Invariants of Lie algebroids

Rui Loja Fernandes

July, 2002

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PART 1

Lie Algebroids: Basic Concepts

Basic Definitions

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A **morphism of Lie algebroids** is a bundle map $\phi : A_1 \to A_2$ which preserves anchors and Lie brackets.

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The **short exact sequence of a leaf** is the short exact sequence of Lie algebroids:

$$0 \longrightarrow \mathfrak{g}_L \longrightarrow A_L \stackrel{\#}{\longrightarrow} TL \longrightarrow 0$$

EXAMPLES	A
Ordinary Geometry (M a manifold)	$TM \to M$
Lie Theory ($\mathfrak{g}$ a Lie algebra)	$\mathfrak{g} \to \{*\}$
Foliation Theory ($\mathcal{F}$ a regular foliation)	$T\mathcal{F} \to M$
Equivariant Geometry ($\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g} \to M$
Presymplectic Geometry (M presymplectic)	$TM \times \mathbb{R} \to M$
Poisson Geometry (M Poisson)	$T^*M \to M$

Lie Algebroid Cohomology

A first example of a global invariant of a Lie algebroid:

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$$d_A Q(\alpha_0, \dots, \alpha_r) \equiv \frac{1}{r+1} \sum_{k=0}^{r+1} (-1)^k \# \alpha_k (Q(\alpha_0, \dots, \widehat{\alpha}_k, \dots, \alpha_r)) + \frac{1}{r+1} \sum_{k<l} (-1)^{k+l+1} Q([\alpha_k, \alpha_l], \alpha_0, \dots, \widehat{\alpha}_k, \dots, \widehat{\alpha}_l, \dots, \alpha_r).$$

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A**-cohomology:**

$$H^\bullet(A) \equiv \frac{\operatorname{Ker} d_A}{\operatorname{Im} d_A}$$

In general, it is very hard to compute...

Examples

	A	$H^\bullet(A)$
Ordinary Geometry (M a manifold)	$TM \to M$	de Rham cohomology
Lie Theory ($\mathfrak{g}$ a Lie algebra)	$\mathfrak{g} \to \{*\}$	Lie algebra cohomology
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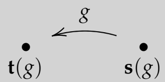

$$\mathcal{G} \underset{\mathbf{s}}{\overset{\mathbf{t}}{\rightrightarrows}} M$$

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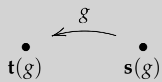

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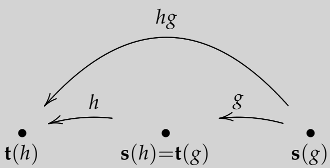

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$$m : \mathcal{G}^{(2)} \to \mathcal{G}$$

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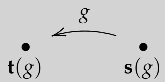

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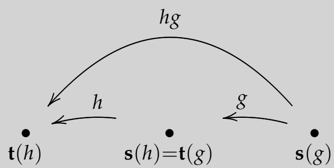

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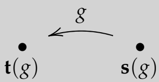

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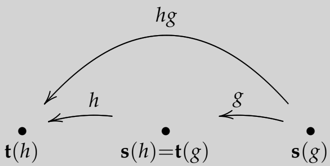

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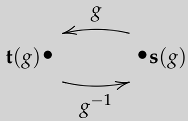

Lie Groupoids

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Caution: $\mathcal{G}$ may not be Hausdorff, but all other manifolds (M, **s** and **t**-fibers,...) are.

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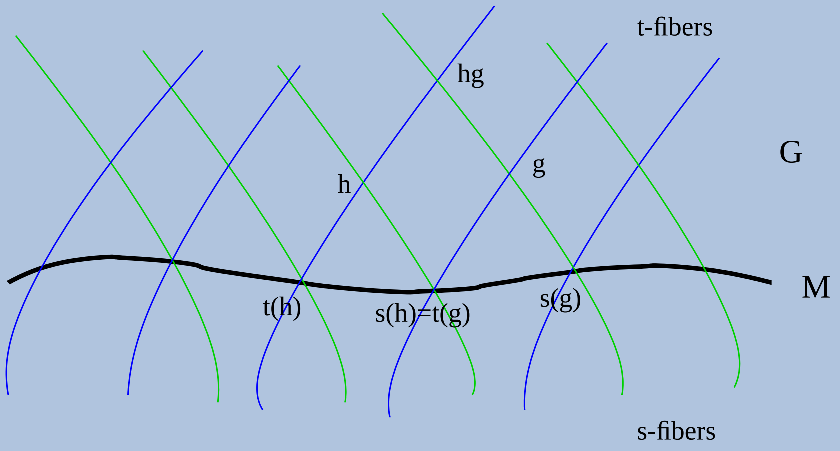

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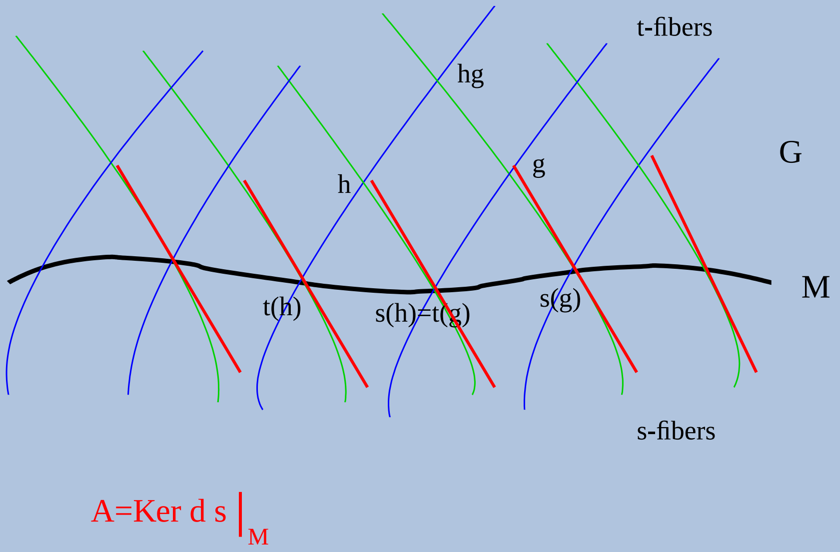

