Singular reduction and integrability

Rui L. Fernandes[1]

Joint work with J.P. Ortega and T. Ratiu

[1] Departamento de Matemática
Instituto Superior Técnico

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Motivation

Global problems in Poisson geometry

Ordinary Geometry

- Points are all equal;
- Basic invariant: fundamental group $\pi_1(M, p)$;
- $f : (M, p) \to (N, q) \Rightarrow f_* : \pi_1(M, p) \to \pi_1(N, q)$;
- To get rid of base points, use fundamental groupoid;

Poisson Geometry

- Points are not all equal;
- Basic invariant: Weinstein groupoid $\Sigma(M)$;
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Smooth Poisson quotients

- (M, π) is a Poisson manifold;
- Lie group G acts on M by Poisson diffeomorphisms;
- Action is proper and free;

Fact

M/G carries a unique Poisson structure π_{red} such that $p : M \to M/G$ is a Poisson map.

Proof.

$$C^\infty(M/G) \simeq C^\infty(M)^G.$$

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Theorem

If (M, π) is an integrable Poisson manifold, then $(M/G, \pi_{red})$ is also an integrable Poisson manifold.

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$$\Sigma(M) := \frac{\{\text{cotangent paths}\}}{\{\text{cotangent homotopies}\}}$$

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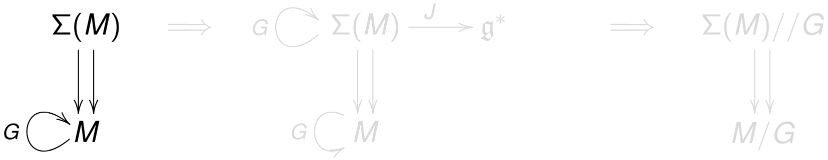

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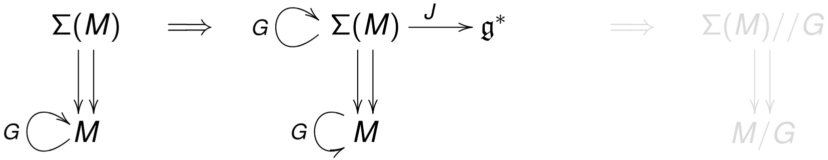

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Symplectization vs Reduction

For general Poisson actions: $\Sigma(M)//G \neq \Sigma(M/G)$.

Theorem

Symplectization and reduction commute if and only if the following groups

$$K_p := \frac{\{a : I \to j^{-1}(0) \mid a \text{ is a cotangent loop such that } a \sim 0_p\}}{\{\text{cotangent homotopies with values in } j^{-1}(0)\}}$$

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- At the Lie groupoid level, $J : \Sigma(M) \to \mathfrak{g}^*$ gives:

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\Sigma(M) & \hookleftarrow & J^{-1}(0)^0 & \xrightarrow{\Phi} & J^{-1}(0)^0/G \\
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M & = & M & \longrightarrow & M/G
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- At the Lie algebroid level, $j : T^*M \to \mathfrak{g}^*$ gives:

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Putting it all together:

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\begin{array}{ccccccc}
K_M & \hookrightarrow & \mathcal{G}(j^{-1}(0)) & \xrightarrow{\widehat{p}} & J^{-1}(0)^0 & & \\
\downarrow{\scriptstyle \widehat{\Phi}} & & \downarrow{\scriptstyle \widehat{\Phi}} & & \downarrow{\scriptstyle \Phi} & & \\
K_{M/G} & \hookrightarrow & \Sigma(M/G) & \xrightarrow{p} & J^{-1}(0)^0/G & = & \Sigma(M)//G
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Hamiltonian actions

Corollary

For $G \times M \to M$ a Hamiltonian action on a symplectic manifold (M, ω) with momentum map $\mu : M \to \mathfrak{g}^*$:

$$K_p := \operatorname{Ker} i_* \subset \pi_1(\mu^{-1}(c), p)$$

where $c = \mu(p)$ and $i : \mu^{-1}(c) \hookrightarrow M$ is the inclusion.

