Lie symmetries of finite-difference equations

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Discretizations of the Helmholtz, heat, and wave equations on uniform lattices are considered in various space–time dimensions. The symmetry properties of these finite-difference equations are determined and it is found that they retain the same Lie symmetry algebras as their continuum limits. Solutions with definite transformation properties are obtained; identities and formulas for these functions are then derived using the symmetry algebra. © *1995 American Institute of Physics.*

I. INTRODUCTION

Finite-difference analysis has recently attracted wide interest, both in mathematics and physics. On the one hand, the advent of supercomputers and the development of efficient numerical schemes have led to the discrete modeling of complex continuous systems. On the other hand, genuine discrete physical systems defined on space–time lattices have been seen to possess rich symmetry properties, that allow for their complete solvability.[1,2]

In this respect, the study of finite-difference equations is central in any development related to the analysis of discrete systems. The theory underlying these equations can be established, at least formally, in parallel to the one associated to differential equations. Nevertheless, little is known actually, on the symmetry properties of difference equations.

The importance of symmetry techniques in the study of partial differential equations need not be stressed.[3] They allow to systematically obtain and classify solutions, and provide at the same time a deep connection with the theory of the special functions of mathematical physics. Indeed, the relation between symmetries of second-order partial differential equations and their solutions via separation of variables is one of the most useful and efficient tool for studying properties of special functions.[4–8]

A systematic study of the symmetry properties of linear finite-difference equations on uniform lattices will be presented below. Though differential techniques are not available in the case of difference equations, for linear equations at least, one can devise an algorithm that allows to determine the symmetry operators. The striking outcome of our investigation is that these symmetry operators obey the same Lie algebra as their continuum counterparts. In other words, the symmetry algebra is left unchanged by the process of discretization.

Notice that this result holds only in the case of finite-difference equations on uniform lattices. A similar analysis of difference equations on exponential lattices has shown that generalized algebraic structures, such as quantum algebras, are needed to describe their symmetry properties.[9–12]

Once the symmetries of a linear difference equation have been obtained, they can be used to find solutions of the equations with definite transformation properties. These solutions turn out to be expressible in terms of hypergeometric series: they appear to be lattice generalizations of many classical special functions and polynomials. Properties and identities for these functions can then be derived using the symmetry algebra.

After devoting a section to the notations and definitions that are used throughout the article, we first examine in Sec. III a discrete version of the two-dimensional Helmholtz equation. The

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Euclidean algebra $\mathscr{E}(2)$ is seen to be the symmetry algebra of this equation. In the following two sections lattice versions of the heat equation in two and three dimensions are, respectively, considered. The Schrödinger algebra is here seen to be the symmetry algebra of these equations. Discrete versions of the wave equations in three and four dimensions are studied in Secs. VI and VII. The Lie algebras so(5) and sl(4) emerge as the respective symmetry algebras. In each case, solutions of the difference equations are constructed using techniques akin to the classical method of separation of variables. Discrete analogs of the Gauss hypergeometric and Bessel functions, and of the polynomials of Hermite, Laguerre, and Gegenbauer are thus found in this way. Finally, Sec. VII comprises concluding comments and remarks.

II. NOTATIONS

We collect here a few formulas in finite difference analysis that will be used in the sequel.[13,14,7]

The shifted factorial $(\alpha)_n$, with α complex and n integer, is defined by[15,16]

$$\begin{gathered}(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), \quad n\neq 0,\\ (\alpha)_0=1.\end{gathered} \tag{2.1}$$

More generally, for n complex, one uses the equivalent definition in terms of Γ functions

$$(\alpha)_n=\Gamma(\alpha)/\Gamma(\alpha+n). \tag{2.2}$$

The symbol $(\alpha)_n$ satisfy many useful identities; we list those that shall be used in the following:[15]

$$(\alpha)_{n+k}=(\alpha)_n(\alpha+n)_k, \tag{2.3a}$$

$$(\alpha)_{n-k}=\frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}, \tag{2.3b}$$

$$(\alpha)_{-n}=\frac{(-1)^n}{(1-\alpha)_n}, \tag{2.3c}$$

$$(-n)_k=(-1)^k\,\frac{n!}{(n-k)!}. \tag{2.3d}$$

It is customary to introduce two discrete versions of the derivative d/dz

$$\Delta_z^+=\frac{1}{\sigma}\,(T_z-1), \tag{2.4a}$$

$$\Delta_z^-=\frac{1}{\sigma}\,(1-T_z^{-1}), \tag{2.4b}$$

where σ is the lattice spacing, and T_z is the shift operator which acts as

$$T_zf(z)=f(z+\sigma) \tag{2.5}$$

on any function of the complex variable z. Both Δ_z^+ and Δ_z^- reduce to the ordinary derivative d/dz as $\sigma\to 0$. Further, notice that $\Delta_z^-T_z=\Delta_z^+$ and similarly, that $\Delta_z^+T_z^{-1}=\Delta_z^-$. The following useful relations are also readily verified

$$\Delta_z^- \left(\frac{z}{\sigma}\right)_n = \frac{n}{\sigma}\left(\frac{z}{\sigma}\right)_{n-1}, \tag{2.6a}$$

$$z\Delta_z^+ \left(\frac{z}{\sigma}\right)_n = n\left(\frac{z}{\sigma}\right)_n. \tag{2.6b}$$

Discrete versions of the exponential function are naturally defined

$$e(\lambda z) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda \sigma^n}{n!}\left(-\frac{z}{\sigma}\right)_n = (1+\lambda\sigma)^{z/\sigma}, \tag{2.7a}$$

$$e^{-1}(\lambda z) = \sum_{n=0}^{\infty} (-1)^n \frac{\lambda \sigma^n}{n!}\left(\frac{z}{\sigma}\right)_n = (1+\lambda\sigma)^{-z/\sigma}, \quad \lambda \in \mathbf{C}. \tag{2.7b}$$

They are the inverse of the other, and eigenfunctions of the finite-difference derivatives

$$\Delta_z^+ e(\lambda z) = \lambda e(\lambda z), \tag{2.8a}$$

$$\Delta_z^- e^{-1}(\lambda z) = -\lambda e^{-1}(\lambda z). \tag{2.8b}$$

In the continuum limit, $\sigma \to 0$, they tend to standard exponentials: $e(\lambda z) \to e^{\lambda z}$, $e^{-1}(\lambda z) \to e^{-\lambda z}$. The exponential functions (2.7) will play an important role in the the considerations that follow.

The generalized hypergeometric series ${}_rF_s$ is defined by

$${}_rF_s(a_1,a_2,\ldots,a_r;b_1,\ldots,b_s;z) \equiv {}_rF_s\left[\begin{matrix} a_1,a_2,\ldots,a_r \\ \\ b_1,\ldots,b_s \end{matrix} ;z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\cdots(a_r)_n}{n!(b_1)_n\cdots(b_s)_n} z^n, \tag{2.9}$$

where a_i and b_j are complex parameters, and b_j are different from negative integers. Since $(-m)_n=0$, for $n=m+1,m+2,\ldots$, the series ${}_rF_s$ terminates if one of the numerator parameters $\{a_i\}$ is a negative integer. By the ratio test, the ${}_rF_s$ series converges absolutely for all z if $r \leqslant s$, and for $|z|<1$ if $r=s+1$. Of particular interest is the case $r=2$ and $s=1$ of Eq. (2.9); it gives the Gauss hypergeometric series

$${}_2F_1(a,b;c;q,z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} z^n, \quad |z|<1. \tag{2.10}$$

In the following sections we shall see that these hypergeometric functions arise in solutions of linear partial finite-difference equations.

III. THE HELMHOLTZ EQUATION IN TWO DIMENSIONS

We start our analysis by examining linear equations on a two-dimensional rectangular lattice, with coordinates x_1 and x_2. We denote by σ_1 and σ_2 the lattice spacings in the two directions, so that x_1/σ_1 and x_2/σ_2 are integers. (Without loss of generality, we assume that the origin belongs to the lattice.) On this lattice, let us consider the following simple finite-difference equation

$$[\Delta_{x_1}^- \Delta_{x_2}^- - \omega^2]\varphi(x_1,x_2) = 0, \tag{3.1}$$

with ω a complex parameter. In the continuum limit σ_1, $\sigma_2 \to 0$, it reduces to the two-dimensional Helmholtz equation $[\partial_{x_1}\partial_{x_2} - \omega^2]\varphi = 0$. We shall call (3.1) the discrete Helmholtz equation in two dimensions.

