Symmetry beyond groups

Rui Loja Fernandes

May, 2004

Main Reference:
A. Weinstein, Groupoids: Unifying Internal and External Symmetry, *Notices Amer. Math. Soc.* **43** (1996).

http://www.math.ist.utl.pt/~rfern/

1. Introduction

Why groupoids?

Usual Credo:

$$\text{Symmetry} = \text{Group Theory}$$

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$$\text{Symmetry} = \textit{Groupoid}\ \text{Theory}$$

Basic Remark:

Many objects which we recognize as *symmetric* admit few or no non-trivial symmetries.

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Groupoids allow one to fix this.

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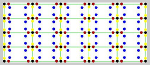
2. Usual credo. . .

$$\text{symmetries} = \text{groups}$$



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- **Inverse**. For all $g \in G$ there exists $g^{-1} \in G$:

$$g g^{-1} = g^{-1} g = e.$$

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The **Euclidean group** is:

$$E(n) = \{\phi : \mathbb{R}^n \to \mathbb{R}^n : d(\phi(x), \phi(y)) = d(x, y), \forall x, y \in \mathbb{R}^n\}$$

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with multiplication *composition* of isometries:

$$\begin{aligned} E(n) \times E(n) &\to E(n) \\ (\phi_1, \phi_2) &\longmapsto \phi_1 \circ \phi_2. \end{aligned}$$

Group of isometries of $\mathbb{R}^n$ (cont.)

Every isometry $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is of the form:

$$\phi(x) = Ax + b,$$

where $b \in \mathbb{R}^n$ and A is an **orthogonal matrix**:

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ISOMETRY = ORTHOGONAL TRANSFORMATION + TRANSLATION

Remark:
A **proper isometry** is an isometry which preserves orientation $\Leftrightarrow \phi(x) = Ax + b$ with $\det A = 1$.

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- The **group of translations**:

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- The **special orthogonal group** ("rotations"):

$$\begin{aligned} SO(n) &= \{\phi \in O(n) : \phi \text{ is proper}\} \\ &\simeq \left\{A : AA^T = A^T A = I,\ \det A = 1\right\}. \end{aligned}$$

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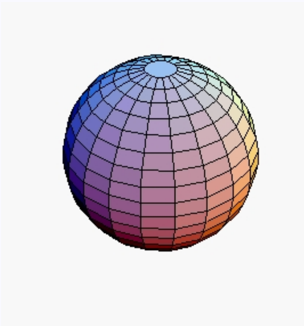

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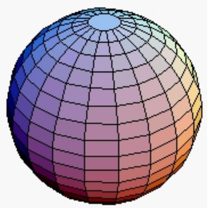

$$G_\Omega = O(n)$$

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Example: Tiling by rectangles of $\mathbb{R}^2$

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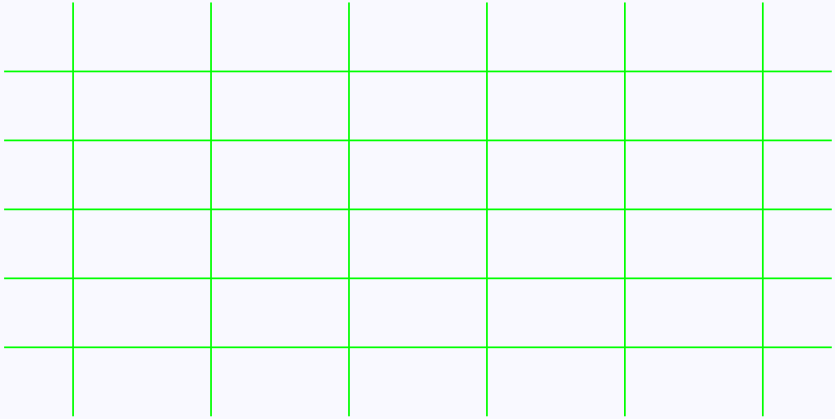
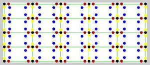
Take $\Omega \subset \mathbb{R}^2$ the tiling of $\mathbb{R}^2$ by $2:1$ rectangles:



What is the group of symmetries G_Ω?

Example: Tiling by rectangles of $\mathbb{R}^2$ (cont.)

The group G_Ω consists of:



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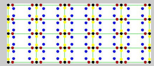
The group G_Ω consists of:

- Translations by elements of the lattice $\Lambda = 2\mathbb{Z} \times \mathbb{Z}$:

$$(x, y) \mapsto (x, y) + (2n, m), \qquad n, m \in \mathbb{Z}.$$



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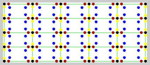

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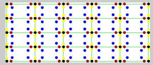

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The tiling has a *lot of symmetry*!