Homotopy long exact sequence of the pair $(M, \mu^{-1}(c))$ gives:

$$\pi_2(M, \mu^{-1}(c), m) \xrightarrow{\partial} \pi_1(\mu^{-1}(c), m) \xrightarrow{i_*} \pi_1(M, m) \xrightarrow{j_*} \pi_1(M, \mu^{-1}(c), m) .$$

So groups vanish if the fibers of the momentum map are simply connected, or if its second relative homotopy groups vanish.

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Example

For the anti-diagonal action of $G = \mathbb{S}^1$ on $M = \mathbb{C}^2 - \{0\}$, which has momentum map $\mu(z, w) = ||z||^2 - ||w||^2$:

$$\mu^{-1}(c) \simeq \begin{cases} \mathbb{C} \times \mathbb{S}^1, \text{ if } c \neq 0, \\ (\mathbb{C} \setminus \{0\}) \times \mathbb{S}^1, \text{ if } c = 0. \end{cases}$$

so that:

$$K_p \simeq \pi_1(\mu^{-1}(c)) = \begin{cases} \mathbb{Z}, \text{ if } c \neq 0, \\ \mathbb{Z} \times \mathbb{Z}, \text{ if } c = 0, \end{cases}$$

and we see that:

$$\Sigma(M)//G \neq \Sigma(M/G).$$

Singular quotients
Orbit type stratification

For a proper action $G \times M \to M$ and $H \subset G$:

- $M^H := \{m \in M : gm = m, \forall g \in H\}$ (*H-fixed point set*);
- $M_H := \{m \in M : G_m = H\}$ (*H-isotropy type*);
- $M_{(H)} := \{m \in M : G_m \in (H)\}$ (*H-orbit type*);

Theorem

The (connected components of the) orbit types determine a smooth stratification of the orbit space:

$$M/G = \bigcup_{(H)} M_{(H)}/G.$$

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Poisson stratifications

What happens if in addition one has Poisson geometry?

Definition

A Poisson stratified space is a smooth stratified space $X = \bigcup_{\alpha \in A} X_\alpha$ such that:

(i) $(C^\infty(X), \{\ ,\ \})$ is a Poisson algebra;

(ii) Each stratum is a Poisson manifold $(X_\alpha, \{\ ,\ \}_\alpha)$;

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If every strata is symplectic, then X is called a symplectic stratified space.

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Example

- $M = \mathfrak{sl}^*(2) \simeq \mathbb{R}^3$: $\{x, z\} = y$; $\{x, y\} = z$; $\{z, y\} = x$.
- Symplectic foliation: $\{(x, y, z) | x^2 + y^2 - z^2 = c\}$.

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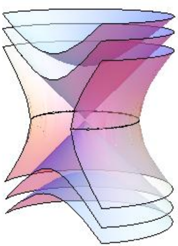

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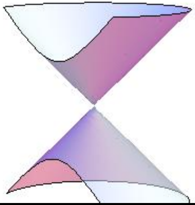

Poisson stratification theorem

Theorem

If $G \times M \to M$ is a proper Poisson action then the orbit type stratification is a Poisson stratification.

Remarks:

- Symplectic leaves of the strata are the orbit reduced spaces obtained from the optimal momentum map.
- G-invariant hamiltonians $H : M \to \mathbb{R}$ give rise to reduced hamiltonian dynamics.
- There is an alternative approach due to J. Śniatycki (2003) using differential spaces (in the sense of Sikorski).

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If $G \times M \to M$ is a proper Poisson action then the orbit type stratification is a Poisson stratification.

Remarks:

- Symplectic leaves of the strata are the orbit reduced spaces obtained from the optimal momentum map.
- G-invariant hamiltonians $H : M \to \mathbb{R}$ give rise to reduced hamiltonian dynamics.
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Example

- $\mathbb{C}P(n) = (\mathbb{C}^{n+1} \setminus \{0\}) / \mathbb{C}^*, \quad \{z_i, z_j\} = a_{ij} z_i z_j.$
  (for a fixed skew-symmetric matrix (a_{ij}))
- $\mathbb{T}^n \times \mathbb{C}P(n) \to \mathbb{C}P(n)$,
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$\mathbb{C}P(n)/\mathbb{T}^n$ is a Poisson stratified space.

