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On relative invariants

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1 Introduction

This paper is concerned with the classification of relative invariants for transformation group actions on manifolds. Classically an invariant (or absolute invariant) of a transformation group is a function whose value is unaffected by the group transformations. A simple example is provided by the area function, which is invariant under the special affine group consisting of area-preserving transformations. The classification of invariants of regular Lie group actions is well known, being a direct consequence of the general Frobenius Theorem, cf. [25]. Our interest is in a slight, but important generalization, where considerably less is known. A relative invariant of a transformation group is a function whose value is multiplied by a certain factor, known as a multiplier, under the group transformations. For example, under the full affine group, area is no longer invariant, but is scaled according to the determinantal multiplier, and hence defines a relative invariant. Ordinary invariants can be viewed as fixed points of the induced representation of the transformation group on the space of real-valued functions on the underlying manifold. Similarly, relative invariants are fixed points of an associated multiplier representation, cf. [2, 22]. The general classification problem for relative invariants is of fundamental importance in a variety of areas, ranging from classical invariant theory, [12, 32], to quantum mechanics, [33, 11], to the theory of special functions, [22, 30], to computer vision, [23], to the study of differential invariants, [25].

Additional impetus for this study comes from the observation that many other types of invariant geometric and algebraic objects can be viewed as certain relative invariants, lending further importance to our results. These include invariant vector fields and differential forms, including invariant volume forms,

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[3], invariant frames and coframes, [25], invariant metrics, [28], invariant differential operators on symmetric spaces, [14, 15, 29], and so on. Indeed, we can identify a multiplier representation with a bundle action of the transformation group on a vector bundle, and the relative invariants then correspond to group-invariant sections of the bundle; the aforementioned cases are all particular types of tensor bundle actions. Unlike the preceding geometric objects, invariant connections, also of great interest to geometers, [17, 28], are associated with a generalization of the underlying multiplier representation, which we name an inhomogeneous multiplier representation. We show how this generalization can be readily treated using the same general framework. An additional class of important applications arises in the theory of prolongation of transformation group actions to jet bundles, which lies at the heart of Lie's theory of symmetry groups of differential equations, cf. [24]. The invariant differential operators which arise in the theory of differential invariants, cf. [25], and the group-invariant arc length and/or volume elements can all be characterized as suitable relative invariants for the prolonged group action. Further applications to symmetry reduction and group-invariant solutions of partial differential equations can be found in [1].

Remarkably, despite this vast wealth of immediate applications, we are not aware of any systematic investigation into the general theory of relative invariants that appears in the literature. Of course, some results, principally dealing with particular types of relative invariants such as invariant metrics, are well known. Nevertheless, the general classification result for relative invariants, along the lines of the Frobenius theorem for absolute invariants, does not appear to be known.

In the paper, we generalize the aforementioned theorem, and completely solve the general classification problem for relative invariants of regular multiplier representations, proving a new algebraic formula that specifies their precise number. A special case of this general theorem, governing the existence of a complete system of relative invariants, appears in the recent book by the second author, [25; Theorem 3.36]. The broad range of applicability of our theorem is illustrated with a detailed treatment of several important applications. In particular, we establish a completely geometric condition for the existence of invariant vector fields. The paper concludes with a discussion of the applications to invariant connections, along with some illustrative examples.

2 Transformation groups and multiplier representations

We begin with a brief review of some basic terminology from the theory of transformation groups; we refer the reader to [25] for the details. We will be considering smooth actions of Lie groups on smooth manifolds. We will state our results for real Lie group actions, although they are equally valid in the complex analytic category. Also, for expedience, we shall assume that the group actions are globally defined, although all of our results can, provided sufficient care is taken, be formulated and proved for local transformation group actions.

Let G be a transformation group acting on a manifold M. Since we will often be applying infinitesimal methods, we will usually require that G be connected. The group action is called *semi-regular* if all its orbits have the same dimension. The action is called *regular* if, in addition, each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is a connected subset thereof. The group action is called *transitive* if the only orbit is M itself; in this case we can identify M with a homogeneous space G/H.

The *isotropy subgroup* of a subset $S \subset M$ is the subgroup $G_S = \{g \in G \mid g \cdot S \subset S\}$ consisting of all group elements g which fix S. A transformation group acts *freely* if the isotropy subgroup of each point is trivial, so $G_x = \{e\}$ for all $x \in M$. The action is *locally free* if G_x is a discrete subgroup of G for all $x \in M$, or, alternatively, that the orbits of the action have the same dimension as G itself (and hence can be locally identified with a neighborhood of the identity in G with G acting via left multiplication).

A transformation group acts *effectively* if different group elements have different actions, so that $g \cdot x = h \cdot x$ for all $x \in M$ if and only if $g = h$. The effectiveness of a group action is measured by its *global isotropy subgroup* $G_0 = \bigcap_{x \in M} G_x = \{g \mid g \cdot x = x \text{ for all } x \in M\}$, which is a normal Lie subgroup of G. Thus, G acts effectively if and only if $G_0 = \{e\}$; slightly more generally, G acts *locally effectively* if G_0 is a discrete subgroup of G. There is a well-defined, effective action of the quotient group $\widehat{G} = G/G_0$ on M, which "coincides" with that of G, in the sense that $g, \tilde{g} \in G$ have the same action on M, so $g \cdot x = \tilde{g} \cdot x$ for all $x \in M$, if and only if they have the same image in $\widehat{G}$, so $\tilde{g} = g \cdot h$ for some $h \in G_0$. We say that a group acts *effectively freely* if and only if $\widehat{G}$ acts freely; this is equivalent to the statement that every local isotropy subgroup equals the global isotropy subgroup: $G_x = G_0$ for all $x \in M$.

Let U be a finite-dimensional vector space. Given a transformation group G acting on a manifold M, there is a naturally induced representation of G on the space $\mathscr{F}(M, U)$ of smooth U-valued functions $F\colon M \to U$, where $g \in G$ maps the function F to the function $\bar{F} = g \cdot F$ defined by $\bar{F}(\bar{x}) = F(g^{-1} \cdot \bar{x})$. Our principal interest lies in an important generalization of this "trivial" representation, known as a multiplier representation, cf. [2, 22, 25].

Definition 2.1 *Given an action of a group G on a space M and a finite-dimensional vector space U, by a* **multiplier representation** *of G we mean a representation $\bar{F} = g \cdot F$ on the space of U-valued functions $\mathscr{F}(M, U)$ of the particular form*

$$\bar{F}(\bar{x}) = \bar{F}(g \cdot x) = \mu(g, x) F(x), \quad g \in G, \; F \in \mathscr{F}(M, U) . \tag{2.1}$$

The **multiplier** *$\mu(g, x)$ is a smooth map $\mu\colon G \times M \to \mathrm{GL}(U)$ to the space of invertible linear transformations on U. The condition that* (2.1) *actually* *defines a representation of the group G requires that the multiplier μ satisfy the* **multiplier equation**

$$\begin{aligned} \mu(g\cdot h,x) &= \mu(g,h\cdot x)\,\mu(h,x), \\ \mu(e,x) &= \mathbb{1}\,, \end{aligned} \qquad \textit{for all } g,h\in G,\ x\in M\,. \tag{2.2}$$

Remark. This definition of multiplier representation is *not* the same as that appearing in the work of Mackey, [21]; the latter are also known as projective representations, [13]. Weyl, [32], requires the multiplier to just depend on the group parameters, $\mu\colon G \to \mathrm{GL}(U)$, in which case (2.2) implies that μ defines a representation of G on U. One justification for this restriction is that, in certain cases such as the linear action of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$, these are, up to gauge equivalence, [25], the only non-trivial multiplier representations; however, in most cases, including the action of $\mathrm{SL}(n)$ on $\mathbb{R}^n$, this restriction is too severe, eliminating many interesting examples.

There is an alternative, fully geometrical approach to the theory of multiplier representations, that allows us to apply standard results from the theory of transformation groups. Let $\pi\colon E \to M$ be a vector bundle over a base manifold M of rank n, which means that E has n-dimensional fibers $E|_x = \pi^{-1}\{x\} \simeq U = \mathbb{R}^n$. Let G be a transformation group acting on E by vector bundle automorphisms; in local coordinates, the group transformations on E have the form

$$g\cdot(x,u) = (g\cdot x, \mu(g,x)u), \quad g\in G,\ x\in M,\ u\in U\,, \tag{2.3}$$

acting linearly on the fiber coordinates u. The condition that (2.3) defines a group action extending the action of G on the base M is equivalent to the condition that μ satisfy the multiplier equation (2.2).

Example 2.2 If G is a transformation group acting on a manifold M, then there is an induced action on the tangent bundle TM, and thus defines a multiplier representation on the space of vector fields $\mathscr{X}(M)$. In local coordinates, we can identify a vector field $\mathbf{v} = \sum_{i=1}^m \eta^i(x)\partial_{x^i}$ with the vector-valued function $\eta(x) = (\eta^1(x),\ldots,\eta^m(x))$. The differential $dg\colon TM \to TM$ of a group transformation $g\cdot x = \chi(g,x)$ then determines the *Jacobian multiplier representation*

$$\bar{\eta}(\bar{x}) = \mu_J(g,x)\eta(x), \quad \text{where } \mu_J(g,x) = \left(\frac{\partial \chi^i}{\partial x^j}\right)\,. \tag{2.4}$$

The multiplier equation (2.2) in this case reduces to the usual chain rule formula for the Jacobian of the composition of two group transformations.

Example 2.3 If G acts on the bundle $E \to M$ according to the multiplier representation with multiplier $\mu(g,x)$, then there is an induced action on the dual bundle E^* whose multiplier is the inverse transpose (or dual) of the original multiplier: $\mu^*(g,x) = \mu(g,x)^{-T}$. In the invariant theory literature, this is known as the *contragredient* multiplier representation. For example, the dual to the Jacobian multiplier representation corresponding to the action of the group on vector fields is the multiplier representation on the space $\Omega^1(M)$ of one-forms corresponding to the pull-back action of G on the cotangent bundle T^*M.

3 Relative invariants

An *invariant* or *absolute invariant* of a transformation group G acting on a manifold M is, by definition, a real-valued function $I\colon M \to \mathbb{R}$ which is unaffected by the group transformations: $I(g \cdot x) = I(x)$ for all $g \in G$ and all x in the domain of definition of I. According to the general Frobenius Theorem, [25], if G acts regularly on M with s-dimensional orbits, then, locally, there exist $m-s$ functionally independent absolute invariants, $I_1, \ldots, I_{m-s}$, with the property that any other invariant can be written as a function of these fundamental invariants: $I(x) = H(I_1(x), \ldots, I_{m-s}(x))$. The analogue of an invariant for a general multiplier representation is known as a relative invariant.

Definition 3.1 *Let G be a transformation group acting on M and let* μ: $M \times G \to \mathrm{GL}(U)$ *be a multiplier. A* **relative invariant of weight** μ *is a function* R: $M \to U$ *which satisfies*

$$R(g \cdot x) = \mu(g,x)R(x) \quad \text{for all } x \in M,\ g \in G\,, \tag{3.1}$$

where defined.

In terms of our vector bundle interpretation of multiplier representations, a relative invariant can be identified with a G-invariant section of the vector bundle E.

Example 3.2 A relative invariant for the Jacobian multiplier representation (2.4) is the same as a G-invariant vector field: $dg(\mathbf{v}) = \mathbf{v}$ for all $g \in G$. Note that the invariant vector fields are *not* usually the infinitesimal generators of the group action, although this is true for abelian transformation groups, cf. [3; Theorem IV.3.4]. For example, in the case that G acts on itself by left multiplication, $h \mapsto g \cdot h$, the G-invariant vector fields are the elements of the left Lie algebra $\mathfrak{g}_L$ of G, whereas the infinitesimal generators of this action are the *right*-invariant vector fields, i.e., the elements of the right Lie algebra $\mathfrak{g}_R$.

Similarly, a G-invariant one-form is a relative invariant for the multiplier representation determined by the dual action of G on the cotangent bundle of M.

Example 3.3 Generalizing the pull-back multiplier representation on the space of one-forms, one can consider the induced action of a transformation group on the exterior powers $\bigwedge^k T^*M$, corresponding to the action of G on the space $\Omega^k(M)$ of differential k-forms on M. Relative invariants for the associated multiplier representation are the invariant differential forms on M. Invariant differential forms serve to define the invariant de Rham cohomology groups, which, in the case of a compact group action, can be used to determine the ordinary de Rham cohomology groups, cf. [6]. For example, the case $k = m = \dim M$, gives the scalar multiplier representation on the space $\Omega^m(M)$ of volume forms, and an invariant volume form is then a relative invariant thereof. The case of right-, left-, and bi-invariant volume forms on Lie groups is a particularly important special case, leading to the left and right Haar measures, as well as the classical result that a Lie group G admits a bi-invariant volume form if and only if it is *unimodular*; see [9; Sect. 2.7] for an extensive survey.