This gives a very successful theory:

- symmetry groups of tilings;
- symmetry groups of crystals;
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- symmetry groups of geometric structures;

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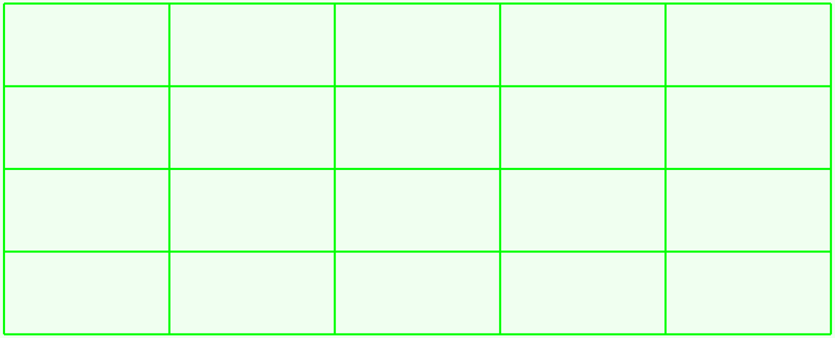
$\vdots$

But . . .

3. Need for a new credo

Instead of tiling, take B a **real** bathroom floor:



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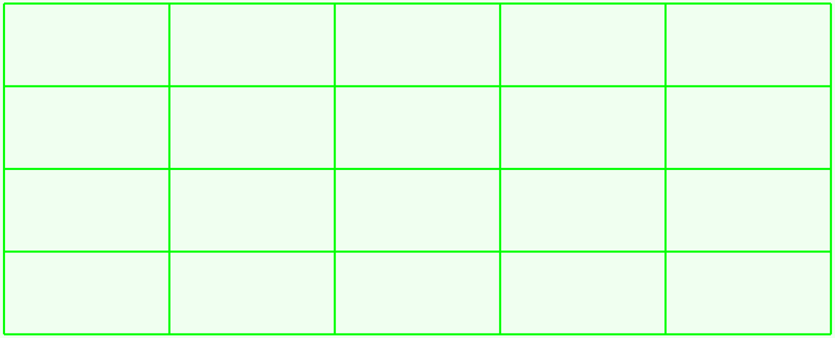

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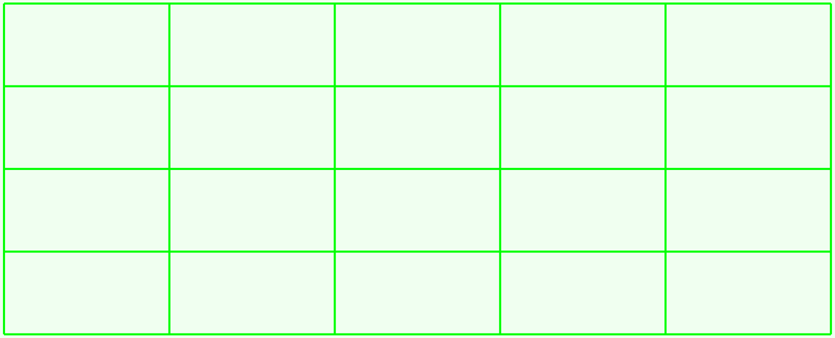

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However, we can still recognize a repetitive pattern...

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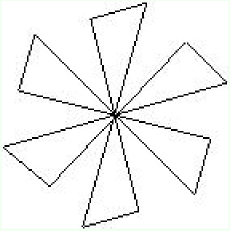

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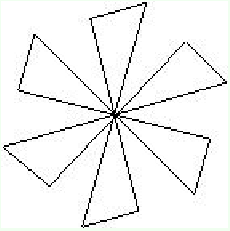
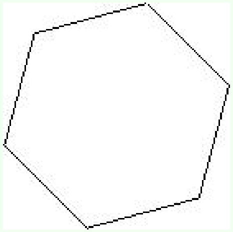


- *The group D_n of symmetries of a regular n-side polyhedron:*



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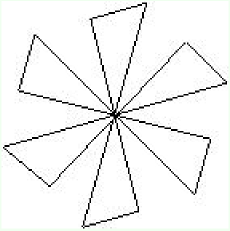

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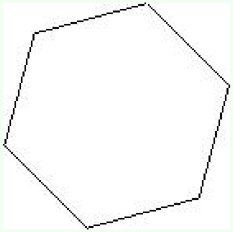
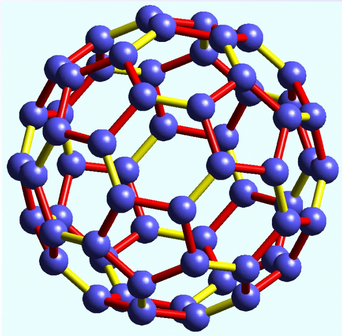


- *The 3 groups of symmetries of the platonic solids.*

For example, the molecule of the fullerene C_{60}:



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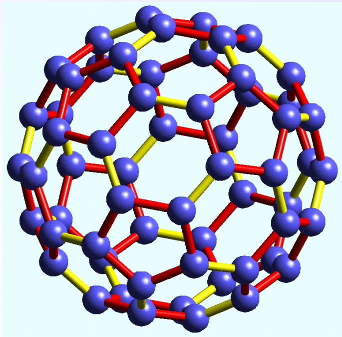