A general class of solutions of Eq. (3.1) can be expressed in terms of the discrete exponentials (2.7). Indeed, using the properties (2.8) one sees that

$$\varphi(x_1,x_2;\alpha,\beta)=(1+\alpha\sigma_1)^{-x_1/\sigma_1}(1+\beta\sigma_2)^{-x_2/\sigma_2} \tag{3.2}$$

solves Eq. (3.1) provided the two complex parameters α and β satisfy the constraint $\alpha\beta=\omega^2$.

By definition, the symmetry operators of a finite-difference equation have the property of transforming solutions into solutions. In the case of differential equations, general algorithms based on local Lie theory allow to construct in an efficient way the corresponding symmetries.[3] These methods are not yet developed in the case of discrete equations, though some partial results in this direction can be found in the recent literature.[17,18,1] One is therefore forced at this point to look for different strategies.

In cases for which a general class of solutions is explicitly known, one can find symmetry transformations by acting with trial operators on solutions to determine which of those operators give back solutions. This method has been successfully applied to q-difference equations in Refs. 9, 10, and will also be used here. For a general nonlinear difference equation this strategy is rather cumbersome and can only give partial results; however, for linear equations it is exhaustive and easy to implement.

Let us consider the solution (3.2) of the discrete-Helmholtz equation. The symmetry operators must be built out of the lattice translation operators T_{x_1} and T_{x_2} and the variables x_1 and x_2. When acting on solutions they will have the effect of multiplying Eq. (3.2) by polynomials $\mathscr{P}_n(x_1,x_2)$ of degree n in x_1 and x_2. To determine the coefficients and degree of these polynomials, one acts directly on $\mathscr{P}_n\varphi$ with the operator

$$\mathscr{O}=[\Delta^-_{x_1}\Delta^-_{x_2}-\omega^2] \tag{3.3}$$

and requires the result to be zero. This produces a set of algebraic defining or consistency relations that have to be solved in order to find a new solution. In this way one recursively constructs the symmetry operators, by considering polynomials with increasing degree n. We refer to Ref. 10 for further details.

In the present case, multiplying any solution (3.2) by a constant gives back a solution. When this constant is either α or β one finds the first two symmetry operators

$$P_1=\Delta^-_{x_1}, \quad P_2=\Delta^-_{x_2}. \tag{3.4}$$

A third symmetry operator is obtained when $n=1$. Following the approach outlined above, one easily finds its explicit expression

$$M=x_2\Delta^+_{x_2}-x_1\Delta^+_{x_1}. \tag{3.5}$$

One can check that Eq. (3.1) does not admit any other symmetry beyond P_1, P_2, and M. Indeed, one obtains only combinations of products of these three operators from considering polynomials $\mathscr{P}_n$ of higher order.

The operators (3.4) and (3.5) are particularly simple and could have been guessed directly, without recourse to the algorithm described above. It has been applied here to illustrate the general method that will be followed in the more complicated situations that will be encountered in the coming sections.

The operators P_1, P_2, and M satisfy the commutation relations

$$[M,\, P_1]=P_1, \quad [M,\, P_2]=-P_2, \quad [P_1,\, P_2]=0 \tag{3.6}$$

and generate the Euclidean algebra $\mathscr{E}(2)$. This is therefore the symmetry algebra of the discrete-Helmholtz equation (3.1). It is also the symmetry algebra of the classical two-dimensional Helmholtz differential equation.[4] This is the first example of a pattern that will repeat itself in the subsequent sections: in passing from the continuum to the lattice, the symmetry properties of the equation are not modified.

Solutions of Eq. (3.1) with definite transformation properties under the action of the symmetry operators P_1, P_2, and M can be easily found, using techniques that resemble the method of separation of variables. First, notice that the solution (3.2) is separated in Cartesian coordinates: it is an eigenfunction of the symmetry operators P_1 and P_2. One can choose to diagonalize instead the operator M, and to look therefore for solutions of the discrete Helmholtz equation of the form

$$\varphi_\nu(x_1,x_2)=\sum_{k=0}^{\infty} c_k \left(\frac{x_1}{\sigma_1}\right)_k \left(\frac{x_2}{\sigma_2}\right)_{k+\nu} \tag{3.7}$$

for which

$$M\varphi_\nu(x_1,x_2)=\nu\varphi_\nu(x_1,x_2), \qquad \nu\in\mathbf{C}. \tag{3.8}$$

Substitution of the *Ansatz* (3.7) into Eq. (3.1), provides a recursion relation for the coefficients c_k that can be solved; one finds

$$\varphi_\nu(x_1,x_2)=(\omega\sigma_2)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+\nu+1)} \left(\frac{x_1}{\sigma_1}\right)_k \left(\frac{x_2}{\sigma_2}\right)_{k+\nu} (\omega^2\sigma_1\sigma_2)^k. \tag{3.9}$$

Recalling the definition of Gauss's hypergeometric function (2.10), one can also rewrite

$$\varphi_\nu(x_1,x_2)=\frac{(\omega\sigma_2)^\nu}{\Gamma(\nu+1)} \left(\frac{x_2}{\sigma_2}\right)_\nu {}_2F_1\left(\frac{x_1}{\sigma_1},\, \frac{x_2}{\sigma_2}+\nu;\nu+1;\omega^2\sigma_1\sigma_2\right). \tag{3.10}$$

Notice that the coordinates x_1 and x_2 appear as parameters of the ${}_2F_1$ function, while its variable is expressed in terms of the lattice spacings σ_1 and σ_2. In the continuum limit, one finds

$$\lim_{\sigma_1,\sigma_2\to 0} \varphi_\nu(x_1,x_2)=\left(\frac{x_1}{x_2}\right)^{\nu/2} I_\nu(2\omega\sqrt{x_1x_2}), \tag{3.11}$$

where I_ν is the classical modified Bessel function of first kind.[16] One can thus consider the solutions (3.9) of the Eq. (3.1) as a discrete generalization of these Bessel functions.

From the algebra (3.6), one realizes that the operators P_1 and P_2 when acting on solutions $\varphi_\nu(x_1,x_2)$, increase and decrease, respectively, the index ν by 1. More specifically, one finds

$$\Delta_{x_1}^{-}\varphi_\nu=\omega\varphi_{\nu+1}, \tag{3.12a}$$

$$\Delta_{x_2}^{-}\varphi_\nu=\omega\varphi_{\nu-1}. \tag{3.12b}$$

Recalling Eq. (3.10), these formulas imply the following identities for Gauss's hypergeometric function (2.10):

$$ {}_2F_1(a,b;c;z) - {}_2F_1(a-1,b;c;z) = \frac{bz}{c}\, {}_2F_1(a,b+1;c+1;z), \tag{3.13a} $$

$$ {}_2F_1(a,b;c;z) - {}_2F_1(a,b-1;c;z) = \left(\frac{c-1}{c-b}\right)[{}_2F_1(a,b;c;z) - {}_2F_1(a,b-1;c-1;z)]. \tag{3.13b} $$

Further, a generating relation for the functions φ_ν, with ν integer, can be found by expanding in terms these functions, the solutions (3.2) of Eq. (3.1)

$$ (1+\alpha\sigma_1)^{-x_1/\sigma_1}(1+\omega^2\sigma_2/\alpha)^{-x_2/\sigma_2} = \sum_{k=-\infty}^{\infty} d_k(\alpha,\omega)\varphi_k(x_1,x_2). \tag{3.14} $$

Applying the operators P_1 or P_2 to both sides of this equation and using Eq. (3.12), one obtains a recursion relation for the coefficients d_k, that can be easily solved: $d_k(\alpha,\omega) = (\omega/\alpha)^k$. Recalling Eq. (3.10) and with obvious redefinitions, one finally obtains the following generating formula:

$$ (1+z/t)^{-m}(1+t)^{-n} = \sum_{k=-\infty}^{\infty} \frac{(n)_k}{k!}\, t^k {}_2F_1(m,n+k;k+1;z). \tag{3.15} $$