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- The map $\mu : \mathbb{C}P(n) \to \Delta^n$,

$$\mu([z_0 : \cdots : z_n]) = \left(\frac{|z_0|^2}{|z_0|^2 + \cdots + |z_n|^2}, \dots, \frac{|z_n|^2}{|z_0|^2 + \cdots + |z_n|^2} \right)$$

  gives identification:

$$\mathbb{C}P(n)/\mathbb{T}^n = \Delta^n := \left\{ (\mu_0, \dots, \mu_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum_{i=0}^{n} \mu_i = 1, \mu_i \geq 0 \right\}.$$

- Poisson bracket on Δ^n:

$$\{\mu_i, \mu_j\}_\Delta = \left(a_{ij} - \sum_{l=0}^{n} (a_{il} + a_{lj}) \mu_l \right) \mu_i \mu_j.$$

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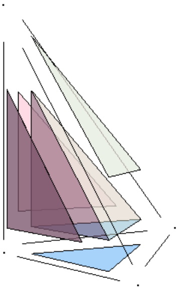

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The proof

Proposition (Vanhaecke, RLF)

If $G \times M \to M$ is a Poisson action of a compact Lie group G, then M^G is a Poisson-Dirac submnifold of M:

$$\{f, h\}_{M^G} = \{\widetilde{f}, \widetilde{h}\}\Big|_{M^G},$$

where $\widetilde{f}, \widetilde{h} \in C^\infty(M)$ are G-invariant extensions of f and h.

Remarks:

- $M^G \hookrightarrow M$ is a backward Dirac map.
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Poisson structure on orbit type $M_{(H)}/G$:

- Fix isotropy type $H \subset G$;
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Proposition

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- Set $L(H) := N(H)/H$;
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Given conjugate isotropy types $(H) = (H')$:

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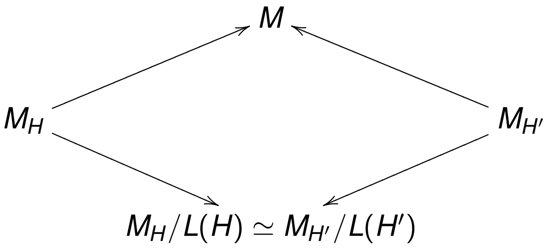

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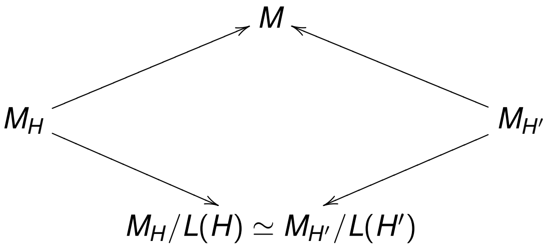

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- $M_{(H)}/G \simeq M_H/L(H)$ carries natural Poisson structure;
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$$M/G = \bigcup_{(H)} M_{(H)}/G \text{ is a Poisson stratification.}$$

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Non-free case

Theorem

If $G \times M \to M$ is a proper Poisson action, and M is an integrable Poisson manifold, then M/G is an integrable Poisson stratified space.

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Commutative diagram of Dirac structures:

- $M = \bigcup_{(H)} M_{(H)}$ is a Dirac stratified space;
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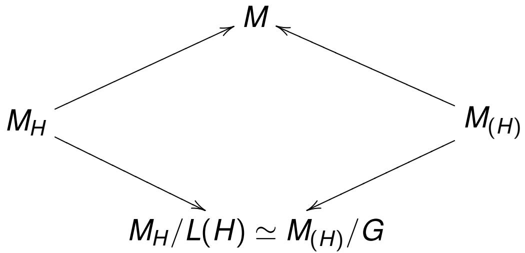

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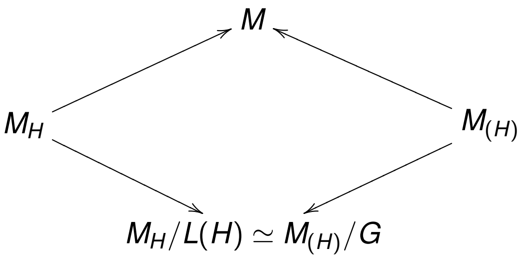

