Journal of Mathematical Sciences, Vol. 109, No. 2, 2002

RIEMANNIAN GEOMETRY

V. V. Trofimov and A. T. Fomenko UDC 514.764.2

Introduction

In recent years, the role of geometric methods in mathematics as well as in a number of fields of natural science has grown considerably. This is related to the following: (a) geometric methods allow us to study various phenomena "in the large," for large values of certain parameters; (b) the geometric language is very convenient for describing many phenomena; various mathematical relationships have been successfully described and have obtained a beautiful qualitative explanation. As an example, we present connection theory in principal bundles. Different variants of field theory are described in its terms. Riemannian geometry is also applicable in general relativity.

1. Historical remarks. In the eighteenth to nineteenth centuries, the historical development of the foundations of Riemannian geometry was considerably determined by the needs of applied problems from mechanics, physics, and engineering. We give only one example. After the age of great geographical discoveries (the period from the middle of the fifteenth century up to the middle of the seventeenth century) followed a time for creating sharp geographical maps. In this connection, there arose the mathematical problem of constructing planar maps that were sufficiently convenient for use. This purely practical need became one of the problems that leds to the creation of an important field of modern mathematics, Riemannian geometry, which deeply influenced the development of mathematics and its applications.

An important reference point in the development of new ideas in geometry was the work of Lobachevskii on the foundations of geometry and the work of Gauss on surface theory. On February 11(23) of 1826, Professor Nikolai Ivanovich Lobachevskii of Kazan' University gave a report, *Short Presentation of the Foundations of Geometry,* at a session of the Department of Physics and Mathematics. This report became a turning point in the history of the development of geometry. The first publication of Lobachevskii, which is devoted to the foundations of geometry, appeared in 1829–1830. Independently of Lobachevskii, the work of the Hungarian mathematician Janos Bolyai containing similar results was published in 1832.

Later on, when the manuscript legacy of the great mathematician Gauss was examined, it became known that Gauss obtained the same conclusion as Lobachevskii and Bolyai, but he did not publish these results.

It was F. Klein who in 1879 solved the mathematical problem on the possibility of the emergence of logical contradictions in the scheme proposed by Lobachevskii. He showed that if there are no contradictions in the Euclidean scheme, then there are no contradictions in the Lobachevskii scheme. Moreover, Beltrami showed that the geometry of geodesics coincides with Lobachevskii's plane geometry on surfaces of constant negative curvature.

In 1828, Gauss published a remarkable work on differential geometry of surfaces, *General Studies on Curved Surfaces,* which is closely related to the geodesic problems he dealt with at that time. Gauss developed the idea of the so-called "intrinsic" geometry, which was completely new for that time.

Gauss' idea on the "intrinsic" geometry of a surface was developed further by Riemann. He came to consider an object that is given by the analytic machinery by Gauss, the first fundamental form, independently, not starting from a surface given in the Euclidean space. Such an object is now called a two-dimensional Riemannian manifold. Some of these two-dimensional manifolds can be realized as

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 76, Geometry-8, 2000.

surfaces in the three-dimensional Euclidean space. In this case, Gauss' theory is applicable for them. Other two-dimensional Riemannian manifolds admit a realization as a surface in the three-dimensional space only locally, which only complicates the general theory.

The generalization of the ideas of Lobachevskii and Gauss to the n-dimensional case was done by Riemann in his famous lecture *On Hypotheses Lying in the Foundations of Geometry,* which he gave in 1854 in Göttingen. Riemann gave a precise general definition of geometry, which is now known as Riemannian.

Finally, in the works of Ricci during 1887–1893, and also in his joint work with Levi-Civita written in 1901, the technique of covariant differentiation was elaborated, and the foundations of tensor analysis were laid.

In 1918, G. Weyl introduced the concept of an affine connection space. At the same time, E. Cartan introduced new differential-geometric ideas. The described period of development of Riemannian geometry was summarized in [55].

The next period in the development of Riemannian geometry is related to topology. There arose new problems on the connections of geometric properties of manifolds with their topological structure. The main results proved before 1970 are contained in [71].

The last period in the development of Riemannian geometry can conditionally be characterized by the fact that earlier one studied general differential operators on simple manifolds, and mathematicians in topology and geometry dealt with simple operators on general manifolds, but now, in order to attain considerable progress in Riemannian geometry, we need to use general differential operators on general manifolds. As an example, we mention the progress in studying manifolds of positive scalar curvature.

2. Problems of Riemannian geometry. The subject of Riemannian geometry is a Riemannian metric on manifolds. Various geometrical concepts and constructions are related to this object: the affine connection, parallel translation, geodesics, and curvature (its different variants). The general problem of Riemannian geometry is studying the relations between these objects and also their connection to other fields of mathematics, for example, with topology, probability theory, etc. We consider in detail some of the main problems of Riemannian geometry. We focus only on the following three problems in this review.

(a) One of the main problems of Riemannian geometry is the problem of studying the behavior of geodesics on Riemannian manifolds. To understand this, it suffices to say that Euclidean geometry is the geometry of the simplest Riemannian manifold, the three-dimensional Euclidean space. Lobachevskii's plane geometry is realized as the geometry of geodesics on the Beltrami sphere. In modern treatment of geometry, we can highlight three aspects in studying geodesics on Riemannian manifolds. First, we consider the problems on existence of one or many closed geodesics on a Riemannian manifold. There are several approaches to the solution of this problem: the Lyusternik–Shnirel'man theory and the Morse theory. The second aspect is related to the study and classification of manifolds all of whose geodesics are closed. Finally, geodesic flows that can be integrated in one sense or another are studied.

We present three important problems of the geometry of geodesics. The central problem related to the studying of geodesics on a Riemannian manifold is to prove that there are infinitely many closed geodesics on any compact Riemannian manifold. This is one of the oldest problems of Riemannian geometry.

A Riemannian manifold is called a *Blaschke manifold* if there exists $l > 0$ such that for any point $m \in M$, the exponential mapping $\exp : D_m \to M$ of the disk D_m of radius l into the manifold M is such that $\exp|_{\operatorname{int} D_m}$ is an embedding and $\exp|_{\partial D_m}$ is a bundle into r-dimensional spheres for a certain r. The Blascke conjecture says that any Blaschke manifold is either the sphere S^n, or the n-dimensional real projective space $\mathbb{R}P^n$, or the n-dimensional complex projective space $\mathbb{C}P^n$, or the n-dimensional quaternion projective space $\mathbb{H}P^n$, or the projective Cayley plane $\mathbb{C}\mathrm{a}P^2$.

Lyusternik and Shnirel'man showed that there are at least three geometrically distinct closed geodesics on a manifold homeomorphic to the sphere S^2. It is required to give a bound on the number of geometrically distinct closed geodesics on a manifold diffeomorphic to the n-dimensional sphere S^n.

(b) An important characteristic of a Riemannian manifold is its curvature, and one of the central problems is to reveal to what extent the topology of a manifold depends on its curvature. In other words, how the local properties of Riemannian manifolds influence their structure as a whole. This problem also arises in modern mathematical physics, since in real physical experiments, we can study only a bounded domain of our space. In the framework of the same problem, we can ask to what extent is a metric determined by its curvature. In dimension $n \geq 3$, the metric is, in general, completely determined by its curvature tensor (if there are no pieces of constant curvature). For $n = 2$, this is no longer true (for this, see [90, 116]). Spaces of constant curvature are also determined by their curvature tensor.

In the problem of describing of Riemannian manifolds in terms of curvature, an important role is played by estimates of the complete curvature tensor, sectional curvatures, etc. In particular, one obtains from them the classification theorems for Riemannian manifolds according to their curvature.

So, the problem is to find those restrictions on the curvature of Riemannian manifolds that allow us to completely describe the topology and the metric of manifolds. For example, let the sectional curvature K_σ of a Riemannian manifold M be constant: $K_\sigma = \text{const} = c$ with respect to any two-dimensional direction σ. If $c > 0$, then the Riemannian manifold M is the quotient S^n/Γ of the sphere S^n by the group Γ with discrete action on the sphere S^n; if $c = 0$, then M is the quotient $\mathbb{R}^n/\Gamma$ of the Euclidean space $\mathbb{R}^n$ by the group Γ with discrete action on $\mathbb{R}^n$; if $c < 0$, then M is the quotient H^n/Γ of the Lobachevskii space H^n by the group Γ with discrete action on H^n.

The next problem arising in the framework of studying the relations between the curvature and the topology can be stated as follows: what restrictions on the curvature of a Riemannian manifold are imposed by its topology? In connection with the problem of describing the topological classification of Riemannian manifolds, there arises the problem on the finiteness of topological types of Riemannian manifolds under various restrictions on curvatures (the complete Riemannian tensor, sectional curvatures, etc.).

In Riemannian geometry, we can highlight conditions that allow us to describe not only the topological type of a Riemannian manifold but also its metric structure. For example, if sectional curvatures K_σ of a Riemannian manifold M satisfy the inequality $\frac{1}{4} \leq K_\sigma \leq 1$, then M is either homeomorphic to the sphere S^n or isometric to the complex projective space $\mathbb{C}P^n$ with the Fubini metric.

In considering the problem on the relation between the topology and curvature, we highlight separately the following four cases: the study of manifolds of positive, nonnegative, nonpositive, and negative sectional curvature; moreover, compact and noncompact manifolds should be considered separately.

(c) The next range of problems is related to the study of specific Riemannian manifolds with rich automorphism groups. The classical examples of Riemannian manifolds, the sphere S^n, Euclidean space $\mathbb{R}^n$, and Lobachevskii space H^n, give examples of Riemannian manifolds that are used as models in proving various properties of general Riemannian manifolds. The geometry of these spaces has been intensively studied. The study of spaces such as homogeneous Riemannian manifolds, symmetric spaces, and Kähler manifolds are divided into separate parts of Riemannian geometry.

3. Methods of Riemannian geometry. The methods of Riemannian geometry include, in particular, the following ones. First, this is the variational theory of geodesics, the Morse theory. Second, there are the comparison theorems. Many problems of Riemannian geometry are related to the solution of sets of partial differential equations. As a rule, sets of partial differential equations arising in geometry are complicated sets of nonlinear partial differential equations. The main operator that finds applications in geometric problems is the Laplace operator

$$\Delta = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right),$$

where $\sum\limits_{i,j} g_{ij} dx^i dx^j$ is a metric and $g = \det \| g_{ij} \|$. Important equations such as the Laplace equation, the heat equation, and the wave equation are related to this operator.

An important role is played by the concept of convex sets in Riemannian manifolds in studying spaces of nonpositive and nonnegative sectional curvature.

4. Content of the review and bibliographical sources. Chapter 1 is devoted to the main concepts of Riemannian geometry, such as the Riemannian metric, affine connection associated with the Riemannian metric, parallel translation, geodesics, covariant derivatives, curvature tensor, torsion tensor, isometries, conformal transformations, and affine transformations.

Chapter 2 contains a review of various properties of geodesics on Riemannian manifolds: the variational theory of geodesics and the theory of closed geodesics on Riemannian manifolds.

Chapter 3 is devoted to one of the central problems of Riemannian geometry, the relation between the curvature and topology. We first consider manifolds of constant curvature, and then we separately study manifolds with various restrictions on the sectional curvature, Ricci curvature, and scalar curvature. Also, we consider the relation between the volume of Riemannian manifolds and their topology.

Chapter 4 is devoted to a discussion of various properties of Riemannian manifolds with rich automorphism groups. We consider homogeneous and symmetric spaces, Kähler manifolds, and also various applications of Riemannian geometry.

Because of the lack of space, we certainly cannot consider all modern applications of Riemannian geometry. We simply mention some of them: the geodesic flow theory on manifolds of negative curvature and the concepts related to them such as Anosov diffeomorphisms, ergodicity, the Bernoulli property, scattering of geodesics, the theory of completely integrable geodesic flows on Riemannian manifolds, explicit finding of geodesics, the theory of volumes of tubular neighborhoods of submanifolds in Riemannian manifolds, the theory of two-dimensional manifolds of bounded curvature, the geometry of the Laplace operator, and geometric applications of the Monge–Ampére equation. For the reader's convenience, we give a brief reference to the literature where one can find a more detailed discussion of separate problems related to Riemannian geometry. For geodesic flows on manifolds of negative curvature, see the second volume of *the Encyclopedia of Mathematical Science* and also Anosov's work [9]. For the theory of completely integrable Hamiltonian flows and, in particular, for completely integrable geodesic flows, see the review [183] (and also, e.g., [111, 114, 201, 202]). Modern methods of explicitly integrable dynamical systems are presented in [50, 125], and in the fourth volume of *the Encyclopedia of Mathematical Science*. Gray's works are devoted to the volume theory of tubes in Riemannian manifolds (see, e.g., [144]). The theory of four-dimensional Riemannian manifolds is presented in [181] in detail (see also [172]). The foundations of the theory of two-dimensional manifolds of bounded curvature were created by A. D. Aleksandrov in [4, 5]. On the basis of these works, Aleksandrov and his disciples constructed the intrinsic and external geometry of various classes of nonregular surfaces. For this, see [4, 5, 12]. For geometric properties of the Laplace operator, see [19]. A detailed presentation of the geometric properties of the Monge–Ampére equation and the corresponding differential operator can be found in [11].

There exist several topics that are close to Riemannian geometry in different senses: Riemannian metrics and connections in bundles and the geometry of pseudo-Riemannian manifolds. In our review, the brief Sec. 5 of Chap. 4 is devoted to the theory of pseudo-Riemannian manifolds, and the geometry of bundles is not considered at all. A separate book can be devoted to each of these fields of geometry.

For a general theory of pseudo-Riemannian manifolds, see [16]; the theory of spaces with indefinite metrics of constant curvature is presented in [213], and the connection between the theory of pseudo-Riemannian manifolds and relativity theory is presented in [92] (see also, e.g., [154, 161]). Modern applications of Riemannian geometry to pseudo-Riemannian geometry include the solution of the "problem of the positivity of energy in general relativity" (see [175, 176]).

Our presentation is of a two-level nature. The first level of presentation is related to main concepts of Riemannian geometry, and the corresponding facts are presented from the very beginning. For example, the first chapter is referred to this level. The second level of presentation is related to more developed problems of Riemannian geometry, and we assume that the reader possesses a sufficient preliminary knowledge; for example, in Chap. 3, which is devoted to relations between the topology and curvature of Riemannian manifolds, we use freely various topological notions. In presenting facts of the second level, we use freely (a) simple concepts of the theory of Lie groups and Lie algebras (see, e.g., [30, 48, 101, 158, 160, 188]), such as a Lie group, the Lie algebra of a Lie group, compact semisimple Lie groups, and homomorphisms; (b) topological concepts such as homotopy groups, homotopy equivalence, space of type $K(\pi, n)$, bundles, coverings, cohomology, genus, and degree of mapping (for these concepts, see [45, 49, 95, 178]); (c) concepts from calculus such as the integral, partition of unity, metric spaces and their completeness, and differential operators and their symbols (for these concepts, see any sufficiently complete textbook on calculus).

In what follows, we use everywhere the following abbreviated rule of summation: if the index (subscript or superscript) occurs twice in a certain algebraic expression, the summation with respect to this index is presupposed. In our presentation, we try to use the coordinate method of description of geometric objects since precisely the coordinate description is used in various mechanical and physical applications of Riemannian geometry (see, e.g., [177]).

We present certain literature sources devoted to Riemannian geometry. The textbooks: [1, 4, 12, 22, 34, 43, 48, 55, 69, 71, 94, 106, 112, 113, 115, 117, 120, 129, 134, 146, 155, 159, 161, 188, 189, 211, 213, 218]. The reviews on various fields of Riemannian geometry: [3, 19, 52, 61, 82, 122, 135, 153, 160, 170, 182, 187, 196, 203, 216]. Application of Riemannian geometry in relativity theory: [92, 117, 154, 161]; in classical mechanics: [10, 64, 177, 185, 190]; in electromagnetic theory: [117, 161, 180]. These books and reviews contain the history of the questions considered and a sufficiently large bibliography. In our work, we do not consider the history of one concept or another, as a rule.

In conclusion, we collect the notation used in the book, which now is conventional:

$\mathbb{R}$, the field of real numbers,
$\mathbb{C}$, the field of complex numbers,
$\mathbb{H}$, the quaternion division ring,
$\mathbb{Z}$, the group of integers,
$\mathbb{Z}_n$, residues modulo n,
S^n, the n-dimensional sphere,
$\mathbb{R}^n$, the n-dimensional Euclidean space,
H^n, the hyperbolic space of dimension n,
$\mathbb{R}P^n$, the real projective space of dimension n,
$\mathbb{C}P^n$, the complex projective space of dimension n,
$\mathrm{O}(n)$, the orthogonal group,
$\mathrm{SO}(n)$, the special orthogonal group,
$\mathrm{U}(n)$, the unitary group,
$\mathrm{SU}(n)$, the special unitary group,
$\mathrm{Sp}(n)$, the symplectic group,
$\mathrm{SL}(r, \mathbb{C})$, complex matrices of size $r \times r$ whose determinant equals 1,
$\mathrm{SL}(n, \mathbb{R})$, real matrices of size $n \times n$, whose determinant equals 1,
$E(n)$, the group of motions of the Euclidean space $\mathbb{R}^n$,
$\mathrm{so}(n)$, the Lie algebra of the Lie group $\mathrm{SO}(n)$,
$\mathrm{u}(n)$, the Lie algebra of the Lie group $\mathrm{U}(n)$,
$\mathrm{su}(n)$, the Lie algebra of the Lie group $\mathrm{SU}(n)$,
$\mathrm{sp}(n)$, the Lie algebra of the Lie group $\mathrm{Sp}(n)$.

Chapter 1

MAIN CONCEPTS OF RIEMANNIAN GEOMETRY

1. Geometry of Riemannian Manifolds

1.1. Concept of a smooth manifold. We recall some of the main concepts of smooth manifold theory (a more detailed discussion of these concepts can be found in textbooks on differential geometry; see, e.g., [48, 134, 188]).

1.1.1. The topological structure is a law according to which the passage to the limit is defined on the space considered. Initially, by topology, one means the study of properties of lines or surfaces that are preserved under arbitrary transformations not violating the continuity. The term was introduced by Listing in *Preparatory Studies in Topology,* where the topology of lines was considered. The topology of surfaces and higher-dimensional spaces was created by Riemann in *Theory of Abelian Functions.* The first axiomatic definition of an abstract topological space was given by Hausdorff in *Foundations of Set Theory.*

A *topological space* is an arbitrary set in which a family of subsets, called open, is highlighted, and, moreover, this family has the following properties: the union of any collection of open sets is an open set; the intersection of finitely many open sets is an open set; the empty set and the whole space are open sets. Any open set containing a given point x is called its neighborhood.

A topological space X is said to be *Hausdorff* if any two of its points $x, y \in X$ $(x \neq y)$ can be included in disjoint neighborhoods. One says that X *has a countable base* if any open set is the union of open sets from a certain countable family.

A mapping $f : X_1 \to X_2$ of a topological space X_1 into a topological space X_2 is said to be continuous if the inverse image of any open set $U \subset X_2$ is an open set in X_1. A one-to-one mapping $f : X_1 \to X_2$ is called a *homeomorphism* if f as well as f^{-1} are continuous mappings. The term "homeomorphism" was introduced by H. Poincaré in *Analysis Situs.*

Example. The arithmetical space $\mathbb{R}^n$ is the set of all possible sequences of n real numbers $x^1, \dots, x^n$; moreover, as open sets, one takes all possible open parallelepipeds of the form $a^i < x^i < b^i, 1 \leq i \leq n$, and all their possible unions.

A Hausdorff topological space M^n with countable base is called a *topological manifold* of dimension n if, for any point $x \in M^n$, there exist a neighborhood V of this point and a homeomorphism h of this neighborhood onto a certain open set from $\mathbb{R}^n$. By definition, a smooth structure on a topological manifold M^n consists of a collection $\{(V_i, h_i)\}$, where V_i is an open set in M^n and h_i is a homeomorphism of the set V_i onto a certain open set of the space $\mathbb{R}^n$; moreover, the following conditions hold:

(1) the collection of sets $\{V_i\}$ forms an open covering of the manifold M^n;
(2) for any pair (i, j), the mapping $h_j h_i^{-1} : h_i(V_i \cap V_j) \to \mathbb{R}^n$ is smooth (see Fig. 1);
(3) the collection $\{(V_i, h_i)\}$ is maximal with respect to the above properties.

Obviously, each collection of pairs (V_i, h_i) satisfying conditions (1) and (2) can be uniquely complemented up to a collection satisfying conditions (1)–(3). A topological n-dimensional manifold M^n equipped with a certain smooth structure is called a *smooth* n-dimensional manifold. Pairs (V_i, h_i) are called *coordinate neighborhoods*. The coordinates $x^1, \dots, x^n$ defined in the space $\mathbb{R}^n$ and transferred to V_i via the mapping h_i are called *local coordinates* in the coordinate neighborhood V_i.

The term "manifold" (Mannigfaligkeit) was initially used by B. Riemann in his lecture *On Hypotheses Lying in the Foundations of Geometry* in the sense of a higher-dimensional space. H. Poincaré used the term "manifold" (variété) in *Analysis Situs.* In the sense used here, this concept was introduced by L. E. Brauer in *Proof of the Invariance of Dimension.*

An open set U of a smooth n-dimensional manifold M^n is also a smooth n-dimensional manifold; namely, the coordinate neighborhoods of the manifold U are the coordinate neighborhoods (V, h) of the

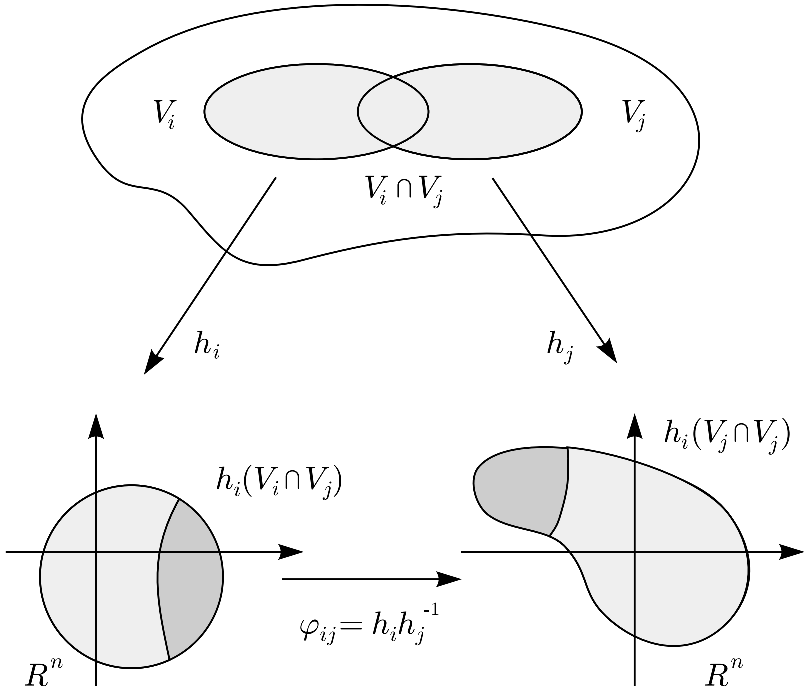


Fig. 1

manifold M^n with $V \subset U$. Such a smooth structure on U is called a *structure induced by the smooth structure on* M^n.

If M_1 and M_2 are two smooth manifolds, then a certain natural smooth structure is well defined on the product $M_1 \times M_2$.

In the definition of a smooth manifold, instead of the space $\mathbb{R}^n$, one can take the half-space

$$\mathbb{R}^n_+ = \{(x^1, \dots, x^n) \mid x^n \geq 0\}.$$

Then we obtain a *manifold with boundary*. The set of points for which $x^n = 0$ is called the *boundary* ∂M of the manifold M. This concept does not depend on the choice of a local coordinate system on the manifold M.

A mapping $\varphi : M_1^m \to M_2^n$ of smooth manifolds M_1^m and M_2^n is said to be *smooth* if, for any coordinate neighborhoods (V, h) and (W, k) of the manifolds M_1^m and M_2^n, respectively, the composition

$$k\varphi h^{-1}\colon\ h(V \cap \varphi^{-1}(W)) \to \mathbb{R}^n$$

is smooth. Two smooth manifolds M_1^n and M_2^n are said to be *diffeomorphic* if there exists a smooth homeomorphism $f : M_1^n \to M_2^n$ such that the inverse homeomorphism f^{-1} is also smooth. In this case, the mapping f is called a *diffeomorphism*. The homeomorphisms $\varphi_{ij} = h_j h_i^{-1}$ are called *gluing functions* or *transition functions* (in a given atlas); see Fig. 1.

A topological manifold is called a *manifold of class* C^r or an *analytic manifold* if all transition functions φ_{ij} belong to the class C^r or are analytic.

A subset V of a manifold M^n is called a *smooth submanifold* of dimension $n - k$ if for any point $p \in V$, there exist an open neighborhood $U(p) \ni p$ with local coordinates $x^1, \dots, x^n$ and a collection of k

smooth functions $g_1, \dots, g_k$ defined on $U(p)$ such that

$$\operatorname{rk} \begin{pmatrix} \dfrac{\partial g_1}{\partial x^1} & \cdots & \dfrac{\partial g_1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_k}{\partial x^1} & \cdots & \dfrac{\partial g_k}{\partial x^n} \end{pmatrix} = k$$

and

$$V \cap U(p) = \{x \in U(p) \mid g_i(x) = 0, \quad 1 \leq i \leq k\}.$$

In other words, the intersection of V with $U(p)$ should coincide with a common zero level surface of the functions $g_1, \dots, g_k$.

1.1.2. One says that a *tangent vector* (or merely, a vector) ξ is given at a point $p \in M^n$ of a smooth manifold M^n if in each chart with local coordinates $(x^1, \dots, x^n) = x$ that contains the point p, we have a sequence of real numbers $\xi = (\xi^1(x), \dots, \xi^n(x))$ (coordinates of the vector ξ) that transforms according to the following rule when passing from one chart to another:

$$\xi^i(x) = \frac{\partial x^i}{\partial y^j} \xi^j(y), \tag{1.1}$$

where $y^1, \dots, y^n$ and $x^1, \dots, x^n$ are "old" and "new" coordinates in the intersection of charts; derivatives are taken at the point p. In (1.1), we assume the summation with respect to $j = 1, \dots, n = \dim M^n$, as was explained in the Introduction.

The addition of vectors and the multiplication by a number are defined in an obvious way. Vectors at a given point $p \in M^n$ form a vector space T_pM^n, called the tangent space of the manifold M^n at the point $p \in M^n$. Its dimension is equal to $\dim M^n = n$. If a vector $\xi(p)$ is given at each point $p \in M^n$, and, moreover, if its coordinates in each chart are a smooth function of the coordinates of a point, then one says that a *vector field* is given.

One can consider each vector field ξ on a manifold M as a differential operator $\xi : C^\infty(M) \to C^\infty(M)$ acting on the space $C^\infty(M)$ of smooth functions on the manifold M by the formula

$$\xi f = \xi^i \frac{\partial f}{\partial x^i},$$

where ξ^i are the coordinates of the vector field ξ in a local coordinate system $(x^1, \dots, x^n)$ on the manifold M.

An ordered pair of vector fields ζ, ξ generates one more vector field, called the *commutator* $[\zeta, \xi]$ of the vector fields ζ and ξ, which is defined by

$$[\zeta, \xi]^i = \zeta^j \frac{\partial \xi^i}{\partial x^j} - \xi^j \frac{\partial \zeta^i}{\partial x^j}.$$

Obviously,

$$[\xi, \eta] = -[\eta, \xi] \quad \text{and} \quad [[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0.$$

A *smooth curve* on a manifold M^n is a smooth mapping of the interval $(-1, 1)$ into the manifold M^n. If we have a curve $x^i = x^i(t)$, $1 \leq i \leq n$, then the tangent vector to this curve is defined by $\xi^i = \dfrac{dx^i(t)}{dt}$, and any tangent vector to the manifold M can be represented in such a form.

Under a smooth mapping $f : M \to N$ of one manifold M into another manifold N, any vector $\xi \in T_xM$ passes to the point $f(x)$ according to the rule

$$(f_*\xi)^i = \frac{\partial y^i}{\partial x^j} \xi^j.$$

Therefore, we have the linear mapping $f_* : T_xM \to T_{f(x)}N$, which is called the *differential* of the mapping f. Here the mapping f in charts $x^1, \dots, x^n$ and $y^1, \dots, y^m$ that include neighborhoods of the points x

and $f(x)$, respectively, is assumed to be written in the form $y^i = f^i(x^1, \dots, x^n)$, $i = 1, \dots, m$, where f^i are smooth functions. The tangent vector $\xi^i = \dfrac{dx^i(t)}{dt}$ to a curve $x(t)$ is transformed according to the rule

$$(f_*\xi)^i = \frac{df(x(t))^i}{dt}, \quad 1 \le i \le n.$$

A vector field is correctly transformed in this way only if f is a diffeomorphism of manifolds.

The union of all tangent spaces T_xM for all points $x \in M$ is called the *tangent bundle* of the manifold M and is denoted by TM. Let $\pi : TM \to M$ be a mapping that to a point (x, ξ) ($x \in M$ and $\xi \in T_xM$ is a vector tangent to the manifold M at the point x) assigns the point x, i.e., the application point of the vector ξ. We can introduce a topology in TM by taking the sets $\pi^{-1}(U)$ for all coordinate neighborhoods U on the manifold M as a basis of neighborhoods of this topology. Moreover, we can introduce the structure of a smooth manifold in TM by defining charts for TM as the inverse images $\pi^{-1}(U)$, where U is a chart on M, and the mapping $f : \pi^{-1}(U) \to \mathbb{R}^{2n}$ as $f(x, \xi) = (x^1, \dots, x^n, \xi^1, \dots, \xi^n)$, where ξ^i are coordinates of the vector ξ in a given coordinate system $x^1, \dots, x^n$ on M^n.

1.2. Riemannian metric and Riemannian manifolds. The higher-dimensional generalization of the idea of the intrinsic geometry of a surface leads to a very elegant and deep theory, Riemannian geometry.

Definition 1.2.1. We say that a *Riemannian metric* is given on a connected smooth manifold if in each chart with local coordinates $(x^1, \dots, x^n) = x$, we have a collection of smooth functions $g_{ij}^{(x)}(p) = g_{ij}(p)$ (coordinates of the metric) that are transformed according to the law

$$g_{ij}^{(y)} = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} g_{kl}^{(x)} \tag{1.2}$$

when passing from one chart to another; here $y^1, \dots, y^n$ and $x^1, \dots, x^n$ are "old" and "new" coordinates in the intersection of the charts. Moreover, it is required that the matrix $G = \|g_{ij}(p)\|$ be positive definite at each point of the manifold, and $g_{ij}(p) = g_{ji}(p)$, i.e., the matrix G is symmetric. In this case, the manifold is said to be Riemannian.

Riemannian manifolds were defined by B. Riemann.

We note that it is not clear in advance whether such an object always exists on an arbitrary smooth manifold. However, the following existence theorem holds; its proof can be found, e.g., in [48, 134].

Theorem 1.2.1. *There exists at least one Riemannian metric on each smooth manifold M^n.*

The assignment of a Riemannian metric on a manifold M^n is equivalent to the assignment of a nondegenerate positive-definite form smoothly depending on the point p on each tangent space T_pM^n to the manifold M^n for $p \in M^n$. Clearly, it defines a symmetric inner product $\langle a, b\rangle$ in each tangent space T_pM^n by $\langle a, b\rangle = g_{ij}a^ib^j$, where $a = (a^1, \dots, a^n) \in T_pM^n$ and $b = (b^1, \dots, b^n) \in T_pM^n$. According to law (1.2) of transformation of the components g_{ij} under the change of coordinates $(x^1, \dots, x^n)$, the inner product $\langle a, b\rangle$ does not depend on the choice of a local coordinate system.

Let a certain curve $\gamma(t) = (x^1(t), \dots, x^n(t))$, $a \le t \le b$, be given in local coordinates $(x^1, \dots, x^n)$ on a Riemannian manifold M^n. Then we can define the length $l(\gamma)$ of an arc of the curve γ from the point $\gamma(a)$ up to the point $\gamma(b)$ by

$$l(\gamma) = \int_a^b \sqrt{g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt = \int_a^b |\dot\gamma(t)|\,dt,$$

where $|\dot\gamma(t)|$ denotes the length of the velocity vector $\dot\gamma(t)$ calculated with respect to the inner product $\langle X, Y\rangle$ in the space T_pM^n. Sometimes, it is useful to write the square of the differential $dl^2 = g_{ij}dx^idx^j$ using the differentials dx^i of coordinates x^i. Often, one writes ds instead of dl.

The length $l(\gamma)$ of the curve γ does not depend on the choice of the local coordinate system and on the choice of the regular parametrization t of the curve. Therefore, the length of a curve is a scalar invariant that is determined only by a Riemannian metric and the trajectory itself, which is considered as the set of points of the curve γ in the manifold M such that the order of passing through these points when the parameter t varies is given.

The simplest example of a Riemannian metric is the Euclidean metric in the arithmetical space $\mathbb{R}^n$. The corresponding Riemannian manifold is called the Euclidean space. In the Cartesian coordinates, the Euclidean metric is given by $g_{ij} = \delta_{ij}$, where δ_{ij} is the standard Kronecker symbol; $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Here

$$dl^2 = \sum_{i=1}^{n} (dx^i)^2.$$

Definition 1.2.2. Local coordinates on a smooth Riemannian manifold M^n are said to be Euclidean if the Riemannian metric g_{ij} in these coordinates becomes Euclidean, that is, δ_{ij}. If such a system exists, then the Riemannian metric in the domain of these coordinates is said to be Euclidean.

Let M be a Riemannian manifold. We define the function $\rho : M \times M \to \mathbb{R} \cup \{\infty\}$ of the form $\rho(m, n) = \inf\limits_{\gamma \in \Gamma} l(\gamma)$ on the manifold M, where Γ is the set of all piecewise-smooth curves that connect the points m and n. Sometimes, one uses the notation $d(x, y)$ for $\rho(x, y)$.

Lemma 1.2.1. *The function ρ is a metric on a Riemannian manifold M, i.e., it satisfies the following relations*:

(a) $\rho(x, y) = \rho(y, x)$;
(b) $\rho(x, y) = 0$ *iff* $x = y$;
(c) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (*triangle inequality*).

The number $d(M) = \sup \rho(x, y)$, $x, y \in M$, is called the *diameter* of the manifold M. The metric ρ induces the initial topology on M, i.e., the topology given by the metric ρ is equivalent to the initial topology of the manifold M, and, therefore, ρ is a continuous function on $M \times M$.

The concept of *angle* is defined on a Riemannian manifold. Let two curves $\gamma_1(t)$ and $\gamma_2(t)$ intersecting at a point $p = \gamma_1(t_0) = \gamma_2(t_0)$ be given on a Riemannian manifold M with metric g_{ij}. Then the angle φ between these curves at the point p is defined by

$$\cos\varphi = \frac{g_{ij} \dfrac{dx_1^i}{dt} \dfrac{dx_2^j}{dt}}{\sqrt{g_{ij} \dfrac{dx_1^i}{dt} \dfrac{dx_1^j}{dt}} \sqrt{g_{ij} \dfrac{dx_2^i}{dt} \dfrac{dx_2^j}{dt}}},$$

where $x_1^i = x_1^i(t)$ and $x_2^i = x_2^i(t)$, $1 \leq i \leq n$, are the coordinate representations of the curves γ_1 and γ_2 in local coordinates $(x^1, \dots, x^n)$ on the manifold M^n.

We can define the volume of a domain in a Riemannian manifold. Let D be a domain in a Riemannian manifold M^n with metric g_{ij}. In the case where the domain D is contained in one chart U, as the volume $\mathrm{vol}(D)$ of the domain D, we take the number

$$\mathrm{vol}(D) = \int\limits_D \sqrt{\det G}\, dx^1 \dots dx^n,$$

where $G = \|g_{ij}\|$ is the matrix of the metric g_{ij}. If the domain D is contained in several charts $U_1, \dots, U_N$, then

$$\mathrm{vol}(D) = \sum_{i=1}^{N} \int\limits_{U_i \cap D} \varphi_i \sqrt{\det G}\, dx_i^1 \dots dx_i^n,$$

where φ_i, $i = 1, \dots, N$, is a partition of unity subordinated to the covering $U_1, \dots, U_N$ (see [48, p. 467]).

Therefore, the main geometric concepts, such as the length of a curve, Euclidean coordinates, the distance between points, angles, areas, and volumes are defined on an arbitrary Riemannian manifold. In Sec. 4, we define the concept of motion of a Riemannian manifold. Therefore, we can develop geometry in Riemannian manifolds in the large. In this geometry, which is called the Riemannian geometry, one can find deep distinctions from the Euclidean geometry. For example, we cannot directly compare vectors applied to distinct points of this manifold; for this, see Sec. 2.4. In contrast to Euclidean manifolds, arbitrary Riemannian manifolds have curvature (see Sec. 3). In Sec. 2.4, we define specific analogs of Euclidean lines.

We can study other structures on smooth manifolds, which, at first glance, look like a Riemannian metric. We mention only one of them. In the definition of a Riemannian manifold, we can reject the requirement that the matrix $G = \|g_{ij}\|$ be positive definite. In this case, we arrive at the concept of a pseudo-Riemannian manifold.

Definition 1.2.3. We say that a *pseudo-Riemannian* metric is given on a smooth manifold if, in each chart with local coordinates $x = (x^1, \dots, x^n)$, we have a collection of smooth functions $g_{ij}^{(x)} = g_{ij}$ that are transformed according to law (1.2) when passing from one chart to another such that the matrix $G = \|g_{ij}\|$ is nonsingular at each point $p \in M^n$ of the manifold M^n and $g_{ij} = g_{ji}$, i.e., the matrix G is symmetric.

Pseudo-Riemannian manifolds were defined by A. Einstein in *Foundations of Relativity Theory*.

In the case of pseudo-Riemannian manifolds, there is no longer an existence theorem that is similar to Theorem 1.2.1. For example, there is no pseudo-Riemannian metric of the form $x^2 + y^2 + z^2 - t^2$ on the four-dimensional sphere S^4 (see, e.g., [48]). However, it is possible to construct a geometry that looks like a Riemannian geometry (see [16]).

1.3. Constructions of Riemannian manifolds. If $f : M \to N$ is a smooth mapping such that the differential $f_* : T_xM \to T_{f(x)}N$ is a monomorphism and there is a Riemannian metric on N, then the relation

$$f^*(g)(A, B) = g(f_*(A), f_*(B)),$$

where $A, B \in T_xM$, defines a Riemannian metric on the manifold M. This metric is called the metric induced by the mapping f.

Let $p : M \to N$ be a smooth covering over the Riemannian manifold N. Then we can define a Riemannian metric on the manifold M by what was said above. The manifold M with this induced metric is called a *Riemannian covering* over the Riemannian manifold N. For example, the Euclidean space $\mathbb{R}^n$ is a simply connected Riemannian covering (i.e., the universal Riemannian covering) over the flat torus T^n (i.e., the torus equipped with a locally Euclidean metric).

We note that if $p : M \to N$ is a covering and the smooth manifold M is equipped with a metric such that all automorphisms of the covering are isometries, then there arises a natural induced metric on the base N with respect to which the mapping $p : M \to N$ becomes a Riemannian covering in the sense of the previous definition (see, e.g., [22]).

Let (M, g_{ij}) and (N, g'_{ij}) be Riemannian manifolds; then their product $M \times N$ is equipped with the Riemannian metric that is defined by the inner product with the matrix $\begin{pmatrix} g_{ij} & 0 \\ 0 & g'_{ij} \end{pmatrix}$ in the tangent space

$$T_{(p,q)}(M \times N) = T_pM \oplus T_qN.$$

The Riemannian manifold $M \times N$ defined in such a way is called the *direct product* of Riemannian manifolds M and N.

Let $N^m \subset M^n$ be a submanifold of a Riemannian manifold (M^n, g_{ij}). Then there naturally arises the induced Riemannian metric $i^*(g)$ on N^m that is given by

$$i^*(g)(A, B) = g(i_*(A), i_*(B))$$

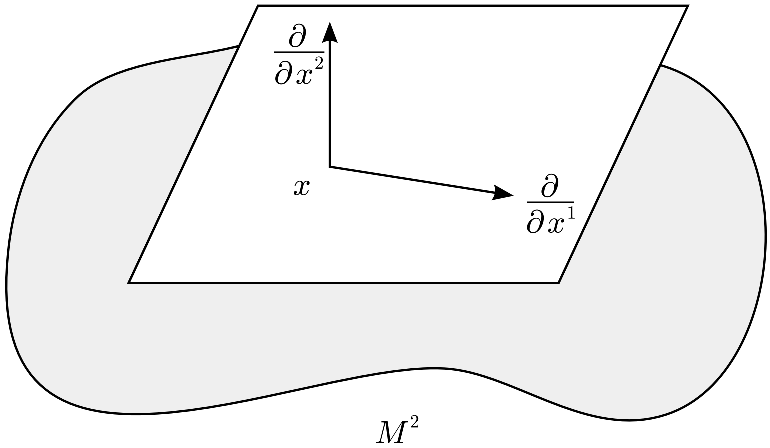


Fig. 2

for any vectors $A, B \in T_xN^m \subset T_xM^n$, where $i : N^m \to M^n$ is the inclusion mapping. In particular, if $M^n = \mathbb{R}^n$, then $T_x\mathbb{R}^n = \mathbb{R}^n$, and $T_xN^m \subset \mathbb{R}^n$ can be considered as the usual tangent space to the surface N^m. In this case, the charts on N^m are called the *"parametric assignment of an m-dimensional surface"* and are written as $\mathbf{r}(u_1, \dots, u_m)$. Then the mapping that defines the chart transforms the base vector $\partial/\partial u_i = (0, \dots, 0, 1, 0, \dots, 0)$ into the vectors $\partial\mathbf{r}/\partial u_i$. Therefore, we obtain a Riemannian metric on N in the form

$$g_{ij} = \left(\frac{\partial \mathbf{r}}{\partial u^i}, \frac{\partial \mathbf{r}}{\partial u^j} \right),$$

where (X, Y) denotes the usual inner product in $\mathbb{R}^n$ (see Fig. 2 for the two-dimensional case).

In the classical case $n = 3$ and $m = 2$, surfaces are defined in the parametric form $(u, v) \to \vec{\mathbf{r}}(u, v)$. In this case, it has been conventional to denote $g_{11} = E$, $g_{12} = F$, and $g_{22} = G$ since Gauss' time. Then the square of the differential of the length ds^2 is called the *first quadratic form* and is written as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where

$$E = \left| \frac{\partial \mathbf{r}}{\partial u} \right|^2, \quad F = \left(\frac{\partial \mathbf{r}}{\partial u}, \frac{\partial \mathbf{r}}{\partial v} \right), \quad \text{and} \quad G = \left| \frac{\partial \mathbf{r}}{\partial v} \right|^2.$$

Therefore, each submanifold in $\mathbb{R}^n$ is a Riemannian manifold. There arises a natural question: is it true that any Riemannian manifold can be obtained by using this construction?

Let U be an open set in the space $\mathbb{R}^n$, and let $g = (g_{ij})$ be a Riemannian metric on U. The mapping $f : U \to \mathbb{R}^N$ of the domain U into a certain Euclidean space $\mathbb{R}^N$ is an isometric immersion if it satisfies the set of equations

$$\sum_{\alpha=1}^{N} \frac{\partial f^\alpha}{\partial x^i} \frac{\partial f^\alpha}{\partial x^j} = g_{ij}(x), \quad 1 \leq i, j \leq n.$$

Any solution to this set of equations is automatically an immersion, and, therefore, it becomes an embedding when we replace the neighborhood U by a smaller one. The earliest publication devoted to locally isometric embeddings was that of Schläfli (see [173]). He asserted that every Riemannian manifold of dimension n admits an isometric embedding in the Euclidean space of dimension $\frac{1}{2}n(n+1)$.

In 1926, Janet [102] published the proof of the Schläfli conjecture, and in 1931, Burstin [33] made the proof more sure (see also the work [38] of E. Cartan).

Theorem 1.3.1 (Janet and Burstin)**.** *Each analytic manifold of dimension n with a distinguished point contains a neighborhood of the distinguished point that admits an isometric analytic embedding in the Euclidean space $\mathbb{R}^{s_n}$, where $s_n = \frac{1}{2}n(n+1)$.*

At present, the theory of isometric embeddings of Riemannian manifolds is an independent field of geometry with its own subject of study and its own methods. In this section, we focus only on the fact, fundamental for Riemannian geometry, that any Riemannian manifold can be considered as a certain higher-dimensional surface in a higher-dimensional Euclidean space.

The problem on the isometric embedding of Riemannian manifolds in the case of C^∞-smoothness as well as in the analytic case was positively solved by Nash.

Theorem 1.3.2 ([144])**.** *Every compact Riemannian manifold M^n of class C^r, $3 \le r \le \infty$, admits an isometric C^r-embedding into $\mathbb{R}^m$, where $m = \frac{1}{2}(3n^2 + 11n)$. If M^n is not compact, then it admits an isometric embedding in $\mathbb{R}^N$, where*

$$N = \frac{1}{2}(3n^2 + 11n)(n+1).$$

In the works of Nash, only manifolds without boundary are considered. However, both results for the compact as well as the noncompact case are easily extended to manifolds with boundary. The dimension of the Euclidean space indicated by Nash is more than three time greater than the dimension we encounter in local theory, and it seems to be overstated. The following improved bound was proved in [82] by M. L. Gromov and V. A. Rokhlin.

Theorem 1.3.3. *Every compact Riemannian manifold of dimension n and of class C^∞ admits an isometric embedding of class C^∞ in the space $\mathbb{R}^q$ with $q = s_n + 4n + 5 = \frac{1}{2}(n^2 + 9n + 10)$.*

In this theorem, the dimension $s_n + 4n + 5$ was further reduced to $s_n + 3n + 5$ (see [79]). In the same work, this result was extended to noncompact manifolds.

2. Geometry of Affine Connection Manifolds

2.1. Affine connection manifolds. In many problems, there arises the problem of differentiation of vector fields on manifolds. The usual differentiation of components of vector fields in local coordinates is the first candidate for this operation. However, elementary examples show that this operation depends on the choice of local coordinates on a manifold. There arises the problem on the invariant differentiation of vector fields in arbitrary coordinates. To solve the problem, we need to create an additional structure (see [48, 71, 113, 134, 149, 161]).

Definition 2.1.1. We say that an *affine connection* is given on a smooth manifold if a collection of smooth functions $\Gamma^i_{jk}(x)$ is given in each chart with local coordinates $x = (x^1, \dots, x^n)$, and when we pass from one chart to another, these functions are transformed according to the law

$$\Gamma^i_{jk}(x) = \frac{\partial x^i}{\partial y^l}\frac{\partial y^s}{\partial x^j}\frac{\partial y^t}{\partial x^k}\Gamma^l_{st}(y) + \frac{\partial x^i}{\partial y^s}\frac{\partial^2 y^s}{\partial x^j \partial x^k}, \tag{2.1}$$

where $y^1, \dots, y^n$ and $x^1, \dots, x^n$ are the "old" and "new" coordinates in the intersection of the charts.

An affine connection space of general form was defined by H. Cartan in the paper "On affine connection spaces and generalizations of relativity theory." The first particular case of this space, which is a Riemannian space, was defined by G. Weyl in the book *Space, Time, and Matter*.

Theorem 2.1.1. *There is at least one affine connection on any smooth manifold.*

An affine connection Γ^i_{jk} on a manifold M^n is said to be symmetric if $\Gamma^i_{jk} = \Gamma^i_{kj}$. If this requirement holds in one coordinate system, then it holds in any other coordinate system, as is easily implied by the transformation law (2.1).

Definition 2.1.2. Local coordinates $(x^1, \dots, x^n)$ are said to be *Euclidean with respect to a given connection* if $\Gamma^i_{jk}(p) \equiv 0$ in these coordinates for all p. In this case, the connection is said to be *flat*.

We note that a Riemannian metric and an affine connection are, in general, independent structures on a manifold M^n. In particular, Euclidean coordinates for an affine connection and Euclidean coordinates for a Riemannian metric are, in general, distinct concepts. In the general case, a Riemannian metric and an affine connection do not determine one another.

2.2. Tensor calculus. In the previous geometric considerations, we have encountered geometric objects that are not described by scalar fields on a manifold. For example, a vector field is described by its coordinates, which are smooth functions, only in local coordinate systems. A similar object arises in the definition of Riemannian and pseudo-Riemannian manifolds. These geometric constructions can be organized in the general scheme, which is known as a tensor field on a smooth manifold. Many physical and mechanical quantities are described by tensor fields. Also, we note that objects arising in geometric constructions are not exhausted by tensors.

Definition 2.2.1. One says that a *tensor field* of type (p, q) is given on a smooth manifold M^n if, in each chart with local coordinates $x = (x^1, \dots, x^n)$, we have a collection of n^{p+q}, $n = \dim M^n$, smooth functions $T^{i_1,\dots,i_p}_{j_1,\dots,j_q}(x)$ that are transformed according to the law (tensor law)

$$T^{i_1\dots i_p}_{j_1\dots j_q}(x) = \frac{\partial x^{i_1}}{\partial y^{l_1}} \dots \frac{\partial x^{i_p}}{\partial y^{l_p}} \frac{\partial y^{s_1}}{\partial x^{j_1}} \dots \frac{\partial y^{s_q}}{\partial x^{j_q}} T^{l_1\dots l_p}_{s_1\dots s_q}(y) \tag{2.2}$$

when passing from one chart to another; here $y^1, \dots, y^n$ and $x^1, \dots, x^n$ are the "old" and "new" coordinates in the intersection of the charts. The number $p + q$ is called the *rank* of the tensor $T^{i_1\dots i_p}_{j_1\dots j_q}$.

A tensor field of type $(1, 0)$ is a vector field, a tensor field of type $(0, 1)$ is called a *covector field*, and a tensor field of type $(1, 1)$ is called a *linear operator field*. Riemannian and pseudo-Riemannian metrics on a manifold are examples of tensors of type $(0, 2)$.

The following fundamental algebraic operations are defined on the set of tensor fields. Let two tensor fields $T^{i_1\dots i_p}_{j_1\dots j_q}$ and $R^{i_1\dots i_p}_{j_1\dots j_q}$ of the same structure be given. Then we can compose a new tensor field, assigning its components $C^{i_1\dots i_p}_{j_1\dots j_q}$ by

$$C^{i_1\dots i_p}_{j_1\dots j_q} = fT^{i_1\dots i_p}_{j_1\dots j_q} + gR^{i_1\dots i_p}_{j_1\dots j_q},$$

where f and g are arbitrary smooth functions on the manifold M^n. Clearly, $C^{i_1\dots i_p}_{j_1\dots j_q}$ is a tensor field. It is called a *linear combination* of tensor fields $T^{i_1\dots i_p}_{j_1\dots j_q}$ and $R^{i_1\dots i_p}_{j_1\dots j_q}$. If $T^{i_1\dots i_p}_{j_1\dots j_q}$ is an arbitrary tensor field, then we can construct a tensor field $C^{i_1\dots i_p}_{j_1\dots j_q}$ by

$$C^{i_1\dots i_p}_{j_1\dots j_q} = T^{\sigma(i_1\dots i_p)}_{\tau(j_1\dots j_q)},$$

where σ and τ are arbitrary permutations. As a result of this operation, we have a tensor field. It is essential here that indices of the same type be permuted. A permutation for distinct types (subscripts and superscripts) is not a tensor operation. We fix two indices of distinct type i_s and j_r of a tensor field $T^{i_1\dots i_p}_{j_1\dots j_q}$ of type (p, q) and construct a new tensor field $C^{i_1\dots i_p}_{j_1\dots j_q}$ of type $(p - 1, q - 1)$ by

$$C^{i_1\dots i_{s-1} i_{s+1}\dots i_p}_{j_1\dots j_{r-1} j_{r+1}\dots j_q} = \sum_{\alpha=1}^{n} T^{i_1\dots i_{s-1},\, i_s=\alpha,\, i_{s+1}\dots i_p}_{j_1\dots j_{r-1},\, j_r=\alpha,\, j_{r+1}\dots j_q}.$$

The obtained tensor field $C^{i_1\dots i_p}_{j_1\dots j_q}$ is called the *compression* of the field $T^{i_1\dots i_p}_{j_1\dots j_q}$ with respect to the indices i_s and j_r.