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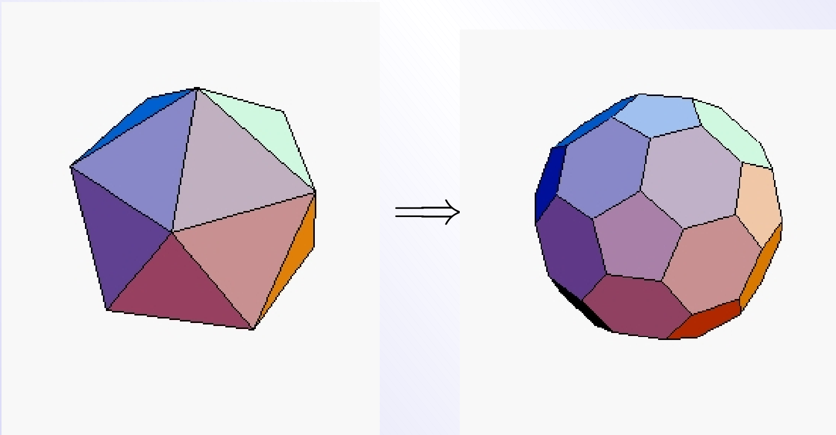

(just truncate the vertexes of the icosahedron).

4. Symmetry groupoids

To distinguish the soccer ball from the icosahedron, to describe the symmetry of a bathroom floor, and in many other problems, we need *groupoids*.

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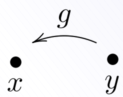

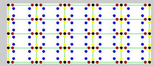

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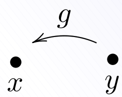
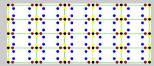


Now, we have:

- *source* and *target maps* $\mathbf{s}, \mathbf{t} : \mathcal{G} \to \mathbb{R}^2$:

$$\mathbf{s}(x, \phi, y) = y, \qquad \mathbf{t}(x, \phi, y) = x.$$



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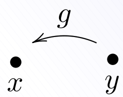

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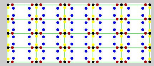

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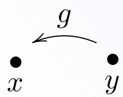

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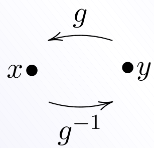
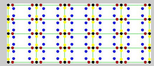
- *identity arrows* $1_x = (x, I, x)$:

- *inverse arrows* $g^{-1} = (y, \phi^{-1}, x)$:





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Definition 4.1. A **groupoid** with base B is a set $\mathcal{G}$ with maps $\mathbf{s}, \mathbf{t} : \mathcal{G} \to B$ and operation satisfying 1–4.

We can *restrict* the symmetry groupoid $\mathcal{G}_\Omega$ of the tiling, to the real bathroom floor $B \subset \mathbb{R}^2$:

$$\mathcal{G}_B = \{(x, \phi, y) : x, y \in B, \phi \in G_\Omega \text{ and } x = \phi(y)\}\,.$$

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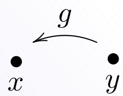

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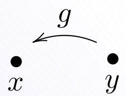

- The **isotropy group** of $x \in B$ is the set of arrows $g \in \mathcal{G}$ from x to x:

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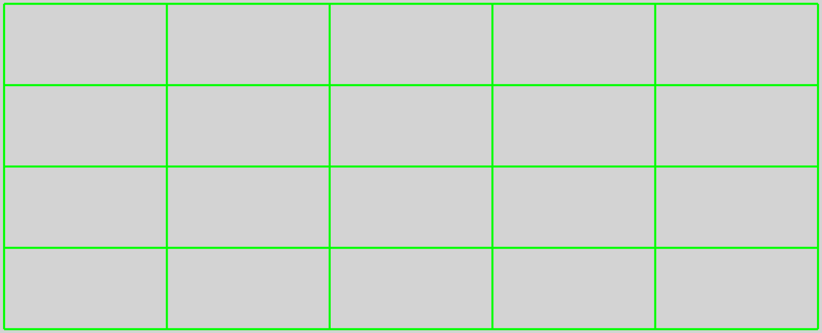

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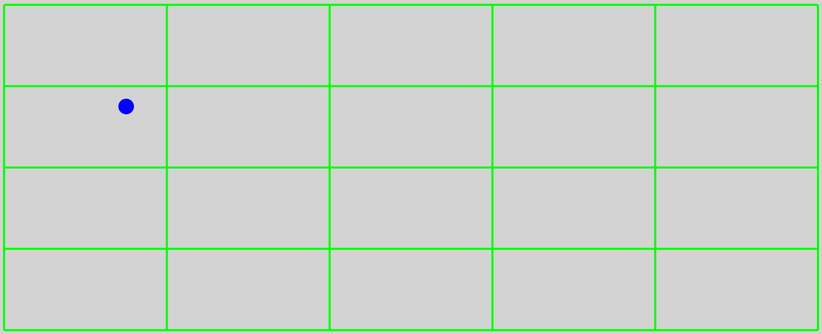

