \subset M$ by choosing p of the one-forms in $\boldsymbol{\omega}$. Since we can rearrange the forms in $\boldsymbol{\omega}$ (or, more generally, take constant coefficient linear combinations) without affecting the symmetry properties of Ω, we shall, without loss of generality, assume that the independence condition (7.3)

* As we shall see below, this is effected by a choice of independent and dependent variables on the original manifold M.

is *always* with respect to the first p one-forms in Ω. With such a choice, we can define transversality of a submanifold with respect to an extended coframe Ω.

DEFINITION 7.4. A p-dimensional submanifold $S = \iota(X)$ is *transverse* with respect to an extended coframe $\Omega = \{\boldsymbol{\omega}, I\}$ where $\boldsymbol{\omega} = \{\omega^1, \ldots, \omega^m\}$, if and only if it is transverse with respect to the first p one-forms $\widetilde{\boldsymbol{\omega}} = \{\omega^1, \ldots, \omega^p\}$.

On a transverse submanifold, $\boldsymbol{\varpi} = \iota^* \widetilde{\boldsymbol{\omega}}$ forms a coframe on the parameter space X. Therefore, we can write the remaining pulled back one-forms as linear combinations of them,

$$\xi^k = \sum_{j=1}^{p} K^k_j(x)\, \varpi^j, \qquad k = p+1, \ldots, m.$$

The coefficients $K = (\ldots K^k_j \ldots)$ provide additional invariants for the overdetermined one-form system Ξ. Replacing the extra one-forms by these invariants reduces Ξ to an invariantly related extended coframe, $\Upsilon = \{\boldsymbol{\varpi}, J, K\}$ on X, having the same symmetry and equivalence properties as Ξ does. We shall call Υ the *restricted G-coframe* on the submanifold S. If $\overline{S}$ is also transverse* we can similarly construct the extended coframe $\overline{\Upsilon} = \{\overline{\boldsymbol{\varpi}}, \overline{J}, \overline{K}\}$ on $\overline{X}$ using the *same* choice of coframe basis $\overline{\boldsymbol{\varpi}} = \overline{\iota}^* \widetilde{\boldsymbol{\omega}}$ relative to the given G-coframe.

LEMMA 7.5. *Let Ω be a G-coframe on M. Two transverse submanifolds $S = \iota(X)$ and $\overline{S} = \overline{\iota}(\overline{X})$ are locally G congruent if and only if the corresponding restricted G-coframes Υ on X and $\overline{\Upsilon}$ on $\overline{X}$ are equivalent.*

Now, even though the original G-coframe Ω is involutive, the restricted G-coframe Υ will almost never be involutive. Thus, one will typically need to replace Υ by its involutive counterpart $\Upsilon^{(s)}$, where s is the *order* of Υ.

DEFINITION 7.6. A submanifold $S \subset M$ is called *regular* of *order s* with respect to the G-coframe Ω if S is transverse and the restricted G-coframe Υ has order s. The *classifying manifold* of S is defined as $\mathcal{C}(S) = \mathcal{C}(\Upsilon^{(s+1)})$. The rank t of S is the dimension of its classifying manifold: $t = \dim \mathcal{C}(S)$.

Using this construction, Theorem 5.16 then gives a complete solution to the congruence problem for submanifolds when the group acts freely.

THEOREM 7.7. *Let G be a free, regular Lie transformation group acting on M, and let Ω be a G-coframe. Let $S = \iota(X)$ and $\overline{S} = \overline{\iota}(\overline{X})$ be regular p-dimensional submanifolds. Then S and $\overline{S}$ are locally G equivalent if and only if they have the same order s, and their classifying manifolds $\mathcal{C}(\overline{S}) = \mathcal{C}(S)$ are identical.*

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* This can always be arranged locally by a suitable choice of the one-forms $\widetilde{\boldsymbol{\omega}}$.

The symmetry group of S is, by definition, its isotropy subgroup $G_S \subset G$. Theorem 7.7 demonstrates that the action of G_S on S can be identified with the symmetry group of the restricted G-coframe Υ on the parameter space X.

THEOREM 7.8. *Let $S \subset M$ be a regular p-dimensional submanifold of order s and rank t with respect to the transformation group G. Then its isotropy group G_S has dimension $p - t = \dim S - \operatorname{rank} S$ and acts freely on S.*

In particular, S has maximal symmetry if and only if it has rank 0, meaning that all the restricted invariants J, K are constant on S. In this case, the dimension of the isotropy group equals the dimension of S and hence G_S acts transitively on (the connected components) of S. In particular, S must lie in a single G orbit of M.

Remark. Later we shall see that the invariants K arising from linear dependencies among the one-forms in Ξ can be identified with the first order differential invariants for the group action. Moreover, the derived invariants correspond to suitable higher order differential invariants. Thus, the classifying manifolds used to solve the equivalence problem are identified with those parameterized by the differential invariants for G.

EXAMPLE 7.9. Consider the Abelian Lie group $G = \mathbb{R}^3$ acting by translations on $M = \mathbb{R}^3$. A G-coframe is given by the coordinate one-forms $\Omega = \{dx, dy, du\}$. The surface $S = \{x^2 + 2yu = 0 \mid y \neq 0\}$ satisfies the transversality condition $dx \wedge dy \mid S \neq 0$. Parametrizing S by $\iota\colon (x, y) \mapsto (x, y, -\frac{1}{2}x^2 y^{-1})$, we see that the restricted one-forms $\Xi = \iota^*\Omega$ satisfy the linear dependency $du = -(x/y)\,dx + \frac{1}{2}(x/y)^2\,dy$, leading to the functionally dependent invariants $-x/y$ and $x^2/(2y^2)$. Therefore, the restricted coframe on S is

$$\Upsilon = \left\{dx,\ dy,\ -\frac{x}{y},\ \frac{x^2}{2y^2}\right\}.$$

However, Υ is not involutive since $d(x/y) = (1/y)\,dx - (x/y^2)\,dy$, so that the derived coframe*

$$\Upsilon^{(1)} = \left\{dx,\ dy,\ -\frac{x}{y},\ \frac{x^2}{2y^2},\ -\frac{1}{y},\ \frac{x}{y^2},\ \frac{x}{y^2},\ -\frac{x^2}{y^3}\right\}$$

is involutive. Therefore, S is a surface having rank 2 and order 1, and hence admits at most a discrete translation symmetry group; in fact, the isotropy subgroup of S is trivial. The classifying manifold is the surface parametrized by the twelve invariants in

$$\Upsilon^{(2)} = \left\{dx,\ dy,\ -\frac{x}{y},\ \frac{x^2}{2y^2},\ -\frac{1}{y},\ \frac{x}{y^2},\ \frac{x}{y^2},\ -\frac{x^2}{y^3},\ 0,\ \frac{1}{y^2},\ -\frac{2x}{y^3},\ \frac{1}{y^2},\ -\frac{2x}{y^3},\ \frac{3x^2}{y^4}\right\}$$

* Actually, since the second invariant $x^2/2y^2$ is a function of the first, its derived invariants are redundant, as their functional dependencies are automatically determined, cf. (5.8).

so that

$$\mathcal{C}(S) = \left\{ (a_1, \ldots, a_{12}) \in Z = \mathbb{R}^{12} \,\middle|\, \begin{array}{l} a_2 = \frac{1}{2}a_1^2, a_4 = a_5 = -a_1a_3, a_6 = a_1^2a_3, a_7 = 0 \\ a_8 = a_{10} = a_3^2, a_9 = a_{11} = 2a_1a_3^2, a_{12} = 3a_1^2a_3^2 \end{array} \right\}.$$

Any translationally equivalent surface $\overline{S}$ must have the same classifying manifold, so that $\overline{S}$ also has order 1 and rank 2, and has the same functional relationships among its corresponding twelve invariants.

We are now ready to discuss the role of the regularized action in the equivalence problem for general group actions. Here we no longer need to assume that G acts freely on M, but we replace it by its freely acting regularization on $\mathcal{B} = G \times M$. Associated with an embedded submanifold $S = \iota(X) \subset M$ is the submanifold $S_G = \iota_G(G \times X) \subset \mathcal{B}$ parametrized by

$$\iota_G(g, x) = (g, \iota(x)), \quad g \in G, \ x \in X.$$

The bundle $G \times X$ is the pull-back under ι of $G \times M$. On $\mathcal{B}$, we consider the G-coframe $\Omega = \{\boldsymbol{\mu}, \mathrm{d}w, w\}$. As a direct consequence of Theorem 7.2, we obtain the following result.

PROPOSITION 7.10. *Two embedded submanifolds S_G and $\overline{S}_G$ parametrized by maps ι_G: $G \times X \to G \times M$, and $\bar{\iota}_G$: $G \times \overline{X} \to G \times M$ are locally G congruent if and only if the pulled-back extended coframes $\Xi = \{\boldsymbol{\xi}, J\} = \iota^*\Omega$ on X and $\overline{\Xi} = \{\overline{\boldsymbol{\xi}}, \overline{J}\} = \bar{\iota}^*\overline{\Omega}$ are locally equivalent.*

Suppose that $S = \iota(X)$ satisfies the transversality condition specified by $\widetilde{\boldsymbol{\omega}}$. Then $S_G = \iota_G(X)$ satisfies the transversality condition defined by $(\pi_M)^*\widetilde{\boldsymbol{\omega}} \cup \boldsymbol{\mu}$. It is clear that we can construct a coframe $\widetilde{\Omega}$ invariantly related to Ω such that the one forms $\omega^1, \ldots, \omega^p \in \widetilde{\Omega}$ generate $(\pi_M)^*\widetilde{\boldsymbol{\omega}}$. Following the procedure above Lemma 7.5 we have the restricted G-coframe Υ on $G \times X$ where $\boldsymbol{\varpi} = \{\varpi^1, \ldots, \varpi^p, \mu^1, \ldots, \mu^r\}$ such that $\varpi^1, \ldots, \varpi^p$ annihilate the tangent space to the fibers of π: $G \times X \to X$. Denoting by similar barred expressions using the map $\bar{\iota}_G$: $G \times \overline{X} \to G \times M$, the equivalence theorem takes the following form.

PROPOSITION 7.11. *Two embedded submanifolds ι: $X \to M$ and $\bar{\iota}$: $\overline{X} \to M$ which satisfy the transversality condition $\widetilde{\boldsymbol{\omega}}$, are equivalent if and only if the extended coframes Υ on $G \times X$ and $\overline{\Upsilon}$ on $G \times \overline{X}$ are equivalent.*

Proof. First suppose X and $\overline{X}$ are equivalent. Thus $\bar{\iota} = g \cdot (\iota \circ \psi^{-1})$, where $g \in G$ and ψ: $X \to \overline{X}$. Define the diffeomorphism Ψ: $G \times X \to G \times \overline{X}$ by

$$\Psi(h, x) = (g \cdot h, \psi(x)), \quad h \in G, \ x \in X. \tag{7.4}$$

Clearly, $\bar{\iota}_G \circ \Psi = \widehat{R}_g \cdot \iota_G$, where $\widehat{R}_g$ denotes the right regularized action (3.2) of G. Then

$$(\bar{\iota}_G \circ \Psi)^* \boldsymbol{\mu} = (\widehat{R}_g \cdot \iota_G)^* \boldsymbol{\mu} = (\iota_G)^* \boldsymbol{\mu},$$
$$(\bar{\iota}_G \circ \Psi)^* w = (\widehat{R}_g \cdot \iota_G)^* w = (\iota_G)^* w.$$

Conversely, if there exists such a Ψ, then Proposition 7.10 implies that $\bar{\iota}_G \circ \Psi = \widehat{R}_g \cdot \iota_G$ for some $g \in G$. In order to finish the proof we need to check that Ψ splits as in (7.4). The conditions on Ψ in the theorem then imply $\Psi^* \overline{\boldsymbol{\mu}} = \boldsymbol{\mu}$ and $\Psi^* \overline{\boldsymbol{\varpi}} = \boldsymbol{\varpi}$ and, hence, Ψ has the form in (7.4). □

Let $L\colon G \times M \to N$ be a regular lifted invariant. Let c be in the image of L and let $\mathcal{L}_c = L^{-1}\{c\}$ be the corresponding invariant level set. Denote the restriction of $\boldsymbol{\mu}$ and w to $\mathcal{L}_c$ by $\widetilde{\boldsymbol{\mu}}$ and $\widetilde{w}$. The submanifold $\mathcal{L}_c$ is G invariant and the local diffeomorphisms of $\mathcal{L}_c$ which preserve the restricted invariants $\widetilde{w}$ and forms $\widetilde{\boldsymbol{\mu}}$ coincide with the action of G on $\mathcal{L}_c$. That is $\{\widetilde{\boldsymbol{\mu}}, \widetilde{w}\}$ forms an (overdetermined) G-coframe on $\mathcal{L}_c$. Now let $R = (\iota_G)^{-1}(\mathcal{L}_c) \subset G \times X$, and similarly for $\overline{R} \subset G \times \overline{X}$.

PROPOSITION 7.12. *Under the above hypothesis, the embedded submanifolds X and $\overline{X}$ are equivalent if and only if there exists a diffeomorphism $\widetilde{\Psi}\colon R \to \overline{R}$ such that*

$$\widetilde{\Psi}^* \overline{I} = I, \qquad \widetilde{\Psi}^* \overline{\boldsymbol{\omega}} = \boldsymbol{\omega},$$

where $I, \overline{I}$ and $\boldsymbol{\omega}, \overline{\boldsymbol{\omega}}$ are the pull-backs of the restricted invariants $\widetilde{w}$ and forms $\widetilde{\boldsymbol{\mu}}$ by ι_G and $\bar{\iota}_G$ respectively.

The proof is similar to that in Proposition 7.11.

Remark. If the function L defining the invariant submanifold $\mathcal{L}_c$ is of rank $r = \dim G$ in the vertical direction for the projection $\pi_M\colon G \times M \to M$ then Theorem 7.12 is resolved by Theorem 7.7.

Remark. The transformations in G_S determine symmetries of the restricted coframe on $G \times X$. However, since at least p of the invariants I are automatically functionally independent, $\dim G_S \leqslant \dim G$, as it should be.

Therefore, regularization can be used to replace the equivalence of p-dimensional submanifolds $S \subset M$ under a nonfree action of G by equivalence of $(p+r)$-dimensional submanifolds $S_G \subset \mathcal{B}$ under the free regularized G action. This approach avoids the use of differential invariants, and will also take care of singular submanifolds, since the lifted submanifold R is always regular. Incidentally, Proposition 7.12 can be used to justify partial normalization, as discussed in Section 16 below, while the preceding remark can be used to justify complete normalization. This alternative method certainly warrants further investigation.

8. Jet Bundles

The results in the preceding sections lead to a complete construction of a moving frame in the case when the group acts freely on the underlying manifold. If the group does not act freely, then an ordinary moving frame does not exist, and one needs to prolong to some jet space of suitably high order before the procedure can be applied. In such cases, the higher order moving frame will naturally lead to the differential invariants for the transformation group. We begin by reviewing the basics of jet bundles, cf. [17, 18].

Given a manifold M, we let $\mathrm{J}^n = \mathrm{J}^n(M, p)$ denote the nth order (extended) jet bundle consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of nth order contact, cf. [16, 17, Chapter 3]. In particular, $\mathrm{J}^0 = M$. We let $\mathrm{j}_n S \subset \mathrm{J}^n$ denote the n-jet of the submanifold S; more explicitly, the parametrization map $\iota\colon X \to S \subset M$ induces a parametrization $\mathrm{j}_n\iota\colon X \to \mathrm{j}_n S \subset M$. The fibers of $\pi^n_0\colon \mathrm{J}^n \to M$ are generalized Grassmann manifolds [16]. A *differential function* of order n is a scalar-valued function $F\colon \mathrm{J}^n \to \mathbb{R}$. Sometimes, it is convenient to work with the infinite jet bundle $\mathrm{J}^\infty = \mathrm{J}^\infty(M, p)$, which is defined as the inverse limit of the finite order jet bundles under the standard projections $\pi^k_n\colon \mathrm{J}^k \to \mathrm{J}^n$, $k > n$. Functions and differential forms on J^∞ are obtained from their finite-order counterparts, where we identify a form ω on J^n with its pull-backs $(\pi^k_n)^*\omega$ on J^k for any $k > n$, and hence with a differential form on J^∞. For further details on infinite jet bundles, see [1, 26].

We introduce local coordinates $z = (x, u)$ on M, considering the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables. Splitting the coordinates into independent and dependent variables has the effect of locally identifying M with an open subset of a bundle $E = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$. Sections $u = f(x)$ of E correspond to p-dimensional submanifolds S that are transverse with respect to the horizontal forms $\mathrm{d}x = \{\mathrm{d}x^1, \ldots, \mathrm{d}x^p\}$, as in Definition 7.3. The induced coordinates on the jet bundle J^n are denoted by $z^{(n)} = (x, u^{(n)})$, with components u^α_J representing the partial derivatives of the dependent variables with respect to the independent variables up to order n. Here $J = (j_1, \ldots, j_k)$ is a symmetric multi-index of order $k = \#J$, with $1 \leqslant j_\nu \leqslant p$. The $(x, u^{(n)})$ define local coordinates on the open, dense subbundle $\mathrm{J}^n E \subset \mathrm{J}^n(M, p)$ determined by the jets of transverse submanifolds, or, equivalently, local sections $u = f(x)$. In the limit, we let $z^{(\infty)} = (x, u^{(\infty)})$ denote the corresponding coordinates on $\mathrm{J}^\infty E \subset \mathrm{J}^\infty(M, p)$, consisting of independent variables x^i, dependent variables u^α, and their derivatives u^α_J, $\alpha = 1, \ldots, q$, $0 \leqslant \#J$, of arbitrary order.

The intrinsic geometry of jet space is governed by a fundamental collection of differential forms.

DEFINITION 8.1. A differential form θ on the jet space $\mathrm{J}^n(M, p)$ is called a *contact form* if it is annihilated by all jets, so that $(\mathrm{j}_n\iota)^*\theta = 0$ for every p-dimensional submanifold $S = \iota(X) \subset M$.

The subbundle of the cotangent bundle $T^*\mathrm{J}^n$ spanned by the contact one-forms will be called the nth order *contact bundle*, denoted by $\mathcal{C}^{(n)}$. The infinite contact bundle $\mathcal{C}^{(\infty)} \subset T^*\mathrm{J}^\infty$ is a codimension p subbundle of $T^*\mathrm{J}^\infty$. (This result is not true for finite-order contact subbundles, which is one of the main reasons for going to infinite order.) In terms of local coordinates $(x, u^{(\infty)})$, every contact one-form can be written as a linear combination of the *basic contact forms*

$$\theta^\alpha_J = \mathrm{d}u^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i}\, \mathrm{d}x^i, \qquad \alpha = 1, \ldots, q,\ 0 \leqslant \#J. \tag{8.1}$$

Combining the horizontal coordinate one-forms $\mathrm{d}x^i$ with the basic contact forms θ^α_J produces the local coordinate coframe on J^∞:

$$\mathrm{d}x = \{\mathrm{d}x^1, \ldots, \mathrm{d}x^p\}, \qquad \boldsymbol{\theta}^{(\infty)} = \{\ldots, \theta^\alpha_J, \ldots\}. \tag{8.2}$$

Therefore, choosing local coordinates on M induces a splitting $T^*\mathrm{J}^\infty = \mathcal{H} \oplus \mathcal{C}^{(\infty)}$ of the cotangent bundle into *horizontal* and contact or *vertical* subbundles, with $\mathcal{H}$ spanned by the horizontal one-forms $\mathrm{d}x$. Let $\pi_H\colon T^*\mathrm{J}^\infty \to \mathcal{H}$ and $\pi_V\colon T^*\mathrm{J}^\infty \to \mathcal{C}^{(\infty)}$ be the induced projections, so that any one-form $\omega = \omega_H + \vartheta$ splits into uniquely defined horizontal and vertical components, where

$$\omega_H = \pi_H(\omega) = \sum_{i=1}^{p} P_i(x, u^{(n)})\, \mathrm{d}x^i \tag{8.3}$$

is a horizontal one-form, and

$$\vartheta = \pi_V(\omega) = \sum_{\alpha, J} Q^\alpha_J(x, u^{(n)})\, \theta^\alpha_J \tag{8.4}$$

is a contact form. If ω is a one-form on J^n then, typically, its horizontal component ω_H is a one-form on J^{n+1}.

