COMPLETELY INTEGRABLE BI-HAMILTONIAN SYSTEMS

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PREFACE

I view this thesis both as a report of the work I have done so far in integrable systems and as a sketch of my own view of (part of) the subject. It is, by its nature, necessary incomplete.

My interest in this field arose through the contact with several people. From João Resina Rodrigues I first learned about Classical Mechanics. The mathematical flavor of the theory I got from Arnold's book, which was suggested to me as a "holiday reading" for the summer of 1985 by Luis T. Magalhães. He was also the person responsible for me coming to study in the U. S. A., and hence the opportunity of pursuing serious graduate studies in Mathematics.

At the school of Mathematics, in Minneapolis, I found a group of people which helped shape my own view of Mathematical Physics. I am in debt to my adviser, Peter J. Olver, who led me through my graduate studies and introduced me to the fascinating world of research. I also would like to acknowledge the discussions with fellow graduate students and other professors, in particular the members of my examination committee, Jack Conn, Rick Moeckel, David Satinger and Thomas Posbergh.

Last, but not least, I would like to mention the support of my parents.

l dedicate this work to my wife Paula, who shared with me both the miseries and glories of life in graduate school.

CONTENTS

INTRODUCTION

Methods of differential geometry are now very popular in mathematical physics. In particular, the Hamiltonian formalism has long been recognized as playing a basic role in both classical and quantum mechanics. In the last few decades, following the intensive study of integrable systems derived from the original works of Kruskal, Gardner, Miura and Greene, there has been a revival of the Hamiltonian formalism, for this is generally accepted as the key to explain the integrability properties.

To motivate the introduction of the Hamiltonian formalism consider a particle of mass m moving in a harmonic potential $V(q) = \sum_{i=1}^{3} {q_i}^2$ so that

$$m\ddot{q} = -grad\, V(q).$$

Introducing the momentum $p_i = mq_i$ and the energy $h_1 = p^2/2m + V(q)$, Newton's equations of the motion are equivalent to Hamilton's equations

$$\dot{x_i} = \{x_i, h_1\}_0,$$

where $x = (q, p)$, and $\{\ ,\ \}_0$ is the Poisson bracket

$$\{f_1, f_2\}_0 = \sum_{i=1}^{3} \left(\frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_2}{\partial q_i} \frac{\partial f_1}{\partial p_i} \right).$$

Now, the Poisson bracket can be used to express remarkable properties of the system. For example, the condition that f is a first integral (a constant of the motion) can be expressed by the commutativity relation

$$\{f, h_1\}_0 = 0.$$

The general Hamiltonian formalism is based on the recognition that the Poisson bracket is the fundamental structure underlying the system, which in turn can be thought of as being "defined" by the Hamiltonian function. Therefore, one proceeds to define a Poisson bracket as a bilinear operation on a set of functions satisfying three basic properties: (i) skew-symmetry, (ii) Jacobi identity and (iii) Leibnitz rule. The Hamiltonian system is then specified by fixing a specific function, the Hamiltonian.

In the example above, at least from the physical point of view, there is no reason why the bracket was so chosen (except, perhaps, from the fact that in this case the hamiltonian is what one usually calls the "energy"). As a matter of fact, we could have taken, for example, the bracket

$$\{f_1, f_2\}_1 = \sum_{i=1}^{3} e^{-({q_i}^2+{p_i}^2)} \left(\frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_2}{\partial q_i} \frac{\partial f_1}{\partial p_i} \right).$$

Although this requires a new choice of hamiltonian, namely $h_0 = \sum_{i=1}^{3} \frac{1}{2} e^{{q_i}^2+{p_i}^2}$, we still have that Newton's equations are equivalent to

$$\dot{x_i} = \{x_i, h_1\}_0 = \{x_i, h_0\}_1.$$

Hence, we have two distinct Hamiltonian formulations for the same system. Less trivial examples of systems admitting a bi-Hamiltonian formulation will be given throughout the text.

The two Poisson brackets above are compatible in the following sense: the sum $\{\ ,\ \}_0 + \{\ ,\ \}_1$ is also a Poisson bracket. Because a Poisson bracket has to satisfy the Jacobi identity, a non-linear equation on the coefficients $J_{ij} = \{x_i, x_j\}$, this compatibility property is far from being a trivial requirement. In fact, it is the clue to a fundamental result due to Magri relating the existence of a bi-Hamiltonian formulation for a system with its integrability. A (finite dimensional) Hamiltonian system is said to be completely integrable if it has enough first integrals in involution (i.e. whose pairwise Poisson brackets vanish). Magri's theorem gives a method to produce first integrals of a non-degenerate, bi-Hamiltonian system. The non-degeneracy hypothesis means that at least one of the Poisson brackets has maximal rank, i.e., the matrix $J = (J_{ij})$ is invertible (in this case the Poisson bracket is derived from a symplectic form). Then one can define a 1-1 tensor N, called the recursion operator of the system, which gives the sequence of first integrals $\{h_0, h_1, h_2, \dots\}$ through the formula:

$$dh_{i+1} = N^* dh_i, \qquad i = 0, 1, \dots .$$

The compatibility condition is reflected on the vanishing of a torsion associated with N, which in turn implies that the right-hand side is actually a closed 1-form.

We can now explain the organization of this thesis.

We review all the basic facts concerning Poisson manifolds and bi-Hamiltonian systems in the first two sections of chapter I. Besides laying down the foundations of the theory, we are interested in explaining how some of the techniques can be extended to the degenerate bi-Hamiltonian systems, through the use of a reduction. For this, it is convenient to combine the master-symmetries introduced by Fuchssteiner together with a result due to Oevel relating conformal symmetries and bi-Hamiltonian structure, as presented in section 3. Then we show by an example, that it can happen that the master-symmetries survive reduction while the recursion operator does not. For the example studied, the non-periodic Toda lattice, we deduce exact deformation relations previously known to hold only up to some equivalence relation. As a byproduct, we obtain a new set of time-dependent symmetries for the Toda lattice, which can be used to integrate the system explicitly. At present we don't know of any general methods to deal with the degenerate case.

The next question we address (chapter II) is the following: does every completely integrable system possess a bi-Hamiltonian formulation ? We generalize, to any number of dimensions, a result due to Brouzet for dimension 4. It gives a necessary and sufficient condition for a completely integrable Hamiltonian system to have a bi-Hamiltonian formulation in a neighborhood of an invariant torus (modulo some natural hypothesis). This condition is expressed in terms of the affine structure defined by the action-angle variables, as a restriction on the graph of the Hamiltonian function. We then interpret it geometrically in terms of Darboux's hypersurfaces of translation, and give examples and counter-examples generalizing Brouzet's original example.

The third and final chapter is dedicated to the investigation of certain non-linear Poisson brackets that appear as second Poisson brackets in some integrable systems. In virtually all examples with physical meaning, the first Poisson bracket of a Poisson pair arises either from a symplectic form or by reduction from one (including, for example, the Lie Poisson brackets). One would like to have similar geometric interpretations for the second Poisson bracket. Also, there is a very close connection between integrable systems and Lie algebraic properties. Examples include Lax pairs, Lie-Poisson brackets, cotangent spaces of Lie groups, etc. We consider here the Poisson Lie groups of Drinfel'd.

A Poisson Lie group is a Lie group G equipped with a Poisson bracket, such that group multiplication $G \times G \to G$ is a Poisson map. The associated infinitesimal objects are the Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$. We propose a corresponding notion for symmetric spaces. A Poisson symmetric Lie group is a pair (G, S), where G is a Poisson Lie group and $S : G \to G$ is an involutive Poisson Lie group anti-morphism. The associated infinitesimal objects are the symmetric Lie bialgebras: a triple $(\mathfrak{g}, \mathfrak{g}^*, s)$ where $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, and $s : \mathfrak{g} \to \mathfrak{g}$ is an involutive Lie bialgebra anti-morphism. Using the structure theory of real semi-simple Lie algebras we obtain examples of symmetric Lie bialgebras. This is essentially equivalent to solving the Yang-Baxter equation under an additional constraint.

Given a Poisson symmetric Lie group, the associated symmetric space G/H, where H denotes the fixed point set of S, is a Poisson manifold. The Poisson brackets arising in this manner seem to be relevant to the theory of integrable systems. In fact we show that several examples of Poisson brackets that have appeared before in the literature, can be obtain in this way. Also, some of the usual techniques for symmetric spaces can be extended to the Poisson case. Closing the chapter we return to the Toda lattice and show how the second Poisson structure arises in this geometric setting.

The original results contained in chapters I and II were announced in [Fe1,Fe2].

CHAPTER I

POISSON PAIRS

In this chapter, we start by recalling some basic facts of Poisson Geometry (section 1), and the theory of Poisson pairs (section 2). The main result here is Magri's theorem, relating bi-Hamiltonian structure and first integrals of the system, through some remarkable relations known as Lenard's recursion relations. In section 3, the notion of symmetry of a differential equation is extended to include the so-called mastersymmetries. For a bi-Hamiltonian system, a result due to Oevel relates mastersymmetries to a conformal symmetry of the system, and yields deformation relations complementing the Lenard's recursion relations. In the final section of the chapter, we illustrate all these constructions with the example of the non-periodic, finite, Toda lattice. Our approach consists in working in the physical variables, and then reducing to Flaschka's variables. Although the recursion operator itself cannot be reduced, deformation relations and mastersymmetries do reduce. In this way, one obtains new time-dependent symmetries of the Toda lattice. Also, we believe that this approach is more natural. For example, deformation relations, previously known to hold up to a certain equivalence relation, are shown to be exact.

1. Poisson geometry

As usual, all objects are assumed to be C^∞.

Recall that a **Poisson bracket** [Li,Li-Ma,We1] on a manifold M is a bilinear skew-symmetric map $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, satisfying the Jacobi identity[1]

$$\bigodot_{f,g,h} \{f, \{g, h\}\} = 0, \quad f, g, h \in C^\infty(M), \tag{1.1}$$

and Leibnitz rule

$$\{f, g \cdot h\} = \{f, g\}h + g\{f, h\}, \quad f, g, h \in C^\infty(M). \tag{1.2}$$

Sometimes it is more convenient to work with bivectors. One defines the **Poisson bivector** Λ by setting

$$\{f, g\} =< \Lambda, df \wedge dg >, \quad f, g \in C^\infty(M). \tag{1.3}$$

The Jacobi identity (1.1) is then equivalent to the vanishing of the Schouten bracket

$$[\Lambda, \Lambda] = 0. \tag{1.4}$$

Associated with a Poisson bracket (or bivector) we have a bundle map $J : T^*(M) \to T(M)$, so that to each smooth function $h \in C^\infty(M)$ there is associated a vector field $X_h = J \cdot dh$. One calls h the **Hamiltonian function**, X_h the **Hamiltonian vector field**, and the equation for the integral curves of X_h

$$\dot{x} = J \cdot dh, \tag{1.5}$$

Hamilton's equations of motion. The **rank** of the Poisson bracket at a point $m \in M$, is the rank of the linear transformation $J(m)$.

EXAMPLE 1.1. Let (M, ω) be a symplectic manifold. For each $h \in C^\infty(M)$ there is defined a vector field X_h on M, by requiring that

$$X_h \lrcorner\, \omega = dh. \tag{1.6}$$

X_h is also called the Hamiltonian vector field associated with h. If we set

$$\{f, g\} =< \omega, X_f \wedge X_g >, \quad f, g \in C^\infty(M) \tag{1.7}$$

[1]The symbol $\bigodot$ denotes a cyclic sum over the indexes

then $\{\ ,\ \}$ is a Poisson bracket, and Hamiltonian vector fields in the new and old senses coincide. The rank of $\{\ ,\ \}$ is everywhere equal to the dimension of M. Conversely, if M is a Poisson manifold whose bracket has rank everywhere equal to the dimension of M, then (1.7) defines a symplectic form on M.

For an arbitrary Poisson manifold M, the distribution $Im\ J$ is integrable. The associated foliation is called the **Kirillov foliation** [We1]. It is, generally speaking, a singular foliation. The restriction of the Poisson bracket to each leaf defines a symplectic form, so the Kirillov foliation is made of symplectic manifolds. A function constant on the leaves of the Kirillov foliation is called a **Casimir**. It Poisson commutes with every other function.

EXAMPLE 1.2. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, and let $\mathfrak{g}^*$ be the space of linear functionals on $\mathfrak{g}$. Then $\mathfrak{g}^*$ becomes a Poisson manifold if we introduce the bracket

$$\{f,g\}(\xi) = < \xi, [d_\xi f, d_\xi g] >, \quad \xi \in \mathfrak{g}^*, f, g \in C^\infty(\mathfrak{g}^*). \tag{1.8}$$

This bracket is known as the **Lie-Poisson bracket**.

The symplectic leaves of the Kirillov foliation of $\mathfrak{g}^*$ coincide with the orbits of the co-adjoint action, so the Casimirs are just the invariants of this action.

A morphism in the Poisson category is a map $\Phi : M \to N$, between Poisson manifolds, preserving the Poisson brackets:

$$\{f,g\}_N \circ \Phi = \{f \circ \Phi, g \circ \Phi\}_M.$$

Direct products exist in the Poisson category.

A **Poisson Lie group** [Dr] is a Lie group G equipped with a Poisson bracket, such that the group operation $G \times G \to G$ is a Poisson map. A **Poisson action** of a Poisson Lie group G on a Poisson manifold M is a group action $\Psi : G \times M \to M$ which is also a Poisson map.

Example 1.3 [Lu]. Let G be a Lie group equipped with the zero Poisson bracket. It is a Poisson Lie group. If M is a Poisson manifold, and if $\Psi : G \times M \to M$ is a group action, then Ψ is a Poisson action iff for each fixed $g \in G$ the map $\Psi(g, \cdot) : M \to M$ is Poisson. Most symmetries of Hamiltonian systems are of this nature, and the associated theory is the well known Meyer-Marsden-Weinstein reduction ([Me,Ma-We,Ma-Ra]).

Example 1.4. Consider the realization of $SL(n, \mathbb{R})$ as the group of $n \times n$ matrices (s_{ij}) with determinant 1. The bracket

$$\{s_{ij}, s_{kl}\} = (\operatorname{sgn}(i-k) - \operatorname{sgn}(l-j))s_{il}s_{kj} \tag{1.9}$$

makes $SL(n, \mathbb{R})$ into a Poisson Lie group.

Let $P(n, \mathbb{R})$ be the set of positive-definite, symmetric, $n \times n$ matrices (p_{ij}) with determinant 1. The Poisson bracket

$$\{p_{ij}, p_{kl}\} = (\operatorname{sgn}(i-k) - \operatorname{sgn}(l-j))p_{il}p_{kj} + (\operatorname{sgn}(j-k) - \operatorname{sgn}(l-i))p_{jl}p_{ik} \tag{1.10}$$

makes $P(n, \mathbb{R})$ into a Poisson manifold

Define a group action Ψ of $SL(n, \mathbb{R})$ on $P(n, \mathbb{R})$ as follows. If $g \in SL(n, \mathbb{R})$ and $p \in P(n, \mathbb{R})$, we set $\Psi(g, p) = g \cdot p \cdot g^T$, where the superscript denotes transposition. This defines a Poisson action.

In chapter 3, we shall return to the geometry of Poisson actions, where we will explain how the brackets (1.9) and (1.10) were obtained.

2. Poisson pairs

Most of the results in this section can be found in the unpublished notes of Magri and Morosi [Ma-Mo]. However, we simplify some of the proofs.

Definition 2.1. A **Poisson pair** on a manifold M is a compatible pair (Λ_0, Λ_1) of Poisson bivectors on M, i.e., Λ_0 and Λ_1 are bivectors on M such that

$$[\Lambda_0, \Lambda_0] = [\Lambda_1, \Lambda_1] = 0 \tag{2.1}$$

$$[\Lambda_0, \Lambda_1] = 0. \tag{2.2}$$

A **bi-Hamiltonian system** is prescribed by specifying two Hamiltonian functions $h_0,\, h_1 \in C^\infty(M)$ satisfying

$$X = J_1 dh_0 = J_0 dh_1, \tag{2.3}$$

where J_i, i=0,1, are the bundle maps determined by Λ_i. The vector field X is said to be a **bi-Hamiltonian vector field**.

One of the basic problems we shall address is the relationship between the compatibility condition (2.2) and the integrability of the bi-Hamiltonian system. First we look at an example:

EXAMPLE 2.2. PERIODIC TODA LATTICE [Ar-Gi]. This system can be described as a 1-dimensional infinite lattice, where each spring has a potential $\phi(r) = e^r - r$, and where one identifies the particles i and i+N. The equations of motion are:

$$\frac{d^2 y_i}{dt^2} = \phi'(y_i - y_{i+1}) - \phi'(y_{i-1} - y_i) \qquad\qquad (y_{i+N} = y_i).$$

Define the new variables:

$$q_i = y_i - y_{i+1} \qquad , \qquad p_i = \dot{y}_i.$$

The equations of motion become

$$\begin{cases} \dot{q}_i = p_i - p_{i+1} \\ \\ \dot{p}_i = e^{q_{i-1}} - e^{q_i} \end{cases} \qquad\qquad i = 1, \ldots, N$$

One checks that the system is bi-Hamiltonian with respect to the Poisson structures

$$\Lambda_0 = \sum_{i=1}^{N} \left(\frac{\partial}{\partial q_{i-1}} - \frac{\partial}{\partial q_i} \right) \wedge \frac{\partial}{\partial p_i}$$

$$\Lambda_1 = \sum_{i=1}^{N} \left(\frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial q_{i+1}} + p_i \left(\frac{\partial}{\partial q_{i-1}} - \frac{\partial}{\partial q_i} \right) \wedge \frac{\partial}{\partial p_i} + e^{q_i} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} \right)$$

and Hamiltonian functions

$$h_0 = \sum_{i=1}^{N} p_i \qquad h_1 = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + q_i.$$

In this example, the Poisson pair is **non-degenerate** since the Poisson structure Λ_0 is symplectic. We shall consider degenerate cases later on in this chapter. If we assume a non-degenerate pair we can make the following definition.

DEFINITION 2.3. The **recursion operator** associated with a non-degenerate pair (Λ_0, Λ_1) is the (1,1)-tensor N defined by

$$N = J_1 J_0^{-1}$$

A first consequence of the compatibility condition is the following result.

PROPOSITION 2.4. *The Nijenhuis torsion of the recursion operator N associated with a Poisson pair (Λ_0, Λ_1) vanishes.*

PROOF. We recall the following expressions for the Nijenhuis torsion and the Schouten bracket, resp.[2]:

$$T_N(X,Y) = L_{NX}(NY) - NL_{NX}(Y) - NL_X(NY) + N^2 L_X(Y), \qquad \forall X, Y \in \mathfrak{X}(M) \tag{2.4}$$

$$\begin{aligned} 2\big[\Lambda_0, \Lambda_1\big](\alpha,\beta) = {} & L_{J_0\beta}(J_1)\alpha + J_1 L_{J_0\alpha}(\beta) + J_1 d\left\langle \alpha, J_0\beta\right\rangle \\ & + L_{J_1\beta}(J_0)\alpha + J_0 L_{J_1\alpha}(\beta) + J_0 d\left\langle \alpha, J_1\beta\right\rangle, \quad \forall \alpha,\beta \in \Omega^1(M) \end{aligned} \tag{2.5}$$

where L denotes the Lie derivative. Now given $X, Y \in \mathfrak{X}(M)$ let $\alpha, \beta \in \Omega^1(M)$ be such that $X = J_0\alpha, Y = J_0\beta$ (α and β exist since Λ_0 is symplectic). Using (2.4) and (2.5) we find

$$T_N(X,Y) = -N^2\big[\Lambda_0, \Lambda_0\big](\alpha,\beta) + 2N\big[\Lambda_0, \Lambda_1\big](\alpha,\beta) - \big[\Lambda_1, \Lambda_1\big](\alpha,\beta),$$

and the result follows. □

[2]The symbols $\mathfrak{X}(M)$ and $\Omega^1(M)$ denote, resp., the spaces of vector fields and differential 1-forms on M.

As in the Hamiltonian case, we define a **locally bi-Hamiltonian vector field** to be a vector field X for which there exists closed 1-forms α and β satisfying

$$X = J_1\alpha = J_0\beta.$$

In the case where both Poisson structures are symplectic, a vector field X is locally bi-Hamiltonian iff it is an infinitesimal automorphism of the Poisson bi-structure, i.e. if it satisfies

$$L_X\Lambda_0 = L_X\Lambda_1 = 0. \tag{2.6}$$

Henceforth we denote by $\mathfrak{X}_{BH}$ the vector space of locally bi-Hamiltonian vector fields.

A second consequence of the compatibility condition is the following result.

PROPOSITION 2.5. *The space of locally bi-Hamiltonian vector fields is a Lie algebra (with the usual bracket) which is invariant under the action of the recursion operator N. If $X, Y \in \mathfrak{X}_{BH}$ then*

$$N[X,Y] = [NX,Y] = [X,NY]. \tag{2.7}$$

In particular, given $X_1 \in \mathfrak{X}_{BH}$ the sequence $X_r = N^{r-1}X_1, r = 1, 2, ...$ forms a set of mutually commuting vector fields.