Example 3.4 A G-invariant metric on a manifold M is determined by a symmetric rank two tensor $ds^2 = \sum h_{ij}(x)\, dx^i\, dx^j$ satisfying $g^*(ds^2) = ds^2$ for all group transformations $g \in G$, so that G acts via isometries. The associated multiplier representation can be identified with the second symmetric tensor power of the dual Jacobian multiplier, governed by the action of G on the symmetric tensor bundle $\bigodot^2 T^*M$. A particularly important case is the problem of existence of bi-invariant metrics on Lie groups; here the manifold M is a Lie group G, while the transformation group is the Cartesian product $G \times G$, acting on G by both left and right multiplication. See [28; Theorem 5.3] for a solution to this problem in the real Riemannian case.

Note that if R is a relative invariant of weight μ and I is any absolute invariant, then $I \cdot R$ is also a relative invariant of weight μ. Similarly, the sum $R_1 + R_2$ of relative invariants of the *same* weight μ is also a relative invariant of weight μ. Thus the space $\mathscr{R}_\mu$ of relative invariants of a given weight μ forms a module over the ring $\mathscr{I}$ of scalar absolute invariants. We therefore define the *dimension* of $\mathscr{R}_\mu$ to be its dimension as an $\mathscr{I}$-module. In other words, if $k = \dim \mathscr{R}_\mu$, then there exist k independent relative invariants $R_1, \ldots, R_k \in \mathscr{R}_\mu$ such that every relative invariant of weight μ has the form $J_1 R_1 + \ldots + J_k R_k \in \mathscr{R}_\mu$ for suitable absolute invariants $J_1, \ldots, J_k \in \mathscr{I}$. As always, our interest is local, so this equation should be interpreted as holding on sufficiently small open subsets of M.

Definition 3.5 *A multiplier representation on an n-dimensional vector space* $U \simeq \mathbb{R}^n$ *is said to admit a* **complete system** *of relative invariants if the number of independent relative invariants equals the dimension of U, so that* $\dim \mathscr{R}_\mu = n = \dim U$.

In the bundle-theoretic interpretation, if $E \to M$ is a rank n vector bundle, then the associated multiplier representation admits a complete system of relative invariants if and only if there exist n pointwise linearly independent invariant (local) sections R_i: $M \to E$, $i = 1, \ldots, n$, so that, at each point $R_1(x), \ldots, R_n(x)$ form a basis for the fiber $E|_x$. For example, the Jacobian multiplier representation corresponding to the tangent bundle TM admits a complete system of relative invariants if and only if there is a (local) G-invariant *frame* on the manifold M, i.e., a system of $m = \dim M$ pointwise linearly independent G-invariant vector fields. A G-invariant frame exists if and only if G acts effectively freely; see [25] and Theorem 6.1 below.

If R is a relative invariant, then its particular values are, of course, not fixed under the group action. However, the zero value is maintained, and hence the system of equations $R(x) = 0$ given by the vanishing of a relative invariant is G-invariant, meaning that G takes solutions to solutions. Eisenhart, [8], actually defines a relative invariant to be a function whose zero set is invariant under the group. This definition is, in fact, equivalent to ours provided the function has maximal rank. (See also [4] for a cohomological interpretation in the context of symmetries of differential equations.)

Proposition 3.6 *Let G be a transformation group acting on the m-dimensional manifold M. Let F: $M \to \mathbb{R}^n$, with $n \leqq m$, be a function whose differential dF has maximal rank n. If the solution set $S = \{x \,|\, F(x) = 0\}$ is non-empty, and hence a submanifold of dimension $m - n$, then S is G-invariant if and only if F is a relative invariant under some multiplier representation of G on M.*

Proof. Since F is assumed to be of maximal rank, we can introduce local coordinates $x = (y,z) = (y^1,\dots,y^n,z^1,\dots,z^{m-n})$ such that $F^i(y,z) = y^i$, hence $S = \{y = 0\}$. In these coordinates, group transformations take the form $g \cdot (y,z) = (\eta(g,y,z), \zeta(g,y,z))$. Since S is G-invariant, $\eta(g,y,z) = 0$ whenever $y = 0$, and hence Proposition 2.10 of [24] (applied to the individual components of $\eta(g,y,z)$) implies that, locally, we can write $\eta(g,y,z) = \mu(g,y,z) \cdot y$ for some function μ: $G \times M \to \mathrm{GL}(n,\mathbb{R})$. The group law $g \cdot (h \cdot x) = (g \cdot h) \cdot x$ immediately implies that μ satisfies the multiplier equation (2.2) and hence defines a multiplier representation of M. Moreover, F is a relative invariant for the multiplier μ, completing the proof. □

4 Infinitesimal generators

In accordance with Lie's general approach to invariant theory, [19; Chapter 23], the study of relative invariants of (connected) Lie group actions is most effectively handled by an infinitesimal approach. If G is a transformation group acting on a manifold M, then its infinitesimal generators form a finite-dimensional Lie algebra of vector fields $\widehat{\mathfrak{g}} \subset \mathscr{X}(M)$ on M. There is a natural Lie algebra[1] epimorphism φ: $\mathfrak{g} \to \widehat{\mathfrak{g}}$ from the Lie algebra of *right-invariant* vector fields on G to the space of infinitesimal generators, mapping the generator of the one-parameter subgroup $\exp(t\mathbf{v})$ to the vector field $\widehat{\mathbf{v}} = \varphi(\mathbf{v})$ whose flow coincides with the action $x \mapsto \exp(t\mathbf{v})x$. In general, we shall use a hat, $\widehat{\mathbf{v}} = \varphi(\mathbf{v})$, to denote the infinitesimal generator corresponding to a given Lie algebra element $\mathbf{v} \in \mathfrak{g}$. If G acts locally effectively, then φ is an isomorphism (and we could unambiguously drop the hats); more generally, the space $\widehat{\mathfrak{g}}$ is isomorphic to the Lie algebra of the effectively acting quotient group $\widehat{G} = G/G_0$. The subspace $\widehat{\mathfrak{g}}|_x \subset TM|_x$ spans the tangent space to the orbit of G passing through the point x.

Turning to the study of relative invariants, the space of infinitesimal generators of the action of G on the vector bundle E consists of a Lie algebra of vector fields on E which is the image of the "extended" Lie algebra epimorphism ψ: $\mathfrak{g} \to \widetilde{\mathfrak{g}} \subset \mathscr{X}(E)$ from the Lie algebra $\mathfrak{g}$ of right-invariant vector fields on G to the space $\mathscr{X}(E)$ of vector fields on E. Linearity of the vector bundle automorphisms defined by the group transformations implies that the

[1] From now on we use $\mathfrak{g} = \mathfrak{g}_R$ to denote the *right* Lie algebra of the group G

local coordinate expressions for the infinitesimal generators are linear in the fiber coordinate u:

$$\widetilde{\mathbf{v}} = \widehat{\mathbf{v}} + [H_{\mathbf{v}}(x)u] \cdot \partial_u = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} + \sum_{\alpha,\beta=1}^{n} h^{\alpha}_{\beta}(x) u^{\beta} \frac{\partial}{\partial u^{\alpha}} , \qquad (4.1)$$

where

$$\widehat{\mathbf{v}} = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} \in \widehat{\mathfrak{g}} \qquad (4.2)$$

is the infinitesimal generator for the action of G on M corresponding to the Lie algebra element $\mathbf{v} \in \mathfrak{g}$.

Definition 4.1 *The linear map* $\sigma\colon \mathfrak{g} \to \mathscr{F}(M, \mathfrak{gl}(U))$ *which takes a Lie algebra element* $\mathbf{v} \in \mathfrak{g}$ *to the associated matrix-valued function* $\sigma(\mathbf{v}) = H_{\mathbf{v}}$ *in* (4.1) *is called the* **infinitesimal multiplier** *for the given multiplier representation.*

It is convenient to identify the infinitesimal generator (4.1) of the multiplier action with an $n \times n$ matrix-valued first order differential operator

$$\mathscr{D}_{\mathbf{v}} = \widehat{\mathbf{v}} - H_{\mathbf{v}} = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i} - H_{\mathbf{v}}(x) , \qquad (4.3)$$

i.e., $\mathscr{D}_{\mathbf{v}}$ is a differential operator acting on sections of E. Here $\widehat{\mathbf{v}} \simeq \widehat{\mathbf{v}}\,\mathbb{1}$ is regarded as a scalar differential operator which acts component-wise on vector-valued functions. (The reason for the change in sign in (4.3) will become apparent once we discuss the infinitesimal conditions for relative invariants – see Theorem 4.3 below.) Note that even if G does not act effectively on M, it may still act effectively on E, in which case Lie algebra elements $\mathbf{v} \in \mathfrak{g}_0$ lying in the global isotropy subalgebra of M are mapped to the zero vector field on M, so $\widehat{\mathbf{v}} \equiv 0$, while the associated differential operator $\mathscr{D}_{\mathbf{v}} = -H_{\mathbf{v}}$ reduces to a pure *multiplication operator*.

The map taking a Lie algebra element $\mathbf{v} \in \mathfrak{g}$ to the corresponding matrix differential operator $\mathscr{D}_{\mathbf{v}}$ is readily seen to be a Lie algebra homomorphism, meaning that the Lie bracket $\mathbf{u} = [\mathbf{v}, \mathbf{w}]$ between two generators $\mathbf{v}, \mathbf{w} \in \mathfrak{g}$ is mapped to the differential operator commutator

$$\mathscr{D}_{\mathbf{u}} = [\mathscr{D}_{\mathbf{v}}, \mathscr{D}_{\mathbf{w}}] = \mathscr{D}_{\mathbf{v}} \cdot \mathscr{D}_{\mathbf{w}} - \mathscr{D}_{\mathbf{w}} \cdot \mathscr{D}_{\mathbf{v}}, \quad \mathbf{u} = [\mathbf{v}, \mathbf{w}] , \qquad (4.4)$$

between the corresponding differential operators. Evaluating (4.4) using the explicit formula (4.3) leads to a direct characterization of infinitesimal multipliers.[2]

Theorem 4.2 *A linear function* $\sigma\colon \mathfrak{g} \to \mathscr{F}(M, \mathfrak{gl}(U))$ *is an infinitesimal multiplier if and only if it satisfies*

$$\sigma([\mathbf{v}, \mathbf{w}]) = \widehat{\mathbf{v}}(\sigma(\mathbf{w})) - \widehat{\mathbf{w}}(\sigma(\mathbf{v})) - [\sigma(\mathbf{v}), \sigma(\mathbf{w})] \quad \textit{for all } \mathbf{v}, \mathbf{w} \in \mathfrak{g} . \qquad (4.5)$$

Theorem 4.3 *Let* G *be a connected group of transformations acting on* M, *and let* $\mu\colon G \times M \to \mathrm{GL}(U)$ *be a multiplier. A function* $R\colon M \to U$ *is a*

[2] See [25] for a discussion of the cohomological interpretation of infinitesimal multipliers

relative invariant of weight μ *if and only if it satisfies the homogeneous linear system of first order partial differential equations*

$$\mathscr{D}_{\mathbf{v}}(R) = \widehat{\mathbf{v}}(R) - H_{\mathbf{v}} R = 0 \quad \textit{for all } \mathbf{v} \in \mathfrak{g}\,. \tag{4.6}$$

Example 4.4 Consider the infinitesimal 2×2 matrix multiplier with infinitesimal generators

$$\partial_x, \quad x\partial_x - \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

projecting to the usual action of the affine group A(1) on $\mathbb{R}$. An easy computation shows that $R(x) = (f(x), h(x))^T$ is a relative invariant if and only if f and h are constant, and, furthermore, $\alpha f + \gamma h = 0 = \beta f + \delta h$. Therefore, this multiplier representation admits a nonzero relative invariant if and only if $\alpha\delta - \beta\gamma = 0$. This indicates that the determination of the number of relative invariants depends on more subtle data than a crude orbit dimension count as in the Frobenius analysis of absolute invariants.

5 An existence theorem for relative invariants

We now turn to the main result of this paper, which is a general existence theorem and dimension count for the number of relative invariants of an arbitrary multiplier representation. Applications to some of the examples discussed above will appear in subsequent sections. We begin with an analysis of the relevant geometrical data for the multiplier representation.

Let G be a transformation group acting by vector bundle automorphisms on a vector bundle $\pi\colon E \to M$. Let $\mathscr{W} \subset \mathscr{X}(E)$ denote the involutive differential system (or distribution, [31]) spanned by the infinitesimal generators $\widetilde{\mathbf{v}}$ of the bundle action of G on E, as in (4.1), and let $\mathscr{V} \subset \mathscr{X}(M)$ denote the involutive differential system spanned by the infinitesimal generators $\widehat{\mathbf{v}}$ of the projected action of G on M. Thus $\mathscr{W}|_z$ is the tangent space to the orbit of G through $z \in E$, while $\mathscr{V}|_x$ is the tangent space to the G orbit through $x \in M$. Let $z \in E$ have projection $x = \pi(z) \in M$. The differential of the projection map of E restricts to a linear epimorphism $d\pi\colon \mathscr{W}|_z \to \mathscr{V}|_x$, which maps $\widetilde{\mathbf{v}} = \widehat{\mathbf{v}} + [H_{\mathbf{v}}(x)u]\partial_u \in \mathscr{W}|_z$ onto the corresponding tangent vector $\widehat{\mathbf{v}} \in \mathscr{V}|_x$. Let $\mathscr{L}|_x = \ker[d\pi] \cap \mathscr{W}|_z$ denote the kernel[3] of the projection $d\pi$ at a point $z \in E$. As in the identification of the infinitesimal generator of the bundle action (4.1) with matrix-valued differential operators (4.3), at each point $x \in M$, an element of $\mathscr{L}|_x$, which is a vector field of the form $[L(x)u]\partial_u$, can be identified with the linear operator $L(x)$, which acts on the fiber $E|_x \simeq U$. In this manner, we identify $\mathscr{L}|_x$ with a linear subspace of the space $\mathfrak{gl}(E|_x) \simeq \mathfrak{gl}(U)$ of linear maps on the fiber. The common kernel of the linear operators in $\mathscr{L}|_x$ will be denoted by $\mathscr{K}|_x = \ker \mathscr{L}|_x \subset E|_x$; in other words, $u \in E|_x$ belongs to $\mathscr{K}|_x$ if and only if $L(x)u = 0$ for all $L(x) \in \mathscr{L}|_x$. As we shall see, the dimension $k = \dim K|_x$ of

[3] Linearity of the infinitesimal generators of G in the fiber coordinates u implies that the kernel only depends on the projected point $x = \pi(z)$

the common kernel is the crucial quantity that determines how many relative invariants there are for the multiplier representation corresponding to the bundle action of G on E.

