LIE ALGEBROIDS AND CLASSIFICATION PROBLEMS IN GEOMETRY

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ABSTRACT. We show how one can associate to a given class of finite type G-structures a classifying Lie algebroid. The corresponding Lie groupoid gives models for the different geometries that one can find in the class, and encodes also the different types of symmetry groups.

CONTENTS

1. INTRODUCTION

The main purpose of this paper is to describe how one can associated to certain classes of geometric structures a *classifying Lie algebroid* $A \to X$, which has the following properties:

(i) to each point on the base X there corresponds a geometric structure of the class;

(ii) two structures in the class are isomorphic if and only if they correspond to points on the same orbit of A;

(iii) the isotropy Lie algebra at a point is the symmetry Lie algebra of the corresponding geometric structure.

We will also be interested in the associated Lie groupoid which allows one to construct different models for the distinct geometries that one can find in the class.

The geometric structures that will be considered here are G-structures. The notion of G-structure is quite general and includes most of the classical geometric structures. Since we want to avoid technical difficulties with infinite dimensional spaces, we restrict ourselves to G-structures of finite type, which means that G is a Lie group of finite type (see Definition 2.9). For finite type G-structures the classification problem can be reduced to a classification problem of $\{e\}$-structures (i.e., coframes) through the method of prolongation.

The method then goes as follows. Suppose that through prolongation we have reduced our classification problem to a problem for a class of coframes. Under suitable regularity assumptions, any coframe is determined by a finite set of functionally independent *structure invariants* (this is the generic case). It will then follow that a regular coframe determines a *transitive* Lie algebroid. By a *class of coframes*, we mean a family of coframes in which all members are determined by the same (not necessarily independent) structure invariants. In order to obtain a classifying Lie algebroid over a finite dimensional base X, we must restrict ourselves to classes of coframes determined by a finite set of structure invariants.

Finally, we will be interested in finding explicit models for the different geometric structures belonging to the classes being studied. The problem of finding a coframe with given structure invariants is known as *Cartan's realization problem* ([1, 2]). We will prove the following result:

Theorem 1.1. *Given the initial data of a Cartan's realization problem, one has that:*

- (i) *a realization exists if and only if the initial data form the structure functions of a Lie algebroid;*
- (ii) *any realization is locally equivalent to a neighborhood of the identity of an $\mathbf{s}$-fiber of a (local) Lie groupoid $\mathcal{G}$ integrating A, equipped with the Maurer-Cartan form;*
- (iii) *any two such realizations are equivalent if and only if they correspond to points on the same orbit of A.*

We call the Lie algebroid in the theorem the *classifying Lie algebroid* of the class of coframes. The theorem then shows that, when this Lie algebroid is integrable (see [3]), each $\mathbf{s}$-fiber of its groupoid, equipped with its Maurer-Cartan form furnishes an explicit *universal* model of the coframe.

This paper is a preliminary announcement of the results obtained by the second author in his thesis. In particular, we will be ignoring here most global aspects, and will mostly consider the local classification. Also, a detailed analysis of specific examples is beyond the scope of this paper.

This paper is organized as follows. In Section 2, we recall some classical results concerning finite type G-structures that we will need, including the method of prolongation of G-structures. In Section 3, which contains the main results, we start by constructing the classifying Lie algebroid for $\{e\}$-structures. Then we explain Cartan's realization problem for these structures, and answer the two basic questions: (i) existence and (ii) classification of realizations. Along the way, we give a brief study of Maurer-Cartan forms on Lie groupoids, since these play a crucial role in the classification. We end this section explaining how to extend these results for any finite type G-structures. In Section 4, we illustrate briefly our approach with two examples: constant curvature Riemannian metrics and Bochner-Kähler metrics.

2. Finite Type G-Structures

In this section we recall the basic facts from the theory of G-structures that we will use. We refer to [4, 6, 7] for details.

2.1. G-Structures. Denote by

$$
\begin{array}{cc}
\mathcal{B}(M) & \curvearrowleft \ \mathrm{GL}_n \\
\pi \downarrow & \\
M &
\end{array}
$$

the bundle of frames on M. This principal bundle carries a canonical 1-form with values in $\mathbb{R}^n$, denoted by $\omega \in \Omega^1(\mathcal{B}(M); \mathbb{R}^n)$, and which is defined by

$$
\omega_p(X) := p^{-1}(\pi_* X), \quad (X \in T_p \mathcal{B}(M)).
$$

This form is called the **tautological form** (or *soldering form*) of $\mathcal{B}(M)$. It is a tensorial form, i.e, it is horizontal and $\mathrm{GL}(n)$-equivariant (with respect to the defining action on $\mathbb{R}^n$). Note that a subspace $H_p \subset T_p \mathcal{B}_G(M)$ is horizontal iff the restriction $\omega : H_p \to \mathbb{R}^n$ is an isomorphism.

Every diffeomorphism φ between two manifolds M and N lifts to an isomorphism of the associated frame bundles:

$$
\mathcal{B}(\varphi) : \mathcal{B}(M) \to \mathcal{B}(N).
$$

The correspondence which associates to each manifold its frame bundle and to each diffeomorphism its lift is functorial.

Now let G be a Lie subgroup of $\mathrm{GL}(n)$. Recall that a G-**structure** is a reduction of the frame bundle $\mathcal{B}(M)$ to a principal G-bundle. This means that $\mathcal{B}_G(M) \subset \mathcal{B}(M)$ is a sub-bundle such that for any $p \in \mathcal{B}_G(M)$ and $a \in \mathrm{GL}(n)$ we have $pa \in \mathcal{B}_G(M)$ if and only if $a \in G$. Given a G-structure $\mathcal{B}_G(M)$, we will still denote by ω the restriction of the tautological form to $\mathcal{B}_G(M)$.

Definition 2.1. Two G-structures $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$ are said to be **equivalent** if there exists an diffeomorphism $\varphi : M \to N$ such that

$$
\mathcal{B}(\varphi)(\mathcal{B}_G(M)) = \mathcal{B}_G(N)
$$

2.2. Equivalence of G-Structures. One of the basic problems we will be interested is deciding if two G-structures are equivalent. The tautological form is the clue to the solution of this equivalence problem. The reason is the following result:

Proposition 2.2. *Two G-structures over M and N are equivalent if and only if there exists a principal G-bundle isomorphism $\psi : \mathcal{B}_G(M) \to \mathcal{B}_G(N)$ such that $\psi^* \omega_N = \omega_M$.*

In order to obtain an invariant of equivalence of G-structures, let us choose some horizontal space H_p at $p \in \mathcal{B}_G(M)$. Given $v \in \mathbb{R}^n$ there exists a unique $\widetilde{v} \in H_p$ such that $\omega(\widetilde{v}) = v$, so one defines:

$$
(2.1) \qquad c_{H_p} : \wedge^2 \mathbb{R}^n \to \mathbb{R}^n,
$$
$$
c_{H_p}(v, w) := \mathrm{d}\omega(\widetilde{v}, \widetilde{w}).
$$