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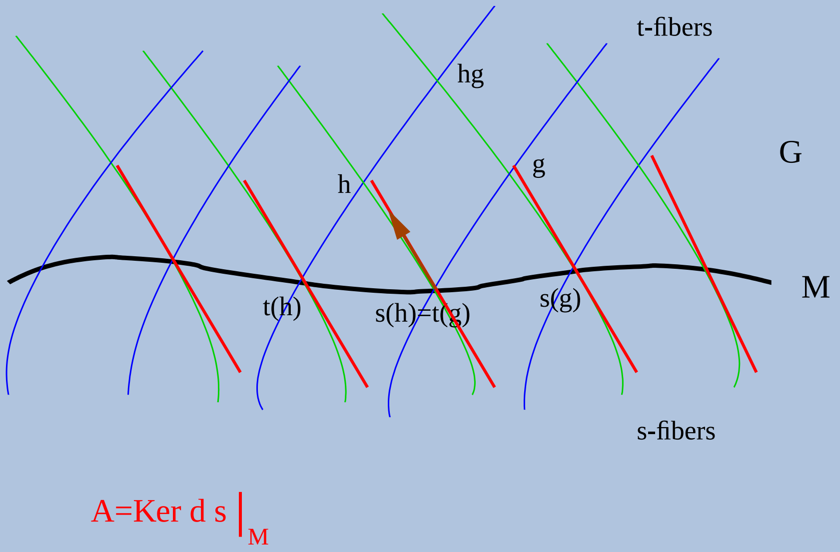

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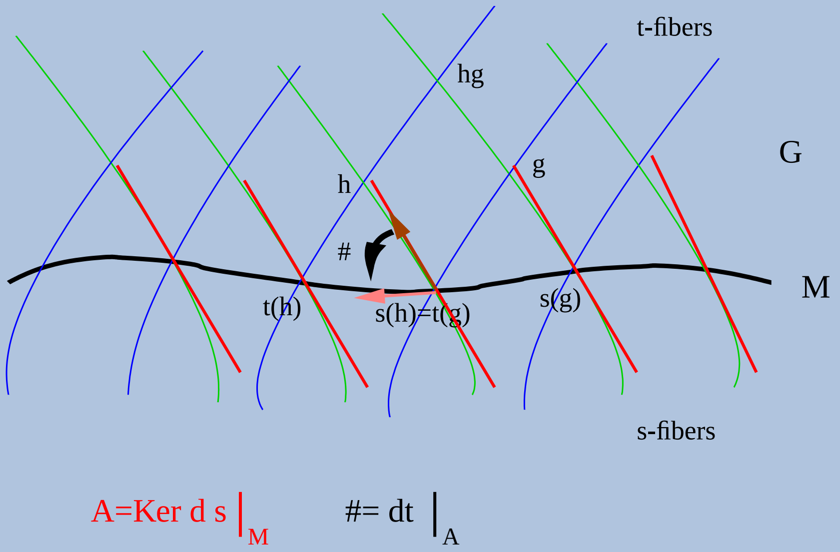

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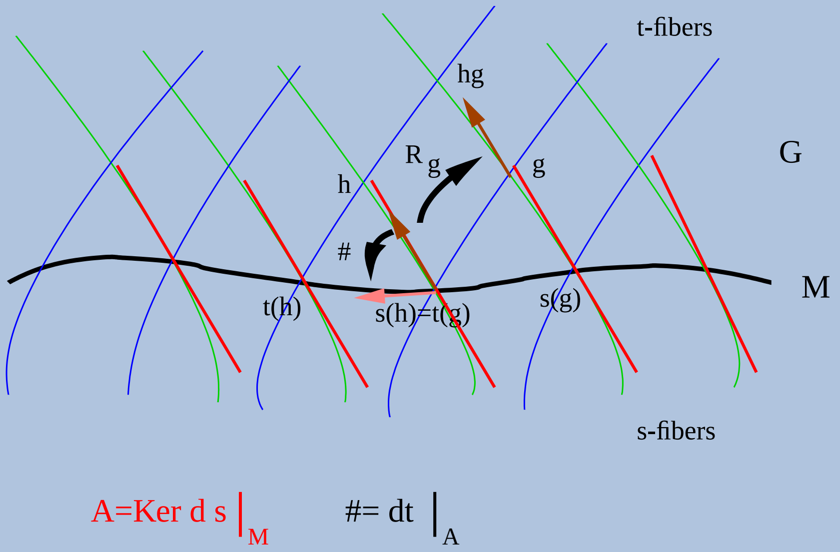

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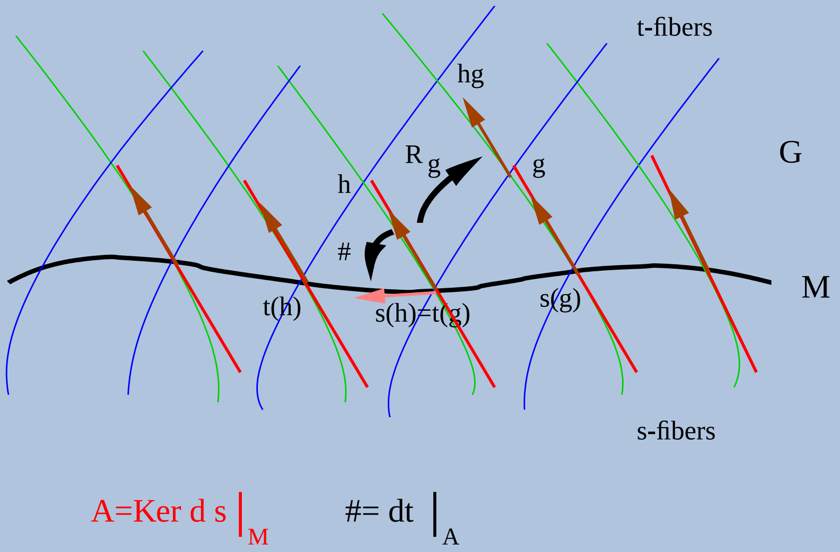

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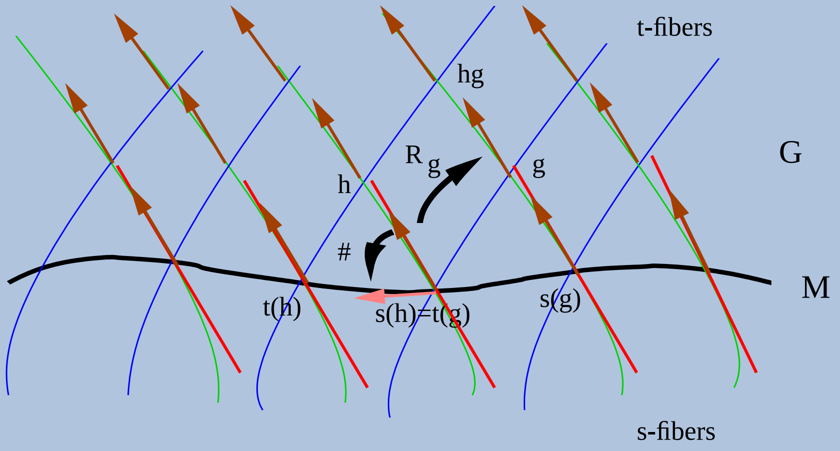

$A = \mathrm{Ker}\, d\mathbf{s}\big|_M \qquad \# = dt\big|_A \qquad [\alpha,\beta] = [X^\alpha, X^\beta]$

Examples

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Poisson Geometry (M Poisson)	$T^*M \to M$	Poisson cohomology	???