Definition 5.1 *The* **kernel rank** ***k*** *of a vector bundle action of a transformation group G on a vector bundle $\pi\colon E \to M$ is the dimension $k = \dim \mathscr{K}|_x$ of the common kernel bundle, as defined above. The multiplier representation is said to be* **regular** *if G acts regularly on the base M, and the kernel rank k is constant.*

In local coordinates, this construction takes the following form: Let G be an r-dimensional Lie group acting (semi-)regularly on M, and let s denote the dimension of its orbits in M. Thus, near any point $x_0 \in M$ we can choose s infinitesimal generators $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \mathfrak{g}$ such that the corresponding vector fields

$$\widehat{\mathbf{v}}_\alpha = \sum_{i=1}^m \xi_\alpha^i(x) \frac{\partial}{\partial x^i}, \quad \alpha = 1, \ldots, s\ , \tag{5.1}$$

form a basis for the subspace $\mathscr{V}|_x \subset TM|_x$ at each point x in a neighborhood of x_0. We complete $\mathbf{v}_1, \ldots, \mathbf{v}_s$ to a basis of the Lie algebra $\mathfrak{g}$, thereby including $r - s$ additional generators $\mathbf{v}_{s+1}, \ldots, \mathbf{v}_r \in \mathfrak{g}$, which map to vector fields

$$\widehat{\mathbf{v}}_\lambda = \sum_{i=1}^m \xi_\lambda^i(x) \frac{\partial}{\partial x^i}, \quad \lambda = s+1, \ldots, r\ . \tag{5.2}$$

If $r = s$, then G acts locally freely, and there are no additional vector fields (5.2). In this case, our results are covered by a known theorem, [25; Theorem 3.36], and so we shall concentrate on the case $s < r$, although the method of proof includes the free case too.

From now on, for convenience, we shall employ the Einstein summation convention on repeated indices. Latin indices i, j, k will run from 1 to m. Greek indices α, β, γ, δ run from 1 to s, while λ, μ, ν, run from $s+1$ to r. Finally ρ, σ, τ run from 1 to r.

Since the first s vector fields (5.1) form a pointwise basis for the space $\mathscr{V}|_x$, the second set of infinitesimal generators can be written as linear combinations (with variable coefficients) of the first, so that, using our summation and index conventions,

$$\widehat{\mathbf{v}}_\lambda = \eta_\lambda^\alpha \widehat{\mathbf{v}}_\alpha, \quad \lambda = s+1, \ldots, r\ , \tag{5.3}$$

where the coefficients $\eta_\lambda^\alpha(x)$ are smooth scalar-valued functions defined on the coordinate chart. On the bundle E, the infinitesimal generators corresponding to the first set of vector fields (5.1) are matrix-valued differential operators of the form

$$\mathscr{D}_\alpha = \widehat{\mathbf{v}}_\alpha - H_\alpha\ , \tag{5.4}$$

where the $H_\alpha(x)$ are smooth $n \times n$ matrix-valued functions of x. Similarly, the second set of infinitesimal generators correspond to differential operators of the form

$$\mathscr{D}_\lambda = \widehat{\mathbf{v}}_\lambda - H_\lambda = \eta_\lambda^\alpha \mathscr{D}_\alpha - L_\lambda\ , \tag{5.5}$$

where, in view of (5.3), (5.4),

$$L_\lambda = H_\lambda - \eta_\lambda^\alpha H_\alpha \,. \tag{5.6}$$

The space $\mathscr{L}|_x$ is the subspace of $\mathfrak{gl}(U)$ spanned by the $r-s$ matrices $L_{s+1}(x), \ldots, L_r(x)$. The associated kernel space $\mathscr{K}|_x$ is just their common kernel:

$$\mathscr{K}|_x = \bigcap_{\lambda=s+1}^{r} \ker L_\lambda(x) = \{u \in U \mid L_{s+1}(x)u = \cdots = L_r(x)u = 0\} \,. \tag{5.7}$$

The kernel rank k of the multiplier representation is the dimension of the subspace $\mathscr{K}|_x$.

A function $R(x)$ forms a relative invariant if and only if it is annihilated by the differential operators (5.4), (5.5), so that

$$\widehat{\mathbf{v}}_\rho(R) = H_\rho R, \quad \rho = 1,\ldots,r \,. \tag{5.8}$$

In view of (5.5), this means that R must satisfy the system of partial differential equations

$$\widehat{\mathbf{v}}_\alpha(R) = H_\alpha R, \quad \alpha = 1,\ldots,s \,, \tag{5.9}$$

along with a system of algebraic equations

$$L_\lambda R = 0, \quad \lambda = s+1,\ldots,r \,. \tag{5.10}$$

In other words, for each $x \in M$, we have $R(x) \in \mathscr{K}|_x$, which explains the significance of the kernel bundle $\mathscr{K}$. At a point x, the dimension of the solution space to the homogeneous linear algebraic system (5.10) is the kernel rank k of the multiplier representation at that point. Our main theorem states that, under appropriate regularity hypotheses, the space of relative invariants has the same dimension k as the pointwise solution space to (5.10).

Theorem 5.2 *Let G be a connected transformation group acting regularly by vector bundle automorphisms on a vector bundle $\pi: E \to M$, with constant kernel rank k. Then the space of relative invariants of the associated multiplier representation has dimension exactly k. Equivalently, there exist precisely k pointwise linearly independent local G-invariant sections of E.*

Proof. As above, let $\mathscr{V}$ denote the involutive differential system spanned by the infinitesimal generators $\widehat{\mathbf{v}}$ of G on M, and let $\mathscr{W}$ denote the corresponding involutive differential system spanned by the infinitesimal generators $\widetilde{\mathbf{v}}$ on E. Frobenius' Theorem implies that we can introduce flat local coordinates $(y,z) = (y^1,\ldots,y^s,z^1,\ldots,z^{m-s})$ on M such that the $z^\lambda = I_\lambda(x)$ provide the local absolute invariants, and the orbits (integral submanifolds) intersect the coordinate chart in the slices $\mathscr{O}_a = \{z = a\}$. Without loss of generality, we may assume that the coordinate chart forms a box, so that $a^\alpha < y^\alpha < b^\alpha$, $c^\lambda < z^\lambda < d^\lambda$. Thus, in the (y,z) coordinates, the differential system $\mathscr{V}$ is spanned by the basis tangent vectors $\partial_{y^1},\ldots,\partial_{y^s}$, which are therefore certain linear combinations

$$\frac{\partial}{\partial y^\alpha} = B_\alpha^\beta \, \widehat{\mathbf{v}}_\beta \tag{5.11}$$

of the generators $\widehat{\mathbf{v}}_\alpha \in \widehat{\mathfrak{g}}$; the coefficient matrix $(B^\beta_\alpha(x))$ in (5.11) is nonsingular. Let

$$\widehat{H}_\alpha = B^\beta_\alpha H_\beta \quad \text{so that} \quad \frac{\partial}{\partial y^\alpha} - \widehat{H}_\alpha = B^\beta_\alpha \mathscr{D}_\beta \,. \tag{5.12}$$

In view of equations (5.11), (5.12), the system of differential equations (5.9) for a relative invariant R is linearly equivalent to a collection of s linear systems of ordinary differential equations

$$\frac{\partial R}{\partial y^\alpha} = \widehat{H}_\alpha R, \quad \alpha = 1, \ldots, s \,, \tag{5.13}$$

in the individual orbit coordinates y^α. Involutivity of the extended differential system $\mathscr{W}$ implies that (5.13) form an involutive system of partial differential equations for R, i.e., that the integrability conditions

$$\frac{\partial}{\partial y^\alpha}(\widehat{H}_\beta R) = \frac{\partial}{\partial y^\beta}(\widehat{H}_\alpha R), \quad \alpha, \beta = 1, \ldots, s \,, \tag{5.14}$$

are identically satisfied for any solution to (5.13). Therefore, given any point $x_0 = (y_0, z_0)$ in our coordinate system, and initial conditions $R(y_0, z_0) = R_0$, there is a unique solution $R(y, z_0)$ to the system (5.13) defined on the integral submanifold $\mathscr{O}_{z_0}$ of $\mathscr{V}$ passing through the initial point x_0. More generally, if we specify initial conditions $R(y_0, z) = S(z)$ on any transversal submanifold $\mathscr{T}_{y_0} = \{(y_0, z)\}$ to the foliation determined by the integral submanifolds of $\mathscr{W}$, there is a unique solution $R(x) = R(y, z)$ for all $x = (y, z)$ in the coordinate chart. This solution will be a (local) relative invariant provided it lies in the kernel bundle, i.e., $R(x) \in \mathscr{K}|_x$ for all x. Clearly, then, we need to choose the initial conditions in the kernel bundle, so that $S(z) \in \mathscr{K}|_{(y_0,z)}$ for each $(y_0, z) \in \mathscr{T}_{y_0}$. We claim that this automatically implies that $R(x) \in \mathscr{K}|_x$, i.e., $R(x)$ satisfies the linear system (5.10) provided $R_0(z) = R(y_0, z)$ does. Note that the claim will automatically imply Theorem 5.2. Indeed, fixing y_0, we can (locally) choose k sections $S_1(z), \ldots, S_k(z)$ which form a basis for the kernel space $\mathscr{K}|_{(y_0,z)}$ at each point in the transversal $\mathscr{T}_{y_0}$. Let $R_i(x) = R_i(y, z)$ be the corresponding solution to the system (5.13) having initial conditions $R_i(y_0, z) = S_i(z)$; by the claim, $R_i(x) \in \mathscr{K}|_x$ for each x, and hence R_i is a relative invariant. Moreover, any other relative invariant $R(x)$ must restrict to a linear combination $R(y_0, z) = J_1(z)S_1(z) + \cdots + J_k(z)S_k(z)$ of the basis functions on the transversal $\mathscr{T}_{y_0}$, the coefficients $J_i(z)$ being scalar functions. By the uniqueness theorem for solutions to the system of ordinary differential equations (5.13), we necessarily have $R(y, z) = J_1(z)R_1(y, z) + \cdots + J_k(z)R_k(y, z)$. The coefficient functions $J_i(z)$ are absolute invariants of G, which proves our result.

Thus the proof of Theorem 5.2 reduces to the following lemma. In fact, the transversal coordinates z only appear as parameters at this point, and can be effectively ignored. (The smooth dependence of the relative invariant on z follows from the smooth dependence on parameters of solutions to ordinary differential equations.)

Lemma 5.3 *Suppose that $R(y,z)$ is a solution to the differential equations* (5.13) *with initial conditions $R(y_0,z)=S(z)$. If $S(z)\in\mathscr{K}|_{(y_0,z)}$, then $R(y,z)\in\mathscr{K}|_{(y,z)}$ for y in a neighborhood of y_0.*

Proof. We need to show that if $S(z)$ satisfies $L_\lambda(y_0,z)S(z)=0$, for $\lambda = r+1,\ldots,s$, then, for all y near y_0, we have $L_\lambda(y,z)R(y,z)=0$, $\lambda=r+1,\ldots,s$. This will immediately follow from the uniqueness theorem for ordinary differential equations once we show that the derivatives of $L_\lambda R$ with respect to the infinitesimal generators $\widehat{\mathbf{v}}_\alpha$ of $\mathscr{V}$ vanish whenever $L_{s+1}R=\cdots=L_rR=0$. Equation (5.11) will then show that all y derivatives of $L_\lambda R$ vanish. One could work directly with the y derivatives, but it is notationally simpler to use the generators $\widehat{\mathbf{v}}_\alpha$. Thus, we must compute

$$\widehat{\mathbf{v}}_\alpha(L_\lambda R)=\widehat{\mathbf{v}}_\alpha(L_\lambda)R+L_\lambda\widehat{\mathbf{v}}_\alpha(R)=(\widehat{\mathbf{v}}_\alpha(L_\lambda)+L_\lambda H_\alpha)R\,, \tag{5.15}$$

the last equality following from (5.9). The desired result will therefore follow once we establish the following fundamental identity:

$$\widehat{\mathbf{v}}_\alpha(L_\lambda)+L_\lambda H_\alpha=H_\alpha L_\lambda+(c^\mu_{\alpha\lambda}-\eta^\beta_\lambda c^\mu_{\alpha\beta})L_\mu\,. \tag{5.16}$$

Here $c^\tau_{\rho\sigma}$ are the structure constants of the Lie algebra $\mathfrak{g}$ relative to the basis $\mathbf{v}_1,\ldots,\mathbf{v}_r$, and the functions η^β_λ are given in (5.3).