These are just a few examples of the identities for the functions (3.10) that can be obtained using the symmetry algebra of the simple equation (3.1). More complicated situations will be analyzed in detail in the following. Nevertheless, let us conclude this section by considering an even simpler finite-difference equation, the one obtained from Eq. (3.1) by letting $\omega=0$

$$ \Delta_{x_1}^- \Delta_{x_2}^- \varphi(x_1,x_2) = 0. \tag{3.16} $$

It is a discrete version of the wave equation in two dimension. In the continuum

$$ \partial_{x_1}\partial_{x_2}\varphi(x_1,x_2) = 0 \tag{3.17} $$

the wave equation possesses an infinite-dimensional symmetry algebra, the direct sum of two copies of the (centerless) conformal, or Witt algebra. It is generated by the elements: $v_m^0 = x_1^m \partial_{x_1}$ and $w_m^0 = x_2^m \partial_{x_2}$, with m integer. Actually, the operators $v_m^k = x_1^m(\partial_{x_1})^{k+1}$ and $w_m^k = x_2^m(\partial_{x_2})^{k+1}$, with k a non-negative integer, clearly map solutions of Eq. (3.17) into other solutions; they generate the algebra $W_{1+\infty} \oplus W_{1+\infty}$,[19] which is the symmetry algebra of Eq. (3.17).

In the discrete case (3.16), similar conclusions can be drawn. In fact, the elements

$$ V_m^k = \sigma_1^m \left(\frac{x_1}{\sigma_1}\right)_m T_{x_1}^m (\Delta_{x_1}^-)^{k+1}, \quad W_m^k = \sigma_2^m \left(\frac{x_2}{\sigma_2}\right)_m T_{x_2}^m (\Delta_{x_2}^-)^{k+1} \tag{3.18} $$

map solutions of Eq. (3.16) into solutions. Each set of operators $\{V_m^k\}$ and $\{W_m^k\}$ generate the algebra $W_{1+\infty}$, so that in the discrete case also, the symmetry algebra is the direct sum $W_{1+\infty} \oplus W_{1+\infty}$.

The situation is different if one abandons ''light-cone'' coordinates and considers the equation

$$ [(\Delta_t^-)^2 - (\Delta_x^-)^2]\varphi(t,x) = 0. \tag{3.19} $$

It still has an infinite number of symmetry operators; however, their general expression is hard to find. They involve polynomials of arbitrary degree in t and x, T_t and T_x. The simplest examples are $P_t = \Delta_t^-$, $P_x = \Delta_x^-$, $M = t\Delta_t^+ + x\Delta_x^+$, and $G = t\Delta_x^- T_t + x\Delta_t^- T_x$. This situation is in contrast with the case of the equation $[\partial_t^2-\partial_x^2]\varphi=0$, which has conformal invariance in the coordinates $t+x$ and $t-x$. This phenomenon is a clear consequence of the fact that the coordinate transformation $(t,x) \rightarrow (t+x,t-x)$ does not preserve the uniform two-dimensional lattice. Light-cone coordinates seem more appropriate in the study of the symmetry properties of finite-difference equations, and for this reason in the following we shall work with them whenever possible.

IV. THE HEAT EQUATION IN TWO DIMENSIONS

In this section we shall study the symmetries of the following finite-difference equation in the two discrete coordinates t and x:

$$[\Delta_t^- - (\Delta_x^-)^2]\varphi(t,x)=0. \tag{4.1}$$

We shall denote by τ and σ the lattice spacings in the time t and space x directions, so that t/τ and x/σ are integers. In the limit $\tau,\ \sigma\rightarrow 0$ this equation becomes the standard heat equation in two dimensions: $[\partial_t-\partial_x^2]\varphi=0$. Other discrete generalizations of the heat equation have been studied in Refs. 20.

The symmetry operators of Eq. (4.1) can be determined by using the method illustrated in the previous section. These are

$$P_t=\Delta_t^-, \tag{4.2a}$$

$$P_x=\Delta_x^-, \tag{4.2b}$$

$$D=2t\Delta_t^+ + x\Delta_x^+ - \tfrac{1}{2}T_x+1, \tag{4.2c}$$

$$G=2t\Delta_x^- T_t + \left(x-\frac{\sigma}{2}\right)T_x, \tag{4.2d}$$

$$K=t^2\Delta_t^+T_t+tx\Delta_x^+T_t+\frac{x^2}{4}\,T_x^2+t\left(T_t^2-\frac{1}{2}\,T_tT_x\right)-\frac{\sigma^2}{16}\,T_x^2 \tag{4.2e}$$

together with the identity I. In the continuum limit, $\tau,\ \sigma\rightarrow 0$, these operators become

$$\begin{gathered} P_t \rightarrow \partial_t, \quad P_x \rightarrow \partial_x, \quad D \rightarrow 2t\partial_t+x\partial_x+\tfrac{1}{2}, \\ G \rightarrow 2t\partial_x+x, \quad K \rightarrow t^2\partial_t+tx\partial_x+\frac{x^2}{4}+\frac{t}{2} \end{gathered} \tag{4.3}$$

and one thus recovers the usual generators of the Schrödinger algebra, the symmetry algebra of the classical heat equation.[4]

It is remarkable that the finite-difference operators (4.2) realize exactly the same Schrödinger algebra with commutation relations

$$\begin{aligned} &[P_t,\ G]=2P_x, \quad &&[P_x,\ G]=I, \\ &[P_t,\ D]=2P_t, \quad &&[P_x,\ D]=P_x, \\ &[P_t,\ K]=D, \quad &&[P_x,\ K]=G/2, \\ &[D,\ G]=G, \quad &&[K,\ G]=0, \\ &[D,\ K]=2K, \quad &&[P_t,\ P_x]=0. \end{aligned} \tag{4.4}$$

Since Eq. (4.1) is an evolution equation, one can check these relations at $t=0$. In fact, any solution of Eq. (4.1) can be written in the form $\varphi(t,x)=U(t)\varphi(0,x)$, with the time evolution operator $U(t)$ and its inverse $U(t)^{-1}$ formally given by

$$U(t)=(1-\tau(\Delta_x^-)^2)^{-t/\tau}, \quad U(t)^{-1}=(1-\tau(\Delta_x^-)^2)^{t/\tau}. \tag{4.5}$$

On the space of solutions, all generators $X(t)$ satisfy

$$X(t)=U(t)X(0)U(t)^{-1} \tag{4.6}$$

provided one replaces Δ_t^- by $(\Delta_x^-)^2$ in the expressions (4.2). The relations (4.4) are invariant under the conjugation (4.6) and therefore they can be most easily determined at $t=0$.

The symmetry algebra (4.4) can now be exploited to find solutions of Eq. (4.1) by requiring these solutions to be eigenfunctions of one of the symmetry generators. It is readily seen that the functions

$$\varphi_\lambda(t,x)=(1-\lambda^2\tau)^{-t/\tau}(1+\lambda\sigma)^{-x/\sigma} \tag{4.7}$$

are the solutions of Eq. (4.1) that diagonalize P_t and P_x, for any value of the complex parameter λ. Indeed

$$P_t\varphi_\lambda(t,x)=\lambda^2\varphi_\lambda(t,x), \quad P_x\varphi_\lambda(t,x)=\lambda\varphi_\lambda(t,x). \tag{4.8}$$

Another set of solutions is obtained by looking for the eigenfunctions $\varphi_n(t,x)$ of D, with eigenvalue $n+1/2$, n a non-negative integer. Their explicit expression is

$$\varphi_n(t,x)=\sigma^n\sum_{k=0}^{[\![n/2]\!]}\frac{(-n)_{2k}}{k!}\left(\frac{t}{\tau}\right)_k\left(\frac{x}{\sigma}-\frac{1}{2}\right)_{n-2k}\left(\frac{\tau}{\sigma^2}\right)^k, \tag{4.9}$$

where $[\![x]\!]$ stands for the integer part of x. In the continuum limit, the functions $\varphi_n(t,x)$ tend to solutions of the heat equation where the variables t and $x/\sqrt{-t}$ are separated. We have

$$\lim_{\tau,\sigma\to 0}\varphi_n(t,x)=(-t)^{n/2}H_n\left(\frac{x}{2\sqrt{-t}}\right), \tag{4.10}$$

with H_n the classical Hermite polynomials.[16] One can thus consider the function (4.9) as a suitable discrete generalization of these polynomials.