This depends on the choice of horizontal space, so it does not define an invariant yet. If H_p and H_p' are two distinct horizontal spaces at $p \in \mathcal{B}_G(M)$, then one checks that:

$$c_{H_p} - c_{H_p'} \in \mathcal{A}(\hom(\mathbb{R}^n, \mathfrak{g})),$$

where $\mathfrak{g} \subset \mathfrak{gl}(n)$ is the Lie algebra of G and $\mathcal{A}$ denotes the anti-symmetrization operator:

$$\mathcal{A} : \hom(\mathbb{R}^n, \mathfrak{g}) \to \hom(\wedge^2 \mathbb{R}^n, \mathbb{R}^n),$$
$$\mathcal{A}(T)(u, v) := T(u)v - T(v)u.$$

Hence, we can set:

Definition 2.3. Given a G-structure $\mathcal{B}_G(M)$ one defines its **first order structure function**:

$$c : \mathcal{B}_G(M) \to \frac{\hom(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)}{\mathcal{A}(\hom(\mathbb{R}^n, \mathfrak{g}))}, \quad c(p) := [c_{H_p}].$$

Since an isomorphism $\psi : \mathcal{B}_G(M) \to \mathcal{B}_G(N)$ maps horizontal spaces to horizontal spaces and it is an equivalence if and only if $\psi^* \omega_N = \omega_M$, we see that

Proposition 2.4. Let $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$ be G-structures. If $\phi : M \to N$ is an equivalence then

$$c_N \circ \mathcal{B}(\phi) = c_M.$$

2.3. **Prolongation.** In order to obtain more refined invariants of equivalence of G-structures one needs to look at higher order terms. This process is known as *prolongation* and takes place on the jet bundles $\mathrm{J}^k \mathcal{B}_G(M)$.

Let $\pi : E \to M$ be a fiber bundle. We denote by $\pi^1 : \mathrm{J}^1 E \to M$ its first jet bundle, which has fiber over $x \in M$:

$$(\mathrm{J}^1 E)_x = \left\{ j_x^1 s | \ s \text{ a section of } E \right\}.$$

This bundle can also be described geometrically as:

$$\mathrm{J}^1 E = \{ H_p : p \in E \text{ and } H_p \subset T_p E \text{ horizontal} \}.$$

If one defines the projection $\pi_0^1 : \mathrm{J}^1 E \to E$ by $\pi_0^1(H_p) = p$, then $\mathrm{J}^1 E$ is an affine bundle over E.

Example 2.5. For a G-structure $\mathcal{B}_G(M)$ the first structure function can also be described as a function $c : \mathrm{J}^1 \mathcal{B}_G(M) \to \hom(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$ by formula (2.1).

It is easy to see that in the case of the frame bundle $\pi : \mathcal{B}(M) \to M$ its first jet bundle $\pi_0^1 : \mathrm{J}^1 \mathcal{B}(M) \to \mathcal{B}(M)$ can be identified with a sub bundle of $\mathcal{B}(\mathcal{B}(M))$: to a horizontal space $H_p \subset T_p \mathcal{B}(M)$ we associate a frame in $\mathcal{B}(M)$ (which we view as an isomorphism $\phi : \mathbb{R}^n \times \mathfrak{gl}(n) \to T_p \mathcal{B}(M)$):

$$\mathbb{R}^n \times \mathfrak{gl}(n) \ni (v, \xi) \overset{\phi}{\longmapsto} (\pi|_{H_p} \circ p)^{-1}(v) + \xi \cdot p \in T_p \mathcal{B}(M).$$

Note that if H_p and H_p' are two horizontal spaces at $p \in \mathcal{B}(M)$, the corresponding frames $\phi, \phi' : \mathbb{R}^n \times \mathfrak{gl}(n) \to T_p \mathcal{B}(M)$ are related by:

$$\phi'(v, \xi) = \phi(v, \xi) + T(v) \cdot p,$$

for some $T \in \hom(\mathbb{R}^n, \mathfrak{gl}(n))$. Conversely, given a frame ϕ associated with some horizontal space H_p and $T \in \hom(\mathbb{R}^n, \mathfrak{gl}(n))$, this formula determines a frame ϕ' which is associated with another horizontal space H_p'. It follows that $\mathrm{J}^1 \mathcal{B}(M)$ is a $\hom(\mathbb{R}^n, \mathfrak{gl}(n))$-structure, where we view $\hom(\mathbb{R}^n, \mathfrak{gl}(n)) \subset \mathrm{GL}(\mathbb{R}^n \oplus \mathfrak{gl}(n))$ as the subgroup formed by those transformations:

$$(v, \xi) \mapsto (v, \xi + T(v)), \text{ with } T \in \hom(\mathbb{R}^n, \mathfrak{gl}(n)).$$

Assume now that $\mathcal{B}_G(M)$ is a G-structure so that $J^1\mathcal{B}_G(M) \subset J^1\mathcal{B}(M)$ is a sub-bundle. An argument entirely similar to one just sketched gives:

Proposition 2.6. *If $\mathcal{B}_G(M)$ is a G-structure then $J^1\mathcal{B}_G(M) \to \mathcal{B}_G(M)$ is a* $\hom(\mathbb{R}^n, \mathfrak{g})$*-structure.*

In order to motivate our next definition we look at a simple example.

Example 2.7. Let us consider the flat G-structure on $\mathbb{R}^n$:

$$\mathcal{B}_G(\mathbb{R}^n) := \mathbb{R}^n \times G \subset \mathcal{B}(\mathbb{R}^n) = \mathbb{R}^n \times \mathrm{GL}(n).$$

Given a vector field X denote by $\phi_X^t : \mathbb{R}^n \to \mathbb{R}^n$ its flow. Observe that X is an infinitesimal automorphism of the G-structure $\mathcal{B}_G(\mathbb{R}^n)$ iff ϕ_X^t lifts to an automorphism $\mathcal{B}(\phi_X^t) : \mathcal{B}_G(\mathbb{R}^n) \to \mathcal{B}_G(\mathbb{R}^n)$. The lifted flow $\mathcal{B}(\phi_X^t)$ is the flow of a lifted vector field on $\mathcal{B}_G(\mathbb{R}^n)$: in coordinates $(x^1, \ldots, x^n)$, so that $X = X^i \frac{\partial}{\partial x^i}$, the lifted vector field is given by:[1]

$$\widetilde{X} = \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial p_j^i},$$

where (p_j^i) are the associated coordinates in $\mathcal{B}(\mathbb{R}^n)$ so that a frame $p \in \mathcal{B}(\mathbb{R}^n)$ is written as:

$$p = (p_1^i \frac{\partial}{\partial x^i}, \ldots, p_n^i \frac{\partial}{\partial x^i}).$$