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PART 2

The Weinstein Groupoid and Integrability

A-Homotopy

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$$g \cdot g'(t) = \begin{cases} g'(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1)g'(1), & \frac{1}{2} < t \le 1. \end{cases}$$

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The quotient gives the **monodromy groupoid**:

$$\tilde{\mathcal{G}} \equiv P(\mathcal{G})/\sim \rightrightarrows M$$

A-Homotopy (cont.)

Lemma 2.2. *The map $D^R : P(\mathcal{G}) \to P(A)$ defined by*

$$(D^R g)(t) \equiv \frac{d}{ds} g(s) g^{-1}(t) \Big|_{s=t}$$

is a homeomorphism onto

$$P(A) \equiv \left\{ a : I \to A | \ \frac{d}{dt} \pi(a(t)) = \# a(t) \right\} \qquad (\textbf{\textit{A-paths}}).$$

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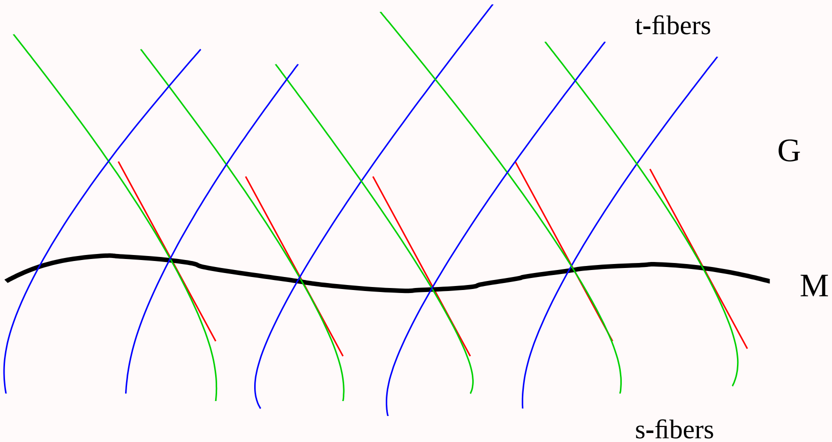

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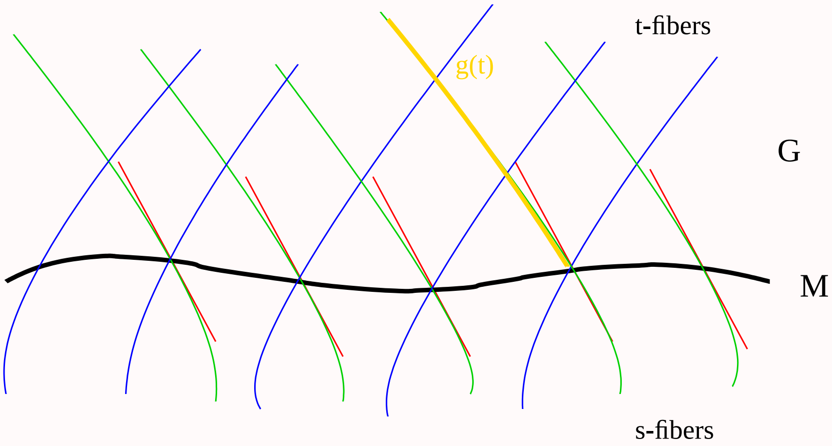

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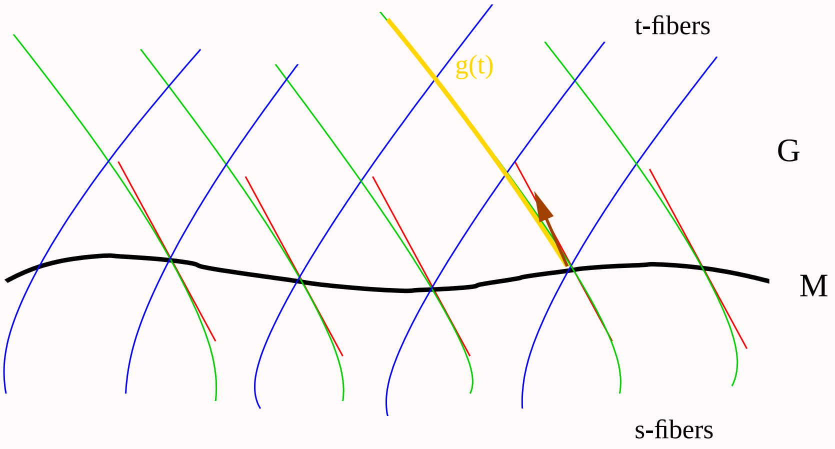

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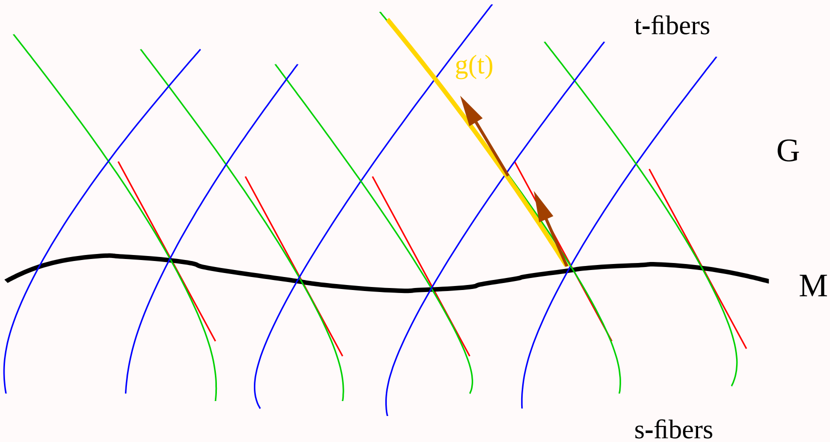