Let two tensor fields $T^{i_1\dots i_p}_{j_1\dots j_q}$ and $R^{i_1\dots i_s}_{j_1\dots j_r}$ of an arbitrary structure be given. Then we can form a new tensor field $C^{i_1\dots i_{p+s}}_{j_1\dots j_{r+q}}$ of type $(p+s, r+q)$ given by its components:

$$C^{i_1\dots i_{p+s}}_{j_1\dots j_{r+q}} = T^{i_1\dots i_p}_{j_1\dots j_q} R^{i_{p+1}\dots i_{p+s}}_{j_{q+1}\dots j_{q+r}}.$$

This operation is called the *tensor product* and is denoted by $C = T \otimes R$. The tensor product is not commutative, i.e., $T \otimes R \neq R \otimes T$.

Let a nondegenerate tensor field of type (0,2), i.e., a field given by a nonsingular matrix field $A = \|a_{ij}\|$ in certain coordinate system, be given on the manifold M^n. Then the matrix of this field is nonsingular in any other coordinate system. The inverse matrix (which is also nonsingular) is denoted by $A^{-1} = \|a^{ij}\|$. Then for any tensor field $T^{i_1\dots i_p}_{j_1\dots j_q}$, where $p \geq 1$, we can construct a new tensor field:

$$C_{\alpha}{}^{i_2\dots i_p}_{j_1\dots j_q} = a_{\alpha i_1} T^{i_1 i_2\dots i_p}_{j_1 j_2\dots j_q}.$$

Similarly, we construct the field

$$Q^{\alpha i_1\dots i_p}_{ j_2\dots j_q} = a^{\alpha j_1} T^{i_1 i_2\dots i_p}_{j_1 j_2\dots j_q}.$$

The first operation is called the *lowering of a superscript*, and the second operation is called the *raising of a subscript*. Since the field A is nondegenerate, these operations are mutually inverse. The operation of raising of subscripts (or lowering of superscripts) allows one to identify canonically the space of all tangent vectors to M with the space of all cotangent vectors on M. In particular, the operations of lowering and raising are defined on Riemannian and pseudo-Riemannian manifolds because we have here the field g_{ij} of the nondegenerate tensor of type (0,2).

With any tensor field $T^{i_1\dots i_p}_{j_1\dots j_q}$, we can associate a new tensor field $T^{i_1\dots i_p}_{(j_1\dots j_q)}$ by

$$T^{i_1\dots i_p}_{(j_1\dots j_q)} = \frac{1}{q!}\sum_{\sigma} T^{\;i_1\dots i_p}_{\sigma(j_1\dots j_q)},$$

where the summation is carried out over all permutations σ of subscripts $(j_1, j_2, \dots, j_q)$. This operation is called the *symmetrization* of the field $T^{i_1\dots i_p}_{j_1\dots j_q}$ with respect to the subscripts $(j_1, \dots, j_q)$. Similarly, one defines the symmetrization with respect to superscripts.

We can associate with any tensor field $T^{i_1\dots i_p}_{j_1\dots j_q}$ a new tensor field $T^{i_1\dots i_p}_{[j_1\dots j_q]}$ by

$$T^{i_1\dots i_p}_{[j_1\dots j_q]} = \frac{1}{q!}\sum_{\sigma} (-1)^{\sigma} T^{\;i_1\dots i_p}_{\sigma(j_1\dots j_q)},$$

where the summation is carried out over all permutations σ of subscripts $(j_1, j_2, \dots, j_q)$, and $(-1)^\sigma = +1$ for even permutations σ and $(-1)^\sigma = -1$ for odd permutations σ. This operation is called the *alternation* of the field T with respect to the subscripts $(j_1, j_2, \dots, j_q)$. Similarly, one defines the alternation with respect to superscripts.

Definition 2.2.1 can obviously be rewritten at "one point." Then we obtain the definition of a tensor of type (p, q) at a point of a smooth manifold M^n. Clearly, tensors at a point $m \in M^n$ form a linear space $T^p_q(M)_m$. In the same way as for $T(M)$, it is easy to show that the union

$$T^p_q(M) = \bigcup_{m \in M} T^p_q(M)_m$$

is a smooth manifold. It is called the *tensor bundle* of type (p, q). There is a natural projection $\pi : T^p_q(M) \to M$ that associates a base point with each tensor.

We now pass to the study of differentiation of tensor fields. We state the problem more precisely. We now restrict ourselves to the case of the space $\mathbb{R}^n$. It is required to find an operation on $\mathbb{R}^n$ (denoted by ∇) that possesses the following properties:

(a) in the Cartesian coordinates (x^i) of the space $\mathbb{R}^n$, the operation $\nabla = \{\nabla_i\}$ coincides with the usual differentiation $\left\{\dfrac{\partial}{\partial x^i}\right\}$;

(b) the operation ∇ is a tensor one, i.e., if T is a tensor field of type (p, q), then ∇T is a tensor field of type $(p, q+1)$.

We try to find an explicit form of the operation ∇ starting from properties (a) and (b). We consider a vector field T^i in $\mathbb{R}^n$. Let (x^i) be the Cartesian coordinates, and let (y^i) be the curvilinear coordinates; then it is easy to calculate that the operation ∇ on vector fields in the Euclidean space $\mathbb{R}^n$ has the form

$$(\nabla T)^i_j(y) = \nabla_j T^i(y) = \frac{\partial T^i(y)}{\partial y^j} + T^k(y)\Gamma^i_{jk},$$

where

$$\Gamma^i_{jk} = \frac{\partial y^i}{\partial x^s}\frac{\partial^2 x^s}{\partial y^j \partial y^k}.$$

Therefore, there arise certain functions Γ^i_{jk}, which measure the deviation of the operation ∇ from the usual Euclidean differentiation in the case of non-Cartesian coordinates. Calculating the explicit form of the operation ∇ on covector fields T_i, we obtain

$$(\nabla T)_{ij} = \nabla_j T_i = \frac{\partial T_i}{\partial x^j} - T_k\Gamma^k_{ij},$$

where the functions Γ^i_{jk} already appeared in calculation of the action of the operation ∇ on vector fields. Proceeding similarly for an arbitrary tensor field, we obtain the following statement. Let $M = \mathbb{R}^n$, let $(x^1, \dots, x^n)$ be the Cartesian coordinates, and let $(y^1, \dots, y^n)$ be arbitrary coordinates (local coordinate system). Then there is a tensor operation ∇ on $\mathbb{R}^n$ that satisfies conditions (a) and (b) and is given by the following formula on arbitrary tensor fields:

$$(\nabla T)^{i_1\dots i_k}_{j_1\dots j_p,\alpha} = \nabla_\alpha T^{i_1\dots i_k}_{j_1\dots j_p} = \frac{\partial}{\partial x^\alpha} T^{i_1\dots i_k}_{j_1\dots j_p} + \sum_{s=1}^{k} T^{i_1\dots(i_s\to q)\dots i_k}_{j_1\dots j_p}\Gamma^{i_s}_{q\alpha} - \sum_{s=1}^{p} T^{i_1\dots i_k}_{j_1\dots(j_s\to q)\dots j_p}\Gamma^{q}_{j_s\alpha}, \tag{2.3}$$

where the functions Γ^i_{jq} are transformed by law (2.1) under the transformation $(x) \to (y)$, i.e., they are transformed according to the law describing the transformation of an affine connection. Here the existence of the desired operation ∇ is asserted. The existence of the Cartesian coordinates allows us to explicitly calculate the functions Γ^i_{jk}, which measure the deviation of the operation ∇ from the usual differentiation. We use essentially the fact that there is a privileged coordinate system in $\mathbb{R}^n$, the *Cartesian* one, in which the operation ∇ coincides with the usual differentiation, i.e., $\nabla_i = \dfrac{\partial}{\partial x^i}$.

Now passing to arbitrary smooth manifolds, we can define the operation ∇ axiomatically taking the properties of the operation ∇ on the space $\mathbb{R}^n$ found above as the base of definition of the operation ∇.

Definition 2.2.2. Let M^n be an affine connection manifold Γ^i_{jk}, and let $T^{i_1\dots i_k}_{j_1\dots j_p}$ be an arbitrary tensor field on M^n. Then the covariant derivative $\nabla_\alpha T^{i_1\dots i_k}_{j_1\dots j_p}$ of the field $T^{i_1\dots i_k}_{j_1\dots j_p}$ is defined by (2.3).

The covariant derivative was defined by Ricci–Curbastro in the paper "On covariant and countervariant differentiation," in which the general concept of a tensor was also introduced.

A tensor field $T^{i_1\dots i_p}_{j_1\dots j_q}$ on a manifold M is said to be *covariantly constant* with respect to a given affine connection if $\nabla_\alpha T^{i_1\dots i_p}_{j_1\dots j_q} = 0$.

The covariant differentiation on an arbitrary affine connection manifold satisfies the following relations:

(1) the operation $\nabla = \{\nabla_i\}$ is linear;

(2) for an arbitrary tensor field $T^{i_1\dots i_p}_{j_1\dots j_q} = T^{(i)}_{(j)}$, the collection of functions $\nabla_k T^{(i)}_{(j)} = (\nabla T)^{(i)}_{(j),k}$ forms a tensor field;

(3) if a tensor field f is a scalar field, then $\nabla f = \{\nabla_i f\} = \left\{\dfrac{\partial f}{\partial x^i}\right\} = \operatorname{grad} f$;

(4) the operation ∇ on the vector fields T_i has the form

$$\nabla_k T^i = \frac{\partial T^i}{\partial x^k} + T^\alpha \Gamma^i_{\alpha k}\,,$$

and the operation ∇ on covector fields T_i has the form

$$\nabla_k T_i = \frac{\partial T_i}{\partial x^k} - T_\alpha \Gamma^\alpha_{ik};$$

(5) the operation ∇ satisfies the Leibniz formula

$$\nabla(T \otimes R) = (\nabla T) \otimes R + T \otimes (\nabla R),$$

where T and R are arbitrary tensor fields.

It turns out that the above properties uniquely define the operation ∇. More precisely, let the operation $\nabla = \{\nabla_i\}$ satisfying the above properties (1)–(5) be given on the manifold M. Then, for an arbitrary tensor field $T^{(i)}_{(j)}$, we have relation (2.3), i.e., ∇ is the covariant differentiation in the sense of Definition 2.2.2. Therefore, the operation $\nabla = \{\nabla_i\}$ can axiomatically be given by using properties (1)–(5).

Let a tensor field $T = T^{i_1\ldots i_p}_{j_1\ldots j_q}$ and a curve $\gamma(t) = \{x^i(t)\}$ be given in the space M^n of an affine connection Γ^i_{jk}. The tensor

$$\nabla_{\dot\gamma} T = \nabla_i T \frac{dx^i(t)}{dt}$$

is called the *covariant derivative* of the tensor $T^{i_1\ldots i_p}_{j_1\ldots j_q} = T$ *along the curve* γ, and the operation

$$\nabla_{\dot\gamma} = \nabla_i \frac{dx^i(t)}{dt}$$

is called the *covariant differentiation along the curve* $\gamma(t)$. The operation $\nabla_{\dot\gamma}$ is sometimes also denoted by $\dfrac{D}{dt}$.

The differentiation $\nabla_{\dot\gamma}$ allows us to look at Γ^i_{jk} from a new viewpoint; they have a clear geometric sense. We consider a local coordinate system $(x^1, \ldots, x^n)$ and denote the vector fields

$$\frac{\partial}{\partial x^i} = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)$$

by ∂_i for brevity. Then the vector fields $v_{\alpha\beta} = \nabla_{\partial_\alpha}(\partial_\beta)$ are defined for arbitrary α and β. We have the relation $v_{\alpha\beta} = \Gamma^i_{\alpha\beta}\partial_i$. Therefore, the functions Γ^i_{jk} are merely the functional coefficients of the expansion of the covariant derivative of the base vector fields ∂_β along the fields ∂_α with respect to the basis ∂_i.

2.3. Riemannian connections. Let a Riemannian metric g_{ij} be given and fixed on a smooth manifold M^n, i.e., let M^n be a Riemannian manifold. Then one highlights one and only one connection compatible with the metric and completely determined by it in the set of all symmetrical connections on M^n (see, e.g., [71, 83, 161]).

Definition 2.3.1. An affine symmetrical connection is said to be *Riemannian* (or *compatible with the Riemannian metric* g_{ij}) iff $\nabla_k g_{ij} = 0$, i.e., the metric tensor g_{ij} is covariantly constant with respect to this connection.

Therefore, the identity $\nabla_k g_{ij} = 0$ holds in all coordinate systems since the operation ∇ is a tensor operation. This and the Leibnitz formula imply that

$$\nabla(g \otimes T) = g \otimes \nabla T$$

for any tensor field T. In particular, ∇ commutes with the operation of raising and lowering of indices.

Theorem 2.3.1. *Let g_{ij} be a Riemannian metric on a manifold M^n. Then there exists a uniquely defined symmetrical connection compatible with the metric g_{ij}, i.e., a Riemannian connection on M^n. Moreover,*

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha}\Big(\frac{\partial g_{k\alpha}}{\partial x^j} + \frac{\partial g_{j\alpha}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\alpha}\Big).$$

If local coordinates are chosen in such a way that all first derivatives of the metric tensor are equal to zero at a given point, then the functions Γ^i_{jk} corresponding to the Riemannian connection are also equal to zero at this point.

We now return to the problem on the relation between Euclidean coordinates in the sense of the metric and Euclidean coordinates in the sense of the affine connection. In the case of the Riemannian connection, these concepts coincide with accuracy up to an affine transformation. Indeed, if the tensor g_{ij} is constant in certain coordinates $(x^1, \dots, x^n)$, then $\Gamma^i_{jk}(x) = 0$ by Theorem 2.3.1, and the coordinates $(x^1, \dots, x^n)$ are also Euclidean for the Riemannian connection Γ^i_{jk}. Conversely, if $\Gamma^i_{jk} = 0$, then Theorem 2.3.1 implies

$$\frac{\partial g_{ij}}{\partial x^k} = g_{\alpha j}\Gamma^\alpha_{ik} + g_{\alpha i}\Gamma^\alpha_{jk},$$

i.e., $\dfrac{\partial g_{ij}}{\partial x^k} = 0$. Therefore, the tensor g_{ij} is constant in the coordinate system $(x^1, \dots, x^n)$ and is reduced to the form δ_{ij} by an affine transformation.

As one of the applications of the Riemannian connection, we give a correct definition of the *divergence* of the vector field T^i; namely, $\operatorname{div} T = \nabla_i T^i$. By properties of the operation ∇, we obtain a scalar-valued function on M^n. One can calculate that

$$\operatorname{div} T = \frac{\partial T^i}{\partial x^i} + T^\alpha \frac{\partial}{\partial x^\alpha}(\ln \sqrt{g}),$$

where $g = \det \|g_{ij}\|$.

2.4. Parallel translation and geodesics in an affine connection space. We consider a smooth manifold M^n and two tangent vectors a and b applied to distinct points $x, y \in M^n$. In many problems, it is required to compare these vectors, which is itself nontrivial since the tangent spaces T_xM^n and T_yM^n are, in general, distinct, and in the general case there is no canonical method for identifying them. In the Euclidean space $M^n = \mathbb{R}^n$, we have the operation of parallel translation, which allows one to compare vectors applied at distinct points. It is useful to consider this operation in the following way. We connect the points x and y by a smooth curve γ in $\mathbb{R}^n$ and translate the vector a from the point x to the point y in a parallel way so that the origin of the vector slides along the curve γ all the time (see Fig. 3). This operation generates a vector field $a(t)$ along the curve $\gamma(t)$ that has constant components with respect to t. In particular, the derivatives with respect to t of the components $a^i(t)$ of the field $a(t)$ equal zero. There is an analog of the derivative, the covariant differentiation introduced above, in an arbitrary affine connection manifold. Therefore, the definition of parallelism given here can be literally extended to the case of an arbitrary affine connection manifold (see [48, 113, 134, 161]).

Definition 2.4.1. Let a smooth vector field T^i be given along a smooth curve $\gamma(t)$ in a manifold M^n. This field is said to be *parallel* along the curve γ with respect to an affine connection Γ^i_{jk} if $\nabla_{\dot\gamma} T \equiv 0$. In other words, the components of this field are covariantly constant along $\gamma(t)$.

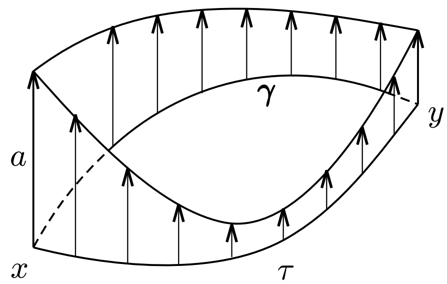


Fig. 3

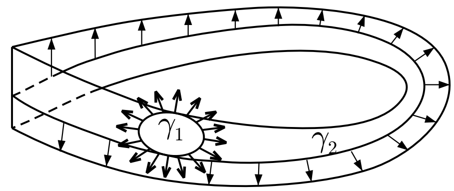


Fig. 4

A vector field T^i is parallel along a curve $\gamma(t)$ iff

$$\frac{dT^i}{dt} + \Gamma^i_{pk} T^p \frac{dx^k}{dt} \equiv 0,$$

where

$$\frac{dT^i}{dt} = \frac{\partial T^i}{dx^k}\frac{dx^k}{dt}$$

and $\frac{dx^k}{dt}$ are components of the velocity vector $\dot{\gamma}$ of the curve $\gamma(t)$.

Definition 2.4.2. The set of equations

$$\frac{dT^i}{dt} + \Gamma^i_{pk} T^p \frac{dx^k}{dt} = 0, \quad i = 1, \dots, n,$$

is called the *equations of parallel translation* along the curve γ on the affine connection manifold (M, Γ^i_{jk}).

The parallel translation of a vector in a Riemannian space was defined by T. Levi-Civita, the Italian researcher in geometry and mechanics, in the paper "The concept of parallelism in an arbitrary manifold and the geometric characteristic of the Riemannian curvature implied by it."

Changing the path γ, we naturally change the equation of parallel translation. When the curve γ is given, the functions $\frac{dx^k}{dt}$ are known. The functions Γ^i_{jk} and $\frac{dx^k}{dt}$ in the equation for T^i are known. The relation $T(0) = a$ holds at the initial instant of time. Since all the functions are smooth, the theory of ordinary differential equations implies that a solution to the set of equations

$$\frac{dT^i}{dt} + \Gamma^i_{pk} T^p \frac{dx^k}{dt} = 0$$

exists, is unique, and is continued up to the point y. The vector $b = T(1)$ arising here at the point y is naturally called the result of the *parallel translation* of the initial vector $a = T(0)$ along the curve γ. In general, the vector b depends on the curve along which the parallel translation is carried out (see Fig. 4). Clearly, the previous constructions generalize the classical parallel translation in the Euclidean space. Therefore, the result of the parallel translation along any curve is uniquely determined by the initial vector and linearly depends on it.

There is one essential distinction of the parallel translation in an arbitrary affine connection manifold from that in the Euclidean space: in the Euclidean space, the parallel translation is independent of the curve along which the translation of a vector is carried out, and in an arbitrary manifold, it depends on such a curve.

In the case of the Riemannian connection, there exist some distinguishing features of the parallel translation. The parallel translation with respect to the Riemannian connection preserves the inner product, i.e., if $a(t)$ and $b(t)$ are parallel vector fields along a curve $\gamma(t)$, then their inner product $(a(t), b(t))$ is constant along the curve $\gamma(t)$. Here, by the inner product we mean the inner product $(a, b) = g_{ij}a^i b^j$ generated by the Riemannian metric g_{ij}. Therefore, under the parallel translation with respect to the Riemannian connection, the length of vectors and angles between them are preserved.

The converse statement is also true: if an affine connection is given on a Riemannian manifold in which the parallel translation along any curve preserves the inner product, then this connection is Riemannian. The parallel translation from the point x to the point y along a curve $\gamma(t)$ with respect to the Riemannian connection is an orthogonal transformation of the tangent space T_xM at the point x into the tangent space T_yM at the point y.

For any affine connection, there exist trajectories that are analogs of straight lines for the Euclidean connection.

Definition 2.4.3. Let an affine connection be given on a manifold M. A smooth curve $\gamma(t)$ is called a *geodesic* of the given affine connection if $\nabla_{\dot\gamma}(\dot\gamma) = 0$, where $\dot\gamma$ is the velocity field of the trajectory $\gamma(t)$. In other words, the velocity vector is parallel translated along the curve itself. The corresponding parameter along the geodesic is said to be *canonical*.

Two canonical parameters along a geodesic differ one from another by an affine transformation.

The problem of searching for geodesics on surfaces was one of the first problems of the application of calculus to geometry. The differential equation of a geodesic on a surface was published for the first time by Euler in his work "Shortest line on an arbitrary surface connecting two arbitrary points." In *Mechanics,* Euler proved that the point moving without acceleration always describes a geodesic. The term "geodesic" was applied initially by Laplace in the second volume of *Celestial Mechanics* to geodesics on the Earth's surface, which was considered as a revolution ellipsoid. Then this term was extended first to all quadrics and then to any surface by Liouville in the paper "Theorem concerning the integration of the geodesic equation."

Since the velocity vector $\dot\gamma$ along a geodesic $\gamma(t)$ remains its velocity vector under the parallel translation along it, we obviously obtain a generalization of the property of straight line to be the most straight in the space $\mathbb{R}^n$. Writing this property in local coordinates $(x^1, \dots, x^n)$, we obtain the following set of equations for geodesics:

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0, \quad i = 1, \dots, n.$$

In the next chapter, we will consider the dependence between the properties of being the most straight and of being the shortest. The written set of equations is obtained from the equations of parallel translations by the substitution $T^i = \dfrac{dx^i}{dt}$, where $\gamma(t) = (x^1(t), \dots, x^n(t))$ in it.

Definition 2.4.4. The differential equations

$$\frac{d^2x^i}{dt^2} + \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

are called the *geodesic equations* of an affine connection Γ^i_{jk} on a manifold M^n.

The geodesics $\gamma(t) = (x^1(t), \dots, x^n(t))$ yield their solutions. Since it is a set of n equations with ordinary derivatives of the second order, it follows that a geodesic is uniquely determined by assignment of an initial point $x_0 \in M^n$ (n parameters) and the initial velocity vector $\dot\gamma(0)$ (additional n parameters).

Proposition 2.4.1. *In a certain open neighborhood of any point $x \in M$ of an affine connection manifold M, for any tangent vector $a \in T_xM$ at this point, there exists a unique geodesic $\gamma(t)$ that emanates from the point x with the initial velocity vector $a = \dot\gamma(0)$.*

Example. In the case of surfaces, i.e., two-dimensional Riemannian manifolds, the general geodesic equations can be rewritten in the following classical form:

$$\begin{cases} 2\dfrac{d}{ds}(Eu_s{}' + Fv_s{}') = E_u(u_s{}')^2 + 2F_u u_s{}' v_s{}' + G_u(v_s{}')^2, \\ 2\dfrac{d}{ds}(Fu_s{}' + Gv_s{}') = E_v(u_s{}')^2 + 2F_v u_s{}' v_s{}' + G_v(v_s{}')^2, \end{cases}$$

where $E = g_{11}$, $F = g_{12}$, and $G = g_{22}$.

Let M be a Riemannian manifold, S be a connected submanifold in M, and $p \in S$. The submanifold S is said to be *geodesic at a point* p if each geodesic in M tangent to S at the point p is a curve in S. The submanifold S is said to be *totally geodesic* if it is geodesic at each of its points.

Using the concept of a geodesic, we can construct certain special coordinates in affine connection manifolds.

Let M^n be an affine connection manifold. We consider an arbitrary point $x \in M^n$ and the tangent space T_xM^n; let $a \in T_xM^n$. Then, by Proposition 2.4.1, there exists a uniquely defined geodesic $\gamma_a(t)$ emanating from the point $x = \gamma_a(0)$ and having the initial vector $\dot{\gamma}_a(0) = a$. It is defined for all sufficiently small t. We construct the mapping $\exp_x : W \to M$, where W is a certain star-shaped open domain in the tangent space T_xM centered at zero, i.e., a domain that, along with any of its points, contains the whole segment connecting this point with zero. We set $\exp_x(ta) = \gamma_a(t)$. The mapping $\exp_x$ is a diffeomorphism of a sufficiently small neighborhood of zero in the space T_xM onto a certain neighborhood of the point x in the manifold M.

Definition 2.4.5. The mapping $\exp_x$ is called the *exponential mapping* at the point $x \in M$ of the affine connection manifold M.

Therefore, the segment of a straight line in T_xM passing through zero in T_xM is mapped into an arc of a geodesic in M. We consider a chart $U \subset M^n$ centered at the point x; let $\varphi : U \to \mathbb{R}^n$ be the coordinate mapping defining curvilinear coordinates $(x^1, x^2, \dots, x^n)$ in U. We assume that $\varphi(x) = 0 \in \mathbb{R}^n$.

Definition 2.4.6. The curvilinear coordinates $(x^1, x^2, \dots, x^n)$ are said to be *normal* if the inverse images of rays passing through zero in $\mathbb{R}^n$ are geodesics in the manifold M. In this case, the neighborhood U is called a *normal coordinate neighborhood*.

In a normal coordinate neighborhood U, each point y can be connected with the center $x = \varphi^{-1}(0)$ by a unique geodesic of U. Such neighborhoods are useful for particular calculations on the manifold M.

The mapping $\exp : T_pM \to M$ defines the normal curvilinear coordinates in an arbitrary affine connection manifold.

Definition 2.4.7. Let $e_1, \dots, e_n$ be an orthonormal basis of the tangent space T_mM^n. Then the *normal coordinates* corresponding to this basis are the coordinates defined by the relation

$$x^j\Big(\exp\Big(\sum_{i=1}^n t_i e_i\Big)\Big) = t_i.$$

As a natural generalization of normal coordinates, we can take the so-called *Fermi coordinates*, which are obtained if we replace the point m by a submanifold P. Let ν be the normal bundle over the submanifold P in M^n. By definition, $\nu = \{(p, v) \mid p \in P,\ v \in T_pP^\perp\}$, where $W^\perp$ denotes the orthogonal complement to the subspace W. The exponential mapping $\exp_\nu$ of the bundle ν is defined by $\exp_\nu(p, v) = \exp_p(v)$ for $(p, v) \in \nu$.

To define a Fermi coordinate system, we need an arbitrary coordinate system $(y^1, \dots, y^q)$ defined in a neighborhood $V \subset P$ of the point p and an orthonormal set of sections $E_{q+1}, \dots, E_n$ of the bundle $\nu|_V$.

Definition 2.4.8. The *Fermi coordinates* $(x^1, \dots, x^n)$ for a submanifold $P \subset V$ at a point p (with respect to the coordinate system $(y^1, \dots, y^q)$ and normal vector fields $E_{q+1}, \dots, E_n$) are the coordinates defined by

$$x^q \exp_\nu \Big(\sum_{j=q+1}^{n} t_j E_j(p') \Big) = y^a(p'), \quad x^i \exp_\nu \Big(\sum_{j=q+1}^{n} t_j E_j(p') \Big) = t_i,$$

where $a = 1, \dots, q$ and $i = q+1, \dots, n$ for $p' \in \nu$.

Remark 2.4.1. In what follows, these coordinates will be needed for the description of the so-called *Jacobi fields*.

In the case where the submanifold P coincides with the point m, we obtain the definition of the usual normal coordinates in a neighborhood of the point $m \in M$.

In what follows, we will need the concept of the set of first conjugate points. We give the corresponding definition. For $p \in M$, we denote by $\tilde{Q}_p$ the tangent set of the first conjugate points for p. By definition, $\tilde{Q}_p = \{v \in T_pM \mid$ the mapping $(\exp_p)_*(tv)\colon T_{tv}(T_pM) \to T_{\exp_p tv}M$ has the maximum rank for $0 \le t < 1$, and it is not maximum for $t = 1\}$. The image $\exp_p(\tilde{Q}_p)$ of the set $\tilde{Q}_p$ under the mapping exp is called the *set of the first conjugate points* and is denoted by $Q_p(M)$.

2.5. Completeness of affine connection spaces. In the above, we speak about small segments of geodesics. In some cases, the geodesic can be continued infinitely to both sides.

Definition 2.5.1. An affine connection on a manifold M is said to be *complete* if geodesics can be infinitely continued. This is equivalent to the fact that the exponential mapping is defined on the whole tangent space.

We consider the concept of completeness for the case of Riemannian manifolds in detail. Let a Riemannian metric g_{ij} be given on a manifold M, and let Γ^i_{jk} be the Riemannian connection, i.e., a symmetric connection compatible with the given metric. This connection defines geodesics on the manifold M. At each point $x \in M$, we consider the tangent space T_xM and the exponential mapping $\exp : T_xM \to M$.

Definition 2.5.2. A Riemannian manifold M is said to be *geodesically complete* if the mapping $\exp : T_xM \to M$ is defined for all points and for all vectors $a \in T_xM$. The corresponding Riemannian connection is said to be *geodesically complete*.

Obviously, the condition of geodesical completeness is equivalent to the fact that each segment $\gamma : [a, b] \to M$ of each geodesic γ on the manifold M is continued to both sides "up to infinity," i.e., up to a smooth mapping $\gamma : \mathbb{R}^1 \to M$. We recall that a geodesic is considered as a continuous mapping of a (closed) interval into the manifold, i.e., together with the given parameter t. In the Riemannian case, as a canonical parameter, we can take the length of a curve (natural parameter). The natural parameter s along the curve is defined with accuracy up to the transformation $\pm s + c$. Therefore, when speaking of the infinite continuability of geodesics, we keep in mind the continuability of the natural parametrization on γ. Along with the concept of the geodesic completeness of a manifold M, it is reasonable to consider the completeness of the manifold M as a metric space. (For the definition of a complete space, see, e.g., [22, 71].)

We recall that the assignment of a Riemannian metric on M transforms M into a metric space. As the distance $\rho(x, y)$ between two points x and y, we take $\inf l(\gamma)$, where $\gamma(0) = x$, $\gamma(1) = y$, and $l(\gamma)$ is the length of a piecewise-smooth trajectory γ connecting the points x and y. The following important theorem, which was discovered by Hopf and Rinow, holds.

Theorem 2.5.1. *For a Riemannian manifold M, the following two conditions are equivalent*:

(a) *M is a complete manifold* (*here we mean the completeness of the manifold M as a metric space*);

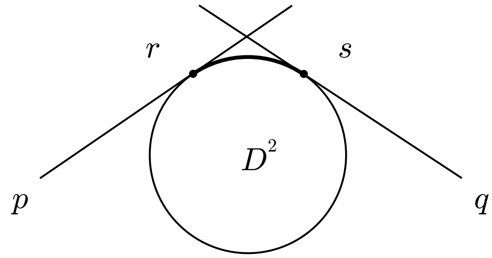


Fig. 5

(b) *all geodesics on M are infinitely continued*; *in other words, the Riemannian connection on M is geodesically complete.*

For the proof of this theorem, see, e.g., [71, 98, 159, 169].

Definition 2.5.3. A Riemannian manifold satisfying the conditions of this theorem is said to be *complete*.

An important property of complete Riemannian manifolds is that in a complete Riemannian space, any two points $x, y \in M$ can be connected by a geodesic whose length is exactly equal to $\rho(x, y)$, i.e., the distance between the points x and y. Such a geodesic is called a *minimal arc* or a *minimal geodesic.*

We note that not on every manifold can any two points be connected by a minimal arc. If we remove a closed disk D^2 from $\mathbb{R}^2$, then points p and q lying on the continuation of the diameter to different sides of the center cannot be connected by a minimal arc since a "true" minimal arc should go along the arc $\overset{\smile}{rs}$, which does not belong to the manifold considered (see Fig. 5).

The example of the disk $D^2 \subset \mathbb{R}^2$ shows that completeness is not implied by the fact that any two points can be connected by a minimal arc.

The example of a sphere and its two diametrically opposite points shows that a minimal geodesic connecting two points is not unique in general. The uniqueness holds only when points x and y are sufficiently close to one another.

Any connected manifold admits a complete Riemannian metric. It is proved in [146] that if every Riemannian metric on M is complete, then the manifold M is compact. The converse theorem on the completeness of a compact metric space is well known. We can construct an example of a noncomplete Riemannian manifold of infinite diameter such that any two points distant from one another by a distance greater than 1 cannot be connected by a curve of minimum length (see [22]).

The completeness property is preserved under the operations considered above. The direct product $M \times N$ of two complete Riemannian manifolds M and N is a complete Riemannian manifold. If M is a complete manifold and $\tilde{M}$ is a Riemannian covering over M, then $\tilde{M}$ is a complete Riemannian manifold.

3. Curvature of Affine Connection Manifolds and Riemannian Manifolds

3.1. Curvature tensor. We consider a manifold M^n with an affine connection Γ^i_{jk}. There arises the following question: how to define a local characteristic of deviation of the connection Γ^i_{jk} from the Euclidean one? In other words, it is required to reveal whether there exist coordinates $(x^1, x^2, \dots, x^n)$ in which $\Gamma^i_{jk} = 0$ at all points. If the connection Γ^i_{jk} is not symmetric, then there are no such coordinates. For an arbitrary affine connection Γ^i_{jk}, the collection of functions $S^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ forms a tensor.

Definition 3.1.1. The tensor S^i_{jk} is called the *torsion tensor* of the affine connection Γ^i_{jk}.

Thus, the first obstruction to the existence of Euclidean coordinates is the torsion tensor S^i_{jk}. Therefore, in what follows, we will be interested in the search for the Euclidean coordinates for symmetric connections.

For smooth functions f, we always have the identity

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i},$$

i.e., the operations $\dfrac{\partial}{\partial x^i}$ and $\dfrac{\partial}{\partial x^j}$ commute. If a connection admits the Euclidean coordinates $(x^1, x^2, \dots, x^n)$, then the tensor fields T are differentiated in the usual way in them since $\nabla_i = \partial/\partial x^i$. Therefore, in the Euclidean coordinates, we always have $\nabla_i \nabla_j T = \nabla_j \nabla_i T$, i.e., $(\nabla_i \nabla_j - \nabla_j \nabla_i)T = 0$. Since T is a tensor field and the operation ∇ is tensor, this relation holds in any curvilinear coordinate system, not only in the Euclidean one. It turns out that the deviation of the connection Γ^i_{jk} from the Euclidean connection is conveniently characterized by the expression $\nabla_i \nabla_j - \nabla_j \nabla_i$. If this operator is different from zero in a certain coordinate system, then the connection Γ^i_{jk} is not Euclidean. Here we essentially used the fact that the operation ∇_i is tensor. We calculate the operation $\nabla_i \nabla_j - \nabla_j \nabla_i$ in local coordinates.

Theorem 3.1.1. *Let T^i be a vector field, and let Γ^i_{jk} be a symmetrical affine connection. Then*

$$(\nabla_k \nabla_l - \nabla_l \nabla_k)T^i = -R^i_{q,kl} T^q,$$

where $R^i_{q,kl}$ is a tensor of rank four. In explicit form, the components of this tensor are given by

$$-R^i_{q,kl} = \frac{\partial \Gamma^i_{ql}}{\partial x^k} - \frac{\partial \Gamma^i_{qk}}{\partial x^l} + \Gamma^i_{pk}\Gamma^p_{ql} - \Gamma^i_{pl}\Gamma^p_{qk}.$$

Definition 3.1.2. The tensor $R^i_{j,pq}$ is called the *Riemannian tensor* (or *Riemannian curvature tensor*) of the given affine connection Γ^i_{jk}.

The curvature tensor was in essence defined by Riemann and was calculated in the paper "An answer to the question suggested by the Paris Academy of Sciences." In this paper, he solved the problem of reducing the heat differential equation

$$\frac{\partial}{\partial x^\alpha}\left(a_{\alpha\beta}\frac{\partial u}{\partial x^\beta}\right) = h\frac{\partial u}{\partial t}$$

to the simplest form, which is equivalent to the problem of transforming the quadratic form $a_{\alpha\beta}dx^\alpha dx^\beta$ into the sum of squares. Riemann showed that the necessary and sufficient condition of reducing this form into the sum of squares consists in the vanishing of components of the tensor $R^\alpha_{\beta,\gamma\delta}$. The components of the curvature tensor were also found by E. B. Christoffel.

Therefore, if the connection is Euclidean, then $R^i_{j,kl} = 0$, i.e., the Riemannian tensor vanishes identically. The action of the operation $\nabla_i \nabla_j - \nabla_j \nabla_i$ on arbitrary vector fields is described as follows:

$$(\nabla_k \nabla_l - \nabla_l \nabla_k)T^{i_1 \dots i_p}_{j_1 \dots j_s} = -R^{i_1}_{q,kl} T^{q i_2 \dots i_p}_{j_1 j_2 \dots j_s} - \dots - R^{i_p}_{q,kl} T^{i_1 \dots i_{p-1} q}_{j_1 \dots j_{s-1} j_s} + R^q_{j_1,kl} T^{i_1 i_2 \dots i_p}_{q j_2 \dots j_s} + R^q_{j_s,kl} T^{i_1 i_2 \dots i_p}_{j_1 \dots j_{s-1} q}.$$

Above, we have given the definition of the Riemannian tensor using the language of local coordinates. We can give an invariant definition, which is useful in certain problems. Let X, Y, and Z be arbitrary smooth vector fields on a manifold M. We construct the "*curvature operator*" R that associates a new vector field $R(X,Y)Z$ with a triple of vector fields X, Y, Z. We will treat fields as differential operators acting on smooth functions by setting

$$Xf = X^i \frac{\partial f}{\partial x^i},$$

where $X = (X^1, X^2, \dots, X^n)$.

Proposition 3.1.1. *Let the operation $R(X,Y)Z$ be defined by*

$$-R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Then the field $R(X,Y)Z$ is linear in each of the arguments X, Y, and Z. The mapping $Z \mapsto R(X,Y)Z$ is given by a tensor of rank 4 coinciding with the Riemannian tensor.

The connection between the invariant and coordinate definitions of the Riemannian tensor is stated as follows. As X and Y, we take the coordinate fields ∂_i and ∂_j, respectively. Since $\nabla_{\partial_i} = \nabla_i$ and $[\partial_i, \partial_j] = 0$, we have

$$-R(\partial_i,\partial_j)Z = (\nabla_i\nabla_j - \nabla_j\nabla_i)Z, \qquad -R(\partial_i,\partial_j) = \nabla_i\nabla_j - \nabla_j\nabla_i, \qquad R(\partial_i,\partial_j)\partial_k = R^l_{k,ij}\partial_l.$$

The Riemannian tensor possesses a number of important algebraic properties that are listed in the following theorem.

Theorem 3.1.2. *For any smooth vector fields X, Y, and Z on an affine connection manifold M, the following identities hold*:

(a) $R(X,Y)Z + R(Y,X)Z = 0$, *or* $R^i_{j,kl} + R^i_{j,lk} = 0$ *in coordinates, i.e., we have the skew symmetry with respect to the arguments X and Y*;

(b) $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$ (*the Jacobi identity*) *or* $R^i_{j,kl} + R^i_{l,jk} + R^i_{k,lj} = 0$ *in coordinates*;

(c) *if the connection is Riemannian, then* $(R(X,Y)Z,W) + (R(X,Y)W,Z) = 0$ *for any vector fields X, Y, Z, and W* (*here* (a,b) *denotes the inner product generated by the metric* g_{ij}); *in coordinates, we have* $R_{ij,kl} + R_{ji,kl} = 0$, *where* $R_{ij,kl} = g_{i\alpha}R^\alpha_{j,kl}$;

(d) *if the connection is Riemannian, then* $(R(X,Y)Z,W) = (R(Z,W)X,Y)$, *i.e.,* $R_{ij,kl} = R_{kl,ij}$.

Definition 3.1.3. The *Ricci tensor* of an affine connection is the trace of the Riemannian tensor, i.e., the tensor $R_{jl} = R^i_{j,il}$ of rank two, obtained by the compression with respect to the pair of indices of the Riemannian tensor. The Ricci tensor is symmetric.

The Ricci tensor was defined by Ricci–Curbastro in the paper "Main formulas of the general theory of manifolds and their curvature."

The previous considerations have a sense in any affine connection manifold. We now pass to the Riemannian case. Therefore, let M be a Riemannian manifold with a Riemannian metric g_{ij}.

Definition 3.1.4. The *scalar curvature* R of a Riemannian manifold is the scalar-valued function $R(x) = g^{kl}R_{kl}$, i.e., the complete compression of the Ricci tensor with tensor inverse to the metric tensor. Clearly, $R(x) = g^{kl}R^i_{k,il}$.

In calculations, it is useful to know the following explicit expression of the Riemannian tensor for the Riemannian connection through the metric tensor:

$$-R_{iq,kl} = -g_{i\alpha}R^\alpha_{q,kl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial x^q \partial x^k} + \frac{\partial^2 g_{qk}}{\partial x^i \partial x^l} - \frac{\partial^2 g_{ik}}{\partial x^q \partial x^l} - \frac{\partial^2 g_{ql}}{\partial x^i \partial x^k}\right) + g_{mp}\left(\Gamma^m_{qk}\Gamma^p_{il} + \Gamma^m_{ql}\Gamma^p_{ik}\right).$$

The Riemannian tensor admits an important geometric interpretation. We consider a local coordinate system centered at a point x; let $(x^1, \dots, x^n)$ be the coordinates in this chart. We denote by K a small square with side ε spanned by the coordinate lines x^i and x^j (see Fig. 6).

We fix a tangent vector $a \in T_xM^n$ at the point x and translate it in a parallel way (with respect to the given connection) along the square K moving in such a way that the interior of the square remains to the left (see Fig. 6). As a result, we obtain a vector $a(\varepsilon)$ at the point x.

Proposition 3.1.2. *We have the relation*

$$\lim_{\varepsilon \to 0} \frac{a^k(\varepsilon) - a^k}{\varepsilon^2} = -R^k_{l,ij}a^l.$$

Therefore, the Riemannian tensor measures the deviation of a vector from its initial position after going around a small closed contour. As we know, the parallel translation in the Euclidean space is independent of the path. It turns out that it is a general fact related to the vanishing of the curvature tensor. More precisely, we have the following theorem.

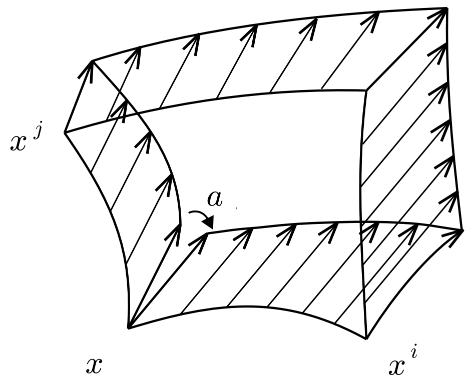


Fig. 6

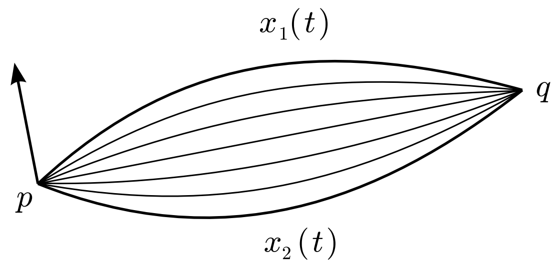


Fig. 7

Theorem 3.1.3. *Let M^n be an affine connection space with zero curvature tensor $R^i_{j,pq} \equiv 0$. Let two curves $x_1(t)$ and $x_2(t) : [0,1] \to M^n$ be homotopic, and, moreover, if $x(t,s)$, $0 \le s \le 1$, is the corresponding homotopy, we set $x(0,s) = x_1(0) = x_2(0) = p$, $x(1,s) = x_1(1) = x_2(1) = q$, $t \in [0,1]$ (see Fig. 7). Then the parallel translation of a vector $\xi \in T_pM^n$ along the curve $x_1(t)$ coincides with the parallel translation of the vector ξ along the curve $x_2(t)$.*

We mention the identity (Bianchi identity)

$$\nabla_m R^\alpha_{i,kl} + \nabla_l R^\alpha_{i,mk} + \nabla_k R^\alpha_{i,lm} = 0,$$

which is important in the characteristic class theory. In terms of vector fields, it is written as follows:

$$\nabla_X R(Y,Z)W + \nabla_Y R(Z,X)W + \nabla_Z R(X,Y)W = 0.$$

We consider a Riemannian manifold M; let $\sigma \subset T_xM$ be an arbitrary two-dimensional subspace in the tangent space T_xM of the manifold M at a point x. Let X and Y be two arbitrary base vectors in the plane σ.

Definition 3.1.5. The *sectional curvature* (or merely, *curvature*) of the Riemannian manifold (M, g_{ij}) in the two-dimensional direction σ is the number

$$K(\sigma) = -\frac{(R(X,Y)X,Y)}{(X,X)(Y,Y) - (X,Y)^2},$$

where $(X,Y) = g_{ij}X^iY^j$ is the inner product with respect to the metric g_{ij}.

Lemma 3.1.1. *The number $K(\sigma)$ is well defined, i.e., it depends only on the two-dimensional subspace $\sigma \subset T_xM$ (plane) in the tangent space T_xM.*

The sectional curvature of a Riemannian manifold was defined (with accuracy up to a multiplier) by Riemann in the paper "On hypotheses lying in the foundations of geometry."

In Riemannian geometry, one divides manifolds having a positive, negative, zero, constant, etc. sectional curvature into separate classes. These fields of geometry differ from each other not only in results but also in the methods of study (see Chapt. 3).

If a Riemannian manifold is two-dimensional, then at each point there exists only one two-dimensional direction, which coincides with the tangent plane. Therefore, the function $K(\sigma)$ becomes a scalar-valued function on the manifold M^2 and is called the *Gauss curvature*. We note that the definition of the Gauss curvature is given here independently of any embedding of M^2 in the space $\mathbb{R}^3$. In other words, the Gauss curvature (as well as other objects described above) belongs to the intrinsic geometry of a Riemannian manifold. The Gauss curvature can be calculated in terms of an embedding of the surface M^2 in the space $\mathbb{R}^3$. (For more details, see textbooks on differential geometry [134,162,206].) An explicit expression of the curvature K through the coefficients of the first quadratic form looks as follows:

$$K = -\frac{1}{4(EG-F^2)^2}\begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{2\sqrt{EG-F^2}}\left\{\left(\frac{E_v - F_u}{\sqrt{EG-F^2}}\right)_v - \left(\frac{F_v - G_v}{\sqrt{EG-F^2}}\right)_u\right\}.$$

We have the relation $R = 2K$ on a two-dimensional Riemannian manifold, where R is the scalar curvature and K is the Gauss curvature. The curvature with respect to a two-dimensional direction has the following clear treatment. We consider a two-dimensional plane $\sigma \subset T_xM$ at a point $x \in M^n$ and draw a geodesic going in the direction of each vector $a \in \sigma$. These geodesics (locally) form a certain two-dimensional surface M^2 whose tangent plane coincides with σ. There arises an induced Riemannian metric on this plane, and we can calculate the Gauss curvature K of this surface M^2 at the point x. It turns out that $K(\sigma)$ coincides with K.

3.2. Structural equations. We can describe connections using the language of differential forms. We recall some facts from the exterior differential form theory. For more details, see modern courses of calculus or detailed courses of differential geometry (see [22,48,113,134,177,188]).

Definition 3.2.1. An *exterior differential form* is a covariant tensor field $a_{i_1\dots i_k}$ (the number k is called the degree of this form) that is skew symmetric, i.e., its coordinates change their sign under an odd permutation of subscripts and do not change it under an even permutation: $a_{\sigma\{i_1 i_2 \dots i_k\}} = \delta_\sigma a_{i_1 i_2 \dots i_k}$, where $\delta_\sigma = +1$ if σ is an even permutation and $\delta_\sigma = -1$ if σ is an odd permutation.

We can use tensor operations in the calculus of exterior differential forms. However, the tensor product does not preserve the class of exterior differential forms. Therefore, a new operation, the exterior product, is introduced.

Definition 3.2.2. The *exterior product* $c_{i_1 i_2 \dots i_{k+l}}$ of exterior differential forms $a_{i_1\dots i_k}$ and $b_{i_1\dots i_l}$ is an exterior differential form defined by

$$c_{i_1\dots i_{k+l}} = a_{[i_1\dots i_k} b_{i_{k+1}\dots i_{k+l}]}.$$

The symbol $c = a \wedge b$ is used for the exterior product. The exterior product possesses the following properties:

(a) $a \wedge (b + c) = a \wedge b + a \wedge c$,
(b) $(b + c) \wedge a = b \wedge a + c \wedge a$,
(c) $a \wedge b = (-1)^{kl} b \wedge a$, where k is the degree of the form a and l is the degree of the form b,
(d) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$.

Definition 3.2.3. Let $a_{i_1\dots i_k}$ be an exterior differential form of degree k. Then the collection of functions $(da)_{i_0 i_1 \dots i_k}$ defined by

$$(da)_{i_0 i_1\dots i_k} = \partial_{[i_0} a_{i_1\dots i_k]},$$

where $\partial_{i_0} = \dfrac{\partial}{\partial x^{i_0}}$, defines an exterior differential form of degree $k+1$, which is called the *exterior derivative* of the form $a_{i_1\dots i_k}$.

The exterior derivative possesses the following properties:

(a) $d(a + b) = da + db$,
(b) $d(a \wedge b) = da \wedge b + (-1)^{\deg a} a \wedge db$,
(c) $d^2 = 0$.

Now let M^n be an affine connection manifold, and let $X_1, \dots, X_n$ be a basis of vector fields on a certain open neighborhood U_p of a point $p \in M^n$, i.e., each vector field X on U_p can be represented in the form $X = \sum\limits_{i=1}^{n} f_i X_i$, where f_i are smooth functions on U_p. Let ω^i and ω^i_j, $1 \le i, j \le n$, be 1-forms on

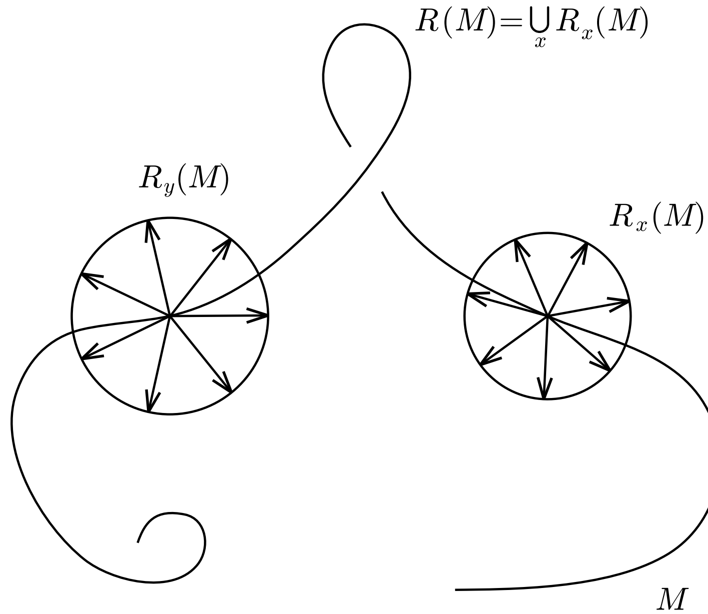


Fig. 8

U_p defined by $\omega^i(X_j) = \delta^i_j$ and $\omega^i_j = \sum_k \Gamma^i_{kj}\omega^k$. Clearly, the forms ω^i_j define the functions Γ^i_{jk} on U_p and, therefore, the connection. On the other hand, as the following theorem shows, the forms ω^i and ω^i_j are described by the curvature and torsion tensor fields.

Theorem 3.2.1 (the Cartan structural equations)**.** *We have the following relations*:

$$d\omega^i + \omega^i_p \wedge \omega^p = \frac{1}{2}S^i_{jk}\omega^j \wedge \omega^k, \qquad d\omega^i_l + \omega^i_p \wedge \omega^p_l = \frac{1}{2}R^i_{l,jk}\omega^j \wedge \omega^k.$$

3.3. Principal frame bundles. In various geometric constructions, it is useful to pass to the principal frame bundle (see [188, 189]).

Definition 3.3.1. Let M^n be a smooth manifold. A pair (x, l), where $l = (l_1, \dots, l_n)$ is a basis of the tangent space T_xM^n, is called a *frame*. On the set $R(M)$ of all frames (see Fig. 8), we have a natural structure of a smooth manifold of dimension $n^2 + n$, where $n = \dim M^n$ (see [189]). There is a natural projection $\pi : R(M) \to M$ that is the principal bundle [22]. Now let M be an affine connection manifold with connection Γ^i_{jk}. In this case, differential forms of degree 1, ω^i and ω^i_j, are defined on the manifold $R(M)$.