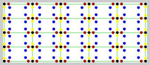

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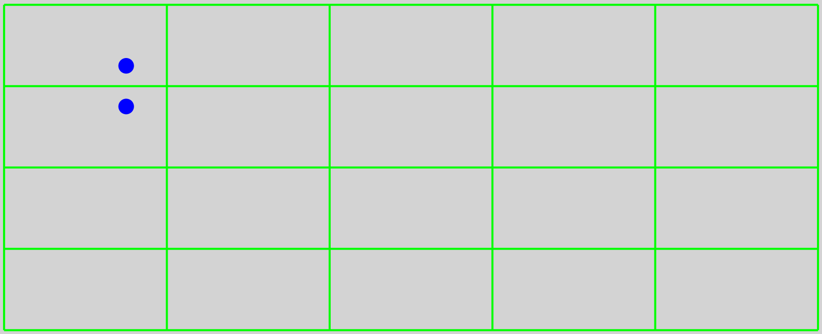

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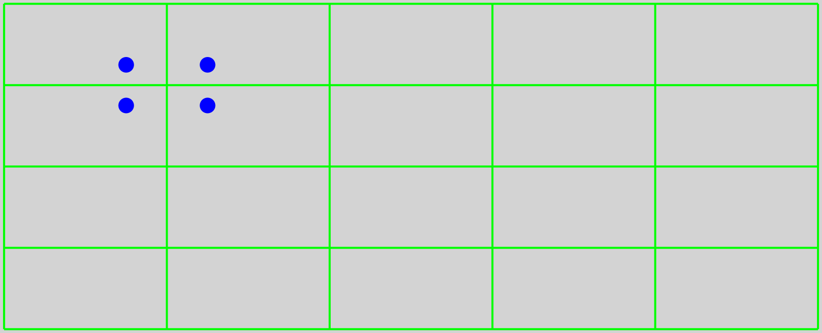

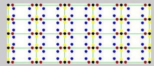

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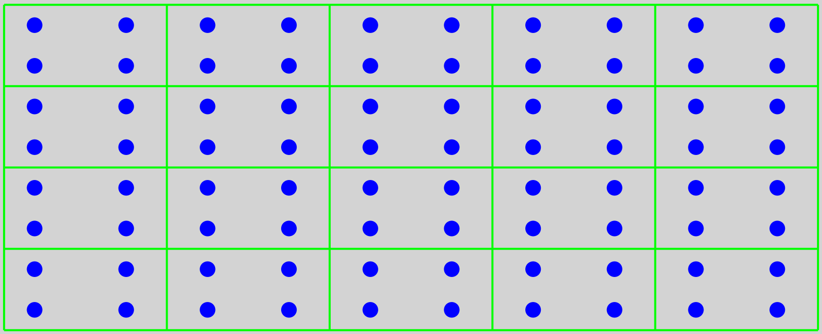

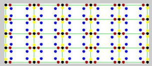

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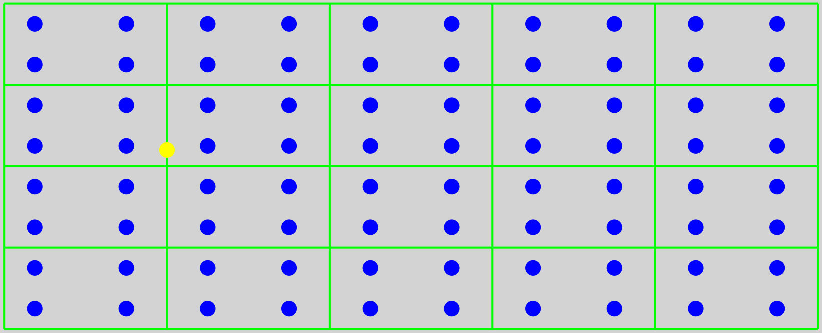

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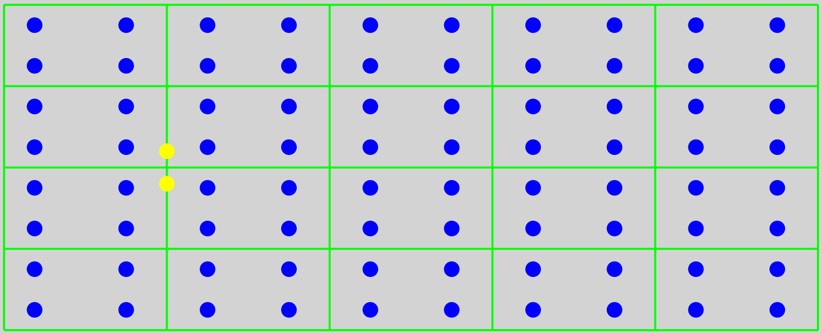