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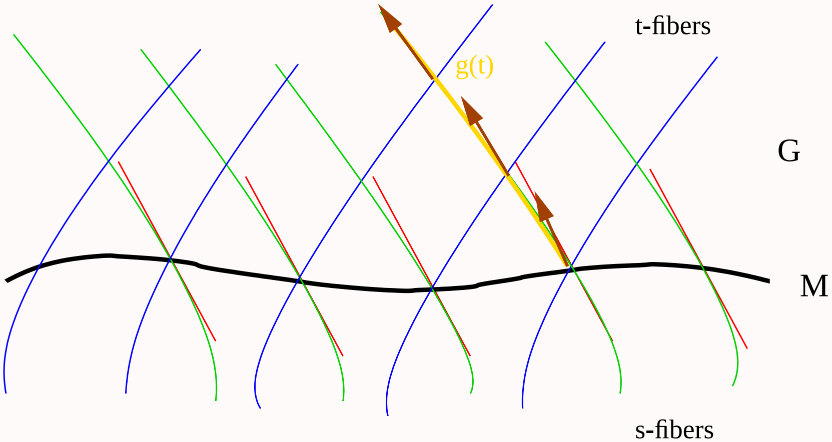

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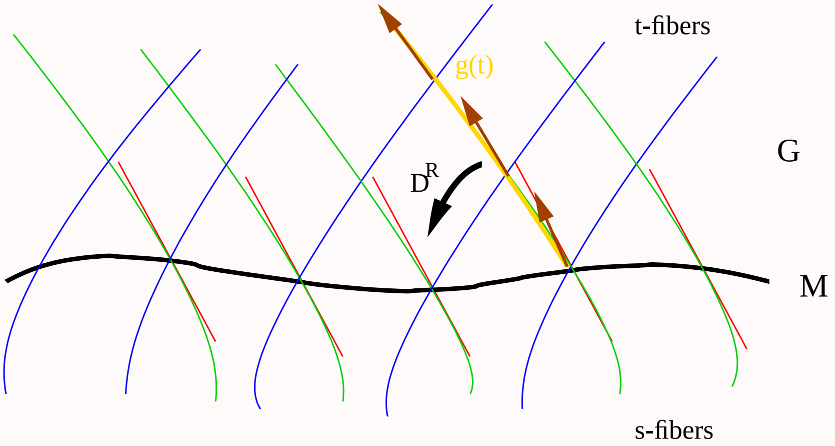

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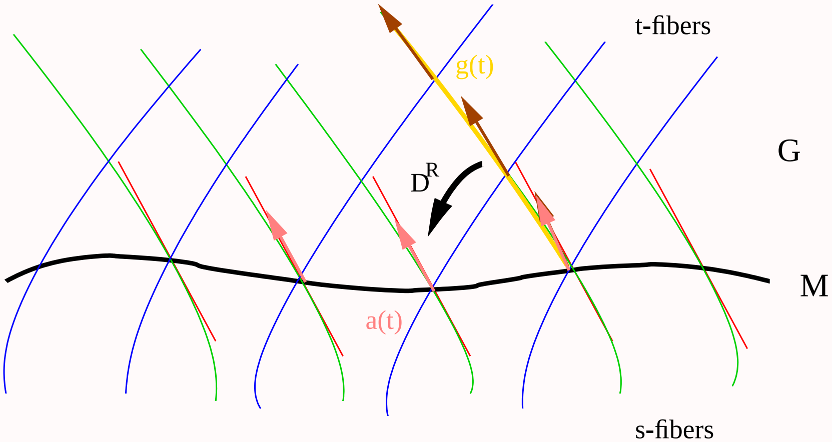

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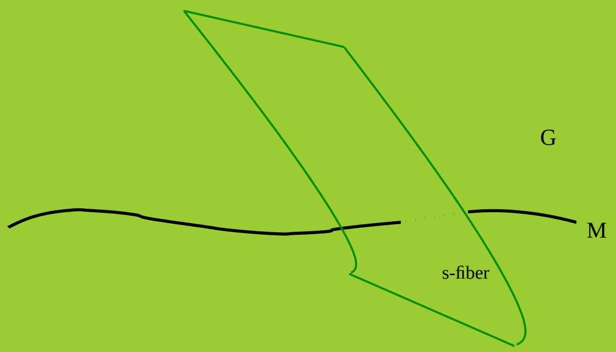

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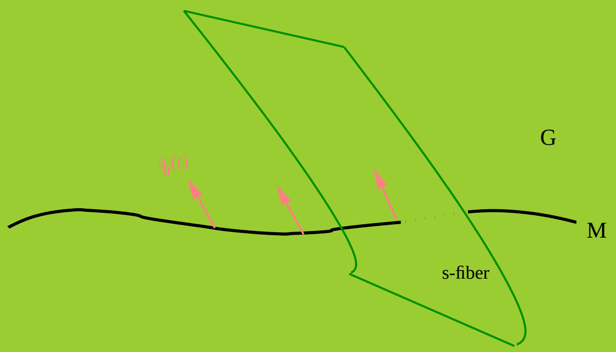

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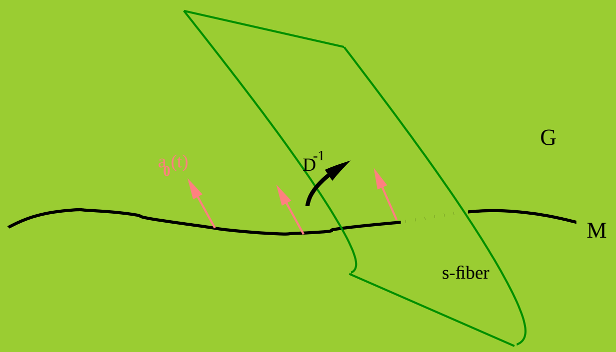