By definition, we set

$$\frac{d}{dt}x(t) = \omega^i(\xi)l_i \quad \text{and} \quad \frac{\nabla}{dt}l_i(t) = \omega^j_i(\xi)l_j,$$

where ξ is a tangent vector to the manifold $R(M)$ at the point $(x, l) \in R(M)$ and

$$\xi = \frac{d}{dt}\bigg|_{t=0} \Big(x(t), l_1(t), \dots, l_n(t)\Big)$$

.

Therefore, we have $n^2 + n$ linearly independent 1-forms ω^i and ω^i_j in the *frame bundle* $R(M)$, which completely describe the connection.

The differential forms ω^i and ω^i_j satisfy the structural equations described in Theorem 3.2.1; here S^i_{jk} and $R^i_{j,kl}$ are now smooth functions on the space $R(M)$ but not tensor fields on the manifold M. On the space $R(M)$, the group $\mathrm{GL}(n)$ of nonsingular transformations of the space $\mathbb{R}^n$ acts in an obvious way. Under this action, the functions S^i_{jk} and $R^i_{j,kl}$ are transformed according to the tensor law described in Sec. 2.2.

The pointwise approach to affine connections is a particular case of the frame one. To observe this, it suffices to consider an n-dimensional submanifold $\widetilde{M}$ in $R(M)$ that consists of base vectors of a certain chart in the manifold M. The restriction of the forms ω^i_j to this submanifold yields the relation $\omega^i_j(d) = \Gamma^i_{pj} dx^p$, i.e., the affine connection Γ^i_{jk} is reconstructed. (Here d is a tangent vector to $R(M)$.) Using the Cartan structure equations restricted to the described submanifold $\widetilde{M}$, we obtain that S^i_{jk} is the usual torsion tensor and $R^i_{j,kl}$ is the usual curvature tensor. In the case of the Riemannian connection, there arise some distinguishing features in the structure of the forms ω^i_j; namely, we have $\omega^i_j = -\omega^j_i$. The distinguishing features of the curvature tensor in the Riemannian case were described above, and we do not repeat them here.

3.4. Gauss and Bonnet theorem. Let $R_{kl,ij}$ be the curvature tensor, and let $l_1, \dots, l_{2p}$ be an orthonormal basis in the tangent plane. We set $R(l_i, l_j)l_l = \sum\limits_k R_{kl,ij} l_k$. As is easily seen, the number

$$K(R) = \frac{1}{2^p(2p)!} \sum_{\alpha,\beta \in S_{2p}} \delta_\alpha \delta_\beta R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2p-1} i_{2p} j_{2p-1} j_{2p}}$$

is independent of the choice of the orthonormal basis (S_{2p} is the set of all permutations of order $2p$ and δ_α is the sign of a permutation α). This number is called the *Lipschitz–Killing curvature* of the tensor R. For a smooth function $f : M \to \mathbb{R}$ on a Riemannian manifold (M, g_{ij}), we define the integral $\int\limits_M f$ by $\int\limits_M f dv$, where dv is the volume element of the manifold M.

We consider a Riemannian manifold M^n of even dimension $n = 2p$ with boundary ∂M^n. The *Gaussian curvature* of the manifold M^n is the Lipschitz–Killing curvature of the curvature tensor $R^i_{j,pq}$ of the manifold M^n. Therefore, the Gaussian curvature is a smooth function $K : M^n \to \mathbb{R}$. We denote by $\nu : \partial M \to TM$ the unit vector of the exterior normal to the boundary ∂M. For $m \in \partial M$, the *second fundamental tensor* is the linear operator

$$B(m) : T_m(\partial M) \to T_m(\partial M)$$

defined by

$$B(m)X = \nabla_X \nu, \quad X \in T_m(\partial M).$$

For each $r = 0, 1, \dots, p-1$, we define a smooth function $\varphi_r : \partial M^{2p} \to \mathbb{R}$ as follows. Let $m \in \partial M^{2p}$, $l_1, \dots, l_{n-1}, l_n = \nu$ be an orthonormal basis in the space $T_m M^{2p}$. We set

$$B(m) l_\alpha = \sum_{\beta=1}^{n-1} b_{\alpha\beta} l_\beta, \quad \alpha = 1, \dots, n-1,$$

and

$$R(l_i, l_j) l_s = \sum_{k=1}^{n} R_{ij,ks} l_k, \quad i, j, s = 1, \dots, n.$$

The number

$$\varphi_r(m) = \sum_{\alpha,\beta \in S_{n-1}} \delta_\alpha \delta_\beta R_{\alpha_1 \alpha_2 \beta_1 \beta_2} \cdots R_{\alpha_{2r-1} \alpha_{2r} \beta_{2r-1} \beta_{2r}} b_{\alpha_{2r+1} \beta_{2r+1}} \cdots b_{\alpha_{n-1} \beta_{n-1}}$$

is independent of the choice of the orthonormal basis $l_1, \dots, l_{n-1}$. We introduce the notation

$$\varphi = \sum_{r=0}^{p-1} d_r \varphi_r,$$

where d_r are positive rational coefficients defined by

$$d_r = \frac{\pi}{r! 2^{2p+r} \Gamma\left(\frac{1}{2}(2p+1)\right) \Gamma\left(\frac{1}{2}(2p-2r+1)\right)}.$$

Denoting by ω_n the volume of the unit n-dimensional sphere, we have

$$\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

In this notation, we have the following theorem.

Theorem 3.4.1 (Gauss and Bonnet)**.** *Let M be a compact Riemannian manifold of an even dimension n with boundary ∂M. Then*

$$\int_M K + \int_{\partial M} \varphi = \frac{1}{2} \omega_n \chi(M),$$

where $\chi(M)$ is the Euler characteristic of the manifold M.

For the proof of this theorem, see [6, 113, 183, 189, 206].

4. Transformations of Affine Connection Manifolds and Riemannian Manifolds

4.1. Isometries. In this section, we define an analog of motions for arbitrary Riemannian manifolds.

Definition 4.1.1. Let M and N be two Riemannian manifolds with Riemannian structures g_{ij} and h_{ij}, respectively. Let f be a smooth mapping of the manifold M into N. The mapping f is called an *isometry* if f is a diffeomorphism of M onto N and $f^* g_{ij} = h_{ij}$ (the operation f^* is defined in Sec. 1.3). The mapping f is called a *local isometry* if, for each point $p \in M$, there exist open neighborhoods U and V of points p and $f(p)$, respectively, such that f is an isometry of the manifold U onto V.

Obviously, if f is an isometry of a Riemannian manifold M onto itself, then f preserves the distance, i.e., $\rho(f(p), f(q)) = \rho(p, q)$. The converse statement also holds.

Theorem 4.1.1. *Let M be a Riemannian manifold, and let f be a distance-preserving mapping. Then f is an isometry.*

For the proof of this statement, see [113]. The set of all isometries forms a group, which is denoted by $I(M)$, and $I_0(M)$ is its connected component of the identity. As for the structure of the isometry group $I(M)$, we have the following important theorem.

Theorem 4.1.2 ([94, 113])**.** *For a Riemannian manifold M, the isometry group $I(M)$ of M is a Lie transformation group with respect to the compact-open topology. The stationary subgroup*

$$I_x(M) = \{g \in I(M) | g(x) = x\}$$

of an arbitrary point $x \in M$ is compact. If the manifold M is compact, then the group $I(M)$ is also compact.

In fact, the idea of the proof of this theorem consists of the fact that isometries are uniquely defined by their values at one point. More precisely, we have the following statement. Let M be a Riemannian manifold, and let f and g be two isometries of the manifold M onto itself such that there exists a point $p \in M$ for which $f(p) = g(p)$ and $f_{*,p} = g_{*,p}$. Then $f = g$.

As a consequence of this statement, we have the following estimate of the dimension of the isometry group.

Proposition 4.1.1 (see, e.g., [134]). *Let M be a compact smooth Riemannian manifold, and let $G = I(M)$ be its isometry group. Then* $\dim G \leq \frac{1}{2}n(n+1)$*, i.e., the transformations $g \in G$ depend on no more than $\frac{1}{2}n(n+1)$ continuous parameters.*

A vector field X on a manifold M is called an *infinitesimal isometry* (or a *Killing vector field*) if the local one-parameter transformation group generated by the field X in a neighborhood of each point of M consists of isometries only. The set $i(M)$ of all infinitesimal isometries of the manifold M forms a Lie algebra with respect to the commutator of vector fields $[X, Y] = XY - YX$.

We can estimate the dimension of the Lie algebra of infinitesimal isometries. The Lie algebra $i(M)$ of infinitesimal isometries of a Riemannian manifold M is of dimension at most $\frac{1}{2}n(n+1)$, where $n = \dim M$. If $\dim i(M) = \frac{1}{2}n(n+1)$, then M is a space of constant curvature (see [113]). Geodesics of a Riemannian manifold can be represented as orbits of infinitesimal isometries. More precisely, let φ_t be a local one-parameter isometry group generated by a vector field X on the Riemannian manifold M. If x is a critical point of the length function $\sqrt{g(X, X)}$ (i.e., all partial derivatives of this function vanish at the point x), then the orbit $\varphi_t(x)$ is a geodesic (see [113]).

4.2. Conformal transformations. Let (M, g) and (M', g') be two n-dimensional Riemannian manifolds with Riemannian metrics g and g', respectively. A diffeomorphism $f : M \to M'$ is called a *conformal mapping* of the manifold M into M' if the Riemannian metric $g^* = f^*g'$ induced by g' via the mapping f is conformally equivalent to g, i.e., there exists a scalar-valued function φ on M such that $g^* = e^{2\varphi}g$. If $\varphi = \text{const}$, then the mapping f is called the *homothety*, and if $\varphi \equiv 0$, then we obtain the concept of an isometry, which is already known.

For example, conformal transformations of the plane were initially considered by Euler in *Arguments on Orthogonal Trajectories*. Euler applied conformal mappings in cartographic works, where he considered conformal mappings of the sphere onto the plane that consist of the stereographic projection of the sphere on the plane and conformal mappings of the plane by using analytic functions. Conformal transformations of the three-dimensional space were initially considered by Liouville.

The set of conformal mappings of a Riemannian manifold forms a group denoted by $\mathrm{Conf}(M)$; its connected component of the identity is denoted by $\mathrm{Conf}_0(M)$. The structure of this group is described by the following important theorem.

Theorem 4.2.1 ([113]). *The group of conformal transformations of an n-dimensional Riemannian manifold M for $n \geq 3$ is a Lie transformation group of dimension at most $\frac{1}{2}(n+1)(n+2)$.*

The case $n = 2$ is exceptional in most problems concerning conformal mappings because of the following. If M is a complex manifold of complex dimension 1 with a local coordinate system $z = x + iy$ and g is a Riemannian metric on M of the form $f(z)(dx^2 + dy^2) = f dz d\bar{z}$, where $f(z)$ is a positive function on the manifold M, then each complex-analytic transformation of the manifold M is conformal.

We consider the case where the group $\mathrm{Conf}_0(M)$ does not coincide with the group $I_0(M)$. In this case, if M is a compact Riemannian manifold of dimension $n > 3$ that is also homogeneous, then M is isometric to the sphere; if M is a complete Riemannian manifold of dimension $n \geq 3$ with a parallel Ricci tensor, then the manifold M^n is isometric to the sphere S^n. If M is a complete Riemannian manifold that is not a local Euclidean space, then a complete homothetic transformation f of the manifold M is an isometry (see [113]).

4.3. Affine transformations of affine connection manifolds. Let $f : M \to N$ be a diffeomorphism of a manifold M onto N, and let the object of an affine connection Γ^i_{jk} on M be given, that is, the parallel translation law $d\xi^i = -\Gamma^i_{jk} dx^j \xi^k$. The diffeomorphism f transfers the connection Γ^i_{jk} from the manifold M to the manifold N. We can give two definitions of the *transferred* affine connection.

Definition 4.3.1. By definition, we set

$$\widehat{\Gamma}^i_{jk} = \frac{\partial y^i}{\partial x^\alpha}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial y^j \partial y^k}\frac{\partial y^i}{\partial x^\alpha};$$

here the diffeomorphism f is given by the collection of functions $y^i = y^i(x^1, \dots, x^n)$ and the inverse diffeomorphism f^{-1} is given by $x^i = x^i(y^1, \dots, y^n)$.

The object of the affine connection defined by this relation is uniquely defined and does not depend on the choice of charts; moreover, $\widehat{\Gamma}^i_{jk}$ is also the object of the linear connection. Definition 4.3.1 is equivalent to the definition consisting of the fact that the coordinates of the object of the affine connection $\widehat{\Gamma}^i_{jk}$ in the chart mapped by the diffeomorphism f coincide with the coordinates of the object of the affine connection Γ^i_{jk}. This is equivalent to the existence of a path with a vector field that is parallel translated such that under the mapping of this path and this vector field, we obtain the vector field that is parallel translated with respect to the connection $\widehat{\Gamma}^i_{jk}$.

Definition 4.3.2. A diffeomorphism of a manifold M onto itself is called an *affine transformation* if it preserves the object of the linear connection, i.e., if $\widehat{\Gamma}^i_{jk}$ is the mapped object of the affine connection, then $\widehat{\Gamma}^i_{jk} = \Gamma^i_{jk}$.

It is easily seen that the preservation of the object of an affine connection is equivalent to the preservation of the parallel translation. The set of affine transformations of the manifold obviously defines a group denoted by $\mathrm{Aff}(M)$; the connected component of the identity is denoted by $\mathrm{Aff}_0(M)$. The structure of the group of affine transformations is described by the following important theorem.

Theorem 4.3.1 ([113])**.** *Let M be an affine connection manifold. Then the group $\mathrm{Aff}(M)$ of affine transformations of the manifold M is a Lie transformation group with respect to the compact-open topology.*

Definition 4.3.3. A vector field on an affine connection manifold M with connection Γ^i_{jk} is called an *infinitesimal affine transformation* if, for each point $x \in M$, the local one-parameter group of transformations φ_t of a neighborhood U of a point $x \in M$ preserves the connection Γ^i_{jk}: more precisely, if each transformation $\varphi_t : U \to M$ is an affine transformation, where U is equipped with an affine connection equal to the restriction of the connection Γ^i_{jk} to U.

Infinitesimal affine transformations of the affine connection manifold M with connection Γ^i_{jk} form a Lie algebra, $\mathrm{aff}(M)$.

Theorem 4.3.2. *If M is an affine connection manifold, then the Lie algebra $\mathrm{aff}(M)$ of infinitesimal affine transformations of the manifold M is of dimension at most $n^2 + n$, where $n = \dim M$. If $\dim\ \mathrm{aff}(M) = n^2+n$, then Γ^i_{jk} is a flat connection, i.e., the torsion and curvature of the connection Γ^i_{jk} vanish identically.*

The completeness of an affine connection can be expressed using the language of infinitesimal transformations. In this connection, we mention the following result of Kobayashi [113].

Theorem 4.3.3. *Let Γ^i_{jk} be a complete affine connection on M. Then each infinitesimal affine transformation X of the manifold M^n is complete, i.e., it generates a global one-parameter group of affine transformations of the manifold M.*

In the case of a Riemannian manifold M, it is clear that $I(M)$ is a closed subgroup in the group $\mathrm{Aff}(M)$. In many cases, the components of the identity of the groups $I(M)$ and $\mathrm{Aff}(M)$ coincide. We mention only the following result: if X is an infinitesimal affine transformation of a complete Riemannian manifold and the lengths of vectors of the field X are bounded, then X is an infinitesimal isometry. This result implies the following Yano theorem.

Theorem 4.3.4 ([113])**.** *We have $I_0(M) = \mathrm{Aff}_0(M)$ on a compact Riemannian manifold M.*

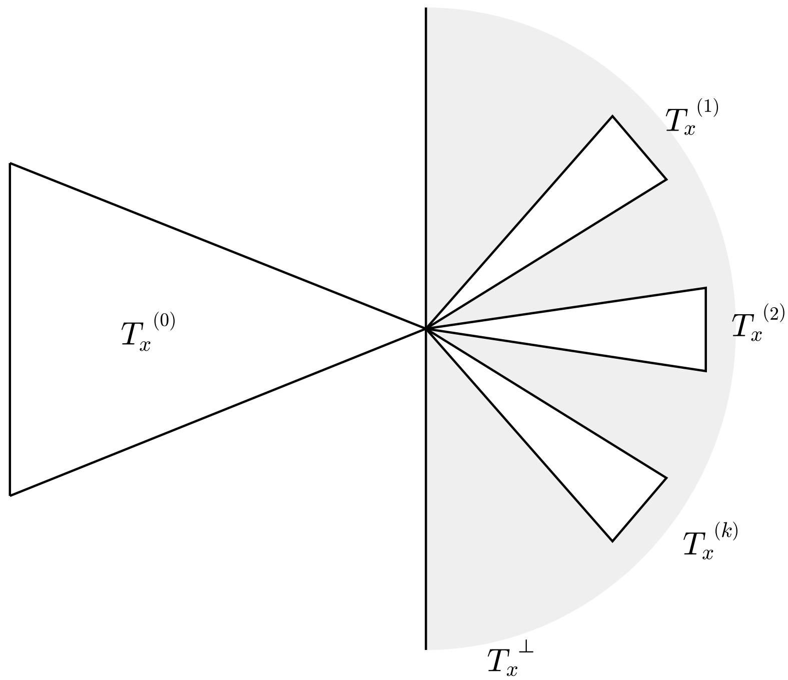


Fig. 9

4.4. Holonomy groups. For each point $x \in M^n$, we denote by $C(x)$ the space of all loops at the point x, i.e., the set of all closed curves with the origin and the end at the point x. If $\tau, \mu \in C(x)$, then the composition $\tau\mu \in C(x)$ is defined in an obvious way (μ follows τ). The parallel translation along each curve $\tau \in C(x)$ defines an isomorphism $\hat{\tau}$ of the space T_xM onto itself.

Definition 4.4.1. The set of all isomorphisms $\hat{\tau}$, $\tau \in C(x)$, of the space T_xM onto itself forms a group $\Phi(x)$, which is called the *holonomy group* of the connection Γ^i_{jk} with support point (or reference point) x. Let $C_0(x)$ be a subset in $C(x)$ consisting of loops that are homotopic to zero. The subgroup of the holonomy group consisting of parallel translations along $\tau \in C_0(x)$ is called the *restricted holonomy group* for the connection Γ^i_{jk} with support point x. This group is denoted by $\Phi_0(x)$. The group $\Phi_0(x)$ is a connected group, because if a loop can be continuously contracted into a point, then the transformations of the holonomy group $\Phi_0(x)$ can be connected by a continuous path with the identity transformation. The holonomy group $\Phi_0(x)$ lies in the group $\mathrm{GL}(n)$ of nonsingular transformations of the space T_xM^n. One asks: is it a Lie subgroup or not? To solve this problem, we can use the following deep theorem belonging to Yamabe.

Theorem 4.4.1. *Each connected subgroup* (*in the abstract sense*) *of a Lie group is a Lie group.*

This implies that $\Phi_0(x)$ is a Lie group. The Lie group $\Phi_0(x)$ is a normal subgroup in $\Phi(x)$, and the quotient group $\Phi(x)/\Phi_0(x)$ is countable. This implies that $\Phi(x)$ is a Lie group whose component of the identity is $\Phi_0(x)$.

Now let M^n be a Riemannian manifold with metric g_{ij}.

Definition 4.4.2. The manifold M^n is said to be *reducible* or *irreducible* in accordance with the reducibility or irreducibility of $\Phi(x)$ as a linear group acting on the space T_xM^n.

Also, a similar definition can be certainly given for affine connection manifolds.

Let $T_x^{(0)}$ be the set of elements of T_xM that are fixed with respect to the group $\Phi(x)$. This is the maximal linear subspace of T_xM on which the group $\Phi(x)$ acts trivially. Let $T_x^{\perp}$ be the orthogonal complement to $T_x^{(0)}$ in the space T_xM. It is invariant under the action of the group $\Phi(x)$, and, therefore, we can decompose it into a direct sum $T_x^{\perp} = \sum\limits_{i=1}^{k} T_x^{(i)}$ of mutually orthogonal irreducible subspaces. The decomposition $T_xM = \sum\limits_{i=0}^{k} T_x^{(i)}$ is called the *de Rham decomposition* of the space T_xM (see Fig. 9).

Theorem 4.4.2 ([113]). *Let M be a Riemannian manifold, $T_xM = \sum\limits_{i=0}^{k} T_x^{(i)}$ be the de Rham decomposition of T_xM, and $T^{(i)}$ be an involutive distribution on M obtained by the parallel translation of the space $T_x^{(i)}$ for each $i = 0, 1, \dots, k$. Let $y \in M$, and let M_i be the maximal integral manifold of $T^{(i)}$ passing through the point y. Then the following assertions hold.*

(1) *A point y admits an open neighborhood V such that $V = V_0 \times V_1 \times \dots \times V_k$, where V_i is an open neighborhood of the point y in M_i and the Riemannian metric in V is the direct product of the Riemannian metrics of all neighborhoods V_i.*
(2) *The maximal integral manifold M_0 is locally Euclidean in the sense that each point of M_0 admits a neighborhood that is isometric to an open set of the n_0-dimensional Euclidean space, where $n_0 = \dim M_0$.*
(3) *If M is simply connected, then the holonomy group $\Phi(x)$ is a direct product $\Phi_0(x) \times \Phi_1(x) \times \dots \times \Phi_k(x)$ of normal subgroups, where $\Phi_i(x)$ acts trivially on the space $T_x^{(j)}$ if $i \neq j$ and is irreducible on $T_x^{(i)}$ for each $i = 1, \dots, k$, and $\Phi_0(x)$ consists of the identity only.*
(4) *If M is simply connected, then the canonical decomposition*

$$T_xM = \sum_{i=0}^{k} T_x^{(i)}$$

is unique up to the enumeration.

We can deduce from this result the statement on the global decomposition of a Riemannian manifold M. It is known as the *de Rham decomposition* (see [113]).

Theorem 4.4.3. *A simply connected complete Riemannian manifold is isometric to a direct product $M_0 \times M_1 \times \dots \times M_k$, where M_0 is the Euclidean space (possibly of zero dimension) and $M_1, \dots, M_k$ are simply connected complete irreducible Riemannian manifolds. Such a decomposition is unique up to an enumeration.*

Using the language of irreducibility, we can give a condition under which the groups $I(M)$ and $\mathrm{Aff}(M)$ coincide (see [113]).

Theorem 4.4.4. *If M is a complete irreducible Riemannian manifold, then $I(M) = \mathrm{Aff}(M)$, except for the case where M is the one-dimensional Euclidean space.*

5. Homogeneous Spaces

5.1. Main definitions and constructions. In *Vergleichende Betrachtungen über neuere geometrische Forshungen*,[1] Felix Klein stated the goal of geometry as follows: "Given a manifold and a transformation group acting on it, develop an invariant theory of this group,"[2] and he considered this principle as the general principle of geometry. Therefore, after the appearance of Riemannian manifolds that do not admit a transitive transformation group, the space admitting a transitive transformation group came to be known as *homogeneous spaces* or *Klein spaces*. The term "homogeneous space" was introduced by E. Cartan. We give the following key definitions of the theory of homogeneous spaces. Let X be a topological

[1]It is also known as *Erlangen Programm* (1872).
[2]Translated from the Russian translation of the German edition.

space, and let G be a topological group. We say that G is a *topological transformation group* of the space X if with each element $g \in G$ of the group G, one associates a homeomorphism $f_g : X \to X$ of the space X onto itself, and, moreover, the following three conditions hold:

(1) the mapping $(g, \gamma) \to f_g(\gamma)$ from $G \times X$ into the space X is continuous;
(2) the identity element e of the group G defines the identical homeomorphism of the space X;
(3) the relation $f_{g_1} f_{g_2}(\gamma) = f_{g_1 g_2}(\gamma)$ holds for $g_1, g_2 \in G$ and $\gamma \in X$, i.e., $(g_1 g_2)(\gamma) = g_1(g_2(\gamma))$ and $fg(x) = gx$.

A topological space X on which a group G acts is called a *G-space*.

We say that a group G acts *transitively* on a space X if, for each pair of points $x_1, x_2 \in X$, there exists an element $g \in G$ of the group G such that $g(x_1) = x_2$. If e is a unique element in the group G that leaves each point $x \in X$ fixed, then we say that the group G acts *effectively* on the space X, and G is called an *effective transformation group*. A space X is said to be *homogeneous* if there exists a transformation group G that acts transitively on X. A subgroup in G that leaves a point $x \in X$ fixed is called the *stationary* subgroup of the point $x \in X$. If H_x is the stationary subgroup of the point $x \in X$ and $gx = y$, then the stationary subgroup of the point y is $H_y = gH_xg^{-1}$. Therefore, the stationary subgroups of any two points of a homogeneous space are isomorphic.

The quotient space G/H is an important example of a homogeneous space. Let G be a topological group, H be a closed subgroup of G, and G/H be the set of left cosets of elements of the group G by the subgroup H. On the space G/H, we define a topology by the canonical mapping $\pi : G \to G/H$. Namely, a subset $U \subset G/H$ is open if $\pi^{-1}(U)$ is open in G. The collection of open sets defined in this way defines a certain Hausdorff topology on the space G/H. If, with each element $g \in G$, we associate a transformation $g : xH \to gxH$, then G becomes a transitive topological transformation group that acts on the space G/H, and, therefore, G/H is a homogeneous space. The group G acts effectively on the space G/H iff H does not contain a normal subgroup of the group G.

The presented construction yields a description of an arbitrary homogeneous space. In fact, a more precise statement holds.

Theorem 5.1.1 ([94])**.** *Let G be a locally compact group with a countable base that acts transitively on a locally compact Hausdorff space X. Let $x \in X$ be any point of the space X, and let H_x be the stationary subgroup of the point x. Then H_x is a closed subgroup in G and the mapping $gH \to g(x)$ is a homeomorphism from G/H onto X.*

Let G be a Lie group, and let M be a smooth manifold. In this case, there is reason to speak of a smooth action of the Lie group G on the manifold M. For this purpose, we need to require the smoothness of the mapping $G \times M \to M$ instead of its continuity. In this case, the homogeneous space M also admits a realization in the form of the quotient space G/H (see, e.g., the work [94] of Helgason).

If M is a Riemannian manifold, then to define a homogeneous Riemannian manifold, we have to require that G be a subgroup of the isometry group of the manifold M. In this case, the metric on the manifold M is said to be *G-invariant* (see [94]).

On the space G/H of left cosets of a Lie group G by a subgroup H, the group G acts by left translations. There arises a natural question on the existence of invariant metrics on the homogeneous spaces G/H. An answer to this question is given by the following theorem.

Theorem 5.1.2 ([94])**.** *Each homogeneous space G/H, where G is a Lie group and H is a compact subgroup, admits an invariant Riemannian metric.*

To obtain geometric assertions for homogeneous Riemannian manifolds that are rich in content, one focuses on various special classes of these spaces. There is one remarkable class of Riemannian homogeneous spaces, the symmetric spaces. The symmetric spaces have become traditional in modern mathematics, starting from the works of E. Cartan. Their role is determined by the fact that various problems of geometry, group theory, differential equations, Hamiltonian mechanics, theoretical physics, etc. are often reduced to one or another problem on symmetric spaces. Since the topological and algebraic structures of these spaces are rich, it is convenient to use them for verifying the effectiveness of many modern methods for studying Riemannian manifolds and related objects. We give here only the definition of these spaces; a slightly more detailed description will be given in Chap. 4. Let G be a connected Lie group, and let σ be an involutive automorphism of the Lie group G (i.e., $\sigma^2 = 1$ and $\sigma \neq 1$). Let G_σ be a closed subgroup in G consisting of all points of the group G that are invariant with respect to the automorphism σ, and let $(G_\sigma)_0$ be the connected component of the identity of the group G_σ. We assume that H is a closed subgroup such that $G_\sigma \supset H \supset (G_\sigma)_0$. In this case, we say that the homogeneous space G/H is *symmetric* (defined according to σ). If σ is also an involutive automorphism of the Lie algebra $\mathfrak{G}$ of the group G that is induced by the automorphism σ, then $\mathfrak{G} = \mathfrak{Y} \oplus \mathfrak{R}$, where $\mathfrak{Y} = \{X \in \mathfrak{G} \mid \sigma(X) = X\}$ coincides with the subalgebra corresponding to the subgroup H and $\mathfrak{R} = \{X \in \mathfrak{G} \mid \sigma(X) = -X\}$. In this case, we have the inclusions $[\mathfrak{Y}, \mathfrak{Y}] \subset \mathfrak{Y}$, $[\mathfrak{Y}, \mathfrak{R}] \subset \mathfrak{R}$, and $[\mathfrak{R}, \mathfrak{R}] \subset \mathfrak{Y}$.

5.2. Reductive homogeneous spaces. Let a group G act on an affine connection manifold M. We say that the affine connection on M is invariant with respect to the action of the group G if all transformations are affine. In Sec. 5.1, we solve the problem on the existence of invariant Riemannian metrics on homogeneous spaces. We now consider a similar question on affine connections. For this purpose, we focus on a special class of homogeneous manifolds.

Definition 5.2.1. Let $\mathfrak{G} \supset \mathfrak{Y}$ be the Lie algebras of two Lie groups G and H, respectively. Let we have a subspace $\mathfrak{M} \supset \mathfrak{G}$ such that $\mathfrak{G} = \mathfrak{M} + \mathfrak{Y}$ (a direct sum of vector spaces), and let $\operatorname{ad}_h \mathfrak{M} = \mathfrak{M}$ for each $h \in \mathfrak{Y}$. In this case, we say that the homogeneous space G/H is *reductive* with respect to the decomposition $\mathfrak{G} = \mathfrak{M} + \mathfrak{Y}$. Obviously, we have the inclusion $[\mathfrak{Y}, \mathfrak{M}] \subset \mathfrak{M}$.

Theorem 5.2.1 ([115]). *On any reductive homogeneous space $M = G/H$, there exists a unique G-invariant affine connection such that for each $X \in \mathfrak{M}$ and any vector field Y on the manifold M, we have*

$$(\nabla_{X^*} Y)_0 = [X^*, Y]_0,$$

where X^* denotes the vector field generated by the action of an element $X \in \mathfrak{M}$, i.e.,

$$X^*(p) = \left.\frac{d}{dt}\right|_{t=0} (\exp tX)p.$$

Definition 5.2.2. The affine connection defined in Theorem 5.2.1 is called the *canonical connection* or *Rashevskii connection* on the homogeneous space.

We now present certain important geometric properties of reductive homogeneous spaces. The parallel translation of vectors at a point $O = H$ along the curve $(\exp tX)(0)$, $0 \leq t \leq s$, coincides with the differential of the mapping $\exp(sX) \in G$ acting on the manifold M. For each point $X \in \mathfrak{M}$, the curve $x(t) = (\exp tX)(0)$ is a geodesic. Conversely, each geodesic emanating from the point O has the form $(\exp tX)(0)$ for a certain element $X \in \mathfrak{M}$. The canonical connection on a reductive space is complete. If a tensor field on M is invariant under the action of G, then it is parallel with respect to the canonical connection ∇ (see [115]).

Example. Let $\mathrm{GL}^+(n, \mathbb{R})$ be the Lie group of real matrices of order n with positive determinant, and let $\mathrm{SO}(n)$ be the group of orthogonal matrices with determinant equal to 1. The homogeneous space $\mathrm{GL}^+(n, \mathbb{R})/\mathrm{SO}(n)$ is reductive since we can take the set of all symmetrical matrices as the invariant complement to the Lie algebra $\mathrm{so}(n)$.

We now present a useful reductivity criterion of homogeneous spaces.

Theorem 5.2.2. *If H is a closed subgroup of a connected semisimple Lie group G, then the homogeneous space G/H is reductive.*

In conclusion, we present a description of invariant affine connections on reductive homogeneous spaces.

Theorem 5.2.3 (Nomizu)**.** *Let G/H be a reductive homogeneous space with the decomposition $\mathfrak{G} = \mathfrak{Y} + \mathfrak{M}$, where $\operatorname{Ad}_H \mathfrak{M} = \mathfrak{M}$. There exists a bijective correspondence between the set of all invariant affine connections on G/H and the set of all bilinear mappings $\alpha : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ that are invariant with respect to Ad_H, i.e.,*

$$\operatorname{Ad}_h \alpha(X, Y) = \alpha(\operatorname{Ad}_h X, \operatorname{Ad}_h Y)$$

for all $X, Y \in \mathfrak{M}$ and $h \in H$. The correspondence is given by

$$\alpha(X, Y) = (\nabla_{X^*} Y^*)_{p_0},$$

and α is called the connection function (Nomizu function) on $\mathfrak{M} \times \mathfrak{M}$.

Chapter 2

GEODESICS ON RIEMANNIAN MANIFOLDS

1. Variational Theory of Geodesics

1.1. The action functional and the length functional. We consider a Riemannian manifold M. Let p and $q \in M$ be two fixed points, and let $\gamma : [0, 1] \to M$ be a piecewise-smooth path with $\gamma(0) = p$ and $\gamma(1) = q$, i.e., there exists a partition $0 = t_0 < t_1 < \ldots < t_k = 1$ of the closed interval $[0, 1]$ such that $\gamma\, |_{[t_i, t_{i+1}]}$, $0 \leq i \leq k-1$, is a smooth mapping and γ is continuous as a whole. The set of all such paths is denoted by $\Omega(M, p, q)$. The piecewise smoothness (but not the smoothness) of trajectories $\gamma(t)$, $\gamma(0) = p$, and $\gamma(1) = q$, is technically useful in proving the theorem on decomposition of the space $\Omega(M, p, q)$ into "cells." With each point $\gamma \in \Omega(M, p, q)$, we associate a certain infinite-dimensional linear space $T_\gamma\Omega$, which can be naturally imagined as the "tangent space" to Ω at a "point" $\gamma \in \Omega = \Omega(M, p, q)$.

Definition 1.1.1. The *tangent space* $T_\gamma\Omega$ to Ω at a point γ is the linear space of all piecewise-smooth vector fields v along the path γ such that $v(0) = 0$ and $v(1) = 0$.

A variation with respect to the parameter u, $-\varepsilon \leq u \leq \varepsilon$, of the path γ that leaves two points p and q fixed is a mapping $\tilde{\alpha} : (-\varepsilon, +\varepsilon) \to \Omega$ ($\varepsilon > 0$ is sufficiently small) such that $\tilde{\alpha}(0) = \gamma$ and there exists a partition $0 = t_0 < t_1 < \ldots < t_k = 1$ for which $\alpha(u, t)$ defined by $\alpha(u, t) = \tilde{\alpha}(u)(t)$ is a smooth mapping into M on each band $t_i \leq t \leq t_{i+1}$ (see Fig. 10).

Since we obtain a piecewise-smooth path $\tilde{\alpha}(u)(t)$ for each fixed u, $-\varepsilon \leq u \leq \varepsilon$, we can consider $\tilde{\alpha}$ as a trajectory in the space $\Omega = \Omega(M, p, q)$. Therefore, we can consider the velocity vector $\tilde{\alpha}(u)$ of this trajectory at the point $\gamma = \tilde{\alpha}(0)$. By definition, we set $v = \dfrac{\partial \tilde{\alpha}}{\partial u}(0, t)$; the field $v = v(t)$ is a piecewise-smooth field along $\gamma(t)$ and, therefore (by the definition of the tangent space $T_\gamma\Omega$), belongs to $T_\gamma\Omega$. It is easy to verify the converse statement: if an arbitrary field $v \in T_\gamma\Omega$ is given, then there always exists a trajectory $\tilde{\alpha}(u) \in \Omega$ such that $\dfrac{\partial}{\partial u}\tilde{\alpha}(0, t) = v(t)$. The field $v(t)$ is usually denoted by $\delta\gamma$ in the calculus of variations.

Let $F(\gamma)$ be a real-valued function on the space Ω. We consider a path $\gamma \in \Omega$ and a field $v = \delta\gamma \in T_\gamma\Omega$. We consider the derivative $\dfrac{\partial}{\partial u} F\,(\tilde{\alpha}(u))|_{u=0}$, assuming that this derivative exists. In the specific example of the functionals $F(\gamma)$ we will deal with, the existence of the derivative will be obvious. We note that the above definition of the derivative $\dfrac{\partial}{\partial u} F(\tilde{\alpha}(u))$ is an exact copy of the "finite-dimensional" definition of the directional derivative of a smooth function on a finite-dimensional manifold. Following this ideology further, we give the definition of a critical path for $F(\gamma)$. We say that a path $\gamma(0) \in \Omega$ is *critical* for $F(\gamma)$

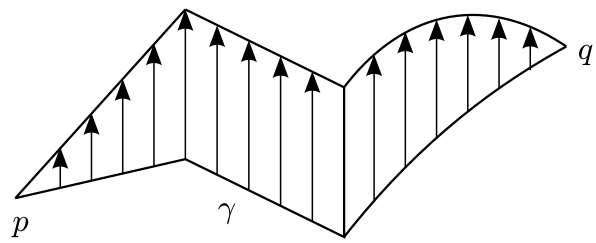


Fig. 10

if

$$\frac{\partial}{\partial u} F(\tilde{\alpha}(u))\Big|_{u=0} \equiv 0$$

for any variation $\tilde{\alpha}(u)$ of the path γ_0 (or the variational derivative $\dfrac{\delta F}{\delta \gamma}$ vanishes).

We now examine the following specific functionals on Ω. These are the action E and the length l of a path γ:

$$E(\gamma) = \int_0^1 \left|\frac{d\gamma}{dt}\right|^2 dt \quad \text{and} \quad l(\gamma) = \int_0^1 \left|\frac{d\gamma}{dt}\right| dt.$$

There is a simple inequality, $l^2 \leq E$, for the functionals E and l, and, moreover, the equality holds here iff $|\dot{\gamma}| = \text{const}$, i.e., in the case where the parameter t is proportional to the arclength (natural parameter) on $\gamma(t)$.

Let $\tilde{\alpha}(u)$ be a variation of the path γ, $v = v(t) = \dfrac{\partial \alpha}{\partial u}(0, t)$ be the vector field $\delta\gamma$ of the variation $\tilde{\alpha}(u)$ (along $\gamma(t)$), $\dot{\gamma}(t)$ be the velocity vector of the trajectory $\gamma(t)$, $a(t) = \nabla_{\dot{\gamma}}(\dot{\gamma})$ be the acceleration vector, and $\Delta\dot{\gamma}(t) = \dot{\gamma}(t^+) - \dot{\gamma}(t^-)$ be a jump of the velocity vector at a point t. Then the following theorem (formula of the first variation) holds.

Theorem 1.1.1 ([17, 71, 113])**.** *We have*

$$\frac{1}{2}\frac{d}{du} E(\tilde{\alpha}(u))\Big|_{u=0} = -\sum_{(t)} \langle v(t), \Delta\dot{\gamma}(t)\rangle - \int_0^1 \langle v(t), a(t)\rangle dt,$$

where $a(t)$ is the variational derivative of the functional and E is a smooth function.

Since the path $\gamma(t)$ is piecewise smooth, we have $\Delta\dot{\gamma}(t) = 0$ for all t, except for finitely many values of t (discontinuity points of the derivative).

The formula of the first variation implies the following statement.

Theorem 1.1.2. *A curve $\gamma_0 \in \Omega$ is a critical point of the functional $E(\gamma)$ iff γ_0 is a geodesic.*

1.2. Jacobi fields. A vector field $v(t)$ along a geodesic γ_0 is called a *Jacobi field* if it satisfies the Jacobi differential equation

$$(\nabla_{\dot{\gamma}_0})^2 v + R(\dot{\gamma}_0, v)\dot{\gamma}_0 = 0.$$

It is convenient to write this equation in coordinates in the following basis: we choose vector fields $l_1(t), \dots, l_n(t)$ parallel along $\gamma_0(t)$ (i.e., $\nabla_{\dot{\gamma}_0} l_\alpha(t) \equiv 0$) that are orthonormal (for each t). Then $v(t) = v^i l_i(t)$, and we obtain

$$\frac{d^2 v^i}{dt^2} + \sum_{j=1}^{n} R^i_j(t) v^j(t) = 0,$$

where

$$R^i_j(t) = \langle R(\dot\gamma_0, l_j)\dot\gamma_0, l_j\rangle.$$

Therefore, a Jacobi field (as a solution to this system) is uniquely defined by the following initial data: $v(0)$ and $\nabla_{\dot\gamma_0} v(0) \in T_{\gamma_0(0)}M^n$.

For a pair of points $A, B \in \gamma_0(t)$, let there exist a nonzero Jacobi field $v(t)$ along $\gamma_0(t)$ such that $v|_A = v|_B = 0$ (i.e., the field $v(t)$ vanishes at the points A and B). Then the points A and B are said to be *conjugate* along the geodesic $\gamma_0(t)$. The *multiplicity* of the pair of conjugate points $A, B \in \gamma_0$ (along γ_0) is the dimension of the linear space of all such Jacobi fields (along γ_0).

We now state a correspondence between the Jacobi fields and the Fermi coordinate systems introduced in Sec. 2.4 of Chap. 1. Let U be an open subset in a Riemannian manifold M; in U, we have the Fermi coordinate system $(x^1, x^2 \dots, x^n)$ with respect to a submanifold $P^q \subset M$ and $p \in U \cap P$. Let $D(U)$ be the Lie algebra of all vector fields on U.

A vector field $A \in D(U)$ is called a *Fermi tangent field* if it has the form

$$A = \sum_{i=1}^{q} c_i \frac{\partial}{\partial x^i},$$

where $c_i = \text{const}$. Similarly, a vector field $X \in D(U)$ of the form

$$X = \sum_{i=q+1}^{n} d_i \frac{\partial}{\partial x^i},$$

where $d_i = \text{const}$, is called a *Fermi normal field*. Let $D(P,p)^T$ and $D(P,p)^\perp$ be the spaces of Fermi tangent fields and Fermi normal fields, respectively. Clearly, $\dim D(P,p)^T = q$ and $\dim D(P,p)^\perp = n - q$, $q = \dim P$.

Let $(x^1, x^2, \dots, x^n)$ be the Fermi coordinate system with respect to a submanifold $P \subset M$. Then we define a function σ^2 by

$$\sigma^2 = \sum_{i=q+1}^{n} x_i^2.$$

The function σ^2 is independent of the choice of the Fermi coordinate system; it depends only on the submanifold $P \subset M$. The following assertion states a correspondence between the Fermi coordinates and the Jacobi fields.

Proposition 1.2.1. (a) *Let γ be a geodesic orthogonal to a submanifold P at a point p, and let $X \in D(P,p)^\perp$ and $A \in D(P,p)^T$. Then the restriction of the fields σX and A to γ are Jacobi fields.*

(b) *Let $(x^1, x^2, \dots, x^n)$ be a normal coordinate system in a neighborhood of a point $m \in M$, and let $\sigma(p)$ be the distance from the point m to a point p. Then the vector fields $\sigma\dfrac{\partial}{\partial x^i}$ are Jacobi fields along any radial geodesic.*

1.3. Hessian of the action functional. We consider a parametric variation $\alpha : U \times [0,1] \to M$, where $U(u_1, u_2)$ is an open neighborhood of the point $(0,0) \in \mathbb{R}^2(u_1, u_2)$, $t \in [0,1]$, $\alpha(0,0,t) = \gamma(t)$, $\dfrac{\partial\alpha}{\partial u_1}(0,0,t) = v_1(t)$, and $\dfrac{\partial\alpha}{\partial u_2}(0,0,t) = v_2(t)$. It is easy to verify that for any pair of fields $v_1, v_2 \in T_\gamma\Omega$, there exists such a variation (see Fig. 11).

Definition 1.3.1. The *Hessian* of the functional E at a critical point $\gamma_0(t) \in \Omega$ is an expression of the form

$$d^2E(v_1, v_2) = \left.\frac{\partial^2 E(\tilde\alpha(u_1, u_2))}{\partial u_1 \partial u_2}\right|_{u_1 = u_2 = 0}.$$

Here $\tilde\alpha(u_1, u_2)(t) = \alpha(u_1, u_2, t)$. We have the following formula for the second variation of the functional E.

Theorem 1.3.1. *Let $\gamma_0 \in \Omega$ be a geodesic (i.e., a critical point of the functional $E(\gamma)$), and let $\tilde{\alpha}(u_1, u_2)$ be a two-parametric variation of the path γ_0, $v_i = \dfrac{\partial \tilde{\alpha}}{\partial u_i}(0,0)$, $i = 1, 2$. Then*

$$\frac{1}{2}\frac{\partial^2 E(\tilde{\alpha})}{\partial u_1 \partial u_2}(0,0) = -\sum_{(i)} \Big\langle v_2(t_i), \Delta(\nabla_{\dot{\gamma}_0} v_1(t_i)) \Big\rangle - \int_0^1 \Big\langle v_2(t), \nabla_{\dot{\gamma}_0} \nabla_{\dot{\gamma}_0} v_1(t) + R(\dot{\gamma}_0, v_1)\dot{\gamma}_0 \Big\rangle dt,$$

where $\Delta(\nabla_{\dot{\gamma}_0} v_1(t)) = \nabla_{\dot{\gamma}_0} v_1(t^+) - \nabla_{\dot{\gamma}_0} v_2(t^-)$ is a jump of the derivative $\nabla_{\dot{\gamma}_0} v_1(t)$ at one of its discontinuity points and R is the curvature tensor.

The geodesics $\gamma_0(t)$ have no break points, and, therefore, we can restrict ourselves to variations $\tilde{\alpha}$ for which $v_1(t)$ and $v_2(t)$ have no break points. Then

$$\frac{1}{2}\frac{\partial^2 E(\tilde{\alpha})}{\partial u_1 \partial u_2}(0,0) = -\int_0^1 \Big\langle v_2(t), \nabla_{\dot{\gamma}_0} \nabla_{\dot{\gamma}_0} v_1 + R(\dot{\gamma}_0, v_1)\dot{\gamma}_0 \Big\rangle dt.$$

We consider $d^2E(v_1, v_2)$; let $W_{\gamma_0} \subset T_{\gamma_0}\Omega$ be the linear subspace in $T_{\gamma_0}\Omega$ that consists of all vector fields v_1 such that $d^2E(v_1, v_2) \equiv 0$ for any $v_2 \in T_{\gamma_0}\Omega$. Sometimes, the subspace W_{γ_0} is called the null-subspace of the Hessian d^2E at the point $\gamma_0 \in \Omega$ or the kernel of the Hessian d^2E. The *degree of degeneration* of the Hessian d^2E is the dimension $\dim W_{\gamma_0}$ (at the critical point $\gamma_0 \in \Omega$).

Theorem 1.3.2. *Let γ_0 be a geodesic on M that connects two points p and q. Then a vector field v belongs to the kernel W_{γ_0} of the Hessian d^2E iff v is a Jacobi field along γ_0 (in particular, $v|_p = v|_q = 0$).*

Therefore, the kernel W_{γ_0} of the Hessian d^2E is different from zero iff the ends p and q of a geodesic γ_0 are conjugate along γ_0. The dimension of the kernel W_{γ_0} (i.e., the degree of degeneracy of the Hessian d^2E) is equal to the multiplicity of the conjugate points p and q along γ_0.

The dimension of the kernel of the Hessian d^2E is always finite since it is equal to the number of linearly independent Jacobi fields along γ_0 (annihilating at the points p and q).

Among the different variations of trajectories γ_0, we highlight the class of so-called *geodesic variations*, i.e., smooth mappings $\alpha : (-\varepsilon, +\varepsilon) \times [0, 1] \to M$ under which $\alpha(0, t) = \gamma_0(t)$ and each trajectory $\tilde{\alpha}(u)$ (we recall that $\tilde{\alpha}(u)(t) = \alpha(u, t)$) is a geodesic (i.e., the perturbed trajectories remain geodesics in the process of perturbation of a geodesic γ). We consider the "velocity vector" of such trajectories $\tilde{\alpha}$ in the space Ω, i.e., the vector field $\dfrac{\partial \alpha}{\partial u}$ along γ_0. A simple calculation shows that this field is a Jacobi field. The converse statement also holds. Any Jacobi field along the geodesic γ_0 can be obtained by using a certain geodesic variation. Indeed, we first suppose that a geodesic γ_0 connects two points p' and q' that are sufficiently close to one another and are located on a certain disk $D^n \subset M^n$ of sufficiently small radius $\varepsilon > 0$. Then we can suppose that any pair of points $\alpha, \beta \in D^n$ is connected by a unique geodesic contained in the domain D^n. We show that there exist Jacobi fields along γ_0 (from p' to q') that have arbitrarily given values at the points p' and q' (see Fig. 12). We consider arbitrary tangent vectors a and b tangent to M at the points p' and q' and construct a Jacobi field along γ_0 with the initial data a at the point p' and b at the point q'. We draw a smooth curve $a(u)$ through the point p' such that $\dfrac{dq(0)}{du} = a$; similarly, we draw a trajectory $b(u)$ through the point q' such that $\dfrac{db(0)}{dt} = b$. We obtain the desired family of geodesics when the points p' and q' are connected by a geodesic (such a geodesic is unique). Changing u, we obtain the desired perturbation of the geodesic γ_0 going from p' to the point q' with the given initial values a and b (see Fig. 13).

The desired Jacobi field along γ_0 going from p' to q' is obtained by the differentiation in the parameter u of the geodesic variation constructed above. Since a Jacobi field is uniquely defined by its values at the points p' and q', any Jacobi field along γ_0 going from the point p' to the point q' can be obtained by the

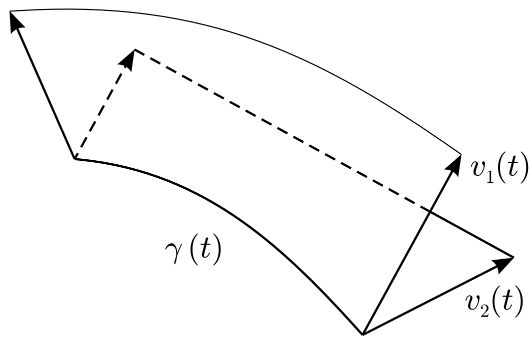


Fig. 11

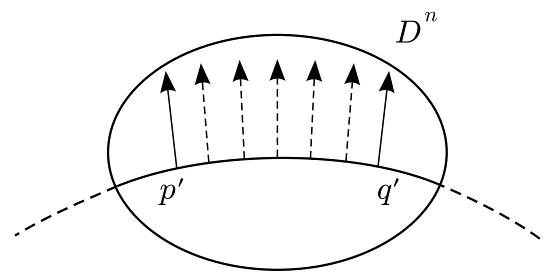


Fig. 12

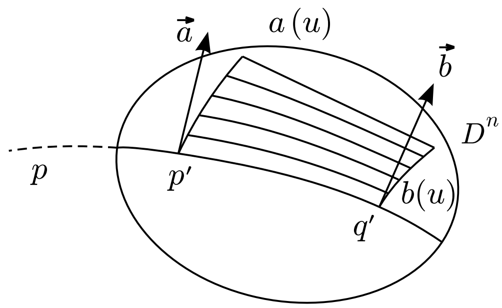


Fig. 13

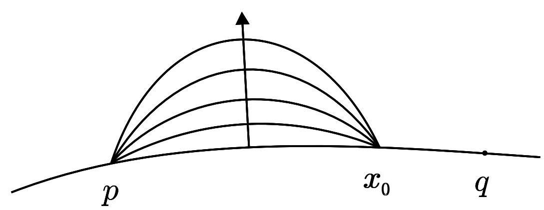


Fig. 14

above method. We note that the linear space of all Jacobi fields along γ_0 going from p' to q' is isomorphic to the $2n$-dimensional linear space $T_{p'}M^n \times T_{q'}M^n$. A more general statement holds: a Jacobi field along a geodesic γ_0 going from a point p to a point q (where the points p and q are not necessarily close to one another) is uniquely define by its values at two other nonconjugate points (along γ_0).

We now show the existence of a geodesic variation generating a given Jacobi field v on the whole geodesic γ_0 going from p to q. For this purpose, we consider a pair of points $p', q' \in \gamma_0$ that are located inside a sufficiently small ball D^n and define the vectors $a = v|_{p'}$ and $b = v|_{q'}$ at the points p' and q'. Further, we construct a geodesic variation that generates a Jacobi field v along γ_0 going from the point p' to the point q' and extend the constructed family of geodesics outside the disk D^n, which yields the desired geodesic variation on the whole geodesic γ_0.

Definition 1.3.2. The *index* of a bilinear functional H on a vector space V is the maximum dimension of a subspace $W \subset V$ on which the functional H is negative definite.

There is an important connection between points conjugate along γ_0 and the properties of the Hessian d^2E, which is more precisely described in the following theorem.