As in the continuum case, many properties of the function $\varphi_n(t,x)$ can now be algebraically determined. First, notice that

$$G^n\varphi_0(t,x)=\varphi_n(t,x), \tag{4.11}$$

where $\varphi_0(t,x)=1$. Then, using the symmetry algebra (4.4), the following formulas are easily obtained:

$$P_t\varphi_n=n(n-1)\varphi_{n-2}, \tag{4.12a}$$

$$P_x\varphi_n=n\varphi_{n-1}, \tag{4.12b}$$

$$D\varphi_n=(n+\tfrac{1}{2})\varphi_n, \tag{4.12c}$$

$$G\varphi_n=\varphi_{n+1}, \tag{4.12d}$$

$$K\varphi_n = \tfrac{1}{4}\varphi_{n+2}. \tag{4.12e}$$

From Eqs. (4.12d) and (4.2d) we straightforwardly find that $\varphi_n(t,x)$ obeys a four-term recurrence relation

$$\varphi_{n+1}(t,x) - \sigma\left(n + \frac{x}{\sigma} - \frac{1}{2}\right)\varphi_n(t,x) - \tau n\left(n + 2\,\frac{t}{\tau} - 1\right)\varphi_{n-1}(t,x)$$

$$+ \tau\sigma n(n-1)\left(n + \frac{x}{\sigma} - \frac{1}{2}\right)\varphi_{n-2}(t,x) = 0. \tag{4.13}$$

A generating relation for the function φ_n can be obtained by expanding the solutions (4.7) of the discrete-heat equation (4.1) in terms of the solutions (4.9) following the steps described in the previous section (for details, see Ref. 20). Instead, we shall adopt here a different strategy and start by computing the action of the operator $e^{\lambda G}$, with λ a complex parameter, on the function $\varphi_0(t,x) = 1$. Writing $G = A + B$, $A = 2t\Delta_x^- T_t$, and $B = (x - \sigma/2)T_x$, and using the classical Campbell–Hausdorff formula, $e^A e^B = e^{A+B+[A,B]/2+\cdots}$, one finds

$$e^{\lambda G}\cdot 1 = e^{\lambda\sigma(x/\sigma - 1/2)T_x} e^{\lambda^2 t T_t}\cdot 1. \tag{4.14}$$

The two exponential operators on the right hand side (rhs) act separately on 1; these actions can be easily expressed in terms of discrete exponentials. One finds

$$e^{\lambda G}\cdot 1 = (1 - \lambda^2\tau)^{-t/\tau}(1 - \lambda\sigma)^{1/2 - x/\sigma}. \tag{4.15}$$

Expanding in series the exponential function and using Eq. (4.11), one also sees that

$$e^{\lambda G}\cdot 1 = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\varphi_n(t,x). \tag{4.16}$$

Putting together the results (4.15) and (4.16), one finally obtains

$$(1 - \lambda^2\tau)^{-t/\tau}(1 - \lambda\sigma)^{1/2 - x/\sigma} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\varphi_n(t,x). \tag{4.17}$$

In the continuum limit, recalling Eq. (4.10), this formula reduces to the following generating relation for the Hermite polynomials[16]

$$e^{2\lambda z - \lambda^2} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,H_n(z). \tag{4.18}$$

V. THE HEAT EQUATION IN THREE DIMENSIONS

As explained at the end of Sec. III, it is convenient to choose "light-cone" coordinates in the spatial directions, and to consider the following finite-difference version of the three-dimensional heat equation

$$[\Delta_t^- - \Delta_{x_1}^- \Delta_{x_2}^-]\varphi(t,x_1,x_2) = 0. \tag{5.1}$$

In the continuum limit, where the lattice spacings go to zero

$$\tau, \sigma_1, \sigma_2 \to 0; \tag{5.2}$$

this equation reduces to $[\partial_t - \partial_{x_1}\partial_{x_2}]\varphi(t,x_1,x_2)=0$.

Solutions of Eq. (5.1) in terms of the discrete exponentials (2.7) are easily found. Indeed, the functions

$$\varphi(t,x_1,x_2;\alpha,\beta_1,\beta_2)=(1-\alpha\tau)^{-t/\tau}(1+\beta_1\sigma_1)^{-x_1/\sigma_1}(1+\beta_2\sigma_2)^{-x_2/\sigma_2} \tag{5.3}$$

satisfy Eq. (5.1), provided the three complex parameters α, β_1, and β_2 obey the constraint: $\alpha=\beta_1\beta_2$. These solutions are eigenfunctions of Δ_t^-, $\Delta_{x_1}^-$, and $\Delta_{x_2}^-$, and these operators are therefore symmetries of Eq. (5.1). However, this equation possesses five more nontrivial symmetries; indeed, one can check that the complete list of symmetry operators is

$$\begin{gathered}
P_t=\Delta_t^-, \quad P_1=\Delta_{x_1}^-, \quad P_2=\Delta_{x_2}^-, \quad M=x_1\Delta_{x_1}^+-x_2\Delta_{x_2}^+,\\
D=2t\Delta_t^++x_1\Delta_{x_1}^++x_2\Delta_{x_2}^++1, \quad G_1=t\Delta_{x_2}^-T_t+x_1T_{x_1},\\
G_2=t\Delta_{x_1}^-T_t+x_2T_{x_2}, \quad K=t^2\Delta_t^+T_t+t(x_1\Delta_{x_1}^++x_2\Delta_{x_2}^+)T_t+x_1x_2T_{x_1}T_{x_2}+tT_t^2
\end{gathered} \tag{5.4}$$

and the identity I. In the continuum limit (5.2), these operators reduce to the generators of the nine-dimensional Schrödinger algebra, the symmetry algebra of the classical heat equation in three dimensions[4]

$$P_t \to \partial_t, \quad P_1 \to \partial_{x_1}, \quad P_2 \to \partial_{x_2}, \quad M \to x_1\partial_{x_1}-x_2\partial_{x_2}, \tag{5.5}$$

$$D \to 2t\partial_t+x_1\partial_{x_1}+x_2\partial_{x_2}+1, \quad G_1 \to t\partial_{x_2}+x_1, \tag{5.5}$$

$$G_2 \to t\partial_{x_1}+x_2, \quad K \to t^2\partial_t+t(x_1\partial_{x_1}+x_2\partial_{x_2})+x_1x_2+t$$

and the identity I.

Both the finite-difference operators (5.4) and the differential operators in Eq. (5.5) realize the commutation relations of the three-dimensional Schrödinger algebra ($i=1,2$):

$$\begin{array}{lll}
[P_1,\,G_1]=I, & [P_2,\,G_1]=0, & [P_t,\,G_1]=P_2,\\
[P_1,\,G_2]=0, & [P_2,\,G_2]=I, & [P_t,\,G_2]=P_1,\\
[P_1,\,M]=P_1, & [P_2,\,M]=-P_2, & [P_t,\,M]=0,\\
[P_1,\,D]=P_1, & [P_2,\,D]=P_2, & [P_t,\,D]=2P_t,\\
[P_1,\,K]=G_2, & [P_2,\,K]=G_1, & [P_t,\,K]=D,\\
[D,\,G_1]=G_1, & [D,\,G_2]=G_2, & [D,\,K]=2K,\\
[M,\,G_1]=G_1, & [M,\,G_2]=-G_2, & [M,\,K]=0,\\
[G_1,\,G_2]=0, & [P_1,\,P_2]=0, & [P_t,P_i]=0,\\
[G_i,\,K]=0, & [M,\,D]=0. &
\end{array} \tag{5.6}$$

In the case of the operators (5.4), these relations are easily verified at $t=0$, by using the formula (4.6), with $U(t)=(1-\tau\Delta_{x_1}^-\Delta_{x_2}^-)^{-t/\tau}$.