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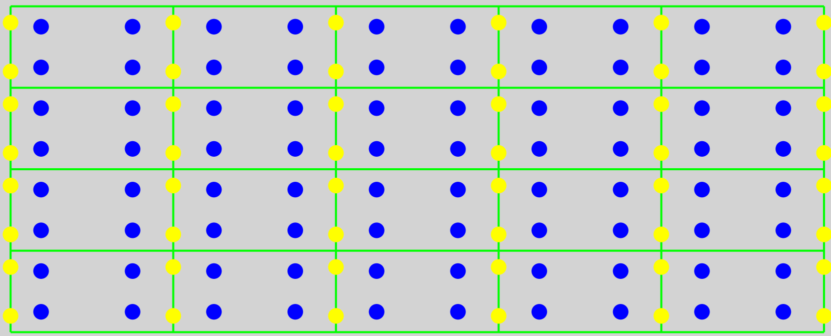

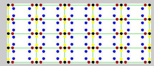

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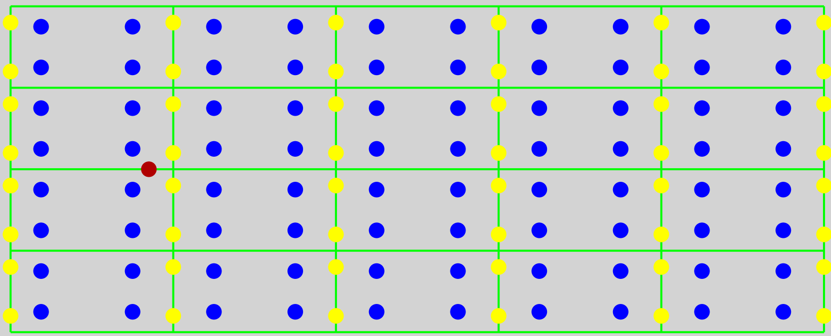

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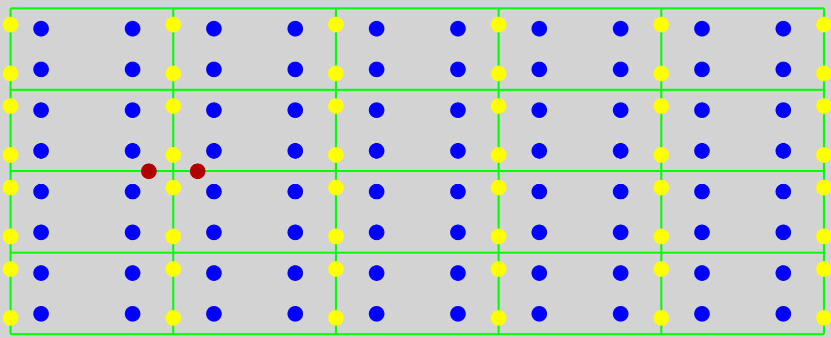

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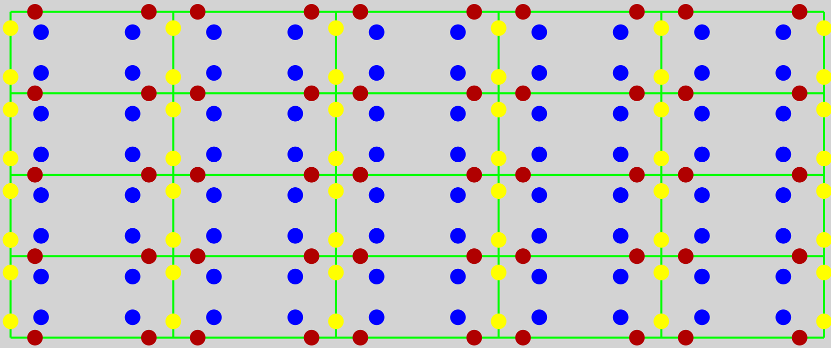

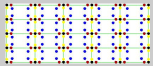

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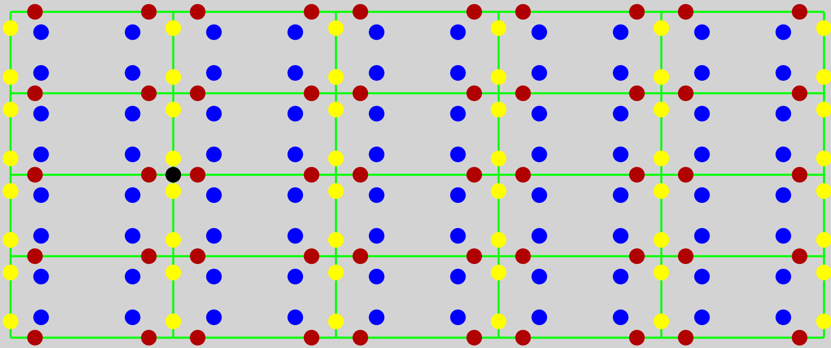