The splitting of $T^*\mathrm{J}^\infty$ induces a bi-grading of the differential forms on J^∞. The differential d on J^∞ naturally splits into horizontal and vertical components, $\mathrm{d} = \mathrm{d}_H + \mathrm{d}_V$, where d_H increases horizontal degree and d_V increases vertical degree. Closure, $\mathrm{d} \circ \mathrm{d} = 0$, implies that $\mathrm{d}_H \circ \mathrm{d}_H = 0 = \mathrm{d}_V \circ \mathrm{d}_V$, while $\mathrm{d}_H \circ \mathrm{d}_V = -\mathrm{d}_V \circ \mathrm{d}_H$. In particular, the horizontal or *total differential* of a differential function $F\colon \mathrm{J}^n \to \mathbb{R}$ is the horizontal one-form

$$\mathrm{d}_H F = \sum_{i=1}^{p} (D_i F)\, \mathrm{d}x^i \tag{8.5}$$

on J^{n+1}, where

$$D_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{J} u^\alpha_{J,i} \frac{\partial}{\partial u^\alpha_J} \tag{8.6}$$

denotes the usual total derivative with respect to x^i, which can be viewed as a vector field on $\mathrm{J}^\infty E$. Similarly, the vertical differential of a function $F(x, u^{(n)})$ is the contact form

$$\mathrm{d}_V F = \sum_{i=1}^{p} \sum_{K} \frac{\partial F}{\partial u_K^\alpha}\, \theta_K^\alpha. \tag{8.7}$$

DEFINITION 8.2. A *total differential operator* is a vector field on J^∞ which lies in the annihilator of the contact bundle $\mathcal{C}^{(\infty)}$.

PROPOSITION 8.3. *Every total differential operator has the form*

$$\mathcal{D} = \sum_{i=1}^{p} Q_i(x, u^{(n)})\, D_i, \tag{8.8}$$

where $Q_1, \ldots, Q_p$ are differential functions.

The preceding construction forms the foundation of the variational bicomplex that is of fundamental importance in the study of the geometry of jet bundles, differential equations and the calculus of variations; see [1, 26, 29] for details.

9. Prolonged Transformation Groups

Any transformation group G acting on M preserves the order of contact between submanifolds. Therefore, there is an induced action of G on the nth order jet bundle $\mathrm{J}^n(M, p)$ known as the nth *prolongation* of G. Alternatively, one can characterize the prolonged group transformations as the unique lifted maps on the jet bundle that preserve the space of contact forms.

DEFINITION 9.1. A map $\Psi\colon \mathrm{J}^n \to \mathrm{J}^n$ is a *contact transformation* if it preserves the order n contact subbundle: $\Psi^*\mathcal{C}^{(n)} \subset \mathcal{C}^{(n)}$.

PROPOSITION 9.2. *If $\psi\colon M \to M$ is a local diffeomorphism, then its nth prolongation is the unique contact transformation $\psi^{(n)}\colon \mathrm{J}^n \to \mathrm{J}^n$ that satisfies $\psi \circ \pi_0^n = \pi_0^n \circ \psi^{(n)}$.*

We denote the prolonged group action on J^n by $G^{(n)}$. Note that if G acts globally on M, then its prolonged action $G^{(n)}$ is also a global transformation group on $\mathrm{J}^n(M, p)$, but, generally only a local transformation group on the coordinate subbundles $\mathrm{J}^n E$ since G may not preserve transversality.

Remark. Our methods also apply, with minor modifications, to more general contact transformation groups. Bäcklund's Theorem, cf. [18], implies that these reduce to prolonged point transformation groups on M, or, in the codimension 1 case, prolonged first order contact transformation groups.

Let us choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ for the Lie algebra $\mathfrak{g}$ of infinitesimal generators on M, and let $\{\mathrm{pr}^{(n)}\,\mathbf{v}_1, \ldots, \mathrm{pr}^{(n)}\,\mathbf{v}_r\}$ denote the corresponding the infinitesimal generators of the prolonged group action $G^{(n)}$. In terms of local coordinates $(x, u^{(\infty)})$ on J^∞, we obtain $\mathrm{pr}^{(n)}\,\mathbf{v}_\kappa$ by truncating the infinitely prolonged vector field

$$\begin{aligned}\mathrm{pr}\,\mathbf{v}_\kappa \;=\; & \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \\ & + \sum_{\alpha=1}^{q}\sum_{j=\#J\geqslant 0} \varphi^\alpha_{J,\kappa}(x,u^{(j)})\,\frac{\partial}{\partial u^\alpha_J}, \qquad \kappa = 1,\ldots,r, \end{aligned} \tag{9.1}$$

at order n. The coefficients of (9.1) are explicitly determined by the standard prolongation formula [18]:

$$\varphi^\alpha_{J,\kappa} = D_J\,Q^\alpha_\kappa + \sum_{i=1}^{p} \xi^i_\kappa\, u^\alpha_{J,i}, \tag{9.2}$$

where

$$Q^\alpha_\kappa(x,u^{(1)}) = \varphi^\alpha_\kappa(x,u) - \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,u^\alpha_i \tag{9.3}$$

is the *characteristic* of $\mathbf{v}_\kappa$.

The moving frame construction in Section 4 can be applied to the prolonged group action $G^{(n)}$ provided it acts (locally) freely on J^n. Therefore, we need to understand the basic geometry of the prolonged action in order to understand the full range of applicability of the higher order moving frame construction.

DEFINITION 9.3. Given G acting on M, we let s_n denote the maximal orbit dimension of the prolonged action $G^{(n)}$ on J^n. The *stable orbit dimension* $s = \max s_n$ is the maximum prolonged orbit dimension. The *stabilization order* of G is the minimal n such that $s_n = s$.

A fundamental stabilization theorem due to Ovsiannikov [22], completely characterizes the stable orbit dimension; see also [18, 20] for further details.

THEOREM 9.4. *A Lie group G acts locally effectively on subsets of M if and only if its stable orbit dimension equals its dimension,* $s = r = \dim G$.

DEFINITION 9.5. The *regular subset* $\mathcal{V}^n \subset \mathrm{J}^n$ is the open subset consisting of all prolonged group orbits of dimension equal to the stable orbit dimension. The *singular subset* is the remainder, $\mathcal{S}^n = \mathrm{J}^n \setminus \mathcal{V}^n$, which is the union of all $G^{(n)}$ orbits of less than maximal dimension.

Note that, by this definition, $\mathcal{V}^n = \emptyset$ and $\mathcal{S}^n = \mathrm{J}^n$ if n is less than the stabilization order of $G^{(n)}$. If G acts analytically, then $\mathcal{V}^n$ is a dense open subset of J^n for n greater than or equal to the stabilization order. The singular subset $\mathcal{S}^n$ can be characterized by the vanishing of the Lie determinant or its generalizations, cf. [18, Chap. 6]. A point $z^{(n)} \in \mathrm{J}^n$ will be called a *regular jet* provided $z^{(n)} \in \mathcal{V}^n$ or, equivalently, the prolonged orbit passing through $z^{(n)}$ has dimension $r = \dim G$, assuming G acts locally effectively on subsets. The stabilization Theorem 9.4 combined with Proposition 2.5 immediately implies the freeness of the prolonged action on the regular subset of jet space.

PROPOSITION 9.6. *If G acts locally effectively on subsets, then G acts locally freely on the regular subset $\mathcal{V}^n \subset \mathrm{J}^n$.*

Remark. It would be nice to know that $G^{(n)}$ acts freely on (a dense open subset of) $\mathcal{V}^n$ provided n is sufficiently large. We do not know a general theorem that guarantees the freeness of prolonged group actions, although it seems highly unlikely, particularly in the analytic category, that a group acts only locally freely on all of $\mathcal{V}^n$ when n is large.

DEFINITION 9.7. A submanifold $S \subset M$ is *order n regular* if $\mathrm{j}_n S \subset \mathcal{V}^n$. A submanifold $S \subset M$ is *totally singular* if $\mathrm{j}_n S \subset \mathcal{S}^n$ for all $n = 0, 1, \ldots$.

The characterization of submanifolds which are singular to all orders is of importance for understanding the range of validity of the moving frame method. The following theorem can be found in [20].

THEOREM 9.8. *A submanifold $S \subset M$ is totally singular if and only if its isotropy subgroup G_S does not act locally freely on S itself.*

EXAMPLE 9.9. Consider the special affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ acting on the plane $M = \mathbb{R}^2$ via $z \mapsto Az + b$, where $\det A = 1$. The totally singular curves are the straight lines, the isotropy subgroup consists of translations, shears, and unimodular scalings in the direction of the line. In terms of the coordinates $z = (x, u)$, the singular subset of J^n is given by

$$\mathcal{V}^n = \{(x, u^{(n)}) \mid u_{xx} = u_{xxx} = \cdots = u_n = 0\}, \tag{9.4}$$

where $u_n = \mathrm{d}^n u/\mathrm{d}x^n$. A curve $u = f(x)$ is totally singular at a point $(x_0, f(x_0))$ if and only if $f^{(n)}(x_0) = 0$ for all $n \geqslant 2$. In particular, an analytic curve that is totally singular at a point is necessarily a straight line.

The full affine group $\mathrm{A}(2)$ is interesting. Here the totally singular curves are the parabolas and the straight lines. The isotropy group of a parabola, say $u = x^2$, is the two-dimensional non-Abelian subgroup $(x, u) \mapsto (\lambda x + \mu, \lambda^2 u + 2\lambda\mu x + \mu^2)$. In this case the singular subset of J^n is also determined by the total derivatives of the Lie determinant

$$\mathcal{V}^n = \left\{(x, u^{(n)}) \,\middle|\, D_x^{n-4}\left[u_{xx}u_{xxxx} - \tfrac{5}{3}u_{xxx}^2\right] = 0\right\}. \tag{9.5}$$

The parabolas and straight lines form the general solution to the affine Lie determinant equation $u_{xx}u_{xxxx} = \frac{5}{3}u_{xxx}^2$.

Let us now quickly review the standard theory of differential invariants for Lie transformation groups; see [18, Chapter 5] for details.

DEFINITION 9.10. A *differential invariant* is a (locally defined) scalar differential function $I\colon \mathrm{J}^n \to \mathbb{R}$ which is invariant under the action of $G^{(n)}$, so that $I(g^{(n)} \cdot z^{(n)}) = I(z^{(n)})$ for all $g \in G$ and all $z^{(n)} \in \mathrm{J}^n$ where the equation is defined.

Assuming G acts locally effectively on subsets, there are

$$i_n = \dim \mathrm{J}^n - \dim G = p + q\binom{p+n}{n} - r \tag{9.6}$$

functionally independent differential invariants of order $\leqslant n$ near any point $z^{(n)} \in \mathcal{V}^n$.

Remark. If n is less that the stabilization order, then we replace r in (9.6) by the maximal orbit dimension of G on J^n and restrict $z^{(n)}$ to lie in the open subset of J^n where the prolonged orbits of $G^{(n)}$ have maximal dimension.

The basic method, due to Lie and Tresse [25], for constructing a complete system of differential invariants is to use invariant differential operators. A total differential operator (8.8) is said to be *G-invariant* if it commutes with the prolonged action of G. The most effective way to construct such operators relies on a suitably G-invariant basis for the horizontal one-forms on the jet space.

DEFINITION 9.11. A differential one-form ω on J^n is called *contact-invariant* if and only if, for every $g \in G$, we have $(g^{(n)})^*\omega = \omega + \theta_g$ for some contact form θ_g. A *horizontal contact-invariant coframe* on J^n is a collection of p linearly independent horizontal one-forms which are contact-invariant under the prolonged action of $G^{(n)}$.

For brevity we shall usually drop the adjective ‘horizontal’ in the description of contact-invariant coframes. Contact-invariant coframes are the jet space counterparts of the differential geometric coframes discussed in Section 5. Note that a contact-invariant coframe only forms a coframe on the horizontal subbundle $\mathcal{H} \subset T^*\mathrm{J}^\infty$. A full coframe on J^∞ requires additional contact forms; see below.

PROPOSITION 9.12. *If I is any differential invariant, its horizontal differential $\mathrm{d}_H I$ is a contact-invariant one-form.*

Thus, if we know p suitably independent differential invariants, we can construct a horizontal contact-invariant coframe. However, this approach is usually not the best method for determining such coframes. If $F(x, u^{(n)})$ is any differential function, we can rewrite its horizontal differential in terms of the horizontal coframe as

$$\mathrm{d}_H F = \sum_{j=1}^{p} (\mathcal{D}_j F)\omega^j. \tag{9.7}$$

The resulting G-invariant total differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are the jet space counterparts of the usual coframe derivatives, cf. (5.5). In local coordinates, suppose

$$\omega^i = \sum_{j=1}^{p} P^i_j(x, u^{(n)})\,\mathrm{d}x^j, \quad i = 1, \ldots, p, \tag{9.8}$$

where the coefficient matrix $P = (P^i_j)$ is nonsingular. The corresponding invariant differential operators are then given by

$$\mathcal{D}_j = \sum_{i=1}^{p} Q^i_j(x, u^{(n)})\,\mathrm{D}_i, \quad j = 1, \ldots, p, \tag{9.9}$$

with inverse coefficient matrix $Q = (Q^i_j) = P^{-1}$. If we consider the coordinate one-forms $\mathrm{d}x = (\mathrm{d}x^1, \ldots, \mathrm{d}x^p)^T$ and total derivatives $\mathbf{D} = (\mathrm{D}_1, \ldots, \mathrm{D}_p)^T$ as column vectors, then (9.8) is written as $\boldsymbol{\omega} = P \cdot \mathrm{d}x$, while (9.9) becomes $\mathcal{D} = Q^T \cdot \mathbf{D} = P^{-T} \cdot \mathbf{D}$. The invariant differential operators form an invariant 'horizontal frame' on J^∞, cf. [15].

Any invariant differential operator maps differential invariants to higher order differential invariants, and thus, by iteration, produces hierarchies of differential invariants of arbitrarily large order. In this way, a complete list of differential invariants can be produced by successively differentiating a finite number of differential invariants, which we call a *generating system* of differential invariants. We use the notation $\mathcal{D}_J = \mathcal{D}_{j_1} \cdots \mathcal{D}_{j_k}$, $1 \leqslant j_\nu \leqslant p$, denote the corresponding kth order invariant differential operators.

THEOREM 9.13. *Suppose that G is a transformation group, and let n be its order of stabilization. Then, in a neighborhood of any regular jet $z^{(n)} \in \mathcal{V}^n$, there exists a contact-invariant coframe $\{\omega^1, \ldots, \omega^p\}$, and corresponding invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$. If $I(z^{(k)})$ is a differential invariant, then so is $\mathcal{D}_J I$ for any multi-index J. Moreover, there exists a generating system of functionally independent differential invariants $I_1, \ldots, I_l$, of order at most $n+1$, such that, locally, every differential invariant can be written as a function of the differentiated invariants $\{\mathcal{D}_J I_\nu \mid \nu = 1, \ldots, l,\ \#J \geqslant 0\}$.*

See [22, p. 320] and [18, Theorem 5.49] for more details. The theorem is misstated in [18] – the order of the fundamental differential invariants should be at most $n+1$, not $n+2$. Except in the case of curves, where $p = 1$, the precise number of differential invariants required in a generating system is not known. A refinement of Theorem 9.13 will be proved below; see Theorem 13.1.

The invariant differential operators coming from a general contact-invariant coframe do not necessarily commute. The commutation formulae (5.6) for ordinary coframe derivatives are an immediate consequence of the closure identity $\mathrm{d}^2 = 0$. Similarly, if $\boldsymbol{\omega}$ is the contact-invariant coframe, then

$$\mathrm{d}_H \omega^k = -\sum_{i<j} A^k_{ij} \omega^i \wedge \omega^j, \quad k = 1, \ldots, p, \tag{9.10}$$

where the coefficients $A^k_{ij} = -A^k_{ji}$ are differential invariants. Thus, applying d_H to (9.7) produces the commutation formulae

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{i=1}^{p} A^k_{ij} \mathcal{D}_k, \quad i, j = 1, \ldots, p, \tag{9.11}$$

for the associated invariant differential operators. If all the coframe forms are constructed from differential invariants, i.e., $\omega^k = \mathrm{d}_H I^k$, $k = 1, \ldots, p$, then $\mathrm{d}_H \omega^k = 0$, and hence the invariant differential operators all commute in this particular case. The commutation formula (9.11) implies that a complete system of higher order differential invariants can be obtained by only including the differentiated invariants $\mathcal{D}_J I_\nu$ corresponding to nondecreasing multi-indices $1 \leqslant j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k \leqslant p$. However, even with this proviso, the differentiated invariants $\mathcal{D}_J I_\nu$ are not necessarily functionally independent.

DEFINITION 9.14. Given a functionally independent generating system of differential invariants $I_1, \ldots, I_k$, a *syzygy* is a functional dependency among the differentiated invariants: $H(\ldots \mathcal{D}_J I_\nu \ldots) \equiv 0$.

There are two types of syzygies, the first arising from the commutation rules for the invariant differential operators, and the second 'essential syzygies' are where the function H only depends on the differentiated invariants $\mathcal{D}_J I_\nu$ having nondecreasing multi-indices J. In Theorem 13.4 below, we shall provide a precise classification of all such syzygies.

EXAMPLE 9.15. An elementary example is provided by the three-parameter group $(x^1, x^2, u) \mapsto (\lambda x^1 + a, \lambda x^2, u + b)$ acting on $M = \mathbb{R}^3$. The one-forms $\omega^1 = (x^2)^{-1}\,\mathrm{d}x^1$, $\omega^2 = (x^2)^{-1}\,\mathrm{d}x^2$ form a contact-invariant coframe, with invariant differential operators $\mathcal{D}_1 = x^2\,\mathrm{D}_1$, $\mathcal{D}_2 = x^2\,\mathrm{D}_2$. We note the commutation formula $[\mathcal{D}_1, \mathcal{D}_2] = -\mathcal{D}_2$. The first-order differential invariants $I_1 = x^2 u_1$ and $I_2 = x^2 u_2$ form a generating system, and $I_{ijk} = (\mathcal{D}_1)^j (\mathcal{D}_2)^k I_i$, $i = 1, 2$, $j + k \geqslant 0$, form a complete system of differential invariants. In this case there is a single essential syzygy, $\mathcal{D}_2 I_1 = \mathcal{D}_1 I_2 - I_1$, from which all higher-order syzygies can be deduced by invariant differentiation.

Remark. For curves in an m-dimensional manifold, one requires $k = m - 1$ generating differential invariants, and a single invariant differential operator $\mathcal{D}$. Moreover, in this case, there are no syzygies among the differentiated invariants $\mathcal{D}^k I_\nu$, cf. [11, 18].

10. Higher Order Regularization

We are now in a position to describe the general version of our moving frame construction. The key idea is to apply the regularization technique to the prolonged group action on the extended jet bundles over the manifold M. All of our earlier constructions (which describe the order zero case) can be immediately applied to this particular type of transformation group action. Moving frames can be computed provided the prolonged action is (locally) free, i.e., on the regular subset of J^n. In this manner, we shall be able to construct a higher-order moving frame associated with all but the totally singular submanifolds of the original space.

We assume that G acts locally effectively on subsets of M. For simplicity, we only discuss the right regularization of the prolonged group action on the jet bundle $\mathrm{J}^n = \mathrm{J}^n(M, p)$ corresponding to p-dimensional submanifolds of M. The left counterparts can be simply obtained by applying the group inversion.

DEFINITION 10.1. The nth order *regularized jet bundle* is the trivial left principal bundle $\pi_n\colon\ \mathcal{B}^{(n)} = G \times \mathrm{J}^n \to \mathrm{J}^n$. The nth order (right) *regularization* of the prolonged group action on J^n is the action of G on $\mathcal{B}^{(n)}$ given by

$$\begin{aligned} R_g^{(n)}(h, z^{(n)}) &= R^{(n)}(g, (h, z^{(n)})) \\ &= (h \cdot g^{-1}, g^{(n)} \cdot z^{(n)}), \quad g \in G,\ (h, z^{(n)}) \in \mathcal{B}^{(n)}. \end{aligned} \tag{10.1}$$

Theorem 3.2 implies that the regularized action on $\mathcal{B}^{(n)}$ is both free and regular.

DEFINITION 10.2. A *lifted differential invariant* is a (locally defined) invariant function $L\colon\ \mathcal{B}^{(n)} \to N$.