One can now use these symmetries to obtain new solutions of Eq. (5.1), with specific transformation properties under the action of the operators (5.4). More specifically, one can check that the following functions labeled by the complex parameters m and n:

$$\varphi_{m,n}(t,x_1,x_2)=\tau^n\sigma_1^m\sum_{k=0}^{\infty}\frac{(-1)^k(-n)_k}{k!(m+1)_k}\left(\frac{\sigma_1\sigma_2}{\tau}\right)^k\left(\frac{t}{\tau}\right)_{n-k}\left(\frac{x_1}{\sigma_1}\right)_{k+m}\left(\frac{x_2}{\sigma_2}\right)_k \tag{5.7}$$

satisfy Eq. (5.1) and are moreover eigenfunctions of the operators M and D with eigenvalues m and $2n+m+1$, respectively. They can be rewritten in terms of the generalized hypergeometric function ${}_3F_2$

$$\varphi_{m,n}(t,x_1,x_2)=\tau^n\left(\frac{t}{\tau}\right)_n\sigma_1^m\left(\frac{x_1}{\sigma_1}\right)_m {}_3F_2\left[\begin{matrix}-n, & m+x_1/\sigma_1, & x_2/\sigma_2 \\ & 1-n-t/\tau, & m+1\end{matrix};\ \frac{\sigma_1\sigma_2}{\tau}\right]. \tag{5.8}$$

When the parameter n is a positive integer, the function ${}_3F_2$ truncates and becomes a hypergeometric polynomial of degree n. For m a negative integer and for unit lattice spacings, $\tau=\sigma_1=\sigma_2=1$, this function is identified as a classical Racah polynomial.

In the continuum limit (5.2), one finds

$$\varphi_{m,n}(t,x_1,x_2)\ \to\ t^n x_1^m\, {}_1F_1(-n;m+1;-x_1x_2/t). \tag{5.9}$$

The ${}_1F_1$ hypergeometric function is the Laguerre polynomial $L_n^{(m)}(-x_1x_2/t)$, and one thus recovers with Eq. (5.9), the solutions of the continuum heat equation separated in the homogeneous variable x_1x_2/t. One can therefore consider that the functions in Eqs. (5.7) and (5.8) represent discrete generalizations of the classical Laguerre polynomials.

From the explicit expression of the generators (5.4), it is clear that each one of these operators maps the functions $\varphi_{m,n}$ into themselves. From this observation and the commutation relations (5.6), one can derive the following identities:

$$P_t\varphi_{m,n}=n\varphi_{m,n-1}, \tag{5.10a}$$

$$P_1\varphi_{m,n}=m\varphi_{m-1,n}, \tag{5.10b}$$

$$P_2\varphi_{m,n}=\frac{n}{m+1}\ \varphi_{m+1,n-1}, \tag{5.10c}$$

$$M\varphi_{m,n}=m\varphi_{m,n}, \tag{5.10d}$$

$$D\varphi_{m,n}=(m+2n+1)\varphi_{m,n}, \tag{5.10e}$$

$$G_1\varphi_{m,n}=\frac{m+n+1}{m+1}\ \varphi_{m+1,n}, \tag{5.10f}$$

$$G_2\varphi_{m,n}=m\varphi_{m-1,n+1}, \tag{5.10g}$$

$$K\varphi_{m,n}=(m+n+1)\varphi_{m,n+1}. \tag{5.10h}$$

One can easily check that this provides a model for the three-dimensional Schrödinger algebra and that the commutation relations (5.6) are verified.

Further properties of the solutions (5.7) can be obtained by considering the action of the operator $e^{\lambda G_2}$, with λ a complex parameter, on the special solutions $\varphi_{m,0}(t,x_1,x_2)\equiv\sigma_1^m(x_1/\sigma_1)_m$. Recalling the explicit expression of G_2 given in Eq. (5.4) and using again the Campbell–Hausdorff formula, one can write

$$e^{\lambda G_2}\varphi_{m,0}=e^{\lambda t\Delta_{x_1}^{-}T_t}\cdot\varphi_{m,0}e^{\lambda x_2 T_{x_2}}\cdot 1. \tag{5.11}$$

In the right-hand side, the action of the last exponential can be easily resummed to find $(1-\lambda\sigma_2)^{-x_2/\sigma_2}$. On the other hand, the action of the first exponential can be expressed in terms of a ${}_2F_1$ hypergeometric function

$$e^{\lambda t\Delta_{x_1}^{-}T_t}\varphi_{m,0}=\sigma_1^m(x_1/\sigma_1)_m\ {}_2F_1\left(-m,\ \frac{t}{\tau};1-m-\frac{x_1}{\sigma_1};\ \frac{\lambda\tau}{\sigma_1}\right). \tag{5.12}$$

We shall take m to be a positive integer; in this case Eq. (5.12) is a Jacobi polynomial of order m in the variable $1-2\lambda\tau/\sigma_1$. Further, by iterating Eq. (5.10g), one sees that

$$G_2^n\varphi_{m,0}=(-1)^n(-m)_n\varphi_{m-n,n}\,. \tag{5.13}$$

Putting together these results, one finally finds the following identity:

$$\sigma_1^m(x_1/\sigma_1)_m(1-\lambda\sigma_2)^{-x_2/\sigma_2}{}_2F_1\left(-m,\ \frac{t}{\tau};1-m-\frac{x_1}{\sigma_1};\ \frac{\lambda\tau}{\sigma_1}\right)=\sum_{n=0}^{\infty}\binom{m}{n}\lambda^n\varphi_{m-n,n}(t,x_1,x_2), \tag{5.14}$$

where $\binom{m}{n}$ is the standard binomial coefficient. In the continuum limit, this relation reduces to the classical generating formula for the Laguerre polynomials

$$(1+\lambda)^m e^{-\lambda z}=\sum_{n=0}^{\infty}\lambda^n L_n^{(m-n)}(z). \tag{5.15}$$

Before concluding this section, let us mention that one can diagonalize the symmetry operator P_t by writing

$$\varphi(t,x_1,x_2)=(1-\omega^2\tau)^{-t/\tau}\psi(x_1,x_2), \tag{5.16}$$

where ψ is any solution of the discrete Helmholtz equation considered in Sec. III

$$[\Delta_{x_1}^{-}\Delta_{x_2}^{-}-\omega^2]\psi(x_1,x_2)=0. \tag{5.17}$$

In particular, recalling the solution φ_ν of this equation, given in Eq. (3.10), one sees that

$$\varphi(t,x_1,x_2)=(1-\omega^2\tau)^{-t/\tau}\frac{(\omega\sigma_2)^\nu}{\Gamma(\nu+1)}\left(\frac{x_2}{\sigma_2}\right)_\nu\ {}_2F_1\left(\frac{x_1}{\sigma_1},\ \frac{x_2}{\sigma_2}+\nu;\nu+1;\omega^2\sigma_1\sigma_2\right) \tag{5.18}$$

satisfies Eq. (5.1). This observation is nontrivial since any symmetry operator of Eq. (5.1) will map Eq. (5.18) into another solution. By choosing appropriate symmetries and solutions ψ of Eq. (5.17), one can construct solutions of Eq. (5.1) satisfying various boundary conditions.

VI. THE WAVE EQUATION IN THREE DIMENSIONS

The discrete generalization of the three-dimensional wave equation that we shall consider in this section is

$$[(\Delta_t^-)^2-\Delta_{x_1}^-\Delta_{x_2}^-]\varphi(t,x_1,x_2)=0. \tag{6.1}$$

We keep the notations of the previous section for the coordinates and lattice spacings; in particular, we use ''light-cone''-like variables for the spatial coordinates. In the continuum limit (5.2), Eq. (6.1) becomes $[\partial_t^2-\partial_{x_1}\partial_{x_2}]\varphi=0$.