Theorem 1.3.3. *The index of the quadratic form d^2E at a critical point $\gamma_0 \in \Omega$ is equal to the number of points on the geodesic $\gamma_0(t)$, $0 < t < 1$, that are conjugate to the initial point $p = \gamma_0(0)$ along $\gamma_0(t)$ (each point of $\gamma_0(t)$ conjugate to $\gamma_0(0) = p$ is counted with its multiplicity). The index $\lambda = \lambda(\gamma_0)$ of the quadratic form d^2E is always finite.*

If two points p and q are not conjugate along γ_0, then we can consider the whole trajectory $\gamma_0(t)$, $0 \le t \le 1$. In this case, $\ker d^2E = 0$ and $\gamma_0 \in \Omega$ is a nondegenerate critical point of index λ.

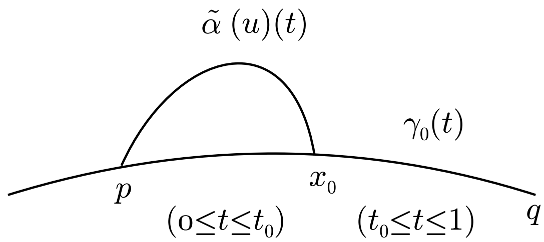


Fig. 15

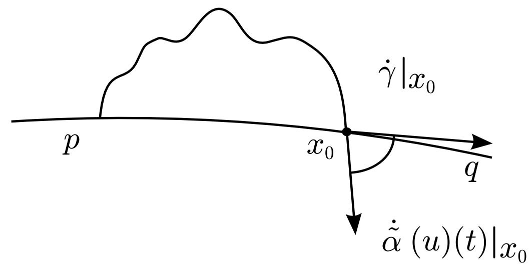


Fig. 16

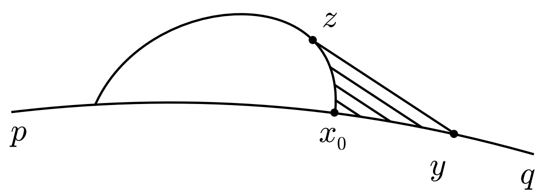


Fig. 17

In particular, this theorem implies that each segment of a geodesic γ_0 contains only finitely many points conjugate to the point $p = \gamma_0(0)$.

Because of the importance of this theorem, we present a clear explanation that shows that conjugate points define variations $\tilde{\alpha}(u)$ in the space Ω along which the quadratic part of the functional E decreases. For a formal proof, see [60, 71, 113, 129, 159].

Let $x_0 \in \gamma_0$ be a point conjugate to $p = \gamma_0(0)$ along $\gamma_0(t)$. Then, along the segment $[p, x_0]$ of the geodesic γ_0, there exist $\lambda(x_0)$ Jacobi fields ($\lambda(x_0) \geq 1$) that annihilate at the points p and x_0 (these fields can certainly vanish at some interior points of the segment $[p, x_0]$). We consider a geodesic variation $\tilde{\alpha}(u)$ of the segment $[p, x_0]$ in the direction of a certain Jacobi field along $[p, x_0]$ that annihilates at p and x_0. This means that there exists an infinitely small "rotation" of the geodesic $[p, x_0]$ that leaves the points p and x_0 fixed (see Fig. 14).

We consider geodesics $\tilde{\alpha}(u)(t)$, $0 \leq t \leq t_0$, that define this geodesic variation, where t_0 corresponds to the point $x_0 \in \gamma_0$. Then we can consider the smooth path $\tilde{\varphi}(u)$ in the space Ω defined as follows: $\tilde{\varphi}(u)(t) = \tilde{\alpha}(u)(t)$ for $0 \leq t \leq t_0$ and $\tilde{\varphi}(u)(t) = \gamma_0(t)$ for $t_0 \leq t \leq 1$ (see Fig. 15).

By the choice of $\tilde{\varphi}(u)$, we can assume in the first approximation that the length of the curve γ_0 going from p to q is equal to the length of $\tilde{\alpha}(u)$ going from p to x_0 plus the length of γ_0 going from x_0 to q, i.e., we can assume that the functional E is not changed under a sufficiently small displacement along the trajectory $\tilde{\varphi}(u)$, $0 \leq u \leq \varepsilon$. Since a Jacobi field is completely defined by its initial data, the angle between the velocity vector of the trajectory γ_0 and the trajectory $\tilde{\alpha}(u)(t)$ at the point x_0 is different from zero (see Fig. 16).

We now construct a new trajectory $\tilde{\psi}(u)$ in the space Ω emanating from the point γ_0 along which the quadratic part of the functional E strictly decreases, i.e., the velocity vector $\dot{\tilde{\psi}}(u)\Big|_{u=0}$ belongs to a subspace on which the Hessian d^2E is negative definite. The construction of the variation $\tilde{\psi}(u)$ is shown in Fig. 17.

Since the strict inequality $\text{length}(x_0, y) + \text{length}(x_0, z) > \text{length}(z, y)$ holds in a sufficiently small triangle x_0yz, we see that the length of the trajectory $\tilde{\psi}(u)(t)$ ($\tilde{\psi} = (pz) + (zy) + (yq)$) is also strictly less than the length of $\tilde{\varphi}(u)(t)$, i.e., the length of γ_0 (from p to q). (Of course, we used here the positive definiteness of the Riemannian metric.) Therefore, each Jacobi field on the segment px_0 that annihilates at the points p and x_0 yields the unit contribution to the index of the Hessian d^2E at the point γ_0.

1.4. Applications of the index theorem. We consider the action functional $E(\gamma)$, where $\gamma \in \Omega M$. This functional is a "Morse function" if all its critical points (i.e., geodesics going from a point p to a point q) are nondegenerate. This is the case iff the points p and q are not conjugate to one another along any geodesic connecting p and q. Further, at each critical point $\gamma_0 \in \Omega M$ of the functional E, there arises an integer, the Morse index of this critical point, i.e., the *index of the geodesic* γ_0 going from the point p to the point q. Therefore, we can expect that at each critical point (i.e., a geodesic γ_0), one cell of dimension equal to the index of this critical point (i.e., the index of the geodesic γ_0) "is suspended." Therefore, there arises a cell partition of the space ΩM into cells whose number and dimension are defined by those of the geodesics connecting the points p and q (if p and q are not conjugate).

Theorem 1.4.1. *Let M^n be a compact (or complete) Riemannian manifold, and let p and q be a pair of points in M^n that are not conjugate along any geodesic. Then the space $\Omega(M^n, p, q)$ is of a cell complex homotopy type such that exactly one cell of dimension λ corresponds to each geodesic going from the point p to the point q and whose index is λ.*

Remark. If a geodesic γ_0 is fixed, then the corresponding cell σ^λ (λ is the index of γ_0) arises as the set of trajectories obtained from γ_0 by perturbation of γ_0 in the direction of all Jacobi fields along γ_0.

2. Periodic Problem of the Calculus of Variations

2.1. Statement of the periodic problem. We consider a compact smooth Riemannian manifold M^n. Let $\Pi(M^n)$ be the space of all closed smooth curves on M^n, i.e., a point of the space $\Pi(M^n)$ is a smooth mapping $\gamma : S^1 \to M^n$, where $S^1 = S^1(t)$, $0 \le t \le 2\pi$, is the circle parametrized by the standard angle coordinate t, and the initial point is not fixed in this case.

The space $\Pi(M^n)$, as well as the space $\Omega(M^n, p, q)$, can be naturally transformed into an "infinite-dimensional manifold"; if $\gamma \in \Pi(M^n)$ is a closed trajectory (by the term "trajectory," we mean a trajectory with a parametrization), the "tangent space" $T_\gamma\Pi(M)$ to the "manifold" $\Pi(M)$ at the point $\gamma \in \Pi(M)$ consists of all smooth vector fields along γ (i.e., periodic vector fields). On the space $\Pi(M)$, both functionals $l(\gamma)$ and $E(\gamma)$ (the length of a path and the action of this path) are defined exactly in the same way as in the case of the space $\Omega(M^n, p, q)$. We describe extremals of the functionals $l(\gamma)$ and $E(\gamma)$. If $\gamma_0 \in \Pi(M)$ is a closed extremal of the functional E, then γ_0 is a closed geodesic related to the parameter, which is proportional to the natural parameter. If $\gamma(t)$ is a periodic extremal for the functional $l(\gamma)$ of the length of γ, then all trajectories $\gamma(t')$ that are obtained from $\gamma(t)$ by smooth changes of the parameter $t \to t'$ are also extremals of the functional l. Therefore, "critical points" of the functional l are not isolated in the space $\Pi(M)$; in particular, in any sense, they cannot be "isolated and nondegenerate" critical points of the functional $l(\gamma)$. Therefore (as in the case of the space $\Omega(M, p, q)$), we focus our main attention on the study of extremals of the functional E. We note that a closed geodesic $\gamma_0(t) \in \Pi(M)$ can be multiple in the sense that when t varies from 0 to 1, the set $\{\gamma(t)\} \subset M^n$, which is a smooth curve, is going around several times. Geodesics $\gamma(t)$ depicted by a smooth curve in M^n that is going around one time are said to be simple (of multiplicity 1) geodesics.

Conversely, if a certain simple closed geodesic is given, then it defines an infinite discrete sequence of closed geodesics that are obtained from it by a multiple going around (with greater velocities than the velocity of the initial geodesic). All these trajectories are distinct points of the space $\Pi(M)$. For example, if the initial trajectory $\gamma_0(t)$ defines a nonzero element of the fundamental group $\pi_1(M)$ (more precisely, its coset is different from the identity element), then the trajectories that are multiple to it and are described above belong to other cosets of the group $\pi_1(M)$.

2.2. Hessian of the action functional. As in the case of geodesics with fixed ends, it is natural to associate a certain integer with each closed geodesic; by analogy with the previous case, it is also called the *degree of degeneration* of a geodesic. We now give its definition. If the degree of degeneration equals zero, then a geodesic is said to be *nondegenerate*.

In the previous section, we have obtained the following formula for the second variation:

$$\frac{1}{2}\frac{\partial^2 E(\tilde{\alpha})}{\partial u_1 \partial u_2}(0,0) = -\int_0^1 \Big\langle v_2, \nabla_{\dot{\gamma}_0}\nabla_{\dot{\gamma}_0} v_1 + R(\dot{\gamma}_0, \dot{\gamma}_1)\dot{\gamma}_0 \Big\rangle dv,$$

where R is the Riemannian curvature tensor, $\dot{\gamma}_0$ is the velocity vector of a geodesic γ_0, and the vector fields v_1 and v_2 describe a two-parameter variation, i.e., a pair of "tangent vectors" to the infinite-dimensional manifold $\Pi(M)$ at the point γ_0. As was noted, the vector fields v_1 and v_2 are defined along the whole trajectory γ_0 and are smooth and periodic. Since the Hessian d^2E defines a bilinear symmetric form on the tangent space $T_{\gamma_0}(\Pi(M))$, we can uniquely define this form by the linear differential operator corresponding to it, which, obviously, has the form

$$D = -(\nabla_{\dot{\gamma}_0})^2 - R(\dot{\gamma}_0,)\dot{\gamma}_0.$$

Here we proceed similarly to the finite-dimensional case, where "to define a bilinear form" means "to define a linear operator D such that the desired form B is defined by $B(x,y) = \langle x, Dy \rangle$." In our case, the action of the operator D on the "tangent vectors" $v \in T_{\dot{\gamma}_0}(\Pi M)$ (i.e., smooth periodic vector fields along the closed geodesic γ_0) is given by

$$D(v) = -(\nabla_{\dot{\gamma}_0})^2 v - R(\dot{\gamma}_0, v)\dot{\gamma}_0 = -\Big[(\nabla_{\dot{\gamma}_0})^2 + R(\dot{\gamma}_0, v)\dot{\gamma}_0\Big](v).$$

A "tangent vector" v, i.e., a periodic vector field, is called a *Jacobi field* if it is annihilated by the operator D, i.e., if it is a solution to the differential equation

$$D(v) = -(\nabla_{\dot{\gamma}_0})^2 v - R(\dot{\gamma}_0, v)\dot{\gamma}_0 = 0.$$

Therefore, Jacobi fields are elements of the kernel of the linear operator D acting on the tangent space $T_{\dot{\gamma}_0}(\Pi M)$.

Definition 2.2.1. The *degree of degeneration* of a closed geodesic γ_0 is the dimension of the kernel of the operator D. A closed geodesic is said to be *nondegenerate* if its degree of degeneration equals zero.

For simplicity, we mainly restrict ourselves to the consideration of closed nondegenerate geodesics. It turns out that an integer called the "index of a geodesic" is naturally associated with each such geodesic. To define this number, we consider the operator D again. The index can be defined in several ways. Indeed, since the index is equal to the number of negative squares of the Hessian after its reduction to the canonical form on the tangent space $T_{\gamma_0}(\Pi M)$, we see that along each "tangent" vector $v \in T_{\gamma_0}(\Pi M)$ corresponding to one of the negative squares of the form d^2E, this form is negative definite, and hence this "tangent vector" is an eigenvector of the operator D corresponding to the eigenvalue $\lambda < 0$. Therefore, the index of the Hessian d^2E can be merely defined as the number of linearly independent solutions to the differential equation $D(v) = \lambda v$, $\lambda < 0$ (this is a set of differential equations with the parameter λ, which is an eigenvalue). Therefore, the solutions to the equation $D(v) = \lambda v$, $\lambda < 0$, are smooth periodic vector fields along the geodesic γ_0 (of course, if these solutions exist at all). This case is different from the case of Jacobi "tangent vectors" where at least the zero solution to the homogeneous system always exists. In the case $\lambda < 0$, a solution may not exist: in this case, we say that the index of a closed geodesic equals zero.

Definition 2.2.2. The *index of a nondegenerate closed geodesic* is the number of linearly independent solutions to the set of differential equations

$$D(v) = -(\nabla_{\dot{\gamma}_0})^2 v - R(\dot{\gamma}_0, v)\dot{\gamma}_0 = 0.$$

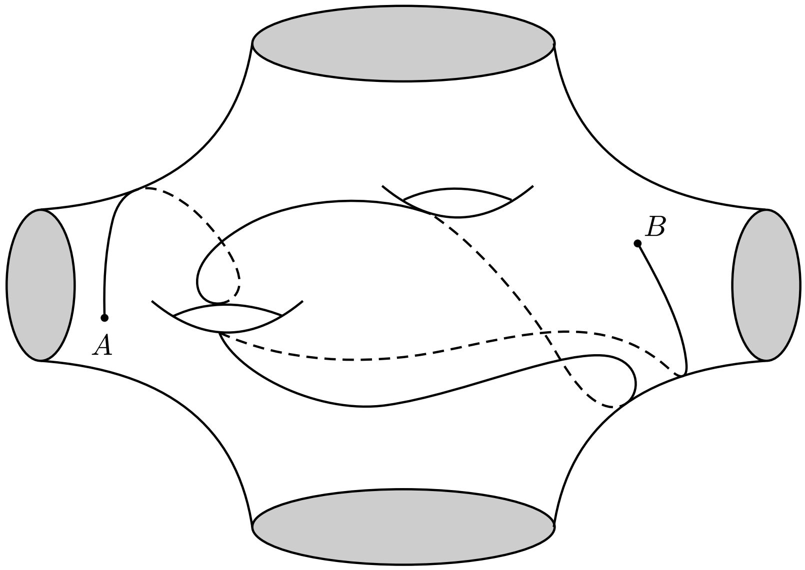


Fig. 18

In the case of geodesics with fixed ends, this definition is also applicable.

The index of a closed geodesic is related to the distribution of points conjugate to a chosen initial point along this geodesic; however, this relation is of a more complex character than that in the case of geodesics with fixed ends, and we do not go into it in detail here. In some sense, the study of the "periodic problem of the calculus of variations" is more complex than the study of geodesics with fixed ends. The character of the difficulties that occur is sufficiently illustrated by the existence of multiple geodesics; for example, the problem of finding the number of simple geodesics is far from trivial.

2.3. Some applications. To simplify the problem of studying closed geodesics, we examine here one example: the case of Riemannian manifolds of negative curvature, i.e., manifolds on which all curvatures with respect to all two-dimensional directions are negative.

Theorem 2.3.1. *Let M be a compact smooth Riemannian manifold of negative curvature. Then, in each free one-dimensional homotopy class, there exists a unique closed geodesic.*

To prove this, we consider a certain fixed class of free closed loops that are homotopic to each other. Let C be the greatest lower bound of all values of the functional on this homotopy class; there exists an infinite sequence of closed loops whose lengths converge to C. By the compactness of the manifold, we can extract from this sequence a sequence of curves that pointwise converge to a certain smooth curve γ_0 that is a closed geodesic.

The proof of the uniqueness is implied by the following statement, which itself is important (see [60]).

Let γ_0 be a closed geodesic on a manifold M of negative curvature (we can assume that the manifold is not compact). Then this geodesic is nondegenerate and its index equals zero, in other words, the differential equations $D(v) = \lambda v$, $\lambda < 0$, have no solutions, and the differential equation $D(v) = 0$ has only the zero solution.

We can deduce from this statement the following important corollary.

Theorem 2.3.2. *Let M be a smooth manifold of negative curvature with respect to all two-dimensional directions. Then there are no two points of the manifold that are conjugate along some geodesic.*

The behavior of geodesics on manifolds of negative curvature was studied by Hadamard and then by Morse. They considered geodesics on surfaces in $\mathbb{R}^3$ that approximately have the form depicted in Fig. 18. More precisely, these surfaces are characterized by the following properties:

(a) the Gaussian curvature of such a surface is everywhere negative (strictly speaking, the vanishing of the curvature is also supposed, but this problem is not studied in detail);
(b) such a surface has $a \geq 2$ expanding "funnels" going to infinity;
(c) such a surface is homeomorphic to the sphere with p handles and a holes, where $2p + a \geq 3$.

First, Hadamard proved the existence of such surfaces in $\mathbb{R}^3$, and then he found the fundamental property of geodesics on surfaces of negative curvature: in a given homotopy class of curves connecting two fixed points A and B (see Fig. 18), there exists exactly one geodesic, and the minimum of the length is attained at it. Starting from this fact, Hadamard proved the existence of infinitely many closed geodesics and also the possibility of approximating any other geodesic remaining in a finite part of the surface by them.

3. Comparison Theorems

In studying the relations between a connection and topological properties of complete Riemannian manifolds, it turns out that comparison theorems of the corresponding differential-geometric or topological properties of two Riemannian manifolds are useful. Choosing the standard space, e.g., one of the spaces of constant curvature as one of them, we apply various constructions in these well-studied spaces and compare them with similar constructions in the space under study. The results obtained in this way are called the *comparison theorems*. They have come to be of independent interest in Riemannian geometry (see [71, 106]).

Here we present the Morse–Shoenberg comparison theorem for indices, the Rauch theorem for comparison of Jacobi fields, the very strong Toponogov theorem on the comparison of angles of a triangle in a complete Riemannian manifold of negative curvature with angles of a triangle in the sphere in the Euclidean space with the same lengths of its sides, and, finally, the Berger comparison theorem.

We recall the main definitions that will be needed in what follows. Let $c : [\alpha, \beta] \to M$ be a certain path in a Riemannian manifold M. It is said to be *normal* if $|\dot{c}(t)| = 1$ for all $t \in [\alpha, \beta]$. The length of a path c is denoted by $l(c)$. The *index* (resp. *quasi-index*) of a quadratic form H on a linear space V is the least upper bound of dimensions of all subspaces in V on which H is negative definite (resp. negative semi-definite). For a normal geodesic $c : [\alpha, \beta] \to M$, we consider the space B_c of all piecewise-smooth fields Y along c that are orthogonal to $\dot{c}$ and satisfy the conditions $Y(\alpha) = Y(\beta) = 0$. The index (quasi-index) of the quadratic form $I : B_c \times B_c \to \mathbb{R}$ defined by

$$I(X, Y) = \int_{\alpha}^{\beta} \Big(\langle X', Y' \rangle - \langle R(X, \dot{c})\dot{c}, Y \rangle \Big)\, dt$$

($\langle X, Y \rangle$ is the inner product on the manifold M) is called the *index* (*quasi-index*) *of the geodesic* c. The index of the geodesic c is denoted by $\operatorname{Ind} c$, and its quasi-index is denoted by $\operatorname{Ind}_0 c$ ($\operatorname{Ind} c \leq \operatorname{Ind}_0 c$). For more details, see Sec. 1.

Example. For a normal geodesic $c : [0, \beta] \to S^n_\rho$ on the sphere S^n_ρ, we have $\operatorname{Ind} c = \nu(n-1)$ for $\nu\pi\rho < \beta < (\nu+1)\pi\rho$, and $\operatorname{Ind}_0 c = \nu(n-1)$ for $\nu\pi\rho \leq \beta < (\nu+1)\pi\rho$, where $\nu \geq 0$ is an integer.

In the Euclidean space $\mathbb{R}^n$, all geodesics are of zero index and zero quasi-index.

Theorem 3.1 (Morse and Shoenberg)**.** *Let M be a Riemannian manifold of dimension n, and let $c : [0, \beta] \to M$ be a normal geodesic. Then the following assertions hold.*

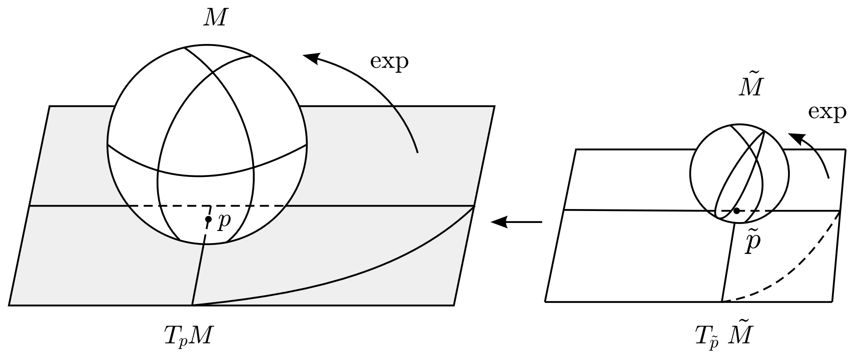


Fig. 19

(a) *If* $K(\sigma) \leq \lambda$ $(\lambda > 0)$ *for all two-dimensional sections* σ *and* $l(c) < (\nu+1)\pi/\sqrt{\lambda}$ (*respectively,* $l(c) \leq (\nu+1)\pi/\sqrt{\lambda}$), *where* $\nu \geq 0$ *is an integer, then* $\operatorname{Ind} c \leq \operatorname{Ind}_0 c \leq \nu(n-1)$ (*respectively,* $\operatorname{Ind} c \leq \nu(n-1)$); *if* $l(c) < \pi/\sqrt{\lambda}$, *then there are no conjugate points on c.*

(b) *If, for all two-dimensional sections* σ, *we have the inequality* $K(\sigma) \leq 0$, *then there are no conjugate points on c and* $\operatorname{Ind} c = \operatorname{Ind}_0 c = 0$.

(c) *If* $K(\sigma) \geq \varkappa > 0$ *for all two-dimensional sections* σ *and* $l(c) \geq \nu\pi/\sqrt{\varkappa}$ (*respectively,* $l(c) > \nu\pi/\sqrt{\varkappa}$), *where* $\nu \geq 1$ *is an integer, then* $\operatorname{Ind}_0 c \geq \nu(n-1)$ (*respectively,* $\operatorname{Ind}_0 c \geq \operatorname{Ind} c \geq \nu(n-1)$), *and, moreover, there is at least one conjugate point on c lying in the semi-open interval* $(0,\beta]$ (*respectively,* $[0,\beta)$).

For the proof of this theorem, see [71].

The classical Sturm theorem for ordinary differential equations of the second order admits a generalization to an arbitrary Riemannian manifold; this is the so-called Rauch comparison theorem.

Theorem 3.2 (Rauch [164])**.** *Let* M *and* M' *be two Riemannian manifolds of the same dimension,* $c : [0,\beta] \to M$ *and* $\tilde{c} : [0,\beta] \to \tilde{M}$ *be normal geodesics,* Y *and* $\tilde{Y}$ *be Jacobi fields along c and* $\tilde{c}$ *satisfying the conditions* $Y(0) = 0$, $\tilde{Y}(0) = 0$, $\langle Y', \dot{c}\rangle|_0 = \langle \tilde{Y}', \dot{\tilde{c}}\rangle|_0 = 0$, *and* $|Y'(0)| = |\tilde{Y}'(0)|$, *and* $\tilde{c}$ *have no conjugate points in* $(0,\beta)$. *If the curvature of the manifold* M *along c does not exceed the curvature of the manifold* $\tilde{M}$ *along* $\tilde{c}$, *i.e.,* $K(\sigma) \leq 0K(\tilde{\sigma})$ *for all two-dimensional sections* σ *tangent to* M *at the point* $c(t)$ *and for all two-dimensional sections* $\tilde{\sigma}$ *tangent to* $\tilde{M}$ *at the point* $\tilde{c}(t)$ $(t \in [0,\beta])$, *then* $|Y(t)| \geq |\tilde{Y}(t)|$ *for* $t \in [0,\beta]$.

The geometric sense of the Rauch comparison theorem consists of the fact that if we identify the tangent spaces T_pM and $T_{\tilde{p}}\tilde{M}$ by a linear isomorphism, then one and the same path on T_pM passes to a longer path on a submanifold whose curvature is less under the mapping $\exp_p$ (see Fig. 19).

The following important theorem on angles of a triangle is proved by Toponogov.

Theorem 3.3. *Let* M^n, $n \geq 3$, *be a complete* (*not necessarily simply connected*) *Riemannian manifold with* $K(\sigma) \geq c$, *and let* S_c *be a simply connected two-dimensional space of constant curvature c. Consider a triangle* $\triangle$ *in* M^n (*three points and three minimal arcs*). *Then there exists a triangle* $\tilde{\triangle}$ *in* S_c *with the same lengths of sides such that the angles of the triangle* $\triangle$ *are not less than the corresponding angles of the triangle* $\tilde{\triangle}$.

In conclusion, we consider a generalized comparison theorem due to Berger, which is based on the Rauch comparison theorem (see [20]). The Rauch comparison theorem deals with one-parameter families of geodesics in a Riemannian manifold M that emanate from the same point $p \in M$. We now consider a family of geodesics whose initial point moves along a certain geodesic $\gamma_0(t)$.

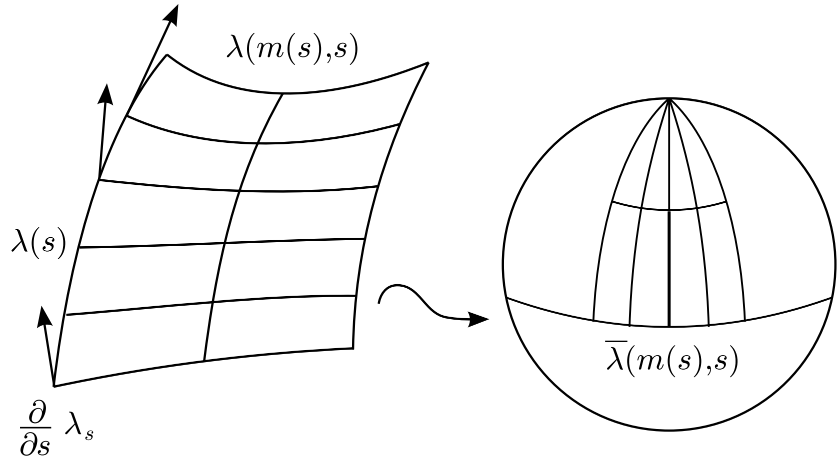


Fig. 20

Theorem 3.4 (Berger). *In a Riemannian manifold M^n with $\delta \le K(\sigma) \le 1$, we consider the family $\lambda(t,s)$, $0 \le s \le l$, $0 \le t \le t_0 \le \frac{\pi}{2}$, of normal geodesics $\lambda_s(t) = \lambda(t,s)$ such that the curve $\gamma(s) = \lambda(0,s)$ is a normal geodesic and the vector field $\frac{d}{ds}\lambda_s$ is parallel along $\gamma(s)$. Denote the similar constructions on a complete simply connected surface of constant curvature δ by the same letters with a bar. Moreover, assume that*

$$\left\langle \frac{d}{ds}\lambda_s, \gamma_s' \right\rangle = \left\langle \frac{d}{ds}\bar{\lambda}_s, \bar{\gamma}_s' \right\rangle$$

(*see Fig.* 20)*. Then, for any function m, $0 \le m(s) \le \frac{\pi}{2}$, the length of the curve $\lambda(m(s),s)$ is not greater than the length of the curve $\bar{\lambda}(m(s),s)$.*

Using his generalized comparison theorem [20], Berger gave a new proof of the Toponogov comparison theorem.

4. Manifolds with Various Restrictions on the Minimum Loci

4.1. Minimum locus. Let M^n be a compact connected n-dimensional Riemannian manifold, and let TM and UM be its tangent bundle and the unit spherical bundle, respectively. Let $\rho_M(x,y)$ be the distance function on M. Let

$$B_r(x,M) = \{y \in M \mid (\rho_M(x,y) < r\}, \quad \bar{B}_r(x,M) = \{y \in M \mid \rho_M(x,y) \le r\},$$

and let K_M be the sectional curvature of the Riemannian manifold M. We assume that geodesics are parametrized by the natural parameter: $|\dot{\gamma}'(t)| = 1$, i.e., normal geodesics are considered.

For a given vector $v \in UM$, the c-value of the manifold M in the direction v is the number $c_p(v)$:

$$c_p(v) = \max\{\lambda \in \mathbb{R} \mid \lambda > 0, \quad \rho(p, \exp_p \lambda v) = \lambda\}.$$

The set

$$A_p = \{v \in T_pM \mid \rho(p, \exp_p v) = |v|\}$$

is called the *fundamental domain* for a point $p \in M$. The boundary $\tilde{c}_p = \partial A_p$ of the fundamental domain A_p is called the *tangent minimum locus*. The *minimum locus* c_p of a point p is the set $c_p = \exp_p \tilde{c}_p$. Sometimes, the minimum locus is also called the *separation set* or the *cut locus*. A minimal geodesic segment or a minimal arc is a geodesic segment that minimizes the arclength between its ends. A minimum point of a point p along a geodesic γ is a point m on γ such that the segment of the curve γ

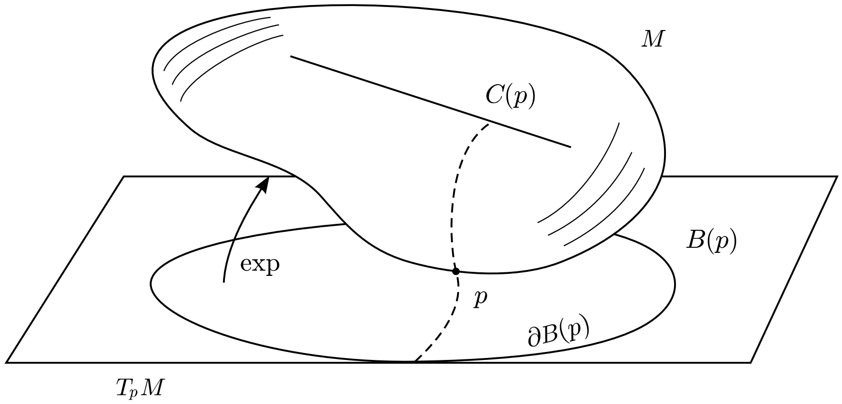


Fig. 21

from p to m is minimal, but any larger segment is no longer minimal. The set of all minimum points is the minimum (cut) locus c_p of the point p.

Since geodesics do not minimize an arclength beyond the first conjugate point, we see that if m is the first point conjugate to p along a geodesic γ, then a minimum point p along γ is located before m. The definition of conjugate points is given in Sec. 1, and the first conjugate point is also defined in Sec. 2.4 of Chap. 1 in terms of the exponential mapping.

A geometric ray emanating from a point p contains at most one minimum point of the point p, although it can contain no such points at all. We mention the following properties. If a point m is not a minimum point of a point p, then there exists no more than one minimal segment γ connecting p with m. If there exists a minimal segment going from the point p to m on which the point m is conjugate to p, then m is a minimum point of the point p. In the case where M is a complete Riemannian manifold, the converse statement holds: if m is a minimum point for p, then either there are two minimal segments or the point m is conjugate to p along a unique segment. A Riemannian manifold M is compact iff for a certain point p there exist its minimum points in any direction. The subsets ∂A_p, A_p, and $\operatorname{int} A_p$ are homeomorphic to S^{n-1}, the n-dimensional closed disk $\bar{D}^n$, and the open disk D^n, respectively.

The number

$$\min\{c_p(v) \mid v \in U_pM\}$$

is called the injectivity radius of a manifold M at a point p and is denoted by i_p; $\min\{i_p \mid p \in M\}$ is called the *injectivity radius* of the manifold M. We denote by d_M the diameter of a manifold M, i.e.,

$$d_M = \max\{d_p \mid p \in M\},$$

where

$$d_p = \max\{c_p(v) \mid v \in U_pM\}.$$

The minimum locus is said to be *spherical* if $i_p = d_p$. The *link* of points $p, q \in M$ is the subset

$$\Lambda(p,q) = \{v \in U_qM \mid \exp_q(\rho(p,q)v) = p\}.$$

A compact Riemannian manifold M is called a *Blaschke manifold at a point* $p \in M$ if, for any point $q \in c_p$, the link $\Lambda(p,q)$ is the intersection of the fiber U_qM with a certain subspace of the space T_qM. A manifold M is called a *Blaschke manifold* if it is a Blaschke manifold at each of its points.

In other words, a Riemannian manifold M^n $(n \geq 1)$ is called a Blaschke manifold if there exists a number $l > 0$ such that for any point $m \in M$, the exponential mapping $\exp : T_mM \to M$ is a diffeomorphism of the interior $\operatorname{int} D$ of the disk D of radius l centered at zero $0 \in T_mM$ and the restriction $\exp|_{\partial D}$ is a principal bundle in r-dimensional spheres (for a certain integer $r \geq 0$) (see Fig. 21 and [152]).

The following list of manifolds gives examples of the so-called canonical Blaschke manifolds:

(a) the unit n-dimensional sphere S^n;
(b) the real n-dimensional projective space $\mathbb{R}P^n$;
(c) the complex n-dimensional projective space $\mathbb{C}P^n$;
(d) the quaternion n-dimensional projective space $\mathbb{H}P^n$;
(e) the Cayley projective plane $\mathbb{C}\mathrm{a}P^2$.

In the first case, $l = \pi$ and $r = n-1$, and in the other cases, $l = \dfrac{\pi}{2}$ and $r = 0, 1, 3, 7$, respectively.

A Riemannian manifold M is a Blaschke manifold if the distance from a point p to the minimum locus c_p along a geodesic does not depend on the choice of the geodesic and the point. The minimum locus of an arbitrary point of the manifold M is either a point or a smooth submanifold of dimension $n-1$, $n-2$, and $n-4$, or, if $n = 16$, then $\dim c_p = 8$. In these cases, we say that M^n is modeled on the sphere, real projective space, complex projective space, quaternion projective space, or Cayley projective plane, respectively.

There arises a natural problem of constructing the Blaschke manifolds. The following regular process was suggested by Weinstein.

Theorem 4.1.1 ([21]). *Let M^n be a manifold of the form $M^n = \bar{D}^n \underset{a}{\cup} E$, where $\bar{D}^n$ is an n-dimensional closed disk, E is the total space of the bundle in k-dimensional disks over an $(n-k)$-dimensional compact manifold, and, moreover, $\partial E \cong S^{n-1}$ with a gluing diffeomorphism $a : \partial \bar{D}^n \to \partial E$. Then we can define a metric on M such that M is a Blaschke manifold at the point p, which is the center of the disk $\bar{D}^n$.*

We consider the properties of Blaschke manifolds. We begin with the following two fundamental properties of Blaschke manifolds.

(1) All geodesics on a Blaschke manifold are closed and have the same length, equal to $2l$.
(2) An arbitrary Blaschke manifold has the same cohomology ring as the canonical Blaschke manifolds.

This assertion admits a converse in a certain sense. If a Blaschke manifold M has the same cohomology ring as S^n or $\mathbb{R}P^n$, then M is homeomorphic to the sphere S^n or the projective space $\mathbb{R}P^n$; if $H^*(M, \mathbb{Z}) = H^*(\mathbb{C}P^n, \mathbb{Z})$, then M is homotopy equivalent to $\mathbb{C}P^n$ (the Bott–Samelson theorem).

For manifolds all of whose geodesics are closed, we can define an important invariant by the following theorem.

Theorem 4.1.2 (Weinstein). *Let (M, q) be a Riemannian manifold all of whose geodesics are closed and have length $2\pi L$. Then the ratio*

$$i(M, g) = \frac{\mathrm{vol}(M, g)}{L^n \, \mathrm{vol}(S^n, \mathrm{can})}$$

is an integer.

The integer $i(M)$ is called the *Weinstein invariant* of the Riemannian manifold M. For canonical Blaschke manifolds, we have

$$i(S^n, \mathrm{can}) = 1, \qquad i(\mathbb{R}P^n, \mathrm{can}) = 2^{n-1},$$
$$i(\mathbb{C}P^n, \mathrm{can}) = C_{2n-1}^{n-1}, \qquad i(\mathbb{H}P^n, \mathrm{can}) = \frac{1}{2n+1} C_{4n-1}^{2n-1},$$
$$i(\mathbb{C}\mathrm{a}P^2, \mathrm{can}) = 39.$$

Let UM be the total space of the bundle of unit normal vectors to a manifold M, and let CM be the space of oriented closed geodesics on M. We have the bundle $\tau : UM \to CM$ over the circle. If M is a Blaschke manifold, then the Weinstein invariant, which is defined as mentioned above, has the following topological description:

$$i(M) = \frac{1}{2} e^{n-1} \cap [CM],$$

where $n = \dim M$, e is the Euler class of the bundle τ, $[X]$ is the fundamental homology class of the manifold X, $[X] \in H^m(X, \mathbb{Z})$, and $m = \dim X$.

As we will show in the next chapter, important information about the structure of Riemannian manifolds as a whole is contained in the volume of a Riemannian manifold. We demonstrate this by examining a Blaschke manifold. If a Blaschke manifold M is homeomorphic to the sphere S^n and the length of closed geodesics on M is the same as on S^n, then $\mathrm{vol}(M) \geq \mathrm{vol}(S^n)$ and we have an equality here iff the manifold M is isometric to the sphere S^n.

4.2. Manifolds with the spherical minimum locus. Let M be a compact connected Riemannian manifold, $K(\sigma) \leq 1$, and for a certain point $p \in M$, let M admit the spherical minimum locus, i.e., $i_p = d_p$. Then M possesses a number of specific geometric properties. If $i_p = d_p = l$, then $l \geq \frac{1}{2}\pi$, i.e., the manifold M cannot be too small. In the extreme case where $l = \frac{1}{2}\pi$, the manifold M is isometric to the real projective space $\mathbb{R}P^n$ ($K_M \equiv 1$ in this case). From the cohomological viewpoint, the manifold M is the projective space; more precisely, if $\frac{1}{2}\pi < l < \pi$, then $H^*(M) = H^*(\mathbb{R}P^n)$, and the universal covering over M is homeomorphic to the sphere S^n. If M is a simply connected manifold, then $l \geq \pi$. If the minimum locus of a point p is not contained in the set $Q_p(M)$ of the first conjugate points, then the tangent minimum locus of the point p does not intersect the tangent set of the first conjugate points, the manifold M^n has the same cohomology groups as the projective space $\mathbb{R}P^n$, and the universal covering is homeomorphic to the sphere S^n. We assume additionally that $l = \pi/\sqrt{\max(K_M)}$; then each geodesic segment starting from the point p and having length $2l$ is a geodesic loop at the point p, and for any point $q \in Q_p(M)$, the multiplicity of q with respect to p (as a conjugate pair) is constant and is equal to $\lambda = 0, 1, 3, 7$, or $n - 1$. For a nonsimply connected manifold, we have $\lambda = 0$; such a manifold has the same cohomology groups as the projective space $\mathbb{R}P^n$, and the universal covering is homeomorphic to the sphere S^n. Let M be a simply connected manifold; then $\lambda = n - 1$ implies that the manifold M is isometric to the sphere of a constant sectional curvature $\max(K_M)$; in the cases $\lambda = 1, 3, 7$, the manifold M has the same integral homology ring as a compact symmetric space of rank 1, i.e., S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathrm{Ca}P^2$. The proof of these assertions can be found, e.g., in [141, 142].

In the work [21] of Besse, it is proved that a manifold M has the spherical minimum locus at a point $p \in M$ iff M is a Blaschke manifold at the point $p \in M$. For more details about Blaschke manifolds, see [21].

Blaschke conjecture. Any Blaschke manifold (i.e., $i_M = d(M)$) is isometric to one of the following manifolds: S^n, $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathrm{Ca}P^2$, whose metric is proportional to the standard one.

We have the following result (see [63]), which allows us to describe the topological structure of the Blaschke manifolds.

Theorem 4.2.1. *Any Blaschke manifold M of dimension ≤ 9 is homeomorphic either to the sphere S^n or the projective space. If, in addition, the manifold M is modeled on the Cayley projective plane $\mathrm{Ca}P^2$, then it is homeomorphic to this plane.*

This theorem yields the following table of Blaschke manifolds:

$\dim M$	1	2	3	4	5	6	7	8	9
	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8	S^9
M		$\mathbb{R}P^2$	$\mathbb{R}P^3$	$\mathbb{R}P^4$	$\mathbb{R}P^5$	$\mathbb{R}P^6$	$\mathbb{R}P^7$	$\mathbb{R}P^8$	$\mathbb{R}P^9$
				$\mathbb{C}P^2$		$\mathbb{C}P^3$		$\mathbb{C}P^4$	
								$\mathbb{H}P^2$	

As for the isometry structure of the Blaschke manifolds, we present the following important result of Berger. If the sphere S^n admits a Riemannian metric with respect to which S^n is a Blashcke manifold, then the manifold S^n is isometric to the sphere with the metric proportional to the standard metric on the sphere.

5. Manifolds All of Whose Geodesics are Closed

We have already considered the so-called Blaschke manifolds S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathbb{C}aP^2$. Another description of this class of Riemannian manifolds (here only the so-called simply connected compact symmetric spaces of rank one are under consideration) is given by the following remarkable theorem obtained by E. Cartan.

Theorem 5.1 ([94]). *On a simply connected symmetric compact space of rank one, all geodesics are closed, and simple closed geodesics are of the same length. More precisely, let the metric on such a manifold M be normalized so that the least upper bound of the curvatures along two-dimensional directions is equal to* 1. *Then the length of all simple closed geodesics equals* 2π.

If M is a sphere, then all geodesics with origin at a point $p \in M$ again intersect when they pass the distance π. If M is one of the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$, or $\mathbb{C}aP^2$ of real dimension $2n$, $4n$, or 16, then projective lines whose dimensions are 2, 4, and 8, respectively, are spheres of constant curvature 1 embedded isometrically. Therefore, each simple closed geodesic lies on its own sphere, S^2, S^4, or S^8, as a large disk.

There arises the following natural problem: is it possible to convert the assertion of Theorem 5.1, i.e., is it true that all Riemannian manifolds all of whose geodesics are closed are exhausted by the symmetric space of rank one?

It is important to note that there exist examples of Riemannian manifolds all of whose geodesics are closed and have the same length but that are not isometric to any of the symmetric spaces of rank 1. The first examples of such surfaces were constructed by Zoll in [223].

Theorem 5.2 ([21,223]). *On the smooth two-dimensional sphere S, there exists a one-parameter family g_t, $0 \leq t \leq \varepsilon$, of metrics such that g_0 is the standard metric of constant curvature* 1, *and for each $t > 0$, the surface (S^2, g_t) is not isometric to (S^2, g_0), but, at the same time, (S^2, g_t) is a surface of revolution on which all geodesics are closed. Further, simple closed geodesics on this surface have no self-intersections, and their length is equal to 2π. For all sufficiently large t, the surface (S^2, g_t) contains a domain where the curvature is negative.*

The proof of this theorem is rather complicated and uses many modern concepts of nonlinear analysis; we refer the reader to [21], which is devoted to the proof of this theorem.

There is an essential distinction between the behavior of geodesics on the sphere S^2 with the standard metric and that on the Zoll surface. On the standard sphere S^2, all geodesics with the origin at a given point are collected together at one and the same point, the antipode of m, after the time π. On the Zoll surface, geodesics with the origin at a certain point m not lying on the axis of revolution have a nontrivial envelope in general (see Fig. 22).

Surfaces on which all geodesics with the origin at a given point m are collected together at one and the same point after a certain time are called *return surfaces*. In 1962, Green proved the following statement.

Theorem 5.3 ([21]). *A return surface is isometric to the sphere with the standard metric of constant curvature.*

The topological structure of manifolds all of whose geodesics are closed is described by the classical Bott theorem. To state this theorem, we need a new concept. Let M be a Riemannian manifold all of whose geodesics are closed. We define the number λ, called the index of the manifold M. Let $g(x)$ be an arbitrary geodesic emanating from a point p. The index λ is equal to the number of points (counted together with their multiplicity) that are conjugate to the point p on the geodesic arc $g(x)$ $(0 < x < w)$.

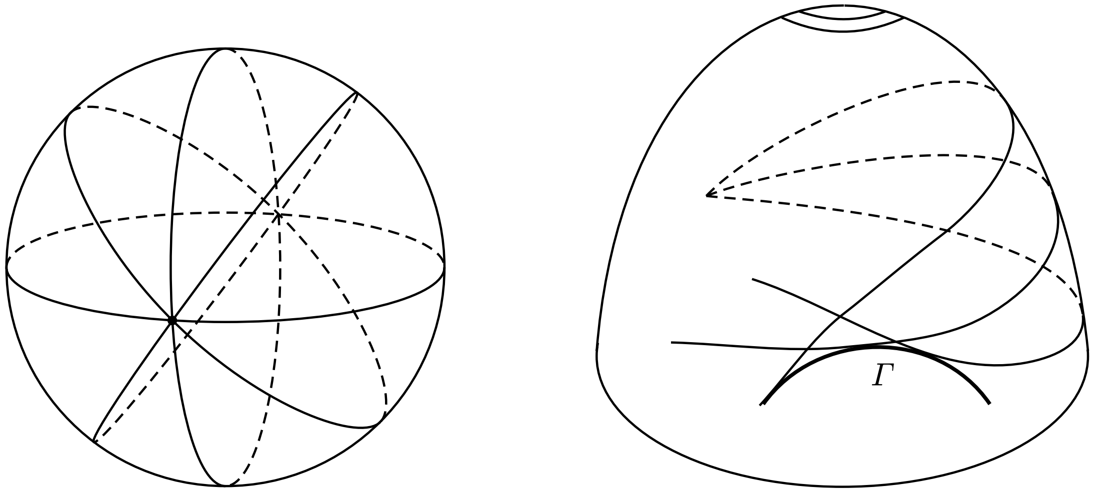


Fig. 22

Theorem 5.4 (Bott [26]). *Let all geodesics of a Riemannian manifold M be closed and simple, and let* $\dim M \geq 2$. *If $\lambda = 0$, then the fundamental group of the manifold M is a group of second order, and the universal covering of the manifold M is a homologic sphere. If $\lambda > 0$, then the manifold M is simply connected, and its integral cohomology ring is a truncated polynomial ring generated by one element θ of dimension $\lambda + 1$.*

We recall that a *truncated polynomial ring* is a ring obtained from the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients by imposing a single relation $x^n = 0$.

The constraints imposed on the ring $H^*(M)$ by this theorem are very strong. For example, the number $\lambda + 1$ should divide $\dim M$. If the number $\lambda + 1$ is odd, then $\lambda + 1 = \dim M$; the Poincaré polynomial of the manifold M over an arbitrary field k has the form

$$P(t) = 1 + t^{\lambda+1} + t^{2(\lambda+1)} + \ldots + t^{n(\lambda+1)}.$$

Very strong constraints for the truncated polynomial ring that is the cohomology ring of a complex are obtained in [2]. The results of [2] imply that if $\theta^2 \neq 0$, then $\dim \theta$ is a power of 2, and if $\dim \theta \geq 0$, then $\theta^3 = 0$.

A very interesting characteristic of manifolds all of whose geodesics are closed can be given in terms of a certain differential operator, which can be defined on an arbitrary Riemannian manifold and is called the *Laplace operator*. Let (M, G_{ij}) be a Riemannian manifold; then, by definition, we set

$$\Delta f = \frac{1}{\sqrt{\det(g_{ij})}} \sum_k \frac{\partial}{\partial x^k} \left(\sum_i g^{ik} \sqrt{\det(g_{ij})} \frac{\partial f}{\partial x^i} \right), \qquad g^{ik} g_{ks} = \delta^i_s.$$

Let M be a Riemannian manifold all of whose geodesics have the period l. Manifolds with such a property can be characterized as those compact Riemannian manifolds for which the square roots of eigenvalues of the Laplace operator Δ form asymptotically an arithmetical progression. The initial value and the difference in this progression are equal to the least common period l and the index of these closed geodesics (modulo 4) (see [21, 200]).

6. Closed Geodesics on Riemannian Manifolds

In the geometry of geodesics, we can consider two large topics. The first topic of problems is related to the study of manifolds all of whose geodesics are closed (see the previous section and [21]). The second topic is related to the study of the existence of one or many closed geodesics on a Riemannian manifold. We now consider the second topic.

6.1. Behavior of geodesics on noncompact manifolds and manifolds of nonnegative and positive curvature. We begin our description with manifolds that in principle have no closed geodesics. The fundamental fact is that this phenomenon arises on manifolds of positive sectional curvature.

Theorem 6.1.1 ([71]). *Let M be a noncompact Riemannian manifold of positive sectional curvature $K(\sigma) > 0$. Then there are no closed geodesics on M.*

A complete description of the behavior of geodesics was obtained by Toponogov. Let the sectional curvature $K(\sigma)$ of a noncompact manifold M satisfy the inequality $K > K(\sigma) > 0$ at each point and for any two-dimensional direction. Then each geodesic on M whose length does not exceed $l = \pi/\sqrt{k}$ is a minimal arc.

In the case where the sectional curvature $K(\sigma)$ of a manifold M satisfies the inequality $K > K(\sigma) \geq 0$ at each point and in each two-dimensional direction, the exact value of r is not indicated, but only the existence of a number r such that any geodesic on the space M whose length does not exceed r is a minimal arc is guaranteed.

These results imply that the injectivity radius of a noncompact manifold M of a sectional curvature $K > K(\sigma) > 0$ is not less than $\pi/\sqrt{k}$, and the greatest lower bound of the injectivity radii taken over all points of the space M is different from zero for noncompact spaces M of sectional curvature $K(\sigma) \geq 0$.

6.2. Existence of several closed geodesics. The simplest case in studying closed geodesics arises in the case of closed surfaces of negative curvature. In this case, each closed geodesic that is not homotopic to zero can be deformed into a closed curve of the minimum length in its free homotopy class. With accuracy up to a parametrization, this closed curve is unique and itself is a closed geodesic (see Theorem 2.3.1).

The problem on the existence of a closed geodesic on a simply connected closed surface is much more complicated. The history of this problem can be found in Klingenberg's work [107].

Considerable progress was obtained by Gromol and Meyer. Let S^1 be the parametrized circle. A mapping $c : S^1 \to M$ is called an H^1-mapping if it is absolute continuous and its derivative $\dot{c}(t)$ (which is defined almost everywhere) is square integrable with respect to the Riemannian metric on the manifold M:

$$\int\limits_{S^1} \langle \dot{c}(t), \dot{c}(t) \rangle_{c(t)} \, dt < \infty.$$

Let $\Lambda(M)$ be the set of all H^1-mappings of the circle S^1 into the Riemannian manifold M. We say that the manifold M satisfies the *condition G_p* if the sequence of $\mathbb{Z}_p$-Betti numbers $b_i(\Lambda(M))$ of the space $\Lambda(M)$ is not bounded.

Theorem 6.2.1 ([107]). *Let M be a compact Riemannian manifold, and, for a certain p, let the condition G_p hold. Then there are infinitely many closed simple geodesics on M.*

The case $p = \infty$, i.e., the case where the field of coefficients coincides with $\mathbb{R}$, is possible in this theorem.

A partial description of manifolds satisfying condition G_∞ is given in the following theorem.