A complete system of functionally independent lifted differential invariants is provided by the components of the order n evaluation map

$$w^{(n)} = g^{(n)} \cdot z^{(n)}. \tag{10.2}$$

Clearly $w^{(n)}\colon\ \mathcal{B}^{(n)} \to \mathrm{J}^n$ is invariant under the lifted action (10.1). As in Section 3, every lifted differential invariant can be locally written as a function of the fundamental lifted differential invariants $w^{(n)}$. In particular, an ordinary differential invariant $I\colon\ \mathrm{J}^n \to \mathbb{R}$ also defines a lifted differential invariant $L = I \circ \pi_n$, and hence can also be locally expressed as a function of the w's; conversely, any lifted invariant $L(g, x, u^{(n)})$ that does not depend on the g coordinates automatically defines an ordinary differential invariant. Our simple replacement Theorem 3.7

immediately applies to the construction of differential invariants from their lifted counterparts.

THEOREM 10.3. *Let $I(z^{(n)})$ be an ordinary differential invariant. Then we can write $I(z^{(n)}) = I(g^{(n)} \cdot z^{(n)})$ as the same function of the lifted differential invariants.*

Remark. In Riemannian geometry, Theorem 10.3 reduces to the striking Thomas Replacement Theorem [24, p. 109], which is proved by appealing to normal coordinates. See [2] for recent applications of Thomas' result.

EXAMPLE 10.4. Consider the (standard) action of the Euclidean group SE(2) on $M = \mathbb{R}^2$. Introducing local coordinates (x, u), the second order prolongation maps a point $(x, u, u_x, u_{xx}) \in \mathrm{J}^2$ to

$$\left(x \cos\phi - u \sin\phi + a,\ x \cos\phi + u \sin\phi + b,\right.$$
$$\left.\frac{\sin\phi + u_x \cos\phi}{\cos\phi - u_x \sin\phi},\ \frac{u_{xx}}{(\cos\phi - u_x \sin\phi)^3}\right), \tag{10.3}$$

reproducing the action (4.12). The second-order lifted invariants (10.2), which we denote as $w^{(2)} = (y, v, v_y, v_{yy})$, are the components of the transformation formulae (10.3). The Euclidean curvature differential invariant can be constructed in terms of the lifted invariants:

$$\kappa = \frac{v_{yy}}{(1 + v_y^2)^{3/2}} = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}. \tag{10.4}$$

The Replacement Theorem 10.3 guarantees that the formula for κ in terms of the usual jet coordinates (x, u, u_x, u_{xx}) is the same functional relation as its formula in terms of the lifted invariants (y, v, v_y, v_{yy}).

The regularization construction extends to the infinite order regularized jet bundle $\pi_\infty\colon \mathcal{B}^{(\infty)} = G \times \mathrm{J}^\infty \to \mathrm{J}^\infty$ in the obvious manner. The pull-back of the contact bundle $\mathcal{C}^{(\infty)} \subset T^*\mathrm{J}^\infty$ defines the contact subbundle[⋆] $\mathcal{C}^{(\infty)} \subset T^*\mathcal{B}^{(\infty)}$. Similarly, the pull-back via $\pi_G\colon \mathcal{B}^{(\infty)} \to G$ of the cotangent bundle of G, spanned by its Maurer–Cartan forms, define a second intrinsic subbundle of $T^*\mathcal{B}^{(\infty)}$, which we also denote by T^*G. The product bundle $T^*G \times \mathcal{C}^{(\infty)}$ forms a codimension p subbundle of the cotangent bundle $T^*\mathcal{B}^{(\infty)}$. Since $\mathcal{B}^{(\infty)} = G \times \mathrm{J}^\infty$ is a Cartesian product, the differential on $\mathcal{B}^{(\infty)}$ naturally splits into jet and group components, $\mathrm{d} = \mathrm{d}_J + \mathrm{d}_G$.

PROPOSITION 10.5. *If ω is a G-invariant differential form on $\mathcal{B}^{(\infty)}$, then so are both $\mathrm{d}_J\omega$ and $\mathrm{d}_G\omega$. In particular, if $L(g, z^{(n)})$ is a lifted differential invariant, then its jet and group differentials, $\mathrm{d}_J L$ and $\mathrm{d}_G L$, are invariant one-forms on $\mathcal{B}^{(\infty)}$.*

⋆ For simplicity, we drop explicit reference to the pull-back via the projection map.

Let us now discuss the local coordinate expressions for the regularized action and its invariants. As above, the introduction of local coordinates $(x, u^{(\infty)})$ on J^∞ produces a local coframe on $\mathcal{B}^{(\infty)}$ consisting of the horizontal forms $\mathrm{d}x$, the system of basic contact forms $\boldsymbol{\theta}$, along with the right-invariant Maurer–Cartan forms $\boldsymbol{\mu}$, all pulled back to $\mathcal{B}^{(\infty)}$. The choice of horizontal complement produces a splitting of the differential on $\mathcal{B}^{(\infty)}$ into horizontal, vertical, and group components, so that we have the more refined decomposition

$$\mathrm{d}\omega = \mathrm{d}_J\omega + \mathrm{d}_G\omega = \mathrm{d}_H\omega + \mathrm{d}_V\omega + \mathrm{d}_G\omega \tag{10.5}$$

for any differential form on $\mathcal{B}^{(\infty)}$. Note also that

$$\begin{aligned} &\mathrm{d}_H \circ \mathrm{d}_H = 0, \qquad \mathrm{d}_V \circ \mathrm{d}_V = 0, \qquad \mathrm{d}_G \circ \mathrm{d}_G = 0, \\ &\mathrm{d}_H \circ \mathrm{d}_V = -\mathrm{d}_V \circ \mathrm{d}_H, \qquad \mathrm{d}_H \circ \mathrm{d}_G = -\mathrm{d}_G \circ \mathrm{d}_H, \\ &\mathrm{d}_V \circ \mathrm{d}_G = -\mathrm{d}_G \circ \mathrm{d}_V. \end{aligned} \tag{10.6}$$

Remark. We have not investigated the topological and variational aspects of the induced ‘regularized variational tricomplex’ governed by the differentials d_H, d_V and d_G.

In particular, the horizontal and the vertical differentials of a function $F(g, x, u^{(n)})$ have the same formulae (8.5), (8.7), as before, where the total derivatives D_i have their usual coordinate formulae, i.e., there are no derivatives with respect to the g coordinates. Note that the horizontal and vertical differentials of a lifted invariant are *not,* in general, G-invariant one-forms on $\mathcal{B}^{(\infty)}$. However, the horizontal differential does satisfy the weaker, but very important, invariance property of Definition 9.11.

PROPOSITION 10.6. *If $L(g, z^{(n)})$ is a lifted invariant, then its horizontal differential $\mathrm{d}_H L$ is a contact-invariant one-form.*

The standard jet space coordinates $(x, u^{(\infty)})$ are not well adapted to the lifted group action on $\mathcal{B}^{(\infty)}$, and we shall replace them by a fundamental system of invariant coordinates based on the fundamental lifted differential invariants. The introduction of local independent and dependent variable coordinates $z = (x, u)$ on M induces a local identification with a trivial bundle $E = X \times U$. This induces a splitting of the fundamental zeroth order lifted invariants $w = (w^1, \ldots, w^m) = g\cdot z$ into two components. In the (x, u) coordinates, we write* $w = (y, v)$, where

* For simplicity, we have chosen to split the lifted invariants into independent and dependent components in the same way as we split the unlifted variables. Actually, one can choose alternative splittings of w into p independent and q dependent components, although one must then accordingly modify the required transversality conditions.

$y = (y^1, \ldots, y^p)$ will be considered as 'lifted independent variables', and $v = (v^1, \ldots, v^q)$, as 'lifted dependent variables'. Let

$$\eta^i = \mathrm{d}_H y^i = \sum_{j=1}^{p} (\mathrm{D}_j y^i)\, \mathrm{d}x^j, \quad i = 1, \ldots, p, \tag{10.7}$$

denote the horizontal differentials of lifted independent variables. The coefficient matrix $\mathbf{D}y = (\mathrm{D}_j y^i)$ is obtained by total differentiation of the lifted invariants y treating the group parameters as constants, so the lifted horizontal forms $\boldsymbol{\eta} = (\mathbf{D}y) \cdot \mathrm{d}x$ are defined on the first-order regularized jet space $\mathcal{B}^{(1)}$. Since the functions y^i are lifted invariants, Proposition 10.6 implies that the one-forms $\boldsymbol{\eta}$ are contact-invariant under the lifted action of G on $\mathcal{B}^{(1)}$. The η's are linearly independent if and only if the y's have nonvanishing total Jacobian determinant:

$$\det \mathbf{D}y = \frac{\mathbf{D}(y^1, \ldots, y^p)}{\mathbf{D}(x^1, \ldots, x^p)} = \det\big(\mathrm{D}_j y^i\big) \neq 0. \tag{10.8}$$

This condition can be geometrically characterized as follows.

PROPOSITION 10.7. *The horizontal one-forms $\boldsymbol{\eta} = (\mathbf{D}y) \cdot \mathrm{d}x$ are linearly independent, $\eta^1 \wedge \cdots \wedge \eta^p \neq 0$, on the open subset $\mathcal{W}^{(1)} \subset \mathcal{B}^{(1)}$ determined by the 1-jets of submanifolds S such that both S and $g \cdot S$ are transverse with respect to the given coordinates on M. Thus, $\mathcal{W}^{(1)} = \{(g, z^{(1)}) \in G \times \mathrm{J}^1 E \mid g^{(1)} \cdot z^{(1)} \in \mathrm{J}^1 E\}$. At such points, we call $\boldsymbol{\eta} = (\eta^1, \ldots, \eta^p)$ the* lifted (horizontal) contact-invariant coframe *for the given coordinate chart.*

The corresponding invariant differential operators are readily found. As in the usual (unlifted) version, (9.7), we write the total differential of any scalar function $F\colon \mathcal{B}^{(n)} \to \mathbb{R}$ in invariant form

$$\mathrm{d}_H F = \sum_{j=1}^{p} (\mathcal{E}_j F)\, \eta^j \tag{10.9}$$

with respect to the prescribed contact-invariant coframe. The corresponding total differential operators are $\mathcal{E} = (\mathbf{D}y)^{-T} \cdot \mathbf{D}$, or, explicitly,

$$\mathcal{E}_j F = \frac{\mathbf{D}(y^1, \ldots, y^{j-1}, F, y^{j+1}, \ldots, y^p)}{\mathbf{D}(y^1, \ldots, y^p)} = \sum_{i=1}^{p} Z_j^i(g, x, u^{(1)})\, \mathrm{D}_i, \tag{10.10}$$

where $Z = (Z_j^i) = (\mathbf{D}y)^{-1}$. Thus, we can identify the *lifted invariant differential operator* $\mathcal{E}_j = D_{y^j}$ with total differentiation with respect to the lifted invariant y^j; in particular, $\mathcal{E}_j y^i = \delta_j^i$. Note that the lifted invariant differential operators do not involve differentiation with respect to the group parameters. A very important point is that, unlike the usual invariant differential operators, the lifted invariant differential operators *always* mutually commute:

$$\big[\mathcal{E}_i, \mathcal{E}_j\big] = 0. \tag{10.11}$$

This follows from the closure of the horizontal differential, $\mathrm{d}_H \circ \mathrm{d}_H = 0$, and is an immediate consequence of the fact that the lifted contact-invariant coframe (10.7) is the horizontal derivative of lifted invariants; see the discussion following (9.11). We let $\mathcal{E}_K = \mathcal{E}_{k_1} \cdot \cdots \cdot \mathcal{E}_{k_l}$ denote the associated higher-order invariant differential operator; Equation (10.11) shows that the order of the invariant differentiation is irrelevant.

The lifted invariant differential operators can be used to compute higher-order lifted differential invariants. The basic result follows immediately from (10.9), the contact-invariance of the forms $\boldsymbol{\eta}$, along with the fact that the prolonged group transformations preserve the contact ideal.

PROPOSITION 10.8. *If $L\colon \mathcal{B}^{(n)} \to \mathbb{R}$ is any lifted differential invariant, then so are its invariant derivatives $\mathcal{E}_K L\colon \mathcal{B}^{(n+k)} \to \mathbb{R}$, where $k = \#K \geqslant 0$.*

If we successively apply the invariant differential operators associated with the first p lifted invariants $y = (y^1, \ldots, y^p)$ to the remaining zeroth order invariants $v = (v^1, \ldots, v^q)$, we recover all the higher-order lifted invariants $v^{(n)}$. Since $w^{(n)} = g^{(n)} \cdot z^{(n)}$, an alternative way of viewing this result is that the process of lifted invariant differentiation produces the explicit formulae for the prolonged group transformations, thereby implementing the standard process of implicit differentiation, cf. [17].

LEMMA 10.9. *The components of the nth order lifted invariant $w^{(n)}$ consist of the basic invariants $w = (y, v)$ together with the higher-order lifted differential invariants*

$$v^\alpha_K = \mathcal{E}_K v^\alpha, \quad \alpha = 1, \ldots, q, \ \#K \leqslant n. \tag{10.12}$$

Proof. For fixed g, the map $w^{(n)}\colon \mathrm{J}^n \to \mathrm{J}^n$ is a contact transformation on the jet bundle, hence the pull-back $(w^{(n)})^*$ maps contact forms to contact forms. Now,

$$(w^{(n)})^* \theta^\alpha_K = \mathrm{d}_J v^\alpha_K - \sum_{i=1}^{p} v^\alpha_{K,i}\, \mathrm{d}_J y^i, \quad \alpha = 1, \ldots, q, \ \#K \leqslant n-1. \tag{10.13}$$

The right-hand side will be a contact form if and only if its horizontal component vanishes, so that

$$\mathrm{d}_H v^\alpha_K = \sum_{i=1}^{p} v^\alpha_{K,i} \eta^i, \quad \alpha = 1, \ldots, q, \ \#K \leqslant n-1. \tag{10.14}$$

Comparing (10.14) with (10.9) completes the proof. □

EXAMPLE 10.10. For the (standard) action of the Euclidean group SE(2) on $M = \mathbb{R}^2$, the zeroth order lifted invariants $w = (y, v)$ are just the group transformation formulae:

$$y = x\cos\phi - u\sin\phi + a, \qquad v = x\cos\phi + u\sin\phi + b.$$

The lifted horizontal contact-invariant form is

$$\eta = \mathrm{d}_H y = (\cos\phi - u_x \sin\phi)\,\mathrm{d}x,$$

which is well-defined provided ϕ does not rotate the curve to have a vertical tangent. Therefore

$$\mathcal{E} = \mathrm{D}_y = \frac{1}{\cos\phi - u_x \sin\phi}\mathrm{D}_x$$

is the lifted invariant differential operator. The higher-order lifted invariants are obtained by successively applying $\mathcal{E}$ to the other zeroth-order lifted invariant v. The first two are

$$v_y = \frac{\sin\phi + u_x \cos\phi}{\cos\phi - u_x \sin\phi}, \qquad v_{yy} = \frac{u_{xx}}{(\cos\phi - u_x \sin\phi)^3}. \tag{10.15}$$

As remarked above, the one-forms $\eta^i = \mathrm{d}_H y^i$ are not strictly invariant under the prolonged group action. However, we can use their invariant counterparts

$$\tau^i = \mathrm{d}_J y^i = \mathrm{d}_H y^i + \mathrm{d}_V y^i = \eta^i + \chi^i, \quad i = 1, \ldots, p, \tag{10.16}$$

to define a fully invariant coframe on the regularized jet bundle $\mathcal{B}^{(\infty)}$. In (10.16), the χ^i are contact forms that are not invariant under the lifted action of G.

There are two methods for constructing invariant contact forms. First, since the horizontal component of the invariant one-form on the right-hand side of (10.13) vanishes by virtue of (10.14), its vertical component

$$\begin{aligned} \vartheta^\alpha_K &= \mathrm{d}_J v^\alpha_K - \sum_{k=1}^{p} v^\alpha_{K,i}\,\mathrm{d}_J y^i \\ &= \mathrm{d}_V v^\alpha_K - \sum_{k=1}^{p} v^\alpha_{K,i}\,\mathrm{d}_V y^i, \quad \alpha = 1, \ldots, q, \end{aligned} \tag{10.17}$$

is an invariant contact form. The resulting collection $\boldsymbol{\vartheta} = \{\ldots, \vartheta^\alpha_K, \ldots\}$ forms a complete set of lifted invariant contact forms on $\mathcal{B}^{(\infty)}$. The forms $\boldsymbol{\tau}, \boldsymbol{\vartheta}$ are the pullbacks of the canonical coframe (8.2) by the map $w^{(\infty)}$ modulo the Maurer–Cartan forms $\boldsymbol{\mu}$.

PROPOSITION 10.11. *The collection of one-forms*

$$\boldsymbol{\tau} = \{\tau^1, \ldots, \tau^p\}, \qquad \boldsymbol{\vartheta} = \{\ldots, \vartheta^\alpha_K, \ldots\}, \qquad \boldsymbol{\mu} = \{\mu^1, \ldots, \mu^r\}, \tag{10.18}$$

provide an invariant local coframe on $\mathcal{B}^{(\infty)} = G \times \mathrm{J}^\infty$.

Invariant contact forms can also be found via the process of invariant differentiation.

THEOREM 10.12. *Let* ϑ^α *define the complete system of invariant zeroth-order contact forms, as in* (10.17) *with* $K = \emptyset$. *The higher-order contact forms*

$$\vartheta^\alpha_K = \mathcal{E}_K \vartheta^\alpha, \quad \alpha = 1, \ldots, q, \ \#K > 0, \tag{10.19}$$

obtained by Lie differentiating the zeroth-order contact forms provide a complete list of lifted invariant contact forms.

Proof. Applying d_H to (10.17) and using (10.6), we find

$$\begin{aligned} \mathrm{d}_H \vartheta^\alpha_K &= -\sum_{j=1}^p \mathrm{d}_V(v^\alpha_{K,j}\eta^j) - \sum_{i,j=1}^p v^\alpha_{K,i,j}\,\eta^j \wedge \mathrm{d}_V y^i + \sum_{i=1}^p v^\alpha_{K,i}\,\mathrm{d}_V \eta^i \\ &= -\sum_{j=1}^p \vartheta^\alpha_{K,j} \wedge \eta^j. \end{aligned} \tag{10.20}$$

The identity (10.19) follows by pairing (10.20) with the total vector field $\mathcal{E}_j$. □

11. Higher Order Moving Frames

The construction of higher-order moving frames proceeds in direct analogy with the zeroth order version. As usual, for simplicity, we only explicitly treat the right versions.

DEFINITION 11.1. An nth order (right) *moving frame* is a map

$$\rho^{(n)}\colon \mathrm{J}^n \longrightarrow G \tag{11.1}$$

which is (locally) G-equivariant with respect to the prolonged action $G^{(n)}$ on J^n, and the right multiplication action $h \mapsto h \cdot g^{-1}$ on G itself.

The corresponding left moving frame of order n is merely $\widetilde{\rho}^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)})^{-1}$. Note that an nth-order moving frame automatically defines a moving frame on the higher-order jet bundles, namely $\rho^{(n)} \circ \pi^k_n\colon \mathrm{J}^k \to G$, $k \geqslant n$. The fundamental existence theorem for moving frames is an immediate consequence of Theorem 4.4.

THEOREM 11.2. *If G acts on M, then an nth-order moving frame exists in a neighborhood of any point $z^{(n)} \in \mathcal{V}^n$ in the regular component of* J^n.

Remark. Proposition 9.6 only guarantees the local G-equivariance of the moving frame; global equivariance requires that $G^{(n)}$ act freely on $\mathcal{V}^n$.

In particular, the minimal order at which any moving frame can be constructed is the stabilization order of the group. Indeed, according to the construction in Section 4, the choice of a cross-section $K^{(n)} \subset \mathrm{J}^n$ to the prolonged group orbits serves to define a moving frame $\rho^{(n)}$ in a neighborhood of any point $z^{(n)} \in K^{(n)}$. The set $\mathcal{L}^{(n)} = (w^{(n)})^{-1}K^{(n)}$ forms the graph of a local G-equivariant section

$\sigma^{(n)}\colon \mathrm{J}^n \to \mathcal{B}^{(n)}$, whose moving frame is $\rho^{(n)} = \pi_G \circ \sigma^{(n)}$. Moreover, composing $\sigma^{(n)}$ with $w^{(n)}$ produces the corresponding differential invariants.