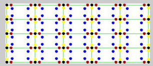

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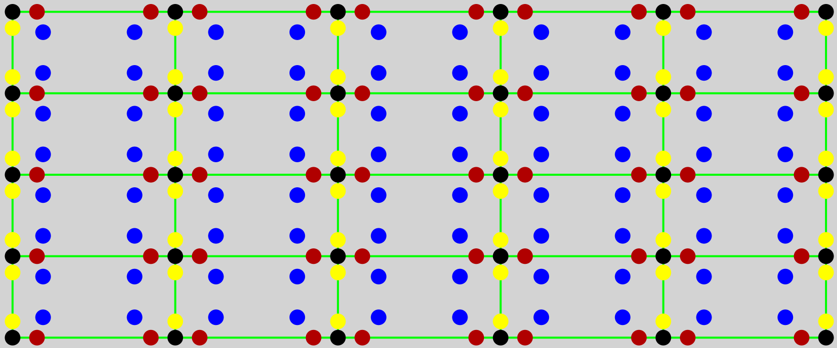

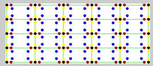

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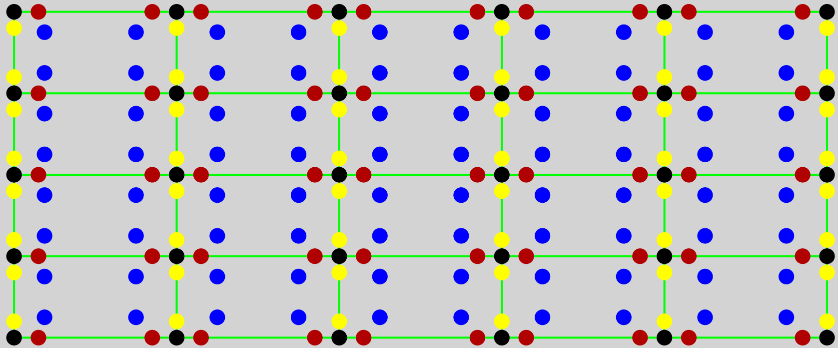

- The only points with non-trivial isotropy are those in $\left(\mathbb{Z} \times \frac{1}{2}\mathbb{Z}\right) \cap B$. For these, the isotropy group is:

$$G = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

5. Other groupoids

Groupoids play an important role in many other contexts, not related with symmetry.

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X any *topological* space
Look at *continuous* curves $\gamma : [0,1] \to X$

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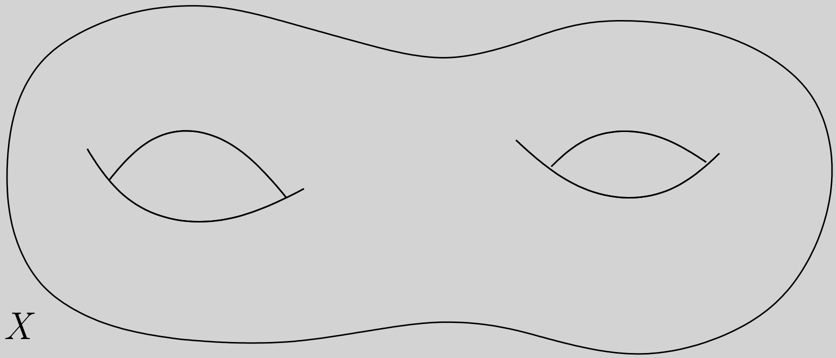

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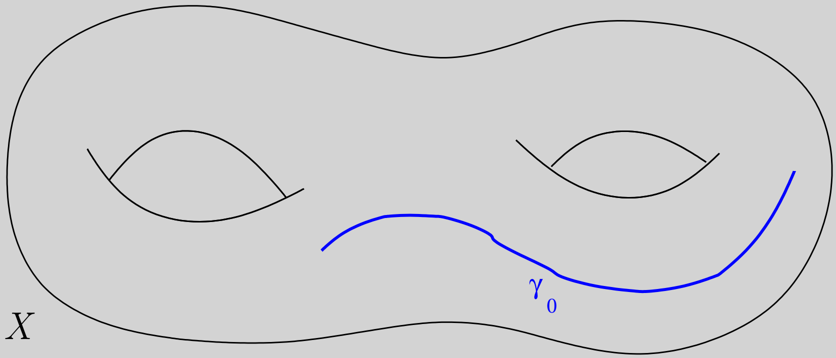

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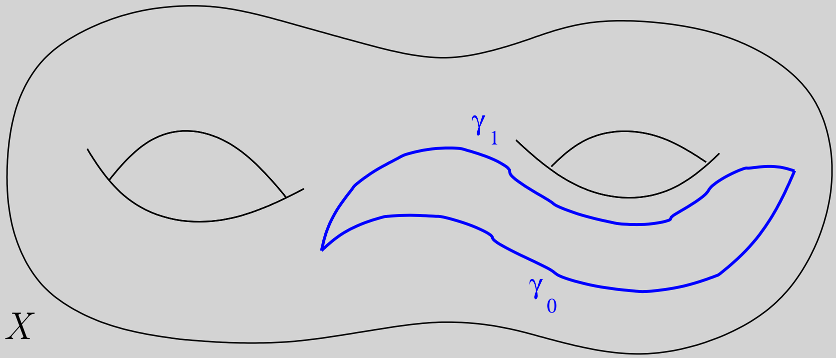