Theorem 6.2.2 ([208]). *Let M be a simply connected compact smooth manifold. Then the fulfillment of condition G_∞ is equivalent to the rational cohomology ring $H^*(M, \mathbb{Q})$ not being a truncated polynomial ring. Condition G_p does not hold for any p if M is of the homotopy type of one of the irreducible symmetric spaces of rank one.*

Ziller [222] defined the $\mathbb{Z}_2$-cohomology of the space $\Lambda(M)$ for all compact symmetric spaces M. As a consequence of this, he revealed that symmetric spaces, except for spaces of rank one, satisfy condition

G_p for $p = \infty$ or $p = 2$. Symmetric spaces for which condition G_2 holds and condition G_∞ does not hold are as follows:

$$\mathrm{SU}(3)/\mathrm{SO}(3), \quad \mathrm{SO}(2n+1)/\mathrm{SO}(2n-1) \times \mathrm{SO}(2), \quad \text{and} \quad G_2/\mathrm{SO}(4).$$

Therefore, Ziller proved the following result.

Theorem 6.2.3 ([222])**.** *If a smooth manifold M is of the homotopy type of a symmetric space of* rank >1, *then any Riemannian metric on M admits infinitely many closed simple geodesics.*

The previous theorems refer to those spaces whose topological structure of the corresponding space of closed curves is not too complicated.

Considerable progress in the solution of one of the most remarkable problems of the global theory of closed geodesics was attained by Klingenberg. He proved the following important theorem.

Theorem 6.2.4. *Let M be a compact Riemannian manifold whose fundamental group is finite. Then there exist infinitely many closed simple nonparametrized geodesics on M.*

Klingenberg'swork [107] is devoted to the proof of this theorem.

For simply connected manifolds, the generalization of Lyusternik and Shnirel'man on three closed geodesics [123] is obtained in [107]. More precisely, the following statement is proved.

Theorem 6.2.5. *Let (M, g_{ij}) be a compact simply connected Riemannian manifold, and let $f : (S^k, \mathrm{can}) \to (M, g_{ij})$ be the mapping of the standard k-dimensional sphere, which represents a nondivisible integral homology class of infinite order. (It is known that such a mapping exists and $k \geq 2$.) Then there exist $2k - 1 \geq 8$ simple closed geodesics on (M, g) that are short in the following sense. Let $\lambda(f)$ be the least upper bound of lengths of curves on (M, g_{ij}) that are images under the mapping f of circles in (S^*, can). Then the lengths of these $2k - 1$ simple geodesics are not greater than $\lambda(f)$.*

For manifolds with an infinite fundamental group, the following statement holds.

Theorem 6.2.6 ([107])**.** *Let M be a compact Riemannian manifold. If the fundamental group $\pi_1(M)$ is infinite, then there exist at least two simple closed geodesics on M.*

If the fundamental group is of a special form, then we can obtain various improved bounds. As an example, we mention the following statement, which is referred to manifolds whose fundamental group is isomorphic to the group $\mathbb{Z}$.

Theorem 6.2.7 ([15])**.** *Let M be a compact Riemannian manifold,* $\dim M \geq 2$, *and let $\pi_1(M) \cong \mathbb{Z}$. If $n(l)$ is the number of geometrically distinct closed geodesics whose lengths do not exceed l, then*

$$\lim_{l\to\infty} n(l)\frac{\ln l}{l} > 0.$$

6.3. Certain estimates on the number of closed geodesics on the sphere and projective space. In this subsection, we present some estimates on the number of closed geodesics on Riemannian manifolds that are homeomorphic to the sphere S^n or the projective spaces $\mathbb{R}P^n$, which improve the estimates in [107]. We first consider the case of manifolds that are homeomorphic to the sphere S^n. Let K be the sectional curvature of a Riemannian manifold.

Theorem 6.3.1 ([13])**.** *Let a manifold M be homeomorphic to the sphere S^n, and let $\frac{4}{9} \leq \delta \leq K \leq 1$. Then the geodesic loop c of maximum length $L \leq 2\pi/\sqrt{\delta}$ is a closed geodesic without self-intersections and* $\mathrm{Ind}_0(c) = 3(n-1)$. *Moreover, c is a geodesic triangle of the maximum perimeter.*

Here $\mathrm{Ind}_0(c)$ stands for the dimension of a space of maximum dimension in the space of vector fields $X(t)$ along the closed geodesic $c(t)$ for which $\langle X(t), \dot{c}(t)\rangle = 0$, $t \in [0,1]$, and $X(0) = X(1)$, such that the form

$$d^2E(X,Y) = \int_0^1 \big(\langle \nabla X, \nabla Y\rangle - \langle R(X,\dot c)\dot c, Y\rangle\big)\, dt$$

is not positive. Here ∇ is the covariant derivative with respect to the given metric $\langle X, Y\rangle$ and R is the curvature tensor.

Theorem 6.3.2 ([13])**.** *If a manifold M is homeomorphic to the sphere S^n, $\frac{1}{4} \le \delta \le K \le 1$, and the curvature K is not constant, then there are no closed geodesics whose lengths belong to the closed interval $[2\pi/\sqrt{\delta}, 4\pi]$.*

We now pass to similar properties of spaces homeomorphic to the projective space $\mathbb{R}P^n$.

Theorem 6.3.3 ([13])**.** *Let g_{ij} be a metric on the projective space $\mathbb{R}P^n$ such that $\frac{1}{4} \le \delta \le K \le 1$, where K is the sectional curvature of the manifold M. Then, for g_{ij}, there exist at least $g(n) = 2n - s - 1$, $0 \le s = n - 2^k < 2^k$, closed geodesics without self-intersections whose lengths are varied in $[\pi, \pi/\sqrt{\delta}] \subset [\pi, 2\pi]$ and which are not homotopic to zero. If all closed geodesics of length $\le 2\pi$ are nondegenerate, then the metric g_{ij} has at least $\frac{1}{2}n(n+1)$ such closed geodesics.*

Let g_0 be a metric of constant curvature 1 on the sphere S^n; a metric g on S^n satisfies the *Morse condition* if $g_0 < g < 4g_0$ (see [136]). In [3], Al'ber stated the assertion that if a metric g satisfies the Morse condition and $0 < K \le 1$ if n is even or $\frac{1}{4} < K \le 1$ if n is odd, then there exist $g(n)$ closed geodesics without self-intersections whose lengths belong to the closed interval $[2\pi, 4\pi]$; for the proof of this fact (see [8]).

Theorem 6.3.4 ([13])**.** (a) *If the diameter $d_p = \max_{q\in M} \rho(p,q)$ is greater than π for all $p \in M$ and the curvature K is not less than $\frac{1}{4}$, then there exist at least $(n-1)$ closed geodesics whose lengths belong to the closed interval $[2\pi, 4\pi]$.*

(b) *If the metric g on the sphere S^n satisfies the Morse condition and $K \le \frac{1}{4}$, then there exist $g(n)$ closed geodesics whose lengths belong to the interval $(2\pi, 4\pi)$.*

(c) *Let g be a metric on the projective space $\mathbb{R}P_n$ such that $g_0 < g < 9g_0$, where g_0 is a metric of constant curvature 1 on $\mathbb{R}P_n$. Then there exist at least $g(n)$ closed geodesics that are not homotopic to zero and whose lengths belong to the interval $(\pi, 3\pi)$.*

The behavior of geodesics of metrics of positive curvature on the sphere S^2 is of great interest. Such metrics can be realized by an appropriate immersion of the sphere S^2 in the Euclidean space (see [212]).

Theorem 6.3.5 ([13])**.** (a) *Let a manifold M be diffeomorphic to the sphere S^n, and let $\frac{1}{9} < \delta \le k \le 1$. Then there are no simple closed geodesics whose lengths belong to the interval $(2\pi/\sqrt{\delta}, 6\pi)$.*

(b) *Let M be diffeomorphic to the sphere S^2, and let $\frac{1}{4} \le k \le 1$. Any shortest closed geodesic on the manifold M has no self-intersections, and its index equals 1.*

(c) *Let M be a convex hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ that contains a ball of radius r and is contained in a ball of radius R. Let $2r > R$; then there exist at least $g(n)$ closed geodesics on the manifold M whose lengths belong to the closed interval $[2\pi r, 2\pi R]$.*

Chapter 3

INFLUENCE OF CURVATURE ON GLOBAL PROPERTIES OF RIEMANNIAN MANIFOLDS

1. Spaces of Constant Curvature

1.1. General definitions and concepts. The simplest and, at the same time, a very specific class of Riemannian manifolds is formed by spaces of constant curvature. The main distinguishing feature of spaces of constant curvature is their homogeneity; it is as complete as that of Euclidean spaces. This homogeneity is expressed in the existence of a motion group that depends on the same number of parameters as in the Euclidean case, i.e., $\dfrac{1}{2}n(n+1)$ in an n-dimensional space.

Definition 1.1.1. A Riemannian manifold M^n $(n \geq 2)$ is called a *space of constant curvature* if its sectional curvatures $K(\sigma)$ in all possible two-dimensional directions σ are the same at every point. In the case $n = 2$, it is required that $K = \text{const}$, where K is the Gaussian curvature.

The theory of spaces of constant curvature was elaborated on by Minding. In the papers "How one does verify whether two given curvilinear surfaces are applicable or not and remarks on surfaces of a constant measure of curvature?" and "Supplements to the theory of minimal arcs on curvilinear surfaces," Minding discovered three types of surfaces of revolution of a constant negative curvature, found trigonometric relations in geodesic triangles on these surfaces, and observed that these relations can be obtained from the same relations for the sphere by multiplying the radius of this sphere by $i = \sqrt{-1}$, i.e., these relations coincide with the Lobachevskii relations. The relation between the studies of Minding and Lobachevskii's geometry was found by Beltrami.

We note that for $n \geq 3$, it is not assumed in advance that the curvatures $K(\sigma)$ are the same at distinct points of the manifold. In spaces of constant curvature, the curvature tensor has the following structure (see [161]):

$$R_{ij,kl} = K(g_{ik}g_{jl} - g_{il}g_{jk}),$$

where K should be supposed for now to be a function depending on the coordinates of a point, i.e., $K = K(x^1, \dots, x^n)$. In fact, the function K is a constant. This affirms the following theorem (see [161]).

Theorem 1.1.1. *Let M be a Riemannian manifold, and let* $\dim M \geq 3$. *If the sectional curvature $K(\sigma)$ of the manifold M is constant at each point, then K is a constant function on the manifold M.*

Therefore, for Riemannian manifolds of dimension greater than 2, in order to assert that the curvature is the same at each of the points of the space, it suffices to require that the sectional curvature be constant in all the directions at each of the given points.

This classical result of Schur generated a number of analogs. We present some of these generalizations.

Theorem 1.1.2. *Let M be an Einstein manifold, i.e., let the Ricci tensor of the manifold M be proportional to the metric tensor. If* $\dim M \geq 3$, *then* $\lambda = \text{const}$, *where* $R_{ij} = \lambda g_{ij}$.

Berger proved the following analog of Schur's theorem.

Theorem 1.1.3. *Let M be a Riemannian manifold with the metric tensor g_{ij} and the Riemann curvature tensor $R_{ij,kl}$. Let*

$$\sum_{i,j,k,l} R_{ijkl}R^{ijkt} = \lambda g_s^t.$$

If $\dim M \geq 5$, *then* $\lambda = \text{const}$.

In [66], Gray proposed a general construction that allows us to obtain the above theorems as a particular case. To formulate Gray's results, we need a new concept, the concept of a double differential form. A *double differential form* of type (p, q) is a tensor field $\omega_{i_1 \dots i_p j_1 \dots j_q}$ that is skew-symmetric with respect to the first p and the last q subscripts. If $D(M)$ is the Lie algebra of vector fields on the manifold M, then we can interpret a double differential form as a multilinear mapping $D(M)^{p+q} \to C^\infty(M)$ that is skew-symmetric with respect to the first p and the last q arguments.

We define a linear operator C acting from the space of double differential forms of type (p, q) into the space of double differential forms of type $(p-1, q-1)$ as follows. Let $E_1, \dots, E_n$ be an orthonormal basis of vector fields on a certain open subset of the manifold M. If ω is a double differential form of type (p, q), then, by definition, we set

$$(C\omega)(X_1, \dots, X_{p-1}, Y_1, \dots, Y_{q-1}) = \sum_{i=1}^{n} \omega(X_1, \dots, X_{p-1}, E_i, Y_1, \dots, Y_{q-1}, E_i);$$

here $n = \dim M$. The operators C^r, $r = 0, 1, 2, \dots$, are defined by induction:

$$C^1\omega = C\omega \quad \text{and} \quad C^{r+1}\omega = C(C^r(\omega)).$$

For $p = 0$ and $q = 0$, we set $C\omega = 0$. A double differential form ω of type (p, p) is said to be *symmetric* if

$$\omega(X_1, \dots, X_p, Y_1, \dots, Y_p) = \omega(Y_1, \dots, Y_p, X_1, \dots, X_p).$$

Let M be a Riemannian manifold with metric g_{ij}, and let ∇ be the corresponding affine connection. For $X \in D(M)$, the form $\nabla_X\omega$ is defined. We define the operator D acting from the space of double differential forms of type (p, q) into the space of double differential forms of type $(p+1, q)$ by

$$(D\omega)(X_1, \dots, X_{p+1}, Y_1, \dots, Y_q) = \sum_{j=1}^{p+1} (-1)^{j+1} (\nabla_{X_j}\omega)(X_1, \dots, \hat{X}_j, \dots, X_{p+1}, Y_1, \dots, Y_q).$$

The operator D is an analog of the operator of exterior differentiation d (see Sec. 3.2 of Chap. 1). A double differential form ω is said to be *Riemannian* if it is symmetric and $D\omega = 0$.

Theorem 1.1.4. *Let A and B be two double Riemannian forms of types (p, p) and (r, r), respectively. Assume that*

(a) *the form B is parallel, i.e., $\nabla_X B = 0$ for all $X \in D(M)$;*
(b) *$C^{r-1}B = \alpha g$ for a certain smooth function α that is not identically equal to zero;*
(c) $p < n = \dim M$;
(d) *there exist a smooth function λ and an integer q such that for all $X_1, \dots, X_{p-q} \in D(M)$, we have*

$$(C^q A)(X_1, \dots, X_{p-q}, X_1, \dots, X_{p-q}) = \lambda (C^{r-p+q} B)(X_1, \dots, X_{p-q}, X_1, \dots, X_{p-q}).$$

Then $\lambda = \text{const}$ on the manifold M.

1.2. Classification problems of spaces of constant curvature. We give the following important definition. A complete connected Riemannian manifold M of constant curvature $K(\sigma)$ is called a *spherical* (for $K(\sigma) > 0$), *Euclidean* (for $K(\sigma) = 0$), or *hyperbolic* (for $K(\sigma) < 0$) space form.

There exists a remarkable relation between the groups with free and completely discontinuous action and the Riemannian manifold of constant curvature. We recall the definition of the completely discontinuous action of a group Γ. We say that the group Γ acts *completely discontinuously* if each of its points admits a neighborhood U such that the set $\{\gamma \in \Gamma \mid \gamma(U) \cap U \neq \varnothing\}$ is finite. More precisely, the relation mentioned above is described by the following theorem of Killing and Hopf (its proof can be found in [213]).

Theorem 1.2.1. *Let M^n be a Riemannian manifold* ($\dim M = n \geq 2$)*, and let k be a real number. Then M is a complete connected manifold of constant curvature K iff it is isometric to one of the following quotient spaces*:

(a) *S^n/Γ, where $\Gamma \subset \mathrm{Iso}(S^n) = \mathrm{O}(n+1)$ if $K > 0$;*
(b) *$\mathbb{R}^n/\Gamma$, where $\Gamma \subset E(n) = \mathrm{Iso}(\mathbb{R}^n) = \mathrm{O}(n) \cdot \mathbb{R}^n$ if $K = 0$;*
(c) *H^n/Γ, where $\Gamma \subset \mathrm{Iso}(H^n)$ if $K < 0$.*

Here Γ *acts freely and completely discontinuously on the corresponding manifold.*

This theorem explains why a complete connected Riemannian manifold of constant curvature K is called a spherical (for $K > 0$), Euclidean (for $K = 0$), or hyperbolic (for $K < 0$) space form.

A local variant of the description of spaces of constant curvature is of interest. In fact, this was done by Riemann. Let M^n be a Riemannian manifold of dimension $n \geq 2$, and let k be a real number. Then the following conditions are equivalent.

(1) The manifold M is of constant curvature K.

(2) In a certain neighborhood of an arbitrary point $x \in M$, there exists a local coordinate system $u^1, \ldots, u^n$ in which the metric ds^2 of the manifold M becomes

$$ds^2 = \frac{\sum\limits_{i=1}^{n} (du^i)^2}{\left[1 + \frac{1}{4} K \sum\limits_{i=1}^{n} (u^i)^2\right]^2}.$$

(3) Each point $x \in M$ has a neighborhood isometric to an open subset of one of the spaces S^n for $K > 0$, $\mathbb{R}^n$ for $K = 0$, or H^n for $K < 0$.

The state of the art concerning the classification of spaces of constant curvature $K > 0$, $K = 0$, and $K < 0$ can be described as follows.

(1) As a consequence of Theorem 1.2.1, this is the problem on completely discontinuous isometry groups acting freely (without fixed points) on the spaces S^n, $\mathbb{R}^n$, and H^n, respectively.

(2) A contiguous question arising in the framework of this study is the problem of classification of all isometry groups acting discontinuously on the sphere S^n, Euclidean space $\mathbb{R}^n$, and Lobachevskii' space H^n.

(3) The solution of these problems is different in the spaces S^n, $\mathbb{R}^n$, and H^n. In the case of the sphere S^n, there is a complete (in some sense) classification up to the solution of comparison systems. In the case of the Euclidean space $\mathbb{R}^n$, there is no such classification. The initial study of Euclidean space forms is merely an application of certain results of geometric crystallography. The main structural theorems on discontinuous groups are the Bieberbach theorems. In the case of the hyperbolic space H^n, we have a very rich isometry group. At present, there is no complete classification of hyperbolic space forms (except for dimension 2).

In the general problem of classification of spaces of constant curvature, we can mention the following three subproblems:

(a) the classification of manifolds with accuracy up to a diffeomorphism;
(b) the classification of manifolds with accuracy up to affine isomorphisms;
(c) the classification of manifolds with accuracy up to an isometry.

Under the assumptions of homogeneity of a Riemannian manifold, the problem of classification of the spaces of constant curvature is considerably simplified.

Theorem 1.2.2 ([213]). *Let* M^n *be a connected homogeneous Riemannian manifold of dimension* n *and constant curvature* K. *If* $K < 0$, *then* M^n *is isometric to the hyperbolic space* H^n. *If* $K = 0$, *then* M^n *is isometric to the direct product* $\mathbb{R}^m \times T^{n-m}$ *of the Euclidean space* $\mathbb{R}^m$ *by the flat torus* T^{n-m}. *If* $K > 0$, *then* M^n *is isometric to the quotient space* S^n/Γ, *where* S^n *and* Γ *have the following meaning. Let* F *be the field* $\mathbb{R}$ *of real numbers, the field* $\mathbb{C}$ *of complex numbers, or the quaternion algebra* $\mathbb{H}$. *Then* S^n *is the sphere* $|x| = 1$ *in the left-Hermitian vector space* V *over* F *of real dimension* $n + 1$; Γ *is a finite multiplicative group of elements with norm* 1 *in* F *that is not contained in any proper subfield* F_1 *in* F *such that* $\mathbb{R} \subset F_1 \subset F$, $F_1 \neq F$, *and, moreover,* Γ *acts on* S^n *by the* F*-inner product of vectors.*

Conversely, all the manifolds listed above are n-dimensional Riemannian homogeneous manifolds of constant curvature.

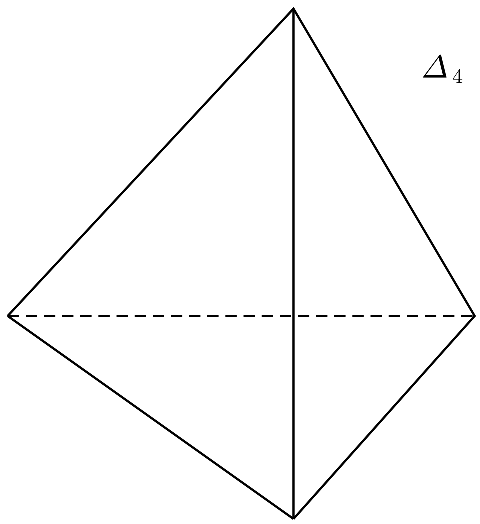


Fig. 23

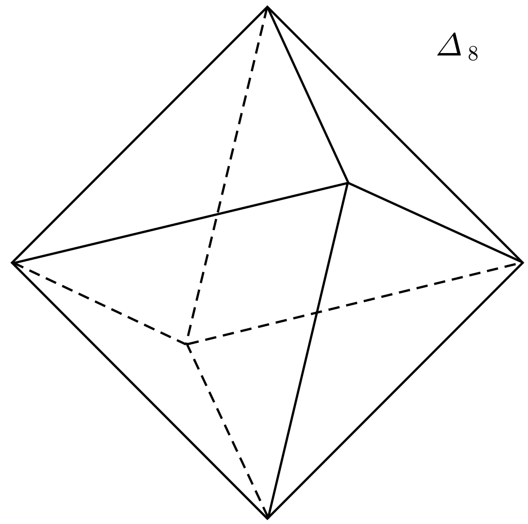


Fig. 24

As a consequence of the above statement, we obtain that a homogeneous Riemannian manifold M^n of dimension n and constant curvature $K > 0$ is determined by its fundamental group $\pi_1(M)$ with accuracy up to an isometry.

1.3. Riemannian manifolds of constant positive curvature. It is known from topology that for even n, any orientation-preserving diffeomorphism $f : S^n \to S^n$ of the sphere S^n onto itself has a fixed point. Therefore, in the case of even dimension, there are a few manifolds of constant positive curvature. More precisely, the following assertion holds.

Theorem 1.3.1 (see, e.g., [213]). *A complete Riemannian manifold M^{2m} of even dimension $2m$ and positive curvature $K > 0$ is isometric either to the sphere S^{2m} of radius $K^{-1/2}$ or to the projective space $\mathbb{R}P^{2m}$.*

In particular, with accuracy up to an isometry, only the sphere S^2 and the projective plane $\mathbb{R}P^2$ are complete two-dimensional surfaces of positive Gaussian curvature.

We now consider the problem on discrete subgroups of the group of motions of the sphere S^{n-1} (the problem on fixed points is ignored for now). The group of motions of the sphere S^{n-1} coincides with $\mathrm{O}(n)$. Since this group is compact, each of its discrete subgroups is finite. We consider the case where $n = 3$. Therefore, let $S^{n-1} = S^2$. The group of motions of this space coincides with $\mathrm{O}(3)$, i.e., we seek finite groups of rotations of the three-dimensional Euclidean space $\mathbb{R}^3$.

It turns out that finite groups of rotations of the space $\mathbb{R}^3$ are exhausted by the following list:

(a) cyclic groups;
(b) the dihedral group;
(c) symmetry groups of the regular tetrahedron, octahedron, and icosahedron.

We note that these groups yield a complete list of finite subgroups in $\mathrm{O}(3)$. They all are simply described in the language of generators and relations (for more details, see the work [213] of Wolf). In the classification presented above, there arise all regular polyhedrons, but two of them are dual to one another. We recall that all these groups, except for one exceptional group, have fixed points, and, therefore, they do not correspond to any space (see Theorem 1.3.1). See Figs. 23–26.

Here there arises the so-called A, D, E-classification. It occurs in various fields of mathematics, for example, in the critical point theory of functions, Lie algebras, category of linear spaces, wave fronts,

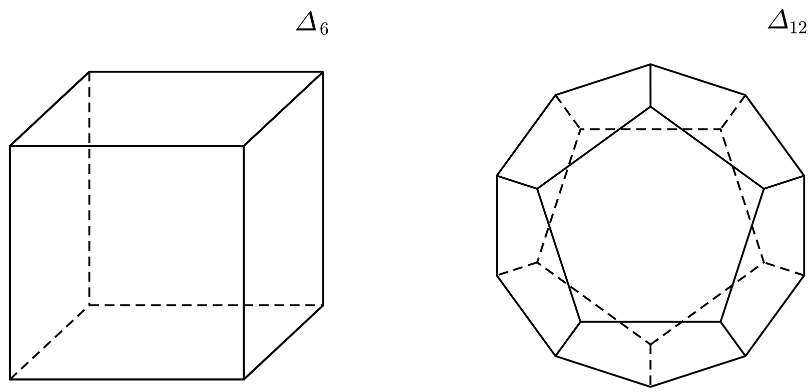


Fig. 25

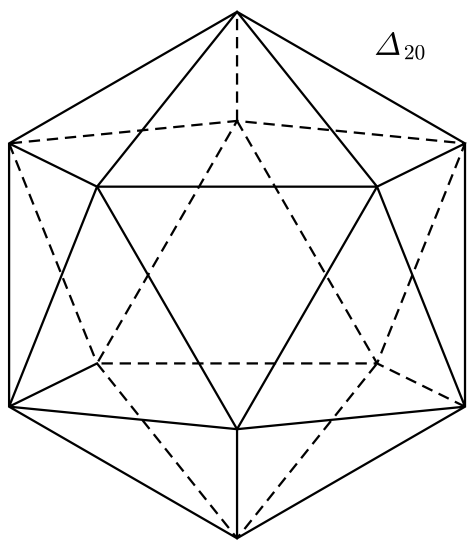


Fig. 26

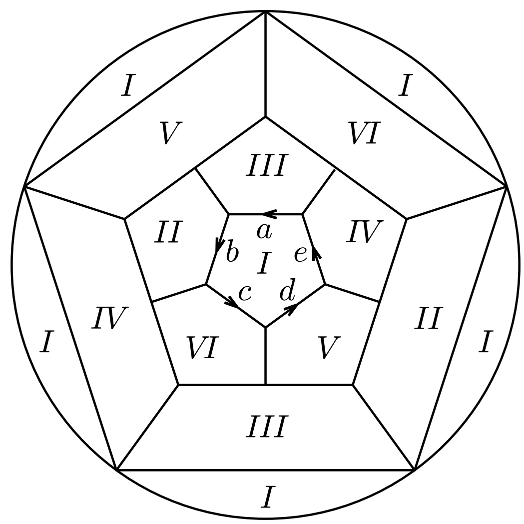


Fig. 27

caustics, regular polyhedrons in the three-dimensional space, and crystallographic groups generated by reflections. The symmetry groups $\Gamma \subset \mathrm{O}(3)$ form two infinite series and three exceptions, the symmetry groups of the tetrahedron (E_6), octahedron (E_7), and icosahedron (E_8), the series being the groups of a regular polygon and regular dihedron, i.e., a two-sided polygon with sides of distinct color.

In odd dimension, there are infinitely many spaces of constant curvature that are not isometric to each other. We consider the case where $\dim M = 3$. The most well-known examples of such a type are lens spaces. Let p and q be two relatively prime positive integers. We represent a cyclic group $\mathbb{Z}_q$ of qth order as a multiplicative group of qth roots of the unit, $\left\{\exp\dfrac{2\pi ik}{q},\ 0 \le k \le q-1\right\}$, that acts freely on the sphere S^3 according to the rule

$$(z_1, z_2) \mapsto \left(z_1 e^{\frac{2\pi ik}{q}}, z_2 e^{\frac{2\pi ik}{q}}\right),$$

where $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$ is described by the equation $z_1\bar{z}_1 + z_2\bar{z}_2 = 1$. The *lens space* $L(p,q)$ is the quotient space $S^3/\mathbb{Z}_q$ with respect to the group operation described above and the induced Riemannian structure.

Also, there exist examples of non-Abelian groups that acts orthogonally and freely on the sphere S^3. As an example, we can take the *Poincaré manifold* (see [49]), which is obtained by identifying the opposite sides of the dodecahedron (pentagons) that are turned by an angle of $\pi/5$ with respect to each other. In Fig. 27, we depict the dodecahedron assigning the Poincaré space.

We call attention to the fact that isomorphic groups can lead, in general, to nonisometric Riemannian spaces. As an example, we can present lens spaces.

We consider the groups that act discontinuously on the sphere by using the methods of group representation theory, and in each dimension the problem is reduced to the problem of solving a certain comparison system. For more details, see the work [213] of Wolf.

1.4. Riemannian manifolds of constant zero curvature. As an example, we first consider two-dimensional manifolds. In the case where $\dim M = 2$, all surfaces of zero curvature $K \equiv 0$ admit a simple classification. The following theorem yields a topological classification of such manifolds.

Theorem 1.4.1 (see, e.g., [213]). *A complete connected surface of constant Gaussian curvature $K \equiv 0$ can topologically be only the Euclidean plane, cylinder, torus, Möbius band, or, finally, Klein bottle* (*see Fig.* 28). *Moreover, any two cylinders, tori, Möbius bands, and Klein bottles are affinely isomorphic.*

For a more detailed description of surfaces of zero curvature, we recall some facts about the group of motions of the Euclidean plane.

The classification of motions of the plane is based on the Chasles theorem describing all motions of the plane. Any motion of the plane is a parallel translation, a turn, or a gliding symmetry. We recall that the motion of the plane consisting of a symmetry in direction of a line and a shift (parallel translation) in direction to this line is called a gliding symmetry. Therefore, the following five types of groups acting on the plane are possible.

(1) The group Γ of motions of the first type consists of only one identity motion. In this case, $\mathbb{R}^2/\Gamma = \mathbb{R}^2$ is the Euclidean plane.

(2) The group Γ of motions of the second type is given by a nonzero vector a and consists of parallel translations by vectors of the form ma, where m is any integer. In this case, $\mathbb{R}^2/\Gamma$ is a cylinder. The class of cylinders isometric to each other is in a one-to-one correspondence with positive reals. The number $|a|$ corresponds to the cylinder described by a vector a.

(3) The group Γ of motions of the third type is given by a line l and a nonzero vector a parallel to l. It consists of parallel translations by vectors of the form ma, $m \in \mathbb{Z}$, and gliding symmetries along the line l by vectors of the form $na + a/2$, where n is an arbitrary integer. In this case, $\mathbb{R}^2/\Gamma$ is the *Möbius band*. The isometry classes of Möbius bands are in a one-to-one correspondence with positive reals. With the Möbius band described by the vector a, one associates the number $2|a|$.

(4) The group Γ of motions of the fourth type is given by a line l, a vector a parallel to this line, and a vector b orthogonal to it. This group consists of parallel translations by a vector of the form $ma + nb$, where m and n are any integers, and gliding symmetries along the line l by vectors of the form $ma + nb + a/2$, where $m, n \in \mathbb{Z}$. In this case, $\mathbb{R}^2/\Gamma$ is the *Klein bottle*. The isometry classes of Klein bottles are in a one-to-one correspondence with ordered pairs of positive reals. With the Klein bottle described by a pair of vectors (a, b), one associates the pair $(2|b|, |a|)$.

(5) The group Γ of motions of the fifth type is given by two noncollinear vectors a and b and consists of parallel translations by all vectors of the form $ma + nb$, where m and n are arbitrary integers. In this case, $\mathbb{R}^2/\Gamma$ is the torus. The isometry classes of tori are in a one-to-one correspondence with pairs (r^2, φ), where r^2 is a positive real and φ belongs to the quotient space $H^2/\operatorname{SL}(2,\mathbb{Z})$ of the upper half-plane H^2 by the natural action of the group $\operatorname{SL}(2,\mathbb{Z})$ on it. The pair $(r^2, \operatorname{SL}(2,\mathbb{Z})(u))$ corresponds to the isometry class of the torus

$$\mathbb{R}^2/\{n_1 r + n_2 r u \mid u_i \in \mathbb{Z}, \quad r > 0,\ i = 1, 2\}.$$

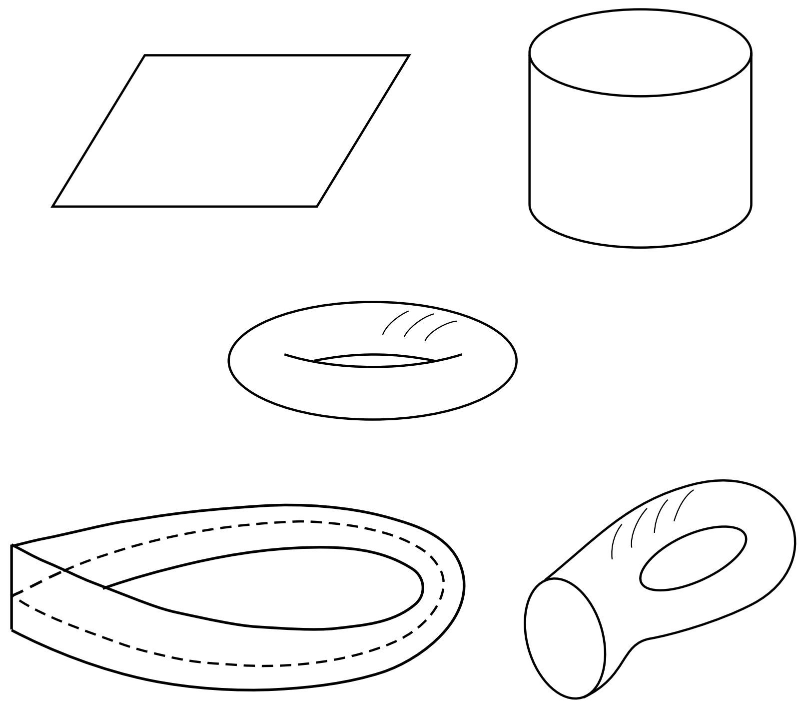


Fig. 28

The action of the group $\mathrm{SL}(2,\mathbb{Z})$ on H^2 is described as follows. Let $H^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$, and let $\mathrm{SL}(2,\mathbb{Z})$ be the group of integral 2×2 matrices with unit determinant. Then with an element

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}),$$

one associates the transformation $f_\alpha : H^2 \to H^2$ given by $f_\alpha(z) = (az+b)/(cz+d)$. The quotient group $G = \mathrm{SL}(2,\mathbb{Z})/\pm E$ is called the *modular group*. In fact, we consider the modular group, since the matrix $\pm E$ acts trivially. We denote by D the subset in H^2 consisting of points $z \in H^2$ such that $|z| \geq 1$ and $|\operatorname{Re}(z)| \leq \frac{1}{2}$. In Fig. 29, we depict the transformations of a domain D under the action of the elements 1, T, TS, $ST^{-1}S$, ST^{-1}, S, ST, STS, $T^{-1}S$, and T^{-1} of the group G, where $S(z) = -1/z$ and $T(z) = z+1$.

The domain D is the fundamental domain of the action of the group G on the half-plane H^2. More precisely, the following assertion holds.

Theorem 1.4.2. (a) *For each $z \in H^2$, there exists an element $g \in G$ such that $g(z) \in D$.*

(b) *Let z and z' be two distinct points of the domain D that are comparable modulo group G. Then either $R(z) = \pm\frac{1}{2}$ and $z = z' \pm 1$ or $|z| = 1$ and $z' = -1/z$.*

(c) *Let $z \in D$, and let $J(z) = \{g \in G \mid gz = z\}$ be the stationary subgroup of the point z. Then $J(z) = 1$, except for the following three cases:*

(a) *$z = i$; in this case, $J(z)$ is a group of second order generated by the element S;*

(b) *$z = \rho = e^{\frac{2\pi i}{3}}$; in this case, $J(z)$ is a group of third order generated by the element ST;*

(c) *$z = -\bar{\rho} = e^{\pi i/3}$; in this case, $J(z)$ is a group of third order generated by the element TS.*

We now recall that the upper half-plane is a model of the Lobachevskii geometry, and the group $\mathrm{SL}(2,\mathbb{R})$ is its group of motions.

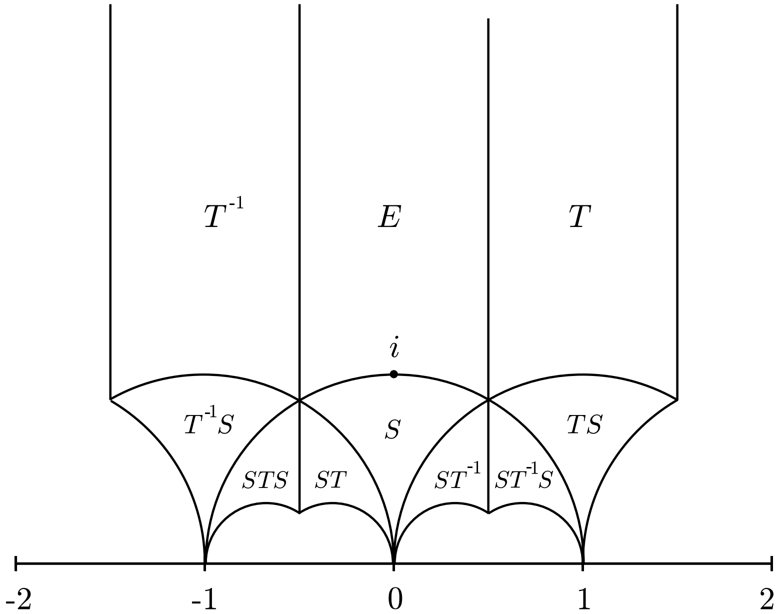


Fig. 29

Proposition 1.4.1. *The set of geometries on the torus (or lattices on the plane) is itself a geometry, a triangle on the Lobachevskii plane with angles $\pi/3$ and $\pi/2$ and one vertex at infinity, with accuracy up to an isometry.*

We now pass to the consideration of manifolds of arbitrary dimension. A closed subgroup $H \subset G$ is said to be *uniform* if the quotient space G/H is compact. Let Γ be a subgroup of the group $\mathcal{E}(n)$ of motions of the Euclidean space $\mathbb{R}^n$. The group Γ acts completely discontinuously with compact quotient space $\mathbb{R}^n/\Gamma$ iff Γ is a discrete uniform subgroup in $\mathcal{E}(n)$. A closed subgroup $\Gamma \subset \mathcal{E}(n)$ acts on $\mathbb{R}^n$ freely iff Γ is torsion-free. A discrete uniform subgroup in the group $\mathcal{E}(n)$ is called a *crystallographic group* on $\mathbb{R}^n$. The structure of crystallographic groups is described in the following two fundamental Bieberbach theorems.

Theorem 1.4.3. *If $\Gamma \in \mathcal{E}(n)$ is a crystallographic group, then the intersection $\Gamma \cap \mathbb{R}^n$ is a normal subgroup of the group Γ of finite index; any minimal set of generators of the group $\Gamma \cap \mathbb{R}^n$ is a basis of the vector space $\mathbb{R}^n$; moreover, with respect to this basis, $\mathrm{O}(n)$-components of the elements of the group Γ have integer coordinates.*

Theorem 1.4.4. *For each integer $n > 0$, there exist only a finite number of classes of pairwise isomorphic crystallographic groups acting on the space $\mathbb{R}^n$. Two crystallographic groups on $\mathbb{R}^n$ are isomorphic iff they are conjugate in the affine group.*

We note that, as for the sphere S^n, on the space $\mathbb{R}^n$, isomorphic groups (even with compact quotients) can also have distinct actions.

Now we can state a theorem that describes the structure of flat compact Riemannian manifolds. The first problem being solved here is the problem on the finiteness of types. Then the problem on the affine classification of such a manifold is solved by a simple topological invariant, the fundamental group. Owing to this remark, there arises a natural problem on an abstract description of fundamental groups of compact flat Riemannian manifolds.

Theorem 1.4.5 ([213])**.** *If M is a flat compact Riemannian manifold of dimension n, then M admits a normal Riemannian covering by a flat n-dimensional torus. There exist only a finite number of flat* *compact connected Riemannian manifolds of a given dimension affinely equivalent to each other. Two flat compact Riemannian manifolds are equivalent iff their fundamental groups are isomorphic*; *in particular, for flat compact manifolds, the affine equivalence is equivalent to the topological equivalence.*

Theorem 1.4.6 ([213])**.** *An abstract group* Γ *is isomorphic to the fundamental group of a compact flat Riemannian manifold* M *of dimension* n *iff*

(1) Γ *contains a normal free Abelian subgroup* Γ_1 *of finite index and rank* n;
(2) Γ_1 *is the maximal Abelian subgroup of the group* Γ;
(3) *in* Γ*, there are no elements of a finite order different from* 1*. In this case, we can consider* Γ *as a subgroup in the group* $\mathcal{E}(n)$ *of motions of the* n*-dimensional Euclidean space* $\mathbb{R}^n$*.*

Then $M = \mathbb{R}^n/\Gamma$*,* $\Gamma_1 = \Gamma \cap \mathbb{R}^n$ *(*$\mathbb{R}^n$ *lies in* $\mathcal{E}(n)$ *as a translation subgroup), and* Γ/Γ_1 *is the automorphism group of the normal Riemannian covering* $\mathbb{R}^n/\Gamma_1 \to M$ *of the manifold* M *by the flat torus.*

Although the last two theorems give certain information about flat manifolds, they do not solve the classification problem for them. In dimension $\dim M = 3$, there exists such a classification, and it can be found in [213].

The study of noncompact flat complete Riemannian manifolds is reduced to the compact case; this is proved by the following theorem (Theorem 1.4.7). The simplest flat Riemannian manifolds are Euclidean spaces $\mathbb{R}^n$. The next in complexity are flat cylinders $\mathbb{R}^n/\Delta$, where Δ is a discrete group of translations of rank less than n. They are followed by flat tori $T^n = \mathbb{R}^n/\Gamma$, where Γ is a discrete subgroup in the group $\mathcal{E}(n)$ of motions of the n-dimensional Euclidean space $\mathbb{R}^n$ consisting of translations only; moreover, its rank equals n. If $\mathbb{R}^k \subset \mathbb{R}^n$ is the linear span of Δ, then $\mathbb{R}^k/\Delta$ is a flat torus, and the space $\mathbb{R}^n/\Delta$ is isometric to $(\mathbb{R}^k/\Delta)\times\mathbb{R}^{n-k}$. Therefore, each flat cylinder contains a compact totally geodesic submanifold on which it is retracted, and this compact submanifold contains the main information about the geometry of the cylinder. This phenomenon is generalized to the case of an arbitrary noncompact flat manifold in the following theorem.

Theorem 1.4.7 ([213])**.** *Let* M *be a flat complete Riemannian manifold. Then there exists a compact totally geodesic submanifold that is a real analytic deformation retract of the manifold* M*.*

The compact submanifold described in this theorem is a flat compact Riemannian manifold, and, therefore, its structure is described by Theorems 1.4.5 and 1.4.6. The theorem presented implies the following two important consequences for the topology of flat manifolds.

(1) The class of fundamental groups of flat complete Riemannian manifolds coincides with the class of fundamental groups of compact flat Riemannian manifolds.
(2) Let M be a complete flat Riemannian manifold. Denote by χ the Euler–Poincaré characteristic for any homology or cohomology theory satisfying the homotopy axiom (as it holds for the singular theory). Then $\chi(M) = 0$ or M is the Euclidean space $\mathbb{R}^n$.

Corollary (1) is immediately implied by Theorem 1.4.7, since a deformation retraction preserves the homotopy type. By the same reason, $\chi(M) = \chi(N)$ under the conditions of corollary (2), where N is a flat compact submanifold. By Theorem 1.4.6, N admits an r-fold covering by the torus T, $r > 0$. Therefore, $\chi(N) = r\chi(T) = 0$, since $\chi(T) = 0$ if $\dim T > 0$.

1.5. Riemannian manifolds of constant negative curvature. We first consider the simplest case of two-dimensional Riemannian manifolds, i.e., surfaces. A metric of constant negative curvature exists on any oriented surface of genus $g > 1$. There is no such metric on the spheres and on the torus, which is implied by the Gauss–Bonnet theorem. Moreover, on each closed surface of genus g, there exists a $6(g-1)$-family of distinct Riemannian metrics of curvature -1. There is a continuum of Riemannian metrics of constant curvature -1 on each noncompact and finitely connected surface of Euler characteristic $\chi < 0$, except for a sphere with three punctured points.

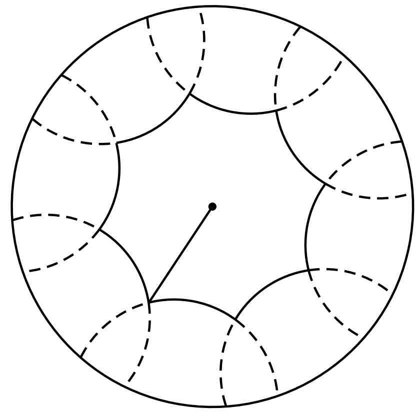


Fig. 30

We now show how to construct an infinite number of hyperbolic space forms. Moreover, we indicate a specific example of a discrete group of motions of the Lobachevskii plane that has a $4g$-gon with sum of angles 2π as the fundamental domain. As such a fundamental domain, we take a regular $4g$-gon with angles $\pi/2g$ centered, e.g., at the center of the unit disk (we consider the Poincaré model of the Lobachevskii plane in the unit disk; see Fig. 30).

Let $A_1, \dots, A_{2g}$ be "shifts" on the Lobachevskii plane that interchange pairs of opposite sides. Each subsequent transformation A_{k+1} is obtained from the preceding transformation A_k by a turn of the direction of a "shift" by the angle $\pi - \pi/2g$ (i.e., by the conjugation via the matrix of rotation B_g by the angle $\pi - \pi/2g$). The transformations $A_1, \dots, A_{2g}$ are related by $A_1 \dots A_{2g} A_1^{-1} \dots A_{2g}^{-1} = \mathrm{id}$ (for more details and an explicit form of the transformations A_k, see [48]).

We now pass to the consideration of general discrete transformation groups of the Lobachevskii plane.

Definition 1.5.1. Let Γ be a discrete transformation group of the Lobachevskii plane which is a subgroup of the isometry group. A subset D of the Lobachevskii plane is called a *fundamental domain* of the group Γ if D is a closed set such that the orbit $\Gamma(D)$ of D coincides with the whole Lobachevskii plane, the covering $\gamma(D)$, $\gamma \in \Gamma$, is such that only finitely many sets of the form $\gamma(D)$ intersect a sufficiently small neighborhood of an arbitrary point, and the image of the set of interior points of D under the action of any transformation from Γ that is different from the identity does not intersect the set of interior points of D.

As a fundamental domain of an arbitrary discrete group Γ on the Lobachevskii plane, we can take a convex polygon with finitely many sides. Above, we explicitly described the fundamental domain of the modular group. Our goal now is to give a geometric description of discrete groups of motions of the Lobachevskii plane. Since the isometry group of the Lobachevskii plane is isomorphic to $\mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2$, this problem is equivalent to the enumeration of discrete subgroups in $\mathrm{SL}(2,\mathbb{R})$. Groups with discrete action on the Lobachevskii plane naturally arise in classifying Riemannian structures on the two-dimensional torus and also in studying the conformal geometry on one-dimensional complex analytic manifolds.

Let H^2 be the Lobachevskii plane, and let Γ be an arbitrary discrete group of motions on H^2. Let D be a convex fundamental polygon for the action of the group Γ. We consider polygons of the form γD, $\gamma \in \Gamma$. They do not overlap each other and cover the whole Lobachevskii plane. Elements of these partitions of the Lobachevskii plane are usually called *"meshes."* Two meshes are said to be *adjacent* if their intersection is a one-dimensional subset, i.e., a curve on the plane. We can assume that if D_1 and D_2 are two adjacent meshes, then $D_1 \cap D_2$ is a common side of these two polygons. To obtain this, it suffices to add to the fundamental polygon a certain number of vertices such that the angle at each of

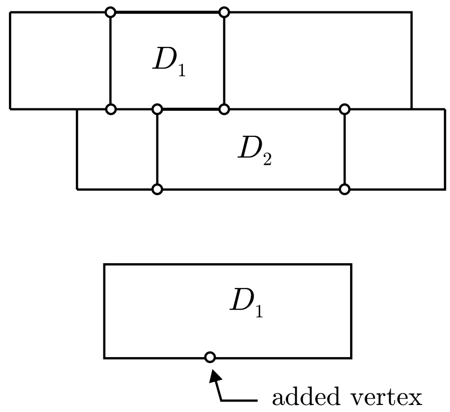


Fig. 31

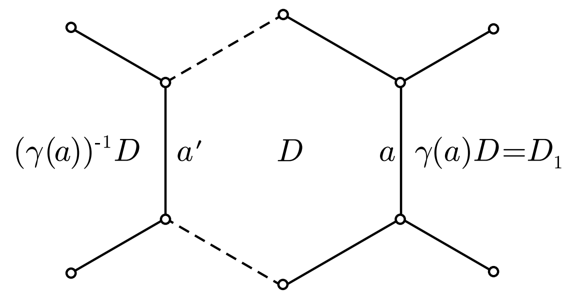


Fig. 32

them equals π. Using this, we can obtain that the intersection of any two adjacent meshes occurs exactly along their common side (see Fig. 31).

For any side a of a mesh D, there exists a unique mesh D_1 that is adjacent to D along the side a. In this case, the mesh D_1 is obtained from the mesh D by applying some transformation $\gamma \in \Gamma$. We denote this transformation by $\gamma(a)$. Since the domain D passes to D_1 under the transformation $\gamma(a)$, we see, therefore, that there exists a certain side $a' \in D$ such that $\gamma(a)a' = a$ (the domain D intersects its image under the action of $\gamma(a)$). This implies that $\gamma(a') = (\gamma(a))^{-1}$ and, in particular, $a'' = (a')' = a$ (see Fig. 32). With each side a, we associate the side a' corresponding to it under the above mapping. There arises an involutive transformation (i.e., a transformation the square of which is the identity mapping) of the set of sides of the domain D. Of course, in this case, it can happen that $a' = a$, but then $(\gamma(a))^2 = e$, and, therefore, $\gamma(a)$ is a mapping of the domain D with respect to this side a or a turn by the angle π with respect to the middle of the side a. Therefore, two meshes $\gamma_1 D$ and $\gamma_2 D$ are adjacent iff $\gamma_2 = \gamma_1\gamma(a)$.

A sequence of meshes $D = D_0, D_1, \dots, D_k$ such that the meshes D_{i-1} and D_i are adjacent for $i = 1, \dots, k$ is called a *mesh chain*. For a mesh D_i, there exists a unique motion γ_i such that $\gamma_i D = D_i$. In this case, there arises an induced mapping of the sides of the fundamental polygon onto the sides of the mesh. Therefore, the sides of the mesh D_i can be denoted by the same symbol as the sides of the polygon $D_0 = D$.

In the mesh chain $D = D_0, D_1, \dots, D_k$ (let $D_i = \gamma_i D_0$), the polygons D_{i-1} and D_i are adjacent, and, therefore, we have $\gamma_i = \gamma_{i-1}\gamma(a_i)$ and $\gamma_k = \gamma(a_1)\gamma(a_2)\dots\gamma(a_k)$. Therefore, with the mesh chain, one associates the sequence of sides $D : a_1, a_2, \dots, a_k$ of the mesh. These arguments prove the following statement.

Theorem 1.5.1. *The group Γ is generated by the elements $\gamma(a)$, where a runs over all sides of the fundamental polygon D.*

We now describe the relations in this group. Let $\gamma(a_1)\dots\gamma(a_k) = e$. We consider the corresponding chain. Then its last element is the mesh D itself, the initial fundamental polygon (see Fig. 33).
Therefore, with relations in the group Γ, one associates the closed chains, which are usually called cycles. Relations of the form $\gamma(a)\gamma(a') = e$ are called *elementary relations of the first type*. These relations generate the cycle D_0, D_1, D_0.

We consider a vertex of a mesh D and all the meshes containing this vertex. Then the sequence of these meshes forms a cycle (see Fig. 34). Such a cycle is called an elementary cycle of the second type, and the relation corresponding to it is called an *elementary relation of the second type*.

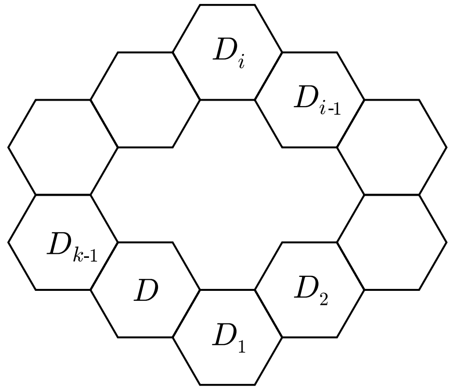


Fig. 33

Theorem 1.5.2. *Elementary relations of the first and second types compose a defining system of group relations for generators* $\gamma(a)$ *of the discrete group* Γ*, i.e., each relation is their consequence.*

We have thus described the structure of any discrete group of motion of the Lobachevskii plane using the geometric language. We now consider the following inverse problem: how does one reconstruct a discrete group Γ by a given fundamental polygon? On the Lobachevskii plane, let a convex polygon with finitely many sides that has no infinitely distant vertices for now be given (see Fig. 35).