The proof of the identity (5.16) depends on a detailed analysis of the commutation relations between infinitesimal generators of both the action of G on M and its bundle action on E. The fact that the infinitesimal generators form a Lie algebra implies that

$$[\widehat{\mathbf{v}}_\rho,\widehat{\mathbf{v}}_\sigma]=c^\tau_{\rho\sigma}\widehat{\mathbf{v}}_\tau=c^\alpha_{\rho\sigma}\widehat{\mathbf{v}}_\alpha+c^\lambda_{\rho\sigma}\widehat{\mathbf{v}}_\lambda=[c^\alpha_{\rho\sigma}+\eta^\alpha_\lambda c^\lambda_{\rho\sigma}]\widehat{\mathbf{v}}_\alpha\,. \tag{5.17}$$

We let

$$\zeta^\alpha_{\rho\sigma}(x)=c^\alpha_{\rho\sigma}+\eta^\alpha_\lambda(x)c^\lambda_{\rho\sigma}\,, \tag{5.18}$$

so that (5.17) takes the abbreviated form

$$[\widehat{\mathbf{v}}_\rho,\widehat{\mathbf{v}}_\sigma]=\zeta^\alpha_{\rho\sigma}\widehat{\mathbf{v}}_\alpha\,. \tag{5.19}$$

Similarly, the differential operators generating the multiplier representation must have the same commutation relations, and hence, by (5.3), (5.5), (5.18),

$$[\mathscr{D}_\rho,\mathscr{D}_\sigma]=c^\tau_{\rho\sigma}\mathscr{D}_\tau=c^\alpha_{\rho\sigma}\mathscr{D}_\alpha+c^\lambda_{\rho\sigma}\mathscr{D}_\lambda=\zeta^\alpha_{\rho\sigma}\mathscr{D}_\alpha-c^\lambda_{\rho\sigma}L_\lambda\,, \tag{5.20}$$

for the *same* structure constants $c^\tau_{\rho\sigma}$.

Remark. Our proof of the fundamental theorem does *not* require that the $c^\tau_{\rho\sigma}$ in (5.17) be constant. However, it is not clear what meaning (if any) one can attach to such an extension to more general involutive differential systems.

We now expand the vector field commutator (5.19) for a subrange of indices, as governed by our index conventions:

$$[\widehat{\mathbf{v}}_\alpha,\widehat{\mathbf{v}}_\lambda]=\widehat{\mathbf{v}}_\alpha(\eta^\beta_\lambda)\widehat{\mathbf{v}}_\beta+\eta^\beta_\lambda[\widehat{\mathbf{v}}_\alpha,\widehat{\mathbf{v}}_\beta]=(\widehat{\mathbf{v}}_\alpha(\eta^\beta_\lambda)+\eta^\gamma_\lambda\zeta^\beta_{\alpha\gamma})\widehat{\mathbf{v}}_\beta\,, \tag{5.21}$$

where we used (5.3), (5.19). Since $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_s$ are pointwise linearly independent, (5.19) and (5.21) imply that

$$\widehat{\mathbf{v}}_\alpha(\eta_\lambda^\beta) + \eta_\lambda^\gamma \zeta_{\alpha\gamma}^\beta = \zeta_{\alpha\lambda}^\beta \,. \tag{5.22}$$

Next, we expand the differential operator commutator (5.20) using (5.17), (5.19):

$$\begin{aligned}\zeta_{\rho\sigma}^\beta \mathscr{D}_\beta - c_{\rho\sigma}^\mu L_\mu = c_{\rho\sigma}^\tau \mathscr{D}_\tau &= [\mathscr{D}_\rho, \mathscr{D}_\sigma] \\ &= [\widehat{\mathbf{v}}_\rho - H_\rho, \widehat{\mathbf{v}}_\sigma - H_\sigma] \\ &= [\widehat{\mathbf{v}}_\rho, \widehat{\mathbf{v}}_\sigma] - \widehat{\mathbf{v}}_\rho(H_\sigma) + \widehat{\mathbf{v}}_\sigma(H_\rho) + H_\rho H_\sigma - H_\sigma H_\rho \\ &= c_{\rho\sigma}^\tau \widehat{\mathbf{v}}_\tau - \widehat{\mathbf{v}}_\rho(H_\sigma) + \widehat{\mathbf{v}}_\sigma(H_\rho) + H_\rho H_\sigma - H_\sigma H_\rho \\ &= \zeta_{\rho\sigma}^\beta \widehat{\mathbf{v}}_\beta - \widehat{\mathbf{v}}_\rho(H_\sigma) + \widehat{\mathbf{v}}_\sigma(H_\rho) + H_\rho H_\sigma - H_\sigma H_\rho \,.\end{aligned}$$

Therefore, by (5.4), we find that

$$\widehat{\mathbf{v}}_\rho(H_\sigma) - \widehat{\mathbf{v}}_\sigma(H_\rho) - H_\rho H_\sigma + H_\sigma H_\rho = c_{\rho\sigma}^\tau H_\tau = \zeta_{\rho\sigma}^\beta H_\beta + c_{\rho\sigma}^\mu L_\mu \,. \tag{5.23}$$

We now analyze (5.23) for the subrange of indices with $\rho = \alpha$, $\sigma = \lambda$. Substituting the formula $H_\lambda = L_\lambda + \eta_\lambda^\beta H_\beta$ from (5.6), we expand the left hand side of (5.23) using (5.3), (5.18):

$$\begin{aligned}&\widehat{\mathbf{v}}_\alpha(L_\lambda) + \widehat{\mathbf{v}}_\alpha(\eta_\lambda^\beta) H_\beta + \eta_\lambda^\beta \widehat{\mathbf{v}}_\alpha(H_\beta) - \eta_\lambda^\beta \widehat{\mathbf{v}}_\beta(H_\alpha) \\ &\quad - H_\alpha L_\lambda - \eta_\lambda^\beta H_\alpha H_\beta + L_\lambda H_\alpha + \eta_\lambda^\beta H_\beta H_\alpha = \zeta_{\alpha\lambda}^\beta H_\beta + c_{\alpha\lambda}^\mu L_\mu \,.\end{aligned} \tag{5.24}$$

On the other hand, if we substitute (5.22) and (5.23), with $\rho = \alpha$, $\sigma = \beta$, we reduce (5.24) to our desired identity (5.16). This proves (5.16) and thereby completes the proof of the fundamental Theorem. □

Theorem 5.2 reduces the determination of the number of relative invariants of a (regular) multiplier representation for a Lie group action to a straightforward algebraic computation based on the infinitesimal generators of the multiplier representation. As such, it can be readily applied to compute the number of, say, invariant vector fields, or invariant differential forms, or invariant metrics, of any regular transformation group action; examples will appear below. The explicit determination of the relative invariants, though, requires the integration of a system of partial differential equations, namely (5.9). A computationally convenient approach to the latter problem is to write the original system of differential equations (5.8) in terms of a local basis (or "frame") of sections $S_1(x), \ldots, S_k(x)$, of the kernel bundle $\mathscr{K}$ near a point x_0 where the kernel rank $k = \dim \mathscr{K}|_x$ is constant. The fact that every relative invariant is a local section of $\mathscr{K}$ allows us to write

$$R(x) = \sum_{i=1}^{k} r_i(x)\, S_i(x) \,, \tag{5.25}$$

where the scalar functions $r_i(x)$ are the components of R relative to the adapted kernel frame. Applying such a change of frame in (5.8) serves to automatically eliminate the algebraic conditions (5.10), while the remaining differential equations (5.9) reduce to a Frobenius system of partial differential equations for the coefficients $r_i(x)$. See Example 7.11 for an illustration of this approach.

A particular case of Theorem 5.2 provides necessary and sufficient conditions for a transformation group to admit a complete system of relative invariants, as in Definition 3.5. This result was first proved, by different methods, in [25; Theorem 3.36].

Theorem 5.4 *A regular transformation group G acting via vector bundle automorphisms on a rank n vector bundle $E \to M$ admits a complete system of n independent relative invariants $R_1(x), \ldots, R_n(x)$ if and only if its orbits in E have the same dimension as its orbits in M.*

Proof. According to Theorem 5.2, the necessary and sufficient conditions that the group admit n independent relative invariants is that the common kernel bundle $\mathscr{K}$ have rank n. This is possible if and only if all the matrices L_λ in (5.5) are identically zero, so that $\mathscr{D}_\lambda = \eta^\alpha_\lambda \mathscr{D}_\alpha$. But, in view of (4.1), this implies that the infinitesimal generators of the bundle action satisfy the same linear relations as their projections on M, i.e., $\widetilde{\mathbf{v}}_\lambda = \eta^\alpha_\lambda \widetilde{\mathbf{v}}_\alpha$, and hence the space $\widetilde{\mathfrak{g}}$ of infinitesimal generators on E has the same pointwise dimension as the space $\widehat{\mathfrak{g}}$ of infinitesimal generators on M. □

Stated another way, a complete set of n independent relative invariants will exist if and only if the subspace of TM spanned by the infinitesimal generators $\widehat{\mathbf{v}}$ of G has the same dimension as the space of matrix differential operators spanned by the associated generators $\mathscr{D}_\mathbf{v} = \widehat{\mathbf{v}} - H_\mathbf{v}$ of the multiplier representation. In particular, since the dimension of the orbits of G in E is necessarily at least as large as the orbit dimension in M, the condition in Theorem 5.4 is automatically satisfied if the dimension of the orbits of G in M is maximal, meaning the same dimension as G itself:

Corollary 5.5 *If G acts locally freely on M, then G admits a complete system of relative invariants.*

6 Invariant vector fields

We now apply Theorem 5.2 to the study of invariant vector fields for a transformation group G acting on a manifold M. As discussed in Examples 2.2 and 3.2, the invariant vector fields can be viewed as relative invariants of the Jacobian multiplier representation on the tangent bundle of M. The infinitesimal generators have the form

$$\mathscr{D}_\mathbf{v} = \widehat{\mathbf{v}} - J_\mathbf{v}, \quad \text{where } J_\mathbf{v}(x) = \left(\frac{\partial \xi^i}{\partial x^j}\right) \tag{6.1}$$

is the "infinitesimal Jacobian matrix" of the coefficients of the vector field $\widehat{\mathbf{v}}$, cf. (4.2). The infinitesimal condition for a vector field $\mathbf{w}$ to be invariant is that it commute with all the infinitesimal generators of the group action: $[\widehat{\mathbf{v}}, \mathbf{w}] = 0$ for all $\mathbf{v} \in \mathfrak{g}$. In local coordinates, $\mathbf{w} = \sum \eta^i(x)\partial_{x^i}$, so the commutator conditions are precisely the infinitesimal invariance conditions (4.6) for a relative invariant of the Jacobian multiplier representation (2.4), i.e., $\widehat{\mathbf{v}}(\eta) - J_{\mathbf{v}}\eta = 0$.

The space $\mathscr{Y} \subset \mathscr{X}(M)$ of G-invariant vector fields forms a module over the ring $\mathscr{I}$ of scalar invariant functions, and we are interested in its dimension $k = \dim \mathscr{Y}$. (As always, our interest is local, on sufficiently small open subsets of M.) The prototypical example is the action of an r-dimensional Lie group on itself by left multiplication: $h \mapsto g \cdot h$. Here the invariant vector fields determine the left Lie algebra $\mathfrak{g}_L$ of G, having the dimension of the group. A basis for the space $\mathscr{Y} = \mathfrak{g}_L$ of left-invariant vector fields on G forms a G-invariant frame on the Lie group G. The existence of left-invariant frames on a Lie group is a consequence of Corollary 5.5 since the right multiplication action of G on itself is transitive, and hence locally free. Indeed, this result is a special case of a theorem governing the existence of G-invariant frames, formulated and proved in [25, Theorem 2.84].

Theorem 6.1 *A regular transformation group G admits an invariant frame if and only if G acts effectively freely on M.*

Remark. The condition of effective freeness is also necessary and sufficient for the existence of a G-invariant coframe, consisting of m one-forms on M that form a pointwise basis for the cotangent space $T^*M|_x$.

In particular, if G acts transitively and effectively freely, then we can locally identify M with a neighborhood of the identity in G, with the action of G on M coinciding with left multiplication, and hence the G-invariant vector fields are identified with the elements of the left Lie algebra $\mathfrak{g}_L$. In the more general intransitive (but still free and regular) case, each G orbit can be identified, as in the transitive case, with an open subset of G itself. In terms of the associated local coordinates (y, z), the invariant frame consists of $r = \dim G$ independent right-invariant vector fields on G, mapped to each orbit, and suitably parametrized by the invariants z, together with $m - r$ additional transverse invariant vector fields.