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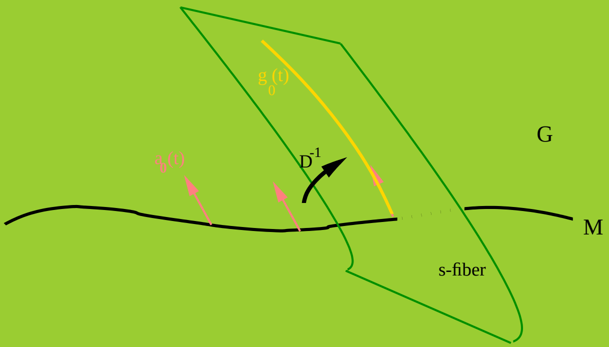

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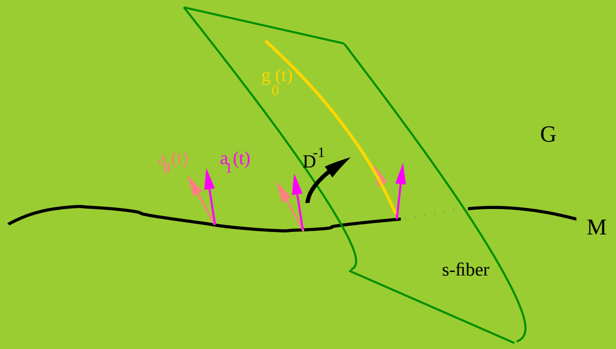

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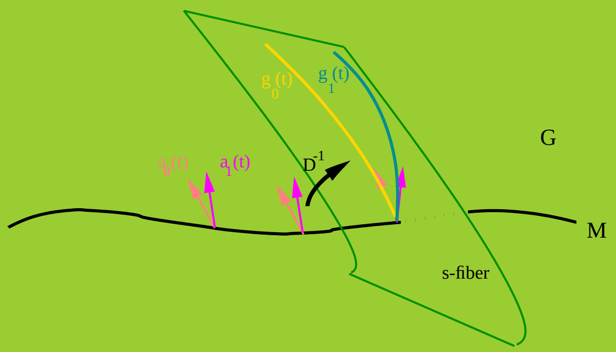

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- $\mathcal{G}(A)$ is a *topological* groupoid with source simply-connected fibers;

Examples

	A	$H^\bullet(A)$	$\mathcal{G}$	$\mathcal{G}(A)$
Ordinary Geometry (M a manifold)	$TM \to M$	de Rham cohomology	$M \times M \rightrightarrows M$	$\pi_1(M) \rightrightarrows M$
Lie Theory ($\mathfrak{g}$ a Lie algebra)	$\mathfrak{g} \to \{*\}$	Lie algebra cohomology	$G \rightrightarrows \{*\}$	Duistermaat-Kolk construction of G
Foliation Theory ($\mathcal{F}$ a regular foliation)	$T\mathcal{F} \to M$	foliated cohomology	$\mathrm{Hol} \rightrightarrows M$	$\pi_1(\mathcal{F}) \rightrightarrows M$
Equivariant Geometry ($\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ an action)	$M \times \mathfrak{g} \to M$	gener. foliated cohomology	$G \times M \rightrightarrows M$	$\mathcal{G}(\mathfrak{g}) \times M \rightrightarrows M$
Poisson Geometry (M Poisson)	$T^*M \to M$	Poisson cohomology	???	Poisson σ-model (Cattaneo & Felder)

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$$\Omega_\sigma(X,Y) = \sigma([X,Y]) - [\sigma(X),\sigma(Y)] \in Z(\mathfrak{g}_L), \qquad \forall X,Y \in \mathfrak{X}(L)$$

Then:

$$N_x(A) = \left\{ \int_\gamma \Omega : [\gamma] \in \pi_2(L,x) \right\}.$$

Example. Take $A = TM \times \mathbb{R}$ the Lie algebroid of a presymplectic manifold (M,ω):

$$0 \longrightarrow M \times \mathbb{R} \longrightarrow TM \times \mathbb{R} \underset{\sigma}{\overset{\#}{\rightleftarrows}} TM \longrightarrow 0$$

For the obvious splitting, the curvature is $\Omega_\sigma = \omega$. We obtain:

$$N_x = \left\{ \int_\gamma \omega : [\gamma] \in \pi_2(L,x) \right\}.$$

Conclusion: $A = TM \times \mathbb{R}$ is integrable iff the group of spherical periods of ω is discrete.

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Let $(M, \{\ ,\ \})$ be a regular Poisson manifold. Fix a symplectic leaf $L \subset M$ and $x \in L$.

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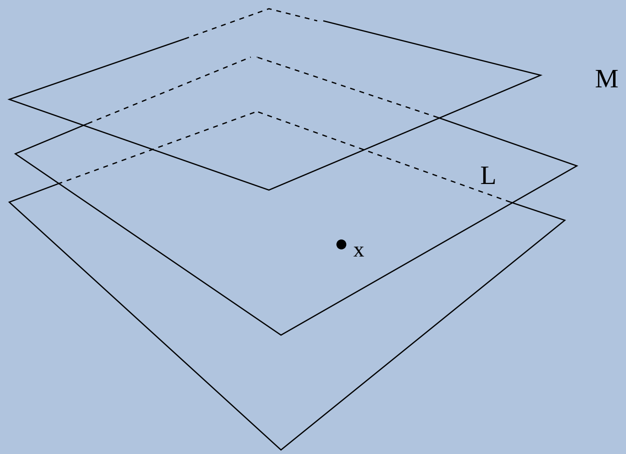

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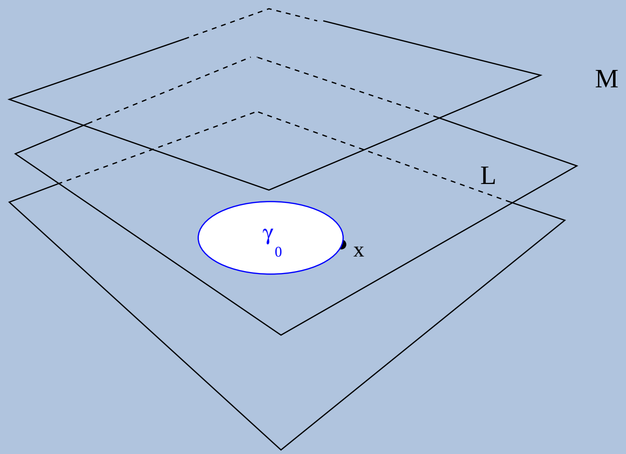