It follows that X is an infinitesimal automorphism iff:

$$\left[\frac{\partial X^i}{\partial x^j} \right]_{i,j=1,\ldots,n} \in \mathfrak{g} \subset \mathfrak{gl}(n).$$

Let us assume now that the lifted flow fixes $(0, I) \in \mathbb{R}^n \times \mathrm{GL}(n)$. The lifted vector field $\widetilde{X}$ vanishes at this point. If we now prolong to the jet bundle $J^1\mathcal{B}_G(\mathbb{R}^n)$, we obtain a flow which is generated by a vector field:

$$j^1\widetilde{X} = \frac{\partial X^i}{\partial x^{j_1} \partial x^{j_2}} \frac{\partial}{\partial p_{j_1,j_2}^i},$$

where $(x^i, p_j^i, p_{j_1,j_2}^i)$ are the induced coordinates on the jet bundle. Note that the coefficients $a_{j_1,j_2}^i = \frac{\partial X^i}{\partial x^{j_1} \partial x^{j_2}}$ of $j^1\widetilde{X}$ satisfy:

$$\left[a_{j_1,j_2}^i \right]_{i,j_1=1,\ldots,n} \in \mathfrak{g} \subset \mathfrak{gl}(n),$$

and are symmetric in the indices j_1 and j_2. Hence, we conclude that:

Lemma 2.8. *The lifts of the symmetries of the flat G-structure $\mathcal{B}_G(\mathbb{R}^n) = \mathbb{R}^n \times G$ to the jet space $\mathrm{J}^1\mathcal{B}_G(\mathbb{R}^n)$ generate a Lie subgroup $G^{(1)} \subset \hom(\mathbb{R}^n, \mathfrak{g})$ with Lie algebra:*

$$\mathfrak{g}^{(1)} := \{ T \in \hom(\mathbb{R}^n, \mathfrak{g}) : T(u)v = T(v)u, \forall u, v \in \mathbb{R}^n \}.$$

This motivates the following definition:

Definition 2.9. Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra. The **first prolongation** of $\mathfrak{g}$ is the subspace $\mathfrak{g}^{(1)} \subset \hom(V, \mathfrak{g})$ consisting of those $T : V \to \mathfrak{g}$ such that

$$T(v_1)v_2 = T(v_1)v_2, \quad \forall v_1, v_2 \in V.$$

The k**-th prolongation** of $\mathfrak{g}$ is the subspace $\mathfrak{g} \subset \hom(V, \mathfrak{g}^{(k-1)})$ defined inductively by

$$\mathfrak{g}^{(k)} = (\mathfrak{g}^{(k-1)})^{(1)}.$$

A Lie algebra $\mathfrak{g}$ is said to be of finite type k if there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^{(k-1)} \neq 0$ and $\mathfrak{g}^{(k)} = 0$.

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[1]We use the convention of summing over repeated indices.

Similarly, at the group level, one introduces:

Definition 2.10. Let G be a subgroup of $GL(V)$. The **first prolongation** of G is the subgroup $G^{(1)}$ of $GL(V \oplus \mathfrak{g})$ consisting of those transformations of the form:

$$(v, \xi) \mapsto (v, \xi + T(v)), \text{ with } T \in \mathfrak{g}^{(1)}.$$

Similarly, the k-**th prolongation** of G is the subgroup $G^{(k)}$ of $\mathrm{GL}(V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(k)})$ defined inductively by

$$G^{(k)} := (G^{(k-1)})^{(1)}.$$

Note that the prolongations $G^{(k)}$ are all Abelian groups. Now, to each G-structure $\mathcal{B}_G(M)$ we can always reduce the structure group of $\mathrm{J}^1 \mathcal{B}_G(M)$ to $G^{(1)}$, obtaining a $G^{(1)}$-structure:

Proposition 2.11. *Let $\mathcal{B}_G(M)$ be a G-structure over M with first structure function $c : \mathrm{J}^1 \mathcal{B}_G(M) \to \mathrm{hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$. Each choice of a complement C to $\mathcal{A}(\mathrm{hom}(\mathbb{R}^n, \mathfrak{g}))$ in $\mathrm{hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$ determines a sub-bundle:*

$$\mathcal{B}_G(M)^{(1)} = \left\{ H_p \in \mathrm{J}^1 \mathcal{B}_G(M) : c_{H_p} \in C \right\},$$

which is a reduction of $\mathrm{J}^1 \mathcal{B}_G(M)$ with structure group $G^{(1)}$. Different choices of complements determine sub-bundles which are related through right translation by an element in $\mathrm{hom}(\mathbb{R}^n, \mathfrak{g})$.

The $G^{(1)}$-structure $\mathcal{B}_G(M)^{(1)} \to \mathcal{B}_G(M)$ is called the **first prolongation** of $\mathcal{B}_G(M)$. Similarly, working inductively, one defines the k-**th prolongation** of $\mathcal{B}_G(M)$:

$$\mathcal{B}_G(M)^{(k)} = (\mathcal{B}_G(M)^{(k-1)})^{(1)},$$

which is $G^{(k)}$-structure over $\mathcal{B}_G(M)^{(k-1)}$.

The relevance of prolongation for the problem of equivalence is justified by the following basic result:

Theorem 2.12. *Let $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$ be G-structures. Then $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$ are equivalent if and only if their first prolongations $\mathcal{B}_G(M)^{(1)}$ and $\mathcal{B}_G(N)^{(1)}$ are equivalent $G^{(1)}$-structures.*

One can now obtain new necessary conditions for equivalence by looking at the structure function of the prolongation $\mathcal{B}_G(M)^{(1)}$ which is a a function

$$c^{(1)} : \mathcal{B}_G(M)^{(1)} \to \frac{\mathrm{hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathbb{R}^n \oplus \mathfrak{g})}{\mathcal{A}(\mathrm{hom}(\mathbb{R}^n \oplus \mathfrak{g}, \mathfrak{g}^{(1)}))}$$

called the **second order structure function** of $\mathcal{B}_G(M)$. Then one can continue this process by constructing the second prolongation and analyzing it's structure function and so on.

Thus, the importance of structures of finite type is that we can reduce the set of necessary conditions for checking that two G-structures are equivalent to a finite amount. In fact, by the method of prolongation, the equivalence problem for finite type G-structures reduces to an equivalence problem for $\{e\}$-structures (coframes). Moreover, one can show that G-structures of finite type always have finite dimensional symmetry groups.

2.4. **Second Order Structure Functions.** By working inductively, all we really must understand are the second order structure functions which we now describe.