It is possible that angles at certain vertices of the polygon are equal to π. Let an involutive permutation $a \to a'$ of sides of this polygon be given. For any side a, there exists a unique motion $\gamma(a)$ such that $\gamma(a)a' = a$ and $\gamma(a)D \cap D = a$. Let the following two conditions hold:

(1) $\gamma(a)\gamma(a') = e$;
(2) for any vertex A of the polygon D, there exists a sequence of sides $a_1, a_2, \dots, a_k$ such that $\gamma(a_1)\gamma(a_2)\dots\gamma(a_k) = e$ and the sequence of polygons $D, \gamma(a_1)D, \gamma(a_1)\gamma(a_2)D, \dots, \gamma(a_1)\dots\gamma(a_k)D$ forms a going around of this vertex A in the sense that each of them contains the vertex A and each element of this chain is adjacent to the preceding element.

In addition, they do not overlap each other and cover (in totality) a certain neighborhood of the point A.

Theorem 1.5.3. *If the conditions* (1) *and* (2) *indicated above are fulfilled, then the motions* $\gamma(a)$ *generate a discrete group of motions of the Lobachevskii plane for which the domain* D *is its fundamental domain.*

We now consider the case where $\dim M = n > 2$. For hyperbolic geometry (as also in the case $n = 2$), $n = 3$ is an exceptional case. For $n > 4$, there are finitely many (say, $\operatorname{vol} M < \text{const}$) hyperbolic manifolds, while for $n = 3$, they form an infinite set. In particular, this makes the three-dimensional hyperbolic geometry rich in content. It was Thurston who obtained the strong results which transform the theory of three-dimensional hyperbolic manifolds into an independent field of geometry, which is rich in content. He has elaborated a new method for studying three-dimensional manifolds based on their cut into pieces admitting a locally homogeneous metric. For the theory of three-dimensional manifolds, see [179].

We begin with examples of three-dimensional manifolds. We describe three series of such examples following Gutsul [88, 89]. In constructing these manifolds, we use prisms in the Lobachevskii space whose dihedral angles between neighboring lateral faces equal $\pi/2$ and whose dihedral angles between a lateral face and any of the bases equal $\pi/3$. Moreover, for the first two series of manifolds, we use prisms whose number of lateral faces is $m = 12 \cdot 2^k$, and for the third series of manifolds, we use prisms with the number of lateral faces equal to $m = 12 \cdot k$, where $k = 2, 3, \dots$.

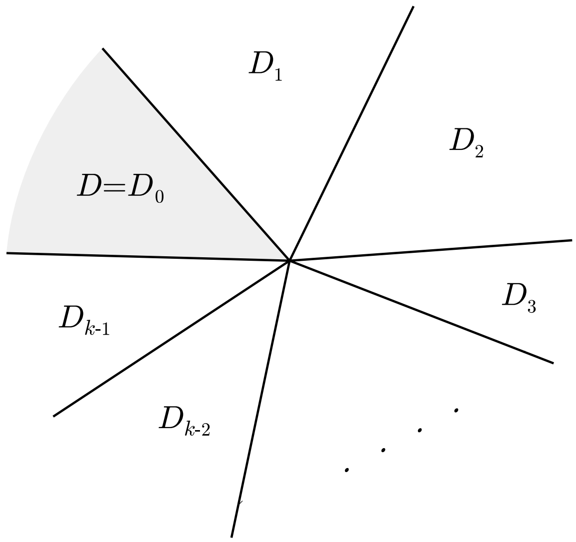


Fig. 34

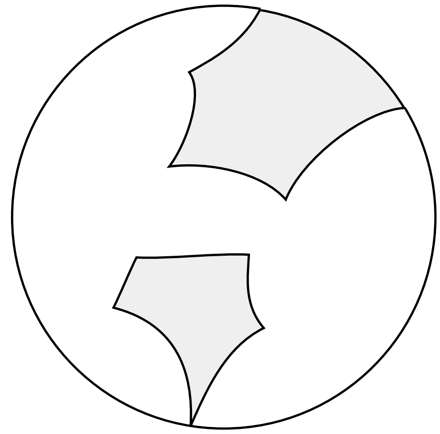


Fig. 35

We consider a certain m-gon prism, where $m = 12 \cdot 2^k$ and $k = 2, 3, \ldots$. We denoted by δ_1 and δ_2 the bases of the prism and by $\beta_1, \beta_2, \ldots, \beta_{12 \cdot 2^k}$ its lateral faces. We first indicate identifications of certain pairs of faces of the prism that are the same for the first two series of manifolds.

We identify the lower base δ_1 of the prism with its upper base δ_2 by a screw motion with the angle of turn $\psi = \pi/3$ and the axis u (u is the axis of the prism; the turn is clockwise if we look from the upper base of the prism).

We identify the lateral faces β_{3i+1}, $i = 0, 1, \ldots, 2 \cdot 2^k - 1$, of the prism with the lateral faces $\beta_{6 \cdot 2^k + 3i+1}$ by the screw motions B_{3i+1}. The angle of turn of each screw motion B_{3i+1} is equal to $\varphi = \pi$, and the axis of rotation is the common perpendicular of the faces β_{3i+1} and $\beta_{6 \cdot 2^k + 3i+1}$. We can easily obtain any screw motion B_{3i+1} in the form of the product of two turns of second order around crossed axes: the turn of second order around the line of intersection of two planes β_{3i+1} and ω (the plane of the middle cross section of the prism) and the turn of second order around the line u (the axis of the prism).

We identify the remaining faces of the prism using two different methods, and owing to this, we obtain two distinct series of manifolds.

We divide the screw motions B_{3i+1} into two categories: even screw motions for $i = 1, 3, \ldots, 2 \cdot 2^k - 1$ (when $3i + 1$ is an even number) and odd screw motions for $i = 0, 2, \ldots, 2 \cdot 2^k - 2$. The faces of the prism that are identified by even screw motions are called the faces of even screw motions, and the faces that are identified by odd screw motions are called the faces of odd screw motions. Also, we divide the remaining lateral faces of the prism into two categories: the faces adjacent to the faces of even screw motions are said to be even, and the faces adjacent to the faces of odd screw motions are said to be odd.

We draw two planes, τ_1 and τ_2, orthogonal the faces β_1 and $\beta_{3 \cdot 2^k + 1}$, respectively, passing threw the axis of the prism. We consider the first method for identifying the remaining faces of the prism. We identify each of the odd faces with the odd face symmetric with respect to the plane τ_1 by a shift; we identify each of the even faces with the face symmetric to it with respect to the plane τ_2 also by a shift. The prism with such face identifications is a fundamental polygon of a certain group Γ_1^k, and the motions identifying the faces of the prism are generators of the group Γ_1^k; moreover, Γ_1^k is a torsion-free group.

Factorizing the Lobachevskii space H^3 by the groups Γ_1^k, we obtain an infinite series of pairwise nonhomeomorphic compact three-dimensional hyperbolic manifolds.

We consider the construction of the second series of manifolds. We draw two planes, τ_3 and τ_4, orthogonal to the planes β_4 and $\beta_{12\cdot 2^{k-2}}$, respectively, passing through the axis of the prism. For faces of screw motions of even dimension, we introduce one more gradation: faces of even screw motions β_{3i+1} and $\beta_{6\cdot 2^k+3i+1}$, where $i = 1, 5, 9, \dots, 2\cdot 2^k - 3$, are called faces of even screw motions of the first category, and faces β_{3i+1} and $\beta_{6\cdot 2^k+3i+1}$, where $i = 3, 7, 11, \dots, 2\cdot 2^k - 1$, are called faces of even screw motions of the second category. We refer an odd face to the first or second category if, under a turn by an angle of $\pi/3$ around the axis of the prism, it is the image of the face of an even screw motion of the first or second category, respectively. We identify all faces of the prism, except for odd faces, according to the same scheme as in constructing the first series of manifolds. Using a shift, we identify each of the odd faces of the first category with another odd face of the first category that is symmetric to it with respect to the plane τ_3, and we identify each of the odd faces of the second category with an odd face of the second category that is symmetric to it with respect to the plane τ_4 using a shift. The prism with such identification of the faces is a fundamental domain of a certain group Γ_2^k, and the motions identifying the faces are generators of the group Γ_2^k, and, moreover, the group Γ_2^k is a torsion-free group. Factorizing the Lobachevskii space H^3 by the groups Γ_2^k, we obtain an infinite series of pairwise nonhomeomorphic (compact) manifolds of constant negative curvature.

We consider the construction of the third series of manifolds. For this purpose, we use $12k$-gon prisms, where $k = 1, 2, \dots$.

We identify the bases of the prisms δ_1 and δ_2 using a screw motion with angle of turn $\pi/3k$ and axis u (the axis of the prism); this is a clockwise turn if we look from the upper base. We identify each lateral surface β_{3i+1} with the lateral surface $\beta_{6k+3i+1}$ using a screw motion B_{3i+1} with angle of turn $\varphi = \pi$ whose axis is the common perpendicular of the faces β_{3i+1} and $\beta_{6k+3i+1}$, and, moreover, $i = 0, 1, \dots, 2k-1$. We divide the remaining faces of the prism into two categories: the faces β_{3i+2} belong to the first category and the faces $\beta_{3(i+1)}$, where $i = 0, 1, \dots, 4k-1$, belong to the second category. We identify each of the faces of the first category with a face of the second category that is the image of a face of the first category under a turn by an angle of $\varphi = \pi\dfrac{3k+1}{6k}$ around the axis of the prism.

The group Γ_3^k generated by motions that identify the faces of the prism is a torsion-free group, and the initial prism is a fundamental polygon of the group Γ_3^k. Therefore, the quotient space H^3/Γ_3^k of the Lobachevskii space H^3 by the group Γ_3^k is a compact three-dimensional manifold of constant negative curvature.

We now describe two series of examples of noncompact manifolds of constant negative curvature whose volume is finite. To construct these examples, we use prisms in the Lobachevskii space H^3 with dihedral angles between lateral faces equal to $\pi/2$ such that dihedral angles between a lateral face and each of the bases of the prism equal to $\pi/4$ and all of whose vertices are infinitely distant points.

We consider a certain $4p$-gon prism and find the motions that identify the faces of the prism ($p \geq 2$). We identify the lower base δ_1 of the prism with its upper base δ_2 using a screw motion with angle of turn $\varphi = \pi/2p$ and axis u (the axis of the prism). We identify each lateral face β_{2i} of the prism with the face β_{2i+2p}, $i = 1, 2, \dots, p-1$, opposite to it using a shift and any lateral face β_{2i-1} with the face $\beta_{2p+2i-1}$ opposite to it using a screw motion with angle of turn $\varphi = \pi$, $i = 1, 2, \dots, p-1$. The prism with the identification of faces described above is a fundamental polygon of a certain discrete group Γ. Factorizing the Lobachevskii space H^3 by the group Γ, we obtain a noncompact manifold of constant negative curvature whose volume is finite.

To construct the second series of noncompact manifolds, we use $4(2p-1)$-gon prisms, where $p \geq 2$. We identify the lower base δ_1 of such a prism with its upper base δ_2 using a screw motion with angle of turn $\varphi = \pi/2$ and axis u (the axis of the prism). We identify each lateral face β_{2i-1}, $i = 1, \dots, 2p-1$, with the face $\beta_{2(2p-1)+2i-1}$ opposite to it by a screw motion whose angle of turn is $\varphi = \pi$ and whose axis is the common perpendicular of these faces. The faces β_{2p} and $\beta_{2(3p-1)}$ are identified by a shift. We identify the faces β_{2i}, $i = 1, 2, \dots, p-1, p+1, 2p-1$, with the faces $\beta_{4(2p-1)-2i+2}$ by screw motions. The prism with the identifications of faces described is a fundamental polygon of a certain discrete group of motions Γ of the Lobachevskii space. Factorizing the space H^3 by the group Γ, we obtain an infinite series of three-dimensional noncompact manifolds of finite volume and constant negative curvature.

Therefore, we now have a sufficiently rich set of examples of three-dimensional manifolds of constant negative curvature.

We now study the general properties of n-dimensional hyperbolic manifolds. We compare these properties with similar properties of two-dimensional manifolds. For $\dim M = 2$, we have compact manifolds of constant negative curvature. A similar statement also holds in the case of an arbitrary dimension: the Lobachevskii space H^n admits compact space forms H^n/Γ; this is implied by the general theorem of Borel [24].

As was mentioned, on a two-dimensional compact surface of genus $g > 1$, there exists a continuum of distinct metrics of constant curvature -1. This is no longer the case in dimension $n > 2$. In this case, the following remarkable Mostow theorem holds (see [137]).

Theorem 1.5.4 (algebraic version). *Let Γ_1 and Γ_2 be two discrete subgroups of the isometry group of the Lobachevskii space H^n, $n \geq 3$, and, moreover, let the manifold H^n/Γ_i, $i = 1, 2$, have a finite volume. If $\varphi : \Gamma_1 \to \Gamma_2$ is a group isomorphism, then Γ_1 and Γ_2 are conjugate subgroups.*

We can state this theorem using the language of hyperbolic manifolds, since the space H^n is a universal covering of a hyperbolic manifold and the fundamental group of this manifold acts as a discrete isometry group of the Lobachevskii space H^n.

Theorem 1.5.4 (geometric version). *Let M_1^n and M_2^n be complete hyperbolic manifolds of finite volume. Then any isomorphism $\varphi : \pi_1(M_1) \to \pi_1(M_2)$ of the fundamental groups $\pi_1(M_1)$ and $\pi_1(M_2)$ is realized by a unique isometry $f : M_1 \to M_2$.*

Since the universal covering a hyperbolic manifold is the Lobachevskii space H^n, any hyperbolic manifold is a manifold of type $K(\pi, 1)$. Therefore, any two such manifolds are homotopy equivalent iff there exists an isomorphism between their fundamental groups. As a consequence, we obtain that two hyperbolic manifolds of finite volume are homeomorphic iff they are homotopy equivalent.

For an arbitrary manifold M, there exists a homeomorphism

$$h : \mathrm{Diff}(M) \to \mathrm{Out}(\pi_1(M))$$

of the diffeomorphism group $\mathrm{Diff}(M)$ of the manifold M into the outer automorphism group

$$\mathrm{Out}(\pi_1(M)) = \mathrm{Aut}(\pi_1(M))/\,\mathrm{Int}(\pi_1(M))$$

(see Fig. 36).

Theorem 1.5.5. *Let M^n be a complete hyperbolic manifold of a finite volume, and let $n \geq 3$. Then the group $\mathrm{Out}(\pi_1(M))$ is of finite order and isomorphic to the isometry group of the manifold M.*

This result shows a sharp distinction of the dimension $n \geq 3$ from $n = 2$ when the group $\mathrm{Out}(\pi_1(M^2))$ is infinite.

The study of hyperbolic manifolds of finite volume is based on the following important Margulis lemma. We present here only a certain particular case of this lemma.

Lemma 1.5.1. *For any $n \geq 2$, there exists $C(n) > 0$ such that for $0 < \varepsilon < c(n)$ and for any point $x \in H^n$, the group $\Gamma_\varepsilon(x)$ generated by those isometries γ of the space H^n for which $\rho(x, \gamma(x)) < \varepsilon$ contains an Abelian subgroup of finite index.*

In conclusion, we present a theorem of Thurston that shows a varirty of hyperbolic three-dimensional manifolds.

Theorem 1.5.6. *Let M be a smooth closed oriented three-dimensional manifold satisfying the following three conditions*:

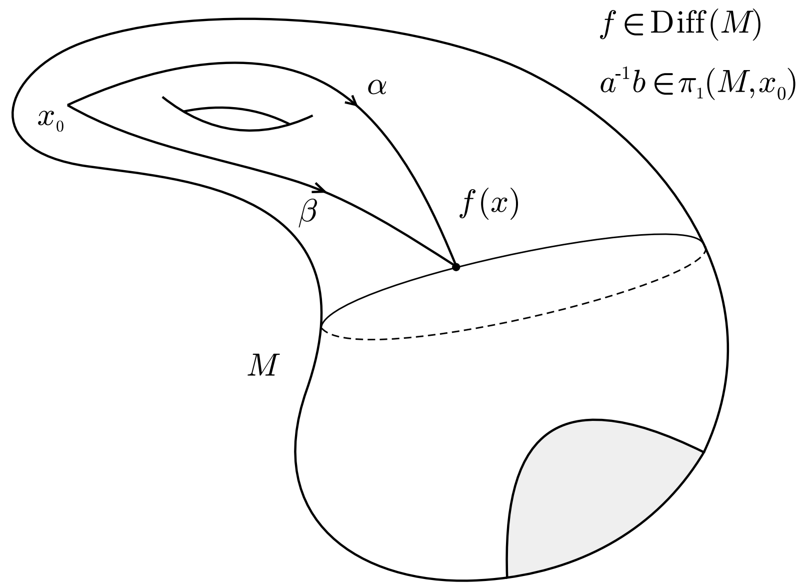


Fig. 36

(1) *its universal covering space is diffeomorphic to* $\mathbb{R}^3$;
(2) *the fundamental group of the manifold* M *does not contain a subgroup isomorphic to* $\mathbb{Z} \oplus \mathbb{Z}$;
(3) *the manifold* M *contains an embedded two-sided surface of genus* $g > 0$ *such that the mapping of the fundamental groups induced by the embedding is injective. Then there exists a metric of constant negative curvature* -1 *on* M.

2. Riemannian Manifolds with Restrictions on the Sectional Curvature

2.1. Compact Riemannian manifolds of nonnegative sectional curvature. The topological classification of closed Riemannian manifolds M^n with $K(\sigma) > 0$, and the more so, with $K(\sigma) \geq 0$ remains an important unsolved problem. We begin with the construction of examples of Riemannian manifolds with $K(\sigma) \geq 0$. Let G be an n-dimensional Lie group, and let H be its closed Lie subgroup. The quotient space G/H is a smooth manifold. Moreover, the natural projection $\pi : G \to G/H$ is a smooth bundle with structural group H. On G, let there exist a bi-invariant Riemannian metric $\langle X, Y \rangle$. We choose the subspace $\mathfrak{M}$ complementary to $\mathfrak{Y}$ and orthogonal to $\mathfrak{Y}$ (here $\mathfrak{Y}$ is the Lie algebra of the Lie group H). For each point $\bar{g} \in G/H$ in the space $T_{\bar{g}}(G/H)$, there exists a unique metric $\langle X, Y \rangle_{\bar{g}}$ such that dp_g maps isometrically the orthogonal complement $L_{g*}\mathfrak{M}$ to $L_{g*}\mathfrak{Y}$ in T_gG into $T_{\bar{g}}(G/H)$. The mapping $\bar{g} \mapsto \langle X, Y \rangle_{\bar{g}}$ is a Riemannian metric on the space G/H, which is called a normal homogeneous metric on G/H. Therefore, G/H is a Riemannian manifold, which is called a *normal homogeneous space*.

Proposition 2.1.1 ([71])**.** *Normal homogeneous spaces are always of nonnegative sectional curvature* $K(\sigma) \geq 0$.

Among these spaces, only several symmetric spaces of rank one are simply connected and have $K(\sigma) > 0$. Along with S^n, they are the projective spaces $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathbb{C}\mathrm{a}P^2$ and also spaces of the type $\mathrm{Sp}(2)/\mathrm{SU}(2)$ (of dimension 7) and $\mathrm{SU}(5)/\mathrm{Sp}(2) \times S^1$ (of dimension 13; see, e.g., [71]).

Among even-dimensional simply connected homogeneous spaces, along with the space mentioned above, only the spaces $\mathrm{SU}(3)/T$, $\mathrm{Sp}(3)/\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, and $F_4/\mathrm{spin}(9)$, where T is the maximal torus (see [209]), admit a metric with $K(\sigma) > 0$.

For $n = 7$, there exist infinitely many pairwise nonhomeomorphic compact manifolds that admit a metric with $K(\sigma) > 0$. These are the spaces of type $\mathrm{SU}(3)/T^1$, where T^1 is a closed connected one-dimensional subgroup in SU(3) having no fixed points. More precisely, these manifolds are described as follows: $M_{p,q} = \mathrm{SU}(3)/U_{p,q}$, where p and q are positive integers and

$$U_{p,q} = \{\exp(2\pi it\,\mathrm{diag}(p, q, -p, -q)) \mid t \in \mathbb{R}\}.$$

Until recently, no other spaces with $K(\sigma) > 0$, except for homogeneous spaces, were known. The paper [56] by Eschenburg constructed examples of inhomogeneous Riemannian manifolds with $K(\sigma) > 0$. We describe these examples (for more details, see [56]).

Let $G = \mathrm{SU}(3)$, and let U be a closed one-parameter subgroup in $G^2 = G \times G$. We define the action of the group U on G by setting

$$(u, g) \mapsto u_1 g u_2^{-1}, \quad g \in G, \quad u = (u_1, u_2) \in U.$$

Let

$$W = 2\pi i(\mathrm{diag}(k, l, -k, -l), \quad \mathrm{diag}(p, q, -p, -q)).$$

The subgroup $\exp tW$, $t \in \mathbb{R}$, is denoted by U_{klpq}. Let $K = \mathrm{U}(2) \subset G$ be a subgroup canonically embedded in U(3). We fix a Riemannian metric on G that is invariant with respect to left translations by elements of the group G and right translations by elements of the group K and induces a metric of a strictly positive sectional curvature $K(\sigma) > 0$ on $M_{pq} = G/U_{pq}$ for arbitrary integers p and q.

For fixed numbers p and q such that $pq(p + q) \neq 0$, let $a_1, \dots, a_k$ be the set of all prime numbers that divide $pq(p + q)$. We set $n_i = 3ia_1a_2 \dots a_k$ and

$$M_i = \mathrm{SU}(3)/U_{1,0,n_ip,n_iq}.$$

A compact topological space is said to be *strictly inhomogeneous* if it is not homotopy equivalent to any homogeneous Riemannian manifold. There is the following topological characterization of the strict inhomogeneity (see [56]). Let $G = \mathrm{SU}(3)$, let $\mathrm{U}(1) \cong U \subset G \times G$ without fixed points, and let $M = G/U$. If

$$H^4(M) = \mathbb{Z}_r, \quad r \equiv 2 \pmod 3,$$

then the manifold M is strictly inhomogeneous. In our case, the space $M = \mathrm{SU}(3)/U_{klpq}$ is of the following cohomologic structure: the ring $H^*(M)$ is generated by elements $w \in H^2(M)$ and $z \in H^5(M)$, and the relations $rw^2 = 0$, $w^3 = 0$, $zw^2 = 0$, and $z^2 = 0$, where $r = |(k^2 + l^2 + kl) - (p^2 + q^2 + pq)|$, hold. The homotopic structure of the space $M = G/U$, where $G = \mathrm{SU}(3)$ and $\mathrm{U}(1) \cong U \subset G \times G$, is described by the relations $\pi_1(M) = 0$, $\pi_2(M) = \mathbb{Z}$, and $\pi_i(M) = \pi_i(\mathrm{SU}(3))$, $i \geq 3$.

Proposition 2.1.2. *The spaces M_i have the following properties*:

(a) $\pi_1(M_i) = 0$;
(b) *M_i is a strictly inhomogeneous Riemannian manifold*;
(c) *the manifolds M_i are pairwise homotopy nonequivalent*;
(d) *$K(\sigma) > 0$ for all integers $p, q > 0$ and any i.*

Before passing to the general facts about manifolds of positive or nonnegative sectional curvature, we mention the Hopf conjecture, which concerns four-dimensional manifolds. The sphere S^4, the real projective space $\mathbb{R}P^4$, and the complex projective plane $\mathbb{C}P^2$ are the only known examples of compact four-dimensional manifolds of strictly positive sectional curvature. Among the known examples of manifolds of nonnegative sectional curvature are two new types: products of manifolds of lower dimension and zero or positive curvature and connected sums $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$.

The following is the Hopf conjecture: on the product $S^2 \times S^2$, there is no metric of strictly positive sectional curvature.

We begin our consideration of manifolds of positive curvature with a remark that indicates distinctions between the cases of even and odd dimension. The following fundamental theorem holds.

Theorem 2.1.1 ([71])**.** *For a connected closed manifold M^{2m}, $m \geq 1$, of sectional curvature $K(\sigma) > 0$, the orientability is equivalent to the simple connectedness. If such a manifold M^{2m} is nonorientable, then its fundamental group $\bar{\pi}_1(M^{2m})$ is isomorphic to the group $\mathbb{Z}_2$. An odd-dimensional Riemannian manifold M^n with $K(\sigma) > 0$ is always orientable.*

The causes of such a dependence of the properties of manifolds on the evenness and oddness will be clear from the scheme of the proof. On an oriented even-dimensional manifold M^{2m}, along a closed geodesic C, there exists a parallel field composed of vectors orthogonal to C. Indeed, let us enclose a bundle of unit vectors orthogonal to C along C. Their ends forming the sphere S^{2m-2} again fill in this sphere. Therefore, there exists a vector that returns to the initial position after the enclosing. By a parallel translation along C, it generates the field required. Similarly, in an nonorientable odd-dimensional manifold M^{2m+1}, there exists a similar field along a closed geodesic C that has no oriented neighborhood. For $K(\sigma) > 0$, the existence of such a field allows us to shorten the geodesic C by varying it along this field.

The inequality $K(\sigma) \geq 0$ also yields strong restrictions on the topology of a manifold. One of the most important topological invariants is the homology group. The simplest bound on the homology groups is given by the following theorem.

Theorem 2.1.2 ([75])**.** *There exists a constant $C = C(n)$ such that each compact n-dimensional Riemannian manifold M of negative curvature $K(\sigma) \geq 0$ satisfies the inequality*

$$\sum_{i=0}^{n} b_i \leq C,$$

where b_i are Betti numbers of the manifold M, i.e.,

$$b_i = b_i(M, F) = \dim_F H_i(M; F),$$

and F is a certain field.

As a consequence of this theorem, we obtain the following: the connected sum of sufficiently many copies of the products of spheres $S^p \times S^{n-p}$, $0 < p < n$, or complex projective spaces does not admit metrics of nonnegative sectional curvature.

The next topological invariant is the fundamental group of a manifold. If a manifold admits a metric with $K(\sigma) \geq 0$, then this yields restrictions on the fundamental group of this manifold.

Theorem 2.1.3. *Let M^n be a complete Riemannian manifold of sectional curvature $K(\sigma) \geq 0$. Then its fundamental group can be generated by no more than $2 \cdot 5^{\frac{1}{2}n}$ elements.*

We note a close result concerning the number of generators of the fundamental group of manifolds whose sectional curvature is bounded from below.

Theorem 2.1.4. *Let M^n be a compact Riemannian manifold of diameter $d(M) < D/2$ and sectional curvature $K(\sigma) \geq -\Lambda^2$. Then its fundamental group $\pi_1(M)$ can be generated by no more than $2 \cdot (3 + 2\operatorname{ch}\Lambda D)^{\frac{1}{2}n}$ elements.*

The few number of examples of manifolds with $K(\sigma) > 0$ leads to the fact that first of all, one study conditions under which manifolds with $K(\sigma) > 0$ are diffeomorphic or homeomorphic to the model spaces, first of all, to the sphere S^n with the standard metric. The first result of such a type is the theorem on the "sphere."

Definition 2.1.1. A metric of a complete Riemannian manifold M^n of sectional curvature $K(\sigma) > 0$ can be assumed to be normalized by multiplying it by a constant such that the sectional curvature $K(\sigma)$ satisfies the inequality $\delta \leq K(\sigma) \leq 1$. In this case, the manifold is called a *manifold with δ-clamped curvature*.

We have the following fundamental theorem on the "sphere," which generated an entire direction in studying Riemannian manifolds.

Theorem 2.1.5. *A complete simply connected Riemannian manifold M^n of δ-clamped curvature is homeomorphic to the sphere S^n for $\delta > \frac{1}{4}$.*

For even n, the bound is sharp here: for $\delta = \frac{1}{4}$, there exist spaces that are not homeomorphic to the sphere S^n. These are exactly symmetric spaces of rank one. For odd n, the theorem remains valid also for $\delta = \frac{1}{4}$. The theorem on the "sphere" stated above with nonexact value of δ was proved by Rauch in [165]. Then it was improved by a number of authors: Berger, Klingenberg, and Toponogov (see [17,18,108,196]). The final statement was obtained by Klingenberg in [109, 110]. A detailed proof of this theorem can be found in [71].

In Theorem 2.1.5, for $n = 2$, the value $\delta > 0$ is not essential; by the Gauss–Bonnet theorem, a two-dimensional manifold M^2 with $K > 0$ has a positive Euler characteristic, and in the case where M^2 is simply connected, it is homeomorphic to the sphere S^2. This manifold is diffeomorphic to the sphere S^2.

As was shown in [99], there exists a sequence of numbers $0 < \delta(n) < 1$ such that $\varliminf_{n\to\infty} \delta(n) \leq 0.68$, and if a complete simply connected manifold M satisfies the condition $\delta(n) < K(\sigma) \leq 1$, then M^n is diffeomorphic to the sphere S^n.

Therefore, each simply connected complete Riemannian manifold M with $K(\sigma)$ sufficiently close to 1 is homeomorphic to S^n. There arises a natural question on the fulfillment of a similar theorem for a nonsimply connected manifold, naturally, with the replacement of S^n by the corresponding spherical form, i.e., one asks is it true that the topology of spherical forms is stable with respect to a small perturbation of the curvature? In the case of nonsimply connected manifolds M^n, the simplest case is $\pi_1(M^n) \cong \mathbb{Z}_2$. In this case, $\delta = 0.56$ is enough for the homeomorphity, and $\delta = 0.7$ is enough for the diffeomorphity of the manifold M^n to the projective space $\mathbb{R}P^n$.

In the general case, we have the following result.

Theorem 2.1.6 ([84, 99])**.** *There exists δ_0, $0 < \delta_0 < 1$, such that for any δ, $\delta_0 < \delta \leq 1$, we have the following. If, on a simply connected complete Riemannian manifold M^n with $\delta < K(\sigma) \leq 1$, a Lie group G acts by isometries, then there exist a diffeomorphism $F : M^n \to S^n$ and a homomorphism $\varphi : G \to \mathrm{O}(n+1)$ such that*

$$(\varphi(g))(x) = F(g(F^{-1}(x))) \quad \textit{for all} \quad g \in G, \quad x \in S^n.$$

This theorem is a stability theorem for spherical space forms and their isometry groups. In particular, it implies that if $\delta < K(\sigma) \leq 1$, then M^n is diffeomorphic to the spherical space form S^n/Γ, where the group Γ is isomorphic to $\pi_1(M^n)$.

The exact value of δ (universal as well as a particular one for each n) is not known. We have only its estimates.

We now consider the so-called extremal theorems. Restrictions on the curvature imply estimates of other geometric characteristics. Such estimates are usually related to the comparison theorems. Extremal theorems are referred to the behavior of manifolds under the imposing of restrictions on them that are related to limit estimates of various geometric quantities.

The inequality $K(\sigma) \geq 1$ implies $d(M) \leq \pi$. Therefore, the following statement is an extremal theorem. If $K(\sigma) \geq 1$ and $d(M) \leq \pi$, then the manifold M^n is isometric to the sphere S^n.

In even dimension, the inequality $K(\sigma) \geq 1$ implies that the length of a closed geodesic is not less than 2π. We have the following extremal theorem by Toponogov. If $K(\sigma) \geq 1$ and there is a closed geodesic on a manifold M^n whose length is 2π (the dimension n is even), then M^n contains a totally geodesic submanifold isometric to the sphere S^2. More precisely, if the index $\operatorname{Ind} l$ of a closed geodesic l of length 2π equals k, then M contains a sphere of dimension k.

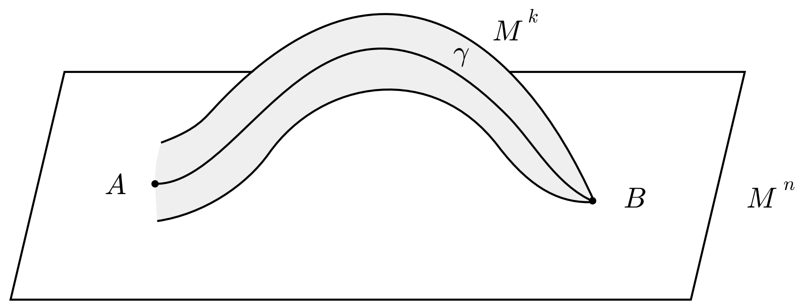


Fig. 37

If the diameter $d = d(M)$ of a manifold M satisfies the inequality $d \geq \pi/2$, then the topology is described only with accuracy up to a homotopy equivalence. More precisely, Nagayushi and Tsukamoto proved the following statement (see [140, 204]). If M^n is a complete Riemannian manifold such that $K(\sigma) \geq 1$ and its diameter d satisfies the inequality $d \geq \pi/2$, then M^n is a homotopic sphere. If $K(\sigma) \geq 1$ and its volume $v(M)$ satisfies the inequality $v(M^n) \geq \frac{1}{2}v(S^n)$, then M^n is either a homotopic sphere or isometric to the projective space $\mathbb{R}P^n$ of constant curvature; if, moreover, M^n contains a periodic geodesic of length $l = \pi$, then the manifold M^n is isometric to the space $\mathbb{R}P^n$ with the standard Riemannian structure.

We mention the following beautiful result that strengthens the previous results.

Theorem 2.1.7 ([85]). *If the conditions $K(\sigma) \geq \delta > 0$ and $d(M^n) \geq \pi/2\sqrt{\delta}$ hold for a complete Riemannian manifold M^n, then the manifold M^n is homeomorphic to the sphere S^n.*

Therefore, a manifold of a positive curvature $K(\sigma) \geq \delta > 0$ cannot be very large.

For a manifold of a positive curvature, the injectivity radius admits a lower bound (cf. Sec. 6.1 of the previous chapter). This bound is described in the following Klingenberg theorem.

Theorem 2.1.8. *In a closed Riemannian space M^n of positive curvature, for even n, the injectivity radius of the exponential mapping is uniformly estimated from below: $i(x) \geq \pi$ for $0 < K(\sigma) \leq 1$, n is even. For an odd n, a similar estimate holds under an additional restriction on the curvature: $i(M^n) \geq \pi$ for $1/4 < K(\sigma) \leq 1$ and n is odd.*

The proof of this statement is based on Theorem 2.1.1, more precisely, on the construction that shortens the deformations and is described above (see [71]). Moreover, one uses the following estimate of the injectivity radius $i(M)$, which holds in an arbitrary Riemannian manifold M all of whose sectional curvatures $K(\sigma)$ satisfy the inequality

$$K(\sigma) \leq 1 : i(M) = \min\{i(x) \mid x \in M\} \geq \min(\pi, l/2),$$

where l is the length of the shortest periodic geodesic that is not degenerate into a point.

2.2. Noncompact Riemannian manifolds of nonnegative sectional curvature. The noncompact case was initially considered by Cohn-Vossen.

Theorem 2.2.1. *In dimension 2, a noncompact complete Riemannian manifold of negative curvature is either diffeomorphic to the plane $\mathbb{R}^2$ or flat.*

The proof of this statement can be found, e.g., in [113].

Definition 2.2.1. A submanifold M^k of a Riemannian manifold M^n is said to be *absolutely convex* if each geodesic segment with ends in M^k lies entirely in M^k (see Fig. 37).

The concept of convexity plays a fundamental role in studying noncompact manifolds of nonnegative curvature. The ideas related to convexity form an independent field of geometry having its own methods.

The structure of noncompact Riemannian manifolds of sectional curvature $K(\sigma) \geq 0$ is revealed by the following fundamental theorem, which plays a key role in studying the structure of Riemannian manifolds with $K(\sigma) \geq 0$.

Theorem 2.2.2. *An open Riemannian manifold M^n satisfying the condition $K(\sigma) \geq 0$ contains a closed absolutely convex totally geodesic submanifold M^k, $0 \leq k < n$, without boundary such that M^n is diffeomorphic to the space $\nu(M^k)$ of the normal bundle of the submanifold M^k in M^n.*

The manifold M^k is called a *soul*. This theorem with the replacement of a diffeomorphism by a homeomorphism was formulated by Cheeger and Gromov. It was independently strengthened by Sharafutdinov [184].

In general, the soul of a manifold is not uniquely defined (even in the homogeneous case). Sharafutdinov proved the following results related to the uniqueness of the soul.

Theorem 2.2.3. *Let S_1 and S_2 be two souls of a complete open manifold M of nonnegative curvature $K(\sigma) \geq 0$. Then there exists a diffeomorphism of the manifold M onto itself that isometrically maps S_1 onto S_2.*

Theorem 2.2.4. *Let S be a soul of a complete open manifold M of nonnegative curvature $K(\sigma) \geq 0$. Then S is a unique soul if the normal bundle $\nu(S)$ admits a nonzero parallel section.*

Theorem 2.2.2 implies the structural theorem concerning the topological structure of manifolds with $K(\sigma) > 0$. The topology of such manifolds is simple.

Theorem 2.2.5. *Any complete open Riemannian manifold of positive curvature $K(\sigma) > 0$ is diffeomorphic to the Euclidean space.*

In the paper by Gromol and Meyer, this theorem was proved first for dimension $\dim M \geq 5$. A complete version of this theorem was obtained by Sharafutdinov in [184]. In proving this theorem, one uses sufficiently deep topological facts.

The topological structure of manifolds admitting metrics with $K(\sigma) \geq 0$ is revealed by the following fundamental theorem of Toponogov.

Theorem 2.2.6. *An open Riemannian manifold M^n of sectional curvature $K(\sigma) \geq 0$ is isometric to the direct metric product $M^{n-k} \times \mathbb{R}^k$, $0 \leq k \leq n$, where the manifold M^{n-k} contains no lines, i.e., geodesics that are minimal arcs on any part of them.*

We present a scheme for proving this theorem. Let M contain a line $\gamma : \mathbb{R} \to M$. We consider the horospheres

$$D_+^a = \bigcup_{t>a} D(\gamma(t), t-a) \quad \text{and} \quad D_-^a = \bigcup_{t>a} D(\gamma(-t), t-a),$$

where $a > 0$; the balls $D(x, r)$ are taken with the intrinsic metric of the manifold M. One verifies that the set $Q = M \setminus (D_+^a \cup D_-^a)$ is absolutely convex in M. In this case, $Q \neq \varnothing$, since $\gamma(0) \in Q$. The geodesics emanating from $\gamma(0)$ that are orthogonal to γ fill in an $(m-1)$-dimensional absolutely convex subset M' that is equidistant from D_+^a and D_-^a. This easily implies that M is isometric to $M' \times \mathbb{R}$. If M' contains a line, the process can be repeated. In this way, we obtain a manifold M_0 containing no lines.

This theorem, together with Theorem 2.2.2, yields a complete classification of noncompact three-dimensional manifolds with $K(\sigma) \geq 0$. We do not present their list here; the reader can find it in [113].

A generalization of Theorem 2.2.6 was obtained by Cheeger and Gromol: instead of the inequality $K(\sigma) \geq 0$, it suffices to require that the Ricci tensor be nonnegative: $\mathrm{Ric} \geq 0$.

Theorem 2.2.6 implies the decomposition of the isometry group $\mathrm{Iso}(M)$ of the manifold M into the direct product

$$\mathrm{Iso}(M) = \mathrm{Iso}(\bar{M}) \times \mathrm{Iso}(\mathbb{R}^k);$$

moreover, the isometry group $\mathrm{Iso}(\bar{M})$ is compact. The manifold M contains a line if the isometry group $\mathrm{Iso}(M)$ is not compact.

In the homogeneous case, the Toponogov theorem can be strengthened.

Theorem 2.2.7. *If M is a homogeneous space of sectional curvature $K(\sigma) \geq 0$, then M is isometric to the direct product*

$$\bar{M}^{n-k} \times \mathbb{R}^k, \quad 0 \leq k \leq n,$$

where M^{n-k} is a compact homogeneous space of nonnegative curvature.

We can give a complete classification of homogeneous spaces with $K(\sigma) \geq 0$. For this purpose, we need certain preparatory considerations. For a given Riemannian manifold S and for $p \in S$, we have a natural representation Γ of the isometry group $\mathrm{Iso}(S)$ in the group $\mathrm{Out}(\pi)$ of all outer automorphisms of the fundamental group $\pi = \pi_1(S, p)$. It is defined as follows. For each element $g \in \mathrm{Iso}(S)$, we choose a path φ connecting a point p with $g(p)$. In each class $\{h\} \in \pi$, we take a representative h. Then $\Gamma(g)$ is the class of the automorphism $\{h\} \to \{-\varphi \cdot g(h) \cdot \varphi\}$. This definition does not depend on the choice of φ. Let C_π be the set of equivalence classes of orthogonal representations of the group π that do not contain trivial summands in a certain fixed Euclidean space V, and let PC_π be the group of permutations of the set C_π. We have a natural homomorphism Δ of the group $\mathrm{Out}(\pi)$ into the group PC_π that is defined as follows. Let r be a representative of the class $[r] \in C_\pi$, and let s be a representative of the class $[s] \in \mathrm{Out}(\pi)$. Then $\Delta([s])([r])$ is the equivalence class for $r \cdot s$. We have the following main classification theorem for homogeneous Riemannian manifolds with $K(\sigma) \geq 0$.

Theorem 2.2.8. *Let S be a compact locally homogeneous space of nonnegative sectional curvature with isometry group $I = I(S)$ and fundamental group $\pi = \pi_1(S, p)$, and let V be a fixed Euclidean space. There exists a one-to-one correspondence between the isometry classes of irreducible complete noncompact locally homogeneous spaces of nonnegative curvature whose soul is isometric to S and bundle of dimension* $\dim V$ *and the elements of the set $C_\pi/(\Delta \circ \Gamma)(I)$.*

In conclusion, we describe the structure of the fundamental group of manifolds with metric of nonnegative sectional curvature. We have the following statement.

Theorem 2.2.9. *Consider a complete Riemannian manifold M with $K(\sigma) \geq 0$ and soul S. Then $\pi = \pi_1(M) \cong \pi_1(S)$. There exists an invariant finite subgroup $\Phi \subset \pi$ such that $\pi^* = \pi/\Phi$ is isomorphic to the crystallographic group, π^* contains an Abelian normal free subgroup $\Gamma \cong \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$ of rank k, $0 \leq k \leq \dim S \leq \dim M$, and the group π^*/Γ is finite. In particular, M is a compact flat manifold if $k = \dim M$.*

As a consequence, we have that in the case where π is a finite group, the Euler characteristic $\chi(M)$ of the manifold M is zero. If, in addition, M is compact, then all real Pontryagin numbers vanish.

2.3. Riemannian manifolds of negative and nonpositive sectional curvature. The theory of manifolds with $K(\sigma) < 0$ (or $K(\sigma) \leq 0$) is a field of Riemannian geometry, which is very rich in content and is related to various directions of modern geometry. It is not possible to give a complete survey of all of these directions; for example, the theory of geodesic flows on manifolds with $K(\sigma) < 0$ became an independent field with its own methods.

2.3.1. Cartan and Hadamard theorem. The following theorem describing the topological structure of Riemannian manifolds with $K(\sigma) < 0$ was proved by Hadamard for surfaces, and later on, E. Cartan generalized it to Riemannian manifolds of higher dimension (see [71]). It turns out that from the topological viewpoint Riemannian manifolds with $K(\sigma) < 0$ are of a very simple structure: they are diffeomorphic to the space $\mathbb{R}^n$.

Theorem 2.3.1. *Let M^n be a complete simply connected Riemannian manifold of dimension $n \geq 2$ such that the sectional curvatures $K(\sigma) \leq 0$ for all directions σ. Then for each point $p \in M^n$, the exponential mapping is a diffeomorphism. In particular, $M^n \cong \mathbb{R}^n$.*

We present a scheme for proving this theorem. Since the manifold M^n is simply connected, the space $\Omega(M^n, p, q)$ is connected. Any two points p and q of the manifold M^n are not conjugate along any geodesic (see Theorem 2.3.2 of Chap. 2). Therefore, every geodesic is of index zero. The Morse theorem implies that the space $\Omega(M, p, q)$ has the cell complex homotopy type whose dimension is zero, and one zero-dimensional cell (a point) corresponds to each geodesic. By the connectedness, $\Omega(M, p, q)$ has only one vertex, and, therefore, the points p and q are connected by a unique geodesic. Therefore, the exponential mapping of the tangent space onto the manifold is one-to-one, which proves the theorem.

Definition 2.3.1. A complete simply connected Riemannian manifold is called an *Hadamard manifold* if all its sectional curvatures are nonpositive.

Let M^n be a complete Riemannian manifold with nonpositive sectional curvatures, and let $\dim M^n = n \geq 2$. Since the universal covering $\tilde{M}$ over M is diffeomorphic to the space $\mathbb{R}^n$ by the Hadamard–Cartan theorem and, in particular, is contractible into a point in itself, we have that M is an Eilenberg–McLane space of type $K(\pi, 1)$, where $\pi = \pi_1(M)$ is the fundamental group of the manifold M (the definition and properties of Eilenberg–McLane spaces can be found, e.g., in [49, p. 101]).

2.3.2. Geometry and fundamental group. Therefore, a complete Riemannian manifold of nonpositive sectional curvature is, in fact, determined by its fundamental group $\pi_1(M)$. Therefore, the geometry of such Riemannian manifolds is reflected in the algebraic properties of the fundamental group. We present several statements of such a type, which show how various algebraic properties of the fundamental group are reflected in geometry of Riemannian manifolds and vice versa.

Let M^n be a compact Riemannian manifold with fundamental group π and nonpositive sectional curvature $K(\sigma) \leq 0$. We have the following important theorem on "flat tori," which relates Abelian subgroups in π and flat submanifolds (see [119]).

Theorem 2.3.2. *There exists an Abelian subgroup of rank k in the group π iff M contains an embedded totally geodesic flat k-dimensional torus.*

If the fundamental group falls into a direct product of subgroups, then the manifold considered is a direct product of submanifolds. More precisely, the following beautiful "splitting" theorem holds (see [119]).

Theorem 2.3.3. *Let the group π have no center. If $\pi = A_1 \times \ldots \times A_N$ is a direct product of subgroups A_i, $1 \leq i \leq N$, then the manifold M is isometric to the direct product $M = M_1 \times \ldots \times M_N$, where $\pi_1(M_k) = A_k$ for $1 \leq k \leq N$.*

The center of the fundamental group π admits a useful geometric description, which is given in the theorem on the "center" (see [119]).

Theorem 2.3.4. *Let C be the center of the group π. Then $C \cong \mathbb{Z}^k$ for a certain $k \geq 0$, and there exists a foliation of the manifold M into totally geodesic flat k-dimensional tori. Moreover, there exists an Abelian covering $T^k \times M' \to M$ over M, where T^k is a flat torus. Let $N = \pi_1(M')$, and let A be the Abelian automorphism group of the covering. Then N is a normal subgroup in π containing the commutant $[\pi, \pi]$ and the sequences*

$$1 \to C \times N \to \pi \to A \to 1$$

and

$$0 \to C \times (N/[\pi, \pi]) \to H_1(M, \mathbb{Z}) \to A \to 0$$

are exact.

The center of the fundamental group can be described by using the language of vector fields on a manifold; more precisely, the following statement holds.

Theorem 2.3.5 ([119])**.** *Let M be a compact Riemannian manifold of sectional curvature $K(\sigma) \leq 0$. Then there exist exactly k linearly independent vector fields on M iff the center of the group $\pi_1(M)$ is of rank k.*

If M is a compact manifold of negative sectional curvature $K(\sigma) < 0$, then any isometry that is homotopic to the identity mapping is itself the identity mapping. In [119], this property was generalized to the case of compact manifolds of nonpositive sectional curvature.

Theorem 2.3.6. *Let M be a compact Riemannian manifold of nonpositive sectional curvature $K(\sigma) \leq 0$, and let there exist a nontrivial isometry f of the manifold M such that $f \sim 1$. Then*

(a) *the manifold M admits a parallel nonzero vector field*;
(b) *the center of the group $\pi_1(M)$ is nontrivial* (*and the assertion of the theorem on the "center" holds*);
(c) *there exists a locally free action of the torus T^k on the manifold M* (*i.e., the stationary subgroup of each point is finite*) *by isometries*; *moreover, $f \in T^k$.*

The isometry group of manifolds of negative sectional curvature is, as a rule, very small. More precisely, as a consequence of Theorem 2.3.6, we obtain the following statement. Let M be the same as in Theorem 2.3.6, and let one of the following assumptions hold:

(a) the curvature $K(\sigma) < 0$;
(b) the Euler characteristic $\chi(M) \neq 0$;
(c) the center of the group $\pi_1(M)$ is trivial;
(d) the first Betti number of the manifold M is zero.

Then an isometry of the manifold M that is homotopic to the identity mapping is the identity mapping. In particular, the isometry group of such a manifold is finite.

Let M be the same as in Theorem 2.3.6, and let $I(M)$ be its isometry group. Then

(a) a connected component $I_0(M)$ of the group $I(M)$ coincides with the torus $I_0(M) = T^k$, where k is the rank of the center of the group $\pi_1(M)$;
(b) if $f \in I(M) \setminus I_0(M)$, then f is not homotopic to the identity mapping.

Some of the results listed above can be extended to the case of complete manifolds with $K(\sigma) \leq 0$ (see [119]). Let M be a complete Riemannian manifold of sectional curvature $K(\sigma) \leq 0$ and of finite volume $\mathrm{vol}(M) < \infty$. Then the assertion of the theorem on the "center" holds. If, moreover, the group $\pi_1(M)$ is finitely generated, then the assertions of Theorems 2.3.5 and 2.3.6 and also all their corollaries hold for the manifold M.

In connection with the consideration of the fundamental group of manifolds of nonpositive curvature, we demonstrate certain features of the de Rham decomposition of such manifolds. Let M be a Riemannian manifold of sectional curvature $K(\sigma) \leq 0$. In this case, the de Rham decomposition obeys additional (to the general case) interesting properties. The dimension of the Euclidean factor is expressed through the rank of the fundamental group. More precisely, the following statement holds.

Theorem 2.3.7. *Let M be a complete Riemannian manifold of finite volume and nonpositive sectional curvature. Then the dimension of the Euclidean factor in the local de Rham decomposition is equal to the rank of the unique maximal Abelian normal subgroup of the fundamental group $\pi_1(M)$ of the manifold M.*

The proof of this theorem can be found in [51].

As a consequence, we obtain a number of properties of such manifolds. Let M_1 and M_2 be Riemannian manifolds of nonpositive curvature and finite volume with isomorphic fundamental groups. Then the dimension of the Euclidean factor in the de Rham decompositions of the manifolds M_1 and M_2 is the same. Let M_1 and M_2 be compact Riemannian manifolds with $K(\sigma) \le 0$ and isomorphic fundamental groups: $\pi_1(M_1) \cong \pi_1(M_2)$; let $H_1 = H_0^{(1)} \times H_1^{(1)} \times \ldots \times H_k^{(1)}$ and $H_2 = H_0^{(2)} \times H_1^{(2)} \times \ldots \times H_j^{(2)}$ be the de Rham decompositions of the universal coverings of M_1 and M_2, respectively, and let $H_0^{(1)}$ and $H_0^{(2)}$ be the Euclidean factors; moreover, let the factors be ordered in such a way that

$$\dim H_i^{(1)} \le \dim H_{i+1}^{(1)}, \quad 1 \le i \le k-1, \quad \text{and} \quad \dim H_r^{(2)} \le \dim H_{r+1}^{(2)}, \quad 1 \le r \le j-1.$$

Then $k = j$ and

$$\dim H_i^{(1)} = \dim H_i^{(2)}, \quad 0 \le i \le k$$

(see [51]).

In the framework of studying the fundamental group, we can introduce the concept of rank of a manifold.

Definition 2.3.2. The *rank* $\operatorname{rk} M$ of a manifold M is the rank of a free Abelian subgroup that is contained in $\pi_1(M)$.

The rank $\operatorname{rk} M$ is equal to the maximum dimension of a flat torus that is isometrically and totally geodesically immersed in M (see [35]). Therefore, $1 \le \operatorname{rk} M \le n$.

The manifold M is said to be *k-splittable*, $1 \le k \le n$, if the group $\pi_1(M)$ contains an invariant free Abelian subgroup of rank k (see [73]). As was proved by Yao, Lawson, Wolf, and Gromol, the following statement holds.