Solutions of Eq. (6.1) in terms of discrete exponentials are readily found

$$\varphi(t,x_1,x_2;\alpha,\beta_1,\beta_2)=(1+\alpha\tau)^{-t/\tau}(1+\beta_1\sigma_1)^{-x_1/\sigma_1}(1+\beta_2\sigma_2)^{-x_2/\sigma_2}, \tag{6.2}$$

with $\alpha^2=\beta_1\beta_2$. The symmetry operators of Eq. (6.1) that transform solutions (6.2) into other solutions are

$$\begin{gathered}
P_1=\Delta_{x_1}^-,\quad P_2=\Delta_{x_2}^-,\qquad P_t=\Delta_t^-,\quad M=x_1\Delta_{x_1}^+-x_2\Delta_{x_2}^+,\\
D=t\Delta_t^++x_1\Delta_{x_1}^++x_2\Delta_{x_2}^+-\tfrac{1}{2}T_t+1,\\
G_1=\left(t-\frac{\tau}{2}\right)\Delta_{x_2}^-T_t+2x_1\Delta_t^-T_{x_1},\\
G_2=\left(t-\frac{\tau}{2}\right)\Delta_{x_1}^-T_t+2x_2\Delta_t^-T_{x_2},\\
K_1=\left(t^2-\frac{\tau^2}{4}\right)\Delta_{x_2}^-T_t^2+4x_1^2\Delta_{x_1}^+T_{x_1}+4tx_1\Delta_t^+T_{x_1}+2x_1(2T_{x_1}-T_t)T_{x_1},\\
K_2=\left(t^2-\frac{\tau^2}{4}\right)\Delta_{x_1}^-T_t^2+4x_2^2\Delta_{x_2}^+T_{x_2}+4tx_2\Delta_t^+T_{x_2}+2x_2(2T_{x_2}-T_t)T_{x_2},\\
K_t=t^2\Delta_t^+T_t+2\left(t-\frac{\tau}{2}\right)(x_1\Delta_{x_1}^++x_2\Delta_{x_2}^+)T_t+4x_1x_2\Delta_t^-T_{x_1}T_{x_2}+tT_t-\frac{\tau}{4}(T_t+T_t^2).
\end{gathered} \tag{6.3}$$

In the continuum limit they tend to the symmetry generators of the classical three-dimensional wave equation[4]

$$\begin{gathered}
P_1\to\partial_{x_1},\quad P_2\to\partial_{x_2},\quad P_t\to\partial_t,\quad M\to x_1\partial_{x_1}-x_2\partial_{x_2},\\
D\to t\partial_t+x_1\partial_{x_1}+x_2\partial_{x_2}+1/2,\quad G_1\to t\partial_{x_2}+2x_1\partial_t,\\
G_2\to t\partial_{x_1}+2x_2\partial_t,\quad K_1\to t^2\partial_{x_2}+4x_1^2\partial_{x_1}+4tx_1\partial_t+2x_1,\\
K_2\to t^2\partial_{x_1}+4x_2^2\partial_{x_2}+4tx_2\partial_t+2x_2,\\
K_t\to(t^2+4x_1x_2)\partial_t+2t(x_1\partial_{x_1}+x_2\partial_{x_2})+t.
\end{gathered} \tag{6.4}$$

To determine the commutation relations that the operators (6.3) obey, one cannot proceed here as in the previous two sections. Equation (6.1) is not a (discrete) evolution equation in t; one has to keep $t\neq 0$ and compute the various commutators directly. After a lengthy calculation, one finds that the nonvanishing commutators are $(i=1,2)$

$$[P_t,\,G_1]=P_2,\quad [P_i,\,G_i]=2P_t,\quad [G_1,\,G_2]=2M,$$

$$
\begin{array}{lll}
[P_t, G_2]=P_1, & [P_1, K_1]=4(D+M), & [P_2, K_2]=4(D-M, \\
[P_t, D]=P_t, & [P_1, K_t]=2G_2, & [P_2, K_t]=2G_1, \\
[P_t, K_i]=2G_i, & [P_1, M]=P_1, & [P_2, M]=-P_2, \\
[P_t, K_t]=2D, & [P_i, D]=P_i, & [G_1, M]=-G_1, \\
[G_1, K_2]=2K_t, & [G_2, K_1]=2K_t, & [G_2, M]=G_2, \\
[G_i, K_t]=K_i, & [M, K_1]=K_1, & [M, K_2]=-K_2, \\
[D, K_i]=K_i, & [D, K_t]=K_t. &
\end{array}
\tag{6.5}
$$

This coincides with the commutation relations realized by the symmetry operators (6.4) of the classical wave equation. They define the Lie algebra so(5). To prove this, it is sufficient to construct from the generators (6.3), the elements of the corresponding Chevalley basis. These are given by the following complex combinations:

$$
\begin{gathered}
e_1=G_1, \quad f_1=G_2, \quad h_1=M, \\
e_2=\frac{i}{\sqrt{2}}P_1, \quad f_2=\frac{i}{\sqrt{2}}K_1, \quad h_2=-(M+D).
\end{gathered}
\tag{6.6}
$$

Indeed, one can check that these operators satisfy the commutation relations

$$
\begin{gathered}
[e_i, f_j]=\delta_{ij}h_i, \quad [h_i, e_j]=a_{ij}e_j, \\
[h_i, h_j]=0, \quad [h_i, f_j]-a_{ij}f_j
\end{gathered}
\tag{6.7}
$$

and the Serre relations

$$
\sum_{m=0}^{1-a_{ij}} (-1)^m \binom{1-a_{ij}}{m} e_i^{1-a_{ij}-m} e_j e_i^m, \quad i \neq j, \tag{6.8a}
$$

$$
\sum_{m=0}^{1-a_{ij}} (-1)^m \binom{1-a_{ij}}{m} f_i^{1-a_{ij}-m} f_j f_i^m, \quad i \neq j; \tag{6.8b}
$$

where a_{ij} is the 2×2 Cartan matrix for the algebra so(5), with $a_{11}=a_{22}=2$, $a_{12}=-2$, and $a_{21}=-1$.

We shall now study of the solutions of Eq. (6.1) that are at the same time eigenfunctions of the operators M and D, with eigenvalues $-m-n$ and $1/2-m$, respectively; here, m and n are in general complex parameters, but we shall be mainly interested in the case n a nonnegative integer. These solutions have the form

$$
\varphi_{m,n}(t,x_1,x_2)=\tau^n\sigma_1^{-m-n}\sum_{k=0}^{[\![n/2]\!]} \frac{(-1)^k(m)_{n-k}}{k!(n-2k)!}\left(\frac{\sigma_1\sigma_2}{\tau^2}\right)^k\left(\frac{t}{\tau}-\frac{1}{2}\right)_{n-2k}\left(\frac{x_1}{\sigma_1}\right)_{k-m-n}\left(\frac{x_2}{\sigma_2}\right)_k. \tag{6.9}
$$

For n even, or n odd, they can be expressed in terms of two different hypergeometric functions ${}_4F_3$. In the continuum limit, these solutions tend to solutions of the classical wave equation that are separated in terms of the homogeneous variable x_1x_2/t^2. They involve the ultraspherical or Gegenbauer classical polynomials $C_n^{(m)}$ (Ref. 16)

$$\varphi_{m,n}(t,x_1,x_2) \rightarrow t^n x_1^{-m-n}\left(-\frac{x_1x_2}{t^2}\right)^{n/2} C_n^{(m)}\left[\frac{1}{2}\left(-\frac{t^2}{x_1x_2}\right)^{1/2}\right]. \tag{6.10}$$

The action of the symmetry operators (6.3) on the solutions $\varphi_{m,n}(t,x_1,x_2)$ can be computed by using the algebra (6.5). One explicitly finds

$$\begin{gathered}
P_t\varphi_{m,n}=m\varphi_{m+1,n-1}, \quad P_1\varphi_{m,n}=-m\varphi_{m+1,n}, \quad P_2\varphi_{m,n}=-m\varphi_{m+1,n-2},\\
M\varphi_{m,n}=-(m+n)\varphi_{m,n}, \quad D\varphi_{m,n}=(1/2-m)\varphi_{m,n},\\
G_1\varphi_{m,n}=(2m+n-1)\varphi_{m,n-1}, \quad G_2\varphi_{m,n}=-(n+1)\varphi_{m,n+1},\\
K_1\varphi_{m,n}=-\frac{(2m+n-1)(2m+n-2)}{(m-1)}\varphi_{m-1,n},\\
K_2\varphi_{m,n}=-\frac{(n+1)(n+2)}{(m-1)}\varphi_{m-1,n+2},\\
K_t\varphi_{m,n}=-\frac{(n+1)(2m+n-1)}{(m-1)}\varphi_{m-1,n+1}.
\end{gathered} \tag{6.11}$$

One can check that these relations define a representation of the so(5) algebra (6.5).