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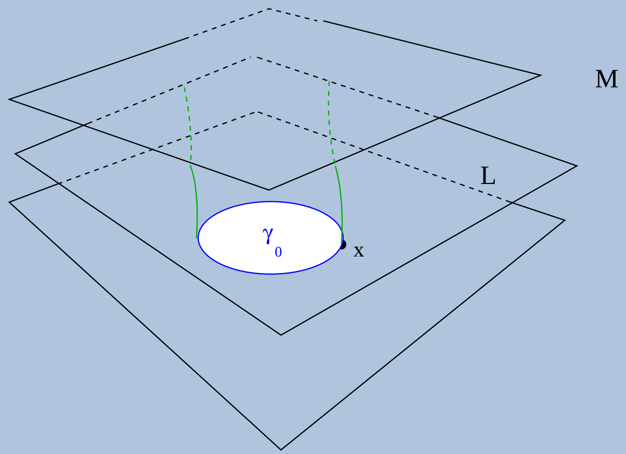

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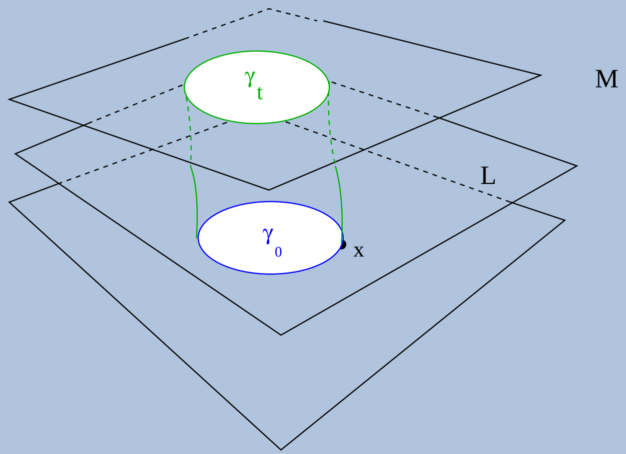

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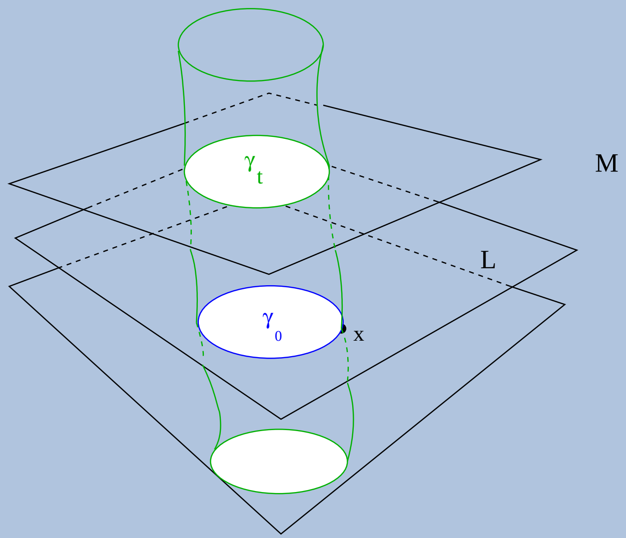

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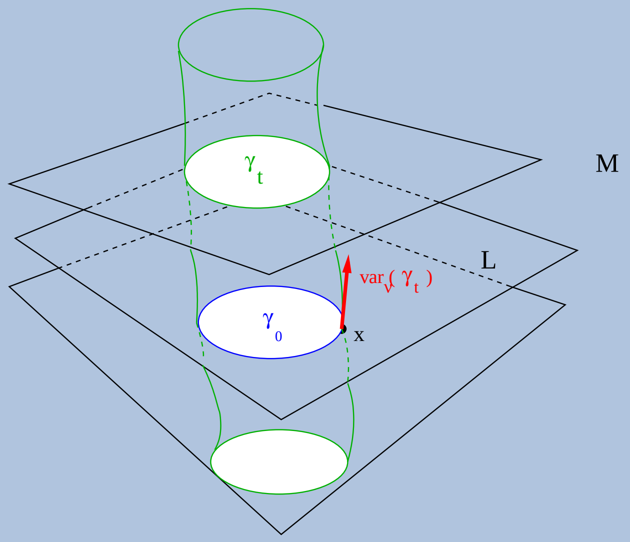

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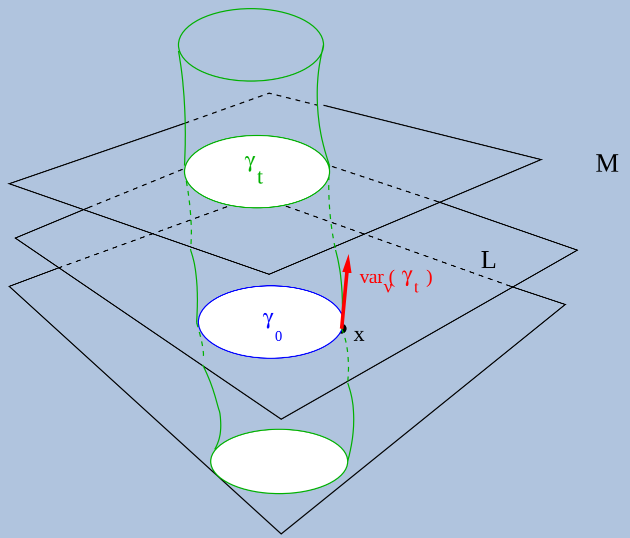

Proposition 2.7. *For a foliated family $\gamma_t : \mathbb{S}^2 \to M$, the derivative of the symplectic areas*

$$\frac{d}{dt} A(\gamma_t)\bigg|_{x=0},$$

depends only on the class $[\gamma_0] \in \pi_2(L, x)$ and $var_\nu(\gamma_t) = [d\gamma_t/dt|_{t=0}] \in \nu(L)_x$.

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Define the **variation of symplectic variations** $A'(\gamma_0) \in \nu_x^*(L)$ by

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- ...