Theorem 2.3.8. *If a manifold M is a k-splittable Riemannian manifold, then its universal covering $\tilde{M}$ is isometric to $\tilde{M}' \times \mathbb{R}^k$, and, moreover, the action of the group $\pi_1(M)$ on $\tilde{M}$ preserves this decomposition. In particular, M is fibered into isometric totally geodesic immersed flat tori.*

2.3.3. Structure of the fundamental group. The fundamental group of manifolds admitting a metric with $K(\sigma) < 0$ ($K(\sigma) \le 0$) has a specific algebraic structure. First of all, we mention the following three fundamental properties of the groups $\pi_1(M)$ of such manifolds M.

(a) Let M be a complete Riemannian manifold of nonpositive sectional curvature. Then each element of the group $\pi_1(M)$ that is different from the identity is of infinite order.
(b) Let M be a compact manifold with Riemannian metric which has a positive curvature everywhere. Then the Abelian subgroup Γ of the fundamental group $\pi_1(M)$ is an infinite cyclic subgroup.
(c) The fundamental group $\pi_1(M)$ of a compact Riemannian manifold of strictly negative curvature cannot be Abelian.

The proof of these statements can be found, e.g., in [49].

Example. Let $M = M_g^2 \times M_h^2$, where M_g^2 and M_h^2 are two compact orientable surfaces of genus g and $h \ge 2$ and of negative curvature. The metric of the product M is of curvature $K(\sigma) \le 0$ everywhere. The properties of the fundamental group of manifolds of negative curvature indicated above imply that there is no metric of strictly negative curvature on M, since the group $\pi_1(M)$ contains a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Exactly in the same way, there is no metric of strictly negative curvature on the torus T^n.

The study of the structure of the fundamental group of manifolds of negative curvature became a separate important direction of geometry. Here many beautiful and important results were obtained, but there is no final answer for now. We describe the structure of fundamental groups of complete Riemannian manifolds of negative sectional curvature in detail.

In [58], Floyd obtained a necessary condition for a group G to be isomorphic to the fundamental group of a compact Riemannian manifold of negative curvature. We now need the construction for completion of the group. Let G be a finitely generated group, and let $\Sigma = \{g_1, \ldots, g_n\}$ be a finite set of its generators.

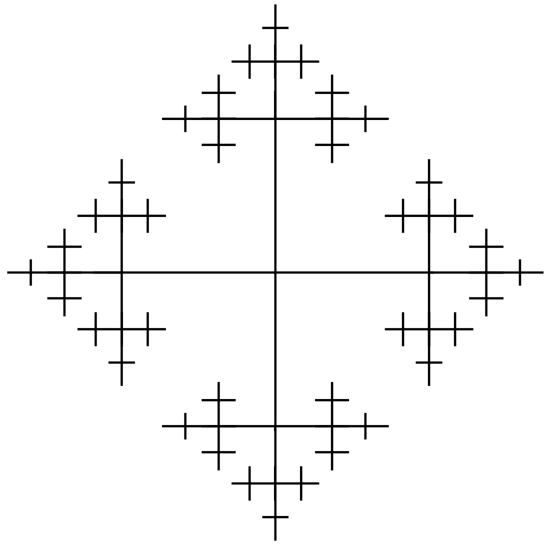


Fig. 38

For $g \in G$, we define the norm $|g|$ as the minimum length of a word composed of the generators $g_1, \dots, g_n$ that yields the element g. On G, we introduce the left-invariant metric $\rho(a,b) = |a^{-1}b|$ for $a, b \in G$. This metric depends on the choice of the set of generators, but with accuracy up to a quasi-isometry, the corresponding metric space does not depend on the choice of the set of generators. Two metric spaces are said to be *quasi-isometric* if there exists a mapping $f : X_1 \to X_2$ of the first metric space (X_1, ρ_1) onto the second metric space (X_2, ρ_2) for which there exists two constants $c, d > 0$ such that

$$c\rho_1(x,y) \leq \rho_2(f(x), f(y)) \leq d\rho_1(x,y) \quad \text{for all} \quad x, y \in X_1.$$

We now define the graph $K(G, \Sigma)$. The vertices of the graph $K(G, \Sigma)$ are in a one-to-one correspondence with the elements of the group G. Two vertices $a, b \in G$ are joined by an edge if $a = bg \pm 1$ for a certain element $g \in \Sigma$. The graph $K(G, \Sigma)$ is called the *Cayley diagram* or the *group diagram*. On $K(G, \Sigma)$, we define the metric $\rho(a,b) = \min(|a|^{-2}, |b|^{-2})$. Let $\bar{K}(G, \Sigma)$ be the completion of the graph $K(G, \Sigma)$ as a metric space. We define the *completion of the group G* by

$$\bar{G} = \bar{G}(\Sigma) = \bar{K}(G, \Sigma) \setminus K(G, \Sigma).$$

For the free product $G = \mathbb{Z} * \mathbb{Z}$ of two groups that are isomorphic to the group of integers $\mathbb{Z}$, we have $\Sigma = \{a, b\}$, the graph $K(G, \Sigma)$ is a tree, and $\bar{G}(\Sigma)$ is a Cantor set. In Fig. 38, we depict the subset of the graph of the group G consisting of vertices whose norm does not exceed 4.

The infinite cyclic group $\mathbb{Z}$ has the completion $\bar{\mathbb{Z}}$ consisting of two points.

We indicate simplest properties of the group completion (see [58]):

(1) The space $\bar{G}$ is a compact metric space.
(2) The group $\operatorname{Aut} G$ acts on the space $\bar{G}$, and, in particular, G acts on $\bar{G}$ (one uses the action of the group G on itself by inner automorphisms).
(3) If $H \subset G$ is a subgroup of finite index, then there exists a quasi-isometric homeomorphism $f : \bar{H} \to \bar{G}$.

Theorem 2.3.9 ([58])**.** *Let M be an n-dimensional $(n \geq 2)$ compact Riemannian manifold of class C^∞ and negative sectional curvature. Then the completion of the fundamental group of this manifold is homeomorphic to the $(n-1)$-dimensional sphere.*

Since the completion of a free group of rank $r \geq 2$ is an infinite metric space everywhere discontinuous, the fundamental group of a compact Riemannian manifold of negative sectional curvature is not free.

In order conclude what the curvature of a Riemannian manifold is using the properties of the fundamental group, we need a new concept. Let G be a certain set, and let X be a closed subset of the space $B(G)$ consisting of all complex-valued functions on G equipped with the uniform norm $\|f\|$. Let all constant functions belong to X, and let this space be invariant with respect to the complex conjugation. By definition, a linear functional m on X is called a *mean* if

(a) $m(\bar{f}) = \overline{m(f)}$ for all $f \in X$;
(b) $\inf\{f(x)\} \leq m(f) \leq \sup\{f(x)\}$ for all real-valued functions $f \in X$.

If G is a group and the functional space X is left-invariant (i.e., $f \in X$ implies the inclusion $f^x \in X$, where $f^x(t) = f(x^{-1}t)$, $x \in G$), then a mean m is said to be *left-invariant* if $m(f^x) = m(f)$ for all elements $x \in G$ and all functions $f \in X$. Similarly, m is called a right-invariant mean if $m(f_x) = m(f)$ for all $x \in G$, where $f_x(t) = f(tx)$ by definition; m is a two-sided invariant mean if it is simultaneously left- and right-invariant.

A group G is said to be *amenable* if there is a left-invariant (or right-invariant) mean on the group $X = B(G)$. The work [68] of Greenleaf is devoted to the study of amenable groups. These groups naturally arise in various fields of geometry, for example, in studying the geometry of the Laplace operator (see, e.g., [31, 32]).

Examples. If G is an Abelian group, then there always exists an invariant mean on $B(G)$ (see [68]). If a finitely generated group has the property that a function equal to the number of words of length n grows more slowly than exponential as n increases, then the group is amenable. For a free group G with two generators, there is no invariant mean on the space $B(G)$. The proper orthogonal group SO(3) contains a free subgroup with two generators and, therefore, is not amenable (in this case, the group SO(3) is considered with the discrete topology). If G is an amenable group and π is its homomorphism onto a group H, then H is also amenable. Each subgroup H of an amenable group G is amenable. If N is the normal divisor of a group G, and, moreover, if it is amenable and N and G/N are amenable, then G are also amenable (see [68]).

Hypothesis. *A discrete group G is not amenable iff it contains no free subgroups with two generators.*

To state the next assertion, we introduce the class C of groups. By definition, this class contains all amenable groups and is closed with respect to products and finite extensions.

Theorem 2.3.10 ([87])**.** *Let M be an n-dimensional $(n \geq 2)$ compact Riemannian manifold of nonzero Euler characteristic, $\chi(M) \neq 0$. Then if the fundamental group $\pi_1(M)$ of the manifold M belongs to the class C, then the sectional curvature of the manifold M at a certain point and in a certain two-dimensional direction is nonpositive.*

If the fundamental group $\pi_1(M)$ of a compact n-dimensional $(n \geq 2)$ Riemannian manifold of negative curvature belongs to the class C, then, as is shown in [87], it is free.

Milnor introduced an important concept of *growth function* of a finitely generated group (see [128]). Let X be a finite set that generates a group G, i.e., $G = F/N$, where F is the free group with basis X. For each positive integer n, let $\gamma(n)$ be the number of elements of the group G that can be represented by elements w in F such that $|w| \leq n$. If we choose one more finite set X' generating the group G, then the functions γ and γ' are equivalent in the following sense. There exist positive integers k and k' such that for all n, we have the inequalities $\gamma(n) \leq \gamma'(kn)$ and $\gamma'(n) \leq \gamma(k'n)$. Milnor showed that the quantity $\gamma(n)^{1/n}$ always converges to a certain number a, $1 \leq a < \infty$. A function γ is exponential if $a > 1$. Milnor related these concepts with the curvature of Riemannian manifolds.

Theorem 2.3.11 ([128])**.** *Let M be a compact Riemannian manifold of sectional curvature $K(\sigma) < 0$. Then the growth function of the fundamental group $\pi_1(M)$ is at least exponential.*

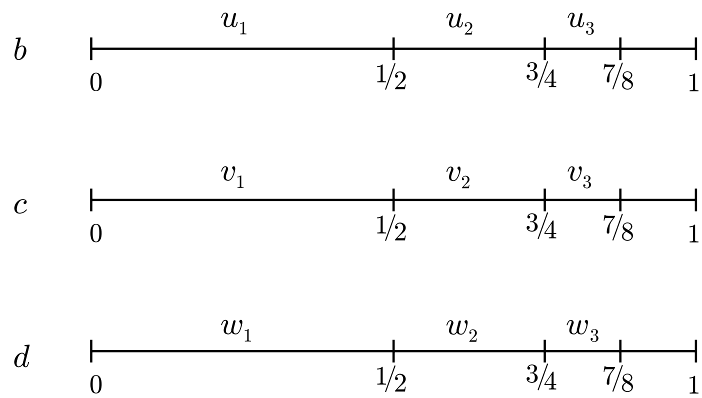


Fig. 39

Therefore, the fundamental group of manifolds of negative curvature is very large: it has an exponential growth.

We present examples of groups with known growth. If a group admits a finitely generated nilpotent subgroup of finite index, then its growth function is equivalent to a polynomial. If, in a polycyclic group, we have no finitely generated nilpotent subgroup of finite index, then its growth is exponential. A nonpolycyclic solvable group is of exponential growth.

In [70], Grigorchuk constructed examples of groups whose growth functions are not equivalent to a power function and also to an exponential function. These groups are defined as transformation groups of the closed interval $[0, 1]$ that preserve the Lebesgue measure. Let the letter T denote the identity transformation of the interval, and let Π denote an interchanging of the halves of a closed interval $[\alpha, \beta]$, i.e.,

$$\Pi(x) = \begin{cases} x + \dfrac{1}{2}(\beta - \alpha), & \alpha < x < \dfrac{1}{2}(\alpha + \beta), \\ x - \dfrac{1}{2}(\beta - \alpha), & \dfrac{1}{2}(\alpha + \beta) < x < \beta. \end{cases}$$

Let U, V, and W be infinite words in the alphabet $\{\Pi, T\}$, $U = u_1 u_2 \dots u_n \dots$, $V = v_1 v_2 \dots v_n \dots$, and $W = w_1 w_2 \dots w_n \dots$. We define the transformations b, c, and d by the diagrams that are depicted in Fig. 39 and set $a = \Pi$.

We consider only those triples (U, V, W) for which, among the letters u_n, v_n, and w_n, we have exactly two letters Π and one letter T for each $n = 1, 2, \dots$. For any triple (U, V, W) of the indicated form, we define the group $G = G(U, V, W)$ generated by the transformations a–d. We introduce the coding

$$0 \leftrightarrow \begin{pmatrix} \Pi \\ \Pi \\ T \end{pmatrix}, \qquad 1 \leftrightarrow \begin{pmatrix} \Pi \\ T \\ \Pi \end{pmatrix}, \qquad 2 \leftrightarrow \begin{pmatrix} T \\ \Pi \\ \Pi \end{pmatrix};$$

it induces a one-to-one correspondence between the triples described above and the sequences $\omega = \omega_1 \omega_2 \dots \omega_n \dots$ of letters in the alphabet $\{0, 1, 2\}$. Let Ω be the space of such sequences, and let G_ω be the group defined by using the sequence ω. Let Ω_0 be the set of sequences in which each of the symbols $0, 1$, and 2 occurs infinitely many times, and let Ω_{-1} be the set of sequences that are constant starting from a certain number and $\Omega_1 = \Omega \setminus \{\Omega_{-1} \cup \Omega_0\}$.

Theorem 2.3.12. *Let $\omega \in \Omega_0 \cup \Omega_1$. Then the growth function of the group G_ω is not equivalent to a power function and an exponential function.*

In particular, the group G_ω, where $\omega = 0\,1\,2\,0\,1\,2\,0\,1\,2\,\dots$, is a group of intermediate growth. For estimates of the growth function, see [70].

2.3.4. Volume of manifolds and its estimates. We now consider the problem of the volume of manifolds of nonpositive curvature. In [35], Buyalo obtained a lower bound on the volume of such manifolds M, which allows one to prove the finiteness of topological types of manifolds with $K(\sigma) \le 0$.

Theorem 2.3.13 ([35]). *Let the sectional curvatures of the metric g_{ij} of a closed Riemannian manifold (M, g_{ij}) satisfy the condition $-1 \le K(\sigma)_g \le 0$. Then there exists a Riemannian metric $\bar g_{ij}$ on M with $-1 \le K(\sigma)_{\bar g} \le 0$ such that the inequalities*

$$\operatorname{vol}(M, \bar g) \ge \alpha_n \exp\left(-\frac{2}{n} d(M, \bar g)\right) \quad \text{and} \quad d(M, \bar g) \le \beta_n d(M, g),$$

where $d(M, \bar g)$ (respectively, $d(M, g)$) is the diameter of the metric $\bar g$ (respectively, g), hold for its volume; the constants $\alpha_n, \beta_n > 0$ depend only on the dimension $n \ge 2$ of the manifold M. Moreover, if the manifold M is not splittable, then we can take the metric g as the metric $\bar g$.

As a consequence of this statement, we have the following important finiteness theorem. For given $n \ge 2$ and $C > 0$, there exists only finitely many pairwise nonhomeomorphic n-dimensional closed Riemannian manifolds M with $-1 \le K(\sigma) \le 0$ and $d(M) \le C$.

The rank of a manifold, which was defined above, can be estimated by using the estimates of its volume. More precisely, we have the following statement (see [35]). There exists a constant $C_n > 0$ such that if the volume of an n-dimensional closed Riemannian manifold M of curvature $-1 \le K(\sigma) \le 0$ satisfies the inequality $\operatorname{vol}(M) \le C_n$, then $\operatorname{rk} M \ge 2$.

In the case where the sectional curvature of a manifold satisfies the inequality $-\varepsilon < K(\sigma) \le -1$, we can prove the following important result of Gromov.

Theorem 2.3.14. *For any $C > 0$ and $n \ne 3$, there exists a number $\varepsilon = \varepsilon(C, n)$ such that each complete n-dimensional Riemannian manifold of sectional curvatures $-\varepsilon < K(\sigma) \le -1$ and volume $\operatorname{vol}(M) < C$ is diffeomorphic to a certain hyperbolic space form, i.e., a complete connected Riemannian manifold of constant sectional curvature $K < 0$.*

To describe the next bound on the volume, we need the concept of a semisimple isometry. Let H be a certain Hadamard manifold. With each isometry $\gamma : H \to H$, one relates its *displacement function* $\delta_\gamma(x) = \rho(x, \gamma(x))$, $x \in H$, where ρ is the distance in H. The set

$$C_\gamma = \{x \in H \mid \delta_\gamma(x) = \inf \delta_\gamma\}$$

is called the *minimal set of the isometry* γ. The function δ_γ is convex, and, therefore, C_γ is also convex.

Definition 2.3.3. An isometry γ is said to be *semisimple* if $C_\gamma \ne \varnothing$.

Recall that a transformation group Γ of the manifold M is said to be *discrete* if all its orbits have no accumulation points, and it is said to be *uniform* if the orbit space M/Γ is compact. A group Γ with a uniform and discrete action on an Hadamard manifold H consists of semisimple elements. We have the following lower bound on the volume of manifolds of positive sectional curvature.

Theorem 2.3.15 ([35]). (a) *Let H be a complete simply connected manifold whose sectional curvatures satisfy the inequality $-1 \le K(\sigma) \le 0$, and let the Ricci curvature be negative definite*: $\operatorname{Ric} < 0$. *Let Γ be a discrete group (possibly not torsion-free) of semisimple isometries of the manifold H. Then $\operatorname{vol}(H/\Gamma) \ge C_n$.*

(b) *If H is a noncompact irreducible symmetric space and Γ is the same as in item* (a), *then* $\operatorname{vol}(H/\Gamma) \ge C_n K^{-n/2}$, *where $K = \sup |K(\sigma)|$.*

Let Γ be a discrete uniform isometry group of an n-dimensional Hadamard manifold H whose sectional curvature satisfies the inequality $-1 \le K(\sigma) \le 0$. We denote by $\delta_\Gamma(x)$ the greatest lower bound

$$\inf\{\delta_\gamma(x) \mid \gamma \in \Gamma,\ \gamma \ne e\}, \quad x \in H,$$

of the displacement functions $\delta_\gamma(x)$. The quantity $\sigma(\Gamma) = \max \delta_\Gamma$ is an important characteristic of the action of the group Γ on H. Obviously, $\mathrm{vol}(H/\Gamma) \geq \alpha_n \sigma^n(\Gamma) > 0$, where the constant α_n depends only on n. In the case where H is the universal covering space of a closed manifold $M = H/\pi_1(M)$, the quantity $\sigma(M) = \sigma(\pi_1(M))$ is equal to the doubled maximum of the injectivity radius of the manifold M.

We can also estimate the diameter using the invariant $R(\Gamma)$, which is defined as follows. The group Γ acts uniformly on H; therefore, for $x \in H$, there exists a number $R > 0$ such that the orbit of the ball of radius $\frac{1}{2}R$ centered at the point x covers entirely the manifold H. Let $R(x)$ be the greatest lower bound of these numbers, and let $R(\Gamma) = \sup_x R(x)$. In the case of the universal covering of H over a closed manifold M, we have the inequality

$$\frac{1}{2}R(\pi_1(M)) \leq \mathrm{diam}\, M \leq R(\pi_1(M)).$$

Using the invariant $R(\Gamma)$, we can also give a characterization of the Euclidean space $\mathbb{R}^n$. There exists a constant ε_n such that if $R(\Gamma) \leq \varepsilon_n$, then H is isometric to the space $\mathbb{R}^n$.

In studying Riemannian manifolds of nonpositive sectional curvature, an important role is played by the assertion known as the Margulis lemma (see [127]). We have considered a particular case of it in Sec. 1.

Theorem 2.3.16. *Let M be a simply connected n-dimensional complete Riemannian manifold of sectional curvature $K(\sigma)$ satisfying the inequality $-1 \leq K(\sigma) \leq 0$. Further, let Γ be a discrete isometry group of the manifold M. Then there exists a positive number ε_n depending only on n such that for any point $x \in M$ and for $0 < \varepsilon \leq \varepsilon_n$, the group $\Gamma_\varepsilon(x)$ generated by $\{\gamma \in \Gamma \mid \rho(x, \gamma(x)) < \varepsilon\}$ is almost nilpotent in the sense that it contains a nilpotent subgroup of finite index.*

2.3.5. Infinitely distant points. The structure of the isometry group. Let M be a simply connected complete Riemannian manifold of nonpositive sectional curvature. Two geodesic rays γ_1 and γ_2 with natural parametrization are said to be *equivalent* if the distance $\rho(\gamma_1(t), \gamma_2(t))$ is uniformly bounded for all $t > 0$. The set of all equivalence classes of geodesic rays with natural parametrization on rays is denoted by $M(\infty)$. The space $M(\infty) \cup M$ is equipped with a topology (the set of all open cones of geodesic rays forms a subbasis of this topology) with respect to which $M(\infty) \cup M$ is a compact topological space homeomorphic to a cell. Each isometry of the manifold M is extended up to a homeomorphism of the space $M \cup M(\infty)$.

Definition 2.3.4. An isometry f of a manifold M is said to be *elliptic* if f has a fixed point in M. An isometry of the manifold M is said to be *parabolic* if f has no fixed points in M and there exists exactly one fixed point in $M(\infty)$. An isometry f of the manifold M is said to be *hyperbolic* if it has no fixed points in M and has exactly two fixed points in $M(\infty)$.

One proves that if the sectional curvatures of a manifold M are bounded from above by a negative number, then the isometry group of the manifold M is divided into three pairwise disjoint classes of elliptic, parabolic, and hyperbolic elements.

We have the following useful statement owing to Gromov, which yields a criterion for a discrete isometry group to have no hyperbolic transformations.

Lemma 2.3.1. *Let M be a simply connected complete Riemannian manifold whose sectional curvature $K(\sigma)$ satisfies the inequality $-1 \leq K(\sigma) < 0$. Let Γ be a discrete isometry group acting without fixed points such that the volume of the space M/Γ is finite. Then there exists a positive number ε' depending on M and Γ such that if $\gamma \in \Gamma$ and $\rho(x, \gamma(x)) < \varepsilon'$ for a certain point $x \in M$, then γ is not a hyperbolic isometry.*

As for elliptic elements, we have the following fundamental result owing to E. Cartan, which finds many applications in geometry. For the proof, see, e.g., [94].

Theorem 2.3.17. *Let M be a complete simply connected Riemannian manifold of negative curvature, and let K be a compact Lie transformation group of the manifold M whose elements are isometries of the manifold M. Then transformations from the group K admit a common fixed point.*

We now consider parabolic elements of the isometry group. For $x \in M(\infty)$, we denote by Γ_x the set of all parabolic elements of the group Γ for which x is a unique fixed point. The set Γ_x either is empty or consists of all elements of the group Γ for which x is a fixed point. The set of all parabolic elements of the group Γ can be represented in the form of a union of disjoint subsets Γ_{x_i}, where $x_i \in M(\infty)$. Let ε be the minimum of two constants ε_n and ε'_n from the Margulis lemma and from the Gromov lemma, and let

$$A_i = \{x \in M \mid \min_{\gamma \in \Gamma_{x_i}} \rho(x, \gamma(x)) < \varepsilon\},$$
$$D = \{x \in M \mid \min_{\gamma \in \Gamma} \rho(x, \gamma(x)) \geq \varepsilon\}.$$

Then the Margulis lemma implies the following important decomposition of the manifold M : $M = D \cup \bigcup_i A_i$.

Let Γ be a completely discontinuous isometry group of a manifold M of nonpositive sectional curvature. We define the limit set $L(\Gamma) \subset M(\infty)$, where $L(\Gamma)$ is the set of accumulation points of the orbit $\Gamma(p)$ in $M(\infty)$ for an arbitrary point $p \in M$. This set does not depend on the choice of the point p.

Definition 2.3.5. A complete Riemannian manifold M is said to be *visible* if the sectional curvature $K(\sigma)$ is not positive, and for any two points $x \neq y$ in $\tilde{M}(\infty)$, there exists at least one geodesic connecting x with y, where $\tilde{M}$ is the universal covering of the manifold M.

To obtain results sufficiently rich in content in the case of manifolds of curvature $K(\sigma) \leq 0$ that to some extent reproduce what is referred to a metric of negative curvature bounded away from zero, it is necessary to require the fulfillment of certain additional conditions. As such a condition, one often uses the concept of visibility. For visible manifolds, we can sufficiently well discover the geometrical properties of such manifolds.

Visible manifolds are divided into three classes. Let M be a complete visible manifold of sectional curvature $K(\sigma) \leq 0$, and let $\tilde{M}$ be its universal covering. The manifold M is said to be *axial* if the limit set $L(\pi_1(M)) \subset \tilde{M}(\infty)$ consists of exactly two points; the manifold M is said to be *parabolic* if the set $L(\pi_1(M))$ consists of one point; the manifold M is said to be *Fuchsian* in the opposite case.

We do not present a complete survey of all geometric properties of the classes of manifolds introduced above, since the study of many classes of Riemannian manifolds was divided into separate directions of geometry with their own subject and methods.

We present three examples illustrating that various restrictions imposed on the set of limit points allow us to obtain interesting geometric results. The first example is related to one if Chen's theorems.

Theorem 2.3.18. *Let M be a simply connected complete Riemannian manifold whose sectional curvatures $K(\sigma)$ are negative, $K(\sigma) < 0$, such that any two points of $M(\infty)$ can be connected by a unique geodesic. If a subgroup G in the isometry group $I(M)$ has no common fixed points in $M(\infty)$ and the limit set $L(G)$ contains more than two points, then G contains a free subgroup with infinitely many generators.*

This theorem implies two important consequences. Let M be a complete Riemannian manifold of sectional curvatures $K(\sigma) < 0$ such that its universal covering $\tilde{M}$ satisfies the assumption of the preceding theorem. Then the fundamental group $\pi_1(M)$ of the manifold M contains a free subgroup with infinitely many generators. Furthermore, if M is a compact Riemannian manifold of negative curvature, then the group $\pi_1(M)$ is of exponential growth.

As the second example, we present certain important properties of the isometry group of visible manifolds.

Theorem 2.3.19. *Let M be a complete visible manifold. If $L(G) = M(\infty)$, then either the isometry group $I(M)$ of the manifold M is discrete or the connected component $I_0(M)$ of the identity of the group $I(M)$ is a noncompact semisimple Lie group.*

As a consequence of this theorem, we indicate two properties of visible manifolds. Let N be a complete visible manifold. If N is of finite volume, then either the group $I(N)$ is discrete or $I_0(N)$ is a noncompact semisimple Lie group. Let N be a two-dimensional complete visible manifold, and let $L(G) = M(\infty)$. Then either the group $I(M)$ is discrete or N is the hyperbolic plane, i.e., N is a hyperbolic space form.

The third example is related to Fuchsian manifolds. Let M be a Fuchsian manifold. Then there exists an infinite subset A in $\pi_1(M)$ such that the subgroup G generated by the set A is a free group and the set A is its free set of generators. We consider a Fuchsian manifold M_1. For any finite group F, there exists a Fuchsian manifold M_2 covering M_1 such that F is a subgroup in the isometry group $I(M_2)$ of the manifold M_2. Various properties of visible manifolds can be found in the works of Eberlein.

2.4. Almost flat Riemannian manifolds. A compact Riemannian manifold M is said to be *ε-flat* if the inequality $|K(\sigma)| \le \varepsilon d(M)^{-2}$ holds; here $K(\sigma)$ is the sectional curvature of the manifold M and $d(M)$ is the diameter of the manifold M. A manifold is said to be *almost flat* if it admits an ε-flat metric for an arbitrary $\varepsilon > 0$.

We consider several examples of almost flat Riemannian manifolds. We recall that a quotient space of a nilpotent Lie group is called a *nilmanifold*.

Proposition 2.4.1. *Any nilmanifold is almost flat.*

As an illustration of this proposition, we consider the construction of almost flat metrics on the quotient space $M = G/\Gamma$, where G is the Lie group of upper triangular matrices:

$$G = \left\{ \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \right\}.$$

For this purpose, on the Lie algebra of upper triangular matrices

$$\mathfrak{y} = \left\{ \begin{pmatrix} 0 & x_{12} & \dots & x_{1n} \\ 0 & 0 & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \right\},$$

we consider the set of inner products

$$\|A\|_q^2 = \sum_{i<j} x_{ij}^2 q^{2(j-i)}.$$

They define the corresponding invariant metric on the Lie group G. Since the estimate

$$\|R_q(A,B)C\|_q \le 24(n-2)^2 \|A\|_q^2 \|B\|_q^2 \|C\|_q^2$$

holds, on any compact coset $M = G/\Gamma$, we have the Riemannian metric that is almost flat.

We have the following main theorem on the structure of almost flat Riemannian manifolds, which belongs to Buser and Karcher.

Theorem 2.4.1. *Let M be a compact Riemannian manifold whose sectional curvatures satisfy the inequality*

$$|K(\sigma)| \le \varepsilon_n d^{-2}(M),$$

where

$$\varepsilon_n = \exp(-\exp(\exp n^2)).$$

Then M is covered by a nilmanifold. More precisely, we have:

(a) *the fundamental group $\pi_1(M)$ contains a normal torsion-free subgroup Γ of rank n;*

(b) *the quotient group $G = \pi_1(M)/\Gamma$ is of order $s \le 2 \cdot (6\pi)^{\frac{1}{2}n(n-1)}$ and is isomorphic to a certain subgroup in $\mathrm{O}(n)$;*

(c) *the finitely sheeted covering of the manifold M with the fundamental group Γ and the automorphism group G is diffeomorphic to the nilmanifold N/Γ;*

(d) *the simply connected and nilpotent Lie group N is uniquely defined by the fundamental group $\pi_1(M)$.*

In addition, Gromov showed that any nilmanifold admits an ε-flat metric for an arbitrary $\varepsilon > 0$. The previous theorem was recently strengthened by Rou.

Theorem 2.4.2. *Let M be a compact Riemannian manifold, d be its diameter, and $K(\sigma)$ be its sectional curvature. There exists a constant $\varepsilon = \varepsilon(n) > 0$ such that the inequality $K < \varepsilon d^{-2}$ implies that the manifold M is diffeomorphic to the quotient space N/Γ, where N is a simply connected nilpotent Lie group and Γ is a certain extension of the lattice $L \subset N$ by a finite group H.*

We indicate the following connection between the injectivity radius and the commutativity. Let M^n be an ε-flat Riemannian manifold, $\varepsilon \le \varepsilon_n$. If the injectivity radius i of the manifold M^n satisfies the inequality $i > 2^{-n^3}(\frac{\varepsilon}{\varepsilon_n})^{1/2} d(M)$, then the subgroup $\Gamma \subset \pi_1(M)$ is Abelian and the manifold M^n is called the torus.

3. Riemannian Manifolds with Restrictions on the Ricci Curvature

3.1. Meyers theorem. As a rule, the restrictions imposed on the Ricci curvature are weaker than the restrictions imposed on the full Riemann curvature tensor. However, the Ricci curvature on manifolds is not arbitrary.

Definition 3.1.1. We say that the Ricci curvature R_{ij} satisfies the inequality $\mathrm{Ric} \ge a$ if $R_{ij}x^ix^j \ge a$ for any tangent vector $v = (x^1, \dots, x^n)$ of unit length.

From the Morse–Schoenberg theorem, it is easy to obtain the following theorem of Meyers (see [138]), which describes the topology of Riemannian manifolds with positive Ricci curvature tensor.

Theorem 3.1.1. *Let M^n be a complete n-dimensional Riemannian manifold whose Ricci curvature satisfies the inequality $\mathrm{Ric} \ge (n-1)a^2 > 0$. Then the manifold M^n is compact, its diameter does not exceed π/a, and the fundamental group is finite.*

For sufficiently small r, we denote by $v(m, r)$ the volume of the sphere $S(m, r)$ contained in a normal coordinate neighborhood. If $(n-1)a^2$ is the greatest lower bound of $R_{ij}x^ix^j$ and b^2 is the least upper bound of $R_{ij}x^ix^j$, where $\|v\| = 1$ and $v(x^1, \dots, x^n)$, then $v(m, r)\left(\dfrac{b}{\sin br}\right)^{n-1}$ is a nondecreasing function of r and $v(m, r)\left(\dfrac{a}{\sin ar}\right)^{n-1}$ is nonincreasing function of r. With accuracy up to a constant, the function $\left(\dfrac{a}{\sin ar}\right)^{n-1}$ coincides with the volume of the sphere in a space of constant curvature. These arguments imply an important theorem on the comparison of volumes of Riemannian manifolds (see [22]).

Theorem 3.1.2. *Let M^n be a complete n-dimensional Riemannian manifold, and let*

$$(n-1)a^2 = \inf R_{ij}x^ix^j,$$

where $v = (x^1, \dots, x^n)$ is a unit vector in the tangent space of the manifold M^n. Then the volume of a ball in normal coordinates does not exceed the volume of the normal coordinate ball of the same diameter in these normal coordinates on a simply connected space form (i.e., on the sphere S^n, Euclidean space $\mathbb{R}^n$, or hyperbolic H^n-space) of constant curvature a^2.

If $a^2 > 0$, then the volume of the manifold M does not exceed the volume of the sphere of radius $1/a$, and, moreover, the equality is attained only if M is isometric to such a sphere.

3.2. Three-dimensional manifolds of positive-definite Ricci curvature. We can completely describe the topological structure of three-dimensional manifolds on which there is a metric with positive-definite Ricci tensor. First of all, this is related to the fact that in dimension three, the full Riemannian curvature tensor is expressed through the Ricci curvature tensor:

$$R_{ijkl} = g_{ik}R_{jl} - g_{il}R_{jk} - g_{jk}R_{il} + g_{jl}R_{ik} - \frac{1}{2}R(g_{ik}g_{jl} - g_{il}g_{jk}).$$

We can always reduce the Ricci tensor R_{ij} at a point to the diagonal form

$$R_{ij} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix},$$

where λ, μ, and ν are eigenvalues of the matrix $\|R_{ij}\|$. Then the essential component R_{ijkl} of the curvature tensor is expressed through $R_{1212} = \frac{1}{2}(\lambda+\mu-\nu)$ in terms of the eigenvalues λ, μ, and ν of the matrix $\|R_{ij}\|$. This implies that in the three-dimensional case, the condition of positivity of the sectional curvature can be expressed through the eigenvalues of the Ricci tensor. In dimension three, a metric is a metric of positive sectional curvature iff $R_{ij} < \frac{1}{2}Rg_{ij}$, where R is the scalar curvature. Therefore, the positivity condition of the Ricci curvature is weaker than the positivity condition of the sectional curvature. The main fact of the theory of compact three-dimensional manifolds of positive-definite Ricci curvature is described in the following theorem.

Theorem 3.2.1 ([91])**.** *Let M be a compact three-dimensional manifold admitting a metric of positive-definite Ricci curvature. Then M also admits a metric of constant positive curvature.*

The method for proving this theorem is based on the following observation. On the manifold M, we fix a metric with positive-definite Ricci tensor. Using the heat equation

$$\frac{\partial}{\partial t}g_{ij} = \frac{2}{n}Rg_{ij} - 2R_{ij},$$

we improve this metric. This equation describes the minimization problem of the "energy" $\displaystyle\int R\,d\mu$. Unfortunately, the heat equation written above has no solutions. Therefore, we replace it by the equation

$$\frac{\partial}{\partial t}g_{ij} = \frac{2}{n}rg_{ij} - 2R_{ij},$$

where

$$r = \frac{\int R\,d\mu}{\int d\mu}$$

and R is the scalar curvature. This equation admits a solution at least for sufficiently small t. There are technical difficulties (one uses the Nash–Moser inverse function theorem). The construction described can be carried out in a Riemannian manifold of arbitrary dimension.

For compact three-dimensional manifolds, it is proved that if the Ricci tensor of the initial metric is positive definite, then this property holds for all t. There exists a limit as $t \to \infty$, and the limit metric is a metric of constant positive curvature. To prove this assertion, we use the maximum principle for parabolic equations.

All manifolds of constant curvature were classified by Wolf (see [213] and also Sec. 1 of this chapter).

Example. The manifold $S^2 \times S^1$ admits a metric with nonnegative Ricci tensor R_{ij} (two eigenvalues equal 1, and the third eigenvalue equals 0). This manifold does not admit a metric of constant curvature, and hence it also does not admit a metric of positive Ricci curvature.

The topology of noncompact three-dimensional manifolds is described by the following important theorem (see [182]).

Theorem 3.2.2. *Let M be a complete noncompact three-dimensional manifold of positive-definite Ricci curvature. Then the manifold M is diffeomorphic to the space $\mathbb{R}^3$.*

The proof is based on the following theorem of Stallings: a contractible three-dimensional manifold is diffeomorphic to the space $\mathbb{R}^3$ iff it is simply connected at infinity and a two-dimensional sphere embedded in M bounds the three-dimensional disk. The verification of the first condition is easy under the assumptions of our theorem. The verification of the second condition is technically more complicated.

There is the following interesting characteristic of three-dimensional symmetric spaces among all Riemannian manifolds. For its description, we define the tensor

$$Q_{ij} = 6S_{ij} - 3RR_{ij} + (R^2 - 2S)g_{ij},$$

where

$$S_{il} = R^2_{il} = R_{ij}g^{jk}R_{kl} \quad \text{and} \quad S = g^{il}S_{il}.$$

Theorem 3.2.3. *The tensor Q_{ij} vanishes identically on any three-dimensional symmetric Riemannian manifold. Any symmetric tensor that is a quadratic form in the Ricci tensor and vanishes identically on any three-dimensional symmetric Riemannian manifold is necessarily proportional to the tensor Q_{ij}.*

3.3. Metrics with prescribed Ricci tensor on two-dimensional manifolds. The search for conditions under which a given symmetric covariant tensor is the Ricci tensor of a certain metric is a fundamental problem. This problem is reduced to the solution of the set of nonlinear partial differential equations

$$\frac{\partial}{\partial x^s}\Gamma^s_{ij} - \frac{\partial}{\partial x^j}\Gamma^s_{is} + \Gamma^s_{ij}\Gamma^t_{st} - \Gamma^s_{it}\Gamma^t_{sj} = R_{ij},$$
$$\Gamma^i_{jk} = \frac{1}{2}g^{is}\left(\frac{\partial g_{js}}{\partial x^k} + \frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}\right).$$

In the two-dimensional case, the situation is considerably simplified, since any metric on a two-dimensional manifold is an Einstein metric. Therefore, if a tensor R_{ij} is the Ricci curvature tensor of a certain Riemannian metric, then it should be $R_{ij} = k\gamma_{ij}$, where γ_{ij} is a certain tensor that is positive definite at each point. Locally, this condition is also sufficient (see [182]). All the results in this field of Riemannian geometry are related to the study of various geometric properties of solutions to sets of nonlinear equations.

Theorem 3.3.1. *Let a tensor R_{ij} be defined on a certain neighborhood of a point $p \in M^2$ in a two-dimensional manifold M^2. There exists a Riemannian metric on M^2 for which R_{ij} is the Ricci tensor iff $R_{ij} = k\gamma_{ij}$ for a certain scalar-valued function k and a positive-definite tensor γ_{ij}.*

The idea of the proof of this theorem consists of seeking the metric g_{ij} in the form $g_{ij} = e^{2u}\gamma_{ij}$. Then we obtain the equation $\Delta u = \varphi^{-k}$, $S_{ij} = \varphi\gamma_{ij}$ for the function u, and the assertion is implied by the local solvability of this equation.

Let dv_γ be a volume element of a metric γ_{ij}. We have the following global analog of Theorem 3.3.1.

Theorem 3.3.2. *Let M^2 be a compact two-dimensional manifold, and let R_{ij} be a tensor satisfying the necessary condition $R_{ij} = k\gamma_{ij}$, $\gamma > 0$. Then R_{ij} is the Ricci tensor of a certain metric on M^2 iff*

$$\int\limits_{M^2} k\, dv_\gamma = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of the manifold M.

We can state a problem of the same type for the full Riemannian curvature tensor. We can consider the Riemannian curvature tensor as a 2-form with values in the set of traceless matrices. In this case, we have the following statement.

Theorem 3.3.3. *The matrix-valued 2-form $R^i_{j,kl}$ is locally the curvature tensor of a certain two-dimensional metric iff the eigenvalues of all matrices are purely imaginary.*

We can restrict ourselves to the problem of the search for not a metric with a given Ricci tensor but an affine connection. In such a statement of the problem, it always has a solution because of the following assertion.

Theorem 3.3.4. *In a certain neighborhood of a point x_0, let a tensor R_{ij} (which is not necessarily symmetric) be given. Then there exists a connection Γ^i_{jk} whose Ricci curvature coincides with the tensor R_{ij}.*

If the tensor R_{ij} is symmetric, then it is natural to seek a solution Γ^i_{jk} in the class of symmetric connections because of the following theorem.

Theorem 3.3.5. *In a certain neighborhood of a point x_0, let a symmetric tensor R_{ij} be given. Then there always exists an affine connection Γ^i_{jk} which is symmetric ($\Gamma^i_{jk} = \Gamma^i_{kj}$) and such that its Ricci curvature coincides with R_{ij}.*

3.4. Metrics with prescribed Ricci tensor on manifolds of dimension $\geq$3. In dimension not less than 3, the fulfillment of the Bianchi identity is an obstruction to the existence of a metric with prescribed Ricci tensor. We introduce the notation

$$\mathrm{Bian}(g,R) = g^{st}\left(\nabla_t R_{sm} - \frac{1}{2}\nabla_m R_{st}\right).$$

We always have the relation $\mathrm{Bian}(g,R) = 0$. The identity $\mathrm{Bian}(g,R) = 0$ for a certain metric is a necessary condition for the existence of a metric with prescribed Ricci tensor.

Example. The tensor

$$R = x^1 dx^1 \otimes dx^1 \pm dx^2 \otimes dx^2 \pm \ldots \pm dx^n \otimes dx^n$$

is not the Ricci tensor of any Riemannian metric near $x^1 = 0$. In fact, there exists a metric for $x^1 > 0$; namely, in the case $n = 3$, it has the form

$$x^1 (f(x^1))^2 dx^1 \otimes dx^1 + 2f(x^1) dx^2 \otimes dx^2 + 2f(x^1) dx^3 \otimes dx^3,$$

where $f(x) = \mathrm{sech}^2(x^{3/2}/\sqrt{18})$ and R is its Ricci curvature.

Example. The tensor

$$\sum_{i=1}^{n}(x^1 dx^i \otimes dx^i) + \frac{1}{2}\sum_{i=1}^{n}(x^i dx^i \otimes dx^1) + \frac{1}{2}\sum_{i=1}^{n}(x^i dx^1 \otimes dx^i)$$

is not the Ricci tensor of a metric of an arbitrary signature near the origin $0 \in \mathbb{R}^n$.

Proposition 3.4.1. *For sufficiently small $\rho > 0$, solutions to the equations $\mathrm{Bian}(g,R) = 0$ in the ball $B_\rho(0)$ of radius ρ centered at zero form a Banach submanifold in the space of Riemannian metrics near a given infinitesimal solution g^0_{ij} if the matrix $\|R_{ij}(0)\|$ is invertible.*

Locally, the invertibility condition of the matrix $\|R_{ij}(0)\|$ appearing in Proposition 3.4.1 completely solves the problem of the existence of a metric with prescribed Ricci tensor (see [182]).

Theorem 3.4.1. *Let R_{ij} be a tensor field in a neighborhood of a point x_0, and let the matrix $\|R_{ij}(x_0)\|$ be invertible. Then there exists a Riemannian metric g_{ij} such that its Ricci curvature tensor coincides with R_{ij} in a neighborhood of the point x_0.*

To find the metric required, we need to solve the elliptic equation

$$\operatorname{Ric}(g) + \operatorname{div}(R^{-1} \operatorname{Bian}(g, R)) = R.$$

It is solved by a method similar to the Picard–Newton method. A successive approximation $g_{ij}^{(n)}$ is constructed in two stages:

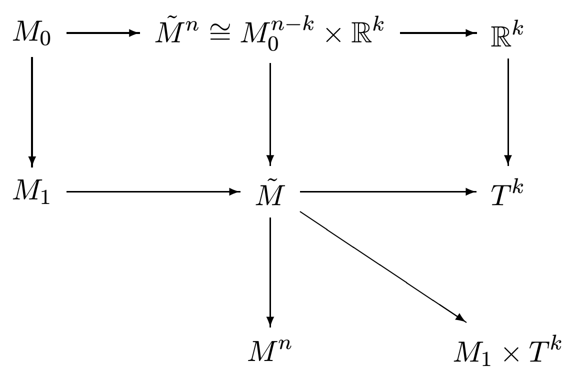
(a) using the initial approximation $g_{ij}^{(n)}$ as in the usual Picard–Newton method, we find the next approximation $\bar{g}_{ij}^{(n)}$;

(b) after that, we project $\bar{g}_{ij}^{(n)}$ to the submanifold of solutions of the Bianchi equation and obtain the next approximation $g_{ij}^{(n+1)}$. The successive approximations $g_{ij}^{(n)}$ converge to the required metric, since the metrics $g_{ij}^{(n)}$ automatically satisfy the relation

$$\operatorname{div}(R^{-1} \operatorname{Bian}(g, R)) = 0.$$

3.5. Riemannian manifolds of nonnegative Ricci curvature. The results obtained in the description of the structure of Riemannian manifolds of nonnegative Ricci curvature are generalization of the theorems referring to Riemannian manifolds of nonnegative sectional curvature. The following theorem describes the structure of such manifolds.

Theorem 3.5.1 ([42])**.** *For a closed Riemannian manifold M^n of positive semi-definite Ricci curvature, we have the following commutative diagram whose vertical mappings are locally isometric coverings, whose horizontal mappings are locally trivial bundles, and whose diagonal mapping is a diffeomorphism (T^k is a flat torus):*



$\tilde{M}$ is the universal covering of M^n and M_0^{n-k} is a closed simply connected space, $0 \le k \le n$.

In studying manifolds with $\operatorname{Ric} < 0$, as in studying manifolds with $K(\sigma) \le 0$, an important role is played by arguments related to the convexity.

If we impose the homogeneity condition on a Riemannian manifold with $\operatorname{Ric} \le 0$, then the classification problem of such manifolds can be considerably more developed than that in studying general Riemannian manifolds with $\operatorname{Ric} \le 0$.

Theorem 3.5.2 ([42])**.** *Let M^n be a complete locally homogeneous space of nonnegative Ricci curvature. Then there exists an absolutely convex compact submanifold $S \subset M^n$ such that the normal bundle $\nu(S)$ with the standard metric is flat and isometric to the manifold M^n.*

Therefore, the classification problem of locally homogeneous complete Riemannian manifolds M^n of nonnegative Ricci curvature is reduced to the problem on flat vector bundles over closed spaces of the same type.

3.6. Riemannian manifolds of negative Ricci curvature. The classification of such manifolds still remains an open problem now. We present here examples of metric manifolds for which $R_{ij} < 0$. For this purpose, we need the decomposition $\mathrm{sl}(n,\mathbb{R}) = \mathfrak{R} \oplus \mathfrak{Y} \oplus \mathfrak{N}$ of the Lie algebra $\mathrm{sl}(n,\mathbb{R})$ of all real trace-free $n \times n$-matrices into vector subspaces; here $\mathfrak{R}$ is the subspace of all skew-symmetric matrices, $\mathfrak{Y}$ is the subspace of all symmetric matrices, and $\mathfrak{N}$ is subspace of diagonal matrices. We denote by $\langle v, w\rangle$ the Killing form of the Lie algebra $\mathrm{sl}(n,\mathbb{R})$, i.e., $\langle v, w\rangle = \frac{1}{2}\operatorname{tr} vw^t$, where w^t is the transposed matrix. With respect to this inner product, the decomposition $\mathrm{sl}(n,\mathbb{R}) = \mathfrak{R} \oplus \mathfrak{Y} \oplus \mathfrak{N}$ is orthogonal. We define the linear operator σ: $\mathrm{sl}(n,\mathbb{R}) \to \mathrm{sl}(n,\mathbb{R})$ by setting $\sigma(v) = \alpha v$ for $v \in \mathfrak{R}$, $\sigma(v) = \beta v$ for $v \in \mathfrak{Y}$, and $\sigma(v) = \gamma v$ for $v \in \mathfrak{N}$. On the Lie algebra $\mathrm{sl}(n,\mathbb{R})$, we define a new inner product $\langle v, w\rangle_{\alpha\beta\gamma} = \langle \sigma(v), w\rangle$. We extend the inner product $\langle v, w\rangle_{\alpha\beta\gamma}$ to the group $\mathrm{SL}(n,\mathbb{R})$ by left translations and obtain a left-invariant metric $g_{ij}^{\alpha\beta}$ on $\mathrm{SL}(n,\mathbb{R})$.

Proposition 3.6.1. *The Ricci curvature R_{ij} of the metric $g_{ij}^{\alpha\beta}$ on the Lie group $\mathrm{SL}(n,\mathbb{R})$ for $n \geq 3$ is negative when α, β, and γ are generic.*

In particular, if Γ is a uniform discrete torsion-free subgroup in the group $\mathrm{SL}(n,\mathbb{R})$, then we obtain examples of compact Riemannian manifolds $M = \mathrm{SL}(n,\mathbb{R})/\Gamma$ on which there exists a set of metrics of negative Ricci curvature, and, moreover, the group $\mathrm{SO}(n)$ acts freely on these manifolds.

We present general facts about the structure of the isometry group of manifolds of negative Ricci curvature. We have the following statement owing to Bochner, which plays the role of the maximum principle for infinitesimal isometries of Riemannian manifolds of negative Ricci curvature.

Theorem 3.6.1 ([113])**.** *Let M be a Riemannian manifold of negative Ricci curvature. If the length of an infinitesimal isometry X of the manifold M attains a relative maximum at a certain point of the manifold M, then the vector field X vanishes identically on M.*

Under certain natural finiteness-type restrictions on the volume, the isometry group of Riemannian manifolds of negative Ricci curvature is very small.

Theorem 3.6.2. *Let M be a complete Riemannian manifold of sectional curvature $K(\sigma) \leq 0$, finite volume, and negative Ricci curvature. Then the connected component $I_0(M)$ of the identity of the isometry group $I(M)$ of the manifold M is trivial, i.e., $I_0(M) = \{e\}$.*

3.7. Manifolds with zero Ricci tensor. If M is a compact Riemannian manifold with zero Ricci tensor, then each of its infinitesimal isometries is a parallel vector field. This result is implied by Theorem 3.6.1 (see [113]).

As a consequence, we can describe the structure of homogeneous Riemannian manifolds of zero Ricci curvature.

Theorem 3.7.1 ([113])**.** *Let M be a compact homogeneous Riemannian manifold having zero Ricci tensor. Then it coincides with a flat torus.*

4. Riemannian Manifolds with Restrictions on the Scalar Curvature

4.1. Riemannian manifolds of negative scalar curvature. Let R be the scalar curvature of a Riemannian metric g_{ij} on a manifold M^n. There arises a natural problem about restrictions on the topology of the manifold M^n that are imposed by the existence of a metric of negative scalar curvature. An answer is given by the following theorem owing to Aubin (see [11]).

Theorem 4.1.1. *There exists a Riemannian metric of negative scalar curvature on any compact Riemannian manifold.*

Therefore, the inequality $R < 0$ contains no topological information about the manifold M.

4.2. Compact Riemannian manifolds of positive scalar curvature. There exist nontrivial topological obstructions to the existence of Riemannian metrics of positive scalar curvature. They are related to the so-called Hirzebruch $\hat{A}$-genus. We recall that the characteristic series of a multiplicative sequence according to which the $\hat{A}$-genus is constructed is the series expansion of the function $\sqrt{z}/2\sinh(\frac{1}{2}z)$ (for more details, see [95, 181]).

We recall that the *spinor structure* on an oriented Riemannian manifold M^n of dimension n is the principal spin(n)-bundle F over M such that the SO(n)-bundle $F \underset{\mathrm{spin}(n)}{\times} \mathrm{SO}(n)$ is SO(n)-equivalent to the principal SO(n)-bundle of oriented orthogonal frames over M.

One can show that the spin(n)-structure exists on M^n iff the second Stieffel–Whitney class $w_2(M)$ of the tangent bundle $T(M)$ vanishes. In this case, the number of nonequivalent spin(n)-structures equals the order of the group $H^1(M, \mathbb{Z}_2)$.