PROOF. The first statement follows easily from the formula

$$[X,Y] = J_i\left(d < \beta, X >\right), \qquad \text{where } Y \equiv J_i\beta.$$

Now suppose $X \in \mathfrak{X}_{BH}$. We claim that then $NX \in \mathfrak{X}_{BH}$. In fact, if $X = J_1\alpha = J_0\beta$ for some closed 1-forms α and β, we have

$$NX = J_0N^*\beta = J_1\beta,$$

so all we have to show is that $N^*\beta$ is closed (here N^* denotes the adjoint of N relative to the natural pairing between vectors and forms). Now, from proposition 2.4 and (2.4) we obtain

$$L_{NX}(N) - NL_X(N) = 0, \qquad \forall X \in \mathfrak{X}(M), \tag{2.8}$$

and dualizing we find

$$L_{NX}(N^*) - L_X(N^*)N^* = 0, \qquad \forall X \in \mathfrak{X}(M), \tag{2.9}$$

We can use (2.9) and the closure of α and β to get

$$\begin{aligned} < dN^*\beta, X > &= L_X(N^*\beta) - d < N^*\beta, X > \\ &= L_X(N^*\beta) - d < \beta, NX > \\ &= L_X(N^*\beta) - L_{NX}(\beta) \\ &= N^* \left(L_X(N^*\alpha) - L_{NX}(\alpha)\right) + \left(L_X(N^*)N^* - L_{NX}(N^*)\right)(\alpha) \\ &= N^* < d(N^*\alpha), X > \\ &= N^* < d\beta, X >= 0. \end{aligned}$$

Finally, to prove (2.7) we observe that by (2.6), we have for every $X, Y \in \mathfrak{X}_{BH}$

$$L_X(N) = L_Y(N) = 0,$$

and, in particular, it follows that

$$L_X(N)(Y) = L_Y(N)(X) = 0,$$

which gives

$$[NX, Y] = N[X, Y] = [X, NY],$$

as desired. □

An important corollary is the following integrability result due to Magri [Ma]:

Theorem 2.6. *Consider a bi-Hamiltonian system*

$$\dot{x} = J_1 dh_0 = J_0 dh_1,$$

on a manifold M, whose first cohomology group is trivial. Then there exists a hierarchy of mutually commuting bi-Hamiltonian functions $h_0, h_1, h_2, ...$, *all in involution to each other, with respect to both Poisson brackets. They generate mutually commuting bi-Hamiltonian flows* $X_i,\ i = 1, 2, \dots,$ *satisfying the Lenard recursion relations*

$$X_{i+j} = J_i dh_j, \tag{2.10}$$

where $J_i \equiv N^i J_0,\ i = 1, 2, \dots$ *are the higher order Poisson tensors.*

If enough of the integrals given by Magri's Theorem are functionally independent then the system is completely integrable (in Arnold's sense, [Ar, chapt.10]). This happens in the case of the periodic Toda lattice where one obtains a complete set of highly non-trivial first integrals.

3. Mastersymmetries and bi-Hamiltonian systems

We recall some basic facts of the theory of mastersymmetries for differential equations. More details can be found in Fuchssteiner [Fu].

Consider a differential equation on a manifold M:

$$\dot{x} = X(x). \tag{3.1}$$

As usual, a vector field Y is a **symmetry** of (3.1) if

$$[Y, X] = 0. \tag{3.2}$$

More generally, a family $Y = Y(x, t)$ of vector fields depending smoothly on t is a **time-dependent symmetry** of (3.1) if

$$\frac{\partial Y}{\partial t} + [Y, X] = 0. \tag{3.3}$$

We view Y as a time-dependent vector field.

We can generalize (3.2) as follows. A vector field Z is called a **generator of degree n** if

$$[[\dots[Z,X],\dots],X],X]] = 0.$$

If Z is a generator of degree n then the time-dependent vector field

$$Y_Z = \exp(ad\ X)\cdot Z = \sum_{k=0}^{n} \frac{t^k}{k!}[[\dots[Z,X],\dots],X],X]] \tag{3.4}$$

satisfies (3.3), and so is a time-dependent symmetry of (3.1). Thus, t-time dependent symmetries which are polynomial in t are in 1-1 correspondence with generators of degree n.

Generators satisfy the following properties:

(i) If Z is a generator of degree n, then $[Z,X]$ is a generator of degree n-1;

(ii) If Z_1 and Z_2 are generators of degree n_1 and n_2, then $[Z_1,Z_2]$ is a generator of degree n_1+n_2;

(iii) A symmetry is a generator of degree 0;

In particular we see that the set of all generators form a Lie subalgebra of the algebra of all vector fields $\mathfrak{X}(M)$. We shall call a generator of degree 1 a **mastersymmetry**. Thus the condition for Z to be a mastersymmetry is:

$$[[Z,X],X] = 0, \quad \text{and} \quad [Z,X] \neq 0.$$

PROPOSITION 3.1. *Let Z be a mastersymmetry. Then:*

(i) $[Z,X]$ is an ordinary symmetry;

(ii) $[Z,[Z,X]]$ is an ordinary symmetry;

PROOF. It is obvious from the definitions and the Jacobi identity. □

In general, given a mastersymmetry all we get is the two symmetries given in the proposition. However, under an additional assumption, we can generate further symmetries as follows.

PROPOSITION 3.2. *Suppose Y is a symmetry of (3.1) which commutes with every other symmetry, and let Z be a mastersymmetry. Then $[Z, Y]$ is also a symmetry.*

PROOF. Use Jacobi identity again. □

On a manifold M, whose first cohomology vanishes, we consider a bi-Hamiltonian system

$$X_1 = J_1 dh_0 = J_0 dh_1, \tag{3.5}$$

where J_i are compatible Poisson tensors, and h_i are the Hamiltonian functions. As before, we assume that J_0 is symplectic, so we can introduce the recursion operator

$$N = J_1 J_0^{-1},$$

the higher order flows

$$X_i = N^{i-1} X_1, \quad (i = 1, 2, \dots),$$

and the higher order Poisson tensors

$$J_i = N^i J_0, \quad (i = 1, 2, \dots).$$

The Hamiltonians $\{h_i\}$ satisfy

$$dh_i = (N^*)^i dh_0, \quad (i = 1, 2, \dots),$$

where N^* denotes the adjoint of N. These relations are equivalent to the Lenard recursion relations:

$$X_{i+j} = J_i dh_j.$$

For a bi-Hamiltonian system mastersymmetries can be obtained from the following result due to Oevel [Oe]:

Theorem 3.3. *Suppose that Z_0 is a conformal symmetry for both J_0, J_1 and h_0, i.e. for some scalars λ, μ, and ν we have*

$$L_{Z_0} J_0 = \lambda J_0, \quad L_{Z_0} J_1 = \mu J_1, \quad L_{Z_0} h_0 = \nu h_0.$$

Then the vector fields

$$Z_i = N^i Z_0$$

satisfy

(a) $[Z_i, X_j] = \big(\mu + \nu + (j-1)(\mu - \lambda)\big) X_{i+j}$;

(b) $[Z_i, Z_j] = (\mu - \lambda)(j - i) Z_{i+j}$;

(c) $L_{Z_i} J_j = \big(\mu + (j - i - 1)(\mu - \lambda)\big) J_{i+j}$;

The set of first integrals $\{h_i\}$ can be obtained from the formula

(d) $Z_i \lrcorner\, dh_j = \big(\nu + (i+j)(\mu - \lambda)\big) h_{i+j}$.

Proof. (a) We compute

$$\begin{aligned}
[Z_i, X_j] &= -L_{X_j}(N^i Z_0) \\
&= -N^i L_{X_j}(Z_0) \\
&= N^i [Z_0, X_j],
\end{aligned}$$

because any bi-Hamiltonian vector field X satisfies $L_X(N) = 0$. Therefore

$$\begin{aligned}
[Z_i, X_j] &= N^i L_{Z_0}(X_j) \\
&= N^i L_{Z_0}(N^{j-1}) X_1 + N^{i+j-1} L_{Z_0}(X_1).
\end{aligned}$$

But:

$$\begin{aligned}
L_{Z_0} N &= L_{Z_0}(J_1) J_0^{-1} - J_1 J_0^{-1} L_{Z_0}(J_0) J_0^{-1} \\
&= \mu J_1 J_0^{-1} - \lambda J_1 J_0^{-1} \\
&= (\mu - \lambda) N,
\end{aligned}$$

$$\begin{aligned} L_{Z_0}X_1 &= L_{Z_0}(J_1 dH_0) \\ &= L_{Z_0}(J_1)dH_0 + J_1 d(L_{Z_0}H_0) \\ &= (\mu+\nu)J_1 dH_0 \\ &= (\mu+\nu)X_1, \end{aligned}$$

so we conclude that

$$\begin{aligned} [Z_i, X_j] &= (\mu-\lambda)(j-1)N^{i+j-1}X_1 + (\mu+\nu)N^{i+j-1}X_1 \\ &= \big(\mu+\nu+(j-1)(\mu-\lambda)\big)X_{i+j}. \end{aligned}$$

(b) From (2.8) we compute

$$\begin{aligned} [Z_i, Z_j] &= [N^i Z_0, N^j Z_0] \\ &= L_{N^i Z_0}(N^j)Z_0 + N^j L_{N^i Z_0}(Z_0) \\ &= N^i L_{Z_0}(N^j)Z_0 - N^j L_{Z_0}(N^i Z_0) \\ &= N^i j(\mu-\lambda)N^j Z_0 - N^j i(\mu-\lambda)N^i Z_0 \\ &= (\mu-\lambda)(j-i)N^{i+j}Z_0 \\ &= (\mu-\lambda)(j-i)Z_{i+j}. \end{aligned}$$

(c) From (2.5), we see that $[\Lambda_1, \Lambda_0] = 0$ is equivalent to:

$$L_{J_0\beta}(J_1)\alpha + J_1 < d\beta, J_0\alpha > + L_{J_1\beta}(J_0)\alpha + J_0 < d\beta, J_1\alpha >= 0, \quad \alpha, \beta \in \Omega(M).$$

Similarly, $[\Lambda_0, \Lambda_0] = 0$ is equivalent to:

$$L_{J_0\beta}(J_0)\alpha + J_0 < d\beta, J_0\alpha >= 0, \quad \alpha, \beta \in \Omega(M),$$

which gives

$$\begin{aligned} < d\beta, J_0\alpha > &= -J_0^{-1}L_{J_0\beta}(J_0)\alpha, \\ < d\beta, J_1\alpha > &= -J_0^{-1}L_{J_0\beta}(J_0)J_0^{-1}J_1\alpha. \end{aligned}$$

Putting all this together, replacing $\beta = J_0^{-1}Z$, and using $N = J_1 J_0^{-1}$, we find

$$L_{NZ}(J_0)\alpha = N L_Z(J_0)\alpha + L_Z(J_0)N^*\alpha - L_Z(J_1)\alpha.$$

This last formula gives a special case of (c):

$$\begin{aligned} L_{Z_1} J_0 &= L_{NZ_0} J_0 \\ &= \lambda N J_0 + \lambda J_0 N^* - \mu J_1 \\ &= (2\lambda - \mu) J_1. \end{aligned}$$

Using induction, one can show that

$$\begin{aligned} L_{Z_i} J_0 &= L_{N^i Z_0} J_0 \\ &= \big(\lambda - i(\mu - \lambda)\big) N^i J_0 \\ &= \big(\lambda - i(\mu - \lambda)\big) J_i. \end{aligned}$$

Finally the general case follows:

$$\begin{aligned} L_{Z_i} J_j &= L_{Z_i}(N^j J_0) \\ &= L_{N^i Z_0}(N^j) J_0 + N^j \Lambda_{Z_i} J_0 \\ &= j N^{i+j-1} L_{Z_0}(N) J_0 + \big(\lambda - i(\mu - \lambda)\big) N^{i+j} J_0 \\ &= \big(\lambda + (j - i)(\mu - \lambda)\big) N^{i+j} J_0 \\ &= \big(\lambda + (j - i)(\mu - \lambda)\big) J_{i+j}. \end{aligned}$$

(d) It is enough to prove the differential version of (d), which can be done as follows:

$$\begin{aligned} d\langle dh_j, Z_i\rangle &= d\langle (N^*)^j dh_0, N^i Z_0\rangle \\ &= L_{Z_0}\big((N^*)^{i+j} dh_0\big) \\ &= L_{Z_0}\big((N^*)^{i+j}\big) dh_0 + (N^*)^{i+j} L_{Z_0} dh_0 \\ &= (i + j)(N^*)^{i+j-1} L_{Z_0}(N^*) dh_0 + \nu (N^*)^{i+j} dh_0 \\ &= (i + j)(\mu - \lambda)(N^*)^{i+j} dh_0 + \nu (N^*)^{i+j} dh_0 \\ &= \big(\nu + (i + j)(\mu - \lambda)\big) dh_{i+j}. \quad \square \end{aligned}$$

¿From the observations on mastersymmetries made above, we obtain:

Corollary 3.4. *Under the hypothesis of the theorem, for each integer* $i = 1, 2, \ldots,$ *the vector fields*

$$Y_{Z_j} = Z_j + t\big(\mu + \nu + (i-1)(\mu - \lambda)\big) X_{i+j}, \quad j = 1, 2, \ldots$$

are time-dependent symmetries of the i^{th}*-order flow.*

Proof. Each $Z_j,\ j = 1, 2, \ldots,$ is a mastersymmetry. Using (3.4) and relation (a) of the the theorem we compute:

$$\begin{aligned} Y_{Z_j} &= Z_j + t[Z_j, X_i] \\ &= Z_j + t\big(\mu + \nu + (i-1)(\mu - \lambda)\big) X_{i+j}. \quad \square \end{aligned}$$

4. Example: The Toda Lattice

In example 2.2 we considered briefly the periodic Toda lattice. In this section we illustrate all the concepts introduced above for the finite, non-periodic, Toda lattice, which leads to a degenerate Poisson pair in the so-called Flaschka's variables.

In the letter [Da1], a construction of mastersymmetries and deformation relations for the Toda lattice was given, as well as its connection with the R-matrix approach, in terms of the Flaschka's variables. On the other hand, theorem 3.3 above relates mastersymmetries to a conformal symmetry of the system, when a recursion operator is available. We shall relate these two constructions using a reduction.

Our approach consists in working in the physical variables, and then reducing to Flaschka's variables. Although the recursion operator itself cannot be reduced (this is also observed in [Mo-To]), the deformation relations and mastersymmetries do reduce. One advantage of this approach is that it yields immediately a hierarchy of time-dependent symmetries for the Toda lattice. Also, we believe that this approach is more natural. For example, the deformation relations, previously known to hold up to a certain equivalence relation [Da1], are shown to be exact.

The finite, non-periodic, Toda lattice, is a system of particles on the line under exponential interaction with nearby particles. It has the following bi-Hamiltonian formulation:

$$
\begin{aligned}
J_0 &= \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \\
J_1 &= \sum_{i=1}^{n-1} 2e^{2(q^i - q^{i+1})} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \\
&\qquad + \frac{1}{2} \sum_{i<j} \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial q^i}
\end{aligned} \tag{4.1}
$$

$$
h_0 = \sum_{i=1}^{n} p_i \qquad h_1 = \sum_{i=1}^{n} \frac{{p_i}^2}{2} + \sum_{i=1}^{n-1} e^{2(q^i - q^{i+1})}
$$

Note that J_0 is symplectic. The recursion operator is then

$$
\begin{aligned}
N &= \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \otimes dq^i \\
&+ \sum_{i=1}^{n-1} 2e^{2(q^i - q^{i+1})} \Big(\frac{\partial}{\partial p_{i+1}} \otimes dq^i - \frac{\partial}{\partial p_i} \otimes dq^{i+1} \Big) \\
&+ \frac{1}{2} \sum_{i<j} \Big(\frac{\partial}{\partial q^i} \otimes dp_j - \frac{\partial}{\partial q^j} \otimes dp_i \Big) + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \otimes dp_i .
\end{aligned} \tag{4.2}
$$

We will now show that the vector field

$$
Z_0 = \sum_{i=1}^{n} \frac{n+1-2i}{2} \frac{\partial}{\partial q^i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial p_i} \tag{4.3}
$$

is a conformal symmetry for both J_0, J_1 and h_0, so we will be able to apply theorem 3.3 and its corollary.

In fact, we compute:

$$
\begin{aligned}
L_{Z_0} J_0 &= \sum_{i,j=1}^{n} \Big[\frac{n+1-2j}{2} \frac{\partial}{\partial q^j}, \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \Big] + \sum_{i,j=1}^{n} \Big[p_j \frac{\partial}{\partial p_j}, \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \Big] \\
&= 0 - \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \\
&= -J_0 .
\end{aligned}
$$

Next observe that Z_0 is a Hamiltonian vector field with respect to J_1 for the Hamiltonian $f = \sum_i q^i$, so we have

$$L_{Z_0} J_1 = 0.$$

Finally, a simple computation shows that $L_{Z_0} h_0 = h_0$.

Therefore theorem 3.3 holds with $\lambda = -1$, $\mu = 0$, $\nu = 1$. It follows that the higher order Poisson tensors for the Toda lattice satisfy the deformation relations:

$$L_{Z_i} J_j = (j - i - 1) J_{i+j}, \tag{4.4}$$

$$L_{Z_i} h_j = (i + j + 1) h_{i+j}, \tag{4.5}$$

where $Z_i \equiv N^i Z_0$ satisfy

$$[Z_i, Z_j] = (j - i) Z_{i+j}. \tag{4.6}$$

If we denote by X_i the Hamiltonian vector field generated by h_i, with respect to J_0, we also have

$$[Z_i, X_j] = j X_{i+j}, \tag{4.7}$$

and from the corollary we obtain the time-dependent symmetries

$$Y_{Z_j} \equiv Z_j + it X_{i+j}, \quad j = 1, 2, \ldots. \tag{4.8}$$

Another multi-Hamiltonian formulation is known for Toda lattice in terms of the Flaschka's variables. Recall that the Flaschka transformation is the map $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ defined by

$$(q^1, \ldots, q^n, p_1, \ldots, p_n) \mapsto (a_1, \ldots, a_{n-1}, b_1, \ldots, b_n),$$

where $a_i = e^{(q^i - q^{i+1})}$, $b_i = p_i$. The Poisson tensors J_0 and J_1 reduce to $\mathbb{R}^{2n-1}$. This can be checked directly by setting

$$
\begin{aligned}
\tilde{J}_0 &= \sum_{i=1}^{n-1} a_i \Big(\frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_i} - \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_{i+1}} \Big) \\
\tilde{J}_1 &= \sum_{i=1}^{n-1} \Big(a_i b_i \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_i} - a_i b_{i+1} \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_{i+1}} \Big) \\
&\quad + \sum_{i=1}^{n-1} 2{a_i}^2 \frac{\partial}{\partial b_{i+1}} \wedge \frac{\partial}{\partial b_i} + \sum_{i=1}^{n-2} \frac{1}{2} a_i a_{i+1} \frac{\partial}{\partial a_{i+1}} \wedge \frac{\partial}{\partial a_i}
\end{aligned} \tag{4.9}
$$

and observing that the projection $\pi : (\mathbb{R}^{2n}, J_i) \to (\mathbb{R}^{2n-1}, \tilde{J}_i)$ is a Poisson morphism.

The bi-Hamiltonian formulation for the Toda lattice in the Flaschka's variables, is exactly the one defined by the Poisson tensors (4.9), and the reduced Hamiltonians

$$
\tilde{h}_0 = \sum_{i=1}^{n} b_i \qquad \tilde{h}_1 = \sum_{i=1}^{n} \frac{{b_i}^2}{2} + \sum_{i=1}^{n-1} {a_i}^2 .
$$

There is however a big difference between the original bi-Hamiltonian formulation and the reduced bi-Hamiltonian formulation. The Poisson structures $\tilde{J}_0$ and $\tilde{J}_1$ are not symplectic, and so there is no obvious recursion operator. In fact, the recursion operator N given by (4.2) cannot be reduced. This is most easily seen using the notion of projectable vector field. Recall that a vector field Z is *projectable* if for every vector field Y tangent to the fibers $\pi^{-1}(x)$, the vector field $L_Y Z$ is also tangent to the fibers. If that is the case, the vector field Z can be reduced to a vector field $\tilde{Z}$ given by $\tilde{Z}(\pi(x)) = d\pi(x) \cdot Z(x)$. Conversely, any vector field on the reduced space is the image by π of a projectable vector field.

Now we claim that N does not map projectable vector fields to projectable vector fields, as it is required for reduction to work. To prove this we note that the fibers are the lines in $\mathbb{R}^{2n}$ parallel to the vector $(1, \ldots, 1, 0, \ldots, 0)$, so a vector field Z is projectable iff for every function $f \in C^\infty(M)$ there exists a function $g \in C^\infty(M)$ such that

$$
\Big[Z, f \sum_i \frac{\partial}{\partial q^i} \Big] = g \sum_i \frac{\partial}{\partial q^i} .
$$

For example, the vector field $\frac{\partial}{\partial q^i}$ is projectable but

$$N\frac{\partial}{\partial q^i} = p_i\frac{\partial}{\partial q^i} - 2e^{2(q^{i-1}-q^i)}\frac{\partial}{\partial p^{i-1}} + 2e^{2(q^i-q^{i+1})}\frac{\partial}{\partial p^{i+1}},$$

is not projectable. We conclude that no recursion operator exists relating the two reduced Poisson tensors.