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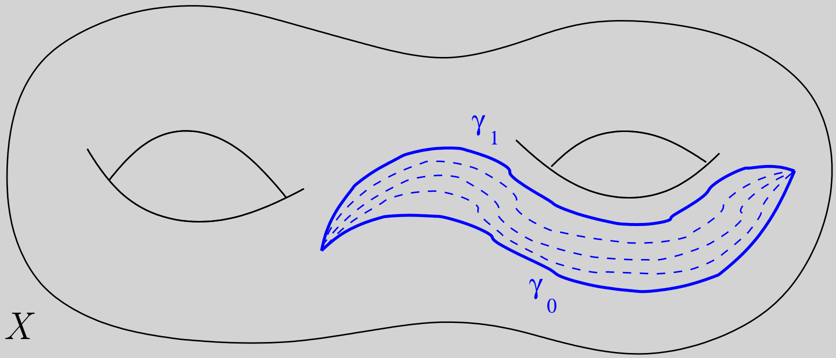

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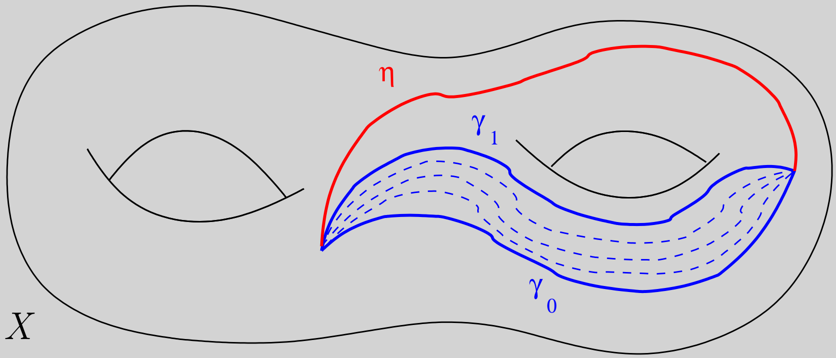

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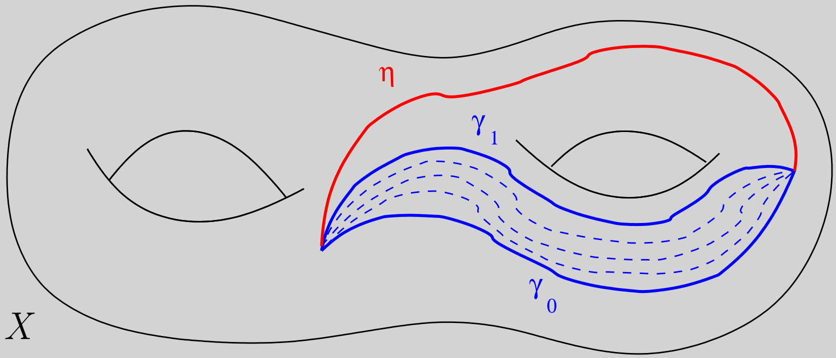

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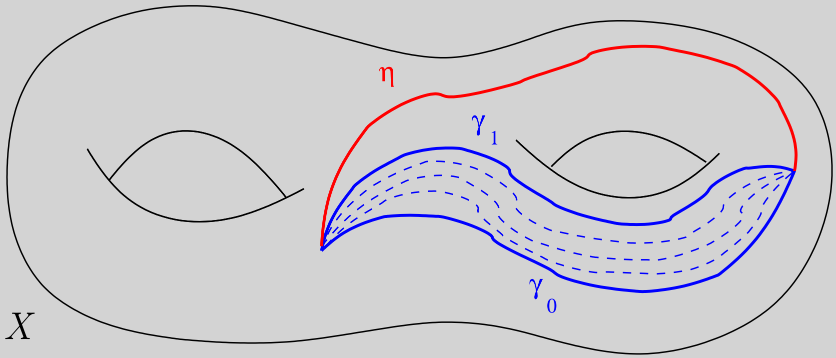

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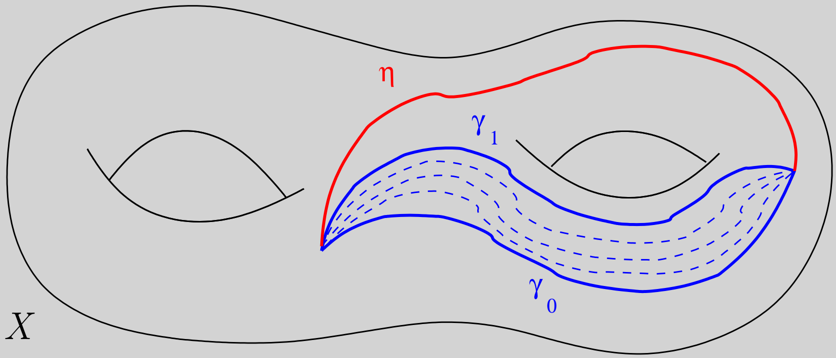