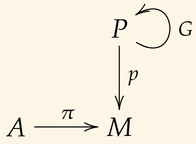
PART 3

Other Invariants: Holonomy, Characteristic Classes and K-Theory

Lie Algebroid Connections

Lie Algebroid Connections

An A-**connection** is a bundle map $h : p^* A \longrightarrow TP$ s.t.:



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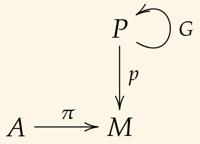
Lie Algebroid Connections



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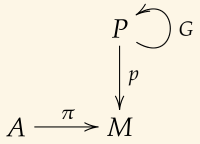

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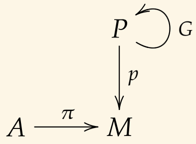

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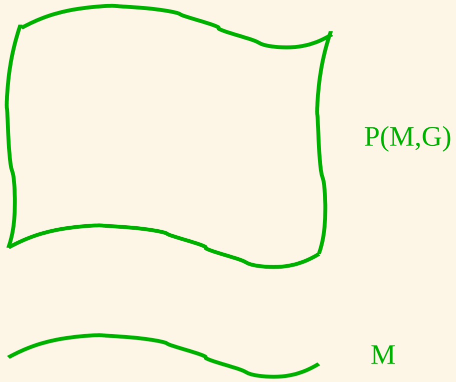

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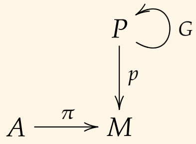

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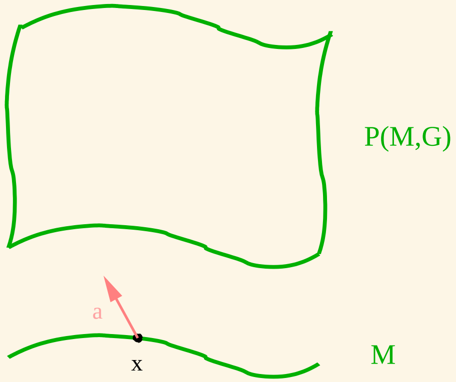

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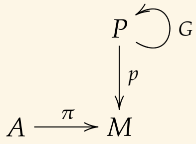

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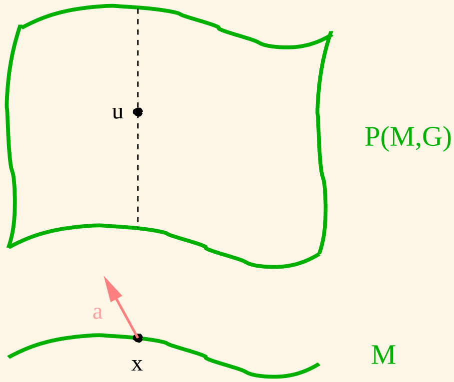

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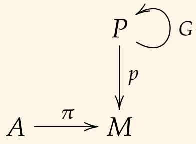

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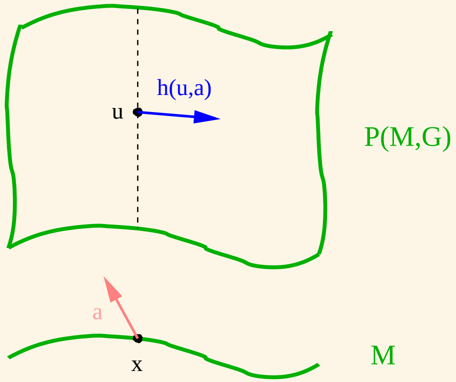

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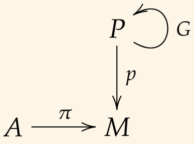

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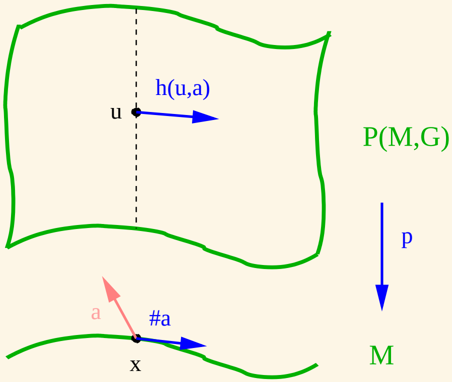

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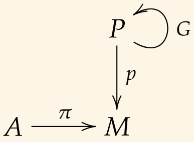

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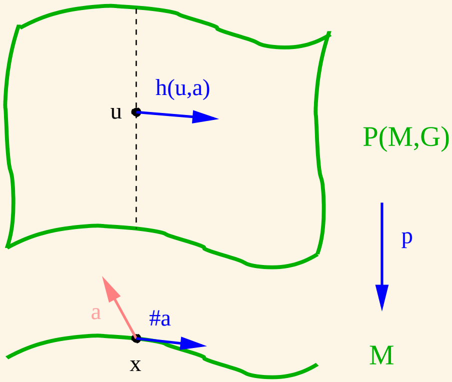

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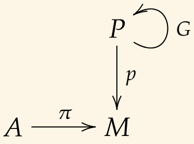

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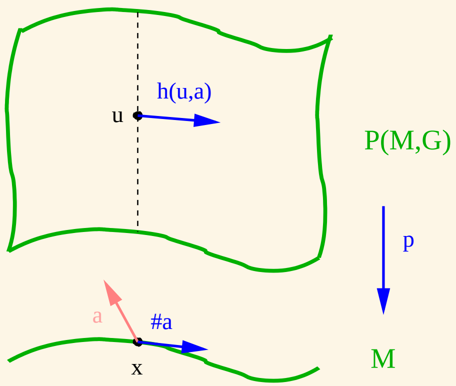

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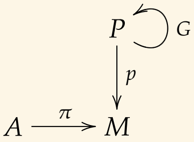

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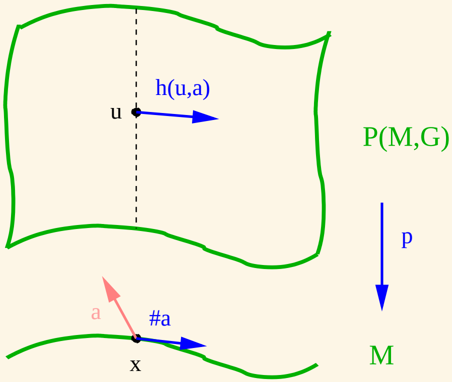

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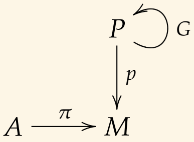

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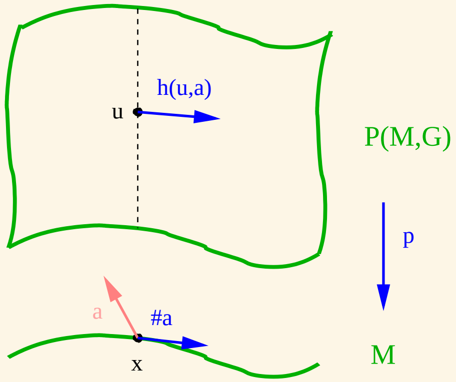