Let $z = H_p \in \mathcal{B}_G^{(1)} = \mathcal{B}_G(M)^{(1)}$ and let $\mathcal{H}_z$ be a horizontal subspace of $T_z \mathcal{B}_G^{(1)}$. Then $c_{\mathcal{H}_z}^{(1)} \in \hom(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathbb{R}^n \oplus \mathfrak{g})$ and we decompose it into three components:

$$\hom(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathbb{R}^n \oplus \mathfrak{g}) = \hom(\wedge^2 \mathbb{R}, \mathbb{R}^n \oplus \mathfrak{g}) \oplus \hom(\mathbb{R}^n \otimes \mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g}) \oplus$$
$$\oplus \hom(\wedge^2 \mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g})$$

Let us describe each of the components of the (representative of the) second order structure function. We denote by u, v elements of $\mathbb{R}^n$ and by A, B elements of $\mathfrak{g}$:

- The first component of $c_{\mathcal{H}_z}^{(1)}$ includes the structure function of $\mathcal{B}_G(M)$:

$$c_{\mathcal{H}_z}^{(1)}(u, v) = c_{H_p}(u, v) + b_{\mathcal{H}_z}(u, v),$$

  for some $b_{\mathcal{H}_z} \in \hom(\wedge^2 \mathbb{R}^n, \mathfrak{g})$.
- The second component of $c_{\mathcal{H}_z}^{(1)}$ has the form:

$$c_{\mathcal{H}_z}^{(1)}(A, u) = -Au + S_{\mathcal{H}_z}(A, u)$$

  for some $S_{\mathcal{H}_z} \in \hom(\mathbb{R}^n \otimes \mathfrak{g}, \mathbb{R}^n)$.
- The last component of $c_{\mathcal{H}_z}^{(1)}$ is given by

$$c_{\mathcal{H}_z}^{(1)}(A, B) = -[A, B]_{\mathfrak{g}}.$$

An important special case occurs when $G^{(1)} = \{e\}$. In this case, a $G^{(1)}$-structure amounts to choosing a horizontal space at each $p \in \mathcal{B}_G(M)$, which in turn is the same as picking a $\mathfrak{g}$-valued (not necessarily equivariant) form ϕ on $\mathcal{B}_G$. The pair (ω, ϕ) is a coframe on $\mathcal{B}_G$. Now, in this case, the projection from $\mathcal{B}_{G^{(1)}}$ onto $\mathcal{B}_G$ is a diffeomorphism, so we may view the second order structure functions as functions on $\mathcal{B}_G$. If we do this, we obtain the **structure equations** of the pair (ω, ϕ):

$$(2.2) \quad \begin{cases} \mathrm{d}\omega = c \circ \omega \wedge \omega - \phi \wedge \omega \\[2mm] \mathrm{d}\phi = b \circ \omega \wedge \omega + S \circ \phi \wedge \omega - \phi \wedge \phi \end{cases}$$

where $\phi \wedge \omega$ is the $\mathbb{R}^n$-valued 2-form obtained from the $\mathfrak{g}$-action on $\mathbb{R}^n$ and $\phi \wedge \phi$ is the $\mathfrak{g}$-valued 2-form obtained from the Lie bracket on $\mathfrak{g}$.

If, additionally, the horizontal spaces can be chosen right invariant (so that $R_{a*} H_p = H_{pa}$ for all $a \in G$), we obtain a principal bundle connection on $\mathcal{B}_G$ with connection form ϕ. In this case, we find that:

- S vanishes identically;
- b is the curvature of the connection;

so we see that, in this case, equations (2.2) reduce to the usual structure equations for a connection.

3. Cartan's Realization Problem

Given a reasonable class of G-structures we now explain how one can associate to it a *classifying Lie algebroid*.

3.1. **Equivalence of Coframes.** Assume that we have prolonged our finite type G-structure as much as necessary so we arrive at an $\{e\}$-structure. Since an $\{e\}$-structure is just the specification of a frame (or a coframe), we must then solve a problem of equivalence of coframes. We recall here how this can be dealt with. For details we refer to [5] and [7].

Let $\theta = \{\theta^1, ..., \theta^n\}$ be a coframe (i.e., a spanning set of everywhere linearly independent 1-forms) on an n-dimensional manifold M. If $\bar{M}$ is another n-dimensional manifold and $\bar{\theta} = \{\bar{\theta}^i\}$ a coframe on $\bar{M}$, the equivalence problem asks:([2])

Problem 3.1 (Equivalence Problem). *Does there exist a (locally defined) diffeomorphism $\phi : M \to \bar{M}$ satisfying*

$$\phi^* \bar{\theta}^i = \theta^i \, ?$$

Exterior differentiation of the 1-forms θ^k give the **structure equations**:

$$(3.1) \qquad\qquad \mathrm{d}\theta^k = \sum_{i<j} C_{ij}^k(x) \theta^i \wedge \theta^j,$$

for some functions $C_{ij}^k \in \mathrm{C}^\infty(M)$ called the **structure functions** of the coframe. These functions play a crucial role in the study of the equivalence problem. For example, since for any coframe $\bar{\theta}$ equivalent to θ one must have

$$\bar{C}_{ij}^k(\phi(x)) = C_{ij}^k(x),$$

the structure functions furnish a a set of invariants of the equivalence problem.

Definition 3.2. A function $I \in \mathrm{C}^\infty(M)$ is called an invariant function of a coframe $\{\theta^i\}$ if for any locally defined self equivalence (symmetry) $\phi : M \to M$ one has

$$I \circ \phi = I.$$

Now, for any function $f \in \mathrm{C}^\infty(M)$ one defines its coframe derivatives $\frac{\partial f}{\partial \theta^k}$ as the coefficients of the differential of f when expressed in terms of the coframe $\{\theta^i\}$,

$$\mathrm{d}f = \sum_k \frac{\partial f}{\partial \theta^k} \theta^k.$$

Using the fact that $\mathrm{d}\phi^* = \phi^*\mathrm{d}$, it follows that if $I \in \mathrm{C}^\infty(M)$ is an invariant function, then so are its coframe derivatives $\frac{\partial I}{\partial \theta^k}$, for all $1 \le k \le n$. It is then natural to consider the sets of *structure invariants*,

$$\mathcal{F}_s = \left\{ C_{ij}^k, \frac{\partial C_{ij}^k}{\partial \theta^l}, \dots, \frac{\partial^s C_{ij}^k}{\partial \theta^{l_1} \cdots \partial \theta^{l_s}} \right\}$$

which give us an infinite number of necessary conditions to solve the equivalence problem. Schematically, we may write

$$(3.2) \qquad\qquad \phi^* \bar{\mathcal{F}}_s = \mathcal{F}_s$$

for all $s \ge 0$.

In order to be able to proceed, we must first reduce these necessary conditions to a finite number.

Definition 3.3. A coframe $\theta = \{\theta^i\}$ is called **fully regular** if for each integer $s = 0, 1, 2, \dots$ the sth order structure map $C^{(s)} : M \to \mathbb{R}^{N_s}$, whose components are the structure invariants in $\mathcal{F}_s$, is regular (i.e., has constant rank).