In spite of the fact that there is no recursion operator for the reduced Toda lattice, higher order Poisson structures are known, and they satisfy certain deformation relations [Da1]. This can be explained by the following result.

THEOREM 4.1. *The vector fields* $Z_i = N^i Z_0$, $i = 0, 1, 2, \ldots,$ *are projectable. The corresponding reduced vector fields satisfy*

$$[\tilde{Z}_i, \tilde{Z}_j] = (j-i)\tilde{Z}_{i+j}. \tag{4.10}$$

In particular, the higher order Poisson tensors can be reduced to Poisson tensors $\tilde{J}_i$, *satisfying the deformation relations*

$$L_{Z_i}\tilde{J}_j = (j-i-1)\tilde{J}_{i+j}. \tag{4.11}$$

There are also reduced Hamiltonians $\{\tilde{h}_i\}$ *and reduced higher order flows* $\tilde{X}_i$ *satisfying:*

$$L_{\tilde{Z}_i}\tilde{h}_j = (i+j+1)\tilde{h}_{i+j}, \tag{4.12}$$

$$[\tilde{Z}_i, \tilde{X}_j] = j\tilde{X}_{i+j}. \tag{4.13}$$

PROOF. All we have to prove is that the vector fields $\tilde{Z}_i$ are projectable, so the all hierarchy can be reduced. The rest of the proposition follows from relations (4.4)-(4.7).

We compute:

$$\begin{aligned}\Big[N, \sum_{i=1}^{n}\frac{\partial}{\partial q^i}\Big] &= \sum_{i=1}^{n-1} 2e^{2(q^i-q^{i+1})}\Big(\frac{\partial}{\partial p_i}\otimes dq^{i+1} - \frac{\partial}{\partial p_{i+1}}\otimes dq^i\Big)\\ &\quad - \sum_{i=2}^{n} 2e^{2(q^{i-1}-q^i)}\Big(\frac{\partial}{\partial p_{i-1}}\otimes dq^i - \frac{\partial}{\partial p_i}\otimes dq^{i-1}\Big) = 0.\end{aligned}$$

Therefore, for any $f \in C^\infty(M)$ we find

$$\begin{aligned}\Big[N^i Z_0, f\sum_i \frac{\partial}{\partial q^i}\Big] &= (N^i Z_0)(f)\sum_i \frac{\partial}{\partial q^i} - f\Big[N^i Z_0, \sum_i \frac{\partial}{\partial q^i}\Big] \\ &= g\sum_i \frac{\partial}{\partial q^i} - fN^i\Big[Z_0, \sum_i \frac{\partial}{\partial q^i}\Big] \\ &= g\sum_i \frac{\partial}{\partial q^i}\end{aligned}$$

so the $Z_i = N^i Z_0,\ i = 0, 1, 2, \ldots,$ are projectable. $\square$

The deformation relations (3.11) where known to hold up to a certain equivalence relation [Da1]. Our proof shows that they are actually exact. We note that the mastersymmetries $\{\tilde{Z}_i\}$, for $i \geq 2$, are different from the mastersymmetries given in [Da1]. However, for $i = 1$ they differ by a multiple of the Hamiltonian vector field X_1, and so the higher order reduced Poisson tensors (4.11), coincide with the ones given in [Da1].

It follows, exactly as in corollary 3.4, the existence of a hierarchy of reduced time-dependent symmetries:

Corollary 4.2. *For each integer $i = 1, 2, \ldots,$ the vector fields*

$$Y_{\tilde{Z}_j} \equiv \tilde{Z}_j + it\tilde{X}_{i+j}, \quad j = 1, 2, \ldots$$

are time-dependent symmetries of the i^{th}-order Toda flow.

We have learned during our research, that corollary 4.2 has also been obtained in [Da2], although through different methods.

CHAPTER II

COMPLETE INTEGRABILITY $\Rightarrow$ BI-HAMILTONIAN STRUCTURE ?

The main purpose of this chapter is to answer the question: does every completely integrable system have a bi-Hamiltonian formulation ?

The study of completely integrable Hamiltonian systems, i.e., systems admitting a complete sequence of first integrals, started with the pioneering work of Liouville [Lio] on finding local solutions by quadrature. We have now a complete picture of the semi-local geometry of such systems, which in its modern presentation is due to Arnol'd [Ar]. A major flaw in the Arnol'd-Liouville theory is that it provides no indication on how to obtain first integrals, and this is one of the reasons for the growing interest on bi-Hamiltonian systems.

We saw in the previous chapter, that for a given bi-Hamiltonian system, Magri's theorem yields a whole hierarchy of first integrals. Under additional assumptions on the algebraic structure of the pair, one obtains a complete sequence of first integrals. Moreover, this assumption may be formulated in a way that still makes sense in the setting of infinite dimensional systems. Therefore, if one wants to extend the notion of complete integrability, the following natural question arises: Given a completely integrable Hamiltonian system, does the complete sequence of first integrals arise from a second Hamiltonian structure via Magri's theorem?

This problem was first studied by Magri and Morosi in their unpublished notes [Ma-Mo], which seems to contain an incorrect answer. More recently, R. Brouzet in his "thése de doctorat", studied the same question when the dimension equals four, and showed that the answer in general is negative [Br]. Theorem 3.1 below shows that a second Hamiltonian structure exists if and only if the Hamiltonian function satisfies a certain geometric condition. This condition may be expressed in an invariant way, using the action-angle variables, as a restriction on the Hamiltonian.

A related problem was considered by De Filippo et al. [Fi]. They showed that a completely integrable system always has a bi-Hamiltonian formulation, but we remark that, in general, neither of the Poisson structures coincides with the given one. The situation we are considering here is more restrictive. We fix the original Hamiltonian structure, and ask under what conditions there exists a second Hamiltonian structure such that the sequence of first integrals is obtained via Magri's result.

This chapter is organized as follows. In section 1, we review the Arnol'd-Liouville theory of completely integrable systems, with an emphasis on action-angle variables. In section 2, starting with a bi-Hamiltonian system for which Magri's theorem gives a complete sequence of first integrals, we construct a set of coordinates which have the property of splitting both the Hamiltonian and the action variables. The construction is based on the systems of first order p.d.e.'s

$$\nabla F = B\nabla G,$$

first studied by Olver [Ol], who used it to give a local classification of bi-Hamiltonian systems. In section 3, we interpret this property as a geometric condition on the graph of the Hamiltonian, and show how it can be used to rule out the existence of a second Hamiltonian structure. Conversely, we show that any completely integrable system whose Hamiltonian satisfies this condition has a bi-Hamiltonian formulation. In section 4, we give several examples illustrating these results. In particular, we give an example of a completely integrable Hamiltonian system, which is not bi-Hamiltonian in the sense just described, but admits a degenerate bi-Hamiltonian structure. This explains why a bi-Hamiltonian formulation is not known for many of the classical examples of completely integrable systems: in general, degenerate Poisson pairs will be required.

1. Arnol'd-Liouville theory revisited

Given a Hamiltonian system (M^{2n}, ω, h) on a symplectic manifold, we say that the system is **completely integrable** if there exists n independent functions $f_1 = h, f_2, \ldots, f_n$, with pairwise vanishing Poisson brackets

$$\{f_i, f_j\} = 0.$$

The geometry of completely integrable systems is described by the so-called Arnol'd-Liouville theorem.

THEOREM 1.1 [Ar]. *Let $\pi : M \to \mathbb{R}^n$ be the fibration $x \to (f_1(x), \ldots, f_n(x))$. Then:*

(i) π is a Lagrangian fibration and each connected component of $\pi^{-1}(c)$ is a cylinder. There exists affine coordinates $(\theta^1, \ldots, \theta^n)$ on $\pi^{-1}(c)$ which straightens out the Hamiltonian flow, i.e., $\dot{\theta}^i = const$;

(ii) If $\pi^{-1}(c)$ is connected and compact then it is a topological torus $\mathbb{T}^n$. There exists a neighborhood U of $\pi^{-1}(c)$ and a trivialization $(s^1, \ldots, s^n, \theta^1, \ldots, \theta^n) : U \to \mathbb{R}^n \times \mathbb{T}^n$ such that $\omega = \sum_i ds^i \wedge d\theta^i$;

For the remaining of the chapter we restrict ourselves to the compact case. For this, it is enough to guarantee that at least one non-degenerate surface level $f_i = c$ is compact (this is often the case with the energy surface h=c).

Suppose that we started with a different complete sequence of commuting first integrals $\tilde{f}_1 = h, \tilde{f}_2, \ldots, \tilde{f}_n$, and assume that

$$det\ (\partial^2 h / \partial s^i \partial s^j) \neq 0. \tag{1.1}$$

This is a non-degeneracy condition which is most often used in the following form: any first integral depends only on the action variables. In fact, (1.1) implies that the non-resonant tori are dense, but any first integral is constant on any such torus. Then by standard arguments in Poisson geometry, one has necessarily

$\tilde{f}_i = \tilde{f}_i(f_1, \ldots, f_n)$, $i = 1, \ldots, n$. Therefore, the Lagrangian foliation is uniquely determined, i.e., it does not depend on the particular choice of complete sequence of first integrals.

The variables (s^i, θ^i) referred to in the theorem, are the so-called **action-angle variables**. Explicit formulas for the action variables can be obtained as follows. Since each torus $\pi^{-1}(c)$ is a Lagrangian submanifold, on some neighborhood U of this level set the symplectic form is exact: $\omega = d\alpha$. The action variables are obtain by integration along a basis $(\gamma_1, \ldots, \gamma_n)$ of 1-cycles of the integral homology $H_1(\mathbb{T}^n, \mathbb{Z})$ of each torus:

$$s_i = \frac{1}{2\pi} \oint_{\gamma_i} \alpha.$$

If we change the basis of the homology group $H_1(\mathbb{T}^n, \mathbb{Z})$, the new action variables will be related with the old variables by an invertible, integral, linear transformation. Also, since we can choose any torus as the origin of our system of coordinates, the addition of a constant vector is also possible. We see that the action variables define a canonical **integral affine structure** on the open subset $U \subset \mathbb{R}^n$, i.e., they are unique up to translations and invertible, integral, linear transformations of $\mathbb{R}^n$ (for more details see [Ar-Gi]).

Note that the Arnol'd-Liouville theory tells nothing about how to find a complete set of first integrals. It is therefore interesting to study its relation with the theory of bi-Hamiltonian systems.

A first, rather trivial remark, is the following: any completely integrable Hamiltonian system X_h is bi-Hamiltonian in a neighborhood of an invariant torus [Fi]. An outline of the proof is as follows. There is a neighborhood where one can find coordinates, say $(x^1, \ldots, x^{2n})$, such that $X_h = \frac{\partial}{\partial x^1}$, so it is easy to construct some bi-Hamiltonian formulation for X_h. However, in general this coordinates will not be canonical, and this artificially constructed Poisson pair has no direct relationship with the original Poisson structure. In practice, some natural Poisson structure is known and one seeks a second one that might give the integrability of the system.

This is the problem we will consider for the remainder of this chapter.

2. Splitting variables for CIS

In this section we look at a completely integrable Hamiltonian system (CIS) (M^{2n}, ω, h) and ask if the complete sequence of integrals arises from a second Poisson structure via Magri's theorem.

The local problem is more or less trivial and is not so interesting from the point of view of the theory of integrability. We shall look rather at a neighborhood of a fixed invariant torus. By the Arnold-Liouville theorem a tubular neighborhood of the torus can be described by choosing action-angle variables (s^i, θ^i), so without loss of generality we can assume that M^{2n} is a product $\mathbb{R}^n \times \mathbb{T}^n$, where the original torus is identified with $\{0\} \times \mathbb{T}^n$ and $h = h(s^1, \ldots, s^n)$, $\omega = \sum_i ds^i \wedge d\theta^i$. The canonical projection $\pi : M \to \mathbb{R}^n$ is a Lagrangian fibration, so each torus $\pi^{-1}(x)$ is a Lagrangian submanifold and $Ker\ d_m\pi = (Ker\ d_m\pi)^{\perp}$. We make the following assumption.

(ND). *det* $(\partial^2 h/\partial s^i \partial s^j) \neq 0$ *in a dense set.*

This non degeneracy condition was explained in the previous section.

We want to investigate the existence of a second Poisson structure, possibly degenerate, giving the complete integrability of the system. Thus we consider an additional assumption:

(BH). *The system is bi-Hamiltonian with diagonalizable recursion operator N, having functionally independent real eigenvalues $\lambda_1, \ldots, \lambda_n$*[3]

¿From the Lenard's recursion relations (I.2.10) in Magri's theorem, it follows that the sequence $\lambda_1, \ldots, \lambda_n$, is a complete sequence of first integrals of the system.

Given a point $m \in M$ we denote by $E_\lambda(m)$ the (real) eigenspace of the recursion operator N belonging to a eigenvalue λ in the spectrum $\sigma(N) = \{\lambda_1, \ldots, \lambda_n\}$. For

[3] It seems possible to relax this condition by assuming only distinct eigenvalues. All the results that follow still hold in this more general setting. Here we consider only the functionally independent case in view of Magri's theorem and in order to keep technical details to a minimum.

diagonalizable (1,1)-tensors with vanishing Nijenhuis torsion we have the following classical result [Ni]:

PROPOSITION 2.1. *For any subset $S \subset \sigma(N)$ the distribution $m \to \oplus_{\lambda\in S} E_\lambda(m)$ is integrable. If $\mu \in \sigma(N)\backslash S$, then μ is an integral of the distribution.*

PROOF. Let X_λ and Y_μ be eigenvectors of N corresponding to eigenvalues λ and μ. From expression (I.2.4) for the Nijenhuis torsion of N we find:

$$(2.1)\quad 0 = T_N(X_\lambda, Y_\mu) = (N-\lambda I)(N-\mu I)[X_\lambda, Y_\mu] + (\lambda-\mu)\big\{(X_\lambda\cdot\mu)Y_\mu - (Y_\mu\cdot\lambda)X_\lambda\big\}$$

If one applies $(N-\lambda I)(N-\mu I)$ to both sides of this equation one gets:

$$[X_\lambda, Y_\mu] \in Ker\ (N-\lambda I)^2(N-\mu I)^2.$$

But N is diagonalizable, so we have

$$\begin{aligned} Ker\ (N-\lambda I)^2(N-\mu I)^2 &= Ker\ (N-\lambda I)^2 \oplus Ker\ (N-\mu I)^2 \\ &= Ker\ (N-\lambda I) \oplus Ker\ (N-\mu I), \end{aligned}$$

and the first part of the proposition follows. If $\lambda \neq \mu$, (2.1) now gives:

$$X_\lambda \cdot \mu = Y_\mu \cdot \lambda = 0,$$

so the second part also follows. $\square$

For $i = 1, \ldots, n$, we denote the foliations associated with the integrable distributions $m \to E_{\lambda_i}(m)$ and $m \to E_{\lambda_1}(m) \oplus \cdots \oplus \widehat{E_{\lambda_i}(m)} \oplus \cdots \oplus E_{\lambda_n}(m)$ by Φ_i and Δ_i, respectively (here $\hat{E}$ means omit the factor E). It is obvious from these definitions that Φ_i and Δ_i are transversal. In fact, we have the following stronger result:

PROPOSITION 2.2. *If $i \neq j$ the foliations Φ_i and Φ_j are ω-orthogonal. In particular, one has $\Phi_i^\perp = \Delta_i$.*

PROOF. If X_λ and X_μ are eigenvectors of N corresponding to distinct eigenvalues λ and μ we find:

$$\begin{aligned}\lambda\omega(X_\lambda, X_\mu) &= \omega(\lambda X_\lambda, X_\mu)\\ &= \omega(NX_\lambda, X_\mu)\\ &= \omega(X_\lambda, NX_\mu)\\ &= \omega(X_\lambda, \mu X_\mu) = \mu\omega(X_\lambda, X_\mu).\end{aligned}$$

Thus $\omega(X_\lambda, X_\mu) = 0$, so E_λ and E_μ are ω-orthogonal. □

The foliations Φ_i are invariant under the Hamiltonian flow:

LEMMA 2.3. *Suppose $\lambda_i \in \sigma(N)$ and X is a vector field tangent to Φ_i. Then $[X_h, X]$ is also tangent to Φ_i.*

PROOF. Because of (BH) the Lie derivative $L_{X_h}N$ vanishes. Therefore, if $X \in E_{\lambda_i}$ one finds:

$$\begin{aligned}N[X_h, X] &= NL_{X_h}X\\ &= L_{X_h}(NX)\\ &= L_{X_h}(\lambda_i X) = (X_h \cdot \lambda_i)X + \lambda_i[X_h, X].\end{aligned}$$

But λ_i is a constant of the motion, so $[X_h, X] \in E_{\lambda_i}$. □

¿From proposition 2.1 we know that each λ_i is an integral of Δ_i. On the other hand, (BH) implies that each λ_i is a constant of the motion, and so by (ND) depends only on the action variables. Since the λ_i's are functionally independent we can use them as new "action" variables $(y_1, \ldots, y_n)$ on $\mathbb{R}^n$ to obtain:

PROPOSITION 2.4. *The foliations Δ_i on M project to $(n-1)$-dimensional foliations on $\mathbb{R}^n$ which are pairwise transversal. In particular, there are coordinate functions (y^i) on $\mathbb{R}^n$, such that each $y^i \circ \pi$ is constant on the leaves of Δ_i.*

In the sequel we will not distinguish between y^i and $y^i \circ \pi$. Our interest in the new coordinates (y^i) lies in the following splitting result.

THEOREM 2.5. *In the new coordinates* (y^i) *one has the following splittings:*

$$h(y^1, \dots, y^n) = h_1(y^1) + \dots + h_n(y^n) \qquad s^i(y^1, \dots, y^n) = s^i_1(y^1) + \dots + s^i_n(y^n).$$

PROOF. Denote by φ^i, $i = 1 \dots, n$, conjugate coordinates in M to the coordinates y^i, so we have the Poisson bracket relations

$$\{y^i, y^j\} = 0 \quad , \quad \{y^i, \varphi^j\} = \delta_{ij} \quad , \quad \{\varphi^i, \varphi^j\} = 0 \quad i, j = 1, \dots, n.$$

Explicitly, one finds

$$\theta^i = \sum_{j=1}^n \frac{\partial y^j}{\partial s^i} \varphi^j. \tag{2.2}$$

We claim that in the new coordinates one has

$$T\Phi_i = span\left\{X_{y^i}, X_{\varphi^i} + \sum_{j=1}^n a_{ij} X_{y^j}\right\} \qquad (a_{ii} = 0). \tag{2.3}$$

To prove this assertion, observe that by proposition 2.4 each y^i is constant on the leaves of Φ_j $(i \neq j)$, while by proposition 2.2 $(T\Phi_i)^\perp = T\Delta_i = T\Phi_1 + \dots + \widehat{T\Phi_i} + \dots + T\Phi_n$, so we have $X_{y^i} \in T\Phi_i$. This shows that

$$T\Phi_i = span\left\{X_{y^i}, bX_{\varphi^i} + \sum_{j=1}^n a_{ij} X_{y^j} + \sum_{j=1}^n c_{ij} X_{\varphi^j}\right\}.$$

where we can assume $a_{ii} = c_{ii} = 0$. Since $X_{y^i} \in T\Phi_i$, each X_{y^j} is orthogonal to $T\Phi_i$ for $i \neq j$, so it follows that:

$$0 = \omega(X_{y^j}, bX_{\varphi^i} + \sum_{k=1}^n a_{ik} X_{y^k} + \sum_{k=1}^n c_{ik} X_{\varphi^k}) = c_{ij} \qquad (i \neq j).$$

We conclude that

$$T\Phi_i = span\left\{X_{y^i}, bX_{\varphi^i} + \sum_{j=1}^n a_{ij} X_{y^j}\right\}.$$

Finally, if the coefficient b vanishes at some $m \in M$, then $T_m\Phi_i \subset Ker\ d_m\pi$, and from proposition (2.2) we get that

$$T_m\Phi_i \subset Ker\ d_m\pi = (Ker\ d_m\pi)^{\perp} \subset (T_m\Phi_i)^{\perp} = T_m\Delta_i,$$

which contradicts the transversality of Φ_i and Δ_i. Thus we can assume $b = 1$ and (2.3) follows.

Using (2.3), we can determine the expression for N in the coordinates (y^i, φ^i). The final result is:

$$\text{(2.3a)} \qquad N = \begin{pmatrix} \Lambda & 0 \\ B & \Lambda \end{pmatrix} \quad \text{where} \quad \Lambda = diag\ (\lambda_1, \ldots, \lambda_n)\ ,\ B_{ij} = (\lambda_j - \lambda_i)a_{ji}$$

Recall now that (BH) assures the existence of a second Hamiltonian $\tilde{h}$ such that

$$N^* d\tilde{h} = dh,$$

and by (ND) we must have $\tilde{h} = \tilde{h}(y^1, \ldots, y^n)$. Thus the first n equations of this system reduce to

$$\text{(2.4)} \qquad y^i \frac{\partial \tilde{h}}{\partial y^i} = \frac{\partial h}{\partial y^i}, \quad i = 1, \ldots, n.$$

By crossing differentiating (2.4) we see that $\partial^2 h/\partial y^j \partial y^i = 0$ $(i \neq j)$, which proves the splitting for h.