Our principal goal is to generalize Theorem 6.1 and determine the geometric conditions underlying the existence of a less than maximal number of invariant vector fields. It is not hard to produce examples of Lie group actions which admit no (non-zero) invariant vector fields. For instance, of the three possible Lie group actions on the one-dimensional manifold $M = \mathbb{R}$, [25; Theorem 2.70], only the translation action $x \mapsto x+b$ admits an invariant vector field. The affine action $x \mapsto ax + b$ and the projective action $x \mapsto (ax + b)/(cx + d)$ admit no non-zero invariant vector field.

Definition 6.2 *A transformation group G acting on a manifold M is said to act* **imprimitively** *if G admits an invariant foliation on M.*

The infinitesimal criterion for the existence of an invariant foliation is contained in the following classical result.

Theorem 6.3 *Suppose that G is a connected Lie group acting regularly on M. The level sets of a regular function $F\colon M \to \mathbb{R}^l$ form a G-invariant foliation if and only if for each $\mathbf{v} \in \mathfrak{g}$ there is a smooth function $\Psi_{\mathbf{v}}\colon \mathbb{R}^l \to \mathbb{R}^l$ such that $\widehat{\mathbf{v}}(F) = \Psi_{\mathbf{v}}(F)$.*

Given a leaf L of a G-invariant foliation, let $G_L = \{g \in G \mid g \cdot L \subset L\}$ denote the associated *leaf isotropy subgroup*, which acts, via restriction, as a transformation group on its leaf L.

Theorem 6.4 *Let G act regularly on M. If G admits a k-dimensional module of G-invariant vector fields, then*

(a) *G acts imprimitively, admitting an invariant foliation with k-dimensional leaves, and*

(b) *each leaf isotropy subgroup G_L acts effectively freely on its leaf L of the foliation.*

Conversely, if G satisfies conditions (a) and (b), and satisfies the additional regularity condition that the orbits of the action of each leaf isotropy subgroups G_L on the corresponding leaf L all have the same dimension, independent of L, then G admits a non-zero k-dimensional module of G-invariant vector fields.

Note. In the transitive case, the indicated foliation has only one leaf, namely M itself, and so, technically, does not form an invariant foliation. However, the integral curves of any individual invariant vector field also determine a G-invariant foliation and so G must still act imprimitively. In particular, a primitive transformation group *never* admits a non-zero invariant vector field!

Example 6.5 According to Lie's classification of transformation groups acting on a two-dimensional manifold, [20], there are, locally, precisely four inequivalent actions of the unimodular group $G = \mathrm{SL}(2, \mathbb{R})$ on $M = \mathbb{R}^2$. See [10] for details of the classification, and [7] for applications of the unimodular actions to the study of differential equations, Painlevé analysis, and classical invariant theory.

There is one intransitive action, generated by the vector fields

$$\widehat{\mathbf{v}}_1 = \partial_x, \quad \widehat{\mathbf{v}}_2 = x\partial_x, \quad \widehat{\mathbf{v}}_3 = x^2\partial_x\,. \tag{6.2}$$

A direct calculation based on the commutator condition $[\widehat{\mathbf{v}}_i, \mathbf{w}] = 0$ proves that every invariant vector field has the form $\mathbf{w} = \eta(y)\partial_y$. Here, the vertical lines $L_a = \{x = a\}$ form a G-invariant foliation. The isotropy group G_a of a vertical line has infinitesimal generators

$$(x-a)\partial_x, \quad (x^2 - a^2)\partial_x\,, \tag{6.3}$$

consisting of the generators of G tangent to L_a, and so forms a two-dimensional subgroup, conjugate to the subgroup of lower triangular matrices (which is the isotropy group for the y-axis). However, since both vector fields (6.3) vanish when $x = a$, the subgroup G_a acts completely trivially, and hence effectively freely, on L_a. Thus, in accordance with Theorem 6.4 the space of invariant vector fields forms a one-dimensional module over the ring $\mathscr{I} = \{f(y)\}$ of invariant functions, reconfirming the direct calculation. Note that in this case, the horizontal lines $\widetilde{L}_a = \{y = a\}$ also form a G-invariant foliation, but the reduced action of the isotropy subgroup $\widetilde{G}_a = G$ is not effectively free, and hence this invariant foliation is not of use in confirming the theorem.

The first transitive action of $G = \mathrm{SL}(2, \mathbb{R})$ is generated by the vector fields

$$\widehat{\mathbf{v}}_1 = \partial_x, \quad \widehat{\mathbf{v}}_2 = x\partial_x + y\partial_y, \quad \widehat{\mathbf{v}}_3 = x^2\partial_x + 2xy\partial_y\,. \tag{6.4}$$

A direct calculation shows that any invariant vector field is a constant multiple of $\mathbf{w} = y\partial_y$. Again, the vertical lines $L_a = \{x = a\}$ provide the invariant foliation. The isotropy subgroup G_a of the leaf L_a is generated by the vector fields

$$(x - a)\partial_x + y\partial_y, \quad (x^2 - a^2)\partial_x + 2xy\partial_y\,. \tag{6.5}$$

Moreover, the restricted action on L_a is effectively free since the infinitesimal generators of $G_a \,|\, L_a$ are obtained by setting $x = a$ in (6.5), leading to two vector fields $y\partial_y$, $2ay\partial_y$, which are constant multiples of a single vector field (which happens to coincide with $\mathbf{w}$ – because the reduced action of G_a on L_a is abelian).

On the other hand, the second transitive action of $G = \mathrm{SL}(2, \mathbb{R})$, which is generated by

$$\widehat{\mathbf{v}}_1 = \partial_x + \partial_y, \quad \widehat{\mathbf{v}}_2 = x\partial_x + y\partial_y, \quad \widehat{\mathbf{v}}_3 = x^2\partial_x + y^2\partial_y\,, \tag{6.6}$$

does not admit an invariant vector field, even though the vertical lines $L_a = \{x = a\}$ form an invariant foliation. Indeed,

$$(x - a)\partial_x + (y - a)\partial_y, \quad (x^2 - a^2)\partial_x + (y^2 - a^2)\partial_y \tag{6.7}$$

are tangent to L_a, and so generate G_a, but their restrictions $(y - a)\partial_y$, $(y^2 - a^2)\partial_y$, are not constant multiples of a single vector field on the one-dimensional leaf of the foliation, and hence the reduced action of G_a on the leaf of L_a is no longer free.

The third transitive action of $\mathrm{SL}(2, \mathbb{R})$ is generated by

$$\widehat{\mathbf{v}}_1 = \partial_x, \quad \widehat{\mathbf{v}}_2 = x\partial_x + y\partial_y, \quad \widehat{\mathbf{v}}_3 = (x^2 - y^2)\partial_x + 2xy\partial_y\,. \tag{6.8}$$

This case can be transformed into the second one, (6.6), by a *complex analytic* change of variables, and so has a similar structure. Alternatively, the reader can analyze this case directly.

Proof of Theorem 6.4. Assume first that G admits a regular k-dimensional module $\mathscr{Y}$ of invariant vector fields, and let $\mathbf{w}_1, \ldots, \mathbf{w}_k$ be a basis. The Lie bracket of two invariant vector fields is also an invariant vector field, and hence $\mathscr{Y}$ forms an involutive differential system on M having, by our regularity hypothesis, constant dimension k. Therefore, by Frobenius' Theorem, there is a k-dimensional foliation of M forming the integral submanifolds of the differential system $\mathscr{Y}$, so that any G-invariant vector field $\mathbf{w} \in \mathscr{Y}$ is tangent to the leaves of the foliation. Since $\mathscr{Y}$ consists of invariant vector fields, this foliation is clearly G-invariant, proving condition (a). Moreover, given a leaf L, the restriction of a basis of the space of invariant vector fields to L forms a frame $\mathbf{w}_1 \mid L, \ldots, \mathbf{w}_k \mid L$ on L which is invariant under the isotropy subgroup G_L of the leaf. Applying Theorem 6.1 to L implies that G_L must act effectively freely on L, which demonstrates condition (b).

To prove the converse, we begin by introducing the adapted flat local coordinates $(y,z) = (y^1, \ldots, y^k, z^1, \ldots, z^{m-k})$ such that the leaves of our G-invariant foliation are (locally) given by the slices $L_a = \{z = a\}$. Let $G_a = G_{L_a}$ denote the isotropy subgroup of the leaf L_a, and let $H_a = \{g \in G_a \mid g \cdot x = x$ for all $x \in L_a\} \subset G_a$ denote the global isotropy subgroup of the restricted action of G_a on L_a. According to our hypothesis, the quotient group $\widehat{G}_a = G_a/H_a$ acts freely on L_a. Moreover, the regularity hypothesis implies that the dimension of the quotient group $t = \dim \widehat{G}_a$, which coincides with the dimension of the orbits of G_a in L_a, is a constant, independent of a. By continuity, we can choose a set of t pointwise linearly independent smooth vector fields $\mathbf{z}_1, \ldots, \mathbf{z}_t$ such that each $\mathbf{z}_\alpha$ is tangent to each leaf, and the restrictions $\mathbf{z}_1 \mid L_a, \ldots, \mathbf{z}_t \mid L_a$ form a basis for the space $\widehat{\mathfrak{g}}_a$ of infinitesimal generators of the action of $\widehat{G}_a$ on L_a.

Note. Clearly at each point $x \in M$, we have $\mathbf{z}_\alpha|_x \in \widehat{\mathfrak{g}}|_x$. However, unless the quotient group $\widehat{G}_a$ itself is independent of a, the $\mathbf{z}_\alpha$'s will *not* generally be infinitesimal generators of the action of G on M.

Let $q = s - t$, where $s = \dim \widehat{\mathfrak{g}}$ denotes the dimension of the orbits of G. Using regularity, we can locally choose infinitesimal generators $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_q \in \widehat{\mathfrak{g}}$ such that, at each point x, the tangent vectors $\widehat{\mathbf{v}}_1|_x, \ldots, \widehat{\mathbf{v}}_q|_x$, $\mathbf{z}_1|_x, \ldots, \mathbf{z}_t|_x$, form a basis for the space $\widehat{\mathfrak{g}}|_x$. In particular, $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_q$ are transverse to the leaf L_a. Every infinitesimal generator $\widehat{\mathbf{v}} \in \widehat{\mathfrak{g}}$ can then be uniquely expressed as a linear combination

$$\widehat{\mathbf{v}} = \sum_{\alpha=1}^{t} \eta_\alpha \mathbf{z}_\alpha + \sum_{\lambda=1}^{q} \zeta_\lambda \widehat{\mathbf{v}}_\lambda\,, \tag{6.9}$$

for certain coefficient functions η_α, ζ_λ. Transversality of the $\widehat{\mathbf{v}}_\lambda$'s and the fact that the leaves form an invariant foliation automatically implies that the functions $\zeta_\lambda = \zeta_\lambda(z)$ depend only on the transverse coordinates – see Theorem 6.3. We claim that, because of the effective freeness of the restricted action, the same is true for the coefficient functions η_α. Indeed, fix a leaf L_a. From the infinitesimal generator (6.9), we construct the modified infinitesimal generator

$$\widehat{\mathbf{v}}^* = \widehat{\mathbf{v}} - \sum_{\lambda=1}^{q} \zeta_\lambda(a)\widehat{\mathbf{v}}_\lambda = \sum_{\alpha=1}^{t} \eta_\alpha(y,z)\mathbf{z}_\alpha + \sum_{\lambda=1}^{q} [\zeta_\lambda(z) - \zeta_\lambda(a)]\widehat{\mathbf{v}}_\lambda\,, \tag{6.10}$$

which lies in $\widehat{\mathfrak{g}}$ since it is a constant coefficient linear combination of infinitesimal generators $\widehat{\mathbf{v}}, \widehat{\mathbf{v}}_\lambda \in \widehat{\mathfrak{g}}$. Moreover, restricting to L_a, i.e., setting $z = a$, we find that

$$\widehat{\mathbf{v}}^* \mid L_a = \sum_{\alpha=1}^{t} \eta_\alpha(y,a)(\mathbf{z}_\alpha \mid L_a) \in TL_a \tag{6.11}$$

lies in the tangent space to the leaf, and hence is an infinitesimal generator of the restricted action of G_a on L_a. We now invoke the following infinitesimal characterization of effectively free group actions.

Lemma 6.6 *A transformation group acts effectively freely with s-dimensional orbits if and only if locally there exist s pointwise linearly independent infinitesimal generators* $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_s$ *such that every other infinitesimal generator is a constant coefficient linear combination thereof*: $\widehat{\mathbf{v}} = c_1\widehat{\mathbf{v}}_1 + \cdots + c_s\widehat{\mathbf{v}}_s$, *for* $c_i \in \mathbb{R}$.

Applying Lemma 6.6 to the action of G_a on L_a, we conclude that $\widehat{\mathbf{v}}^* \,|\, L_a$ must be a *constant coefficient* linear combination of the infinitesimal generators $\mathbf{z}_\alpha \,|\, L_a$ of G_a, and hence the coefficients $\eta_\alpha(y, a) = \eta_\alpha(a)$ in (6.11) must be independent of the leaf coordinates y. But this holds for every a, and hence, as claimed, $\eta_\alpha = \eta_\alpha(z)$ depends only on z.