Note that, in the fully regular case, locally we can always find a finite set of functionally independent structure invariants $\{h_1, \dots, h_d\}$ which generate the full set of structure invariants. This means that for every $s \ge 0$, any element in $f \in \mathcal{F}_s$ can be written as:

$$f = H(h_1, ..., h_d),$$

---

[2]We will use unbarred letters to denote objects on M and barred letters to denote objects on $\bar{M}$.

for some function $H : \mathbb{R}^d \to \mathbb{R}$. Finally, observe that in order to verify (3.2), it suffices to verify that

$$\phi^* \bar{h}_i = h_i$$

for $1 \le i \le d$. The integer d is called the **rank of the coframe**.

We can now summarize all essential data obtained from a regular coframe of rank d in the following convenient manner. First of all, we have a set of invariant functions which determine a map

$$h : M \to \mathbb{R}^d, \quad h(x) := (h_1(x), ..., h_d(x)).$$

Next, since $\{h_i\}$ are independent and generate $\mathcal{F}_t$ for all $t \ge 0$ (in particular $\mathcal{F}_0$), we can think of $h_1, ..., h_d$ as coordinates on an open subset X of $\mathbb{R}^d$. Then the structure functions may be seen as functions $C_{ij}^k \in \mathrm{C}^\infty(X)$. Finally, differentiating h_a we obtain

$$\mathrm{d}h_a = \sum_i F_i^a \theta^i,$$

where, for the same reason, F_i^a are invariant functions and hence can be seen as elements of $\mathrm{C}^\infty(X)$. In other words, our final structure equations are:

$$\mathrm{d}\theta^k = \sum_{i<j} C_{ij}^k(h) \theta^i \wedge \theta^j,$$

$$\mathrm{d}h_a = \sum_i F_i^a(h) \theta^i.$$

As we will see next, the functions $F_i^a, C_{ij}^k \in \mathrm{C}^\infty(X)$ form the initial data of a *Cartan's realization problem*. It will then be clear that to any fully regular coframe we can associate a transitive, flat Lie algebroid.

3.2. Cartan's Realization Problem.
Not every set of functions $F_i^a, C_{ij}^k \in \mathrm{C}^\infty(X)$ determines a class of coframes. Determining when this is true is the content of:

Problem 3.4 (Cartan's Realization Problem). *One is given:*

- *an integer $n \in \mathbb{N}$,*
- *an open set $X \subset \mathbb{R}^d$,*
- *a set of functions $C_{ij}^k \in \mathrm{C}^\infty(X)$ with indexes $1 \le i, j, k \le n$,*
- *and a set of functions $F_i^a \in \mathrm{C}^\infty(X)$ with $1 \le a \le d$*

and asks for the existence of

(1) an n-dimensional manifold M
(2) a coframe $\{\theta^i\}$ on M
(3) a function $h : M \to X$

satisfying the structure equations

$$(3.3) \qquad \mathrm{d}\theta^k = \sum_{i<j} C_{ij}^k(h) \theta^i \wedge \theta^j,$$

$$(3.4) \qquad \mathrm{d}h_a = \sum_i F_i^a(h) \theta^i.$$

A solution (M, θ^i, h) to Cartan's problem is called **realization**. If for each $h_0 \in X$ there exists a realization (M, θ^i, h) and $x_0 \in M$ such that $h(x_0) = h_0$, the initial set of data is said to specify a **class of coframes**. The classification problem can now be stated as:

Problem 3.5 (Classification Problem). *What are the possible solutions to a Cartan's realization problem? When are two solutions to a Cartan's realization problem equivalent?*

3.3. **Existence of Solutions.** Let us consider the question of existence of solutions to a Cartan's realization problem. One obtains some obvious necessary conditions for existence by differentiating the structure equations above and using $\mathrm{d}^2 = 0$. The resulting equations is a complicated set of non-linear pde's, but which have a very simple geometric interpretation: they are the differential equations that define a Lie algebroid. In fact, denoting by $A = X \times \mathbb{R}^n \to X$ the trivial vector bundle over X of rank n, with basis of sections $\{e_1, \ldots, e_n\}$, we define a bracket

$$(3.5) \qquad [e_i, e_j]_A := C_{ij}^k e_k,$$

and an anchor map $\# : A \to TX$ by:

$$(3.6) \qquad \#(e_i) = F_i^a \frac{\partial}{\partial x^a}.$$

Then one checks that:

$$\mathrm{d}^2 \theta^k = 0 \iff \text{Jacobi identity for } [\ ,\ ]_A$$
$$\mathrm{d}^2 h_a = 0 \iff \# : \Gamma(A) \to \mathfrak{X}(M) \text{ is a morphism.}$$

Thus, we have obtained:

Proposition 3.6. *For a Cartan's realization problem to have solutions for every $h_0 \in X$ it is necessary that $C_{ij}^k, F_i^a \in \mathrm{C}^\infty(X)$ be the structure functions of a flat Lie algebroid A over X.*

It turns out that the above necessary condition is also sufficient, as we shall show in the sequel.

3.4. **The Maurer-Cartan form on a Lie groupoid.** We will need a generalization of the usual Maurer-Cartan form on a Lie group to a Lie groupoid.

By a differential 1-form on a manifold M with values in a Lie algebroid $A \to X$ we mean a bundle map

$$\begin{array}{ccc} TM & \xrightarrow{\ \eta\ } & A \\ \downarrow & & \downarrow \\ M & \xrightarrow{\ h\ } & X \end{array}$$

which is compatible with the anchors, i.e, such that

$$\begin{array}{ccc} TM & \xrightarrow{\ \eta\ } & A \\ & {\scriptstyle h_*}\searrow & \downarrow {\scriptstyle \#} \\ & & TX \end{array}$$

Since there is no canonical way of differentiating forms with values in a vector bundle, we introduce an arbitrary connection ∇ on $A \to X$, and for $\xi_1, \xi_2 \in \mathfrak{X}(M)$ we define

$$d_\nabla \eta(\xi_1, \xi_2) = \nabla_{\xi_1} \eta(\xi_2) - \nabla_{\xi_2} \eta(\xi_1) - \eta([\xi_1, \xi_2]).$$

It is important to note that, in general, $\mathrm{d}_\nabla^2 \neq 0$ so that d_∇ *is not* a differential. If $\phi : TM \to A$ is another A-valued 1-form, we define

$$[\eta, \phi]_\nabla(\xi_1, \xi_2) = T_\nabla(\eta(\xi_1), \phi(\xi_2)) + T_\nabla(\phi(\xi_1), \eta(\xi_2))$$

where T_∇ is the *torsion* of ∇,

$$T_\nabla(\alpha, \beta) = \nabla_{\#\alpha}\beta - \nabla_{\#\beta}\alpha - [\alpha, \beta]_A, \quad (\alpha, \beta \in \Gamma(A)).$$

We have the following crucial result which follows from a more or less straightforward computation:

Lemma 3.7. *An A-valued 1-form η satisfies the* **generalized Maurer-Cartan equation**:

$$(3.7) \qquad\qquad d_\nabla \eta + \frac{1}{2}[\eta, \eta]_\nabla = 0$$

if and only if $\eta : TM \to A$ is a Lie algebroid morphism.