DEFINITION 11.3. The *fundamental nth-order normalized differential invariants* associated with a moving frame $\rho^{(n)}$ of order n (or less) are given by

$$I^{(n)}(z^{(n)}) = w^{(n)} \circ \sigma^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)}. \tag{11.2}$$

THEOREM 11.4. *If $J(x, u^{(n)})$ is any nth-order differential invariant, then, locally, J is a function of the normalized nth order differential invariants, i.e., $J = H \circ I^{(n)}$.*

Remark. The fundamental normalized differential invariants are not necessarily functionally independent. Indeed, typically we normalize some of the components of the fundamental lifted invariant $w^{(n)}$ by setting them equal to constants; the corresponding normalized differential invariants will then, of course, also be constant. However, Theorem 4.5 does imply that the nth order normalized differential invariants contain *all* of the nth order differential invariants. In particular, any lower order differential invariants, including those on jet bundles where G does not yet act freely, will appear as functional combinations of the $I^{(n)}$.

As in the order zero case, given an arbitrary differential function $F\colon \mathrm{J}^n \to \mathbb{R}$, then $L = F \circ w^{(n)}\colon \mathcal{B}^{(n)} \to \mathbb{R}$ defines a lifted differential invariant, and hence $J = L \circ \sigma^{(n)} = F \circ I^{(n)}$ defines a differential invariant, called the *invariantization* of F with respect to the given moving frame. Thus a moving frame provides a natural way to construct a differential invariant from any differential function! Theorem 10.3 just says that if F itself is a differential invariant, then $F \circ w^{(n)}$ is independent of the group parameters, and hence $J = F$. Thus, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants. One case of interest is the jth total derivative $D_j F(x, u^{(n+1)})$ of a differential function $F(x, u^{(n)})$. The corresponding lifted invariant coincides with the jth invariant derivative of the lifted invariant $L = F \circ w^{(n)}$, so that $D_j F \circ w^{(n+1)} = \mathcal{E}_j L$. Consequently, the invariantization of $D_j F$ is given by

$$\mathcal{E}_j L \circ \sigma^{(n)} = D_j F \circ I^{(n+1)}. \tag{11.3}$$

As in the order zero case, in applications to equivalence problems, one restricts the moving frame to a submanifold.

DEFINITION 11.5. An *nth order moving frame restricted* to a p-dimensional submanifold $\iota\colon X \to S \subset M$ whose n-jet $\mathrm{j}_n S$ lies in the domain of definition of $\rho^{(n)}$ is defined as the composition

$$\lambda^{(n)} = \rho^{(n)} \circ \mathrm{j}_n \iota\colon\ X \longrightarrow G. \tag{11.4}$$

Equivalently, a nth order moving frame is a smooth map $\lambda^{(n)}\colon X \to G$ which factors through an equivalent map from J^n to G:

$$\begin{array}{ccc} & \mathrm{J}^n & \\ {}^{\mathrm{j}_n\iota}\nearrow & & \searrow^{\rho^{(n)}} \\ X & \xrightarrow{\quad\lambda^{(n)}\quad} & G \end{array} \tag{11.5}$$

generalizing the order zero construction (4.17). Theorems 4.4 and 9.8 serve to characterize the submanifolds which admit moving frames.

THEOREM 11.6. *A submanifold $S \subset M$ admits a (locally defined) nth-order moving frame if and only if S is regular of order n, i.e., $\mathrm{j}_n S \subset \mathcal{V}^n$.*

Thus, in the analytic category, a submanifold S admits a moving frame (of some sufficiently high order) if and only if its isotropy subgroup G_S acts freely on S.

The practical implementation of the higher-order moving frame construction relies on the higher order version of the normalization method. Consider a point $z^{(n)} \in \mathcal{V}^n$ contained in the regular subset of the nth jet space. According to Proposition 4.10, in a neighborhood of $z^{(n)}$, we can choose a regular system of r lifted differential invariants $L(g, x, u^{(n)})$ having maximal rank $r = \operatorname{rank} \mathrm{d}_G L = \dim G$. The Implicit Function Theorem allows us to solve the normalization equations

$$L_1(g, z^{(n)}) = c_1, \quad \ldots, \quad L_r(g, z^{(n)}) = c_r, \tag{11.6}$$

for the group parameters g in terms of $z^{(n)}$ provided the normalization constants $c = (c_1, \ldots, c_r)$ belong to the image of L. Typically, one chooses L to be r suitable components of the fundamental lifted differential invariant $w^{(n)} = g^{(n)} \cdot z^{(n)}$ that have as low an order as possible, subject to the maximal rank condition, although this is by no means essential to the implementation of the method. The solution to the normalization equations (11.6) determines an nth-order moving frame $g = \rho^{(n)}(x, u^{(n)})$. Substituting the formula for the moving frame into the remaining lifted invariants produces a complete system of differential invariants on the open neighborhood of $z^{(n)} \in \mathcal{V}^n$ where $\rho^{(n)}$ is defined.

In terms of the invariant local coordinates $w^{(n)} = (y, v^{(n)})$ on $\mathcal{B}^{(n)}$, the fundamental normalized differential invariants $I^{(n)} = (\sigma^{(n)})^* w^{(n)}$ associated with the given moving frame $g = \rho^{(n)}(x, u^{(n)})$ are

$$\begin{aligned} J^i(x, u^{(n)}) &= y^i(\rho^{(n)}(x, u^{(n)}), x, u), \quad i = 1, \ldots, p, \\ I^\alpha_K(x, u^{(l)}) &= v^\alpha_K(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}), \quad \alpha = 1, \ldots, q, \ k = \#K \geqslant 0. \end{aligned} \tag{11.7}$$

In the second formula, $l = \max\{n, k\}$. As above, some of these may be constant and/or functionally dependent due to normalizations. However, (11.7) do include a complete system of differential invariants, meaning that, provided $k \geqslant n$, any other

kth-order differential invariant can be locally expressed as a function of the J^i and I^α_K for $\#K \leqslant k$.

EXAMPLE 11.7. Consider the elementary similarity group $G = \mathbb{R}^+ \ltimes \mathbb{R}^2$ acting on $M = \mathbb{R}^2$ via

$$(x, u) \longmapsto (\alpha x + a, \alpha^3 u + b). \tag{11.8}$$

If we choose $y = \alpha x + a$ as the lifted independent variable, then $\eta = \mathrm{d}_H y = \alpha \mathrm{d}x$ is the corresponding horizontal invariant form, with invariant differential operator $\mathcal{E} = \mathrm{D}_y = \alpha^{-1}\mathrm{D}_x$. Successively applying $\mathcal{E}$ to the dependent-order zero lifted invariant v produces the complete system of higher-order lifted invariants: $v_n = \mathcal{E}^n v = \alpha^{3-n} u_n$, where $u_n = \mathrm{D}^n_x u$, and $n \geqslant 1$. Therefore, on $\mathcal{B}^{(4)}$, say, the lifted invariants $w^{(4)}$ are

$$\begin{aligned} &y = \alpha x + a, \qquad v = \alpha^3 u + b, \qquad v_y = \alpha^2 u_x, \qquad v_{yy} = \alpha u_{xx}, \\ &v_{yyy} = u_{xxx}, \qquad v_{yyyy} = \alpha^{-1} u_{xxxx}. \end{aligned} \tag{11.9}$$

The simplest first order moving frame is found by normalizing $y = v = 0$, $u_x = 1$, whereby

$$a = -\frac{x}{\sqrt{u_x}}, \qquad b = -\frac{u}{u_x^{3/2}}, \qquad \alpha = \frac{1}{\sqrt{u_x}}, \tag{11.10}$$

which is well-defined on the subset $\widetilde{\mathcal{V}}^1 = \{u_x > 0\}$. The resulting normalized fourth-order differential invariants $I^{(4)}$ are obtained by substituting (11.10) into the lifted invariants:

$$\begin{aligned} &J^1 = 0, \qquad I_0 = 0, \qquad I_1 = 1, \qquad I_2 = \frac{u_{xx}}{\sqrt{u_x}}, \\ &I_3 = u_{xxx}, \qquad I_4 = \sqrt{u_x}\, u_{xxxx}. \end{aligned}$$

The moving frame (11.10) applies to curves $u = f(x)$ provided the tangent is not horizontal, so $u_x \neq 0$. If the curve has a horizontal tangent, then one can construct a second-order moving frame by using the alternative normalization $v_{yy} = 1$, with

$$a = -\frac{x}{u_{xx}}, \qquad b = -\frac{u}{u_{xx}^2}, \qquad \alpha = \frac{1}{u_{xx}}, \tag{11.11}$$

which is well-defined on the subdomain $\mathcal{V}^2 = \{u_{xx} \neq 0\}$ and, hence, applies to curves with horizontal tangent at a point, but not those with inflection points. (Curves with horizontal inflection points can be handled by a yet higher-order normalization.) The moving frame (11.11) leads to a slightly different normalized fourth-order differential invariant $I^{(4)}$:

$$\begin{aligned} &\widetilde{J} = 0, \qquad \widetilde{I}_0 = 0, \qquad \widetilde{I}_1 = \frac{u_x}{u_{xx}^2}, \qquad \widetilde{I}_2 = 1, \\ &\widetilde{I}_3 = u_{xxx}, \qquad \widetilde{I}_4 = u_{xx} u_{xxxx}, \end{aligned}$$

all of which are, naturally, functions of the previous normalized differential invariants on their common domain of definition. Note that the two moving frames correspond to different choices of cross-section of J^2, namely $\{(0, 0, 1, k)\}$ for (11.10) and $\{(0, 0, \tilde{k}, 1)\}$ for (11.11).

Remark. In his thesis, I. Lisle [15, Ex. 4.4.21], introduces a 'naïve elimination method' for determining differential invariants that is essentially the same as the normalization method used here. Our theory of normalization demonstrates how Lisle's method can be formalized into a practical and elegant alternative to the more traditional methods for computing differential invariants.

12. Higher Order Moving Coframes

The final ingredients in our general theory are the jet space counterparts of the moving coframe forms. These will produce the normalized invariant differential operators that can be used to recursively construct complete systems of higher-order differential invariants, and will govern the equivalence and symmetry properties of submanifolds.

DEFINITION 12.1. The *moving coframe* of order n associated with an order n moving frame $\rho^{(n)}\colon J^n \to G$ is the extended differential system $\Sigma^{(n)} = \{\boldsymbol{\zeta}^{(n)}, \mathrm{d}I^{(n)}, I^{(n)}\}$ consisting of the pull-back $\boldsymbol{\zeta}^{(n)} = (\rho^{(n)})^* \boldsymbol{\mu}$ of the Maurer–Cartan forms to J^n, along with the nth-order normalized differential invariants $I^{(n)}$ and their differentials.

LEMMA 12.2. *The nth order moving coframe $\Sigma^{(n)}$ forms a $G^{(n)}$-coframe on $\mathcal{V}^n$.*

In other words, $\Sigma^{(n)}$ is involutive and its symmetry group coincides with the nth prolongation of G acting on J^n. Lemma 12.2 is a direct consequence of Lemma 6.4. Any other $G^{(n)}$-coframe on J^n is invariantly related to the moving coframe, meaning that its functions are combinations of the differential invariants, and the one-forms are linear combinations of the moving coframe forms, with differential invariant coefficients. A particularly useful $G^{(n)}$-coframe can be constructed using the method in Theorem 6.6.

THEOREM 12.3. *Let $\rho^{(n)}\colon \mathcal{V}^n \to G$ be a right moving frame on the nth jet bundle over M. The extended coframe $\Gamma^{(n)} = \{\boldsymbol{\gamma}^{(n)}, I^{(n)}\}$ consisting of the normalized differential invariants*

$$I^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} = (\sigma^{(n)})^* w^{(n)}, \tag{12.1}$$

and the one-forms

$$\boldsymbol{\gamma}^{(n)} = \rho^{(n)}(z^{(n)}) \cdot \mathrm{d}z^{(n)} = (\sigma^{(n)})^* \, \mathrm{d}_J w^{(n)}, \tag{12.2}$$

forms a $G^{(n)}$-coframe on J^n and, hence, is invariantly related to the moving coframe $\Sigma^{(n)}$.

EXAMPLE 12.4. Consider the elementary similarity group (11.8). The moving coframe $\Sigma^{(2)}$ is obtained by applying the moving frame map (11.10) to the right invariant Maurer–Cartan forms

$$\mu^1 = \frac{\mathrm{d}\alpha}{\alpha}, \qquad \mu^2 = \mathrm{d}a - \frac{a}{\alpha}\,\mathrm{d}\alpha, \qquad \mu^3 = \mathrm{d}b - \frac{3b}{\alpha}\,\mathrm{d}\alpha. \tag{12.3}$$

The resulting moving coframe forms are

$$\zeta^1 = -\frac{\mathrm{d}u_x}{2u_x}, \qquad \zeta^3 = -\frac{\mathrm{d}u}{u_x^{3/2}}, \qquad \zeta^2 = -\frac{\mathrm{d}x}{\sqrt{u_x}}, \qquad \zeta^4 = \mathrm{d}\left(\frac{u_{xx}}{\sqrt{u_x}}\right),$$

where the final form is the differential of the fundamental second-order differential invariant $I_2 = u_x^{-1/2}u_{xx}$. The second-order extended coframe $\Gamma^{(2)}$ is obtained by applying the moving frame map (11.10) to the jet differentials of the lifted invariants

$$\mathrm{d}_J y = \alpha\,\mathrm{d}x, \qquad \mathrm{d}_J v = \alpha^3\,\mathrm{d}u, \qquad \mathrm{d}_J v_y = \alpha^2\,\mathrm{d}u_x, \qquad \mathrm{d}_J v_{yy} = \alpha\,\mathrm{d}u_{xx},$$

leading to

$$\gamma^1 = \frac{\mathrm{d}x}{\sqrt{u_x}}, \qquad \gamma^2 = \frac{\mathrm{d}u}{u_x^{3/2}}, \qquad \gamma^3 = \frac{\mathrm{d}u_x}{u_x}, \qquad \gamma^4 = \frac{\mathrm{d}u_{xx}}{\sqrt{u_x}}.$$

The invariant relation

$$\gamma^1 = -\zeta^2, \qquad \gamma^2 = -\zeta^3, \qquad \gamma^3 = -2\zeta^1, \qquad \gamma^4 = \zeta^4 - I_2\zeta^1,$$

between the two $G^{(2)}$-coframes follows from (6.4), using the coefficients of the prolonged infinitesimal generators for the given transformation group.

As in Section 10, we use the local coordinates $(x, u^{(\infty)})$ on J^∞ and lifted coordinates $(g, y, v^{(\infty)})$ on $\mathcal{B}^{(\infty)}$. The pull-back of the lifted contact-invariant coframe $\boldsymbol{\eta} = \mathrm{d}_H y$ under the moving frame section will produce a contact-invariant coframe, from which we can construct the required invariant differential operators.

DEFINITION 12.5. The *normalized contact-invariant coframe* is the pull-back of the lifted contact-invariant coframe:

$$\boldsymbol{\omega} = (\sigma^{(n)})^*\boldsymbol{\eta} = (\sigma^{(n)})^*\,\mathrm{d}_H y. \tag{12.4}$$

LEMMA 12.6. *The horizontal one-forms $\boldsymbol{\omega} = (\sigma^{(n)})^*\mathrm{d}_H y$ are linearly independent at a point $z^{(n)}$ in the domain of definition of the moving frame map if and only if $z^{(n)} = \mathrm{j}_n S|_z$ is the n-jet of a transverse submanifold $S \subset M$.*

Proof. In terms of our bundle coordinates, the transversality of S implies $z^{(n)} \in \mathrm{J}^n E$. According to Proposition 10.7, the one-forms $\boldsymbol{\eta}$ will be linearly independent at a point $(g, z^{(n)}) \in G \times \mathrm{J}^n E \subset \mathcal{B}^{(n)}$ if and only if $g^{(1)} \cdot \pi_1^n(z^{(n)}) \in \mathrm{J}^1 E$, which automatically implies $g^{(n)} \cdot z^{(n)} \in \mathrm{J}^n E$. Therefore, $\boldsymbol{\omega}$ will be linearly independent if and only if $k^{(n)} = \rho^{(n)}(z^{(n)}) \cdot z^{(n)} \in \mathrm{J}^n E$. But, by construction, $k^{(n)} \in K^{(n)}$ is the cross-section representative of the orbit of $G^{(n)}$ through $z^{(n)}$ and, hence, lies in the coordinate chart $\mathrm{J}^n E$ used to construct the moving frame. □

In local coordinates, the normalized one-forms $\boldsymbol{\omega}$ are therefore obtained by using the moving frame to replace the group parameters in (10.7), so

$$\omega^i = \sum_{j=1}^{p} \mathrm{D}_j y^i(\rho^{(n)}(x, u^{(n)}), x, u^{(n)})\, \mathrm{d}x^j = \sum_{j=1}^{p} P_j^i(x, u^{(n)})\, \mathrm{d}x^j, \tag{12.5}$$

whose coefficient matrix $P = (\sigma^{(n)})^* \mathbf{D} y$ is the pull-back of the total Jacobian matrix of the independent lifted invariants.

Remark. The coefficients P_j^i *cannot* be obtained by invariantly differentiating the normalized invariants $J^i = (\sigma^{(n)})^* y^i$; in other words, $\omega^i \neq \mathrm{d}_H J^i$. Indeed, in many cases, the y's are normalized to be constant, whereas the ω's are clearly not zero. This is because the operations of total differentiation and normalization *do not commute*.

The invariant differential operators associated with the horizontal coframe (12.5) are obtained by normalizing the lifted invariant differential operators (10.10), so that the $\mathcal{E}_i$ on $\mathcal{B}^{(\infty)}$ project, by $\sigma^{(n)}$, to G-invariant total differential operators on J^∞. In coordinates,

$$\mathcal{D}_i = \sum_{j=1}^{p} Q_i^j(x, u^{(n)})\, \mathrm{D}_j = \sum_{j=1}^{p} Z_i^j(\rho^{(n)}(x, u^{(n)}), x, u^{(1)})\, \mathrm{D}_j, \tag{12.6}$$

where $Q = P^{-1} = (\sigma^{(n)})^* Z$ can be constructed directly from (10.10).

EXAMPLE 12.7. Consider the similarity group (11.8). Under the first-order moving frame map (11.10), the lifted horizontal form $\eta = \mathrm{d}_H y = \alpha\, \mathrm{d}x$ reduces to $\omega = u_x^{-1/2}\, \mathrm{d}x$. Similarly the lifted invariant differential operator $\mathcal{E} = \alpha^{-1} \mathrm{D}_x$ reduces to $\mathcal{D} = \sqrt{u_x}\, \mathrm{D}_x$, which maps differential invariants to higher-order differential invariants. However, $\mathcal{D}$ does not directly produce the normalized invariants. For example, v_{yy} normalizes to $I_2 = u_x^{-1/2} u_{xx}$, but $v_{yyy} = \mathcal{E}(v_{yy})$ normalizes to $I_3 = u_{xxx}$, which is not the same as $\mathcal{D} I_2 = u_{xxx} + u_x^{-1} u_{xx}^2$. The second-order moving frame (11.11) produces a different horizontal one-form, namely $\tilde{\omega} = u_{xx}^{-1} \mathrm{d}x$, whose invariant differential operator $\widetilde{\mathcal{D}} = u_{xx}\, \mathrm{D}_x$ produces yet another hierarchy of differential invariants, which, naturally, are functions of the normalized differential invariants. The explicit formulae relating these different hierarchies of differential invariants will be found in the next section.

13. Recurrence Formulae, Commutation Relations, and Syzygies

We have now introduced the basic ingredients in the regularized theory of moving frames. In this section, we discuss several important consequences of our constructions. These include recurrence formulae and general classification results for differential invariants, commutation formulae for the associated invariant differential operators, and, finally, a general syzygy classification. The results are all illustrated at the end of the section by a particular example arising in classical invariant theory.