Turning to the induced Jacobian multiplier representation on the tangent bundle, we let $\mathscr{D}_{\mathbf{v}} = \widehat{\mathbf{v}} - J_{\mathbf{v}}$ denote the differential operator associated with a vector field $\widehat{\mathbf{v}}$, cf. (6.1). In particular, we let $\mathscr{E}_\alpha = \mathbf{z}_\alpha - J_\alpha$ and $\mathscr{D}_\lambda = \widehat{\mathbf{v}}_\lambda - J_\lambda$ denote the differential operators associated with our previously defined vector fields. (As with the vector field $\mathbf{z}_\alpha$, the differential operator $\mathscr{E}_\alpha$ does not necessarily correspond to an infinitesimal generator of the Jacobian multiplier representation of the transformation group G.) In particular, the differential operator associated with the infinitesimal generator (6.9) has the form

$$\mathscr{D}_{\mathbf{v}} = \sum_{\alpha=1}^{t} \eta_\alpha \mathscr{E}_\alpha + \sum_{\lambda=1}^{q} \zeta_\lambda \mathscr{D}_\lambda - \widetilde{L}_{\mathbf{v}} \,, \tag{6.12}$$

where, by the chain rule, the residual Jacobian matrix $\widetilde{L}_{\mathbf{v}}$ is an appropriate linear combination of the Jacobians of the coefficient functions η_α, ζ_λ. Since the vector fields $\mathbf{z}_\alpha$, $\widehat{\mathbf{v}}_\lambda$ are pointwise linearly independent, the kernel space (5.7) coincides with the common kernel of the residual Jacobians: $\mathscr{K}|_{(y,z)} = \bigcap_{\mathbf{v}\in\mathfrak{g}} \ker \widetilde{L}_{\mathbf{v}}$. Moreover, because the coefficient functions η_α, ζ_λ in (6.9) depend only on z, the y derivative entries in the residual Jacobian matrices must all vanish, and so these matrices have the block form $\widetilde{L}_{\mathbf{v}} = (0, \bar{L}_{\mathbf{v}})$ whose first k columns, corresponding to the y coordinates, are all zero. This implies that the first k basis vectors of $TM|_{(y,z)}$ (i.e., the ∂_{y^α}) all lie in the common kernel, and therefore, $\dim \mathscr{K}|_{(y,z)} \geqq k$. Thus the kernel rank is at least k, and therefore Theorem 5.2 implies that G admits (at least) k independent invariant vector fields: $\dim \mathscr{Y} \geqq k$. □

In the transitive case, there is an alternative approach to the invariant vector field problem that relies directly on the (local) identification of the manifold M with a homogeneous space. Assuming G acts transitively on M, we fix a point $x_0 \in M$, and let $H = G_{x_0}$ be the associated isotropy subgroup. We can identify M with an open subset of the homogeneous space G/H. Let $N = N(H) = \{g \,|\, gHg^{-1} = H\}$ denote the normalizer subgroup of H, and $\mathfrak{n} \subset \mathfrak{g}$ its Lie algebra, which is the normalizer subalgebra of the Lie algebra $\mathfrak{h}$ of the isotropy subgroup H, i.e., $\mathfrak{n} = \{\mathbf{v} \in \mathfrak{g} \mid [\mathbf{v}, \mathbf{w}] \in \mathfrak{h} \text{ for all } \mathbf{w} \in \mathfrak{h}\}$. The following characterization of the normalizer subalgebra $\mathfrak{n}$ is standard.

Lemma 6.7 *Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra of the Lie algebra $\mathfrak{g}$, and $\mathfrak{n} \subset \mathfrak{g}$ denote its normalizer subalgebra. Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for $\mathfrak{g}$ such that $\mathbf{v}_{m+1}, \ldots, \mathbf{v}_r$ form a basis for the $(r-m)$-dimensional subalgebra $\mathfrak{h}$. Let c^i_{jk} denote the associated structure constants. A generator $\mathbf{v} = a^1\mathbf{v}_1 + \cdots + a^r\mathbf{v}_r$ will lie in $\mathfrak{n}$ if and only if its coefficients satisfy*

$$\sum_{j=1}^{m} c^i_{j\alpha} a^j = 0, \quad i = 1, \ldots, m, \ \ \alpha = m+1, \ldots, r\,. \tag{6.13}$$

Note that the last $r - m$ coefficients of the generator $\mathbf{v}$ are irrelevant, since they just indicate its projection into $\mathfrak{h}$. In practice, we identify the quotient space $\mathfrak{g}/\mathfrak{h}$ with the space of generators of the form $\mathbf{v} = a^1\mathbf{v}_1 + \cdots + a^m\mathbf{v}_m$, whereby (6.13) are the necessary and sufficient conditions for $\mathbf{v}$ to lie in $\mathfrak{g}/\mathfrak{h}$.

If G acts on a manifold M, then we let $N_x = N(G_x)$ denote the normalizer of the isotropy subgroup G_x of the point $x \in M$. We let $\mathfrak{g}_x \subset \mathfrak{g}$ denote the Lie algebra of G_x, and $\mathfrak{n}_x \subset \mathfrak{g}$ the Lie algebra of N_x. Since the space $\widehat{\mathfrak{g}}|_x$ of infinitesimal generators at the point x is identified with the quotient space $\mathfrak{g}/\mathfrak{g}_x$ of the Lie algebra by the isotropy subalgebra, we can identify the subspace $\widehat{\mathfrak{n}}_x|_x$ of generators coming from the normalizer subalgebra with the quotient Lie algebra $\mathfrak{n}_x/\mathfrak{g}_x$.

Proposition 6.8 *Suppose G acts transitively on M. Let $N_x = N(G_x)$ denote the normalizer subgroup at the point $x \in M$, and let $\mathfrak{n}_x$ be its Lie algebra. If $\mathbf{w}$ is an invariant vector field on M, then $\mathbf{w}|_x \in \widehat{\mathfrak{n}}_x|_x$.*

Proof. Fix the point x_0, and choose a basis $\mathbf{v}_1, \ldots, \mathbf{v}_r$ for $\mathfrak{g}$ such that $\mathbf{v}_{m+1}, \ldots, \mathbf{v}_r$ form a basis for the isotropy subalgebra $\mathfrak{g}_0 = \mathfrak{g}_{x_0}$, whereby $\mathbf{v}_i|_{x_0} = 0$, $i = m+1, \ldots, r$. Since G acts transitively, the first m infinitesimal generators $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_m$ form a frame on M in a neighborhood of x, and so we can write any invariant vector field

$$\mathbf{w} = \sum_{i=1}^{m} h^i(x)\widehat{\mathbf{v}}_i$$

as a linear combination thereof. As usual, the infinitesimal invariance conditions

$$[\widehat{\mathbf{v}}_\rho, \mathbf{w}] = 0, \quad \rho = 1, \ldots, r\,, \tag{6.14}$$

decouple into a system of differential equations, corresponding to the first m generators, plus a system of algebraic equations, cf. (5.10), corresponding to the generators of $\mathfrak{g}_x$. Writing the latter in terms of the frame, $\widehat{\mathbf{v}}_\alpha = \sum_{i=1}^m \eta^i_\alpha \widehat{\mathbf{v}}_i$, a straightforward computation shows that

$$[\widehat{\mathbf{v}}_\rho, \mathbf{w}] = \sum_{i=1}^{m} \left[\widehat{\mathbf{v}}_\rho(h^i) + \sum_{j=1}^{m} \left(c^i_{\rho j} + \sum_{\beta=m+1}^{r} c^\beta_{\rho j} \eta^i_\beta \right) h^j \right] \widehat{\mathbf{v}}_i, \quad \rho = 1, \ldots, r\,.$$

Therefore, setting $\rho = \alpha$, where $m+1 \leqq \alpha \leqq r$, and subtracting off the corresponding conditions for $1 \leqq \rho \leqq m$, we find that the algebraic relative invariance conditions (5.6) take the form

$$\sum_{j=1}^{m}\left[c_{\alpha j}^{i}+\sum_{\beta=m+1}^{r}c_{\alpha j}^{\beta}\eta_{\beta}^{i}-\sum_{i=1}^{m}\eta_{\alpha}^{k}\left(c_{kj}^{i}+\sum_{\beta=m+1}^{r}c_{kj}^{\beta}\eta_{\beta}^{i}\right)\right]h^{j}=0,\qquad \begin{array}{l} i=1,\ldots,m\,,\\ \alpha=m+1,\ldots,r\,. \end{array} \tag{6.15}$$

In particular, at x_0, we have $\eta_\alpha^i(x_0) = 0$, and so (6.15) reduces to $\sum_{j=1}^m c_{\alpha j}^i h^j = 0$. Lemma 6.7 completes the proof. □

Theorem 6.9 *Let $M = G/H$ with G acting by left multiplication. Let $N = N(H)$ be the normalizer. Then there is a well-defined action of N/H on G/H induced by the right action of N on G. The invariant vector fields for the left action of G on G/H are then the infinitesimal generators of this right action of N/H.*

Proof. The right action of the normalizer on G, namely $g \mapsto g \cdot n$, for $n \in N$, induces a well-defined action of the quotient group N/H on the homogeneous space G/H:

$$(nH, gH) \mapsto gH \cdot nH = gn \cdot (n^{-1}Hn) \cdot H = gnH, \quad nH \in N/H,\ gH \in G/H\ .$$

The right action of N/H clearly commutes with the left action of G on G/H, and hence the infinitesimal generators of N/H are the G-invariant vector fields on G/H. On the other hand, Proposition 6.8 implies that the kernel rank of the multiplier representation of G on the space of vector fields on G/H equals the dimension of N/H, and hence Theorem 5.2 implies that these generators provide a complete collection of G-invariant vector fields on G/H. □

Example 6.10 Let us return to the transitive actions of $\mathrm{SL}(2,\mathbb{R})$ on $M = \mathbb{R}^2$ discussed above in Example 6.5. For the first transitive action, generated by the vector fields (6.4), the isotropy subgroup $\widehat{\mathfrak{g}}_{a,b}$ at a generic point (a,b), $b \neq 0$, where the orbits are two-dimensional, is generated by

$$\widetilde{\mathbf{v}}_3 = (x-a)^2\partial_x + 2(x-a)y\partial_y\ .$$

We use the adapted basis $\widehat{\mathbf{v}}_1 = \partial_x$, $\widehat{\mathbf{v}}_2 = x\partial_x + y\partial_y$ and $\widetilde{\mathbf{v}}_3$, as in the proof of Proposition 6.8. The normalizer subalgebra $\widehat{\mathfrak{n}}_{a,b}$ consists of all vector fields $\mathbf{w} = a^1\widehat{\mathbf{v}}_1 + a^2\widehat{\mathbf{v}}_2 + a^3\widetilde{\mathbf{v}}_3$, with a_i constant, satisfying $[\mathbf{w}, \widetilde{\mathbf{v}}_3] = \lambda\widetilde{\mathbf{v}}_3$. Using Lemma 6.7, we find that $\widehat{\mathfrak{n}}_{a,b}$ is spanned by $\widehat{\mathbf{v}}_2 - a\widehat{\mathbf{v}}_1$ and $\widetilde{\mathbf{v}}_3$. Therefore, $\dim N_{a,b}/G_{a,b} = 1$ and, as we saw above, there is one independent invariant vector field, $\mathbf{w} = y\partial_y$, which can be seen to generate the right action of $N_{a,b}/G_{a,b}$ on M.

For the second transitive action, generated by the vector fields (6.6), the isotropy subgroup $\widehat{\mathfrak{g}}_{a,b}$ at a generic point (a, b), $a \neq b$, is generated by

$$\widetilde{\mathbf{v}}_3 = (x-a)(x-b)\partial_x + (y-a)(y-b)\partial_y\ .$$

However, in this case, a similar elementary computation shows that the normalizer subalgebra coincides with the isotropy subalgebra, $\widehat{\mathfrak{n}}_{a,b} = \widehat{\mathfrak{g}}_{a,b}$, and hence, as we saw above, there are no non-zero invariant vector fields on M. The third case, (6.8), is left to the reader.

Of course, Theorem 6.4 includes Theorem 6.9 as a special case. The invariant foliation of G/H is provided by the orbits of the right action of N/H, and it is easy to see directly that N/H acts locally freely on G/H. However, this approach does not appear to readily generalize to the intransitive case, since the invariant vector fields can have components which are transverse to the G orbits in M. However, there is a restricted, but useful, generalization, which, in the intransitive case, characterizes the invariant vector fields which are tangent to the group orbits. The proof is straightforward; see [1] for applications to symmetry reduction of differential equations.

Theorem 6.11 *Let G act regularly on M. Assume that the dimension of the quotient normalizer group N_x/G_x is a constant, k, independent of x. Then the space of invariant vector fields which are everywhere tangent to the orbits of G forms a k-dimensional module over the space of invariant functions. Moreover, if* $\mathbf{w}$ *is such an invariant vector field, at each point it lies in the space $\widehat{\mathfrak{n}}_x|_x$, and, in fact, its restriction to the orbit through x can be identified with a generator of the right action of N_x/G_x on the orbit.*

Corollary 6.12 *If $\dim N_x = \dim G_x$ for every $x \in M$, then every nonzero invariant vector field must be transverse to the orbits of G.*

There are several interesting generalizations of the problem of existence of invariant vector fields that are worth a more detailed investigation. One is to determine the invariant multi-vector fields of a given transformation group, a problem that arises in the analysis of the cohomology of the quotient space M/G of a manifold by a regular group action, [1]. A *multi-vector field* is, by definition, a section of an exterior power of the tangent bundle, $\bigwedge^k TM$, and is the dual object to a differential k-form. The algebraic conditions for the existence of invariant multi-vector fields under non-free group actions are straightforwardly determined using Theorem 5.2, but the underlying geometry remains to be determined.