Now, let $\mathcal{G}$ be a Lie groupoid with Lie algebroid A. Right translation by an element $g \in \mathcal{G}$ is a diffeomorphism between $\mathbf{s}$-fibers: $R_g : \mathbf{s}^{-1}(\mathbf{t}(g)) \to \mathbf{s}^{-1}(\mathbf{s}(g))$. Hence, by a **right invariant 1-form** on $\mathcal{G}$ we mean a $\mathbf{s}$-foliated 1-form ω on $\mathcal{G}$ such that for all $g \in \mathcal{G}$:

$$\omega(\xi) = \omega(\mathrm{d}_h R_g(\xi)), \quad \forall h \in \mathbf{s}^{-1}(\mathbf{t}(g)), \xi \in T_h^{\mathbf{s}}\mathcal{G}.$$

For short, we write this condition as $(R_g)^*\omega = \omega$.

A Lie groupoid carries a natural canonical $\mathbf{s}$-foliated right invariant differential 1-form with values in its Lie algebroid:

Definition 3.8. The **Maurer-Cartan form on a Lie groupoid** $\mathcal{G}$ is the A-valued $\mathbf{s}$-foliated right invariant 1-form defined by

$$\omega_{\mathrm{MC}}(\xi) = (\mathrm{d}R_{g^{-1}})_g(\xi)$$

for $\xi \in T_g^{\mathbf{s}}\mathcal{G}$.

Note that the Maurer-Cartan 1-form $\omega_{\mathrm{MC}} : T^{\mathbf{s}}\mathcal{G} \to A$ covers the target map $\mathbf{t} : \mathcal{G} \to M$. As one could expect, this 1-form is a solution of the generalized Maurer-Cartan equation. Moreover, it satisfies a universal property analogous to the case of Lie groups:

Proposition 3.9. *Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid A and let ω_{MC} be its right invariant Maurer-Cartan form. If $\eta : TM \to A$ is a solution of the Maurer-Cartan equation covering a map $h : M \to X$, then for each $x \in M$ and $g \in \mathcal{G}$ such that $h(x) = \mathbf{s}(g)$ there exists a unique locally defined diffeomorphism $\phi : M \to \mathbf{s}^{-1}(h(x))$ satisfying:*

$$\phi(x) = g \ \text{ and } \ \phi^*\omega_{MC} = \eta.$$

Let us sketch a proof of this proposition. Since it is a local result we may assume that M is simply connected. The source 1-connected Lie groupoid integrating TM is then the pair groupoid $M \times M \rightrightarrows M$. Since a Maurer-Cartan form is nothing but a Lie algebroid morphism, we can integrate $\eta : TM \to A$ to a unique Lie groupoid morphism (see [3]):

$$
\begin{array}{ccc}
M \times M & \xrightarrow{\ H\ } & \mathcal{G} \\
\Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow \\
M & \xrightarrow[\ h\ ]{} & X.
\end{array}
$$

If we fix a point x_0 in M we may always write

$$H(x, y) = \phi(x)\phi(y)^{-1}$$

where $\phi : M \to \mathbf{s}^{-1}(h(x_0)) \subset \mathcal{G}$ is defined by $\phi(x) := H(x, x_0)$. But then ϕ satisfies

$$\phi(x_0) = \mathbf{1}_{h(x_0)}, \quad \phi^*\omega_{\mathrm{MC}} = \eta.$$

So ϕ is the desired local diffeomorphism.

As a corollary of Proposition 3.9 we obtain:

Corollary 3.10. *Let $\mathcal{G} \rightrightarrows X$ be a Lie groupoid with Maurer-Cartan form ω_{MC}. If $\phi : \mathbf{s}^{-1}(x) \to \mathbf{s}^{-1}(y)$ is a symmetry of ω_{MC} (i.e., $\phi^*\omega_{MC} = \omega_{MC}$) then x and y belong to the same orbit of $\mathcal{G}$ and ϕ is locally of the form $\phi = R_g$ for some $g \in \mathcal{G}$.*

The proof sketched above also shows that if we impose a topological condition we obtain the following global version of the universal property of Maurer-Cartan forms:

Theorem 3.11. *Let $\mathcal{G} \rightrightarrows X$ be a source 1-connected Lie groupoid with Lie algebroid A and let $\eta \in \Omega^1(M, A)$ be an A-valued differential 1-form covering $h : M \to X$. Then, there exists an embedding $\phi : M \to \mathbf{s}^{-1}(h(x_0))$ satisfying*

$$\phi(x_0) = 1_{h(x_0)} \ and \ \phi^*\omega_{MC} = \eta.$$

if and only if

(i) (local obstruction) η satisfies the generalized Maurer-Cartan equation and

(ii) (global obstruction) the Lie groupoid morphism H integrating η is trivial when restricted to the fundamental group $\pi_1(M, x_0)$.

3.5. **The Classification Theorem.** We can now give a complete solution to the classification problem 3.5:

Theorem 3.12. *Let (n, X, C_{ij}^k, F_i^a) be the initial data of a Cartan's realization problem. Then:*

(i) a realization exists if and only if C_{ij}^k and F_i^a are the structure functions of a Lie algebroid;

(ii) any realization is locally equivalent to a neighborhood of the identity of an $\mathbf{s}$-fiber of a groupoid $\mathcal{G}$ equipped with the Maurer-Cartan form;

(iii) two such realizations are equivalent if and only if they correspond to points on the same orbit of A.

Proof. We already know that for Cartan's problem to have a solution the C_{ij}^k and F_i^a form the structure functions of the flat Lie algebroid $A = X \times \mathbb{R}^n$, with bracket and anchor given by (3.5) and (3.6). Assume, for simplicity, that A is integrable and that $\mathcal{G}$ is a Lie groupoid integrating A. Denote by ω_{MC} the Maurer-Cartan form of $\mathcal{G}$ and by $\{\omega_{\mathrm{MC}}^1, \ldots, \omega_{\mathrm{MC}}^n\}$ its components with respect to the basis $\{e_1, \ldots, e_n\}$. Then it is clear that, for each $x_0 \in X$, $(\mathbf{s}^{-1}(x_0), \{\omega_{\mathrm{MC}}^i\}, \mathbf{t})$ is a realization of the Cartan problem with initial data (n, X, C_{ij}^k, F_i^a). This proves (i).