An important point, encountered in Example 12.7, is that the normalized invariant differential operators, unlike their lifted counterparts, do *not* directly produce the normalized differential invariants. For example, consider the normalized differential invariant $I^\alpha = (\sigma^{(n)})^* v^\alpha$ corresponding to the lifted zeroth-order invariant v^α as in (11.7). Applying an invariant differential operator to I^α produces a higher-order differential invariant $\mathcal{D}_K I^\alpha$, but this is *not*, in general, equal to its normalized counterpart $I^\alpha_K = (\sigma^{(n)})^* v^\alpha_K = (\sigma^{(n)})^*[\mathcal{E}_K v^\alpha]$. For example, if we normalize $v^\alpha = c^\alpha$, then $I^\alpha = c^\alpha$ is constant, and so its derivatives are all zero, but the higher-order I^α_K are generally *not* trivial. The goal is to determine a recursive formula for constructing the I^α_K directly without having to appeal to the lifted invariants. Our starting point is formula (10.14), to which we apply the moving frame pull-back $(\sigma^{(n)})^*$. A difficulty is that, while $(\sigma^{(n)})^*$ trivially commutes with the differential d, it does *not* commute with the operations d_H and d_V. Therefore, we rewrite

$$\mathrm{d}_H v^\alpha_K = \mathrm{d} v^\alpha_K - \mathrm{d}_V v^\alpha_K - \mathrm{d}_G v^\alpha_K \tag{13.1}$$

before applying $(\sigma^{(n)})^*$. We find

$$\begin{aligned}\sum_{i=1}^p I^\alpha_{K,i}\,\omega^i &= (\sigma^{(n)})^*(\mathrm{d}_H v^\alpha_K)\\ &= \mathrm{d} I^\alpha_K - (\sigma^{(n)})^*(\mathrm{d}_V v^\alpha_K) - (\sigma^{(n)})^*(\mathrm{d}_G v^\alpha_K)\\ &= \mathrm{d}_H I^\alpha_K - \pi_H\big[(\sigma^{(n)})^*(\mathrm{d}_G v^\alpha_K)\big]\\ &= \sum_{i=1}^p (\mathcal{D}_i I^\alpha_K)\,\omega^i - \pi_H\big[(\sigma^{(n)})^*(\mathrm{d}_G v^\alpha_K)\big].\end{aligned} \tag{13.2}$$

The next to last equality is obtained by applying the horizontal projection π_H, noting that the left-hand side is a horizontal form. Moreover, the pull-back of any lifted contact form, such as $\mathrm{d}_V v^\alpha_K$, remains a contact form on J^∞. The second summand in the final line of (13.2) provides the correction terms that relate the differential invariants $I^\alpha_{K,i}$ and $\mathcal{D}_i I^\alpha_K$.

To find the explicit formula for these correction terms, we adapt Theorem 3.10 to the case of the nth order regularized action of G on $\mathcal{B}^{(n)}$. Since v^α_K is a component of the lifted invariant $w^{(n)} = g^{(n)} \cdot z^{(n)}$, Equation (3.8) implies that, at a point $w^{(n)} \in \mathcal{B}^{(n)}$, we can write the group differential in terms of the Maurer–Cartan forms on $\mathcal{B}^{(n)}$:

$$
\begin{aligned}
\mathrm{d}_G y^i &= \sum_{\kappa=1}^{r} \xi^i_\kappa(w)\, \mu^\kappa, \quad i = 1, \dots, p, \\
\mathrm{d}_G v^\alpha_K &= \sum_{\kappa=1}^{r} \varphi^\alpha_{K,\kappa}(w^{(k)}) \mu^\kappa, \quad \alpha = 1, \dots, r, \quad k = \# K.
\end{aligned} \tag{13.3}
$$

The coefficients in (13.3) are the invariant counterparts of the coefficients $\xi^i_\kappa(z)$, $\varphi^\alpha_{K,\kappa}(z^{(k)})$ of the prolonged infinitesimal generator $\mathrm{pr}\,\mathbf{v}_\kappa$, as given in (9.1). Substituting (13.3) into (13.2) and its counterpart for $\mathrm{d}_H y^i$ using (11.7) leads to the key system of identities

$$
\begin{aligned}
&\omega^i = \mathrm{d}_H J^i - \sum_{\kappa=1}^{r} \xi^i_\kappa(I^{(0)}) \zeta^\kappa_H, \quad i = 1, \dots, p, \\
&\sum_{i=1}^{p} I^\alpha_{K,i}\, \omega^i = \mathrm{d}_H I^\alpha_K - \sum_{\kappa=1}^{r} \varphi^\alpha_{K,\kappa}(I^{(k)})\, \zeta^\kappa_H, \quad \alpha = 1, \dots, r,\ k = \# K.
\end{aligned} \tag{13.4}
$$

Here $\boldsymbol{\zeta}^{(n)}_H = \{\zeta^1_H, \dots, \zeta^r_H\} = \pi_H(\boldsymbol{\zeta}^{(n)}) = \pi_H((\rho^{(n)})^* \boldsymbol{\mu})$. The coefficients in (13.4) are obtained by invariantizing the coefficients of the prolonged infinitesimal generators of the group action (9.1), meaning that we replace the jet coordinates $z^{(k)}$ by the fundamental normalized differential invariants $I^{(k)}$. Note that if G acts transitively on J^k, then there are no nonconstant kth-order differential invariants, and hence in such cases the coefficients of order k or less will be automatically constant. The first terms on the right-hand side of (13.4) can be re-expressed in terms of the contact-invariant coframe $\boldsymbol{\omega}$ using the associated invariant differential operators, as in (9.7), so

$$
\mathrm{d}_H J^i = \sum_{j=1}^{p} (\mathcal{D}_j J^i) \omega^j, \qquad \mathrm{d}_H I^\alpha_K = \sum_{j=1}^{p} (\mathcal{D}_j I^\alpha_K) \omega^j. \tag{13.5}
$$

On the other hand, the horizontal components of the Maurer–Cartan forms can themselves be written in terms of our contact-invariant coframe,

$$
\zeta^\kappa_H = \sum_{j=1}^{p} K^\kappa_j [I^{(n+1)}(x, u^{(n+1)})] \omega^j, \quad \kappa = 1, \dots, r, \tag{13.6}
$$

where the coefficients are certain differential invariants of order $n+1$. Substituting (13.5), (13.6), into (13.2) produces the fundamental *recurrence formulae* for the differential invariants:

$$
\mathcal{D}_j J^i = \delta^i_j + M^i_j, \qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{K,j} + M^\alpha_{K,j}. \tag{13.7}
$$

The ‘correction terms’, that account for the noncommuting of the processes of normalization and ‘horizontalization’, are explicitly given by

$$\begin{aligned} M^i_j &= \sum_{\kappa=1}^{r} \xi^i_\kappa(I^{(0)}) K^\kappa_j(I^{(n+1)}), \quad i, j = 1, \ldots, p, \\ M^\alpha_{K,j} &= \sum_{\kappa=1}^{r} \varphi^\alpha_{K,\kappa}(I^{(k)}) K^\kappa_j(I^{(n+1)}), \quad \alpha = 1, \ldots, q, \ \#K = k. \end{aligned} \tag{13.8}$$

There are similar recurrence formulae for higher-order differentiated invariants,

$$\mathcal{D}_J I^\alpha_K = I^\alpha_{J,K} + M^\alpha_{K,J}, \tag{13.9}$$

where the higher-order correction terms can be determined by iterating the basic recurrence formulae (13.7).

The coefficients K^κ_j in (13.6) can, in fact, be explicitly determined from a subset of the identities (13.7). Suppose, for simplicity, that we are normalizing r components of $w^{(n)}$ to be constant. The corresponding invariants, J^i and I^α_K will then also be constant, and hence the horizontal derivative term on the right hand side of (13.4) will vanish. For these particular forms, (13.4) reduces to a system of r linear equations relating the horizontal moving coframe forms $\zeta^1_H, \ldots, \zeta^r_H$ to the contact-invariant coframe forms $\omega^1, \ldots, \omega^p$. The coefficients of these linear equations are differential invariants of order $\leqslant n + 1$. (On the right-hand side, the coefficients are of order $\leqslant n$, while $(n + 1)$st order differential invariants can appear on the left.) Since $G^{(n)}$ acts freely, its infinitesimal generators are linearly independent on the domain of definition of $\rho^{(n)}$, and hence transversality of the cross-section used to normalize the differential invariants implies that the coefficient matrix for this linear system is invertible. Solving for one-forms $\boldsymbol{\zeta}_H$ produces the required system of coefficients in (13.6).

Remark. In the method of moving coframes [9], one normalizes the lifted differential invariants arising from the linear dependencies among the horizontal components of the moving coframe forms. In this case, the coefficients in (13.6) will be the chosen normalization constants and/or differential invariants. Typically, one is able to normalize all the coefficients to be constant up until the final step, at which point the fundamental differential invariants appear as coefficients.

The key observation is that the correction term (13.8) is a (typically nonlinear) function of the differential invariants of order $\leqslant k$, *provided* $k \geqslant n + 1$, where n is the order of the chosen moving frame. This immediately implies provides a new proof, and a refined version of, Theorem 9.13.

THEOREM 13.1. *Suppose G acts freely on $\mathcal{V}^n \subset \mathrm{J}^n$. Then, locally, every differential invariant on $\mathcal{V}^\infty = (\pi^\infty_n)^{-1}\mathcal{V}^n$ can be found by successively applying the invariant differential operators $\mathcal{D}_1, \ldots, \mathcal{D}_p$ to a generating set of differential invariants of order at most $n + 1$, namely the independent components of $I^{(n+1)}$.*

The commutation formulae (9.11) for the invariant differential operators (12.6) can now be explicitly determined from the moving frame formulae. In view of (10.6), (12.4), we can compute

$$\begin{aligned} \mathrm{d}_H \boldsymbol{\omega} &= \pi_H(\mathrm{d}\boldsymbol{\omega}) = \pi_H[(\sigma^{(n)})^*(\mathrm{d}\,\mathrm{d}_H y)] \\ &= \pi_H[(\sigma^{(n)})^*(\mathrm{d}_G\, \mathrm{d}_H y)] = -\pi_H[(\sigma^{(n)})^*(\mathrm{d}_H\, \mathrm{d}_G y)]. \end{aligned} \tag{13.10}$$

Here we used the fact that d_V produces (lifted) contact forms, which do not contribute to the horizontal two-form $\mathrm{d}_H \boldsymbol{\omega}$. Applying d_H to (13.3), and noting that $\mathrm{d}_H \boldsymbol{\mu} = 0$, we find

$$\mathrm{d}_H\, \mathrm{d}_G y^k = \sum_{\kappa=1}^{r} [\mathrm{d}_H \xi^k_\kappa(w)] \wedge \mu^\kappa = \sum_{j=1}^{p} \sum_{\kappa=1}^{r} \mathcal{E}_j(\xi^k_\kappa(w))\, \eta^j \wedge \mu^\kappa, \tag{13.11}$$

where $\xi^k_\kappa(z)$ is the coefficient of $\partial/\partial x^k$ in the infinitesimal generator $\mathbf{v}_\kappa$. Combining (11.3), (13.6), (13.10) and (13.11), proves that

$$\mathrm{d}_H \omega^k = \sum_{i,j=1}^{p} \sum_{\kappa=1}^{r} K^\kappa_i [I^{(n+1)}](\mathrm{D}_j \xi^k_\kappa)[I^{(1)}]\, \omega^i \wedge \omega^j,$$

where $(\mathrm{D}_i \xi^k_\kappa)[I^{(1)}]$ is obtained by substituting the first-order normalized differential invariant into the total derivative $\mathrm{D}_i \xi^k_\kappa(z^{(1)})$ of the coefficient ξ^k_κ of the infinitesimal generator $\mathbf{v}_\kappa$. Therefore, by (9.10), the commutation coefficients in (9.11) are explicitly given by

$$A^k_{ij} = \sum_{\kappa=1}^{r} K^\kappa_j [I^{(n+1)}](\mathrm{D}_i \xi^k_\kappa)[I^{(1)}] - K^\kappa_i [I^{(n+1)}](\mathrm{D}_j \xi^k_\kappa)[I^{(1)}]. \tag{13.12}$$

Our fundamental recurrence formulae (13.7) also provide a resolution of the syzygy problem for differential invariants. First, in the normalization context, the solution is now trivial. According to our general construction, given a moving frame of order n, the normalized differential invariants (11.7) provide a complete system of differential invariants of order $k \geqslant n$. Assume, for simplicity, that the normalization consists of setting $r = \dim G$ components* of the nth order lifted invariants $w^{(n)}$ to be constant. Then the remaining components will pull-back to functionally independent differential invariants. Therefore, all syzygies among the normalized differential invariants (11.7) occur through the normalization equations and, hence, are of order at most n, the order of the moving frame.

The more subtle question is to understand the syzygies among the differentiated invariants $\mathcal{D}_J I_\nu$, arising from a generating system of differential invariants. If we choose the generating system to be the nonconstant normalized differential invariants of order $\leqslant n+1$, then the resulting syzygies will be of two kinds. Those

* In the more general situation where we normalize certain functional combinations of the components of $w^{(n)}$, one must modify the subsequent constructions accordingly.

involving differential invariants of order $\leqslant n$ will depend on the precise structure of the normalizations and the invariants themselves. Once these are understood, the higher order syzygies are more regular. Before attempting to formulate a general theorem, let us consider a simple example. Suppose our moving frame has order n and that the normalized differential invariant I_K^α of order $n = \#K$ is constant. Suppose that the normalized differential invariants $I_{K,i}^\alpha$ and $I_{K,j}^\alpha$ of order $n+1$ are not constant, and can be taken as part of the generating set of differential invariants. Since the correction terms in (13.7) have order k for $k \geqslant n+1$, we have

$$\mathcal{D}_j I_{K,i}^\alpha = I_{K,i,j}^\alpha + M_{K,i,j}^\alpha, \qquad \mathcal{D}_i I_{K,j}^\alpha = I_{K,i,j}^\alpha + M_{K,j,i}^\alpha,$$

where the correction terms $M_{K,i,j}^\alpha$ and $M_{K,j,i}^\alpha$ are differential invariants of order $\leqslant n+1$ that are not necessarily equal. Therefore, we deduce a syzygy between the differentiated invariants

$$\mathcal{D}_j I_{K,i}^\alpha - \mathcal{D}_i I_{K,j}^\alpha = M_{K,i,j}^\alpha - M_{K,j,i}^\alpha,$$

where the right-hand side is a differential invariant of order $n+1$. The constant normalized differential invariant I_K^α is a ‘phantom differential invariant’ that provides the seed for the syzygy. A general syzygy theorem for differential invariants can now be straightforwardly proved using these basic observations.

DEFINITION 13.2. A *phantom differential invariant* is a constant normalized differential invariant.

THEOREM 13.3. *A generating system of differential invariants consists of*

(a) *all nonphantom differential invariants J^i and I^α coming from the nonnormalized zeroth order lifted invariants y^i, v^α, and*
(b) *all nonphantom differential invariants of the form $I_{J,i}^\alpha$ where I_J^α is a phantom differential invariant.*

Proof. The key remark is that the coefficients K_j^κ in the formulae (13.6) are all either constant or one of the generating differential invariants mentioned in the theorem. The invariantized vector field coefficients $\varphi_{K,\kappa}^\alpha$, on the other hand, are of order at most $\#K$. Therefore, if $I_{K,j}^\alpha$ is a normalized differential invariant that does not belong to the generating set, then rewriting the recurrence formula (13.7) as

$$I_{K,j}^\alpha = \mathcal{D}_j I_K^\alpha - M_{K,j}^\alpha$$

expresses it in terms of the generating invariants and lower order invariants. A simple induction completes the proof. □

THEOREM 13.4. *All syzygies among the differentiated invariants arising from the generating system constructed in Theorem* 13.3 *are differential consequences of the following three fundamental types*:

(i) $\mathcal{D}_j J^i = \delta^i_j + M^i_j$, *when* J^i *is nonphantom,*
(ii) $\mathcal{D}_J I^\alpha_K = c + M^\alpha_{K,J}$, *when* I^α_K *is a generating differential invariant, while* $I^\alpha_{J,K} = c$ *is a phantom differential invariant, and*
(iii) $\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}$, *where* I^α_{LK} *and* I^α_{LJ} *are generating differential invariants the multi-indices* $K \cap J = \emptyset$ *are disjoint and nonzero, while* L *is an arbitrary multi-index.*

Remark. One can often use the syzygies to substantially reduce the generating system of differential invariants. In such cases, one must accordingly modify the remaining syzygies.

EXAMPLE 13.5. We now illustrate the preceding formulae with a nontrivial example. Let $M = \mathbb{R}^3$, with coordinates x^1, x^2, u. Consider the action of GL(2) defined by the order zero invariants

$$y^1 = \alpha x^1 + \beta x^2, \qquad y^2 = \gamma x^1 + \delta x^2,$$
$$v = (\alpha\delta - \beta\gamma)u = \lambda u, \tag{13.13}$$

where $\lambda = \alpha\delta - \beta\gamma$. This action plays a key role in the classical invariant theory of binary forms, when u is a homogeneous polynomial, [18]. The lifted contact-invariant coframe and associated invariant differential operators are

$$\begin{aligned} \eta^1 &= \mathrm{d}_H y^1 = \alpha\, \mathrm{d}x^1 + \beta\, \mathrm{d}x^2, \qquad & \mathcal{E}_1 &= \lambda^{-1}(\delta\, \mathrm{D}_1 - \gamma\, \mathrm{D}_2), \\ \eta^2 &= \mathrm{d}_H y^2 = \gamma\, \mathrm{d}x^1 + \delta\, \mathrm{d}x^2, \qquad & \mathcal{E}_2 &= \lambda^{-1}(-\beta\, \mathrm{D}_1 + \alpha\, \mathrm{D}_2), \end{aligned} \tag{13.14}$$

where D_i is the total derivative with respect to x^i. The lifted differential invariants are thus $v_{jk} = (\mathcal{E}_1)^j(\mathcal{E}_2)^k v$; in particular

$$\begin{aligned} v_1 &= \delta u_1 - \gamma u_2, \qquad v_2 = -\beta u_1 + \alpha u_2, \\ v_{11} &= \frac{\delta^2 u_{11} - 2\gamma\delta u_{12} + \gamma^2 u_{22}}{\lambda}, \\ v_{12} &= \frac{-\beta\delta u_{11} + (\alpha\delta + \beta\gamma)u_{12} - \alpha\gamma u_{22}}{\lambda}, \\ v_{22} &= \frac{\beta^2 u_{11} - 2\alpha\beta u_{12} + \alpha^2 u_{22}}{\lambda}. \end{aligned}$$

If we normalize using the cross-section

$$y^1 = 1, \qquad y^2 = 0, \qquad v_1 = 1, \qquad v_2 = 0, \tag{13.15}$$

we are led to the first-order moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{x^1 u_1 + x^2 u_2} \begin{pmatrix} u_1 & u_2 \\ -x^2 & x^1 \end{pmatrix}. \tag{13.16}$$

This moving frame is well-defined on surfaces $u = f(x, y)$ provided the relative invariant $x^1 u_1 + x^2 u_2 \neq 0$. (Different choices of cross-section lead to other types of constraints. For example, if $u \neq 0$, then we could normalize $v = 1$ instead of, say, $v_2 = 0$.) The resulting normalized differential invariants are $I^{(2)} = (J^1, J^2, I, I_1, I_2, I_{11}, I_{12}, I_{22}) = (\sigma^{(2)})^* w^{(2)}$, where

$$\begin{aligned} J^1 &= 1, \qquad J^2 = 0, \qquad I = \frac{u}{x^1u_1 + x^2u_2}, \qquad I_1 = 1, \qquad I_2 = 0, \\ I_{11} &= \frac{(x^1)^2u_{11} + 2x^1x^2u_{12} + (x^2)^2u_{22}}{x^1u_1 + x^2u_2}, \\ I_{12} &= \frac{-x^1u_2u_{11} + (x^1u_1 - x^2u_2)u_{12} + x^2u_1u_{22}}{x^1u_1 + x^2u_2}, \\ I_{22} &= \frac{(u_2)^2u_{11} - 2u_1u_2u_{12} + (u_1)^2u_{22}}{x^1u_1 + x^2u_2}. \end{aligned} \tag{13.17}$$

The normalized coframe and associated invariant differential operators are

$$\begin{aligned} \omega^1 &= \frac{u_1\,dx^1 + u_2\,dx^2}{x^1u_1 + x^2u_2} = \frac{d_H u}{x^1u_1 + x^2u_2}, \qquad \mathcal{D}_1 = x^1\,D_1 + x^2\,D_2, \\ \omega^2 &= \frac{-x^2\,dx^1 + x^1\,dx^2}{x^1u_1 + x^2u_2}, \qquad \mathcal{D}_2 = -u_2\,D_1 + u_1\,D_2. \end{aligned} \tag{13.18}$$