The analogous splitting for s^i is proved by showing that X_{s^i} is a bi-Hamiltonian vector field in the set of points where all λ_j's are nonzero. Then we can repeat the argument of the last paragraph to show that the splitting holds on this set. But, by (ND), it must hold everywhere. Now, if all the λ_i's are nonzero the second Poisson structure is symplectic, and X_{s^i} is bi-Hamiltonian provided $L_{X_{s^i}} N = 0$. In the original variables (s^i, θ^i) one has $X_{s^i} = \partial/\partial\theta^i$ so the Lie derivative of N vanishes if one can show that its entries do not depend on the θ^i's. This is proved in two steps:

(i) In the variables (y^i, φ^i) the entries of N do not depend on the φ^i's;

Since $\lambda_i = y^i$, we see from (2.3a) that the assertion will follow provided that $a_{ij} = a_{ij}(y^1, \dots, y^n)$. By lemma 2.3 and (2.3) we have

$$\Big[X_h, X_{\varphi^i} + \sum_{j=1}^n a_{ij} X_{y^j}\Big] = -X_{\{h,\varphi^i\}} + \sum_{j=1}^n \big(X_h(a_{ij}) X_{y^j} - a_{ij} X_{\{h,y^j\}}\big) \in T\Phi_i$$

Since $\{h, \varphi^i\} = \frac{\partial h}{\partial y^i}$, $\{h, y^j\} = 0$, we conclude that

$$\sum_{j=1}^n \Big(-\frac{\partial^2 h}{\partial y^i \partial y^j} + X_h(a_{ij})\Big) X_{y^j} \in T\Phi_i.$$

But we have shown already that $\partial^2 h/\partial y^i \partial y^j = 0$ $(i \neq j)$, so by (2.3) we also have $X_h(a_{ij}) = 0$ $(i \neq j)$. Finally from (ND) we conclude that $a_{ij} = a_{ij}(y^1, \dots, y^n)$.

(ii) In the variables (s^i, θ^i) the entries of N do not depend on the θ^i's.

Because of the form of the transformation $(s^i, \theta^i) \to (y^i, \varphi^i)$ (cf. (2.2)), we see from (i) that the entries of N when written in the variables (s^i, θ^i) are at most linear in the θ^i's. But those entries are well defined functions on the tori, so in fact they do not depend on the θ^i's at all. $\square$

Remark 2.6. In Morandi et al. [Mo], a partial version of theorem 2.5 was obtained (cf. prop. 3.22). They use the Lagrangian approach, and prove separability of the Hamiltonian with respect to both Hamiltonian structures. However, they fail to recognize the importance of the action-angle variables, which as we will see in the next sections play a central role.

3. Geometric interpretation

Let $(x^1, \dots, x^{n+1})$ be affine coordinates in a (n+1)-dimensional affine space $\mathbb{A}^{n+1}$. A hypersurface in $\mathbb{A}^{n+1}$ is called a **hypersurface of translation** if it admits a parameterization of the form:

$$(3.1) \quad (y^1, \dots, y^n) \to x^l(y^1, \dots, y^n) = a_1^l(y^1) + \cdots + a_n^l(y^n) \quad (l = 1, \dots, n+1)$$

This generalizes Darboux's definition [Dar] for n=2: a surface of translation is a surface obtained by parallel translating a curve along another curve.

The results of the previous section lead to the following geometric picture:

THEOREM 3.1. *A completely integrable Hamiltonian system is bi-Hamiltonian if and only if the graph of the Hamiltonian function is a hypersurface of translation, relative to the affine structure determined by the action variables.*

PROOF. The 'only if' part was the subject of the previous section. Now assume that (M^{2n}, ω, h) is a completely integrable system and that *graph* h is a hypersurface of translation relative to the action variables (s^i), so it has a parameterization of the form (3.1), with $x^i = s^i$, $i = 1, \ldots, n$ and $x^{n+1} = h$. We can choose the parameters (y^i) so that the Hamiltonian takes the simple form

$$h(y^1, \ldots, y^n) = y^1 + \cdots + y^n.$$

If $(\varphi^1, \ldots, \varphi^n)$ are coordinates conjugate to the $(y^1, \ldots, y^n)$, we define a second Poisson structure by the formula

$$\Lambda_1 = \sum_{i=1}^{n} y^i \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial \varphi^i}.$$

One checks easily that the two Poisson structures are compatible, and that the recursion operator is given by

$$N = \sum_{i=1}^{n} y^i \Big(\frac{\partial}{\partial y^i} \otimes dy^i + \frac{\partial}{\partial \varphi^i} \otimes d\varphi^i \Big)$$

It is now clear from the expression of the Hamiltonian function in the y-coordinates that $L_{X_h} N = 0$, so the vector field X_h is bi-Hamiltonian. $\square$

REMARK 3.2. Note that the notion of the graph of the Hamiltonian being a hypersurface of translation is associated with the system in an invariant way, being defined relative to the affine structure determined by the action variables (see the remarks on unicity of the action variables following theorem 1.1). It should not be confused with separability of the Hamiltonian system, or of the Hamilton-Jacobi equation, which are coordinate dependent. For example, consider a system with Hamiltonian $h(s^1, s^2) = s^1(1 + (s^2)^2)$ expressed in action-angle variables. It will be shown below that *graph* h is not an hypersurface of translation. But if we introduce new canonical coordinates $(y^1, y^2, \varphi^1, \varphi^2)$ with $y^1 = h(s^1, s^2)$, the Hamiltonian system splits in the new coordinates into two independent, two-dimensional, Hamiltonian systems.

It arises the problem of recognizing when is *graph* h a hypersurface of translation. For this purpose, we introduce the Hessian metric g on $\mathbb{R}^n$, which is defined with respect to the affine coordinates (s^i) by the formula:

$$g = \sum_{i,j} \frac{\partial^2 h}{\partial s^i \partial s^j}\ ds^i ds^j \tag{3.2}$$

In the (non-affine) coordinates (y^α) the metric g is given by:

$$g = \sum_{\alpha,\beta} \Big(\frac{\partial^2 h}{\partial y^\alpha \partial y^\beta} - \frac{\partial h}{\partial s^k} \frac{\partial^2 s^k}{\partial y^\alpha \partial y^\beta} \Big) dy^\alpha dy^\beta,$$

and so, by (3.1), diagonalizes:

$$\frac{\partial^2 h}{\partial y^\alpha \partial y^\beta} = 0 \quad , \quad \frac{\partial^2 s^k}{\partial y^\alpha \partial y^\beta} = 0 \quad (\alpha \neq \beta). \tag{3.3}$$

Defining the coordinate vector fields $Y_\alpha = \partial / \partial y^\alpha$, these conditions on the metric can be written in the form

$$g(Y_\alpha, Y_\beta) = 0 \qquad (\alpha \neq \beta), \tag{3.4}$$

$$Y_\alpha(Y_\beta(s^k)) = 0 \qquad (\alpha \neq \beta). \tag{3.5}$$

We conclude that the existence of vector fields $Y_\alpha,\ \alpha = 1, \ldots, n$, satisfying (3.4) and (3.5) is a necessary and sufficient condition for *graph* h to be an hypersurface of translation. Note that we can find the n-tuples of vector fields satisfying

(3.4) by solving an eigenvalue problem. This will define the (Y_α)'s up to multiplicative factors C_α, and equations (3.5) then form a system of first order linear p.d.e's for each of these factors. Our recognition problem is then reduced to the investigation of the local solvability of these equations. Suppose for example that $Y_\alpha = C_\alpha \sum_i A_{\alpha i}\partial/\partial s^i$. Then (3.5) gives:

$$Y_\alpha(C_\beta)A_{\beta k} + C_\beta Y_\alpha(A_{\beta k}) = 0 \quad (\alpha \neq \beta). \tag{3.6}$$

Since we are interested in non-zero, local, solutions of this equation, we obtain the following integrability conditions

$$Y_\alpha(A_{\beta k})A_{\beta l} - A_{\beta k}Y_\alpha(A_{\beta l}) = 0 \quad (\alpha \neq \beta, k \neq l). \tag{3.7}$$

A more invariant way of describing the above conditions can be given as follows. Denote by ∇ the law of covariant differentiation associated with the connection defined by the affine structure on $\mathbb{R}^n$. Then the Hessian metric defined by h is given invariantly by the expression $g(X,Y) = \nabla_X\nabla_Y h$. To solve our recognition problem we seek an orthonormal basis $(Y_1,\ldots,Y_n)$ for the tangent bundle $T(M)$, satisfying

$$\nabla_{Y_\beta}Y_\alpha = 0 \quad (\alpha \neq \beta),$$

i.e., such that each vector field Y_α can be obtained by parallel transport along the integral curves of any other vector field Y_β, $(\alpha \neq \beta)$.

4. Examples and counter-examples

Counter-examples. A possible way to construct CIS with no bi-Hamiltonian formulation is to consider the standard model $M \simeq \mathbb{R}^n \times \mathbb{T}^n$ with $\omega = \sum_{i=1}^n ds^i \wedge d\theta^i$, and choose any Hamiltonian function $h = h(s^1,\ldots,s^n)$ whose graph is not a hypersurface of translation. For example, take

$$h(s^1,\ldots,s^n) = s^1 + s^1(s^2)^2 + (s^3)^2 + \cdots + (s^n)^2.$$

The Hessian matrix has eigenvectors

$$Y_{1,2} = C_{1,2}\Big(2s^2\frac{\partial}{\partial s^1} + \big(s^1 \pm \sqrt{(s^1)^2 + 4(s^2)^2}\,\big)\frac{\partial}{\partial s^2}\Big),$$
$$Y_j = C_j\frac{\partial}{\partial s^j}, \qquad\qquad j = 3, \ldots, n.$$

It is easy to check that conditions (3.7) are not satisfied. For example, if we let $\alpha = 1, \beta = 2, k = 1, l = 2$ we obtain

$$Y_1(A_{21})A_{22} - A_{21}Y_1(A_{22}) = -4(s^2)^2 + 12s^1(s^2)^2\sqrt{(s^1)^2 + 4(s^2)^2} \neq 0.$$

We conclude that *graph h* is not a hypersurface of translation. Brouzet's original counter-example corresponds to the case n=2 (see Brouzet, 1990).

A counter-example with some physical meaning is obtained by considering the perturbed Kepler problem. In spherical coordinates (r, θ, ϕ), where θ denotes the co-latitude and ϕ the azimuth, the Hamiltonian takes the form

$$h = \frac{1}{2}\Big({p_r}^2 + \frac{{p_\theta}^2}{r^2} + \frac{{p_\phi}^2}{r^2\sin^2\theta}\Big) - \frac{1}{r} + \frac{\varepsilon}{2r^2}.$$

Two additional integrals, Poisson commuting with H, are the total angular momentum

$$l^2 \equiv {p_\theta}^2 + \frac{{p_\phi}^2}{\sin^2\theta},$$

and the component of the angular momentum along the polar axis

$$m \equiv p_\phi.$$

For $E < 0$, each common level set $\{h = E, l = L, m = M\}$ is an embedded 3-torus on the phase space. The action variables are obtained by integration along a basis $(\gamma_1, \gamma_2, \gamma_3)$ of 1-cycles for this torus:

$$s_i = \frac{1}{2\pi}\oint_{\gamma_i} p_r dr + p_\theta d\theta + p_\phi d\phi$$

We obtain

$$
\begin{aligned}
s_\phi &= \frac{1}{2\pi}\oint p_\phi d\phi = M \\
s_\theta &= \frac{1}{2\pi}\oint p_\theta d\theta = \frac{1}{2\pi}\oint \sqrt{L^2 - \frac{M^2}{\sin^2\theta}}d\theta = L - M \\
s_r &= \frac{1}{2\pi}\oint p_r dr = \frac{1}{2\pi}\oint \sqrt{2E + \frac{2}{r} - \frac{L^2+\varepsilon}{r^2}}dr = \frac{1}{\sqrt{-2E}} - \sqrt{L^2+\varepsilon}
\end{aligned}
$$

We conclude that the Hamiltonian, when written in action variables, takes the form

$$
h = -\frac{1}{2\left(s_r + \sqrt{(s_\phi + s_\theta)^2 + \varepsilon}\ \right)^2}.
$$

A more or less tedious computation, similar to the one in the previous example, shows that conditions (3.7) once again are not satisfied. Also we note that for the unperturbed Kepler problem ($\varepsilon = 0$) the graph of the Hamiltonian is a surface of translation, and so it has a bi-Hamiltonian formulation (on the other hand, one can show that the relativistic Kepler problem also does not have a bi-Hamiltonian formulation).

These counter-examples help one understand why a bi-Hamiltonian formulation is not known for many of the classical integrable systems. In general, one will require some degenerate Poisson pair in a higher dimension manifold. This is the situation, for example, in the R-matrix approach (see next chapter). In the case of the perturbed Kepler problem, the Hamilton-Jacobi equation can be solved by separation of variables, and it follows from the work of Rauch-Wojciechowski [Ra] that this system admits a degenerate bi-Hamiltonian formulation in a higher dimensional manifold.

An example. Consider a symmetric top rotating freely about a fixed point. As in the general theory of tops (see for example [Bo] and references therein), it can be realized as an Hamiltonian system for the Lie-Poisson structure on $\mathfrak{e}(3)$, the Lie algebra of the group of motions of three dimensional Euclidean space. For the usual coordinates $(m_1, m_2, m_3, p_1, p_2, p_3)$ on $\mathfrak{e}(3)$, the Poisson bracket is defined by the relations

$$\{m_i, m_j\} = \varepsilon_{ijk} m_k, \qquad \{m_i, p_j\} = \varepsilon_{ijk} p_k, \qquad \{p_i, p_j\} = 0,$$

and the Hamiltonian is given by

$$h = \frac{1}{2I_1}({m_1}^2 + {m_2}^2) + \frac{1}{2I_3}{m_3}^2.$$

Note that this bracket is degenerate, the algebra of Casimirs being generated by $C_1 = \sum_i {p_i}^2$ and $C_2 = \sum_i p_i m_i$. Since m_3 provides an additional first integral commuting with h, it follows that the system is completely integrable when restricted to any symplectic leaf of the Kirillov foliation. Let us consider the symplectic leaf $C_1 = 1, C_2 = 0$. It can be identified with the tangent bundle $T\mathbb{S}^2$ of the unit sphere on $\mathbb{R}^3$, and this suggests introducing new variables $(\theta, \varphi, p_\theta, p_\varphi)$ given by

$$\begin{array}{lll} p_1 = \cos\theta\cos\varphi & p_2 = \cos\theta\sin\varphi & p_3 = \sin\theta \\ m_1 = p_\varphi \tan\theta\cos\varphi - p_\theta\sin\varphi & m_1 = p_\varphi\tan\theta\sin\varphi + p_\theta\cos\varphi & m_3 = p_\varphi \end{array}$$

This coordinates are canonical and the Hamiltonian function takes the form

$$h = {p_\varphi}^2\Big(\frac{1}{2I_1}\tan^2\theta + \frac{1}{2I_3}\Big) + \frac{1}{2I_1}{p_\theta}^2.$$

For $E/m^2 > 1/2I_3$, the level surfaces $\{h = E, m_3 = L\}$ are embedded 2-tori in the fixed symplectic leaf. The action variables are computed in the usual way:

$$\begin{aligned} s_1 &= \frac{1}{2\pi}\oint_{\gamma_1} p_\phi d\phi = \frac{1}{2\pi}\oint L\ d\phi = L \\ s_2 &= \frac{1}{2\pi}\oint_{\gamma_2} p_\theta d\theta = \frac{1}{2\pi}\oint \sqrt{2EI_1 - L^2\frac{I_1}{I_3} - L^2\tan^2\theta}\ d\theta \\ &= \sqrt{2EI_1 - L^2\frac{I_1 - I_3}{I_3}} - L \end{aligned}$$

The expression for the Hamiltonian in the action variables is

$$h = \frac{I_1 - I_3}{2I_1 I_3}{s_1}^2 + \frac{1}{2I_1}(s_1 + s_2)^2,$$

so according to theorem 3.1 the system possesses a bi-Hamiltonian formulation. It is easy to see that the second Poisson structure (cf. proof of theorem 3.1) is given by:

$$\begin{aligned}\Lambda_0 =& \frac{I_1 - I_3}{2I_1 I_3} {s_1}^2 \frac{\partial}{\partial s_1} \wedge \frac{\partial}{\partial \theta_1} + \left(\frac{I_1 - I_3}{2I_1 I_3} {s_1}^2 - \frac{1}{2I_1} (s_1 + s_2)^2 \right) \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial s_2} \\ &+ \frac{1}{2I_1} (s_1 + s_2)^2 \frac{\partial}{\partial s_2} \wedge \frac{\partial}{\partial \theta_2}.\end{aligned}$$

which can, in principle, be written in terms of the original variables.

CHAPTER III

POISSON SYMMETRIC SPACES

We have seen in the previous two chapters several examples of Poisson pairs. In most instances, the first Poisson structure of the pair arises from a symplectic form or by reduction from it, while the second Poisson structure is given *a priori.* Although no general method to construct Poisson pairs is known, it is often the case that the second Poisson bracket also has some geometric structure underlying it. This chapter is devoted to the study of certain geometric objects yielding non-linear Poisson brackets.

We have mentioned in chapter I the notions of Poisson Lie group and Poisson action, first introduced by Drinfel'd in [Dr]. We shall now proceed along these lines to develop several new concepts in Poisson geometry. For example, a Poisson homogeneous space is a homogeneous space $M = G/H$ where G is a Poisson Lie group, M a Poisson manifold, and $\pi: G \to M$ a Poisson map. These objects have been study in [K-R-R,L-Q,Lu-We,STS]. We go a step further: we call a pair (G, S) a Poisson symmetric Lie group if G is a Poisson Lie group and $S: G \to G$ is an involutive Poisson Lie group anti-morphism. If H is the fixed point set of S, then $M = G/H$ is a Poisson homogeneous space, and we call M a Poisson symmetric space. For example, every Poisson Lie group is a Poisson symmetric, for the usual construction $G \simeq G \times G/H$, where H is the diagonal and $S(g_1, g_2) = (g_2, g_1)$, goes through (see example 4.9 below). As this example shows, the requirement "anti-morphism" rather than simply "morphism" is essential. Another nice feature of the theory is that the usual duality for symmetric spaces extends to the Poisson case. The compact Poisson Lie groups of Majid [M] and Lu and Weinstein [Lu-We] are examples of compact Poisson symmetric spaces, and the results of Deift and Li [D-L] on solutions of the Yang-Baxter equation, have here a natural interpretation. As several examples will show, Poisson brackets arising in this way are related with Poisson pairs for integrable systems.

The organization of this chapter is as follows. In sections 1 and 2 we review some basic results on Poisson Lie groups and Poisson actions. In section 3 we introduce Poisson homogeneous spaces. In section 4 we define Poisson symmetric Lie groups and the corresponding infinitesimal objects, the symmetric Lie bialgebras. Associated with them are the Poisson symmetric spaces. We then show that the duality for symmetric spaces extends to the Poisson case. In section 5 we recall some facts concerning r-matrices, which we view as an algebraic tool underlying this Poisson geometry in the semisimple case. We use them in section 6, together with some structure theory of real Lie algebras, to give several examples of orthogonal symmetric Lie algebras. Section 7 contains a study of the Poisson properties of the Cartan immersion, and in the final section we present several examples related to Poisson pairs and integrable systems.

In this chapter we assume that all Lie groups are connected.

1. Poisson Lie groups and Lie bialgebras

Let G be a Poisson Lie group (sec. I.1.1). Denoting by L_g (resp. R_g) left translation (resp. right translation) on G, the requirement that multiplication $G \times G \to G$ be a Poisson map can be written in the form

$$\Lambda^G_{g\cdot h} = (L_g)_*\Lambda^G_h + (R_h)_*\Lambda^G_g, \qquad g, h \in G, \tag{1.1}$$

where Λ^G denotes the Poisson bivector on G.

It follows from (1.1) that the rank of the Poisson Lie bracket at the identity $e \in G$ is zero, so linearization at e [We1] furnishes the Lie algebra $\mathfrak{g} = Lie(G)$ with a linear Poisson structure. Equivalently, the dual space $\mathfrak{g}^*$ has a Lie algebra structure. The map $\varphi: \mathfrak{g} \to \mathfrak{g} \bigwedge \mathfrak{g}$ dual to the Lie bracket on $\mathfrak{g}^*$ is a 1-cocycle for the adjoint action.

DEFINITION 1.1 [Dr]. Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called a **Lie bialgebra** if there exists a Lie algebra structure on $\mathfrak{g}^*$ such that the map $\varphi: \mathfrak{g} \to \mathfrak{g} \bigwedge \mathfrak{g}$ dual to the Lie bracket on $\mathfrak{g}^*$ is a 1-cocycle.

Therefore, to every Poisson Lie group corresponds a tangent Lie bialgebra. Conversely, we have the following result [KS].

THEOREM 1.2. *To each Poisson Lie group G corresponds a tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. Conversely, if G is connected and simply connected, each Lie bialgebra structure on $\mathfrak{g}$ defines a Poisson Lie structure on G whose tangent Lie bialgebra is the given one. Moreover, homomorphisms of connected, simply connected, Poisson Lie groups are in one-to-one correspondence with homomorphisms of Lie bialgebras.*

Let G be a Poisson Lie group, with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. The connected, simply connected, Poisson Lie group G^* whose Lie bialgebra is $(\mathfrak{g}^*, \mathfrak{g})$ is called the **dual Poisson Lie group** of G.