A second generalization, motivated by Helgason's approach to geometric analysis and representation theory on Lie groups and symmetric spaces, is to the existence problem for invariant differential operators on manifolds under transformation groups. The literature on invariant differential operators and their applications is vast, and we refer the reader to [14, 15, 29], for many additional references. Both the first order case, which is related to the existence of "compatible" multiplier representations, and the problem for higher order operators, particularly those of Laplace–Beltrami type, are of interest, and can be handled by our general methods.

Yet another generalization is motivated by the determination of invariant differential operators of (prolonged) group actions on jet bundles, as discussed in [25; Chapter 5]. Such operators can be viewed as suitable relative invariants of the prolonged group action, and are essential in the construction of complete systems of differential invariants. It can be shown that, for a prescribed jet bundle, the number of invariant differential operators corresponds to the number of vector fields which are invariant modulo a suitable vertical sub-bundle, and thus forms a particular case of the following general problem: Given a group G acting on a manifold M, and a G-invariant foliation defined by an involutive differential system $\mathscr{W} \subset TM$, determine the number of G-invariant sections of the quotient bundle $TM/\mathscr{W}$. Or, stated another way, determine the number of vector fields $\mathbf{w}$ on M which are G-invariant modulo the sub-bundle $\mathscr{W}$, i.e., $[\widehat{\mathbf{v}}, \mathbf{w}] \in \mathscr{W}$ for all $\mathbf{v} \in \mathfrak{g}$. Thus, this problem is a natural generalization of the problem of invariant vector fields, but we have been unable to determine simple geometrical conditions governing the number of such vector fields.

7 Invariant connections and inhomogeneous relative invariants

Let $E \to M$ be a rank n vector bundle over a smooth manifold M, and let G be a transformation group acting on E by vector bundle automorphisms, thereby determining a multiplier representation of G. In this section, we discuss the problem of finding G-invariant connections on E. The most important case is when the bundle is the tangent bundle, $E = TM$, but the same methods can be easily adapted to more general vector bundles. As we shall see, the invariance conditions for a connection lead to a generalization of the concept of relative invariant, in that the infinitesimal conditions (4.6) will contain an additional inhomogeneous term. The resulting "inhomogeneous multiplier representation" will correspond to the action of G on an affine bundle over the base manifold. The inhomogeneous terms will cause no difficulties for extending our general methods, and we will easily establish an analogue of the main existence result of Theorem 5.2. This will be applied to provide an immediate result on the structure and dimension of the space of G-invariant connections. The result will be illustrated by some elementary examples.

Definition 7.1 *A* **connection** *on a vector bundle* $\pi: E \to M$ *is given by a horizontal sub-bundle* $\mathscr{H} \subset TE$ *of the tangent bundle of E which is equivariant with respect to the action induced by the scaling in the fibers of E.*

A vector bundle automorphism $\Phi: E \to E$ induces a map $\Phi_*: TE \to TE$ on the tangent bundle, and hence maps a connection $\mathscr{H}$ to an **equivalent** connection $\overline{\mathscr{H}} = \Phi_*(\mathscr{H})$; in particular Φ determines a **symmetry** of the connection $\mathscr{H}$ provided $\Phi_*(\mathscr{H}) = \mathscr{H}$. See [27; p. 397] for details.

In local coordinates the connection is prescribed by a collection of $m = \dim M$ vector fields on E of the form

$$\mathbf{V}_i = \frac{\partial}{\partial x^i} - \Gamma^\alpha_{i\beta}(x) u^\beta \frac{\partial}{\partial u^\alpha}, \quad i = 1, \ldots, m. \tag{7.1}$$

Here and below, we again invoke the summation convention; now Latin indices run from 1 to m, whereas Greek indices run from 1 to n. Thus, in local coordinates, a connection is uniquely prescribed by the mn^2 connection coefficients $\Gamma^\alpha_{i\beta}(x)$. As in (4.3), we can identify any vector field $\widetilde{\mathbf{v}} \in \mathscr{X}(E)$ depending linearly on the fiber coordinates with a matrix-valued differential operator $\mathscr{D}_{\mathbf{v}}$. For the vector fields (7.1), the associated operators are

$$\mathscr{D}_i = \frac{\partial}{\partial x^i} + \Gamma_i, \quad i = 1, \ldots, m , \tag{7.2}$$

where $\Gamma_i = (\Gamma^\alpha_{i\beta})$ is the i^{th} $n \times n$ matrix of connection coefficients. The condition that the vector bundle automorphism $\Phi(x, u) = (\chi(x), \psi(x)u)$ map a connection spanned by the vector fields (7.1) to a connection spanned by vector fields $\overline{\mathbf{V}}_1, \ldots, \overline{\mathbf{V}}_m$ is that $\Phi_*(\mathbf{V}_i) = J^j_i \overline{\mathbf{V}}_j$ for some smooth invertible matrix-valued function $J = (J^j_i)\colon M \to \mathrm{GL}(m, \mathbb{R})$, which, owing to the form of the spanning vector fields, must be the Jacobian matrix $J^j_i = \partial\chi^j/\partial x^i$ of the base transformation. A straightforward computation then produces the corresponding transformation rule for the connection coefficients.

Lemma 7.2 *A vector bundle map $\Phi\colon E \to E$ with local coordinate formula $\Phi(x, u) = (\chi(x), \psi(x)u)$ maps the connection $\mathscr{H}$ with connection coefficients $\Gamma^\alpha_{i\beta}$ to the connection $\overline{\mathscr{H}}$ whose connection coefficients are given by*

$$\Phi^*(\overline{\Gamma}^\alpha_{i\beta}) = \psi^\alpha_\gamma \widetilde{J}^j_i \widetilde{\psi}^\varepsilon_\beta \Gamma^\gamma_{j\varepsilon} - \frac{\partial\psi^\alpha_\gamma}{\partial x^j} \widetilde{J}^j_i \widetilde{\psi}^\gamma_\beta . \tag{7.3}$$

Here $\widetilde{\psi}(x) = (\widetilde{\psi}^\alpha_\gamma(x)) = \psi(x)^{-1}$, and $\widetilde{J}(x) = (\widetilde{J}^j_i(x)) = J(x)^{-1}$ is the inverse of the Jacobian matrix of χ.

We note that the transformation rules (7.3) can be compactly re-expressed in terms of the associated *connection form*

$$\Omega = (du^\alpha + \Gamma^\alpha_{i\beta} u^\beta\, dx^i) \otimes \frac{\partial}{\partial u^\alpha} , \tag{7.4}$$

as

$$\Phi^* \overline{\Omega} = \Omega .$$

Here $\Phi^*(\omega \otimes \mathbf{w}) = (\Phi^*\omega) \otimes (\Phi^{-1})_* \mathbf{w}$ for ω a one-form and $\mathbf{w}$ a vector field on E. See [17, 27] for details.

If the manifold M admits a G-invariant metric ds^2, then the associated Levi-Civita connection is automatically G-invariant, but this is not the only way that G-invariant connections can arise. Indeed, it is easy to give examples in which the group G admits invariant connections, which are metric connections, but yet the group admits no invariant metric.

Example 7.3 Consider the case where the group $\mathrm{GL}(n, \mathbb{R})$ acts in the usual linear fashion on $M = \mathbb{R}^n$. The flat connection on TM is invariant and is the Levi-Civita connection for the standard Euclidean metric on M. However, $\mathrm{GL}(n, \mathbb{R})$ is certainly not acting by isometries on M. This demonstrates that G-invariant Levi-Civita connections may exist, in which the group G is not the isometry group of the metric.

It is convenient to introduce a multi-index notation for the connection coefficients, letting capital Latin letters denote triples of indices, $A = (i, \alpha, \beta)$, whereby $\Gamma^A = \Gamma^\alpha_{i\beta}$. Summing on repeated multi-indices, the transformation rules (7.3) take on the more compact form

$$\overline{\Gamma}^A(\bar{x}) = \mu^A_B(x)\Gamma^B(x) + \nu^A(x), \quad \text{when } \bar{x} = \Phi(x)\,, \tag{7.5}$$

where

$$\mu^A_B = \psi^\alpha_\gamma \widetilde{J}^j_i \widetilde{\psi}^\delta_\beta, \quad \nu^A = -\frac{\partial \psi^\alpha_\gamma}{\partial x^j} \widetilde{J}^j_i \widetilde{\psi}^\gamma_\beta, \quad A = (i, \alpha, \beta), \quad B = (j, \gamma, \delta)\,. \tag{7.6}$$

If the transformation Φ belongs to a Lie group G acting on the bundle E, the connection transformation rules (7.5) look like an inhomogeneous version of our multiplier representation equation (2.1). This serves to motivate the following extension of the concept of a multiplier representation.

Definition 7.4 *Let G be a transformation group acting on a manifold M, and let W be a finite-dimensional vector space. An* **inhomogeneous multiplier representation** *of G is a representation $\bar{F} = g \cdot F$ on the space of W-valued functions $\mathscr{F}(M, W)$ of the particular form*

$$\bar{F}(\bar{x}) = \bar{F}(g \cdot x) = \mu(g,x)F(x) + \nu(g,x), \quad g \in G,\ F \in \mathscr{F}(M, W)\,, \tag{7.7}$$

where μ: $G \times M \to \mathrm{GL}(W)$ and ν: $G \times M \to W$ are smooth maps.

The condition that (7.7) actually defines a representation of the group G requires that the functions μ and ν satisfy the **inhomogeneous multiplier equations**

$$\begin{aligned} &\mu(g \cdot h, x) = \mu(g, h \cdot x)\mu(g, x), && \mu(e, x) = \mathbb{1}, && \text{for all} \quad g, h \in G\,, \\ &\nu(g \cdot h, x) = \mu(g, h \cdot x)\nu(h, x) + \nu(g, h \cdot x), && \nu(e, x) = 0, && \qquad\qquad x \in M\,. \end{aligned} \tag{7.8}$$

We remark that these conditions can be intrinsically formulated by the action of G on an affine bundle $A \to M$ whose fibers are isomorphic to the n-dimensional affine space W; details are left to the reader.

Definition 7.5 *An* **inhomogeneous relative invariant** *for an inhomogeneous multiplier representation (μ, ν) of a transformation group G is a function S: $M \to W$ which satisfies*

$$S(g \cdot x) = \mu(g,x)S(x) + \nu(g,x) \quad \textit{for all } x \in M,\ g \in G\,, \tag{7.9}$$

where defined.

For example, a connection admits a transformation group G on E as a symmetry group if and only if its connection coefficients form a relative invariant for G under the inhomogeneous multiplier representation whose multiplier is given by equations (7.6). As with ordinary relative invariants, we call (μ, ν) the *weight* of the inhomogeneous relative invariant. It is important to note that, unlike an ordinary multiplier representation which always has the trivial zero relative invariant, an inhomogeneous multiplier representation may have *no* inhomogeneous relative invariants. Note that if S is an inhomogeneous relative invariant of weight (μ, ν) and R is any relative invariant of weight μ, then $S + R$ is also an inhomogeneous relative invariant of weight (μ, ν).

Proposition 7.6 *Let (μ, v) be an inhomogeneous multiplier for a regular transformation group G acting on an m-dimensional manifold M. Assume that the space of homogeneous relative invariants of weight μ has dimension k, and let $R_1, \ldots, R_k$ be a basis thereof. If G admits one inhomogeneous relative invariant S_0, then the most general inhomogeneous relative invariant of weight (μ, v) has the form $S = S_0 + I_1 R_1 + \cdots + I_k R_k$ for absolute invariants $I_1, \ldots, I_k$.*

Therefore, by Theorem 5.2, if the inhomogeneous multiplier representation admits one inhomogeneous relative invariant S_0, the space of inhomogeneous relative invariants forms a k-dimensional affine module over the ring of absolute invariants, where $k = \dim \mathscr{K}$ denotes the dimension of the common kernel, given by (5.7).