Finally, let us observe that an interesting class of solutions of Eq. (6.1) is provided by functions of the form

$$\varphi(t,x_1,x_2)=(1-\sigma_2)^{-x_2/\sigma_2}\psi(t,x_1). \tag{6.12}$$

Indeed, the function (6.12) satisfies Eq. (6.1) if ψ solves

$$[\Delta_{x_1}^- - (\Delta_t^-)^2]\psi(t,x_1)=0. \tag{6.13}$$

By letting

$$(t,x_1) \rightarrow (x,t) \tag{6.14}$$

one recognizes in Eq. (6.13) the two-dimensional discrete heat equation discussed in Sec. IV. Of the symmetries (6.3), only P_t, P_1, $M+D$, G_1, and K_1 survives after the dimensional reduction of Eq. (6.1) to Eq. (6.13). Under the conjugation

$$X \rightarrow (1-\sigma_2)^{x_2/\sigma_2}X(1-\sigma_2)^{-x_2/\sigma_2} \tag{6.15}$$

the explicit expressions of these symmetry operators become identical to those given in Eq. (4.2), provided the redefinition (6.14) is taken into account. One can now express a solution of Eq. (6.1) in terms of the discrete polynomials of Eq. (4.9). Recalling their explicit expression, one immediately sees from Eq. (6.12) that

$$\varphi(t,x_1,x_2)=(1-\lambda^2\sigma_2)^{-x_2/\sigma_2}(\lambda\tau)^n\sum_{k=0}^{[\![n/2]\!]}\frac{(-n)_{2k}}{k!}\left(\frac{x_1}{\sigma_1}\right)_k\left(\frac{t}{\tau}-\frac{1}{2}\right)_{n-2k}\left(\frac{\sigma_1}{\lambda^2\tau^2}\right)^k \tag{6.16}$$

satisfies Eq. (6.1), for any complex λ. This solution is a simultaneous eigenfunction of the symmetry operators P_2 and $M+D$, with eigenvalues λ^2 and $n+1/2$, respectively.

VII. THE WAVE EQUATION IN FOUR DIMENSIONS

In this section, we shall move to four dimensions and study the following discrete version of the wave equation, using complex light-cone coordinates:

$$[\Delta_{x_1}^-\Delta_{x_2}^- - \Delta_{x_3}^-\Delta_{x_4}^-]\varphi(x_1,x_2,x_3,x_4)=0. \tag{7.1}$$

In the limit in which all the lattice spacings σ_i, $i=1,2,3,4$, go to zero, this equation reduces to $[\partial_{x_1}\partial_{x_2} - \partial_{x_3}\partial_{x_4}]\varphi = 0$. Equation (7.1) admits the following 15 symmetry operators:

$$\begin{gathered}
P_1=\Delta_{x_1}^-, \quad P_2=\Delta_{x_2}^-, \quad P_3=\Delta_{x_3}^-, \quad P_4=\Delta_{x_4}^-,\\
M_1=x_1\Delta_{x_1}^+ + x_4\Delta_{x_4}^+, \quad M_2=x_2\Delta_{x_2}^+ + x_4\Delta_{x_4}^+, \quad M_3=x_3\Delta_{x_3}^+ - x_4\Delta_{x_4}^+,\\
G_1=x_1\Delta_{x_3}^- T_{x_1} + x_4\Delta_{x_2}^- T_{x_4}, \quad G_2=x_1\Delta_{x_4}^- T_{x_1} + x_3\Delta_{x_2}^- T_{x_3},\\
G_3=x_2\Delta_{x_3}^- T_{x_2} + x_4\Delta_{x_1}^- T_{x_4}, \quad G_4=x_2\Delta_{x_4}^- T_{x_2} + x_3\Delta_{x_1}^- T_{x_3},\\
K_1=x_1x_2\Delta_{x_4}^- T_{x_1}T_{x_2} + x_1x_3\Delta_{x_1}^+ T_{x_3} + x_2x_3\Delta_{x_2}^+ T_{x_3} + x_3^2\Delta_{x_3}^+ T_{x_3} + x_3T_{x_3}^2,\\
K_2=x_1x_2\Delta_{x_3}^- T_{x_1}T_{x_2} + x_1x_4\Delta_{x_1}^+ T_{x_4} + x_2x_4\Delta_{x_2}^+ T_{x_4} + x_4^2\Delta_{x_4}^+ T_{x_4} + x_4T_{x_4}^2,\\
K_3=x_3x_4\Delta_{x_1}^- T_{x_3}T_{x_4} + x_2x_3\Delta_{x_3}^+ T_{x_2} + x_2x_4\Delta_{x_4}^+ T_{x_2} + x_2^2\Delta_{x_2}^+ T_{x_2} + x_2T_{x_2}^2,\\
K_4=x_3x_4\Delta_{x_2}^- T_{x_3}T_{x_4} + x_1x_3\Delta_{x_3}^+ T_{x_1} + x_1x_4\Delta_{x_4}^+ T_{x_1} + x_1^2\Delta_{x_1}^+ T_{x_1} + x_1T_{x_1}^2.
\end{gathered} \tag{7.2}$$

In the continuum limit these operators become

$$\begin{gathered}
P_i \to \partial_{x_i}, \quad i=1,2,3,4,\\
M_1 \to x_1\partial_{x_1} + x_4\partial_{x_4}, \quad M_2 \to x_2\partial_{x_2} + x_4\partial_{x_4}, \quad M_3 \to x_3\partial_{x_3} - x_4\partial_{x_4},\\
G_1 \to x_1\partial_{x_3} + x_4\partial_{x_2}, \quad G_2 \to x_1\partial_{x_4} + x_3\partial_{x_2},\\
G_3 \to x_2\partial_{x_3} + x_4\partial_{x_1}, \quad G_4 \to x_2\partial_{x_4} + x_3\partial_{x_1},\\
K_1 \to x_1x_2\partial_{x_4} + x_1x_3\partial_{x_1} + x_2x_3\partial_{x_2} + x_3^2\partial_{x_3} + x_3,\\
K_2 \to x_1x_2\partial_{x_3} + x_1x_4\partial_{x_1} + x_2x_4\partial_{x_2} + x_4^2\partial_{x_4} + x_4,\\
K_3 \to x_3x_4\partial_{x_1} + x_2x_3\partial_{x_3} + x_2x_4\partial_{x_4} + x_2^2\partial_{x_2} + x_2,\\
K_4 \to x_3x_4\partial_{x_2} + x_1x_3\partial_{x_3} + x_1x_4\partial_{x_4} + x_1^2\partial_{x_1} + x_1.
\end{gathered} \tag{7.3}$$

These elements generates the Lie algebra sl(4), the symmetry algebra of the continuum wave equation in four dimensions. The same algebra is also the symmetry algebra of the finite-difference equation (7.1). Indeed, it is easy to construct a Chevalley basis for sl(4) from the operators (7.2):

$$e_1=G_1, \quad e_2=iP_1, \quad e_3=G_2, \quad f_1=G_4, \quad f_2=iK_4, \quad f_3=G_3, \tag{7.4}$$

$$h_1=M_1-M_2-M_3, \quad h_2=-(2M_1+M_3+1), \quad h_3=M_1-M_2+M_3.$$

One can check that these elements obey the commutation relations (6.7) and (6.8), where in the present case, the 3×3 Cartan matrix a_{ij} has for nonzero entries $a_{ii}=2$ and $a_{ii\pm1}=-1$.