Next we observe that, if $(M, \{\theta^i\}, h)$ is another realization of (n, X, C_{ij}^k, F_i^a), the A-valued 1-form

$$\theta = \sum_{i=1}^n \theta^i e_i \in \Omega^1(M, A)$$

satisfies the generalized Maurer-Cartan equation. In fact, equation (3.4) is equivalent to (θ, h) being an A-valued differential form and equation (3.3) is equivalent to the Maurer-Cartan equation. Hence, if $p_0 \in M$ is such that $h(p_0) = x_0$ then by the universal property of Maurer-Cartan forms (Theorem 3.9) we can find a neighborhood V of p_0 in M and a diffeomorphism

$$\phi : V \to \phi(V) \subset \mathbf{s}^{-1}(x_0)$$

such that $\phi(p_0) = \mathbf{1}_{x_0}$ and $\phi^*\omega_{\mathrm{MC}} = \theta$. This shows that any realization of Cartan's problem is locally equivalent to a neighborhood of the identity of an $\mathbf{s}$-fiber of $\mathcal{G}$ equipped with the Maurer-Cartan form, so (ii) follows.

Finally, it follows from Corollary 3.10 that two realizations are equivalent if and only if they correspond to points on the same orbit of A, so (iii) also follows. $\square$

The Classification Theorem 3.12 shows that there is a one to one correspondence between flat Lie algebroids of rank n and classes of locally defined coframes on n-dimensional manifolds. We call the Lie algebroid associated with a class the **classifying Lie algebroid** of the class of coframes. Its integration furnishes explicit models to the realization problem.

Remark 3.13. Even when A is not integrable, the proof above shows that all we need is a local groupoid integrating it, and this always exists (see [3]).

3.6. **Realizations of Finite Type G-Structures.** We now briefly describe the realization problem for G-structures of finite type. We focus on the case where $G^{(1)} = \{e\}$. There are two reasons for this: (i) this is the case that appears in most geometric applications, and (ii) more general finite type G-structures can then be handle by induction once this case is understood.

Suppose that we would like to determine all G-structures belonging to a certain class, where G is a fixed Lie group satisfying $G^{(1)} = \{e\}$. The considerations of Section 2.4 show that we must solve the following realization problem:

Problem 3.14 (Realization Problem for G-Structures with $G^{(1)} = \{e\}$). *Given the data:*

- *An open set $X \subset \mathbb{R}^d$,*
- *an integer $n \in \mathbb{N}$,*
- *a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n$ satisfying $\mathfrak{g}^{(1)} = 0$,*
- *a Lie group $G \subset GL(n)$ with Lie algebra $\mathfrak{g}$, and*
- *maps $c : X \to \hom(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)$, $b : X \to \hom(\mathbb{R}^n \wedge \mathbb{R}^n, \mathfrak{g})$, $S : X \to \hom(\mathbb{R}^n \otimes \mathfrak{g}, \mathfrak{g})$, $\Theta : X \to \hom(\mathbb{R}^n, \mathbb{R}^d)$, and $\Phi : X \to \hom(\mathfrak{g}, \mathbb{R}^d)$*

one asks for the existence of

(1) a manifold M^n,
(2) a G-structure $\mathcal{B}_G(M)$ on M with tautological form $\omega \in \Omega^1(\mathcal{B}_G, \mathbb{R}^n)$
(3) a maximal rank one form $\phi \in \Omega^1(\mathcal{B}_G, \mathfrak{g})$ transversal to ω, and
(4) a map $h : \mathcal{B}_G \to X$

such that:

$$\tag{3.8} \mathrm{d}\omega = c(h) \circ \omega \wedge \omega - \phi \wedge \omega$$

$$\tag{3.9} \mathrm{d}\phi = b(h) \circ \omega \wedge \omega + S(h) \circ \omega \wedge \phi - \phi \wedge \phi$$

$$\tag{3.10} \mathrm{d}h = \Theta(h) \circ \omega + \Phi(h) \circ \phi$$

Just like in the case of $G = \{e\}$, one checks that a necessary condition for solving this problem is that the structure functions c, b, S, Θ, Φ determine a flat Lie algebroid $A \to X$ with fiber $\mathbb{R}^n \oplus \mathfrak{g}$. The Lie bracket on constant sections, is given by

$$[(u, \alpha), (v, \beta)]_A(x) = (w, \gamma),$$

where,

$$w := c(x)(u \wedge v) - \alpha \cdot v + \beta \cdot u,$$
$$\gamma := b(x)(u \wedge v) + S(x)(u \otimes \beta - v \otimes \alpha) - [\alpha, \beta]_{\mathfrak{g}}.$$

The anchor map $\# : A \to TX$ is determined by:

$$\#(u, \alpha) := \Theta(x)u + \Phi(x)\alpha.$$

Observe that the natural inclusion $\mathfrak{g} \hookrightarrow \Gamma(A)$, $\alpha \mapsto (0, \alpha)$, is a Lie algebra homomorphism, since we have:

$$[(0, \alpha), (0, \beta)]_A = (0, [\alpha, \beta]_{\mathfrak{g}}).$$

Therefore, we obtain an inner action of $\mathfrak{g}$ on A by setting

$$\rho : \mathfrak{g} \to \mathrm{Der}(A), \quad \rho(\alpha)(\sigma) := [(0, \alpha), \sigma]_A \quad (\sigma \in \Gamma(A)).$$

Conversely, if the initial data of the realization problem determines a flat Lie algebroid $A = X \times (\mathbb{R}^n \oplus \mathfrak{g}) \to X$ such that the natural inclusion $\mathfrak{g} \hookrightarrow \Gamma(A)$ defines an infinitesimal inner $\mathfrak{g}$-action, then for each $h_0 \in X$ we can find a G-structure whose structure functions assume the values $c(h_0), b(h_0), S(h_0), \Theta(h_0)$ and $\Phi(h_0)$.

Note that now, in order to exhibit explicit universal models, we must integrate not only the Lie algebroid but also the infinitesimal inner $\mathfrak{g}$-action. Therefore, assume that A is integrable and that we can integrate the $\mathfrak{g}$-action to a proper and free action of G on some groupoid $\mathcal{G}$ integrating A. Each $\mathbf{s}$-fibers of $\mathcal{G}$ will be the total space of a G-structure determined by its tautological form; the $\mathbb{R}^n$ component of the Maurer-Cartan form. The isotropy group of $\mathcal{G}$ at a point will be the symmetry group of the corresponding G-structure. Moreover, for any frame p in another realization $\mathcal{B}_G(M)$, there will exist a neighborhood which is isomorphic to a neighborhood of the identity in an $\mathbf{s}$-fiber of $\mathcal{G}$. Details on these constructions will appear elsewhere.

4. Examples

In order to illustrate our method, we will present two concrete examples. The first one is a toy example: metrics of constant curvature in $\mathbb{R}^2$. The second example is a more serious application to Bochner-Kähler metrics and is based on the work of Bryant [1].