EXAMPLE 1.3. Suppose that G is equipped with the zero Poisson bracket. The tangent Lie bialgebra is $(\mathfrak{g}, \mathfrak{g}^*)$ where $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{g}^*$ is abelian. The dual Poisson Lie group is the vector space $\mathfrak{g}^*$ with the usual Lie-Poisson bracket[4].

EXAMPLE 1.4. In example I.1.4, we have made $SL(n, \mathbb{R})$ into a Poisson Lie group. If we equip the corresponding Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ with the non-degenerate, bilinear form

$$(x, y) \equiv tr\ (xy), \qquad x, y \in \mathfrak{sl}(n, \mathbb{R}),$$

then the corresponding Lie bialgebra $\mathfrak{sl}(n, \mathbb{R})^*$ is identified with the vector space $\mathfrak{sl}(n, \mathbb{R})$ with a new Lie bracket

$$[x, y]_* = [Ax, y] + [x, Ay], \tag{1.2}$$

[4]Caution!! In our non-commutative world "Poisson Lie bracket" and "Lie-Poisson bracket" have completely different meanings.

where $A\colon \mathfrak{g} \to \mathfrak{g}$ is the linear map defined by

$$Ax = \begin{cases} -x & \text{if } x \in \mathfrak{n}_+ \\ 0 & \text{if } x \in \mathfrak{h} \\ x & \text{if } x \in \mathfrak{n}_- \end{cases}$$

and $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$, $\mathfrak{h}$), denotes the set of upper triangular (resp. lower triangular, diagonal) $n \times n$ matrices of trace zero.

The bracket (1.2) has the following explicit form

$$[x_1 + h_1 + y_1, x_2 + h_2 + y_2]_* = -2[x_1, x_2] + ad\ h_1 \cdot (-x_2 + y_2)$$
$$- ad\ h_2 \cdot (-x_1 + y_1) + 2[y_1, y_2], \quad x_i \in \mathfrak{n}_+, h_i \in \mathfrak{h}, y_i \in \mathfrak{n}_-.$$

so $\mathfrak{g}^*$ is isomorphic to the semi-direct product $\mathfrak{h} \ltimes (\mathfrak{n}_+ \oplus \mathfrak{n}_-)$ relative to the homomorphism $\phi\colon \mathfrak{h} \to \mathrm{Der}(\mathfrak{n}_+ \oplus \mathfrak{n}_-)$ defined by $h \mapsto (ad(-h), ad\ h)$. Thus the dual Poisson Lie group is isomorphic to the Lie group $H \ltimes (N_+ \times N_-)$, where H, N_+ and N_- are respectively the diagonal matrices of determinant 1, upper triangular and lower triangular matrices with 1's in the diagonal, and whose product is given by

$$(D_1, U_1, L_1) \cdot (D_2, U_2, L_2) = (D_1 D_2, D_1^{-1} U_2 D_1 U_1, L_1 D_1 L_2 D_1^{-1}).$$

where $D_i \in H$, $U_i \in N_+$, and $L_i \in N_-$.

Let G be a Poisson Lie subgroup. If $H \subset G$ is a closed Lie subgroup with Lie subalgebra $\mathfrak{h} \in \mathfrak{g}$, then H is a Poisson submanifold of G iff the annihilator $\mathfrak{h}^\perp$ of $\mathfrak{h}$ is an ideal in $\mathfrak{g}^*$. In this case H is called a **Poisson Lie subgroup**. It is a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{h}, \mathfrak{g}^*/\mathfrak{h}^\perp)$.

Example 1.5. The group $U(n, \mathbb{R})$ of upper triangular matrices of determinant 1 is a Poisson Lie subgroup of $SL(n, \mathbb{R})$. In fact, its Lie algebra $\mathfrak{u} = \mathfrak{h} + \mathfrak{n}_+$ has annihilator $\mathfrak{u}^\perp = n_+$ which is an ideal in $\mathfrak{g}^*$. The corresponding Lie bialgebra is $(\mathfrak{u}, \mathfrak{u}^*)$ where $\mathfrak{u}^*$ is isomorphic to the semi-direct product $\mathfrak{h} \triangleright \mathfrak{n}_-$.

On any Poisson manifold (M, Λ) the vector space $\Omega^1(M)$ of differential 1-forms on M carries a Lie algebra structure defined by [We2]

$$[\omega_1, \omega_2] = d[\Lambda(\omega_1, \omega_2)] + < d\omega_1, J\omega_2 > - < d\omega_2, J\omega_1 >, \qquad \omega_1, \omega_2 \in \Omega^1(M).$$

The bundle map $J: \Omega^1(M) \to \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism:

$$[J\omega_1, J\omega_2] = -J[\omega_1, \omega_2] \qquad \omega_1, \omega_2 \in \Omega^1(M).$$

If now G is a Poisson Lie group, the left-invariant 1-forms on G form a Lie subalgebra of $\Omega^1(G)$ isomorphic to $\mathfrak{g}^*$ [We2], and any $\xi \in \mathfrak{g}^*$ can be identified with the corresponding left-invariant 1-form ξ^L on G. Define $\psi: \mathfrak{g}^* \to \mathfrak{X}(G)$ by setting

$$\psi(\xi) \equiv J\xi^L, \qquad \xi \in \mathfrak{g}^*. \tag{1.3}$$

Then ψ is a Lie algebra homomorphism, and integrating it we obtain a (local) action $\Psi: G^* \times G \to G$, called the **left dressing action** of G^* on G.

THEOREM 1.6 [STS2]. *The symplectic leaves of the Kirillov foliation of a Poisson Lie group G are the orbits of the dressing action of G^* on G.*

We shall see in the next section that the dressing action is an example of a Poisson action.

EXAMPLE 1.7. From example 1.3 we know that $\mathfrak{g}^*$, with the Lie-Poisson bracket, is a Poisson Lie group whose dual group is G. The left dressing action of G on $\mathfrak{g}^*$ coincides with the co-adjoint action. Theorem 1.5 reduces to a well known result about the Lie-Poisson bracket: the leaves of the Kirillov foliation of $\mathfrak{g}^*$ are the orbits of the co-adjoint action.

REMARK 1.8. One can also define the **right dressing action** of G^* on G. If one identifies $\xi \in \mathfrak{g}^*$ with a right-invariant 1-form ξ^R on G, then the right dressing action is obtained by integrating the infinitesimal action $\psi: \mathfrak{g}^* \to \mathfrak{X}(G)$ defined by

$$\psi(\xi) \equiv -J\xi^R, \qquad \xi \in \mathfrak{g}^*. \tag{1.3.a}$$

Theorem 1.6 holds also for the right dressing action.

2. Poisson actions

Let $\Psi: G \times M \to M$ be an action of a Poisson Lie group (G, Λ^G) on a Poisson manifold (M, Λ^M). For each $g \in G$, and each $m \in M$, we define $\Psi_g: M \to M$ and $\Psi_m: G \to M$ by the formulas

$$\Psi_g: m \mapsto \Psi(g,m) = g \cdot m, \qquad \Psi_m: g \mapsto \Psi(g,m) = g \cdot m.$$

If $x \in \mathfrak{g}$, which we view as a left-invariant vector field on G, we denote the corresponding infinitesimal generator on M by X_x, so $X: \mathfrak{g} \to \mathfrak{X}(M)$ is a Lie algebra anti-morphism. Finally, if $f \in C^\infty(M)$ we set $\nabla f(m) = d_e(f \circ \Psi_m) \in \mathfrak{g}^*$.[5]

Proposition 2.1. *The following conditions are equivalent:*

(a) Ψ *is a Poisson action;*

(b) For all $m \in M$ *and* $g \in G$

$$\Lambda^M_{m \cdot g} = (\Psi_m)_* \Lambda^G_g + (\Psi_g)_* \Lambda^M_m; \tag{2.1}$$

(c) For all $f_1, f_2 \in C^\infty(M)$ *and* $x \in \mathfrak{g}$

$$X_x \cdot \{f_1, f_2\} = \{X_x \cdot f_1, f_2\} + \{f_1, X_x \cdot f_2\} + < [\nabla f_1, \nabla f_2]_*, x >; \tag{2.2}$$

Proof. The proof follows immediately from the definitions. □

Remark. As we have noted before (example I.1.3), Hamiltonian actions correspond to the case where G has the trivial Poisson bracket, so the Lie algebra $\mathfrak{g}^*$ is commutative. In this case, the last term on the r.h.s. of (2.2) vanishes, and we obtain a well known formula for Hamiltonian actions.

The proof of the following corollary can be found in [Lu-We].

[5] Unless otherwise stated, we will assume left group actions. Occasionally, we will need right actions. Virtually all results to be stated below hold for both type of actions, with obvious modifications.

Corollary 2.2. *The dressing action of the dual Poisson Lie group G^* on G is a Poisson action.*

Recall from the previous section that Poisson Lie subgroups correspond to ideals $\mathfrak{h}^\perp$ in $\mathfrak{g}^*$. Under the weaker assumption that the annihilator $\mathfrak{h}^\perp$ is a subalgebra of $\mathfrak{g}^*$, we have the following result which will be applied later in the study of homogeneous spaces.

Proposition 2.3 [STS2]. *Let $\Psi\colon G \times M \to M$ be a Poisson action. Let $H \subset G$ be a Lie subgroup with Lie algebra $\mathfrak{h}$, and let $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ be the annihilator of $\mathfrak{h}$. If the orbit space M/H is a smooth manifold, there exists a Poisson bracket on M/H such that the projection $\pi\colon M \to M/H$ is a Poisson map iff $\mathfrak{h}^\perp$ is a subalgebra of $\mathfrak{g}^*$. Such a Poisson structure is unique.*

Proof. Identify $C^\infty(M/H)$ with the space of H-invariant elements of $C^\infty(M)$. A function $f \in C^\infty(M)$ is H-invariant iff $X_x \cdot f = 0$, for all $x \in \mathfrak{h}$. In this case $\nabla f \in \mathfrak{h}^\perp$. Therefore, if $f_1, f_2 \in C^\infty(M)$ are H-invariant, it follows from (2.2) that

$$X_x \cdot \{f_1, f_2\} =< [\nabla f_1, \nabla f_2]_*, x >, \qquad x \in \mathfrak{h}.$$

This shows that $\{f_1, f_2\}$ is H-invariant iff $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is a Lie subalgebra. $\square$

In order to determine the Kirillov foliation of the quotient Poisson manifold M/G one needs to introduce, just as in the symplectic case, the notion of a momentum map. The correct definition in this context is due to Lu.

Definition 2.4 [Lu]. A smooth map $P\colon M \to G^*$ is called a **momentum map** for the Poisson action $\Psi\colon G \times M \to M$ if

$$X_x(m) = J(m) \cdot T_m^* P \cdot x^L, \qquad \forall x \in \mathfrak{g},\ \forall m \in M,$$

where $x^L \in \Omega(G^*)$ denotes the left invariant 1-form whose value at e is x.

Examples 2.5.

(a) If G has the zero Poisson bracket and M is a symplectic manifold then $G^* = \mathfrak{g}^*$ (cf. example 1.3) and the definition above coincides with the usual definition of a momentum map $P\colon M \to \mathfrak{g}^*$.

(b) The identity map is a momentum map for the dressing action $G \times G^* \to G$.

The usual results for Hamiltonian systems with symmetry still hold in this more general context [Lu]. Noether's theorem can be stated as follows.

Theorem 2.6. *Let $H \in C^\infty(M)$ be an Hamiltonian, invariant for a Poisson action of G on M with momentum map $P\colon M \to G^*$. Then P is a constant of the motion.*

We have introduced in the previous section the dressing action of G^* on G. Since the notion of duality is reflexive, we also have a dressing action of G on G^*. When we say that a momentum map $P\colon M \to G^*$ for some Poisson action is G-equivariant, we mean for the given action of G on M and for the dressing action of G on G^*. The main result of Poisson reduction has now the following formulation.

Theorem 2.7. *Let $\Psi\colon G \times M \to M$ be a Poisson action with G-equivariant momentum map $P\colon M \to G^*$. Assume that $\mu \in G^*$ is a regular value of P, and that the residual symmetry group G_μ acts regularly on the submanifold $P^{-1}(\mu)$. Then there is a natural immersion ϕ making $P^{-1}(\mu)/G_\mu$ into a Poisson submanifold of M/G in such a way that the following diagram commutes:*

$$
\begin{array}{ccccc}
 & & M & & \\
 & \overset{i}{\nearrow} & & \overset{\pi}{\searrow} & \\
P^{-1}(\mu) & & & & M/G \\
 & \underset{\pi_\mu}{\searrow} & & \underset{\phi}{\nearrow} & \\
 & & P^{-1}(\mu)/G_\mu & &
\end{array}
$$

If M is symplectic, $P^{-1}(\mu)/G_\mu$ is a leaf of the symplectic foliation of M/G.

Any Hamiltonian system on M having G as a symmetry group, reduces to Hamiltonian systems in the other spaces of the diagram.

3. Poisson homogeneous spaces

Recall that $M = G/H$ is a homogeneous space if G is a Lie group and $H \subset G$ is a closed subgroup. We denote by $\pi: G \to M$ the canonical projection $g \mapsto gH$.

DEFINITION 3.1. A **Poisson homogeneous space** is a homogeneous space $M = G/H$ such that G is a Poisson Lie group, M is a Poisson manifold and $\pi: G \to M$ is a Poisson morphism.

If $M = G/H$ is a Poisson homogeneous space the natural left action of G on M is a Poisson action.

¿From proposition 2.3 we obtain an infinitesimal criteria for a homogeneous space to be Poisson.

PROPOSITION 3.2. *Let $M = G/H$ be a homogeneous space and let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of H. Then M is a Poisson homogeneous space iff $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra and $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is a Lie subalgebra.*

EXAMPLE 3.3. A Lie group G is trivially a homogeneous space for we have $G \simeq G \times G/H$ where $H \subset G \times G$ is the diagonal. Similarly, this makes a Poisson Lie group G into a Poisson homogeneous space, when we consider on $G \times G$ the Poisson bivector $\Lambda^G \oplus (-\Lambda^G)$. Then $G \times G$ is a Poisson Lie group with Lie bialgebra $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}^* \oplus \mathfrak{g}^*_{opp})$. If $\mathfrak{h} = \{(x, x) : x \in \mathfrak{g}\}$ is the Lie algebra of the diagonal H, then $\mathfrak{h}^\perp = \{(\xi, -\xi) : \xi \in \mathfrak{g}^*\}$ is a Lie subalgebra of $\mathfrak{g}^* \oplus \mathfrak{g}^*_{opp}$. From proposition 3.2 we conclude that $G \times G/H$ is Poisson homogeneous. The map $(g_1, g_2) \mapsto g_1 g_2^{-1}$ induces a Poisson isomorphism from $G \times G/H$ onto G.

Example 3.4. Let G be a Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. We define a new Lie algebra $\mathfrak{d}$ with underlying vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ and with Lie bracket:

$$[x_1 + \xi_1, x_2 + \xi_2] = [x_1, x_2] - ad^*\xi_2 \cdot x_1 + ad^*\xi_1 \cdot x_2 + [\xi_1, \xi_2]_* + ad^*x_1 \cdot \xi_2 - ad^*x_2 \cdot \xi_2 \tag{3.1}$$

There is an analogous bracket on $\mathfrak{d}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ being a Lie bialgebra, implies that $(\mathfrak{d}, \mathfrak{d}^*)$ is also a Lie bialgebra, called the **double** of $(\mathfrak{g}, \mathfrak{g}^*)$. The corresponding Poisson Lie group D is called the double of G. From proposition 3.2 one concludes that $D/G^* \simeq G$ and $D/G \simeq G^*$ are Poisson homogeneous spaces.

Example 3.5. In example I.1.4 we have introduced a Poisson bracket on the homogeneous space $P(n, \mathbb{R}) = SL(n, \mathbb{R})/SO(n, \mathbb{R})$. This space is Poisson homogeneous, since the annihilator $\mathfrak{so}(n, \mathbb{R})^\perp$ is the space $\mathfrak{p}$ of symmetric matrices of trace zero, and an elementary computation using (2.1) shows that $\mathfrak{p} \subset \mathfrak{g}^*$ is a subalgebra.

The Kirillov foliation of a Poisson homogeneous space is obtained as follows.

Proposition 3.6. *Let $M = G/H$ be a Poisson homogeneous space and let $H^\perp$ be the Lie subgroup of G^* whose Lie algebra is $\mathfrak{h}^\perp$. Then the dressing action $G^* \times G \to G$ induces an action of $H^\perp$ on M whose orbits are the leaves of the Kirillov foliation of M.*

Proof. We have the infinitesimal action $\psi\colon \mathfrak{h}^\perp \times G \to \mathfrak{X}(G)$ given by

$$\psi(\xi, g) = J(g) \cdot \xi^L, \qquad g \in G, \xi \in \mathfrak{h}^\perp, \tag{3.2}$$

induced from the left dressing action (1.3). This action factors through to an infinitesimal action of $\mathfrak{h}^\perp$ on $M = G/H$ iff

$$d_g\pi \cdot \psi(\xi, g) = d_{g\cdot h}\pi \cdot \psi(\xi, g \cdot h), \qquad \forall h \in H. \tag{3.3}$$

We prove (3.3) as follows. The map $(d_g\pi)^*\colon T^*_{g\cdot H}(G/H) \to T^*_g(G)$ is a bijection onto $\{\xi^L(g) : \xi \in \mathfrak{h}^\perp\}$, and we have

$$(d_{g\cdot h}\pi)^*(d_g\pi)^{*-1} \cdot \xi^L(g) = \xi^L(g \cdot h), \qquad \forall h \in H.$$

Therefore, from relation (3.2) it follows that for every $h \in H$:

$$\begin{aligned} d_{g\cdot h}\pi \cdot \psi(\xi, g\cdot h) &= d_{g\cdot h}\pi \cdot J(g\cdot h)\cdot \xi^L(g\cdot h) \\ &= d_{g\cdot h}\pi \cdot J(g\cdot h)\cdot (d_{g\cdot h}\pi)^* \cdot (d_g\pi)^{*-1}\cdot \xi^L(g) \\ &= d_g\pi \cdot J(g)\cdot \xi^L(g) = d_g\pi\cdot \psi(\xi, g), \end{aligned}$$

where the last equality holds since the Poisson tensor J projects down to G/H. □

4. Poisson symmetric spaces

By a **symmetric Lie group** we mean a pair (G, S) where G is a Lie group and $S: G \to G$ is an involutive automorphism. Denote by H^S the subgroup of elements of G fixed under S. It is a closed subgroup, so the connected component of the identity is a Lie subgroup $H \subset G$. The homogeneous space $M = G/H$ is a **symmetric space**.

The infinitesimal version of a symmetric space is a **symmetric Lie algebra**, i.e. a pair $(\mathfrak{g}, s)$ where $\mathfrak{g}$ is a Lie algebra and $s: \mathfrak{g} \to \mathfrak{g}$ is an involutive Lie algebra automorphism. Let $\mathfrak{h}$ and $\mathfrak{p}$ denote the $+1$ and -1 eigenspaces of s. Then the vector space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ holds, and since $s: \mathfrak{g} \to \mathfrak{g}$ is a homomorphism we have the relations

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}. \tag{4.1}$$

In the Poisson category we propose the following definition.

Definition 4.1.

(i) A **Poisson symmetric Lie group** is a pair (G, S) where G is a Poisson Lie group and $S: G \to G$ is an involutive Poisson Lie group anti-morphism.

(ii) A **symmetric Lie bialgebra** is a triple $(\mathfrak{g}, \mathfrak{g}^*, s)$ where $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra and $s: \mathfrak{g} \to \mathfrak{g}$ is an involutive Lie bialgebra anti-morphism.

Let G be a Poisson Lie group, and $S: G \to G$ a group homomorphism. The requirements for (G, S) to be Poisson symmetric are

$$S^2 = id, \qquad \{f\circ S, g\circ S\} = -\{f, g\}\circ S, \quad f, g \in C^\infty(G). \tag{4.2}$$

Similarly, let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra, with associated 1-cocycle $\varphi: \mathfrak{g} \to \mathfrak{g} \bigwedge \mathfrak{g}$. If $s: \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra homomorphism, then $(\mathfrak{g}, \mathfrak{g}^*, s)$ is a symmetric Lie bialgebra iff the relations

$$s^2 = id, \qquad \varphi \circ s = -s \circ \varphi, \tag{4.3}$$

hold. Here we have denoted by the same letter the endomorphism of $\bigwedge^* \mathfrak{g}$ extending the map $s: \mathfrak{g} \to \mathfrak{g}$.

If (G, S) is a Poisson symmetric Lie group, let $(\mathfrak{g}, \mathfrak{g}^*)$ be its tangent Lie bialgebra and set $s = Lie(S): \mathfrak{g} \to \mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{g}^*, s)$ is a symmetric Lie bialgebra, for conditions (4.1) imply (4.2). Conversely, a slight modification of the proof of theorem 1.2 gives the following correspondence.