The invariant differential operators are well known: $\mathcal{D}_1$ is the scaling process and $\mathcal{D}_2$ the Jacobian process in classical invariant theory. The prolonged infinitesimal generator coefficient matrix and its invariantized counterpart are, up to second order,

$$\begin{pmatrix} x^1 & x^2 & 0 & 0 \\ 0 & 0 & x^1 & x^2 \\ u & 0 & 0 & u \\ 0 & 0 & -u_2 & u_1 \\ u_2 & -u_1 & 0 & 0 \\ -u_{11} & 0 & -2u_{12} & u_{11} \\ 0 & -u_{11} & -u_{22} & 0 \\ u_{22} & -2u_{12} & 0 & -u_{22} \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ I & 0 & 0 & I \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -I_{11} & 0 & -2I_{12} & I_{11} \\ 0 & -I_{11} & -I_{22} & 0 \\ I_{22} & -2I_{12} & 0 & -I_{22} \end{pmatrix}. \tag{13.19}$$

The invariant linear relations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \zeta^1_H \\ \zeta^2_H \\ \zeta^3_H \\ \zeta^4_H \end{pmatrix} = \begin{pmatrix} -\zeta^1_H \\ -\zeta^3_H \\ -\zeta^4_H \\ \zeta^2_H \end{pmatrix}, \tag{13.20}$$

follow from (13.6) and the subsequent remarks. The left-hand side in (13.20) is obtained by pulling back the lifted contact-invariant one-forms

$$\begin{aligned} d_H y^1 &= \eta^1, \qquad d_H y^2 = \eta^2, \qquad d_H v_1 = v_{11}\eta^1 + v_{12}\eta^2, \\ d_H v_2 &= v_{12}\eta^1 + v_{22}\eta^2, \end{aligned}$$

corresponding to our choice (13.15) of normalizations; the matrix on the right-hand side is the minor consisting of first, second, fourth and fifth rows of the invariantized matrix (13.19), again governed by the normalizations. We rewrite (13.20) in the matrix form

$$\begin{pmatrix} \zeta_H^1 \\ \zeta_H^2 \\ \zeta_H^3 \\ \zeta_H^4 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ I_{12} & I_{22} \\ 0 & -1 \\ -I_{11} & -I_{12} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix}. \tag{13.21}$$

The coefficients K_j^κ in (13.6) are the entries of the coefficient matrix in (13.21). The commutator between the two invariant differential operators,

$$[\mathcal{D}_1, \mathcal{D}_2] = -I_{12}\mathcal{D}_1 + (I_{11} - 1)\mathcal{D}_2, \tag{13.22}$$

now follows from our general formula (9.11), (13.12). Indeed, the $(D_i \xi_\kappa^k)[I^{(1)}]$ are obtained by first computing the total derivatives of the independent variable coefficient matrix (which consists of the first two rows of (13.19))

$$\begin{pmatrix} x^1 & x^2 & 0 & 0 \\ 0 & 0 & x^1 & x^2 \end{pmatrix}$$

and then invariantizing by substituting the normalized differential invariants (13.17) for the jet coordinates. In this particular case, the latter process is trivial since the total derivatives are all either 1 or 0.

The correction terms to the recurrence formula can be easily obtained by multiplying the invariantized matrix (13.19) by the coefficient matrix (13.21); the resulting matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ I & 0 & 0 & I \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ -I_{11} & 0 & -2I_{12} & I_{11} \\ 0 & -I_{11} & -I_{22} & 0 \\ I_{22} & -2I_{12} & 0 & -I_{22} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ I_{12} & I_{22} \\ 0 & -1 \\ -I_{11} & -I_{12} \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -I(1+I_{11}) & -I\,I_{12} \\ -I_{11} & -I_{12} \\ -I_{12} & -I_{22} \\ (1-I_{11})I_{11} & (2-I_{11})I_{12} \\ -I_{11}I_{12} & (1-I_{11})I_{22} \\ (I_{11}-1)I_{22}-2I_{12}^2 & -I_{12}I_{22} \end{pmatrix} \tag{13.23}$$

contains the correction terms in (13.7) – the columns correspond to normalized differential invariants and the rows to invariant differential operators. Specifically, we have

$$\begin{aligned}
&\mathcal{D}_1 J^1 = \delta_1^1 - 1 = 0, && \mathcal{D}_2 J^1 = \delta_2^1 - 0 = 0,\\
&\mathcal{D}_1 J^2 = \delta_1^2 - 0 = 0, && \mathcal{D}_2 J^2 = \delta_2^2 - 1 = 0,\\
&\mathcal{D}_1 I = I_1 - I(1+I_{11}) = 1 - I(1+I_{11}), && \mathcal{D}_2 I = I_2 - I\,I_{12} = -I\,I_{12},\\
&\mathcal{D}_1 I_1 = I_{11} - I_{11} = 0, && \mathcal{D}_2 I_1 = I_{12} - I_{12} = 0,\\
&\mathcal{D}_1 I_2 = I_{12} - I_{12} = 0, && \mathcal{D}_2 I_2 = I_{22} - I_{22} = 0,\\
&\mathcal{D}_1 I_{11} = I_{111} + (1-I_{11})I_{11}, && \mathcal{D}_2 I_{11} = I_{112} + (2-I_{11})I_{12},\\
&\mathcal{D}_1 I_{12} = I_{112} - I_{11}I_{12}, && \mathcal{D}_2 I_{12} = I_{122} + (1-I_{11})I_{22},\\
&\mathcal{D}_1 I_{22} = I_{122} + (I_{11}-1)I_{22} - 2I_{12}^2, && \mathcal{D}_2 I_{22} = I_{222} - I_{12}I_{22}.
\end{aligned}$$

Here $I_{ijk} = (\sigma^{(1)})^* v_{ijk}$ are the third-order normalized differential invariants. An alternative method for computing the correction matrix (13.23) that avoids the intermediate system (13.21) is to first perform a Gauss–Jordan *column* reduction on the invariantized coefficient matrix (13.19) making the chosen normalization rows – in the present case rows 1, 2, 4, 5 – into an identity matrix, and then multiply by the pulled-back coefficient matrix corresponding to the horizontal derivatives of the normalized lifted invariants, as given on the left-hand side of (13.20); the result will be minus the correction matrix. In the present case, (13.23) is *minus* the matrix product

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ I & 0 & I & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -I_{11} & -2I_{12} & I_{11} & 0 \\ 0 & -I_{22} & 0 & I_{11} \\ I_{22} & 0 & -I_{22} & 2I_{12} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ I_{11} & I_{12} \\ I_{12} & I_{22} \end{pmatrix}. \tag{13.24}$$

According to Theorem 13.3, we can take $I, I_{11}, I_{12}, I_{22}$ as our generating system of differential invariants. The third row of this system of identities produces the syzygies of the second type. Actually, this means that we can use I to generate

I_{11}, I_{12}, leaving only I_{22} as an additional fundamental invariant. There are three fundamental syzygies of the third type:

$$\begin{aligned}
\mathcal{D}_1 I_{12} - \mathcal{D}_2 I_{11} &= -2I_{12}, \\
\mathcal{D}_1 I_{22} - \mathcal{D}_2 I_{12} &= 2(I_{11}-1)I_{22} - 2I_{12}^2, \\
\mathcal{D}_1^2 I_{22} - \mathcal{D}_2^2 I_{11} &= 2I_{22}\mathcal{D}_1 I_{11} + (5I_{12}-2)\mathcal{D}_1 I_{12} + (3I_{11}-5)\mathcal{D}_1 I_{22} - \\
&\quad - (2I_{11}-5)(I_{11}-1)I_{12} + 4(I_{11}-1)I_{12}^2.
\end{aligned}$$

The final syzygy comes from extending our recurrence formulae on to the next order, by appending the appropriate columns to the prolonged vector field coefficient matrix (13.19). Using I to generate I_{11} and I_{12} will modify the syzygies accordingly.

14. Equivalence, Symmetry, and Rigidity

We now reach the culmination of the paper. The fundamental problems that have motivated the development of the theory of moving frames are equivalence and symmetry of submanifolds under a Lie transformation group G, as introduced in Section 7. If G acts freely on M, then, as we saw, the basic order zero theory, as described in Theorems 7.7 and 7.8, provides the solution. However, in the nonfree case, we need to prolong in order to make the group act (locally) freely. Since two submanifolds are equivalent under the action of G on M if and only if their n-jets are equivalent under the prolonged action of $G^{(n)}$ on J^n, we can then readily adapt our earlier results.

When we restrict the $G^{(n)}$-coframe on J^n to a submanifold, the resulting linear dependencies among the restricted one-forms lead to additional invariants. In the order zero context, these invariants are not directly predicted by the moving coframe, but appear to depend on the submanifold itself. An important fact is that, in the jet bundle context, they are merely the restrictions of *higher order differential invariants*! Thus, even in the order zero case, the jet bundle constructions lead to significant new information.

We start with the $G^{(n)}$-coframe $\Gamma^{(n)} = \{\boldsymbol{\gamma}^{(n)}, I^{(n)}\}$ on J^n constructed in Theorem 12.3. Let $\iota\colon X \to M$ parametrize a submanifold $S = \iota(X)$, so that $\mathrm{j}_n\iota\colon X \to \mathrm{J}^n$ parametrizes the corresponding n-jet $\mathrm{j}_n S$. We assume that $\mathrm{j}_n S$ lies in the domain of definition of our chosen order n moving frame $\rho^{(n)}$, which implies that S is order n regular. Let $\Xi^{(n)} = (\mathrm{j}_n\iota)^*\Gamma^{(n)}$ denote the restriction of the nth order coframe to S. As in the order zero case, the one-form system $\Xi^{(n)}$ is overdetermined on X, and we need to reduce it to an extended coframe. Now since $\mathrm{j}_n\iota$ annihilates the contact forms, only the horizontal components of the forms in $\Gamma^{(n)}$ will contribute to the one-forms in $\Xi^{(n)}$. Therefore, the linear dependencies among these one-forms will arise from the linear dependencies among the horizontal components of the one-forms in the $G^{(n)}$-coframe. The one-forms $\boldsymbol{\gamma}^{(n)} = (\sigma^{(n)})^*\,\mathrm{d}_J w^{(n)}$ are, by definition, the pull-backs of the jet differentials of the lifted invariants. We have already used the horizontal components $\mathrm{d}_H y$ of the 'independent variable' lifted invariants to construct the contact-invariant coframe $\boldsymbol{\omega} = (\sigma^{(n)})^* \mathrm{d}_H y$. The remaining 'dependent variable' lifted invariants will lead to additional contact-invariant horizontal forms $\boldsymbol{\delta}^{(n)} = (\sigma^{(n)})^* \mathrm{d}_H v^{(n)}$, which must be invariant linear combinations of the contact-invariant coframe $\boldsymbol{\omega}$. According to (13.2),

$$\delta_K^\alpha = (\sigma^{(n)})^* \mathrm{d}_H v_K^\alpha = \sum_{i=1}^{p} I_{K,i}^\alpha \omega^i, \quad \alpha = 1, \ldots, q, \ \#K \geqslant 0. \tag{14.1}$$

The coefficient $I_{K,i}^\alpha$ is the normalized differential invariant of order $\#K + 1$. Therefore, *the linear dependencies among the horizontal forms* $\boldsymbol{\gamma}_H^{(n)} = \{\boldsymbol{\omega}, \boldsymbol{\delta}^{(n)}\}$ *are the differential invariants of order* $n + 1$. With this in mind, we make the following definition.

DEFINITION 14.1. The nth order *differential invariant coframe* on J^n is the extended horizontal coframe

$$\Delta^{(n)} = \{\boldsymbol{\omega}, I^{(n)}\} \tag{14.2}$$

consisting of the contact-invariant coframe and the nth order normalized differential invariants.

PROPOSITION 14.2. *The horizontal components of the* nth *order moving coframe* $\Sigma_H^{(n)} = \{\boldsymbol{\zeta}_H^{(n)}, \mathrm{d}_H I^{(n)}, I^{(n)}\}$ *or its normalized counterpart* $\Gamma_H^{(n)} = \{\boldsymbol{\gamma}_H^{(n)}, I^{(n)}\}$ *are invariantly related to the differential invariant coframe* $\Delta^{(n+1)} = \{\boldsymbol{\omega}, I^{(n+1)}\}$ *of order* $n + 1$.

Proof. Formula (14.1) shows that $\Gamma_H^{(n)}$ is invariantly related to $\Delta^{(n+1)}$. Moreover, since $\Sigma^{(n)}$ is invariantly related to $\Gamma^{(n)}$, the same is true for $\Sigma_H^{(n)}$. In particular, the fact that $\mathrm{d}_H I^{(n)}$ can be written as a linear combination of $\boldsymbol{\omega}$ with $(n + 1)$st order differential invariant coefficients is immediate from (9.7). □

We now restrict the coframes to a regular submanifold $S = \iota(X)$. Let $\Upsilon^{(n)} = \{\boldsymbol{\varpi}, J^{(n)}\} = (\mathrm{j}_n\iota)^* \Delta^{(n)}$ denote the restriction of the differential invariant coframe to S. Transversality implies that $\boldsymbol{\varpi} = (\mathrm{j}_n\iota)^* \boldsymbol{\omega}$ will form a coframe on the parameter space X, while $J^{(n)} = (\mathrm{j}_n\iota)^* I^{(n)}$ corresponds to the pull-back of the nth order normalized differential invariants to X.

PROPOSITION 14.3. *Let* $S = \iota(X)$ *be a submanifold whose* n *jet lies in the domain of definition of the given moving frame. Then* $\Xi^{(n)} = (\mathrm{j}_n\iota)^* \Gamma^{(n)}$ *is invariantly related to the restricted* $(n + 1)$*st-order differential invariant coframe* $\Upsilon^{(n+1)} = (\mathrm{j}_{n+1}\iota)^* \Delta^{(n+1)}$.

Remark. A key point is that, by construction, the invariant relation does not depend on the particular submanifold S and hence we can replace $\Xi^{(n)}$ by $\Upsilon^{(n+1)}$ without altering the equivalence relations between different submanifolds.

If $\Upsilon^{(n+1)}$ is not involutive, then we need to extend it by appending additional derived invariants. A second key fact is that the derived invariants are merely the differential invariants of the next higher-order restricted to S. This is an immediate consequence of (13.4) and (13.6).

PROPOSITION 14.4. *The* kth-*order derived coframe* $(\Upsilon^{(n+1)})^{(k)}$ *for the restricted differential invariant coframe* $\Upsilon^{(n+1)}$ *is invariantly related to the coframe* $\Upsilon^{(n+k+1)}$.

Remark. We can now interpret the additional invariants that arose in the order zero construction – they are the differential invariants associated with the freely acting transformation group on M.

DEFINITION 14.5. The kth order *differential invariant classifying manifold* $\mathcal{C}^{(k)}(S)$ associated with a submanifold $\iota\colon X \to M$ is the manifold parametrized by the normalized differential invariants of order k, namely $J^{(k)} = I^{(k)} \circ \mathrm{j}_k\iota$. The submanifold S is *order* k *regular* if $\mathcal{C}^{(k)}(S)$ is an embedded submanifold of its classifying space $Z^{(k)}$ (which can, in fact, be identified with $\mathrm{J}^k E$).

DEFINITION 14.6. The *differential invariant order* of S with respect to an nth order moving frame $\rho^{(n)}$ is the minimal integer $s \geqslant n$ such that the extended coframe $\Upsilon^{(s)}$ is involutive. The *differential invariant rank* of S is $t = \operatorname{rank} \Upsilon^{(s)} = \dim \mathcal{C}^{(s)}(S)$.

Remark. The differential invariant order defined here is slightly different from the order defined earlier. For instance, if the $(n+1)$st-order differential invariants $I^{(n+1)}$ provide a complete system of invariants on S, then S will have differential invariant order $n+1$, but will be an order zero submanifold with respect to the restricted coframe $\Upsilon^{(n+1)}$.

THEOREM 14.7. *Let* $S \subset M$ *be a regular* p*-dimensional submanifold of differential invariant rank* t *with respect to the transformation group* G*. Then its isotropy group* G_S *is a* $(p-t)$*-dimensional subgroup of* G *acting locally freely on* S.

In particular, the maximally symmetric submanifolds are those of rank 0, where all the differential invariants are constants. See [5, 14], for a general characterization of such submanifolds as group orbits in the case when $M = G/H$ is a homogeneous space.

In the fully regular case, the ranks $t_k = \operatorname{rank} \mathrm{d}J^{(k)} = \dim \mathcal{C}^{(k)}(S)$ of the kth order fundamental differential invariants on S are all constant for* $k \geqslant n$, and satisfy

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leqslant p, \tag{14.3}$$

* The differential invariant ranks for $k < n$ will not play any significant role.

where t is the differential invariant rank and s the differential invariant order. Generically, a p-dimensional submanifold will have differential invariant order n equal to the stabilization order of the group, provided there are at least p functionally independent differential invariants of order $\leqslant n$; if G admits less than p independent nth-order differential invariants, then the generic differential invariant order will be $n+1$. According to (9.6), the latter situation occurs only when

$$q\binom{p+n}{n} < r = \dim G \leqslant p + q\binom{p+n}{n} = \dim \mathrm{J}^n. \tag{14.4}$$

If S is fully regular, then its differential invariant order is always bounded by either $n+p-1$ or, possibly, $n+p$; the latter case only occurs if all nth-order differential invariants are constant, and there is but one independent differential invariant appearing at each order $n+1 \leqslant k \leqslant n+p$.

In this context, it is instructive to reconsider the higher-order submanifold discussed in Example 7.9.

EXAMPLE 14.8. Consider the Lie group $G = \mathbb{R}^3$ acting by translations on $M = \mathbb{R}^3$. For a moving frame of order zero, the generating differential invariants for surfaces $u = f(x, y)$ are just the derivatives u_x, u_y. Any nonplanar solution to the nonlinear partial differential equation* $u_y = \frac{1}{2}u_x^2$ will define a surface of rank 2 and differential invariant order 2. (The function $u(x, y) = -x^2/2y$ discussed in Example 7.9 above is a particular case.) Indeed, the second-order differential invariants are u_{xx}, $u_{xy} = u_x u_{xx}$, and $u_{yy} = u_x^2 u_{xx}$. The single independent invariant u_{xx} is, however, not a function of the first-order invariant u_x, since their Jacobian matrix is

$$\frac{\partial(u_x, u_{xx})}{\partial(x, y)} = \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xxx} & u_{xxy} \end{pmatrix} = \begin{pmatrix} u_{xx} & u_x u_{xx} \\ u_{xxx} & u_x u_{xxx} + u_{xx}^2 \end{pmatrix} = u_{xx}^3 \neq 0,$$

since u is nonplanar. Thus one must use the third-order differential invariant classifying manifold to characterize such solutions.

Remark. The nonplanar solutions to the differential equation in Example 14.8 provide examples of nonreducible partially invariant submanifolds, where we are using Ovsiannikov's terminology [22]. Ondich [21] discusses conditions that a partially invariant solution be 'nonreducible', meaning that it is not invariant under a (continuous) subgroup of the symmetry group G, and hence has maximal rank p. In the moving frame approach, then, one can completely characterize nonreducible partially invariant solutions to partial differential equations as those whose graphs are submanifolds of higher order and maximal rank.

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* Any other first-order nonlinear equation $u_y = F(u_x)$ relating the two differential invariants will also work.

The fundamental equivalence theorem for submanifolds under general transformation group actions is a direct consequence of the corresponding Equivalence Theorem 7.2 for submanifolds under free actions.

THEOREM 14.9. *Let $S, \overline{S} \subset M$ be regular p-dimensional submanifolds whose n-jets lie in the domain of definition of a moving frame map $\rho^{(n)}$. Then S and $\overline{S}$ are* (*locally*) *congruent, $\overline{S} = g \cdot S$ if and only if they have the same differential invariant order s and their classifying manifolds of order $s+1$ are identical: $\mathcal{C}^{(s+1)}(\overline{S}) = \mathcal{C}^{(s+1)}(S)$.*

Finally, we discuss rigidity theorems for submanifolds under transformation groups. These come in two varieties. Roughly speaking, a rigidity result says that, under certain conditions, a submanifold is uniquely determined by its k-jet for some finite order k.