Theorem 7.7 *Let G be a connected group of transformations acting on M. A function $S(x)$ is an inhomogeneous relative invariant for the associated inhomogeneous multiplier representation if and only if it satisfies the following inhomogeneous linear system of partial differential equations*:

$$\mathscr{D}_{\mathbf{v}}(S) = \widehat{\mathbf{v}}(S) - H_{\mathbf{v}} S = K_{\mathbf{v}}, \quad \textit{for every } \mathbf{v} \in \mathfrak{g} . \tag{7.10}$$

Here σ: $\mathfrak{g} \to \mathscr{F}(M, \mathfrak{gl}(W))$, $\sigma(\mathbf{v}) = H_{\mathbf{v}}$, is the infinitesimal multiplier corresponding to the ordinary multiplier representation of weight μ, and τ: $\mathfrak{g} \to \mathscr{F}(M, W)$, $\tau(\mathbf{v}) = K_{\mathbf{v}}$, is its inhomogeneous counterpart. Thus σ satisfies the infinitesimal multiplier condition (4.5), whereas τ satisfies

$$\tau([\mathbf{v}, \mathbf{w}]) = \widehat{\mathbf{w}}(\tau(\mathbf{v})) - \widehat{\mathbf{v}}(\tau(\mathbf{w})) - \sigma(\mathbf{v})\tau(\mathbf{w}) + \sigma(\mathbf{w})\tau(\mathbf{v}) . \tag{7.11}$$

In local coordinates, the condition that there exist inhomogeneous relative invariants takes the following form. Suppose $\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_s$ span the differential system $\widehat{\mathfrak{g}}$ at each point. As in (5.3), we rewrite the additional generators as functional linear combinations of the first s generators. Using (5.5) and its inhomogeneous counterpart, we reduce the existence problem to a system of algebraic equations of the form

$$L_\lambda S = N_\lambda, \quad \lambda = s+1, \ldots, r , \tag{7.12}$$

where

$$L_\lambda = H_\lambda - \eta_\lambda^\alpha H_\alpha, \quad N_\lambda = K_\lambda - \eta_\lambda^\alpha K_\alpha . \tag{7.13}$$

The associated inhomogeneous multiplier representation admits an inhomogeneous relative invariant if and only if the inhomogeneous linear system (7.12) admits a solution S. If this occurs, then the space of inhomogeneous relative invariants is an affine module over the space of absolute invariants of dimension equal to that of the common kernel of the matrices $L_{s+1}, \ldots, L_r$.

The analogue of Theorem 5.2 for inhomogeneous relative invariants can now be formulated. The proof proceeds similarly: the fact that the solution space to the inhomogeneous linear system (7.12) is preserved under the system of differential equations (7.10) follows from an inhomogeneous version of the fundamental identity (5.16) whose precise statement and verification we leave to the reader.

Theorem 7.8 *Let* (μ, ν) *be an inhomogeneous multiplier for a connected regular transformation group* G *acting on a manifold* M*. Assume that the multiplier representation with multiplier* μ *is regular, and let* k *be the kernel rank. If the inhomogeneous linear system* (7.12) *is solvable at each point* x*, then the space of inhomogeneous relative invariants of weight* (μ, ν) *forms a* k*-dimensional affine module over the space of absolute invariants.*

The infinitesimal invariance conditions for a G-invariant connection are found by differentiating (7.5) with respect to the group parameters. Alternatively, one can use the Lie derivative condition $\widehat{\mathbf{v}}(\Omega) = 0$ on the connection form (7.4). After a straightforward calculation, we deduce the following explicit formula. Here δ_γ^α denotes the standard Kronecker symbol.

Proposition 7.9 *Let* G *be a connected group of transformations acting on the vector bundle* $E \to M$ *with infinitesimal generators as in* (4.1). *A connection* $\mathscr{H} \subset TE$ *is* G*-invariant if and only if its connection coefficients satisfy the infinitesimal invariance conditions*

$$\widehat{\mathbf{v}}(\Gamma_{i\beta}^\alpha) + \left\{ \delta_i^j (h_\beta^\varepsilon \delta_\gamma^\alpha - h_\gamma^\alpha \delta_\beta^\varepsilon) + \delta_\gamma^\alpha \delta_\beta^\varepsilon \frac{\partial \xi^j}{\partial x^i} \right\} \Gamma_{j\varepsilon}^\gamma = - \frac{\partial h_\beta^\alpha}{\partial x^i}, \quad \textit{for all } \mathbf{v} \in \mathfrak{g}\,. \tag{7.14}$$

Here $\widehat{\mathbf{v}} \in \widehat{\mathfrak{g}}$ *denotes the associated infinitesimal generator of the action of* G *on* M, *and* $\sigma(\mathbf{v}) = H_{\mathbf{v}}(x) = (h_\gamma^\alpha(x))$ *the infinitesimal multiplier. In particular, for an invariant connection on* TM, $H_{\mathbf{v}} = J_{\mathbf{v}}$ *is the infinitesimal Jacobian multiplier* (6.1).

We write the infinitesimal conditions (7.14) in vector form

$$\widehat{\mathscr{D}}_{\mathbf{v}}(\Gamma) = \widehat{\mathbf{v}}(\Gamma) - \widehat{H}_{\mathbf{v}} \Gamma = \widehat{K}_{\mathbf{v}}\,, \tag{7.15}$$

where $\widehat{\sigma}(\mathbf{v}) = \widehat{H}_{\mathbf{v}}$, $\widehat{\tau}(\mathbf{v}) = \widehat{K}_{\mathbf{v}}$, are the infinitesimal multipliers of the inhomogeneous multiplier representation (7.3), with entries

$$\widehat{H}_B^A = \delta_i^j (h_\gamma^\alpha \delta_\beta^\varepsilon - h_\beta^\varepsilon \delta_\gamma^\alpha) - \delta_\gamma^\alpha \delta_\beta^\varepsilon \frac{\partial \xi^j}{\partial x^i}, \quad \widehat{K}^A = - \frac{\partial h_\beta^\alpha}{\partial x^i}, \quad \begin{array}{l} A = (i, \alpha, \beta)\,, \\ B = (j, \gamma, \varepsilon)\,. \end{array} \tag{7.16}$$

In particular, if $\mathbf{v}_1, \ldots, \mathbf{v}_r$ form a basis for $\mathfrak{g}$, and, on M, satisfy linear relations of the form (5.3), then, as in (7.12), the algebraic constraints on invariant connections have the form

$$\widehat{L}_\lambda \Gamma = \widehat{N}_\lambda, \quad \lambda = s+1, \ldots, r\,, \tag{7.17}$$

where

$$\widehat{L}_\lambda = \widehat{H}_\lambda - \eta_\lambda^\alpha \widehat{H}_\alpha, \quad \widehat{N}_\lambda = \widehat{K}_\lambda - \eta_\lambda^\alpha \widehat{K}_\alpha\,. \tag{7.18}$$

Example 7.10 Consider the action of $\mathrm{SL}(2, \mathbb{R})$ on $M = \mathbb{R}^2$ generated by the vector fields in (6.6). Choosing $\widehat{\mathbf{v}}_1 = \partial_x + \partial_y$ and $\widehat{\mathbf{v}}_2 = x\partial_x + y\partial_y$ to serve as generators of the differential system, we have the linear relation

$$\widehat{\mathbf{v}}_3 = -xy\,\widehat{\mathbf{v}}_1 + (x+y)\,\widehat{\mathbf{v}}_2\,.$$

A straightforward computation based on equations (7.16), (7.18), shows that the algebraic constraints (7.17) that determine the invariant connection coefficients are the inhomogeneous linear system

$$(x-y)\Gamma^1_{11} = -2, \quad (y-x)\Gamma^1_{12} = 0, \quad (y-x)\Gamma^1_{21} = 0, \quad 3(y-x)\Gamma^1_{22} = 0\,,$$
$$3(x-y)\Gamma^2_{11} = 0, \quad (x-y)\Gamma^2_{12} = 0, \quad (x-y)\Gamma^2_{21} = 0, \quad (y-x)\Gamma^2_{22} = -2\,. \tag{7.19}$$

In this case, (7.19) has a unique solution

$$\Gamma^2_{11} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^1_{22} = 0, \quad \Gamma^1_{11} = -\Gamma^2_{22} = -\frac{2}{x-y}\,, \tag{7.20}$$

which are the Christoffel symbols for the hyperbolic metric $ds^2 = (x-y)^{-2}dx\,dy$. Theorem 7.8 then implies that this group action admits a *unique* invariant connection, generated by the differential operators

$$\partial_x + \begin{pmatrix} 2(x-y)^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \partial_y - \begin{pmatrix} 0 & 0 \\ 0 & 2(x-y)^{-1} \end{pmatrix}\,, \tag{7.21}$$

cf. (7.2). Note that we do not need to check that this connection satisfies the associated differential equations

$$\widehat{\mathbf{v}}_i(\Gamma) = \widehat{H}_i(\Gamma) + \widehat{K}_i, \quad i = 1,2\,, \tag{7.22}$$

since this follows automatically from Theorem 7.8. The metric ds^2 also admits the generators (6.6) as infinitesimal isometries, and so in this case the connection (7.21) arises as a G-invariant metric connection.

Example 7.11 Consider the action of the group $\mathrm{SL}(2,\mathbb{R})$ on M generated by the vector fields (6.4). Using the relation

$$\widehat{\mathbf{v}}_3 = -x^2\widehat{\mathbf{v}}_1 + 2x\widehat{\mathbf{v}}_2\,,$$

the resulting inhomogeneous linear system (7.17) is (canceling a common factor of 2)

$$y\Gamma^1_{12} + y\Gamma^1_{21} = -1, \quad y\Gamma^1_{22} = 0, \quad -y(\Gamma^1_{11} - \Gamma^2_{12} - \Gamma^2_{21}) = 0\,,$$
$$y\Gamma^1_{22} = 0, \quad y\Gamma^1_{12} - y\Gamma^2_{22} = 1, \quad y\Gamma^1_{21} - y\Gamma^2_{22} = 1\,.$$

In this case, the general solution is

$$\Gamma^1_{11} = r_1 + r_2, \quad \Gamma^1_{12} = -\frac{1}{2y}, \quad \Gamma^1_{21} = -\frac{1}{2y}, \quad \Gamma^1_{22} = 0\,,$$
$$\Gamma^2_{11} = r_1, \quad \Gamma^2_{12} = r_2, \quad \Gamma^2_{21} = r_3, \quad \Gamma^2_{22} = -\frac{3}{2y}\,,$$

which depends on the three free variables r_1, r_2, r_3. This shows that the kernel rank is 3, and hence Theorem 7.8 implies that there exists a three parameter family of connections on TM which are invariant under the infinitesimal action

(6.4). These are found by setting the undetermined coefficients to be smooth functions, $r_i = r_i(x, y)$, and substituting into the remaining differential equations (7.22). (This effectively implements the use of the frame-adapted coordinates (5.25) for sections of the common kernel space.) The residual differential equations for the unknowns r_i are easily found to be the Frobenius system

$$\frac{\partial r_i}{\partial x} = 0, \quad y\frac{\partial r_i}{\partial y} + r_i = 0, \quad i = 1,2,3 .$$

Thus the most general invariant connection is generated by the matrix-valued differential operators

$$\frac{\partial}{\partial x} - \frac{1}{2y}\begin{pmatrix} c_2 + c_3 & -1 \\ c_1 & c_2 \end{pmatrix}, \quad \frac{\partial}{\partial y} - \frac{1}{2y}\begin{pmatrix} -1 & 0 \\ c_3 & -3 \end{pmatrix}, \tag{7.23}$$

where c_1, c_2 and c_3 are real constants. An easy exercise, based on Theorem 5.2, shows that there are no metrics which admit the group having infinitesimal generators (6.4) as a symmetry group. Therefore, this provides another example of a transformation group admitting invariant connections but no invariant metrics.

Remark. When G acts transitively on M, which can thus be identified with a homogeneous space G/H, considerably more is known about invariant connections. See [17; Sect. II.11, Sect. X.2] for an extensive survey. It would be interesting to understand how our simple algebraic approach might shed additional light on the deep geometrical theorems discussed there.

8 Conclusions and further research

In this paper we have determined a general result governing the precise number of relative invariants of multiplier representations of connected Lie group actions. Applications to the study of invariant vector fields and invariant connections have been explicitly indicated. Many additional applications are possible, including the determination of the space of invariant differential forms, differential invariants, invariant differential operators, both on the manifold and its higher order jet spaces, etc. The practical determination of the number of relative invariants is, in specific examples, a straightforward algebraic computation based on Theorem 5.2. Thus, the more interesting problem is to analyze how the geometry of the transformation group action determines the number of relative invariants, along the lines of our Theorem 6.4. Of particular interest is the problem of existence of invariant differential forms. Preliminary investigations indicate that this is more complicated than the invariant vector field case, and we have not, as yet, been able to determine reasonable geometric conditions for their existence. Indeed, for transitive actions on symmetric spaces, the spaces of invariant differential forms are governed by the Lie algebra cohomology spaces, [6], and so our approach can be regarded as a complement to this established theory. The geometry associated with invariant multi-vector fields and with invariant vector fields modulo foliations, as discussed at the end of Sect. 6 are also worthy of investigation. Another interesting problem is the determination of invariant differential operators for prolonged group actions on jet spaces, as these provide the basic mechanism for constructing higher order differential invariants. General results, [25; Chapter 5], show that a complete system of such operators always exists at the stabilization order of the group, but, in many interesting cases, lower order operators can be found. Theorem 5.2 provides readily verifiable algebraic conditions that permit such lower order operators to exist, but the associated jet space geometry remains obscure. Even in the case of invariant connections, the above presentation only indicated how to perform the required algebraic manipulations for determining their number, but the underlying geometry, including its relation to the existence of invariant metric tensors, requires a more thorough investigation.

Finally, the extension of these results to the study of relative invariants under the action of an infinite Lie pseudo-group, [5, 26], is eminently worth a detailed investigation. For instance, the theory of differential invariants of Lie pseudo-group actions and their associated invariant differential operators, [18], would be one immediate application of the appropriate generalization of Theorem 5.2.

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