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References

- R.L. Fernandes, J.-P. Ortega and T.S. Ratiu,
  The Momentum Map in Poisson geometry
  Preprint arXiv:0705.0693
- R.L. Fernandes, D. Iglesias Ponte,
  Symmetries and Reduction of Poisson Lie Groupoids
  in preparation
- J.-P. Ortega and T.S. Ratiu,
  Momentum Maps and Hamiltonian Reduction,
  Progress in Mathematics, volume 222.
  Birkhaüser Verlag, 2004.

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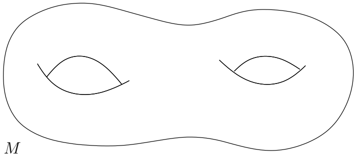

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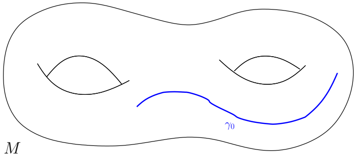

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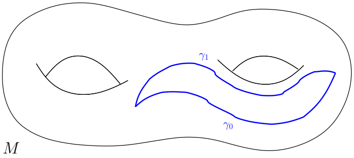

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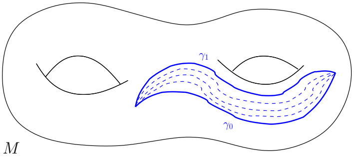

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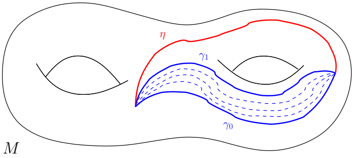

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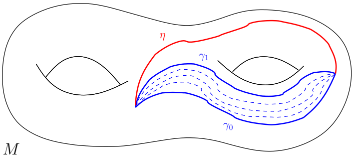

The fundamental groupoid of *M* is:

$$\Pi_1(M) := \{\text{paths } \gamma\}/\{\text{homotopies}\} = \{[\gamma] \mid \gamma : [0,1] \to M\}\,.$$

The fundamental groupoid of a manifold

The fundamental groupoid

$$\Pi_1(M) = \{[\gamma] \mid \gamma : [0,1] \to M\}$$

has the following structure:

- source and target: $s([\gamma]) = \gamma(0)$, $t([\gamma]) = \gamma(1)$;
- product: $[\gamma] \cdot [\eta] = [\gamma \cdot \eta]$;
- units: $1_x = [\gamma]$, where $\gamma(t) = x$;
- inverses: $[\gamma]^{-1} = [\overline{\gamma}]$, where $\overline{\gamma}(t) = \gamma(1-t)$.

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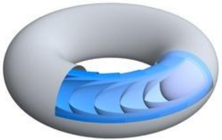
The Weinstein groupoid

Take any Poisson manifold (M, π) :



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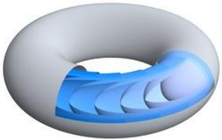

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- A cotagent path is a path $a(t) \in T_{\gamma(t)}M$ such that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \pi^\sharp(a(t));$$

- A cotangent homotopy is a family of cotangent paths $a_\varepsilon(t)$, such that the solution $b = b(\varepsilon, t)$ of:(*)

$$\partial_t b - \partial_\varepsilon a = T_\nabla(a, b), \qquad b(\varepsilon, 0) = 0,$$

satisfies $b(\varepsilon, 1) = 0$.

(*) After some choice of ∇; T denotes the torsion.

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The Weinstein groupoid

$\Sigma(M) \rightrightarrows M$ is a topological groupoid.

Definition

A Poisson manifold (M, π) is called integrable if $\Sigma(M)$ is smooth, i.e., it is a Lie groupoid.

In this case, $\Sigma(M)$ carries a natural symplectic structure Ω which is compatible with multiplication:

$$m^*\Omega = \pi_1^*\Omega + \pi_2^*\Omega,$$

where $m, \pi_1, \pi_2 : \Sigma(M) \times \Sigma(M) \to \Sigma(M)$.

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