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- The **curvature** of $\nabla : \Gamma(A) \times \Gamma(E) \longrightarrow \Gamma(E)$ is

$$R_\nabla(\alpha, \beta) = \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha - \nabla_{[\alpha,\beta]}.$$

A-derivatives

An A-connection on $P = P(M,G)$ induces on any associated vector bundle $E \to M$ an A-**derivative** operator:

$$\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E).$$

Axioms for an A-derivative:

For any $\alpha \in \Gamma(A)$, $s \in \Gamma(E)$, $f \in C^\infty(M)$:

(i) $\nabla_{f\alpha} s = f \nabla_\alpha s$;

(ii) $\nabla_\alpha (fs) = f \nabla_\alpha s + \#\alpha(f) s$;

A-derivatives work like the usual covariant derivatives (case $A = TM$):

- The **curvature** of $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ is

$$R_\nabla(\alpha, \beta) = \nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha - \nabla_{[\alpha,\beta]}.$$

- The **torsion** of $\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is

$$T_\nabla(\alpha, \beta) = \nabla_\alpha \beta - \nabla_\beta \alpha - [\alpha, \beta].$$

Characteristic Classes

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- **Characteristic classes of representations** of a Lie algebroid [Crainic, 2001].

K-theory

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Flat A-connections $\Leftrightarrow$ **Representations** of A

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Axioms for a representation of A:

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Proposition 3.2. *Every representation of A determines a representation of $\mathcal{G}(A)$. The converse also holds, provided A is integrable.*

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- Representations lead to **Morita equivalence** in the context of Lie algebroids [Ginzburg, 2001; Crainic & RLF, 2002].

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TO BE CONTINUED. . .

THE LEIBNIZ IDENTITY.

For any sections $\alpha, \beta \in \Gamma(A)$ and function $f \in C^{\infty}(M)$:

$$[\alpha, f\beta] = f[\alpha, \beta] + \#\alpha(f)\beta.$$

The Tangent Lie Algebroid.

M - a manifold

- bundle: $A = TM$;
- anchor: $\# : TM \to TM$, $\# =$id;
- Lie bracket: $[\ ,\ ] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, usual Lie bracket of vector fields;
- characteristic foliation: $\mathcal{F} = \{M\}$.

The Lie Algebroid of a Lie Algebra.

$\mathfrak{g}$ - a Lie algebra

- bundle: $A = \mathfrak{g} \longrightarrow \{*\}$;
- anchor: $\# = 0$;
- Lie bracket: $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$,

  given Lie bracket;
- characteristic foliation: $\mathcal{F} = \{*\}$.

The Lie Algebroid of a Foliation.

$\mathcal{F}$ - a regular foliation

- bundle: $A = T\mathcal{F} \to M$;
- anchor: $\# : T\mathcal{F} \hookrightarrow TM$, inclusion;
- Lie bracket: $[\ ,\ ] : \mathfrak{X}(\mathcal{F}) \times \mathfrak{X}(\mathcal{F}) \to \mathfrak{X}(\mathcal{F})$,

  usual Lie bracket restricted to vector fields tangent to $\mathcal{F}$;
- characteristic foliation: $\mathcal{F}$.

The Action Lie Algebroid.

$\rho : \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ - an infinitesimal action of a Lie algebra

- bundle: $A = M \times \mathfrak{g} \longrightarrow M$;
- anchor: $\# : A \longrightarrow TM, \#(x, v) = \rho(v)|_x$;
- Lie bracket: $[,] : C^\infty(M, \mathfrak{g}) \times C^\infty(M, \mathfrak{g}) \longrightarrow C^\infty(M, \mathfrak{g})$

  $$[v, w](x) = [v(x), w(x)] + (\rho(v(x)) \cdot w)|_x - (\rho(w(x)) \cdot v)|_x;$$

- characteristic foliation: orbit foliation.

THE LIE ALGEBROID OF A PRESYMPLECTIC MANIFOLD.

M - an presymplectic manifold with closed 2-form ω

- bundle: $A = TM \times \mathbb{R} \to M$;
- anchor: $\# : A \to TM, \#(v, \lambda) = v$;
- Lie bracket: $\Gamma(A) = \mathfrak{X}(M) \times C^\infty(M)$

$$[(X,f),(Y,g)] = ([X,Y], X(g) - Y(f) - \omega(X,Y));$$

- characteristic foliation: $\mathcal{F} = \{M\}$.

The Cotangent Lie Algebroid.

M - a Poisson manifold with Poisson tensor π

- bundle: $A = T^*M$;
- anchor: $\# : TM^* \to TM, \#\alpha = i_\pi \alpha$;
- Lie bracket: $[\ ,\ ] : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M)$,

  Kozul Lie bracket:

$$[\alpha, \beta] = \mathcal{L}_{\#\alpha}\beta - \mathcal{L}_{\#\beta}\alpha - d\pi(\alpha, \beta);$$

- characteristic foliation: the symplectic foliation.

The Pair Groupoid.

M - a manifold

- **arrows**: $\mathcal{G} = M \times M$;
- **objects**: M;
- **target** and **source**: $\mathbf{s}(x, y) = x$, $\mathbf{t}(x, y) = y$;
- **product**: $(x, y) \cdot (y, z) = (x, z)$;

The Lie Groupoid of a Lie Group.

G - a Lie group

- **arrows**: $\mathcal{G} = G$;
- **objects**: $M = \{*\}$;
- **target** and **source**: $\mathbf{s}(x) = \mathbf{t}(x) = *$;
- **product**: $g \cdot h = gh$;

THE HOLONOMY GROUPOID.

$\mathcal{F}$ - a regular foliation in M

- **arrows**: $\mathcal{G} = \{[\gamma] : \text{holonomy equivalence classes}\}$;
- **objects**: M;
- **target** and **source**: $\mathbf{s}([\gamma]) = \gamma(0)$, $\mathbf{t}([\gamma]) = \gamma(1)$;
- **product**: $[\gamma] \cdot [\gamma'] = [\gamma \cdot \gamma']$;

THE ACTION GROUPOID.

$G \times M \to M$ - an action of a Lie group on M

- **arrows**: $\mathcal{G} = G \times M$;
- **objects**: M;
- **target** and **source**: $\mathbf{s}(g, x) = x$, $\mathbf{t}(g, x) = gx$;
- **product**: $(h, y) \cdot (g, x) = (hg, x)$;