THEOREM 4.2. *There is a one-to-one correspondence between simply connected Poisson symmetric Lie groups (G, S) and symmetric Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*, s)$.*

The results of the previous section on Poisson homogeneous spaces imply the following proposition.

PROPOSITION 4.3. *Let $M = G/H$ be the symmetric space associated with a Poisson symmetric Lie group (G, S). Then there is a unique Poisson structure on M such that $\pi: G \to M$ is a Poisson map. The leaves of the Kirillov foliation of M are the orbits of the action of $H^\perp$ on M induced from the dressing action of G^* on G. The symmetry $S_0: M \to M : gH \mapsto S(g)H$ is a Poisson anti-morphism.*

PROOF. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be the Lie bialgebra of G, and $s = Lie(S): \mathfrak{g} \to \mathfrak{g}$. Then the dual map $s^*: \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie algebra anti-morphism.

The decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, gives the dual space decomposition $\mathfrak{g}^* = \mathfrak{p}^\perp \oplus \mathfrak{h}^\perp$, and we have $s^*|_{\mathfrak{p}^\perp} = id$, $s^*|_{\mathfrak{h}^\perp} = -id$. Since $s^*: \mathfrak{g}^* \to \mathfrak{g}^*$ is an anti-morphism we obtain the relations

$$[\mathfrak{p}^\perp, \mathfrak{p}^\perp]_* \subset \mathfrak{h}^\perp, \quad [\mathfrak{h}^\perp, \mathfrak{p}^\perp]_* \subset \mathfrak{p}^\perp, \quad [\mathfrak{h}^\perp, \mathfrak{h}^\perp]_* \subset \mathfrak{h}^\perp. \tag{4.4}$$

In particular, $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is a Lie subalgebra, so the result follows from propositions 3.2 and 3.6. $\square$

Henceforth, we shall call a symmetric space $M = G/H$ as in the proposition, a **Poisson symmetric space**.

Example 4.4. Any Poisson Lie group is a Poisson symmetric space. In example 3.3 we have made $G\times G$ into a Poisson Lie group. The map $S\colon G\times G \to G\times G$ defined by $S(g_1, g_2) = (g_2, g_1)$ is an involutive Poisson Lie group anti-morphism. The fixed point set of S is the diagonal $H \subset G\times G$, so $G\times G/H$ is Poisson symmetric. As was observed in example 3.3, this space is Poisson isomorphic to G. The corresponding symmetric Lie bialgebra is $(\mathfrak{g}\oplus\mathfrak{g}, \mathfrak{g}^*\oplus\mathfrak{g}^*_{opp}, s)$ where $s\colon \mathfrak{g}\oplus\mathfrak{g} \to \mathfrak{g}\oplus\mathfrak{g}$ is the involution $s(x_1, x_2) = (x_2, x_1)$.

Example 4.5. The Poisson homogeneous space $P(n,\mathbb{R}) = SL(n,\mathbb{R})/SO(n,\mathbb{R})$ of example 3.5 is a Poisson symmetric space: the map $S\colon SL(n,\mathbb{R}) \to SL(n,\mathbb{R})$ defined by[6] $g \mapsto (g^T)^{-1}$ is an involutive Poisson Lie group anti-morphism, whose fixed point set is $SO(n,\mathbb{R})$.

Let $\mathfrak{g}$ be a real Lie algebra. Recall that if $(\mathfrak{g}, s)$ is a symmetric Lie algebra with decomposition $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$, then the **dual symmetric Lie algebra** is the pair $(\tilde{\mathfrak{g}}, \tilde{s})$, where $\tilde{\mathfrak{g}}$ is the subalgebra $\mathfrak{h}\oplus i\mathfrak{p}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ and $\tilde{s}\colon \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is the involution $x + iy \mapsto x - iy$. Duality works in the Poisson category. If $(\mathfrak{g}, \mathfrak{g}^*, s)$ is a symmetric Lie bialgebra then its dual is the symmetric Lie bialgebra $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}^*, \tilde{s})$, where $(\tilde{\mathfrak{g}}, \tilde{s})$ is the dual symmetric Lie algebra to $(\mathfrak{g}, s)$ and $\tilde{\mathfrak{g}}^*$ is the subalgebra $\mathfrak{p}^\perp \oplus i\mathfrak{h}^\perp$ of $\mathfrak{g}^*_{\mathbb{C}}$ with Lie product $i[\ ,\ ]_*$. Thanks to relations (4.4), $\tilde{\mathfrak{g}}^*$ is well defined. The extensions to $\mathfrak{g}_{\mathbb{C}}$ of the 1-cocycles $\tilde{\varphi}$ and φ associated with the Lie bialgebras $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}^*)$ and $(\mathfrak{g}, \mathfrak{g}^*)$ are related by

$$\tilde{\varphi} = i\varphi. \tag{4.5}$$

[6] The superscript T denotes matrix transposition.

It follows from (4.2) that $\tilde{s}^2 = id$ and $\tilde{\varphi} \circ \tilde{s} = -\tilde{s} \circ \tilde{\varphi}$, so $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}^*, \tilde{s})$ is a symmetric Lie bialgebra.

We finish this section by recalling some facts from the theory of symmetric spaces (for details see [He,Ko-No]).

Let M be a manifold with an affine connection. A symmetry S_m at a point $m \in M$ is a diffeomorphism of a neighborhood U of m into itself sending $\exp(X) \mapsto \exp(-X)$. If $(x^1, \ldots, x^n)$ are normal coordinates then S_m sends $(x^1, \ldots, x^n) \mapsto (-x^1, \ldots, -x^n)$. In particular, $S_m^2 = id$ and $T_m S_m = -id$. The manifold M is said to be **affine symmetric** if for each $m \in M$ the symmetry S_m is a globally defined affine transformation of M.

Fix a point $m_0 \in M$ on an affine symmetric manifold, and denote by $A(M)$ the group of affine transformations of M. Then if $G = A_0(M)$ is the connected component of the identity of $A(M)$ and $H \subset G$ is the subgroup of affine transformations fixing m_0, a standard argument shows that $M = G/H$. Moreover, if we define $S\colon G \to G$ by

$$S(g) = S_{m_0} \circ g \circ S_{m_0}^{-1},$$

the pair (G, S) is a symmetric Lie group, and $H_0^S \subset H \subset H^S$. The connection on M coincides with the unique G-invariant, torsion free, affine connection on G/H [Ko-No, Ch.XI].

A Riemannian manifold M is said to be **Riemannian symmetric** if it is affine symmetric with respect to the Levi-Civita connection. In this case we can take G and K to be, respectively, the connected component of the identity of the group of isometries and the isotropy subgroup of G at a point $m_0 \in M$. The corresponding symmetric space $M = G/K$ satisfies

$$Ad_{\mathfrak{g}}(K) \quad \text{is compact.}$$

When we express this condition in terms of the associated symmetric Lie algebra $(\mathfrak{g}, s)$, we see that $\mathfrak{k}$, the set of fixed points of s, is a compactly embedded subalgebra of $\mathfrak{g}$, i.e. $(\mathfrak{g}, s)$ is an **orthogonal symmetric Lie algebra** [He, Ch.IV].

A symmetric Lie algebra $(\mathfrak{g}, s)$, with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, is said to be **effective** if $\mathfrak{k} \cap \mathrm{Center}(\mathfrak{g}) = \{0\}$. The symmetric Lie algebras associated with Riemannian symmetric spaces $M = G/K$ are effective because K has no normal subgroups of G distinct from $\{e\}$. Conversely, every effective, orthogonal, symmetric Lie algebra is associated with a Riemannian symmetric space. For effective, orthogonal, symmetric Lie algebras there is the following classical decomposition [He, Ch.V].

PROPOSITION 4.6. *Let $(\mathfrak{g}, s)$ be an effective, orthogonal, symmetric Lie algebra. Then there are ideals $\mathfrak{g}_+$, $\mathfrak{g}_0$, and $\mathfrak{g}_-$, such that:*

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-.$$

Denote by s_+, s_0, and s_-, resp., the restrictions of s to each of this factors. Then $(\mathfrak{g}_+, s_+)$, $(\mathfrak{g}_0, s_0)$ and $(\mathfrak{g}_-, s_-)$ are effective, orthogonal, symmetric Lie algebras satisfying:

(i) $(\mathfrak{g}_+, s_+)$ is of **compact type***, i.e., $\mathfrak{g}_+$ is compact and semisimple;*

(ii) $(\mathfrak{g}_0, s_0)$ is of **euclidean type***, i.e., $\mathfrak{k}_0$ is compact and $\mathfrak{p}_0$ is an abelian ideal;*

(iii) $(\mathfrak{g}_-, s_-)$ is of **noncompact type***, i.e., $\mathfrak{g}_-$ is semisimple and $\mathfrak{k}_- \oplus \mathfrak{p}_-$ is a Cartan decomposition;*

If $(\mathfrak{g}, s)$ is an orthogonal symmetric Lie algebra of compact type then its dual symmetric Lie algebra $(\tilde{\mathfrak{g}}, \tilde{s})$ is an orthogonal symmetric Lie algebra of noncompact type, and vice-versa.

The classification into compact, euclidean and noncompact type is justified because the associated Riemannian spaces have, respectively, non-negative, zero and non-positive sectional curvature.

5. R-matrices

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra, so the map $\varphi\colon \mathfrak{g} \to \mathfrak{g} \bigwedge \mathfrak{g}$ dual to the Lie bracket on $\mathfrak{g}^*$ is a 1-cocycle. We shall assume the cohomology $H_1(\mathfrak{g}, V)$, where V is the $\mathfrak{g}$-module $\mathfrak{g} \wedge \mathfrak{g}$, vanishes. In particular, φ is exact: $\varphi = \delta r$

A 0-cochain r is just an element of $\mathfrak{g} \bigwedge \mathfrak{g}$, which can also be viewed as a skew-symmetric linear transformation $r\colon \mathfrak{g}^* \to \mathfrak{g}$. The dual of the coboundary $\varphi = \delta r$ is then given by:

$$\varphi^*(\xi_1 \wedge \xi_2) = ad^*(r\xi_1) \cdot \xi_2 - ad^*(r\xi_2) \cdot \xi_1 \equiv [\xi_1, \xi_2]_*.$$

Define the element $[r, r] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ by[7]:

$$< [r, r], \xi_1 \wedge \xi_2 \wedge \xi_3 >= \bigodot_{\xi_1, \xi_2, \xi_3} \langle \xi_1, [r(\xi_2), r(\xi_3)].$$

Then it is easy to show that $[\ ,\ ]_*$ defines a Lie bracket on $\mathfrak{g}^*$ iff r satisfies the **Yang-Baxter equation**:

$$(\mathrm{YB}_1) \qquad [r, r] \quad \text{is } ad_{\mathfrak{g}}\text{-invariant;}$$

We conclude that:

Proposition 5.1. *Let $\mathfrak{g}$ be a Lie algebra such that $H_1(\mathfrak{g}, V) = 0$. The Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$ are in one-to-one correspondence with skew-symmetric solutions of the Yang-Baxter equation.*

For a skew-symmetric solution r of (YB_1) the corresponding Poisson Lie bivector on G is given by:

$$(5.1) \qquad \Lambda_g = (L_g)_* r - (R_g)_* r,$$

The case

$$(\mathrm{YB}_1(0)) \qquad [r, r] = 0,$$

is known as the **classical Yang-Baxter equation**. In this case, $\Lambda_g^L \equiv (L_g)_* r$ and $\Lambda_g^R \equiv (R_g)_* r$ are, respectively, left and right invariant Poisson bivectors on G [Dr].

[7]The bracket defined here is an algebraic analog of the Schouten bracket.

Conversely, every (left or right) invariant Poisson bivector on G takes this form, for some solution r of $\mathrm{YB}_1(0)$.

Suppose now that we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via an invariant, non-degenerate, symmetric form $(\ ,\ )$ (for a classification of Lie algebras admitting such a bilinear form see [Me-Re]). Under the identification $\mathfrak{g}^* \simeq \mathfrak{g}$, the 0-cochain $r\colon \mathfrak{g}^* \to \mathfrak{g}$ corresponds to a skew-symmetric linear map $A\colon \mathfrak{g} \to \mathfrak{g}$, and the Lie bracket $[\ ,\ ]_*$ corresponds to the Lie bracket $[\ ,\ ]_A$ on $\mathfrak{g}$ defined by:

$$[x,y]_A = [Ax,y] + [x,Ay].$$

This leads to the following definition [STS1]:

Definition 5.2. An **r-matrix** is a linear transformation $R\colon \mathfrak{g} \to \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ with the property that the modified bracket $[x,y]_R \equiv [Rx,y] + [x,Ry]$ defines a second Lie algebra structure. The pair $(\mathfrak{g}, R)$ is called a **double Lie algebra**.

Henceforth, we shall write $\mathfrak{g}_R$ to denote the Lie algebra with underlying vector space $\mathfrak{g}$ and Lie bracket $[\ ,\ ]_R$.

The non-trivial condition to be verified in definition 5.2 above is the Jacobi identity for $[\ ,\ ]_R$. This identity can be written in the form:

$$\bigodot_{x,y,z} \big[[Rx,Ry] - R[x,y]_R, z\big] = 0.$$

We conclude:

Proposition 5.3. *Let $\mathfrak{g}$ be a Lie algebra such that $H_1(\mathfrak{g}, V) = 0$, and suppose that there exists an invariant, non-degenerate, symmetric form on $\mathfrak{g}$. The Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$ are in one-to-one correspondence with skew-symmetric solutions $A\colon \mathfrak{g} \to \mathfrak{g}$ of*

$$(\mathrm{YB}_2) \qquad \bigodot_{x,y,z} \big[[Ax,Ay] - A[x,y]_A, z\big] = 0.$$

In this case the bracket (5.1) is known as the **Sklyanin bracket**, and it takes the form

$$\{f_1, f_2\}(g) = (A(\tilde{\nabla} f_1(g)), \tilde{\nabla} f_2(g)) - (A(\nabla f_1(g)), \nabla f_2(g)), \tag{5.2}$$

where we have denoted by $\tilde{\nabla}$ and ∇, resp., the left and right differentials on G:

$$(\tilde{\nabla} f(g), x) = \frac{d}{dt} f(g \exp(tx))|_{t=0} \qquad (\nabla f(g), x) = \frac{d}{dt} f(\exp(tx)g)|_{t=0},$$

for all $g \in G,\ x \in \mathfrak{g}$. In this case, the dressing action is obtained as follows. By the duality, we can identify an element $x \in \mathfrak{g}_A$ with a right invariant differential form $\alpha_x \in \mathfrak{g}^*$. The infinitesimal dressing action $G \times \mathfrak{g}_A \to \mathfrak{X}(G) : (g, x) \to \psi_x(g)$ is the image of $-\alpha_x$ under the Poisson tensor on G (cf. (1.3)), and a small computation using (5.2) shows that

$$\psi_x(g) = T_e L_g (Ad\ g^{-1} \circ A \cdot x - A \circ Ad\ g^{-1} \cdot x). \tag{5.3}$$

Integrating we obtain the (global) dressing action.

If $\mathfrak{g}$ is a semisimple Lie algebra the hypothesis made above are satisfied: the Killing form provides an invariant bilinear form (,), and Whitehead's lemma gives $H_1(\mathfrak{g}, V) = 0$. In this case the Yang-Baxter equation takes a very special form.

Proposition 5.4. *Let $\mathfrak{g}$ be a real semisimple Lie algebra with canonical decomposition*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_p \oplus \mathfrak{g}_{p+1} \oplus \cdots \oplus \mathfrak{g}_{p+q}, \tag{5.4}$$

where each $\mathfrak{g}_i$ is simple, and in the first p factors $(g_i)_{\mathbb{C}}$ is simple and in the last q factors each $(g_i)_{\mathbb{C}}$ is a sum of two simple ideals. Then (YB_1) is equivalent to

$$[Ax, Ay] - A[x, y]_A = Q[x, y], \tag{YB}$$

where $Q\colon \mathfrak{g} \to \mathfrak{g}$ is a linear map such that

$$Q|_{\mathfrak{g}_i} = \alpha_i I_i, \qquad 1 \le i \le p,\ \alpha_i \in \mathbb{R}$$

$$Q|_{\mathfrak{g}_j} = \alpha_j I_j + \beta_j J_j, \qquad p+1 \le j \le p+q,\ \alpha_i, \beta_j \in \mathbb{R}$$

with I_i denoting the identity map on $\mathfrak{g}_i$, and J_j the complex structure on $\mathfrak{g}_j$.

PROOF. Denote by q the element corresponding to $[r,r]$ under the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ provided by the Killing form $(\ ,\ )$:

$$\begin{aligned} q(x,y,z) &= \bigodot_{x,y,z}([Ax,Ay],z) \\ &= ([Ax,Ay] - A[x,y]_A, z), \qquad x,y,z \in \mathfrak{g}. \end{aligned} \tag{5.5}$$

Then $q \in \bigwedge^3 \mathfrak{g}^*$ and (YB_1) gives

$$q([w,x],y,z) + q(x,[w,y],z) + q(x,y,[w,z]) = 0, \qquad \forall x,y,z,w \in \mathfrak{g},$$

i.e., q is an invariant 3-cocycle. A result of Koszul [Ko, sec.11] shows that if $H_1(\mathfrak{g},V) = H_2(\mathfrak{g},V) = 0$ the invariant 3-cocycles are in 1-1 correspondence with invariant symmetric bilinear forms, the correspondence being given by

$$q(x,y,z) = B([x,y],z). \tag{5.6}$$

If $\mathfrak{g}$ is simple, then two things can happen:

(i) $\mathfrak{g}_{\mathbb{C}}$ is simple, and there is a scalar $\alpha \in \mathbb{R}$ such that

$$B(x,y) = \alpha(x,y), \qquad \forall x,y \in \mathfrak{g};$$

(ii) $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where each $\mathfrak{g}_i$ is a simple ideal in $\mathfrak{g}_{\mathbb{C}}$ isomorphic, as a real Lie algebra, to $\mathfrak{g}$. The complex structure on $\mathfrak{g}_1$ (or $\mathfrak{g}_2$) induces a complex structure J in $\mathfrak{g}$, and there are scalars $\alpha, \beta \in \mathbb{R}$ such that

$$B(x,y) = \alpha(x,y) + \beta(Jx,y), \qquad \forall x,y \in \mathfrak{g};$$

Returning now to the general case where $\mathfrak{g}$ is semisimple, we have the decomposition (5.4), and there must exist Q as in the statement of the proposition, such that

$$\begin{aligned} q(x,y,z) &= B([x,y],z) \\ &= (Q[x,y],z), \qquad x,y \in \mathfrak{g} \end{aligned}$$

which together with expression (5.5) for q leads to

$$[Ax, Ay] - A[x, y]_A = Q[x, y],$$

as desired. $\square$

Let us consider in more detail the case where A is a solution of

$$\text{(YB}(\alpha)) \qquad [Ax, Ay] - A[x, y]_A = -\alpha[x, y],$$

for some scalar parameter α. The transformation $A \to (1/\sqrt{\alpha})A$ maps solutions of YB(α) to solutions of YB(1). Therefore, over the complex field, YB(α) is equivalent to either YB(0) or YB(1), while over the real field there is the third possibility YB(-1).

In the case where A solves YB(1) the factorization results of [STS1] allow one to give an explicit description of the dressing action.

PROPOSITION 5.5. *Set* $\mathfrak{g}_\pm \equiv Im(A \pm I)$, $\mathfrak{k}_\pm \equiv Ker(A \mp I)$. *Then:*

(a) $A \pm I \colon \mathfrak{g}_A \to \mathfrak{g}$ *are Lie algebras homomorphisms;*

(b) $\mathfrak{g}_\pm$ *are subalgebras of* $\mathfrak{g}$*;*

(c) $\mathfrak{k}_\pm$ *are ideals in* $\mathfrak{g}_\pm$*;*

Moreover, let $\theta \colon \mathfrak{g}_+/\mathfrak{k}_+ \to \mathfrak{g}_-/\mathfrak{k}_- : (A+I)x + \mathfrak{k}_+ \longmapsto (A-I)x + \mathfrak{k}_-$ *be the Cayley transform of* A. *Then:*

(d) Every $x \in \mathfrak{g}$ *has a unique factorization*

$$x = x_+ - x_-, \qquad x_\pm \in \mathfrak{g}_\pm, \quad \theta(\bar{x}_+) = \bar{x}_-,$$

and $\mathfrak{g}_A$ *can be identified with the subalgebra*

$$\tilde{\mathfrak{g}}_A = \{(x_+, x_-) \in \mathfrak{g}_+ \oplus \mathfrak{g}_- : \theta(\bar{x}_+) = \bar{x}_-\}. \tag{5.7}$$

This result has a (local) Lie group counterpart. Let $G_\pm \subset G$ be (local) Lie subgroups corresponding to the subalgebras $\mathfrak{g}_\pm \subset \mathfrak{g}$. Then each $g \in G$ has a unique factorization:

$$g = g_+ g_-^{-1}, \qquad g_\pm \in G_\pm, \qquad \Theta(\bar{g}_+) = \bar{g}_-, \tag{5.8}$$

where Θ denotes the lift of θ. This furnishes a concrete realization for G_A: it is the (local) Lie group with base manifold G and group operation defined by

$$g \wedge h = g_+ h g_-^{-1}. \tag{5.9}$$

Proposition 5.6. *Using the model $G_A \simeq G$ the dressing action $G \times G_A \to G$ is given by the formula*

$$(h, g) \mapsto g_+^{-1} h (h^{-1} g h)_+. \tag{5.10}$$

The usefulness of this proposition is limited by ones ability to solve the factorization problem (5.8). For evolution equations, where the groups in question are loop groups, this is a Riemann-Hilbert type factorization problem, while for the finite dimensional problems we are considering this is an algebraic factorization.