The solutions of Eq. (7.1) that are simultaneously eigenfunctions of the translation operators P_i involve once more the discrete exponentials (2.7), $i=1,2,3,4$

$$\varphi(x_i;\alpha_i)=(1+\alpha_1\sigma_1)^{-x_1/\sigma_1}(1+\alpha_2\sigma_2)^{-x_2/\sigma_2}(1+\alpha_3\sigma_3)^{-x_3/\sigma_3}(1+\alpha_4\sigma_4)^{-x_4/\sigma_4}. \tag{7.5}$$

Other interesting solutions can be found by diagonalizing the operators M_1, M_2, and M_3. The following functions:

$$\varphi_{l,m,n}(x_i)=\sigma_1^{-l}\sigma_2^{-m}\sigma_3^{n-1}\sum_{k=0}^{\infty}\frac{(l)_k(m)_k}{k!(n)_k}\left(\frac{\sigma_3\sigma_4}{\sigma_1\sigma_2}\right)^k\left(\frac{x_1}{\sigma_1}\right)_{-l-k}\left(\frac{x_2}{\sigma_2}\right)_{-m-k}\left(\frac{x_3}{\sigma_3}\right)_{n+k-1}\left(\frac{x_4}{\sigma_4}\right)_k \tag{7.6}$$

satisfy Eq. (7.1) and are at the same time eigenfunctions of M_1, M_2, and M_3 with complex eigenvalues $-l$, $-m$, and $n-1$, respectively. One can rewrite $\varphi_{l,m,n}$ more conveniently in terms of ${}_4F_3$ hypergeometric functions

$$\begin{aligned}\varphi_{l,m,n}(x_i)=&\sigma_1^{-l}\left(\frac{x_1}{\sigma_1}\right)_{-l}\sigma_2^{-m}\left(\frac{x_2}{\sigma_2}\right)_{-m}\sigma_1^{n-1}\left(\frac{x_3}{\sigma_3}\right)_{n-1}\\ &\times\ {}_4F_3\left[\begin{matrix}l, & m, & x_3/\sigma_3+n-1, & x_4/\sigma_4 & \dfrac{\sigma_3\sigma_4}{\sigma_1\sigma_2}\\ n, & l-x_1/\sigma_1+1, & m-x_2/\sigma_2+1 & ; & \end{matrix}\right].\end{aligned} \tag{7.7}$$

In the continuum limit, $\sigma_i\to0$, the ${}_4F_3$ function becomes the hypergeometric function ${}_2F_1$ of Gauss, and one has

$$\varphi_{l,m,n}(x_i)\ \to\ x_1^{-l}x_2^{-m}x_3^{n-1}\,{}_2F_1\left(l,m;n;\ \frac{x_3x_4}{x_1x_2}\right). \tag{7.8}$$

This is the solution of the classical wave equation studied in Refs. 21 and 22.

The solutions (7.7) of the discrete wave equation naturally provide a sl(4)-module. The action of the symmetry operators (7.2) on these functions can be computed using the commutation relations of the algebra sl(4). Explicitly, one finds

$$\begin{gathered}P_1\varphi_{l,m,n}=-l\varphi_{l+1,m,n},\\ P_2\varphi_{l,m,n}=-m\varphi_{l,m+1,n}, \quad P_3\varphi_{l,m,n}=(n-1)\varphi_{l,m,n-1},\\ P_4\varphi_{l,m,n}=\frac{lm}{n}\varphi_{l+1,m+1,n+1}, \quad M_1\varphi_{l,m,n}=-l\varphi_{l,m,n}, \quad M_2\varphi_{l,m,n}=-m\varphi_{l,m,n},\\ M_3\varphi_{l,m,n}=(n-1)\varphi_{l,m,n}, \quad G_1\varphi_{l,m,n}=(n-1)\varphi_{l-1,m,n-1},\\ G_2\varphi_{l,m,n}=\frac{m(l-n)}{n}\varphi_{l,m+1,n+1}, \quad G_3\varphi_{l,m,n}=(n-1)\varphi_{l,m-1,n-1},\\ G_4\varphi_{l,m,n}=\frac{l(m-n)}{n}\varphi_{l+1,m,n+1}, \quad K_1\varphi_{l,m,n}=\frac{(n-l)(n-m)}{n}\varphi_{l,m,n+1},\end{gathered} \tag{7.9}$$

$$K_2\varphi_{l,m,n}=(n-1)\varphi_{l-1,m-1,n-1}\,, \quad K_3\varphi_{l,m,n}=(n-m)\varphi_{l,m-1,n}\,, \quad K_4\varphi_{l,m,n}=(n-l)\varphi_{l-1,m,n}\,.$$

One can check that this action reproduces the commutation relations that define the Lie algebra sl(4); the functions (7.7) therefore constitute a basis for an irreducible representation of this algebra.

Finally notice that the relations (7.9) can be used to find identities involving the hypergeometric function

$$F \equiv {}_4F_3\left[\begin{matrix} a_1, & a_2, & a_3, & a_4 & \\ b_1, & b_2, & b_3 & ; & z \end{matrix}\right]. \tag{7.10}$$

For example, inserting Eq. (7.7) in the first formula in (7.9), one obtains

$$(a_1-b_2)[F(b_2+1)-F]=a_1[F(a_1+1;b_2+1)-F], \tag{7.11}$$

where $F(b_2+1)$ stands for the function (7.10) with the parameter b_2 replaced by b_2+1, and similarly for $F(a_1+1;b_2+1)$. Other more complicated identities can be obtained from the rest of the relations (7.9).

VIII. CONCLUDING REMARKS

Symmetry techniques provide some of the most useful methods to analyze and classify solutions of partial differential equations. Separation of variables, in particular, allows to prove many properties of special functions that are important in applications to physical problems.

In the case of finite-difference equations the corresponding techniques have not been studied so well, and only partial results are known. Symmetry properties of discrete canonical systems are studied in Ref. 23. The methods developed there are however limited and difficult to apply to specific examples. Lie algebras in connection with finite-difference equations are also discussed in Ref. 24. However, the symmetries considered there are of a generalized form, since they affect the underlying space–time lattice, which is not kept fixed.

In the case of linear partial equations the situation is clearer. A systematic analysis of the symmetry properties has been given in Refs. 10 for equations defined on nonuniform exponential lattices (see also Ref. 25), while in the present article, we examined the symmetries of difference equations on uniform rectangular lattices. In the first case, generalized algebraic structures, like quantum algebras, are needed to describe the underlying symmetry structures, while for equations on uniform lattices, ordinary Lie algebras appear as symmetry algebras. (In this respect, notice that realizations of Lie algebras in terms of finite-difference operators have been constructed in Refs. 26 and 27.)

Actually, one finds that the symmetries of the various finite-difference equations we have analyzed are the same as those of their continuum differential versions. In other words, it appears that the Lie symmetries of linear partial differential equations are preserved by the process of discretization.

Although this conclusion might be true only for linear equations, it is nevertheless striking and can be of great help in applications. We have used this result to find solutions of various finite-difference equations with specific transformation properties under the action of symmetry operators. These solutions can be expressed in terms of generalized hypergeometric functions and reduce to ordinary special functions in the continuum limit. Therefore, they can be considered as discrete generalizations of these classical functions.

In particular, we have met discrete versions of the classical polynomials of Hermite, Laguerre, and Gegenbauer, and discrete analogs also of the Bessel functions. Other discrete generalizations of these special functions have been introduced in the literature,[13] as particular solutions of discrete versions of Sturm–Liouville problems in one dimension. However, these functions are of little use in the study of partial finite-difference equations. As shown by the examples that we have examined, it is clear that true separation of variables is not generally allowed on uniform lattices, so that one cannot reduce the various difference equations to one-dimensional problems.

On the other hand, the discrete functions we have discussed emerge directly as solutions of the difference equations and should therefore prove important in applications. Furthermore, the symmetry algebras can be used to study properties of these functions in analogy with the classical special functions. We have indeed derived in this way various formulas and identities for these discrete functions.

We would like to point out that these formulas and identities are just a few examples of the many properties that can be algebraically derived. Indeed, since the discrete functions we discussed are directly connected with classical Lie algebras, a more systematic study of their properties can be carried out using standard Lie group techniques, in strict analogy with what is done for the usual special functions. Further investigations along these lines are presently under development and will be reported elsewhere.

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