DEFINITION 14.10. A submanifold S is *order k congruent* to a submanifold $\overline{S}$ at a point $z \in S$ if there is a group transformation $g \in G$ such that S and $g \cdot \overline{S}$ have order k contact at the point z.

We shall call S order k congruent to $\overline{S}$ if this occurs for every $z \in S$. Note that the group transformation $g = g(z)$ may vary from point to point. If $G^{(k)}$ acts freely on J^k, then the group transformation $g(z)$ determining the contact is uniquely determined. The first rigidity theorem, which generalizes results in Griffiths [12], Green [11], and Jensen [14], states that order k congruence implies congruence provided k is sufficiently large. The *rigidity order* of S is the minimal k for which this applies. For example, the rigidity order of a circle under the Euclidean group is two, since the only curves that are second-order congruent to a circle are translates of it. On the other hand, a generic curve in the plane has rigidity order 3 under the Euclidean group.

THEOREM 14.11. *Let $S \subset M$ be a regular p-dimensional submanifold which has differential invariant order s with respect to a given moving frame. Then S has rigidity order at most $s+1$. In other words, a submanifold $\overline{S}$ is order $s+1$ congruent to S at every point $z \in S$ if and only if $S = g \cdot \overline{S}$ for a fixed $g \in G$.*

Proof. Note first that $\overline{S}$ and $\overline{S}_g = g \cdot \overline{S}$ have identical classifying manifolds. Moreover, if S and $\overline{S}_g$ have order $s+1$ contact at a common point z, then their $(s+1)$-jets coincide and, hence, their order $s+1$ differential invariant classifying manifolds agree at the point z. Therefore, the two submanifolds are order $s+1$ congruent at every point if and only if their order $s+1$ differential invariant classifying manifolds are identical: $\mathcal{C}^{(s+1)}(\overline{S}) = \mathcal{C}^{(s+1)}(S)$. Therefore, the result is an immediate consequence of Theorem 14.9. □

The simplest case is when the order of the moving frame equals the stabilization order of the group G. Generically, the rigidity order of a regular submanifold will be either $n+1$ or $n+2$, depending on whether (14.4) holds. Barring higher order singularities, the maximal rigidity order will be $n+p+1$. Jensen [14] appears to assert that the rigidity order is at most $n+1$, but does not consider (nongeneric) submanifolds of higher order, as in Example 14.8, or having other types of singularities.

A second type of rigidity theorem shows that one can uniquely characterize the group transformation mapping congruent submanifolds by knowing their order of contact.

DEFINITION 14.12. A p-dimensional submanifold $S \subset M$ is said to be order k *rigid* if the only congruent submanifold $\overline{S} = g \cdot S$ which has kth order contact with S at a point is S itself.

In other words, if $\overline{S} = g \cdot S$, then the condition $\mathrm{j}_k\overline{S}|_{z_0} = \mathrm{j}_k S|_{z_0}$ at $z_0 \in S \cap \overline{S}$ implies $g \in G_S$ and so $\overline{S} = S$. The second rigidity theorem can now be stated.

THEOREM 14.13. *Let G act freely on $\mathcal{V}^n \subset \mathrm{J}^n$. Let S be an order n regular p-dimensional submanifold which has differential invariant order $s \geqslant n$. Then S is rigid of order $s+1$.*

Proof. We let $\rho^{(n)}$ be a moving frame defined in a neighborhood of S. Let $\overline{S} = g \cdot S$ have contact at order $s+1$ at $z_0 \in S \cap \overline{S}$. Let $z_0^{(s+1)} = \mathrm{j}_{s+1}\overline{S}|_{z_0} = \mathrm{j}_{s+1}S|_{z_0} \in \mathcal{V}^{s+1}$. Congruence implies that S and $\overline{S}$ have identical differential invariant classifying manifolds $\mathcal{C}^{(s+1)}(\overline{S}) = \mathcal{C}^{(s+1)}(S)$, which are parametrized by their $(s+1)$-jets. Theorem 5.16 implies uniqueness of the group transformation g defining the congruence map once we specify that it fix the common point $z_0^{(s+1)}$. Finally, freeness of the action of G on $\mathcal{V}^{s+1}$ implies that $g = e$, which proves rigidity. □

Remark. If G only acts locally freely on $\mathcal{V}^{s+1}$, then Theorem 14.13 reduces to a local rigidity result, i.e., $(s+1)$st-order contact of $\overline{S} = g \cdot S$ and S implies that the congruence transformation g must lie in a discrete subgroup of G. However, since the higher order differential invariants completely determine the higher order jets of the submanifolds, one can eliminate the discrete ambiguity provided $G^{(k)}$ acts freely on the appropriate subset of J^k for k sufficiently large.

EXAMPLE 14.14. Consider the translation action $z \mapsto z + a$ of $G = \mathbb{R}^2$ on $M = \mathbb{R}^2$. The derivative coordinates $u_x, u_{xx}, u_{xxx}, \ldots$ provide a complete system of differential invariants. The classifying curve of a generic curve $u = f(x)$ is parametrized by (u_x, u_{xx}). However, singularities may require us to prolong to higher order in order to assure rigidity. For example, the curve C given by $u = x^4 - 2x^2$ has second order contact at $z_0 = (1, 0)$ with its translate by $a = (2, 0)$. Moreover, the first two differential invariants $\{u_x, u_{xx}\}$ have rank 1 on C. However, C is not regular of differential invariant order 2 because its second-order classifying curve intersects itself at the point $u_x = 0$, $u_{xx} = 8$, which permits second order nonrigidity. The curve C is locally rigid at order 1, and completely rigid at order 3.

15. Examples

We now demonstrate the preceding theory with several additional examples. Only space precludes discussing a more extensive range of examples in this paper. However, all of the classical examples, including Euclidean, affine and projective geometry, as well as an extensive variety of new transformation group actions (e.g., conformal geometry) not previously treated by the classical moving frame techniques, can be directly handled by our regularized techniques.

EXAMPLE 15.1. We return to the multiplier representation

$$(x, u) \longmapsto \left(\frac{\alpha x + \beta}{\gamma x + \delta}, \frac{u}{\gamma x + \delta}\right), \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}(2), \tag{15.1}$$

of the general linear group GL(2) on $\mathbb{R}^2$ that was studied in depth in Part I [9], and plays a fundamental role in classical invariant theory and the calculus of variations. The right-lifted invariants of order zero are just

$$y = \frac{\alpha x + \beta}{\gamma x + \delta}, \qquad v = \frac{u}{\gamma x + \delta}. \tag{15.2}$$

Choosing y as the lifted independent variable, its jet differential

$$\eta = \mathrm{d}_J y = \frac{\alpha\delta - \beta\gamma}{(\gamma x + \delta)^2}\, \mathrm{d}x \tag{15.3}$$

determines the lifted horizontal invariant form. The corresponding invariant differential operator is

$$\mathcal{E} = \mathrm{D}_y = \frac{(\gamma x + \delta)^2}{\alpha\delta - \beta\gamma}\, \mathrm{D}_x. \tag{15.4}$$

Applying $\mathcal{E}$ recursively to the dependent lifted invariant v leads to the lifted differential invariants $v_k = \mathcal{E}^k v$, the first few of which are

$$\begin{aligned} v_y &= \frac{(\gamma x + \delta) u_x - \gamma u}{\alpha\delta - \beta\gamma}, \qquad v_{yy} = \frac{(\gamma x + \delta)^3 u_{xx}}{(\alpha\delta - \beta\gamma)^2}, \\ v_{yyy} &= \frac{(\gamma x + \delta)^5 u_{xxx} + 3\gamma(\gamma x + \delta)^4 u_{xx}}{(\alpha\delta - \beta\gamma)^3}, \\ v_{yyyy} &= \frac{(\gamma x + \delta)^7 u_{xxxx} + 8\gamma(\gamma x + \delta)^6 u_{xxx} + 12\gamma^2(\gamma x + \delta)^5 u_{xx}}{(\alpha\delta - \beta\gamma)^4}. \end{aligned} \tag{15.5}$$

These formulae coincide with the transformation laws for the prolonged group action. On the regular subdomain $V = \{uu_{xx} > 0\} \subset \mathrm{J}^2$, we can choose the cross-section defined by the normalizations

$$y = 0, \qquad v = 1, \qquad v_y = 0, \qquad v_{yy} = 1. \tag{15.6}$$

Solving for the group parameters gives

$$\alpha = \sqrt{uu_{xx}}, \qquad \beta = -x\sqrt{uu_{xx}}, \qquad \gamma = u_x, \qquad \delta = u - xu_x. \tag{15.7}$$

These serve to parametrize a right GL(2) moving frame of order two:

$$\rho^{(2)}(x, u, u_x, u_{xx}) = \begin{pmatrix} \sqrt{uu_{xx}} & -x\sqrt{uu_{xx}} \\ u_x & u - xu_x \end{pmatrix}. \tag{15.8}$$

The left-moving frame computed in [9] is obtained by inverting:

$$\begin{aligned} \widetilde{\rho}^{(2)}(x, u, u_x, u_{xx}) &= \rho^{(2)}(x, u, u_x, u_{xx})^{-1} \\ &= \frac{1}{\sqrt{u^3 u_{xx}}} \begin{pmatrix} u - xu_x & x\sqrt{uu_{xx}} \\ -u_x & \sqrt{uu_{xx}} \end{pmatrix}. \end{aligned} \tag{15.9}$$

Substituting the moving frame normalizations (15.7) into the higher order lifted differential invariants leads to the normalized differential invariants; the first non-constant ones are obtained by normalizing v_{yyy} and v_{yyyy}:

$$I = \frac{uu_{xxx} + 3u_x u_{xx}}{\sqrt{uu_{xx}^3}}, \qquad J = \frac{u^2 u_{xxxx} + 8uu_x u_{xxx} + 12u_x^2 u_{xx}}{uu_{xx}^2}. \tag{15.10}$$

Incidentally, the Replacement Theorem 10.3 implies that we can also write I and J using the *same* formulae in the lifted invariants, e.g., $I = v^{-1/2} v_{yy}^{-3/2}(vv_{yyy} + 3v_y v_{yy})$. Applying the normalizations (15.7) to the lifted horizontal form (15.3) leads to the contact-invariant one-form and the associated invariant differential operator:

$$\omega = \sqrt{\frac{u_{xx}}{u}}\, \mathrm{d}x, \qquad \mathcal{D} = \sqrt{\frac{u}{u_{xx}}}\, \mathrm{D}_x. \tag{15.11}$$

The jet differentials of the second-order lifted invariants are

$$\begin{aligned} \mathrm{d}_J y &= \frac{\alpha\delta - \beta\gamma}{(\gamma x + \delta)^2}\, \mathrm{d}x, \qquad \mathrm{d}_J v = \frac{\mathrm{d}u}{\gamma x + \delta} - \frac{\gamma u\, \mathrm{d}x}{(\gamma x + \delta)^2}, \\ \mathrm{d}_J v_y &= \frac{(\gamma x + \delta)\, \mathrm{d}u_x - \gamma\, \mathrm{d}u + \gamma u_x\, \mathrm{d}x}{\alpha\delta - \beta\gamma}, \\ \mathrm{d}_J v_{yy} &= \frac{(\gamma x + \delta)^3\, \mathrm{d}u_{xx} + 3\gamma(\gamma x + \delta)^2 u_{xx}\, \mathrm{d}x}{(\alpha\delta - \beta\gamma)^2}. \end{aligned} \tag{15.12}$$

The right-invariant Maurer–Cartan forms on GL(2) are the entries of the matrix product $\mathrm{d}A \cdot A^{-1}$, namely

$$\begin{aligned} \mu^1 &= \frac{\delta\,\mathrm{d}\alpha - \gamma\,\mathrm{d}\beta}{\alpha\delta - \beta\gamma}, \qquad \mu^2 = \frac{-\beta\,\mathrm{d}\alpha + \alpha\,\mathrm{d}\beta}{\alpha\delta - \beta\gamma}, \\ \mu^3 &= \frac{\delta\,\mathrm{d}\gamma - \gamma\,\mathrm{d}\delta}{\alpha\delta - \beta\gamma}, \qquad \mu^4 = \frac{-\beta\,\mathrm{d}\gamma + \alpha\,\mathrm{d}\delta}{\alpha\delta - \beta\gamma}. \end{aligned} \tag{15.13}$$

The eight one-forms (15.12), (15.13) form a coframe on $\mathcal{B}^{(2)} = \mathrm{GL}(2) \times \mathrm{J}^2$ whose symmetry group coincides with the right-lifted action of GL(2). The group differentials can be written as invariant linear combinations of the Maurer–Cartan forms:

$$\begin{aligned} &\mathrm{d}_G y = y\mu^1 + \mu^2 - y^2\mu^3 - y\mu^4, \qquad \mathrm{d}_G v = -yv\mu^3 - v\mu^4, \\ &\mathrm{d}_G v_y = -v_y\,\mu^1 + (yv_y - v)\mu^3, \\ &\mathrm{d}_G v_{yy} = -2v_{yy}\mu^1 + 3yv_{yy}\mu^3 + v_{yy}\mu^4, \end{aligned} \tag{15.14}$$

and can replace the Maurer–Cartan forms in the lifted coframe. The coefficients in (15.14) are given directly by formula (3.8). As in Example 6.7, we write down the coefficient matrix corresponding to the prolonged infinitesimal generators of GL(2); we find, to order 4,

$$\begin{pmatrix} x & 1 & -x^2 & -x \\ 0 & 0 & -xu & -u \\ -u_x & 0 & xu_x - u & 0 \\ -2u_{xx} & 0 & 3xu_{xx} & u_{xx} \\ -3u_{xxx} & 0 & 5xu_{xxx} + 3u_{xx} & 2u_{xxx} \\ -4u_{xxxx} & 0 & 7xu_{xxxx} + 8u_{xxx} & 3u_{xxxx} \end{pmatrix}. \tag{15.15}$$

The lifted version is obtained by replacing x and u by y and v:

$$\begin{pmatrix} y & 1 & -y^2 & -y \\ 0 & 0 & -yv & -v \\ -v_y & 0 & yv_y - v & 0 \\ -2v_{yy} & 0 & 3yv_{yy} & v_{yy} \\ -3v_{yyy} & 0 & 5yv_{yyy} + 3v_{yy} & 2v_{yyy} \\ -4v_{yyyy} & 0 & 7yv_{yyyy} + 8v_{yyy} & 3v_{yyyy} \end{pmatrix}. \tag{15.16}$$

The first four rows of (15.16) then give the coefficients in (15.14). The normalized matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & 1 \\ -3I & 0 & 3 & 2I \\ -4J & 0 & 8I & 3J \end{pmatrix} \tag{15.17}$$

is obtained by substituting (15.6), (15.10) into (15.16); in other words, (15.17) is the invariantization of the infinitesimal generator coefficient matrix (15.15).

Since GL(2) acts transitively on the open subset of J^2 under consideration, we can find the moving coframe on J^2 either by normalizing the jet differentials (15.12) or the Maurer–Cartan forms (15.13). The former become $\boldsymbol{\gamma} = (\rho^{(2)})^* \, d_J w$, so that

$$\begin{aligned}
\gamma^1 &= (\sigma^{(2)})^* \, d_J y = \sqrt{\frac{u_{xx}}{u}}\, dx, \\
\gamma^2 &= (\sigma^{(2)})^* \, d_J v = \frac{du - u_x\, dx}{u}, \\
\gamma^3 &= (\sigma^{(2)})^* \, d_J v_y = \frac{du_x - u_{xx}\, dx}{\sqrt{u u_{xx}}} + \frac{u_x (du - u_x\, dx)}{\sqrt{u^3 u_{xx}}} + \sqrt{\frac{u_{xx}}{u}}\, dx, \\
\gamma^4 &= (\sigma^{(2)})^* \, d_J v_y = \frac{du_{xx} - u_{xxx}\, dx}{u_{xx}} + \frac{u u_{xxx} + 3 u_x u_{xx}}{u_{xx}}\, dx,
\end{aligned} \tag{15.18}$$

where we have explicitly written out the contact and horizontal components, the latter being invariant linear combinations of the invariant one-form ω. Indeed, in view of (15.6), (15.10),

$$\begin{aligned}
\gamma^1_H &= (\sigma^{(2)})^* \, d_H y = \omega, \\
\gamma^2_H &= (\sigma^{(2)})^* \, d_H v = (\sigma^{(2)})^* (v_y \, d_H y) = 0, \\
\gamma^3_H &= (\sigma^{(2)})^* \, d_H v_y = (\sigma^{(2)})^* (v_{yy} \, d_H y) = \omega, \\
\gamma^4_H &= (\sigma^{(2)})^* \, d_H v_{yy} = (\sigma^{(2)})^* (v_{yyy} \, d_H y) = I\omega.
\end{aligned} \tag{15.19}$$

On the other hand, substituting (15.17) in the general identity (6.4) produces the explicit linear dependencies:

$$\gamma^1 = -\zeta^2, \qquad \gamma^2 = \zeta^4, \qquad \gamma^3 = \zeta^3, \qquad \gamma^4 = 2\zeta^1 - \zeta^4. \tag{15.20}$$

Combining (15.19), (15.20) yields the corresponding formulae for the horizontal components of the moving coframe:

$$\zeta^1_H = \tfrac{1}{2} I\, \omega, \qquad \zeta^2_H = -\omega, \qquad \zeta^3_H = \omega, \qquad \zeta^4_H = 0, \tag{15.21}$$

reconfirming our moving coframe computation in Part I.

Substituting (15.17), (15.21) into the general formula (13.7), (13.8), produces the explicit formula connecting the normalized and derived differential invariants. The easiest way to compute the correction terms is to multiply the matrix (15.17) by the column vector $\left(\tfrac{1}{2} I, -1, 1, 0\right)^T$ whose entries are given in (15.21); the result is a column vector

$$\left(1, 0, 1, -1 - I, 3 - \tfrac{3}{2} I^2, 8I - 2IJ\right)^T$$

whose entries are the correction terms. (Alternatively, one can use column operations as in Example 13.5.) For example, the last two entries imply

$$\mathcal{D} I = J - \tfrac{3}{2} I^2 + 3, \qquad \mathcal{D} J = K - 2IJ + 8, \tag{15.22}$$

where $K = (\sigma^{(2)})^* v_{yyyyy}$ is the fifth-order normalized differential invariant. Note that we can iterate to find higher-order correction terms, e.g.,

$$\mathcal{D}^2 I = \mathcal{D} J - 3I\, \mathcal{D} I = K - 5IJ + \tfrac{9}{2} I^3 - 9I + 8.$$

EXAMPLE 15.2. Consider the intransitive action of the orthogonal group O(3) on surfaces in three-dimensional space $M = \mathbb{R}^3$. Assume that the surface is given as the graph of a function $u = f(x^1, x^2)$. The order zero invariants are

$$\begin{pmatrix} y^1 \\ y^2 \\ v \end{pmatrix} = R \begin{pmatrix} x^1 \\ x^2 \\ u \end{pmatrix}, \qquad R = (R^i_j) \in \mathrm{O}(3). \tag{15.23}$$

The lifted contact-invariant coframe and associated invariant differential operators are

$$\begin{aligned} \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix} &= \begin{pmatrix} R^1_1 + R^1_3 u_1 & R^1_2 + R^1_3 u_2 \\ R^2_1 + R^2_3 u_1 & R^2_2 + R^2_3 u_2 \end{pmatrix} \begin{pmatrix} \mathrm{d}x^1 \\ \mathrm{d}x^2 \end{pmatrix}, \\ \begin{pmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{pmatrix} &= \begin{pmatrix} R^1_1 + R^1_3 u_1 & R^1_2 + R^1_3 u_2 \\ R^2_1 + R^2_3 u_1 & R^2_2 + R^2_3 u_2 \end{pmatrix}^{-T} \begin{pmatrix} \mathrm{D}_1 \\ \mathrm{D}_2 \end{pmatrix}. \end{aligned} \tag{15.24}$$

The lifted invariants are $v_{jk} = (\mathcal{E}_1)^j (\mathcal{E}_2)^k v$; in particular

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} R^1_1 + R^1_3 u_1 & R^1_2 + R^1_3 u_2 \\ R^2_1 + R^2_3 u_1 & R^2_2 + R^2_3 u_2 \end{pmatrix}^{-T} \begin{pmatrix} R^3_1 + R^3_3 u_1 \\ R^3_2 + R^3_3 u_2 \end{pmatrix}.$$

The normalization

$$y^1 = 0, \qquad v_1 = 0, \qquad v_2 = 0, \tag{15.25}$$

leads to a particularly simple first-order moving frame. Introduce the column vectors

$$z = (x_1, x_2, u)^T, \qquad n = \frac{N}{|\,N\,|} = \frac{(-u_1, -u_2, 1)^T}{\sqrt{1 + u_1^2 + u_2^2}},$$

which respectively define the point on the surface, and the corresponding unit normal. Then

$$R = (t\, \hat{t}\, n)^T, \qquad \text{where } t = \frac{z \wedge n}{|z \wedge n|}, \ \hat{t} = n \wedge t = \frac{z - (z \cdot n) n}{|z \wedge n|}, \tag{15.26}$$

define distinguished, orthogonally equivariant, unit tangent vectors. The moving frame (15.26) applies to surfaces provided that the unit normal is not parallel to the point z. Pulling back the remaining lifted invariants leads to the first-order differential invariants

$$\begin{aligned} J &= (\sigma^{(1)})^* y^2 = \frac{(z \cdot n)^2 - |\,z\,|^2}{|\,z \wedge n\,|} = -\,|\,z \wedge n\,| = -\sqrt{|\,z\,|^2 - (z \cdot n)^2}, \\ I &= (\sigma^{(1)})^* v = z \cdot n. \end{aligned} \tag{15.27}$$

(It's interesting that we don't obtain the invariant $|z|$ directly; it is of course a function of the fundamental invariants (15.27).)