6. Orthogonal symmetric Lie bialgebras

In this section we consider an orthogonal symmetric Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, s)$ with $\mathfrak{g}$ semisimple. It follows from the results of the previous section that the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is defined by a skew-symmetric r-matrix $A: \mathfrak{g} \to \mathfrak{g}$, and that $\mathfrak{g}^* \simeq \mathfrak{g}_A$ under the identification provided by the Killing form $(\ ,\ )$. It is easy to see that condition (4.3) relating s and the 1-cocycle φ is now equivalent to the anti-commutation relation

$$sA = -As. \tag{6.1}$$

Lemma 6.1. *Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the canonical decomposition into eigenspaces of s. Then*

$$A(\mathfrak{k}) \subset \mathfrak{p}, \qquad A(\mathfrak{p}) \subset \mathfrak{k}. \tag{6.2}$$

Proof. Use (6.1). $\square$

If $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}^*, \tilde{s})$ is the orthogonal symmetric Lie bialgebra dual to $(\mathfrak{g}, \mathfrak{g}^*, s)$, then the Lie bialgebra $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}^*)$ is also defined by an r-matrix $\tilde{A}: \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$. In fact, it follows from (4.5) that the extensions of A and $\tilde{A}$ to $\mathfrak{g}_{\mathbb{C}}$ are related by

$$\tilde{A} = iA. \tag{6.3}$$

Note that relations (6.2) imply that $\tilde{A}(\mathfrak{k}) \subset i\mathfrak{p}$ and $\tilde{A}(i\mathfrak{p}) \subset \mathfrak{k}$, so $\tilde{A}$ maps $\tilde{\mathfrak{g}} = \mathfrak{k} \oplus i\mathfrak{p}$ into itself. Also, if A solves (YB) with coefficient Q, as in proposition 5.4, then $\tilde{A}$ solves (YB) with coefficient $-Q$.

Our objective, for the remainder of this section, is to explore the structure theory of real semisimple Lie algebras to obtain solutions of (YB) satisfying (6.1), and hence examples of symmetric Lie bialgebras.

The semisimple orthogonal symmetric Lie algebras decompose into irreducible factors [He, Ch.VIII]. The irreducible orthogonal symmetric Lie algebras of the compact type are:

(I) Pairs $(\mathfrak{u}, \theta))$ where $\mathfrak{u}$ is a compact simple Lie algebra and θ is an involutive automorphism of $\mathfrak{u}$;

(II) Pairs $(\mathfrak{u} \oplus \mathfrak{u}, \theta)$ where $\mathfrak{u}$ is a compact simple Lie algebra and $\theta(x_1, x_2) = (x_2, x_1)$;

The irreducible orthogonal symmetric Lie algebras of the noncompact type are:

(III) Pairs $(\mathfrak{g}, \theta)$ where $\mathfrak{g}$ is a noncompact simple Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ is a simple Lie algebra, and θ is a Cartan involution;

(IV) Pairs $(\mathfrak{g}^{\mathbb{R}}, \theta)$ where $\mathfrak{g}^{\mathbb{R}}$ is a complex simple Lie algebra $\mathfrak{g}$, viewed as a real Lie algebra, and θ is conjugation with respect to a maximal compactly embedded subalgebra;

Types I and III, as well as types II and IV, are dual to each other. For each of this types we would like to find a skew-symmetric solution A of (YB), satisfying (6.1), so they become symmetric Lie bialgebras.

Let $\mathfrak{g}$ be a simple complex Lie algebra. Solutions of the Yang-Baxter equation were classified in [Be-Dr]. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, with associated root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, and root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha.$$

Also, let $\{(\mathfrak{h}_\alpha, x_\alpha) : \alpha \in \Delta\}$ be a Weyl basis, so $h_\alpha \in \mathfrak{h}$, $x_\alpha \in \mathfrak{g}^\alpha$, and they satisfy

$$\begin{aligned} [x_\alpha, x_{-\alpha}] &= h_\alpha, \qquad [h, x_\alpha] = \alpha(h) x_\alpha, \quad h \in \mathfrak{h}, \\ [x_\alpha, x_\beta] &= 0, \qquad \alpha + \beta \notin \Delta,\ \alpha + \beta \neq 0, \\ [x_\alpha, x_\beta] &= N_{\alpha,\beta} x_{\alpha+\beta} \qquad \alpha + \beta \in \Delta, \end{aligned} \tag{6.4}$$

where $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$. The simplest solutions of $YB(1)$, in the Belavin-Drinfel'd classification, take the form

$$Ax = \begin{cases} x & \text{if } x \in \bigoplus_{\alpha>0} \mathfrak{g}^\alpha \\ 0 & \text{if } x \in \mathfrak{h} \\ x & \text{if } x \in \bigoplus_{\alpha<0} \mathfrak{g}^\alpha \end{cases} \tag{6.5}$$

This solution is skew-symmetric relative to the Killing form on $\mathfrak{g}$ since $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ if $\alpha + \beta \neq 0$.

Type IV. Let $\mathfrak{g}$ be a simple complex Lie algebra as above. Relations (6.4) show that

$$\mathfrak{u} \equiv \sum_{\alpha \in \Delta} i\mathbb{R} h_\alpha + \sum_{\alpha \in \Delta} \mathbb{R}(x_\alpha - x_{-\alpha}) + \sum_{\alpha \in \Delta} i\mathbb{R}(x_\alpha + x_{-\alpha}), \tag{6.6}$$

is a compact real form of $\mathfrak{g}$. Let $\mathfrak{g}^\mathbb{R}$ denote the real Lie algebra obtained from $\mathfrak{g}$ by restricting the scalars to $\mathbb{R}$, and let $\theta \colon \mathfrak{g}^\mathbb{R} \to \mathfrak{g}^\mathbb{R}$ be conjugation with respect to $\mathfrak{u}$. Then $(\mathfrak{g}^\mathbb{R}, \theta)$ is a symmetric Lie algebra of type IV. The Killing forms on $\mathfrak{g}^\mathbb{R}$ and $\mathfrak{g}$ are related by

$$(x, y)_{\mathfrak{g}^\mathbb{R}} = 2Re(x, y)_\mathfrak{g}, \tag{6.7}$$

so (6.5) defines a skew-symmetric solution of YB on $\mathfrak{g}^\mathbb{R}$. Moreover, we have

$$A\theta = -\theta A.$$

In fact, if we set $\mathfrak{p} = i\mathfrak{u}$, so $\mathfrak{g}^{\mathbb{R}} = \mathfrak{u} \oplus \mathfrak{p}$, a small computation shows that

$$A(\mathfrak{u}) \subset \mathfrak{p}, \qquad A(\mathfrak{p}) \subset \mathfrak{u},$$

which is equivalent to (6.7). Therefore, A makes $(\mathfrak{g}^{\mathbb{R}}, \theta)$ into a symmetric Lie bialgebra.

Type II. The symmetric Lie algebras of type II, being dual to type IV, are also transformed into symmetric Lie bialgebras. If $(\mathfrak{g}^{\mathbb{R}}, \theta)$ is of type IV, as above, its dual is the symmetric Lie algebra $(\tilde{\mathfrak{g}}, \tilde{\theta})$ where $\tilde{\mathfrak{g}} = \mathfrak{u} \oplus J\mathfrak{p}$ (J denoting the complex structure on $(\mathfrak{g}^{\mathbb{R}})_{\mathbb{C}}$) and $\tilde{\theta}\colon \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ is the involution

$$\tilde{\theta}(x + Jy) = x - Jy, \qquad x \in \mathfrak{u},\ y \in \mathfrak{p}.$$

The dual r-matrix $\tilde{A}$, as obtained from (6.3), is given by

$$\tilde{A}(x + Jy) = -Ay + JAx, \qquad x \in \mathfrak{u},\ y \in \mathfrak{p}. \tag{6.8}$$

If we set $u_1 \equiv \{x + Jix : x \in \mathfrak{u}\}$ and $u_2 \equiv \{x - Jix : x \in \mathfrak{u}\}$, then u_1 and u_2 are ideals in $\tilde{\mathfrak{g}}$ isomorphic to $\mathfrak{u}$, so we have $\tilde{\mathfrak{g}} \simeq \mathfrak{u} \oplus \mathfrak{u}$. Under this isomorphism, we see that

$$\tilde{\theta}(x_1, x_2) = (x_2, x_1), \qquad \tilde{A}(x_1, x_2) = (-iAx_2, iAx_1), \qquad x_1, x_2 \in \mathfrak{u}. \tag{6.9}$$

This gives an explicit form for the r-matrix on any symmetric Lie algebra of type II which makes it into a symmetric Lie bialgebra.

Type III. Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra of type III. Our plan is to use the root space decomposition of $\mathfrak{g}_{\mathbb{C}}$ relative to a carefully chosen Cartan subalgebra $\mathfrak{h}$, so solution (6.5) of YB in $\mathfrak{g}_{\mathbb{C}}$ can be reduced to $\mathfrak{g}$.

Since $\theta\colon \mathfrak{g} \to \mathfrak{g}$ is a Cartan involution, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ. The Killing form is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$, so $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact form of $\mathfrak{g}_{\mathbb{C}}$. We denote by σ and τ the conjugations of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{g}$ and $\mathfrak{u}$. Then

$$\theta = \sigma\tau = \tau\sigma. \tag{6.10}$$

Let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{g}$ which is θ-invariant, so that:

$$\mathfrak{a} = \mathfrak{a}_{\mathfrak{k}} \oplus \mathfrak{a}_{\mathfrak{p}}, \quad \text{where} \quad \mathfrak{a}_{\mathfrak{k}} \equiv \mathfrak{a} \cap \mathfrak{k},\ \mathfrak{a}_{\mathfrak{p}} \equiv \mathfrak{a} \cap \mathfrak{p}.$$

Then $\mathfrak{h} = \mathfrak{a}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ and we have the root space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C}}^{\alpha},$$

where $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ is the root system of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}$. We can choose a Weyl basis $\{(\mathfrak{h}_\alpha, x_\alpha) : \alpha \in \Delta\}$ such that $\tau(x_\alpha) = -x_\alpha$. Then there are constants $k_\alpha \in \mathbb{C}$ such that

$$\sigma(x_\alpha) = k_\alpha x_{\alpha^\sigma}, \tag{6.11}$$

where $\alpha \mapsto \alpha^\sigma$ is the involution of Δ defined by

$$\alpha^\sigma(h) = \overline{\alpha(\sigma(h))}, \qquad \forall h \in \mathfrak{h}. \tag{6.12}$$

Proposition 6.2. *Suppose that we can choose a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{g}$ and an ordering of Δ such that $\alpha \mapsto \alpha^\sigma$ is an order preserving map. Then solution (6.5) of YB on $\mathfrak{g}_{\mathbb{C}}$ restricts to a solution on $\mathfrak{g}$ satisfying:*

$$A\theta = -\theta A.$$

Proof. Since $\alpha \mapsto \alpha^\sigma$ is order preserving we have:

$$\begin{aligned} \sigma A x_\alpha &= \sigma(\pm x_\alpha) \\ &= \pm k_\alpha x_{\alpha^\sigma} \\ &= A(k_\alpha x_{\alpha^\sigma}) = A\sigma x_\alpha, \qquad \pm\alpha > 0. \end{aligned}$$

Then $A\sigma = \sigma A$, so A restricts to $\mathfrak{g}$. On the other hand, we compute

$$\begin{aligned} \tau A x_\alpha &= \tau(\pm x_\alpha) \\ &= \mp x_\alpha \\ &= A x_{-\alpha} = -A\tau x_\alpha, \qquad \pm\alpha > 0. \end{aligned}$$

so we also have $\tau A = A\tau$. From (6.10) we conclude that $A\theta = -\theta A$. □

REMARK 6.3. In contrast with the complex case, the Cartan subalgebras of a real Lie algebra $\mathfrak{g}$ in general are not conjugate under $\mathrm{Int}(\mathfrak{g})$. The equivalent classes of Cartan subalgebras vary between two extreme cases. The case where the vector part $\mathfrak{a}_\mathfrak{p}$ is maximal abelian in $\mathfrak{p}$ and the case where the toral part $\mathfrak{a}_\mathfrak{k}$ is maximal abelian in $\mathfrak{k}$. In Araki's method ([A],[Wa,Ch.I]) of classification of real simple Lie algebras, Cartan subalgebras with maximal vector part are used, and to each pair $(\mathfrak{g}, \theta)$ is associated the Satake diagram. It is easy to see that the hypothesis of the proposition holds iff the Satake diagram of $(\mathfrak{g}, \theta)$ has no black holes.

EXAMPLE 6.4. Let $\mathfrak{g}$ be a normal real form of $\mathfrak{g}_\mathbb{C}$, so a Weyl basis can be chosen such that

$$\mathfrak{g} = \sum_{\alpha\in\Delta} \mathbb{R} h_\alpha + \sum_{\alpha\in\Delta} \mathbb{R} x_\alpha.$$

In this case we have

$$\mathfrak{k} = \sum_{\alpha>0} \mathbb{R}(x_\alpha - x_{-\alpha}), \qquad \mathfrak{p} = \sum_{\alpha\in\Delta} \mathbb{R} h_\alpha + \sum_{\alpha>0} \mathbb{R}(x_\alpha + x_{-\alpha}),$$

so the Cartan subalgebra $\mathfrak{a} = \sum_{\alpha\in\Delta} \mathbb{R} h_\alpha = \mathfrak{a}_\mathfrak{p}$ is maximal abelian in $\mathfrak{p}$. The map $\alpha \mapsto \alpha^\sigma$ is the identity map and hence proposition 6.2 holds. This agrees with the (obvious) fact that A restricts to $\mathfrak{g}$. This type of solutions of YB were considered in [D-L].

EXAMPLE 6.5. Let $\mathfrak{su}(p,p)$ be the real form of $\mathfrak{sl}(2p,\mathbb{C})$ consisting of all complex matrices of the form

$$\begin{pmatrix} A & B \\ \bar{B}^T & C \end{pmatrix} \qquad A, C \in \mathfrak{u}(p), \quad tr\ A + tr\ C = 0.$$

It is defined by the conjugation $\sigma(X) = -J\bar{X}^T J$, where $J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. The map $\theta(X) = \bar{X}^T$ is a Cartan involution of $\mathfrak{s}u(p,p)$ with eigenspaces

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix} \right\} \simeq \mathfrak{su}(p) \times \mathfrak{su}(p) \times \mathfrak{u}(1), \qquad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ \bar{B}^T & 0 \end{pmatrix} \right\}.$$

The subalgebra $\mathfrak{a}$ of matrices of the form

$$\begin{pmatrix} iD_1 & D_2 \\ D_2 & iD_1 \end{pmatrix} \qquad D_1, D_2 \text{ real diagonal matrices,}$$

is a θ-invariant Cartan subalgebra. It's complexification $\mathfrak{h} = \mathfrak{a}_{\mathbb{C}}$ consists of those matrices $h(X,Y) = \begin{pmatrix} X & Y \\ Y & X \end{pmatrix}$ in $\mathfrak{sl}(2p,\mathbb{C})$ with X and Y diagonal. Conjugation by the matrix $\frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}$ transforms $\mathfrak{h}$ into the standard Cartan subalgebra of $\mathfrak{sl}(2p,\mathbb{C})$. Hence, if we let ξ_i and η_i be the linear forms on $\mathfrak{h}$ defined by

$$\xi_i\left(h(X,Y)\right) = X_{ii} + Y_{ii}, \qquad \eta_i\left(h(X,Y)\right) = X_{ii} - Y_{ii},$$

the roots of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ are the differences between the elements

$$\xi_1, \ldots, \xi_p, \eta_p, \ldots, \eta_1. \tag{6.13}$$

Let us take as a basis $\{\alpha_1, \ldots, \alpha_{2p-1}\}$ for Δ the difference between consecutive elements of the sequence (6.13). Then we check using definition (6.12) that

$$\alpha_i^\sigma = \alpha_{2p-i}, \quad i = 1, \ldots, p, \qquad \alpha_p^\sigma = \alpha_p.$$

Therefore, for the order determined by this basis, the map $\alpha \mapsto \alpha^\sigma$ is order preserving, and proposition 6.2 gives a solution of YB in $\mathfrak{su}(p,p)$ making it into a Lie bialgebra.

Type I. By taking the dual to Lie bialgebras of type III we obtain Lie bialgebras of type I.

Example 6.6. Let us consider the dual Lie bialgebra $(\mathfrak{u}, \tilde{\theta})$ to the symmetric Lie bialgebra $(\mathfrak{g}, \theta)$ introduced in example 6.4. Then $\mathfrak{u}$ is the compact real form (6.6) of $\mathfrak{g}_{\mathbb{C}}$, $\tilde{\theta}\colon \mathfrak{u} \to \mathfrak{u}$ is the automorphism

$$\tilde{\theta}(x_\alpha - x_{-\alpha}) = x_\alpha - x_{-\alpha}, \qquad \tilde{\theta}(ih_\alpha) = -ih_\alpha, \qquad \tilde{\theta}(i(x_\alpha + x_{-\alpha})) = -i(x_\alpha + x_{-\alpha}),$$

and $\tilde{A}$ is the r-matrix defined by relations

$$\tilde{A}(x_\alpha - x_{-\alpha}) = i(x_\alpha + x_{-\alpha}), \qquad \tilde{A}(ih_\alpha) = 0, \qquad \tilde{A}(i(x_\alpha + x_{-\alpha})) = -(x_\alpha - x_{-\alpha}).$$

Remark 6.7. Other types of solutions of the Yang-Baxter equation for real Lie algebras were found in [L-Q,K-R-R]. However, this solutions do not satisfy the anti-commutation relation (6.1), and hence do not make them into symmetric Lie bialgebras. In [K-R-R] a Poisson structure is given on any hermitian symmetric space G/K, such that it makes it into a Poisson homogeneous space. However, this does not make it into a Poisson symmetric space.

7. The Cartan immersion

In this section we consider a Poisson symmetric space $M = G/K$ associated with an orthogonal symmetric Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \theta)$, so we have for some skew-symmetric r-matrix $A\colon \mathfrak{g} \to \mathfrak{g}$, anti-commuting with θ, the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}_A$. The Poisson-Lie bracket on G, here denoted by $\{\ ,\ \}_G$, takes the form (5.2), and according to proposition 4.3 reduces to a Poisson bracket $\{\ ,\}_M$ on G/K, such that the projection $\pi\colon G \to G/K$ is a Poisson morphism.

Now consider the map $\iota\colon M \to G$ defined by

$$\iota(gK) = g\theta(g^{-1}).$$

This map is well defined and we have the following well known proposition.

PROPOSITION 7.1. *The map $\iota : M \to G$ is a totally geodesic immersion. If P denotes its image, then $P = \exp(\mathfrak{p})$ and coincides with the connected component of $\{g\theta(g^{-1}) : g \in G\}$ containing the identity. Moreover, If M is of non-compact type, ι is an embedding and* $\exp : \mathfrak{p} \to P$ *is a diffeomorphism.*

We shall now study the Poisson properties of the Cartan immersion ι and the exponential map.

First note that P, being the connected component of the identity of the fixed point set of the diffeomorphism $g \mapsto \theta(g^{-1})$, is a regular submanifold of G. We then have the following proposition.

PROPOSITION 7.2. *The bilinear map $\{\ ,\ \} : C^\infty(G) \times C^\infty(G) \to C^\infty(G)$ defined by*

$$\begin{aligned} \{f_1, f_2\} =& (A(\tilde{\nabla} f_1), \tilde{\nabla} f_2) - (A(\nabla f_1), \nabla f_2) \\ & + (\theta A(\nabla f_1), \tilde{\nabla} f_2) - (\theta A(\tilde{\nabla} f_1), \nabla f_2) \end{aligned} \tag{7.2}$$

can be reduced to $C^\infty(P)$, and makes P into a Poisson manifold.

PROOF. The proof is divided into two parts.

(i) $\{\ ,\ \}$ restricts to P.

We have to show that if $f_1, f_2 \in C^\infty(P)$ and $\tilde{f}_1, \tilde{f}_2 \in C^\infty(P)$ are extensions of these functions to G, then the restriction of $\{\tilde{f}_1, \tilde{f}_2\}$ to P does not depend on the particular extensions that were choosen.