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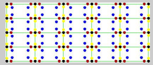

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- *inverse* is the opposite curve:

$$[\gamma]^{-1} = [\overline{\gamma}], \qquad \text{where } \overline{\gamma}(t) = \gamma(1-t).$$

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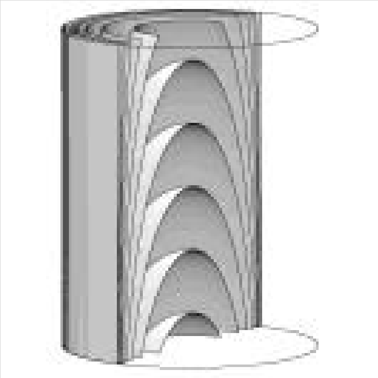
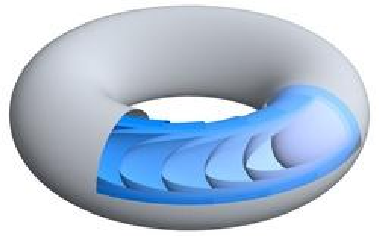
Examples:

- If $X = SO(2)$ one has $\pi(X, x) = \mathbb{Z}$.
- If $X = SO(n)$ one has $\pi(X, x) = \mathbb{Z}_2 = \{+1, -1\}$.

Groupoids and control theory

X a *foliated space*:





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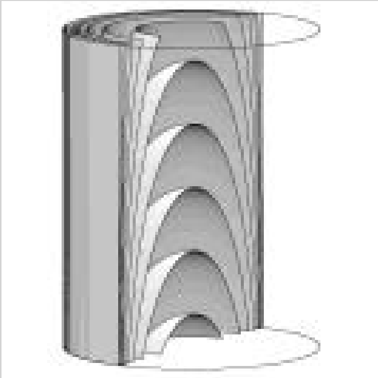

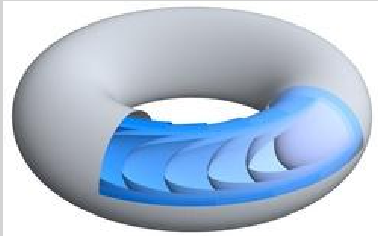

Now, we can only deform curves lying on leaves of the foliation $\mathcal{F}$.

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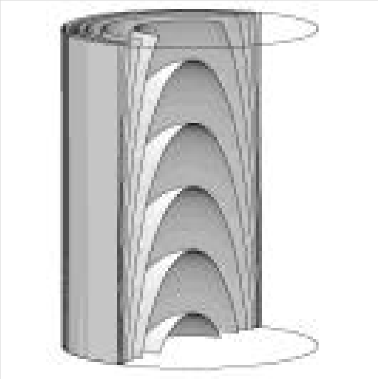

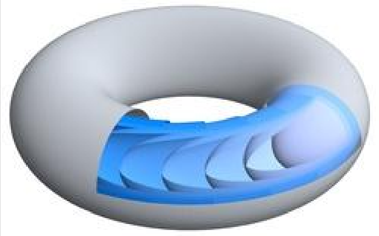

Now, we can only deform curves lying on leaves of the foliation $\mathcal{F}$. We obtain the *monodromy groupoid* of the foliation:

$$\Pi(\mathcal{F}) = \{[\gamma] \mid \gamma : [0,1] \to L,\ L \text{ is a leaf of } \mathcal{F}\}\,.$$

Groupoids and control theory

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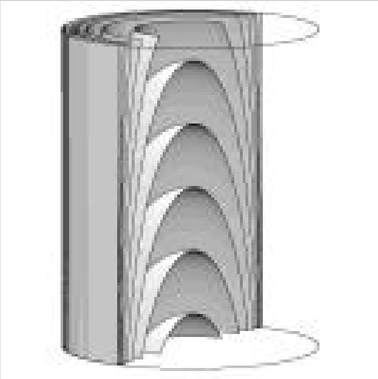

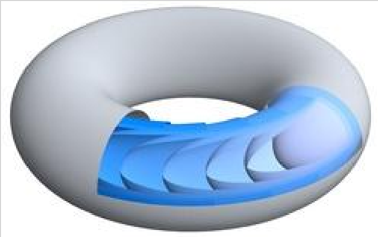

Now, we can only deform curves lying on leaves of the foliation $\mathcal{F}$. We obtain the *monodromy groupoid* of the foliation:

$$\Pi(\mathcal{F}) = \{[\gamma] \mid \gamma : [0,1] \to L,\ L \text{ is a leaf of } \mathcal{F}\}.$$

- Orbits are the leaves of $\mathcal{F}$;
- Isotropy groups are the fundamental groups of the leaves.

In control theory:

$$\text{ORBITS} = \text{ACCESSIBLE SETS}$$

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This is where the *real math* starts and where this talk stops. . .