4.1. **Metrics of Constant Curvature in $\mathbb{R}^2$.** Suppose we would like to classify all Riemann metrics of constant curvature in a neighborhood of the origin in $\mathbb{R}^2$. The G-structure to be considered is $\mathcal{B}_{O_2}(\mathbb{R}^2)$, the orthogonal frame bundle of $\mathbb{R}^2$. In this case, the first order structure function vanishes identically and $O_2^{(1)} = \{e\}$. The structure equation has the following simple form:

$$\begin{cases} \mathrm{d}\omega = -\phi \wedge \omega \\ \mathrm{d}\phi = k\omega \wedge \omega, \end{cases}$$

where ϕ is the connection 1-form on $\mathcal{B}_{O_2}(\mathbb{R}^2)$ corresponding to the Levi-Civita connection and k is the Gaussian curvature of ϕ. If the curvature k is constant, then $\mathrm{d}k = 0$. In terms of the canonical base of $\mathbb{R}^2$ the structure equation becomes

$$\mathrm{d}\omega^1 = -\phi \wedge \omega^2,$$
$$\mathrm{d}\omega^2 = \phi \wedge \omega^1,$$
$$\mathrm{d}\phi = k\omega^1 \wedge \omega^2,$$
$$\mathrm{d}k = 0.$$

The Gaussian curvature is the only invariant function. It follows that the classifying Lie algebroid for this class is the flat bundle $A = \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$, with a basis of sections $\{e_1, e_2, e_3\}$ and structure given by:

$$[e_1, e_2](k) = ke_3,$$
$$[e_1, e_3](k) = -e_2,$$
$$[e_2, e_3](k) = e_1,$$
$$\#e_i = 0.$$

Note that this Lie algebroid is just a bundle of Lie algebras with fibers isomorphic to:

$\mathfrak{sl}_2$	if $k < 0$	Hyperbolic Geometry
$\mathfrak{se}_2$	if $k = 0$	Euclidean Geometry
$\mathfrak{so}_3$	if $k > 0$	Spherical Geometry

The inner action of $\mathfrak{o}_2$ on A is the fiberwise action obtained from the adjoint representation. It follows that to each value of $k \in \mathbb{R}$ there corresponds the germ at 0 of a constant curvature metric on $\mathbb{R}^2$. Moreover, two constant curvature metrics on $\mathbb{R}^2$ are locally equivalent if and only if they have the same curvature.

4.2. **Bochner-Kähler Structures.** Let (M, σ, Ω) be a Kähler manifold. Its curvature tensor can be decomposed into three irreducible components: the scalar curvature, the traceless Ricci curvature and the Bochner curvature. We say that (M, σ, Ω) is Bochner-Kähler if its Bochner curvature vanishes identically.

Bryant in [1] performs a differential analysis which shows that the local classification of Bochner-Kähler metrics can be reduced to the following structure equations

$$\mathrm{d}\omega = -\phi \wedge \omega,$$
$$\mathrm{d}\phi = -\phi \wedge \phi + S\omega^* \wedge \omega - S\omega \wedge \omega^* - \omega \wedge \omega^* S + (\omega^* \wedge S\omega)I_n,$$
$$\mathrm{d}S = -\phi S + S\phi + T\omega^* + \omega T^* + \frac{1}{2}(T^*\omega + \omega^*T)I_n,$$
$$\mathrm{d}T = -\phi T + (UI_n + S^2)\omega,$$
$$\mathrm{d}U = T^*S\omega + \omega^*ST.$$

where ω is the $\mathbb{C}^n$-valued tautological form on the unitary coframe bundle of M, ϕ is the $\mathfrak{u}(n)$-valued connection form associated to the Levi-Civita connection on M, $S, T,$ and U are functions with values, respectively, in $i\mathfrak{u}(n)$, $\mathbb{C}^n$ and $\mathbb{R}$ and $I_n \in GL_n(\mathbb{C})$ is the identity.

The corresponding classifying Lie algebroid can be described as follows: as a vector bundle, it is the trivial bundle over $X = i\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$ with fiber type $\mathbb{C}^n \oplus \mathfrak{u}(n)$. The Lie bracket is defined on constant sections by

$$[(z_1, \alpha_1), (z_2, \alpha_2)]_A(s, t, u) := (\alpha_2 z_1 - \alpha_1 z_2, -[\alpha_1, \alpha_2]_{\mathfrak{u}(n)} + (z_1^* z_2 - z_2^* z_1)s +$$
$$- s(z_1 z_2^* - z_2 z_1^*) - (z_1(z_2^* s) - z_2(z_1^* s)) + (z_1^*(sz_2) - z_2^*(sz_1))I_n),$$

while the anchor is given by:

$$\#(z, \alpha)(s, t, u) := (-\alpha s + s\alpha + tz^* + zt^* + \frac{1}{2}(t^*z + z^*t)I_n,$$
$$- \alpha t + (uI_n + s^2)z, t^*sz + z^*st).$$

Finally, the inner action of $\mathfrak{u}(n)$ on $A = X \times (\mathbb{C}^n \oplus \mathfrak{u}(n))$ is composed of the defining action of $\mathfrak{u}(n)$ on $\mathbb{C}^n$ and the adjoint action on $\mathfrak{u}(n)$. This action covers an infinitesimal $\mathfrak{u}(n)$-action on $X = i\mathfrak{u}(n) \oplus \mathbb{C}^n \oplus \mathbb{R}$, which on the first factor corresponds to the adjoint action, on the second factor is just the defining action of $\mathfrak{u}(n)$ and is trivial on the last factor.

In order to understand the local classification of Bochner-Kähler metrics and find local models for such metrics one must integrate this classifying Lie algebroid. The orbit space will correspond to the different classes of Bochner-Kähler metrics, while the different isotropy types will correspond to their symmetry groups. We leave this analysis for a future work.

References

1. R. Bryant, Bochner-Kähler metrics, *J. Amer. Math. Soc.* **14** (2001), no. 3, 623–715.
2. E. Cartan, Sur la structure des groupes inifinis de transformations, *Ann. Éc. Norm.* **3** (1904), 153–206. (Reprinted in Cartan's Collected Works, Part II.)
3. M. Crainic and R.L. Fernandes, Integrability of Lie brackets, *Ann. of Math. (2)* **157** (2003), 575–620.
4. S. Kobayashi, *Transformation Groups in Differential Geometry*, Reprint of the 1972 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.
5. P. Olver, *Equivalence, Invariants and Symmetry*, Cambridge University Press, New York, USA, 1995.
6. I.M. Singer and S. Sternberg, The infinite groups of Lie and Cartan, Part I (the transitive groups), *J. Analyse Math.* **15** (1965), 1–114.
7. S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, New Jersey, USA, 1964.

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