Since the map $(\tilde{f}_1, \tilde{f}_2) \mapsto \{\tilde{f}_1, \tilde{f}_2\}$ is bilinear, skew-symmetric, and satisfies the Leibnitz identity, we have a well defined bi-vector $\Lambda \in \bigwedge^2 T(G)$ by setting

$$\{\tilde{f}_1, \tilde{f}_2\} = < \Lambda, d\tilde{f}_1 \wedge d\tilde{f}_2 > .$$

Let $J : T^*(G) \to T(G)$ be the corresponding bundle map. All we have to show is that $Im\ J(g) \in T_g P$ whenever $g = \exp(p) \in P$. From (7.2) we compute

$$J(g) \cdot df(g) = (T_e L_g \cdot A \cdot Ad\, g^{-1} - T_e R_g \cdot A + T_e L_g \cdot \theta A - T_e R_g \cdot \theta A \cdot Ad\, g^{-1}) \cdot x, \tag{7.3}$$

where $x \equiv \nabla f(g)$. Now if $g = \exp(p) \in P$, a more or less tedious computation using

$$\theta \cdot Ad\ g = Ad\ g^{-1} \cdot \theta, \tag{7.4}$$

shows that

$$J(g) \cdot df(g) = (T_e L_g - T_e R_g) \cdot y + (T_e L_g + T_e R_g) \cdot z, \tag{7.5}$$

where

$$\begin{aligned} y &= \frac{1}{2}(\theta + I)(A + I)(I + Ad\ g^{-1})x \in \mathfrak{h}, \\ z &= \frac{1}{2}(\theta - I)(A + I)(I - Ad\ g^{-1})x \in \mathfrak{p}. \end{aligned}$$

The first term on the right hand side of (7.5) can be written in the form

$$\begin{aligned} \frac{d}{dt} \exp(-ty) \exp(p) \exp(ty) \mid_{t=0} &= \frac{d}{dt} \exp(Ad(\exp(-ty)) \cdot p) \mid_{t=0} \\ &= \frac{d}{dt} \exp(\sum_{i=o}^{\infty} \frac{1}{i!} ad^i(-ty) \cdot p) \mid_{t=0} \in T_{\exp(p)}P, \end{aligned}$$

since $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$. On the other hand, the second term on the right-hand side of (7.5), can be written as

$$\begin{aligned} \frac{d}{dt} \exp(tz) \exp(p) \exp(tz) \mid_{t=0} &= \frac{d}{dt} a(t)\theta(a(t)^{-1}) \mid_{t=0} \\ &= \frac{d}{dt} \iota(a(t)) \mid_{t=0} \in T_{\exp(p)}P, \end{aligned}$$

where $a(t) \equiv \exp(tz)\exp(p/2)$. Therefore, $J(g) \cdot df(g) \in T_g P$ as required.

(ii) The restriction of $\{\ ,\ \}$ to P is a Poisson bracket

We have to verify the Jacobi identity, or equivalently, the vanishing of the Schouten bracket

$$[\Lambda, \Lambda]_g = 0,$$

whenever $g = \exp(p) \in P$.

Using expression (I.2.5) for the Schouten bracket, we compute

$$(7.6) \qquad [\Lambda,\Lambda](\alpha,\beta,\gamma) = \bigodot_{\alpha,\beta,\gamma} <\beta,[J\alpha,J\gamma]> -L_{J\alpha}<\beta,J\gamma>,$$

We can identify right-invariant forms on G with elements of $\mathfrak{g}^* \simeq \mathfrak{g}_A$. Then if x, y and z correspond to right-invariant forms α, β and γ, we see that

$$(7.7) \qquad [\Lambda,\Lambda]_g(\alpha,\beta,\gamma) = \bigodot_{x,y,z}(y,[J(g)\cdot x, J(g)\cdot y]) - L_{J\alpha}(y, J(g)\cdot z),$$

where now J denotes the Poisson tensor in the right-invariant frame:

$$(7.8) \qquad J(g) = Ad\ g \cdot A \cdot Ad\ g^{-1} - A + Ad\ g\cdot\theta\cdot A - \theta\cdot A\cdot Ad\ g^{-1}.$$

It is shown in the appendix to this chapter, that using (7.8) in (7.7) one obtains

$$\begin{aligned}
[\Lambda,\Lambda]_g(\alpha,\beta,\gamma) &= \bigodot_{x,y,z}(Ad\ g^{-1}\cdot x, [A\cdot Ad\ g^{-1}\cdot y, A\cdot Ad\ g^{-1}\cdot z]) - (x,[Ax,Ay]) \\
(7.9) \qquad &+ \bigodot_{x,y,z}(\theta x, A[Ad\ g^{-1}\cdot y, Ad\ g^{-1}\cdot z]_A - [A\cdot Ad\ g^{-1}\cdot y, A\cdot Ad\ g^{-1}\cdot z]) \\
&+ \bigodot_{x,y,z}(\theta\cdot Ad\ g^{-1}\cdot x, A[y,z]_A - [Ay,Az]).
\end{aligned}$$

Since A solves (YB), the first factor on the right-hand side of (7.9) vanishes. Therefore, if we let $Q\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ be as in proposition 5.3 we see that (7.9) reduces to

$$[\Lambda,\Lambda]_g(\alpha,\beta,\gamma) = \bigodot_{x,y,z}(\theta x, Q(Ad\ g^{-1}\cdot y, Ad\ g^{-1}\cdot z)) - (\theta\cdot Ad\ g^{-1}\cdot x, Q(y,z).)$$

The proof of proposition 5.3 shows that

$$Q(Ad\ g\cdot y, Ad\ g\cdot z) = Ad\ g\cdot Q(y,z),$$

so we have

$$[\Lambda,\Lambda]_g(\alpha,\beta,\gamma) = \bigodot_{x,y,z}((Ad\ g\cdot\theta - \theta\cdot Ad\ g^{-1})x, Q(y,z)) = 0,$$

where we used (7.4). □

REMARK 7.3. Poisson brackets on Lie groups similar to (7.2) were studied in [Li-Par], but the results there don't apply directly in our scenario, since it can be shown that the Poisson bracket (7.2), in general, is not defined on the Lie group G.

Our next result shows that P, equipped with the Poisson bracket of proposition 7.2, is a "model" for the Poisson manifold G/K.

THEOREM 7.4. *The immersion* $\iota\colon G/K \to P$ *is a Poisson map.*

PROOF. Let $\hat{\iota}\colon G \to P$ be the map $g \mapsto g\theta(g^{-1})$. Since $\pi\colon G \to G/K$ is a Poisson map and the diagram

$$\begin{array}{ccc} G & \xrightarrow{\hat{\iota}} & P \\ \pi\big\downarrow & \nearrow_{\iota} & \\ G/K & & \end{array}$$

commutes, all we have to show is that $\hat{\iota}\colon G \to P$ is a Poisson map. Let $f \in C^\infty(P)$ and extend it to a smooth function on G. Then we compute

$$\begin{aligned} (\tilde{\nabla}(f\circ\hat{\iota})(g), x) &= d_g(f\circ\hat{\iota})\cdot T_eL_g\cdot x, \\ &= d_e(f\circ R_{\theta(g^{-1})}\circ L_g)\cdot(I-\theta)\cdot x, \qquad \forall x\in\mathfrak{g}, g\in G. \end{aligned} \tag{7.10}$$

$$\begin{aligned} (\nabla(f\circ\hat{\iota})(g), x) &= d_g(f\circ\hat{\iota})\cdot T_eR_g\cdot x \\ &= d_{\hat{\iota}(g)}f\cdot T_e(\hat{\iota}\circ L_g)\cdot x \\ &= (\nabla f(\hat{\iota}(g)) - \theta\tilde{\nabla}f(\hat{\iota}(g)), x) \qquad \forall x\in\mathfrak{g}, g\in G. \end{aligned} \tag{7.11}$$

It follows from (7.10) that if $x \in \mathfrak{h}$ the scalar product $(\tilde{\nabla}(f\circ\hat{\iota})(g), x)$ vanishes, and so $\tilde{\nabla}(f\circ\hat{\iota})(g) \in \mathfrak{h}^\perp = \mathfrak{p}$. But then lemma 6.1 and (7.11) gives:

$$\begin{aligned} \{f_1\circ\hat{\iota}, f_2\circ\hat{\iota}\}_G(g) &= -(A(\nabla(f_1\circ\hat{\iota})(g)), \nabla(f_2\circ\hat{\iota})(g)) \\ &= -(A(\nabla f_1(\hat{\iota}(g)) - \theta(\tilde{\nabla}f_1(\hat{\iota}(g)))), \nabla f_2(\hat{\iota}(g)) - \theta(\tilde{\nabla}f_2(\hat{\iota}(g)))). \end{aligned}$$

Now set $x = \nabla f_1(\hat{\iota}(g))$, $y = \nabla f_2(\hat{\iota}(g))$, $\tilde{x} = \tilde{\nabla}f_1(\hat{\iota}(g))$ and $\tilde{y} = \tilde{\nabla}f_2(\hat{\iota}(g))$. Then

$$\begin{aligned} \{f_1\circ\hat{\iota}, f_2\circ\hat{\iota}\}_G(g) &= -(Ax, y) - (A\theta\tilde{x}, \theta\tilde{y}) + (Ax, \theta\tilde{y}) + (A\theta\tilde{x}, y) \\ &= (\theta A\tilde{x}, \tilde{y}) - (Ax, y) + (\theta Ax, \tilde{y}) - (\theta A\tilde{x}, y) \\ &= \{f_1, f_2\}(\hat{\iota}(g)), \end{aligned}$$

so $\hat{\iota}\colon G \to P$ is a Poisson map. □

REMARKS 7.5.

(i) The results in [D-L] resemble (a special case of) theorem 7.4, but the authors don't seem to have in mind the complete geometric picture we are aiming to present here.

(ii) The proof above shows that the Poisson bracket on P can also be written in the form

$$\{f_1, f_2\}_P = (A(\tilde{\nabla} f_1 - \theta \nabla f_1), \tilde{\nabla} f_2 - \theta \nabla f_2) \tag{7.12}$$

In [Gu], the bracket (7.12) is studied for the case $\theta = id$. It is shown there that, in this case, the map $g \mapsto g^n$ perserves the bracket for any fixed integer n. The infinitesimal version of this bracket is also given.

In our approach the case $\theta = id$ is trivial. Nevertheless we shall prove the following result:

PROPOSITION 7.6. *The bracket on G defined by*

$$\{f_1, f_2\} = (A(\tilde{\nabla} f_1 - \nabla f_1), \tilde{\nabla} f_2 - \nabla f_2), \tag{7.13}$$

is quadratic in exponential coordinates. In particular, it is invariant under scalings $\exp(x) \mapsto \exp(\alpha x)$ *for any scalar* $\alpha \in \mathbb{R}$.

PROOF. Consider the bracket on $\mathfrak{g}$ defined by

$$\{f_1, f_2\}(x) = (A[\nabla f_1(x), x], [\nabla f_2(x), x]). \tag{7.14}$$

This bracket is homogeneous of degree 2 and is invariant under scalings $x \mapsto \alpha x$. We claim that (7.13) and (7.14) are exp-related. In fact, we have the following formula for the differential of the exponetial map:

$$T_x \exp = T_e L_{\exp(x)} \frac{I - \exp(-ad\ x)}{ad\ x} \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (ad\ x)^n, \qquad x \in \mathfrak{g}.$$

It follows that if $f : G \to \mathbb{R}$ is smooth, we have

$$\begin{aligned} [\nabla(f \circ \exp)(x), x] &= (I - Ad(\exp(x)))\tilde{\nabla} f(\exp(x)) \\ &= \tilde{\nabla} f(\exp(x)) - \nabla f(\exp(x)), \end{aligned}$$

so the claim follows. □

8. Hamiltonian systems on symmetric spaces

We shall now consider Hamiltonian systems on a Poisson symmetric space, defined by certain special functions. For computational purposes it is easier to work on the image P of the Cartan embedding, rather than on the symmetric space G/K itself, and use the formulas for the brackets deduced in the previous section.

The basic result here is the following.

THEOREM 8.1. *Let $h: P \to \mathbb{R}$ be a function for which there exists a central extension $\tilde{h}: G \to \mathbb{R}$. Then Hamilton's equations on P take the "Lax pair form"*

$$\dot{p} = T_e L_p \cdot (R\nabla\tilde{h}) - T_e R_p \cdot (R\nabla\tilde{h}), \tag{8.1}$$

where $R = A(I - \theta)$. Moreover, if $f: P \to \mathbb{R}$ also has a central extension to G then

$$\{h, f\}_P = 0, \tag{8.2}$$

so any such function is a first integral of the system.

PROOF. If $\tilde{h}$ is central, $\tilde{\nabla}\tilde{h} = \nabla\tilde{h}$, so (8.1) follows from (7.3) and (8.2) follows from (7.12). □

The term "Lax Pair" is justified because if G is a matrix group then the equation of the motion (8.1) takes the form

$$\dot{L} = [L, R(\nabla\tilde{h})], \qquad L \in G.$$

The previous proposition can be interpreted as a non-linear version of the Adler-Kostant-Symes scheme [Ad,K,Sy].

We now consider the class of examples furnish by the r-matrices on normal real forms $\mathfrak{g}$ given in example 6.4. We let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and we use the Iwasawa decomposition of G ([He, ch.VI]), i.e. we decompose G as NAK where K, A, and N are the analytic subgroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n} \equiv \sum_{\alpha>0} \mathbb{R} \cdot x_\alpha$ (some notation as in example 6.4). Then the factorization results of Semenov-Tyan-Shansky, mention at the end of section 5 can be used, in the manner explained in [STS1] (see thm. 14), to prove the following result.

Proposition 8.2. *For the r-matrix naturally associated with a real normal form* $\mathfrak{g}$*, the solutions of Hamilton's equations (8.1), determined by an Hamiltonian* $h : P \to \mathbb{R}$ *admitting a central extension to* G*, take the form*

$$p(t) = b(t) \cdot p_0, \tag{8.3}$$

where $b(t)$ *is obtained by solving the factorization problem*

$$\exp(2t\nabla h(p_o)) = b(t)k(t), \qquad b(t) \in B \equiv NA,\ k(t) \in K.$$

In formula (8.3) the dot represents the action of G on P induced from the left action of G on G/K, i.e,

$$(g, p) \mapsto g \cdot p \equiv gp\theta(g^{-1}), \qquad g \in G,\ p \in P.$$

The Poisson symmetric spaces associated with the r-matrices of the proposition above are of non-compact type (see section 6). It should be clear that one can use the duality for Poisson symmetric spaces to study the solutions of dual systems on spaces of compact type.

Example 8.3. Toda lattice revisited. Let us consider the case of the Poisson bracket on $SL(n, \mathbb{R})/SO(n, \mathbb{R})$ studied in examples 1.4 (ch. I), and 1.3, 3.5, 4.5 (ch. III). It follows from the results of section 6 that it is associated with the r-matrix on the normal real form of $\mathfrak{sl}(n, \mathbb{R})$. The Poisson bracket is given by formula (I.1.10), and coincides with the Poisson bracket obtained from the r-matrix using the formalism of the previous section (theorem 7.3).

Now the explicit form (I.1.10) for the bracket, shows that it can be reduced to the set of positive definite tridiagonal matrices of the form

$$L = \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & & & & \\ & & & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & & & \\ & & & & a_{n-1} \\ & & & a_{n-1} & b_n \end{pmatrix}$$

so this set is a Poisson submanifold. The reduced bracket is given by the relations:

$$\{a_i, a_{i+1}\} = -a_i a_{i+1} \qquad \{a_i, b_i\} = 2a_i b_i$$
$$\{a_i, b_{i+1}\} = -2a_i b_{i+1} \qquad \{b_i, b_{i+1}\} = -4{a_i}^2$$

so we see that it coincides, up to a factor, with the quadratic Poisson bracket for the Toda lattice in Flaschka's variables (cf. I.4.9).

The powers of the traces, $h_k(L) = tr\ L^{k+1}$ furnish n-1 functionally independent functions on $P(n, \mathbb{R})$ which extend to central functions on $SL(n, \mathbb{R})$. The Hamiltonian system defined by h_0 is precisely the Toda lattice, and it follows from theorem 8.1, that this is a completely integrable system, a well known result due independently to Flaschka, Hénon and Manakov[Fl,H].

In this example, the factorization of proposition 8.2 is as follows. Any element $L \in SL(n, \mathbb{R})$ factors as $L = RQ$ where $Q \in SO(n, \mathbb{R})$ and $R \in B(n)$, the group of upper triangular matrices with determinant 1. The solution of the Toda flow is then obtained from

$$L(t) = R(t) L_0 R(t)^T,$$

where $R(t)$ is the solution of the factorization problem

$$\exp(2tL_0) = R(t)Q(t), \qquad R(t) \in B(n),\ Q(t) \in SO(n, \mathbb{R}).$$

This method of integration of the Toda flow was known to several authors (see e.g. [Pe]), and it was generalized to the "Fat Toda lattice" by Deift and co-workers. They also gave a dynamical interpretation of the QR-algorithm to compute eigenvalues of a symmetric matrix, as a time 1 flow of the Fat Toda lattice [D].

Appendix

In this appendix we furnish some details omitted in the proof of proposition 7.2, namely, we will show how (7.9) follows from (7.8) and (7.7).

We rewrite (7.7) in the form

$$[\Lambda,\Lambda]_g(\alpha,\beta,\gamma) = \bigodot_{x,y,z} (y,[J(g)\cdot x, J(g)\cdot z]) - (y, L_{J\alpha}(J(g))\cdot z), \tag{A.1}$$

and we compute the Lie derivative in the second term as follows. Let $\gamma(t)$ be the integral curve of the vector field $J\alpha$ through the point $g\in G$. Then a simple computation gives

$$\begin{aligned}\frac{d}{dt}Ad(\gamma(t))|_{t=0} &= Ad\ g\cdot ad(Ad\ g^{-1}\cdot J(g)x)\\ \frac{d}{dt}Ad(\gamma^{-1}(t))|_{t=0} &= -ad(Ad\ g^{-1}\cdot J(g)x)\cdot Ad\ g^{-1},\end{aligned}$$

so we have

$$\begin{aligned}L_{J\alpha}(J(g)) = {} & Ad\ g\cdot ad(w)\cdot A\cdot Ad\ g^{-1} - Ad\ g\cdot A\cdot ad(w)\cdot Ad\ g^{-1}\\ & + Ad\ g\cdot ad(w)\cdot\theta A + \theta A\cdot ad(w)\cdot Ad\ g^{-1}\end{aligned} \tag{A.2}$$

where we have set

$$w = Ad\ g^{-1}\cdot J(g)x.$$

Replacing (A.2) in (A.1) we find

$$\begin{aligned}[\Lambda,\Lambda]_g(\alpha,\beta,\gamma) = {} & \bigodot_{x,y,z}(Ad\ g^{-1}\cdot J(g)x, [Ad\ g^{-1}\cdot y, Ad\ g^{-1}\cdot z]_A)\\ & - (x,[J(g)\cdot y, J(g)\cdot z]) + (Ad\ g\cdot\theta A\cdot z, [J(g)\cdot x, y] + [x, J(g)\cdot y])\end{aligned} \tag{A.3}$$

The first term on the right hand sides gives:

$$\bigodot_{x,y,z}(Ad\ g^{-1}\cdot x, [A\cdot Ad\ g^{-1}\cdot y, A\cdot Ad\ g^{-1}\cdot z]) + \text{Rem } 1, \tag{A.4}$$

where

$$\text{Rem } 1 = \bigodot_{x,y,z} ((\theta A - Ad\ g^{-1} \cdot \theta A \cdot Ad\ g^{-1}) \cdot x, [Ad\ g^{-1} \cdot y, Ad\ g^{-1} \cdot z]_A).$$

The second term gives

$$\bigodot_{x,y,z} (x, [Ay, Az]) + \text{Rem } 2, \tag{A.5}$$

where

$$\text{Rem } 2 = \bigodot_{x,y,z} (x, [J_\theta(g) \cdot y, J_\theta(g) \cdot z] - [J(g) \cdot y, J_\theta(g) \cdot z] - [J_\theta(g) \cdot y, J(g) \cdot z]),$$

and we have denoted by $J_\theta(g)$ the terms in $J(g)$ that contain θ. There is a remarkable cancelation of terms when we add Rem 1, Rem2 and the third term on the right hand side of (A.3). In fact, the sum Rem 1 + Rem 2 + 3^{rd} term gives

$$\begin{aligned} &\bigodot_{x,y,z} (x, \theta A[Ad\ g^{-1} \cdot y, Ad\ g^{-1} \cdot z]_A - Ad\ g \cdot \theta A[y, z]_A) \\ &\quad + (x, [\theta A \cdot Ad\ g^{-1} \cdot y, \theta A \cdot Ad\ g^{-1} \cdot z]) + (x, [Ad\ g \cdot \theta A \cdot y, Ad\ g \cdot \theta A \cdot z]). \end{aligned} \tag{A.6}$$

Finally, using (A.4), (A.5) and (A.6) we conclude that

$$\begin{aligned} [\Lambda, \Lambda]_g(\alpha, \beta, \gamma) &= \bigodot_{x,y,z} (Ad\ g^{-1} \cdot x, [A \cdot Ad\ g^{-1} \cdot y, A \cdot Ad\ g^{-1} \cdot z]) - (x, [Ax, Ay]) \\ &\quad + \bigodot_{x,y,z} (x, Ad\ g \cdot \theta A[y, z]_A - [Ad\ g \cdot \theta Ay, Ad\ g \cdot \theta Az]) \\ &\quad + \bigodot_{x,y,z} (x, \theta A[Ad\ g^{-1} \cdot y, Ad\ g^{-1} \cdot z]_A - [\theta A \cdot Ad\ g^{-1} \cdot y, \theta A \cdot Ad\ g^{-1} \cdot z]). \end{aligned} \tag{A.7}$$

which gives immediately (7.9).

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