The contact-invariant coframe and invariant differential operators are obtained by pulling back the horizontal differentials of the y^i, so

$$\begin{aligned}\boldsymbol{\omega} &= A\,\mathrm{d}x,\\ \mathcal{D} &= A^{-T}\,\mathbf{D},\end{aligned} \qquad \text{where } A = \begin{pmatrix} t\cdot t_1 & t\cdot t_2\\ \hat{t}\cdot t_1 & \hat{t}\cdot t_2\end{pmatrix}. \tag{15.28}$$

Here $t_1 = (1, 0, u_1)$, $t_2 = (0, 1, u_2)$ are the coordinate tangent vectors to the surface, and A is the transpose of their coefficient matrix with respect to the moving frame tangent vectors t, $\hat{t}$. A generating system of differential invariants requires the corresponding normalized second-order invariants:

$$\begin{pmatrix} I_{11} & I_{12}\\ I_{12} & I_{22}\end{pmatrix} = \frac{1}{|N|}\mathcal{D}^2 u = \frac{A^{-T}(\nabla^2 u)A^{-1}}{\sqrt{1+u_1^2+u_2^2}}. \tag{15.29}$$

Here $\nabla^2 u$ is the usual Hessian matrix of u, so $\mathcal{D}^2 u$ represents an 'equivariant Hessian'. However, using (13.7), the recurrence relations (or syzygies)

$$\begin{aligned}\mathcal{D}_1 J &= I\,I_{12}, & \mathcal{D}_2 J &= 1 + I\,I_{22},\\ \mathcal{D}_1 I &= -J\,I_{12}, & \mathcal{D}_2 I &= -J\,I_{22},\end{aligned}$$

show that only I, J, and I_{11} are required to form a generating system of differential invariants.

EXAMPLE 15.3. Consider the action of the rotation group SO(3) on $M = \mathbb{R}^4$ corresponding to the lifted zeroth order invariants

$$y = R\,x, \qquad v = u \quad \text{with } R \in \mathrm{SO}(3), \tag{15.30}$$

where $y = (y^1, y^2, y^3)$, $x = (x^1, x^2, x^3)$. In this case, the differential invariants were found in [18, Chapter 5] by an ad hoc approach; the moving frame method allows us to be systematic. The lifted invariant one-forms and corresponding invariant differential operators are

$$\eta^i = \mathrm{d}_H y^i = \sum_{j=1}^{3} R^i_j\,\mathrm{d}x^j, \qquad \mathcal{E}_i = \sum_{j=1}^{3} R^i_j\,\mathrm{D}_j, \quad i = 1, 2, 3.$$

The lifted invariants are then

$$\begin{aligned} y^i &= \sum_{j=1}^{3} R^i_j x^j, & v &= u,\\ v_i &= \sum_{j=1}^{3} R^i_j u_j, & v_{ij} &= \sum_{k,l=1}^{3} R^i_k R^j_l u_{kl}, \quad \dots .\end{aligned}$$

To determine a first-order moving frame, we consider the cross-section

$$y^2 = 0, \qquad y^3 = 0, \qquad v_3 = 0. \tag{15.31}$$

The normalization equations (15.31) can be solved provided $x \wedge \nabla u \neq 0$, where $\nabla u = (u_1, u_2, u_3)$. The solution is $R = (a\ b\ c)^T$, where the column vectors

$$\begin{aligned} a &= \frac{x}{|x|}, \qquad b = a \wedge c = \frac{(x \cdot \nabla u)x - |\,x\,|^2\, \nabla u}{|\,x\,|\,|\,x \wedge \nabla u\,|}, \\ c &= \frac{x \wedge \nabla u}{|\,x \wedge \nabla u\,|}, \end{aligned} \tag{15.32}$$

define a rotationally equivariant orthonormal frame. The resulting first-order invariants are

$$\begin{aligned} J^1 &= |\,x\,|, \qquad J^2 = J^3 = 0, \qquad I = u, \qquad I_1 = \frac{x \cdot \nabla u}{|\,x\,|}, \\ I_2 &= -\frac{|\,x \wedge \nabla u\,|}{|\,x\,|}, \qquad I_3 = 0. \end{aligned} \tag{15.33}$$

Of course, one can eliminate the denominators since they are invariant themselves. The corresponding contact-invariant coframe and invariant differential operators are

$$\begin{aligned} &\widetilde{\omega}^1 = x \cdot \mathrm{d}x, \qquad \widetilde{\omega}^2 = \left[(x \cdot \nabla u)x - |\,x\,|^2\, \nabla u\right] \cdot \mathrm{d}x, \\ &\widetilde{\omega}^3 = (x \wedge \nabla u) \cdot \mathrm{d}x, \\ &\widetilde{\mathcal{D}}_1 = x \cdot \mathbf{D}, \qquad \widetilde{\mathcal{D}}_2 = \nabla u \cdot \mathbf{D}, \qquad \widetilde{\mathcal{D}}_3 = (x \wedge \nabla u) \cdot \mathbf{D}, \end{aligned} \tag{15.34}$$

where the tildes indicate that we have dropped the invariant denominators arising from a direct pull-back via (15.32). We leave it to the reader to deduce the commutator formulae. A complete generating system of differential invariants requires second order invariants:

$$\begin{aligned} I_{11} &= x^T(\nabla^2 u)x, \qquad I_{12} = x^T(\nabla^2 u)\nabla u, \qquad I_{13} = x^T(\nabla^2 u)(x \wedge \nabla u), \\ I_{22} &= \nabla u^T(\nabla^2 u)\nabla u, \qquad I_{23} = \nabla u^T(\nabla^2 u)(x \wedge \nabla u), \\ I_{33} &= (x \wedge \nabla u)^T(\nabla^2 u)(x \wedge \nabla u). \end{aligned} \tag{15.35}$$

However, using either the recurrence relations (keeping in mind that we modified the invariant differential operators and second-order invariants from their normalized versions) or directly computing, we see that only three differential invariants,

$$J^1 = |\,x\,|, \qquad I = u, \qquad I_{33} = (x \wedge \nabla u)^T\, (\nabla^2 u)\, (x \wedge \nabla u),$$

are required to generate all the rest.

16. Partial Regularization

The one draw-back to the regularized method as presented so far is that one needs to compute a sufficient number of higher-order lifted differential invariants before commencing the normalization procedure. This can be quite computationally intensive – for instance, in the case of projective geometry of curves in the plane, cf. [6], one needs to prolong to sixth-order derivatives in order to specify a complete set of normalizations. In the classical Cartan approach, as well as our earlier method of moving coframes, cf. [9], one avoids having to to perform a complete prolongation before starting to normalize. A similar option exists in the regularized method; one can, provided some care is taken, normalize lower-order lifted invariants by solving for some of the group parameters, and then using these simplified expressions to compute higher order, *partially regularized* lifted invariants. The optimal strategy is to normalize globally defined lifted invariants, but regularize locally defined ones. This allows one to construct, with a minimal amount of computation, a partially regularized moving frame that applies to all submanifolds. The full normalization can then be accomplished for particular classes of submanifolds satisfying appropriate regularity conditions.

An essential complication is that the lifted invariant differential operators that are used to construct the higher order invariants *cannot be directly normalized*! Indeed, unlike their fully lifted or their fully normalized counterparts, partially normalized invariant differential operators will often contain additional terms involving derivatives with respect to the remaining group parameters. As pointed out by I. Anderson (personal communication), the additional terms can be interpreted as coming from the reduction of the flat connection on the regularized bundle to the appropriate partially normalized principal subbundle. These terms are correctly predicted by the moving coframe approach, but are less transparent when using a direct approach based on the lifted invariants. The resulting theory has yet to be fully developed, and lack of space precludes a detailed treatment in the present paper.

We shall content ourselves with treating one final illustrative example, that of curves in the plane under the special affine group; see [9] for details. We shall demonstrate how a regularized version of our moving coframe method can be used to perform a globally defined partial regularization that includes nonconvex curves. Also, for variety, and since the classical results are in terms of the left moving frame, we will use the left regularized action in this example. Let $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$ act on $M = \mathbb{R}^2$ according to

$$g \cdot (x, u) = (\alpha x + \beta u + a, \gamma x + \delta u + b), \quad \alpha\delta - \beta\gamma = 1. \tag{16.1}$$

The zeroth order left lifted invariants are the components of $g^{-1} \cdot (x, u)$, i.e.,

$$y = \delta(x - a) - \beta(u - b), \qquad v = -\gamma(x - a) + \alpha(u - b). \tag{16.2}$$

In the fully regularized approach, we compute the higher-order lifted invariants by successively differentiating v with respect to y using the lifted invariant differential operator

$$\mathcal{E} = \frac{1}{\delta - u_x \beta} \mathrm{D}_x \tag{16.3}$$

associated with the invariant horizontal form $\eta = \mathrm{d}_H y = (\delta - \beta u_x)\,\mathrm{d}x$. The first few are

$$\begin{aligned}
v_y &= \mathcal{E} v = -\frac{\gamma - \alpha u_x}{\delta - \beta u_x}, \qquad v_{yy} = \mathcal{E} v_y = -\frac{u_{xx}}{(\delta - \beta u_x)^3},\\
v_{yyy} &= \mathcal{E} v_{yy} = -\frac{(\delta - \beta u_x) u_{xxx} + 3\beta u_{xx}^2}{(\delta - \beta u_x)^5},\\
v_{yyyy} &= \mathcal{E} v_{yyy} = -\frac{u_{xxxx}(\delta - \beta u_x)^2 + 10 u_{xx} u_{xxx} \beta(\delta - \beta u_x) + 15 u_{xx}^3 \beta^2}{(\alpha + \beta u_x)^7}.
\end{aligned}$$

By choosing the cross-section $\{(0,0,0,1,0)\} \subset \mathrm{J}^3$ we obtain the classical equi-affine moving frame

$$\begin{aligned}
\beta &= -\frac{1}{3} u_{xx}^{-5/3} u_{xxx}, \qquad \alpha = u_{xxx}^{-1/3},\\
\gamma &= -u_x u_{xx}^{-1/3}, \qquad a = x, \qquad b = u.
\end{aligned} \tag{16.4}$$

The first differential invariant is found by applying the moving frame normalizations to the next lifted invariant v_{yyyy}, leading to the equi-affine curvature

$$\kappa = \frac{3 u_{xx} u_{xxxx} - 5 u_{xxx}^2}{3 u_{xxx}^{8/3}}. \tag{16.5}$$

In the partial normalization approach, we try to normalize lifted invariants as they appear, and thereby avoid the long computations required to initially produce the general higher-order lifted invariants. For example, we can normalize the zeroth order lifted invariants $y = v = 0$ by setting $a = x$, $b = u$. In the moving coframe method, we substitute these normalizations into the independent left invariant Maurer–Cartan forms

$$\begin{aligned}
&\mu^1 = \delta\,\mathrm{d}\alpha - \beta\,\mathrm{d}\gamma, \qquad \mu^2 = \delta\,\mathrm{d}\beta - \beta\,\mathrm{d}\delta, \qquad \mu^3 = \alpha\,\mathrm{d}\gamma - \gamma\,\mathrm{d}\alpha,\\
&\nu^1 = \delta\,\mathrm{d}a - \beta\,\mathrm{d}b, \qquad \nu^2 = -\gamma\,\mathrm{d}a + \alpha\,\mathrm{d}b.
\end{aligned} \tag{16.6}$$

The linear dependency between the horizontal components

$$\nu_H^1 = (\delta - \beta u_x)\,\mathrm{d}x, \qquad \nu_H^2 = (-\gamma + \alpha u_x)\,\mathrm{d}x,$$

produces the first order lifted invariant v_y, which can, of course, be constructed directly. Normalizing $\nu_H^2 = 0$ produces the partial normalizations

$$a = x, \qquad b = u, \qquad \gamma = \alpha u_x, \qquad \delta = \beta u_x + \frac{1}{\alpha}, \tag{16.7}$$

the final formula being a consequence of unimodularity. The partially normalized Maurer–Cartan forms

$$\nu^1 = \alpha^{-1}\,\mathrm{d}x - \beta(\mathrm{d}u - u_x\,\mathrm{d}x), \qquad \nu^2 = \alpha(\mathrm{d}u - u_x\,\mathrm{d}x),$$

include the basic invariant contact form, while

$$\nu_H^1 = \omega = \alpha^{-1}\,\mathrm{d}x \tag{16.8}$$

is a contact-invariant horizontal form. Now, the key complication is that even though one might be tempted to directly normalize the invariant differential operator (16.3), the resulting total differential operator $\widehat{\mathcal{E}} = \alpha\,\mathrm{D}_x$, which is dual to the horizontal form (16.8), is *not an invariant differential operator*! In other words, applying $\widehat{\mathcal{E}}$ to the higher-order partially normalized differential invariants *does not produce lifted differential invariants*. For example, the linear dependency between the horizontal component of $\mu^3 = \alpha^2\,\mathrm{d}u_x$ and ω leads to the second-order partially normalized differential invariant

$$J = \alpha^3 u_{xx},$$

which agrees with the reduction of the lifted invariant v_{yy} under the partial normalizations (16.7). However, $\alpha\,\mathrm{D}_x J = \alpha^4 u_{xxx}$ does not agree with the reduction of v_{yyy} under (16.7), which is

$$K = \alpha^4 u_{xxx} + 3\alpha^5\beta u_{xx}^2. \tag{16.9}$$

Indeed, $\alpha^4 u_{xxx}$ is not even a lifted invariant! Thus, we cannot use the directly normalized total differential operator to compute higher order partially normalized invariants. One resolution of this difficulty relies on adapting the moving coframe method [9]. The remaining partially normalized Maurer–Cartan forms are

$$\begin{aligned} \mu^1 &= \alpha^{-1}\,\mathrm{d}\alpha - \alpha\beta\,\mathrm{d}u_x, \qquad \mu^2 = -\alpha^{-2}\beta\,\mathrm{d}\alpha + \alpha^{-1}\,\mathrm{d}\beta - \beta^2\,\mathrm{d}u_x,\\ \mu^3 &= \alpha^2\,\mathrm{d}u_x. \end{aligned} \tag{16.10}$$

If $L(\alpha, \beta, x, u^{(n)})$ is any function, then

$$\begin{aligned} \mathrm{d}L &\equiv (\mathrm{D}_x L)\,\mathrm{d}x + L_\alpha\,\mathrm{d}\alpha + L_\beta\,\mathrm{d}\beta\\ &\equiv (\alpha\,\mathrm{D}_x L + \beta J L_\alpha)\omega + (\alpha L_\alpha - \beta L_\beta)\mu^1 + \alpha L_\beta \mu^2, \end{aligned}$$

where $\equiv$ indicates that we have omitted the unimportant vertical (contact) components. We conclude that if L is any lifted invariant, then so are

$$\begin{aligned} &\mathcal{D}L = \alpha\,\mathrm{D}_x L + \beta J L_\alpha = \alpha\,\mathrm{D}_x L + \alpha^3\beta u_{xx} L_\alpha,\\ &\mathcal{F}_1(L) = \alpha L_\alpha - \beta L_\beta, \qquad \mathcal{F}_2(L) = \alpha L_\beta. \end{aligned}$$

For example, $\mathcal{D}J = K$, $\mathcal{F}_1 J = 3J$, $\mathcal{F}_2 J = 0$, while

$$\begin{aligned} &\mathcal{D}K = L = \alpha^5 u_{xxxx} + 10\alpha^6\beta u_{xx}u_{xxx} + 15\alpha^7\beta^2 u_{xx}^3,\\ &\mathcal{F}_1 K = 5K, \qquad \mathcal{F}_2 K = 10JK, \end{aligned}$$

where L is obtained by substituting (16.7) into v_{yyyy}. Therefore, higher-order partially normalized differential invariants are given by successively applying the invariant differential operator

$$\mathcal{D} = \alpha\, \mathrm{D}_x + \beta J \partial_\alpha = \alpha\, \mathrm{D}_x + \alpha^3 \beta\, u_{xx} \partial_\alpha \tag{16.11}$$

to the fundamental invariant $J = \alpha^3 u_{xx}$. (Since the operators $\mathcal{F}_1 = \alpha\partial_\alpha - \beta\partial_\beta$ and $\mathcal{F}_2 = \alpha\partial_\beta$ preserve the order of differential invariants, they will not produce anything new.) Note the appearance of additional 'connection terms' involving derivatives with respect to the remaining group parameters in (16.11); these have no counterpart in either the fully lifted theory or the fully normalized version. They can be interpreted as arising from the total derivative component of the reduction of the flat connection on $\mathcal{B}^{(\infty)}$, to the subbundle specified by the normalizations (16.7). As usual, further reductions rely on imposing genericity assumptions on the curve. In the standard case, one assumes that $u_{xx} \neq 0$, which allows us to perform the nonglobal normalization $J = 1$, $K = 0$, leading to the standard moving frame (16.4). See [9] for further details.

17. Conclusions

In this paper we have provided a general theoretical foundation for the method of moving frames for finite-dimensional Lie transformation groups. The regularization procedure is also of great practical applicability, and gives a powerful tool for investigating the differential invariants, equivalence and symmetry properties of submanifolds under quite general transformation groups. Further applications that warrant further research and development include:

(1) An immediate application of the moving frame method would be to the classification of the differential invariants associated with many of the transformation groups arising in physics. As remarked in [18], to date such classifications have not been completed, even for some of the most fundamental groups of physical importance.

(2) In [11] M. Green gives various intriguing numerical formulae for the number of differential invariants for curves in a homogeneous space. These formulae were generalized in [18], but the extension to surfaces and higher dimensional submanifolds remains open. The resolution of the syzygy problem given here should provide insight into resolving such generalizations.

(3) The completion of the theory of partial regularization of Section 16 and the determination of explicit connection formulae would greatly aid in the practical application of the method to concrete problems.

(4) The variational tricomplex given by the operators d_H, d_V and d_G on the regularized bundle could have important applications to the study of differential equations, variational problems, and conservation laws under the action of symmetry groups, and thus deserves a detailed investigation.

(5) Applications to Ovsiannikov's method of partially invariant solutions using the remarks after Example 14.8 appear to be quite promising.
(6) The commutation formulae and syzygy classifications will have important applications to Lisle's 'frame method' for symmetry classification of partial differential equations [15].
(7) An inductive approach to complicated equivalence problems was described in [18], and is based on the solution to a simpler problem based on a subgroup of the full group. Lisle [15] successfully uses an inductive approach to determining the invariant differential operators, which indicates that a general implementation of inductive methods for moving frames would not be difficult. Inductive formulae have the advantage of expressing invariant quantities for the larger group in terms of those associated with the subgroup.
(8) Finally, a theoretical justification of the moving frame method for infinite pseudo-groups, as illustrated in [9], corresponding to the finite-dimensional theory described here, would be of great significance. Such a theory would, we believe, be an important aid in further developing the general theory and applications of Lie pseudo-groups.

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