Before passing to negative results, we present several simple constructions, which allow us to construct examples of manifolds on which there exists a Riemannian metric of positive scalar curvature. Let M and N be two Riemannian manifolds with metrics ds_M^2 and ds_N^2 of scalar curvatures R_M and R_N, respectively, and let $f : M \to \mathbb{R}^+$ be a smooth function on M. On the manifold $M \times N$, we consider the metric $ds^2 = ds_M^2 + f^2 ds_N^2$. Then the scalar curvature R of this metric is given by

$$R = R_M + \frac{1}{f^2} R_N - \frac{n(n-1)}{f^2} \|\nabla f\|^2 - \frac{2n}{f} \Delta f,$$

where $\Delta f = \sum \nabla_{e_i, e_i} f$ is the Laplace operator on the manifold M and $n = \dim N$. In particular, if M is compact and $-t\nabla^2 + k \geq k_0$ for a certain $t \leq 2$, then $M \times S^1$ has a metric of positive scalar curvature not less than k_0.

If (X, ds^2) is a Riemannian manifold with metric ds^2, $n = \dim X$, and R is the scalar curvature of the manifold X, then for any function $f > 0$, the scalar curvature of the metric $d\hat{s}^2 = f^{\frac{4}{n-2}} ds^2$ has the form

$$\hat{R} = f^{\frac{n+2}{2-n}} \left[-\frac{4(n-1)}{n-2} \nabla^2 f + kf \right].$$

In particular, if X is compact and $-t\nabla^2 + k > 0$ for a certain $t \leq \dfrac{4(n-1)}{n-2}$, then X admits a metric of positive scalar curvature.

We now pass to the topological characteristics of manifolds on which there exists a metric of positive scalar curvature. We begin with the Lichnerowich theorem [121].

Theorem 4.2.1. *Let M be a spinor manifold such that $\hat{A}(M) \neq 0$. Then on M, there is no Riemannian metric whose scalar curvature is nonnegative and positive at least at one point.*

Examples of spinor manifolds for which $\hat{A}(M) \neq 0$ are well known in the theory of spin-manifolds (see [131]). In [97], also by using the index theorem, it is shown that an exotic sphere that does not bound a spin-manifold does not admit a metric of positive scalar curvature.

The condition $\hat{A}(M) = 0$ admits a simple topological interpretation: it holds iff the union of a certain number of copies of M is cobordant to a spinor manifold that admits a nontrivial action of the group S^1.

We give several definitions that will be needed for describing obstructions to the existence of metrics of positive scalar curvature. A mapping $f : M \to N$ of a Riemannian manifold M into a Riemannian manifold N is said to be *ε-contractible* if $\|f_* v\| \leq \varepsilon \|v\|$ for any vector $v \in T_x M$. A complete oriented n-dimensional Riemannian manifold M^n is said to be *ε-spherical* if there exists an ε-contractible mapping $f : M \to S^n$ that is constant outside a certain compact subset in M and whose degree is different from zero. We recall that the *degree* of a mapping f is the number

$$\deg f = \sum_{p \in f^{-1}(q)} \mathrm{sign}(\det f_*)_p,$$

where q is a regular value of the mapping f (see, e.g., [48]). A Riemannian manifold is said to be *spherical* if it is ε-spherical for any $\varepsilon > 0$. As an example, we note that a complete simply connected Riemannian manifold M whose sectional curvatures are not positive is spherical. Each Euclidean space $\mathbb{R}^n$ is spherical. The direct product of two spherical Riemannian manifolds is again a spherical manifold.

A compact manifold M is said to be *narrow* if, for any $\varepsilon > 0$ and for any Riemannian metric on M, there exists a spinor ε-spherical manifold $\tilde{M}$ covering M (in the metric lifted on $\tilde{M}$).

Any compact manifold admitting a metric of nonpositive sectional curvature is narrow. We indicate certain properties of narrow manifolds (see [81]). The property of a manifold to be narrow is an invariant of the homotopy type. The product of two narrow manifolds is a narrow manifold. The connected sum of any spinor manifold and a narrow manifold is a narrow manifold. We have the following main theorem.

Theorem 4.2.2 ([81])**.** *There is no metric of positive scalar curvature on a narrow manifold. Any metric of nonpositive scalar curvature on a narrow manifold is necessarily Ricci-flat.*

As a consequence of this theorem, we obtain the following statement. A compact manifold M on which there exists a metric of nonpositive sectional curvature cannot be equipped with a Riemannian metric of positive scalar curvature. In fact, any metric of nonnegative scalar curvature is necessarily flat. The previous results can be slightly improved. Let $f : M \to N$ be a smooth mapping, and let q be its regular value. Then the $\hat{A}$-*degree* of the mapping f is the value of the $\hat{A}$-genus on $f^{-1}(q)$, i.e., $\hat{A}(f^{-1}(q))$, where $\hat{A}(X)$ is the complete $\hat{A}$-class of the manifold X (see [111]). A complete connected oriented Riemannian manifold M is said to be *ε-spherical in dimension n* ($0 \leq n \leq \dim M$) if there exists an ε-contractible mapping $f : M \to S^n$ that is constant outside a certain compact set and has a nonzero $\hat{A}$-degree. If $\dim M = n$, then we obtain an ordinary ε-sphericity of the manifold M. A compact Riemannian manifold is said to be *narrow in dimension* n if, for any $\varepsilon > 0$, there exists an oriented spinor covering $\tilde{M}$ over M that is ε-spherical in dimension n.

The property to be narrow in dimension n is an invariant of the homotopy type. If X is a narrow manifold in dimension n and Y is narrow in dimension m, then $X \times Y$ is a narrow manifold in dimension $n+m$. A compact spinor manifold N such that $\hat{A}(N) \neq 0$ is a narrow manifold in dimension 0. In this case, $N \times M$, where M is a narrow manifold in dimension n, is narrow in dimension n. The following theorem is a generalization of Theorem 4.2.2.

Theorem 4.2.3. *A compact manifold M that is narrow in dimension $n \geq 0$ cannot have a metric of positive scalar curvature. In other words, any metric of nonnegative scalar curvature on the manifold M is flat.*

As a consequence, we obtain that if M is a compact manifold admitting a spinor mapping of nonzero $\hat{A}$-degree onto a manifold M_0 with metric of nonnegative sectional curvature, then the manifold M cannot have a metric of positive scalar curvature. In particular, the product $M_0 \times N$, where N is any compact spinor manifold with $\hat{A}(N) \neq 0$, cannot have a metric of positive scalar curvature (see [81]).

Theorem 4.2.4. *Let M be a compact manifold of type $k(\pi, 1)$, and let M contain a narrow submanifold $M_0 \subset M$ of type $k(\pi, 1)$ and codimension 2, and, moreover, let a homeomorphism $\pi_1(M_0) \to \pi_1(M)$ induced by an embedding be injective. If the boundary of a tubular neighborhood of the submanifold M_0 in M is a narrow manifold, then there is no metric of positive scalar curvature on M.*

The following statement is a direct consequence of this theorem. There is no (nonflat) Riemannian metric of negative scalar curvature on manifolds M of the following form:

(1) $M = M_1 \# M_2$, where M_2 is a spinor manifold and M_1 is as in the previous theorem;

(2) $M = M_1 \times M_2$, where M_2 is a spinor manifold with $\hat{A}(M_2) \neq 0$ and M_1 is as in the previous theorem.

We now briefly consider methods for proving the above theorems. The so-called Dirac operator serves as the main working tool. We recall its definition. Let M be a Riemannian manifold. We denote by $\mathrm{Cl}(M)$ the Clifford bundle over M. The fiber of the bundle $\mathrm{Cl}(M)$ over a point $x \in M$ is the Clifford algebra $\mathrm{Cl}(T_xM)$ of the tangent space T_xM of the manifold M at the point $x \in M$ (see [25]). We have the canonical embedding $T(M) \subset \mathrm{Cl}(M)$. The Riemannian connection and the Riemannian metric are extended to the bundle $\mathrm{Cl}(M)$ so that the covariant differentiation preserves the metric and

$$\nabla(\varphi \cdot \psi) = (\nabla\varphi) \cdot \psi + \varphi \cdot (\nabla\psi)$$

for all sections φ, ψ of the bundle $\mathrm{Cl}(M)$ (the dot stands for the product in the Clifford algebra).

Further, let $S \to M$ be the bundle of left modules over the bundle $\mathrm{Cl}(M)$ (i.e., for any point $x \in M$, the fiber S_x is a module over the algebra $\mathrm{Cl}_x(M)$, and the multiplication depends smoothly on the point). We assume that S is equipped with a metric and a connection ∇ compatible with this metric; moreover, let the following conditions hold:

(a) the module multiplication $l : S_x \to S_x$ by an arbitrary unit vector $e \in T_xM$ is an isometry at each point $x \in M$;
(b) $\nabla(\varphi \cdot \sigma) = (\nabla\varphi) \cdot \sigma + \varphi \cdot (\nabla\sigma)$ for all sections φ of the bundle $\mathrm{Cl}(M)$ and all sections σ of the bundle S.

We denote by $\Gamma(S)$ the space of all sections of the bundle S. Under the above conditions, we define the first-order differential operator $D : \Gamma(S) \to \Gamma(S)$, called the *Dirac operator*, by setting

$$D \equiv \sum_{i=1}^{n} e_k \cdot \nabla_{e_k},$$

where $e_1, \dots, e_n$ is an orthonormal basis of the space T_xM. We have the canonical orthogonal decomposition $S = S^+ \oplus S^-$ with respect to which the operator D becomes

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

If the manifold M is compact, then the operator D^+ is a Fredholm operator of index $\operatorname{ind} D^+ = \dim\ker D^+ - \dim\operatorname{coker} D^+$.

The Dirac operator was initially considered in the middle of the 60s in connection with the development of the index theory aimed at a generalization of the Rokhlin result on the divisibility of the signature of a smooth four-dimensional spinor manifold by 16. In contrast to the Laplace operator, the dimension of the kernel of the Dirac operator is not a topological invariant and depends on the geometry of the manifold considered.

Partial information about the spectrum of the Dirac operator is obtained from the following theorem owing to Wolf.

Theorem 4.2.5. *If (M^n, g) is a complete oriented Riemannian spinor manifold, then the Dirac operator and its square are essentially self-adjoint operators on the Hilbert space of square integrable spinor fields.*

We mention the following two properties of the spectrum of the Dirac operator. If (M^n, g) is a complete Riemannian manifold, then the spectrum $\mathrm{Spec}(D)$ of the Dirac operator is real and does not contain the residual spectrum. If M is a compact Riemannian manifold, then $\mathrm{Spec}(D)$ consists of only real eigenvalues of finite multiplicity.

The first eigenvalue of the Dirac operator on a Riemannian manifold of positive scalar curvature R yields a lot of information. Let $\lambda^{\pm}$ be the minimum positive (negative) eigenvalue of the operator D, and let

$$R_0 = \min\{R(x) \mid x \in M^n\}.$$

Then the following (Friedrichs) estimate holds:

$$\frac{1}{2}\sqrt{\frac{n}{n-1}R_0} \le |\lambda^{\pm}|.$$

Theorem 4.2.6. *If*

$$\frac{1}{2}\sqrt{\frac{n}{n-1}R_0} \quad or \quad -\frac{1}{2}\sqrt{\frac{n}{n-1}R_0}$$

are eigenvalues of the Dirac operator, then M is an Einstein space.

Therefore, the greatest lower bound of $\frac{1}{2}\sqrt{\frac{n}{n-1}R_0}$ is attained only on an Einstein space. The problem on description of compact Einstein spaces M that admit a spinor field ψ such that

$$D\psi = \pm\frac{1}{2}\sqrt{\frac{R_0 n}{n-1}}\psi$$

is still open.

We consider the case $\dim M = 3$. An Einstein space M^3 is a manifold of a constant sectional curvature, i.e., $M^3 = S^3/\Gamma$. The number $\pm\frac{1}{2}\sqrt{\frac{n}{n-1}R_0}$ is an eigenvalue of the Dirac operator for the space S^3/Γ iff S^3/Γ is a homogeneous space, i.e., $\Gamma \subset I(S^3)$. As was indicated in Sec. 1.3 of Chap. 3, there exist five possibilities for Γ: a cyclic group, a dihedral group, E_6, E_7, or R_8. Finally, we have the following picture. On $M^3 = S^3$, both values $\pm\frac{1}{2}\sqrt{\frac{R_0 n}{n-1}}$ are eigenvalues of the operator D. The space $M^3 = \mathbb{R}P^3$ admits two spinor structures. With respect to one of them, $\frac{1}{2}\sqrt{\frac{nR_0}{n-1}}$ is an eigenvalue, and this is not the case for $-\frac{1}{2}\sqrt{\frac{nR_0}{n-1}}$. With respect to the second structure, the case is the opposite. On all other homogeneous spaces S^3/Γ, the number $-\frac{1}{2}\sqrt{\frac{nR_0}{n-1}}$ is not an eigenvalue, while $\frac{1}{2}\sqrt{\frac{nR_0}{n-1}}$ is an eigenvalue of one and only one spinor structure.

In dimension four, we have the following description of $\mathrm{Spec}(D)$.

Theorem 4.2.7. *If M^4 is a compact Riemannian spinor manifold of positive scalar curvature and if $\sqrt{R_0/3}$ or $-\sqrt{R_0/3}$ is an eigenvalue of the Dirac operator, then the manifold M^4 is isometric to the sphere S^4.*

In dimension five, we restrict ourselves to an example of an Einstein manifold of nonconstant sectional curvature at which the greatest lower bound indicated above is attained as an eigenvalue.

Example. We consider the Stieffel manifold $V_{4,2} = \mathrm{SO}(4)/\mathrm{SO}(2)$ and the corresponding decomposition of the Lie algebra $\mathrm{so}(4) = \mathrm{so}(2) + \mathfrak{M}$. Let B be the Killing metric on $\mathrm{so}(4)$. Then we define the metric $g(X, Y)$ on $V_{4,2}$ by setting

$$g(a+\xi, b+\eta) = \frac{1}{2}B(\xi,\eta) + \frac{2}{3}B(a,b),$$

where $a, b \in \mathrm{so}(2)$ and $\xi, \eta \in \mathfrak{M}$. This Riemannian metric is an Einstein metric of scalar curvature $R = 26/3$. On the space $(V_{4,2}, g)$, the equation

$$D\psi = \pm\frac{1}{2}\sqrt{\frac{5R}{4}}\psi$$

admits a nontrivial solution.

The eigenvalues of the Dirac operator on five-dimensional manifolds of constant sectional curvature were studied by Sulanke. In particular, he proved that there exists an inhomogeneous space of constant sectional curvature for which $\pm\frac{1}{2}\sqrt{\frac{5R}{4}}$ is an eigenvalue of the Dirac operator D.

We denote by $\operatorname{Spec}^{\Delta}(M)$ the spectrum of the Laplace operator on a manifold M and by $\operatorname{Spec}^{D}(M)$ the spectrum of the Dirac operator D. In certain cases, knowledge of $\operatorname{Spec}^{D}(M)$ allows us to say much more about the geometry of the Riemannian manifold considered. We present several statements of such a type.

Let M^n and $\bar{M}^n$ be two closed spinor manifolds for which

$$\operatorname{Spec}^{\Delta}(M^n) = \operatorname{Spec}^{\Delta}(\bar{M}^n) \quad \text{and} \quad \operatorname{Spec}^{D}(M^n) = \operatorname{Spec}^{D}(\bar{M}^n).$$

If the sectional curvature of the manifold M^n is constant, then the sectional curvature of the manifold $\bar{M}^n$ is also constant and is equal to the sectional curvature of the manifold M^n. Therefore, M^n and $\bar{M}^n$ are locally isometric.

Let $\operatorname{Spec}^{D^2}(M)$ be the spectrum of the square D^2 of the operator D. Let M^4 and $\bar{M}^4$ be four-dimensional closed Riemannian spinor manifolds of constant scalar curvatures R and $\bar{R}$, respectively. If $\operatorname{Spec}^{D^2}(M^4) = \operatorname{Spec}^{D^2}(\bar{M}^4)$ and $\operatorname{Spec}^{\Delta}(M^4) = \operatorname{Spec}^{\Delta}(\bar{M}^4)$, then the manifolds M^4 and $\bar{M}^4$ are of the same Euler characteristic.

The consideration of the Dirac operator allows us to define a new metric invariant, the so-called index coefficient of a pair of Riemannian metrics, which possesses some interesting geometric properties. Let g_0 and g_1 be two metrics on a manifold M. On $M \times \mathbb{R}$, we define a metric ds^2 of the form $ds^2 = g_t + dt^2$, where $g_t = g_0$ for $t = 0$, $g_t = g_1$ for $t = 1$, and g_t is an arbitrary smooth homotopy between g_0 and g_1 for $0 \le t \le 1$.

The *index coefficient* $i(g_0, g_1)$ of a pair of metrics g_0, g_1 on a manifold M is the number $i(g_0, g_1) = \operatorname{ind}(D^+)$, where D is the canonical Dirac operator on the manifold $M \times \mathbb{R}$. This definition is correct. The index coefficient of metrics has the following properties:

(a) $i(g_0, g_1) = -i(g_1, g_0)$;
(b) $i(g_0, g_1) + i(g_1, g_2) + i(g_2, g_0) = 0$;
(c) if a metric g_0 is homotopic to g_1, and, moreover, in the process of homotopy, we pass through metrics of positive scalar curvature, then $i(g_0, g_1) = 0$.

Therefore, the invariant $i(g_0, g_1)$ is constant on connected components of the space of metrics of positive scalar curvature. The invariant $i(g_0, g_1)$ is not trivial, which is asserted by the following theorem.

Theorem 4.2.8. *For metrics g of positive scalar curvature on the sphere S^7, the invariant $i(g, g_{\mathrm{can}})$ assumes infinitely many integer values.*

In particular, the space of metrics of positive scalar curvature on the sphere S^7 has infinitely many connected components.

For an arbitrary fixed metric g on a compact spinor manifold M and for an arbitrary diffeomorphism $F \in \operatorname{Diff}(M)$, we set $i_g(F) = i(g, F^*(g))$.

Theorem 4.2.9. *For any metric g of positive scalar curvature on a manifold M, the mapping i_g : $\operatorname{Diff}(M) \to \mathbb{Z}$ is a group homomorphism.*

As a consequence of this theorem, we have that each metric g_{ij} of positive scalar curvature assigns a homomorphism $i_g : \Gamma(M) \to \mathbb{Z}$, where

$$\Gamma(M) = \operatorname{Diff}(M) / \operatorname{Diff}_0(M)$$

is the group of components of the diffeomorphism group of the manifold M. In particular, if the group $\Gamma(M)$ is finite, then $i_g = 0$ for an arbitrary metric g.

4.3. Noncompact Riemannian manifolds of positive scalar curvature. We now consider the case of noncompact Riemannian manifolds that admit a metric of positive scalar curvature. The results of the preceding subsection admit a generalization to this case (for the corresponding proofs, see [81]).

Let M_1 and M_2 be two Riemannian manifolds. A smooth mapping $f : M_1 \to M_2$ is said to be (ε, Λ^2)*-contractible* or ε*-contractible on* 2*-forms* if $\|f^*\varphi\| \le \varepsilon \|\varphi\|$ for any 2-form φ on the manifold M_2. A connected manifold M is said to be Λ^2-*narrow* if, for any given metric on M and any $\varepsilon > 0$, there exists a spinor covering $\tilde{M} \to M$ such that $\tilde{M}$ is (ε, Λ^2)-spherical in the metric lifted from M. A lift of a Λ^2-*narrow* manifold *in dimension* n is defined similarly to a narrow manifold in dimension n. The following two important characteristics of the Λ^2-narrowness of a manifold hold. Let M and N be connected oriented manifolds, and let $f : M \to N$ be a proper spinor mapping of zero $\hat{A}$-degree. If N is Λ^2-narrow in dimension n, then the manifold M also has this property. The second characteristic is related to the existence of Λ^2-narrow submanifolds. More precisely, let M be a connected spinor manifold. If M contains an open Λ^2-narrow submanifold $U \subset M$ such that the homomorphism $\pi_1(U) \to \pi_1(M)$ is injective, then M is Λ^2-narrow.

We indicate several interesting examples of Λ^2-narrow manifolds. If M_0 is an arbitrary compact narrow manifold, then $M = M_0 \times \mathbb{R}$ is Λ^2-narrow. Any hyperbolic manifold M of finite volume is Λ^2-narrow.

We have the following main theorem.

Theorem 4.3.1. *Let M be a Λ^2-narrow manifold. Then M cannot have a complete Riemannian metric of positive scalar curvature. In other words, any metric of nonnegative scalar curvature on M is necessarily Ricci-flat.*

As a consequence, we obtain that there is no complete Riemannian metric of positive scalar curvature on the following manifolds M:

(1) $M = M_0 \times \mathbb{R}$, where M_0 is a narrow manifold;
(2) M is a connected manifold such that M contains a compact narrow submanifold $M_0 \subset M$ of codimension one and the homomorphism $\pi_1(M_0) \to \pi_1(M)$ induced by embedding is injective;
(3) M is a hyperbolic Riemannian manifold of finite volume;
(4) $M = M_1 \times M_2$, where M_1 is any manifold described in items (1)–(3) and M_2 is a compact spinor manifold with $\hat{A}(M_2) \neq 0$ (for example, $\mathbb{R} \times M_2$).

4.4. Manifolds of lower dimension. We consider the following two cases: $\dim M = 3$ and $\dim M = 4$. Any three-dimensional manifold admits the decomposition

$$M = M_1 \# \cdots \# M_l \# (S^1 \times S^2) \# \cdots \# (S^1 \times S^2) \# K_1 \# \cdots \# K_n,$$

where the group $\pi_1(M_j)$ is finite for $1 \leq j \leq l$ and each k_j is a manifold of type $k(\pi, 1)$ (see Fig. 40).

Theorem 4.4.1 ([81])**.** *Let M be a compact three-dimensional manifold having $k(\pi, 1)$ summands in the above decomposition. Then M admits no metrics of positive scalar curvature.*

We note that a manifold of the form

$$M_1 \# \cdots \# M_l \# (S^1 \times S^2) \# \cdots \# (S^1 \times S^2)$$

admits a metric of positive scalar curvature under the condition that each M_j is diffeomorphic to S^3/Γ_j for a certain subgroup $\Gamma_j \subset \mathrm{O}(4)$ acting on the sphere S^3 in the standard way.

A compact surface Σ embedded in a manifold M is said to be *incompressible* if the fundamental group $\pi_1(\Sigma)$ is infinite and the induced mapping $\pi_1(\Sigma) \to \pi_1(M)$ is injective. If we replace the embedding by a proper imbedding in this definition, then we obtain the definition of a *tight* surface in M. Any incompressible surface is tight.

Theorem 4.4.2 ([81])**.** *A three-dimensional manifold containing a tight surface does not admit a complete Riemannian metric of positive scalar curvature. Moreover, there is no Riemannian metric whose scalar curvature R satisfies the inequality $R \geq \mathrm{const} > 0$.*

A three-dimensional manifold containing an incompressible surface cannot have a complete nonflat Riemannian metric of nonnegative scalar curvature.

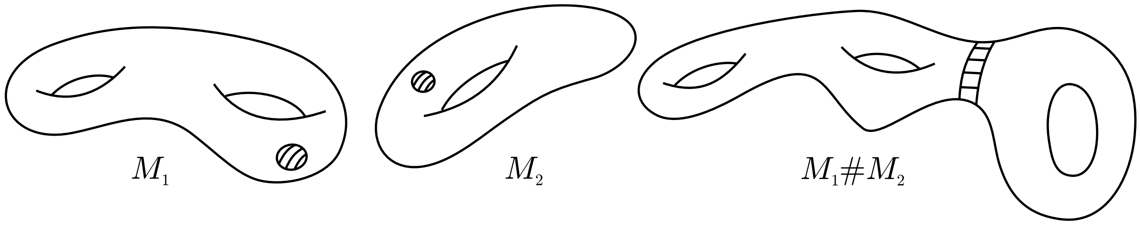


Fig. 40

Theorem 4.4.3. *On a compact four-dimensional manifold of type $k(\pi, 1)$ that contains an incompressible surface, there is no metric of positive scalar curvature.*

Let M be a three-dimensional manifold. We say that a smoothly embedded curve $\gamma \subset M$ is *small* if it has infinite order in the group $H_1(M)$ and the normal circle γ' in the normal bundle to γ has infinite order in $H_1(M \setminus \gamma)$. A similar definition can also be given in the four-dimensional case. A compact incompressible surface $\Sigma \subset M$ embedded in a four-dimensional manifold M is said to be small if the order of the quotient $\pi_1(M)/\pi_1(\Sigma)$ is finite and the normal circle from the normal bundle of Σ has infinite order in the group $H_1(\tilde{M} \setminus \Sigma)$, where $\tilde{M}$ is a covering over M with $\pi_1(\tilde{M}) \cong \pi_1(\Sigma)$.

Theorem 4.4.4. *Let M be an open three-dimensional manifold such that the group $H_1(M)$ is finitely generated. If M contains a small curve, then M has no Riemannian metric whose scalar curvature R satisfies the inequality $R \geq \text{const} > 0$.*

We note that Theorem 4.4.4 implies Theorem 4.4.1.

A similar result holds for four-dimensional manifolds.

Theorem 4.4.5. *A compact four-dimensional manifold containing a small surface cannot have a Riemannian metric of positive scalar curvature.*

In conclusion, we present one more important result that concerns restrictions on the fundamental group of manifolds admitting a metric of positive curvature.

Theorem 4.4.6. *Let M be a three-dimensional manifold whose fundamental group $\pi_1(M)$ contains a subgroup isomorphic to the fundamental group of a compact surface of positive genus. Then the manifold M cannot have a complete metric of positive scalar curvature.*

4.5. Metrics of prescribed scalar curvature. It was noted long ago that there are global restrictions imposed by the topology of a manifold on its differential-geometric structures. The simplest restrictions come from the Gauss–Bonnet theorem for two-dimensional Riemannian manifolds:

(a) if the Euler characteristic $\chi(M^2)$ of a manifold M^2 satisfies the inequality $\chi(M^2) > 0$, then the Gaussian curvature K of the manifold M^2 should be positive somewhere on M^2;

(b) if $\chi(M^2) = 0$, then the Gauss curvature K should change its sign or should be identically equal to zero;

(c) if $\chi(M^2) < 0$, then the Gauss curvature K of the manifold M^2 should be negative somewhere on M^2.

We can assume that any smooth function K on a two-dimensional manifold is locally the Gaussian curvature of a certain Riemannian metric. In a coordinate neighborhood $U(x, y)$, we seek a metric of a special form $ds^2 = dx^2 + G(x, y)dy^2$ in which the lines $y = \text{const}$ are geodesics. Such a coordinate system always exists in a neighborhood of any point. The function G and the curvature K are related by the equation

$$\frac{\partial^2 \sqrt{G}}{\partial x^2} + K(x, y)\sqrt{G} = 0$$

for which we take the initial conditions

$$\sqrt{G}\Big|_{x=0} = 1 \quad \text{and} \quad \frac{\partial\sqrt{G}}{\partial x}\Big|_{x=0} = 0.$$

This ordinary differential equation has a positive solution $\sqrt{G} > 0$ (y is a parameter) near $x = 0$, which yields the desired metric. We assume that a smooth real-valued function $K : M \to \mathbb{R}$ is given on a closed two-dimensional manifold M. One asks: does there exist a Riemannian metric on N for which K is its Gaussian curvature? The Gauss–Bonnet condition cannot be stated in advance on M, since we have no area element on M for now. The following theorem completely solves the problem on conditions under which a given function coincides with the Gaussian curvature of a certain metric.

Theorem 4.5.1 ([105]). *Let M be a compact two-dimensional Riemannian manifold of Euler characteristic $\chi(M)$. Then a function $K \in C^\infty(M)$ is the Gaussian curvature of a certain metric iff*

(a) *the function K is positive somewhere and $\chi(M) > 0$;*
(b) *the function K changes sign or $K \equiv 0$ if $\chi(M) = 0$;*
(c) *the function K is negative somewhere if $\chi(M) < 0$.*

4.6. Functional $\lambda_1(g)$. On a Riemannian manifold M, there always exists a differential operator, the so-called Laplace operator, which always deserves more attention. Let g_{ij} be a Riemannian structure on M, and let $(x^1, \dots, x^n)$ be a local coordinate system in a neighborhood $U \subset M$. We define g^{jk} by the relation $g_{ij}g^{jk} = \delta_i^k$, $g = \det(g_{ij})$. Each function f on M generates the vector field $\operatorname{grad} f$ whose restriction to U is given by

$$\operatorname{grad} f = g^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^i}.$$

If X is a vector field on M, then its *divergence* is a function on M that is given by

$$\operatorname{div} X = \frac{1}{\sqrt{g}}\sum_i \frac{\partial}{\partial x^i}(\sqrt{g}X^i)$$

on U, where

$$X = X_i\frac{\partial}{\partial x^i}$$

on U. The Laplace operator Δ is defined by $\Delta f = \operatorname{div}\operatorname{grad} f$ on smooth functions f. In local coordinates, we have

$$\Delta f = \frac{1}{\sqrt{g}}\sum_k \frac{\partial}{\partial x_k}\left(\sum_i g^{ik}\sqrt{g}\frac{\partial f}{\partial x^i}\right).$$

The operator Δ is a differential operator whose principal symbol is of the form $q(\lambda) = q^{-1}(\lambda, \lambda)$, where g^{-1} stands for the norm on T^*M induced by the Riemannian metric g_{ij}. In local coordinates, $q(\lambda) = g^{ij}\lambda_i\lambda_j$. Therefore, Δ is an elliptic differential operator. We indicate the simplest properties of the operator Δ on a Riemannian manifold (M, g_{ij}). The operator Δ is symmetric, i.e.,

$$\int_M u(x)[\Delta v](x)\,dx = \int_M [\Delta u](\dot{x})v(x)\,dx$$

for any functions u and v such that one of them is compactly supported. If f is a diffeomorphism of the manifold M, then the operator Δ is invariant with respect to f iff f is an isometry.

The operator Δ can be extended to a positive self-adjoint unbounded operator on the space $L^2(M, \mu)$, where μ is a measure generated by the Riemannian metric. The operator thus obtained is also denoted by Δ. Since the resolvent of the operator Δ is completely continuous, its spectrum is discrete (see, e.g., [221]).

In some cases, the spectrum of the operator Δ completely determines the Riemannian manifold considered. For example, let us consider the simplest case of a one-dimensional manifold. Let (M, g) and

(M', g') be two one-dimensional Riemannian manifolds. Then if the spectra $\mathrm{Spec}(M, g)$ and $\mathrm{Spec}(M', g')$ of the Laplace operator on M and M' coincide, we have $(M, g) = (M', g')$. Let Γ and Γ' be two lattices in the space $\mathbb{R}^2$. If $\mathrm{Spec}(\mathbb{R}^2/\Gamma) = \mathrm{Spec}(\mathbb{R}^2/\Gamma')$, then $(\mathbb{R}^2/\Gamma, g_0/\Gamma) = (\mathbb{R}^2/\Gamma', g_0/\Gamma')$, i.e., the two-dimensional torus is uniquely determined by the spectrum of the Laplace operator. We consider the Klein bottle $k(a, b)$ obtained by factorization of the plane $\mathbb{R}^2$ by the action of the group $(x, y) \mapsto (x + a, b - y)$. Let $\mathrm{Spec}\, k(a, b) = \mathrm{Spec}\, k(a', b')$. Then $k(a, b) = k(a', b')$.

In general, it is not true that if the spectra of two manifolds coincide, then these manifolds are isometric. Milnor constructed two nonisometric tori of dimension 16 with the same spectrum. Kneser proved that the number of pairwise nonisomorphic tori with the same spectrum is finite. We present the Milnor example. In $\mathbb{R}^8$, we consider the lattice Γ_1. Let $e_1, \ldots, e_8$ be the standard basis of the space $\mathbb{R}^8$. Then Γ_1 is generated by the vectors $e_1 - e_8, e_2 - e_8, \ldots, e_7 - e_8, e_1 + e_2, e_3 + e_4, e_5 + e_6$, and $\omega = \frac{1}{2}(e_1 + e_2 + \ldots + e_8)$. All the generators of the lattice Γ_1 are of length $\sqrt{2}$. In $\mathbb{R}^{16}$, we consider the lattice Γ_2. Let $e_1, \ldots, e_{16}$ be the standard basis of the space $\mathbb{R}^{16}$. Then elements of the lattice Γ_2 are of the form $a_1 e_1 + \ldots + a_{16} e_{16}$ and $(a_1 + \frac{1}{2})e_1 + \ldots + (a_{16} + \frac{1}{2})e_{16}$, where $a_1, \ldots, a_{16} \in \mathbb{Z}$. The length of the vectors of the second type is not equal to $\sqrt{2}$, since $\frac{1}{2}[(2a_1 + 1)^2 + \ldots + (2a_{16} + 1)^2]^{1/2} \geq 2$. In the space $\mathbb{R}^{16} = \mathbb{R}^8 \oplus \mathbb{R}^8$, we consider two lattices, $\Gamma_1 \oplus \Gamma_1$ and Γ_2. The first lattice is generated by a vector of length $\sqrt{2}$, and at the same time, the second lattice does not satisfy this condition. Therefore, the tori $\mathbb{R}^{16}/\Gamma_1 \oplus \Gamma_1$ and $\mathbb{R}^{16}/\Gamma_2$ are not isometric, but at the same time, $\mathrm{Spec}(\mathbb{R}^{16}/\Gamma_1 \oplus \Gamma_1) = \mathrm{Spec}(\mathbb{R}^{16}/\Gamma_2)$ (see [19]).

An important role in the problem of existence of metrics with given scalar curvatures is played by the functional $\lambda_1(g)$, the minimum eigenvalue of the Laplace operator Δ of a metric g_{ij} $(\Delta\psi = \lambda_1(g)\psi)$. The functional $\lambda_1(g)$ obeys a number of surprising properties, which show that it is closely related to the geometry of the Riemannian manifold (M, g_{ij}).

Theorem 4.6.1. *Let M be a compact Riemannian manifold, and let* $\dim M = n \geq 3$. *Then*

(a) *the sign $\lambda_1(g)$ is a conformal invariant*;

(b) *on M, one can introduce a metric of positive (resp. zero and negative) scalar curvature that is pointwise conformally equivalent to the metric g iff $\lambda_1(g) > 0$ (resp. $\lambda_1(g) = 0$ and $\lambda_1(g) < 0$)*;

(c) *there exist topological obstructions to the assignment of a metric with $\lambda_1(g) > 0$ and $\lambda_1(g) = 0$ on M*;

(d) *on any manifold M, one can introduce a Riemannian metric g with $\lambda_1(g) < 0$*;

(e) *if M admits a metric g_t with $\lambda_1(g_t) > 0$, then it also admits a metric with $\lambda_1(g) = 0$*;

(f) *critical points of the functional $\lambda_1(g)$ on the space of all metrics g with* $\mathrm{vol}(M, g) = 1$ *are Einstein metrics, i.e.,*

$$R_{ij}(x) = \frac{1}{n} R(x) g_{ij}(x).$$

Let $CE(g)$ stand for the set of smooth functions that are scalar curvatures of metrics conformal to a metric g.

Theorem 4.6.2 ([104]). (a) *If $\lambda_1(g) < 0$, then $CE(g)$ coincides with the set of smooth functions that are negative somewhere on M.*

(b) *If $\lambda_1(g) = 0$, then $CE(g)$ coincides with the set of smooth functions that either change their sign on M or are identically equal to zero.*

(c) *If $\lambda_1(g) > 0$, then $CE(g)$ contains those smooth functions K for which there exists a constant $C > 0$ such that* $\min K < Ck < \max K$. *Moreover, if $CE(g)$ contains a constant function, then $CE(g)$ coincides with the set of smooth functions that are positive somewhere on M (K is the scalar curvature of the metric g).*

As an example, we note that on the sphere S^n $(n \geq 3)$, any function $K \in C^\infty(M)$ is the scalar curvature of a certain metric.

4.7. Riemannian manifolds with zero scalar curvature. The following theorem yields a simple example of a topological obstruction to the existence of metrics of zero scalar curvature. We denote by $\hat{A}(M)$ the Hirzebruch genus. Let $b_1(M)$ be the first Betti number, and let all manifolds considered be compact and connected.

Theorem 4.7.1. *Let M be a spinor manifold such that $\hat{A}(M) \neq 0$ and $b_1(M) = \dim M$. Then this manifold does not admit a metric of zero scalar curvature.*

As an example of such a manifold, we can take the manifold M, which is the connected sum of the n-dimensional torus and an n-dimensional spin manifold M_1 of genus $\hat{A}(M_1) \neq 0$ (as M_1, a $K3$-surface can be taken).

4.8. Conformally equivalent metrics. Let (M, g_{ij}) be a compact Riemannian manifold of dimension $n \geq 3$, and let R be its scalar curvature. There arises a natural question: does there exist a metric g' on the manifold that is conformally equivalent to g such that its scalar curvature $R' \equiv \text{const}$ (see [215])? An answer to this question is given by the following theorem of Yamabe (see [113]).

Theorem 4.8.1. *Each Riemannian metric on a compact Riemannian manifold M of dimension $n \geq 3$ can be conformally deformed into a Riemannian metric of constant scalar curvature.*

The sign of the scalar curvature is preserved under conformal transformations in the following sense. There is no conformal mapping of a compact Riemannian manifold whose scalar curvature is nonpositive everywhere onto the manifold whose scalar curvature is nonnegative everywhere, except for the case where both scalar curvatures vanish identically. If two compact Riemannian manifolds of dimension $n \geq 3$ are of zero scalar curvature, then any conformal transformation is a homothety.

Let (M, g_{ij}) and (M', g'_{ij}) be two compact Riemannian manifolds of scalar curvatures R_g and $R_{g'}$, respectively, which are nonnegative everywhere and do not vanish identically. Then a conformal mapping $f : M \to M'$ is a homothety iff $R_{f_* g} = f_* R_g = e^{-2k} R_g$ for a certain constant k (see [148]). In this case, $f_* g' = e^{2k} g$. In particular, a conformal mapping of compact Riemannian manifolds of scalar curvature that is nonpositive everywhere and does not vanish identically is an isometry iff this mapping preserves the scalar curvature.

We now consider the case of Riemannian manifolds of constant scalar curvature. In this case, both numbers R_g and $R_{g'}$ are either simultaneously zero or of the same sign. Let (M, g_{ij}) and (M', g'_{ij}) be two compact Riemannian manifolds of nonpositive constant scalar curvature. Then each conformal transformation $f : M \to M'$ is a homothety such that $R_{g'} f_* g' = R_g g$. These assertions easily imply (see [148]) that a conformal transformation of a Riemannian manifold of nonpositive constant scalar curvature is always an isometry.

For compact Riemannian manifolds (M^n, g_{ij}), we define a conformal invariant $\nu(g)$ as follows. Let

$$W^i_{jkl} = R^i_{jkl} - \frac{1}{n-2}(L^i_k g_{jl} + g^i_k L_{jl} - L^i_l g_{jk} - g^i_l L_{jk}),$$

where

$$L_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}, \quad L^i_j = g^{is} L_s \quad (W = 0 \quad \text{if} \quad \dim M = n \leq 2).$$

By definition, we set

$$\nu(g) = \frac{2}{n} \int\limits_M |W|^{\frac{n}{2}} d\nu_g,$$

where

$$|W|^{\frac{n}{2}} = \langle W, W \rangle^{\frac{n}{4}} = (g_{ip} g^{jq} g^{kr} g^{ls} W^i_{jkl} W^p_{qrs})^{\frac{n}{4}}.$$

The functional $\nu(q)$ has the following properties:

(a) $\nu(e^{2l} g) = \nu(g)$ for any smooth function f on the manifold M;

(b) for any diffeomorphism φ, we have $\nu(\varphi^* g) = \nu(g)$;
(c) if $\dim M = n \leq 3$, then $\nu(g) = 0$;
(d) if $\dim M = n \geq 4$, then $\sup\limits_g \nu(g) = \infty$.

The functional $\nu(g)$ is estimated from below by the first Pontryagin class:

$$|p_1(M)| \leq \frac{\nu(g)}{8\pi^2}, \quad \dim M = 4.$$

The functional $\nu(g)$ allows us to define a new invariant of Riemannian manifolds:

$$\nu(M) = \inf\{\nu(g) | g \in R(M)\},$$

where $R(M)$ is the space of all Riemannian metrics on the manifold M.

Using the invariant $\nu(M)$, we can distinguish conformally flat metrics because of the following result. If g is a conformally flat metric, then $\nu(g) = 0$. Therefore, if a manifold M can be equipped with a conformally flat metric, then $\nu(M) = 0$. However, the relation $\nu(M) = 0$ does not imply, in general, that the manifold M admits a conformally flat metric.

Examples. If g_0 is a Fubini–Study metric on $\mathbb{C}P^2$, then $\nu(\mathbb{C}P^2) = \nu(g_0) = 24\pi^2$. If $M = k\mathbb{C}P^2 = \mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2$ (k times), then $\nu(k\mathbb{C}P^2) = 24\pi^2$.

Let g and $\bar{g}$ be two Riemannian metrics on the sphere S^2 with Gaussian curvatures K and $\bar{K}$, respectively. Then for the metric $\hat{g} = g \times \bar{g}$ on the product $S^2 \times S^2$, we have

$$\nu(g) = \frac{128}{3}\pi^2 + \frac{2}{3} \int\limits_{S^2 \times S^2} (K - \bar{K})^2 d\nu.$$

The vanishing of the invariant $\nu(M)$ admits a topological interpretation because of the following Kobayashi theorem.

Theorem 4.8.2. *If the group S^1 acts freely on a manifold M, then $\nu(M) = 0$.*

5. Volume of Riemannian Manifolds and Their Topology

5.1. Simplicial volume and its estimates. We have already considered the problem of estimation of the volume in terms of various geometric and topological characteristics of Riemannian manifolds. As recent studies show, these problems are related to deep topological problems. Let M be a complete Riemannian manifold with metric g_{ij} such that its sectional curvatures $K(\sigma)$ satisfy the inequality $|K(\sigma)| < 1$.

Definition 5.1.1. The *minimum volume* of the manifold M is the number $\operatorname{mvol}(M) = \inf\limits_{|K(\sigma)| \leq 1} \operatorname{vol}(M, g)$.

Examples. If M is a closed connected surface, then $\operatorname{mvol}(M) = 2\pi|\chi(M)|$, where $\chi(M)$ is the Euler characteristic of the manifold M. If $n = \dim M > 2$, then for a closed manifold M, the Gauss–Bonnet formula implies the inequality $\operatorname{mvol}(M) \geq C_n|\chi(M)|$ for a certain constant $C_n > 0$. For an even n, this inequality is noninformative. A similar inequality also holds for the Pontryagin numbers:

$$\operatorname{mvol}(M) \geq C_n'|p(M)|.$$

To estimate the volume of a Riemannian manifold, we introduce a new concept-simplicial volume. Let X be an arbitrary topological space, and let $C_* = C_*(X)$ be the chain complex of the space X with coefficients in the field $\mathbb{R}$. Any chain $c \in C_*$ is a linear combination $c = \sum c_i\sigma_i$ of singular simplexes σ_i of the space X with real coefficients $c_i \in \mathbb{R}$. We define the norm $\|c\|$ on the space C_* by setting $\|c\| = \sum\limits_i |c_i|$. On the homology space $H_* = H_*(X^i, \mathbb{R})$, this norm generates the pseudonorm $\|\alpha\| = \inf\limits_z \|z\|$, where z runs over all singular cocycles representing a cohomology class $\alpha \in H_*(X, \mathbb{R})$.

Definition 5.1.2. Let M be a closed manifold; its *simplicial volume* $\|M\|$ is the norm of its fundamental class. If M is not orientable, then we set $\|M\| = \frac{1}{2}\|\tilde{M}\|$, where $\tilde{M}$ is a two-sheeted covering over M.

Example. We have the relations $\|\mathbb{R}^1\| = \infty$ and $\|\mathbb{R}^n\| = 0$ for $n \geq 2$ (see [76]). If M is a closed surface of constant negative curvature, then $\|M\| = 2|\chi(M)|$.

The simplicial volume has the following properties.

(1) If M_1 is a closed manifold and M_2 is arbitrary, then

$$C\|M_1\|\|M_2\| \geq \|M_1 \times M_2\| \geq C^{-1}\|M_1\|\|M_2\|,$$

where $C > 0$ is a constant depending only on $n = \dim(M_1 \times M_2)$.

(2) Let $n \geq 3$; for the connected sum of n-dimensional manifolds M_1 and M_2, we have

$$\|M_1 \# M_2\| = \|M_1\| + \|M_2\|.$$

The problem on the existence of manifolds of a nonzero simplicial volume is solved by the Milnor–Sullivan–Thurston inequality.

Theorem 5.1.1 ([130]). *If a closed manifold M admits a flat bundle ξ of dimension $n = \dim M$, then $\|M\| \geq |\chi|$, where χ is the Euler characteristic of the bundle ξ.*

This theorem is useful only in the case where n is even, since the Euler characteristic vanishes in odd dimension. The following estimate holds in any dimension.

Theorem 5.1.2 ([76]). *Let M be a complete Riemannian manifold of finite volume $\operatorname{vol}(M) < \infty$. If the sectional curvatures $K(\sigma)$ of the manifold M satisfy the inequality $-\infty < -K \leq K(\sigma) \leq -1$, then $\operatorname{vol}(M) \leq C_n\|M\|$.*

Two Riemannian manifolds are said to be *étally isometric* if their universal coverings are isometric. The ratio $\|M\|/\operatorname{vol}(M)$ is an invariant of the étale isometry. More precisely, the following statement holds.

Theorem 5.1.3 ([76]). *If the universal coverings of two closed Riemannian manifolds M_1 and M_2 are isometric, then $\|M_1\|/\operatorname{vol}(M_1) = \|M_2\|/\operatorname{vol}(M_2)$.*

The usual volume of a Riemannian manifold admits the following estimate in terms of the Betti numbers $b_i(M) = \dim H^i(M, \mathbb{R})$ of the manifold M. Let M be a complete connected real analytic manifold such that $-K^2 \leq K(\sigma) \leq 0$, and let the Ricci tensor of the manifold M be negative at a certain point $x \in M$. Then

$$\sum_{i=0}^{n} b_i(M) \leq \operatorname{const} K^n \operatorname{vol}(M),$$

where $n = \dim M$.

A similar estimate also holds for the simplicial volume. Let M be the connected sum of manifolds of one of the following two types:

(a) compact locally symmetric spaces of nonzero simplicial volume;

(b) complete manifolds of finite volume whose sectional curvature satisfies the inequality $-K_1 \leq K_M(\sigma) \leq -K_2 < 0$.

Then

$$\sum_{i=0}^{n} b_i(M) \leq C\|M\|,$$

where the constant C depends only on the dimension $n = \dim M$ and the ratio K_1/K_2.

As usual, we denote by $\operatorname{Ric}(M)$ the Ricci tensor of a manifold M. The following inequality owing to Gromov holds (see [76]).

Theorem 5.1.4. *Let M be a complete n-dimensional Riemannian manifold such that*

$$\operatorname{Ric}(M)(\tau,\tau) \geq -\frac{1}{n-1}\langle \tau,\tau\rangle$$

for all tangent vectors $\tau \in T(M)$. Then $\|M\| \leq C(n)\operatorname{vol}(M)$ for a certain constant $C(n)$ such that $0 < C(n) < n!$

This result implies the following estimate for the minimum volume of a Riemannian manifold: $\|M\| \leq (n-1)^n n! \operatorname{mvol}(M)$. If M is a complete Riemannian manifold such that $\operatorname{Ric}(\tau,\tau) \geq -K^2\langle \tau,\tau\rangle$ for all tangent vectors $\tau \in T(M)$ and M is homeomorphic to a compact hyperbolic manifold or, more generally, to the connected sum of manifolds of the types (a) and (b) described above, then

$$\sum_{i=0}^{n} b_i(M) \leq C(n)K^n \operatorname{vol}(M).$$

Finally, we give one more consequence of Theorem 5.1.4. Let M and M' be complete Riemannian manifolds of dimension n, and let $f : M \to M'$ be a continuous proper mapping. If

$$\operatorname{Ric}(M)(\tau,\tau) \geq -\frac{1}{n-1}\langle \tau,\tau\rangle, \quad -\infty < -K \leq K_{M'}(\sigma), \quad \operatorname{vol}(M') < \infty,$$

then

$$|\deg f| \leq C(n)\frac{\operatorname{vol}(M)}{\operatorname{vol}(M')}.$$

We have presented above the estimate of the simplicial volume of the product of two manifolds. Similarly, we can estimate the minimum volume of the product of two Riemannian manifolds: $p(M_1)\|M_2\| \leq C(n)\operatorname{mvol}(M_1 \times M_2)$, where $n = \dim M$ and $p(M_1)$ is the Pontryagin number of the manifold M_1.

The following theorem yields the criterion for vanishing of the simplicial volume of a Riemannian manifold (see [76]).

Theorem 5.1.5. *Let M be a complete n-dimensional manifold such that $\operatorname{Ric}(\tau,\tau) \geq -\langle \tau,\tau\rangle$ for all tangent vectors $\tau \in TM$, and let the unit ball centered at an arbitrary point $x \in M$ satisfy the inequality $\operatorname{vol}(B_x(1)) \leq \varepsilon$ for a sufficiently small positive number $\varepsilon = \varepsilon(n)$. Then the simplicial volume $\|M\|$ of the manifold M is zero. In particular, if $\operatorname{vol}(M) \leq \varepsilon(n)$, then $\|M\| = 0$.*

For an arbitrary Riemannian manifold M, we have two important invariants: the simplicial volume $\|M\|$ and the injectivity radius $i(M)$. In some sense, both these invariants measure the "value" of the manifold M; therefore, there naturally arises the question about their interrelation. In fact, this interrelation exists because of the following result.

Theorem 5.1.6 ([76])**.** *Let the sectional curvatures $K(\sigma)$ of a manifold M satisfy the inequality $|K(\sigma)| \leq 1$, and let $U_\varepsilon \subset M$ be the set of all points $x \in M$ of the manifold M for which the injectivity radius $i(M)$ of the manifold M satisfies the inequality $i(M) \geq \varepsilon = \varepsilon_n$. Then $\|M\| \leq C(n)\operatorname{vol}(U_\varepsilon)$. In particular, if the inequality $i(M) \leq \varepsilon_n$ holds for all points $x \in M$, then $\|M\| = 0$.*

5.2. Volume of hyperbolic manifolds. The volume of hyperbolic manifolds is an important invariant of them. For $\dim M^n = 2$, by the Gauss–Bonnet theorem, the volume $\operatorname{vol}(M)$ of a complete compact hyperbolic surface M can assume only discrete values: $\operatorname{vol}(M) = 2\pi k$, $k = -1, -2, -3, \ldots$; moreover, only finitely many pairwise nondiffeomorphic surfaces correspond to each k. The finiteness theorem for an arbitrary dimension was proved by Wang (see [210]).

Theorem 5.2.1. *For $n \geq 4$ and for any $a > 0$, there exist only finitely many pairwise nondiffeomorphic (or, which is equivalent, pairwise nonisometric) n-dimensional complete hyperbolic manifolds of volume not exceeding a.*

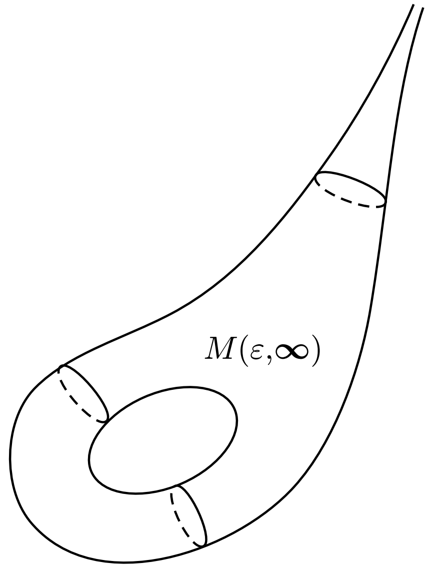


Fig. 41

In Sec. 5.1, we have introduced the concept of the simplicial volume of a Riemannian manifold. For hyperbolic manifolds, the concept of simplicial volume practically coincides with the concept of the usual volume of a Riemannian manifold because of the following result.

Theorem 5.2.2 ([193]). *For a closed orientable hyperbolic manifold M^n, we have the relation* $\mathrm{vol}(M) = \mu_n\|M\|$*, where μ_n is the volume of the n-dimensional regular simplex.*

This theorem implies that the volume of a hyperbolic manifold naturally behaves under mappings. More precisely, if $f : M_1 \to M_2$ is an arbitrary smooth mapping of closed oriented hyperbolic n-dimensional manifolds, then $\mathrm{vol}(M_1) \geq |\deg f|\,\mathrm{vol}(M_2)$.

Theorem 5.2.2 admits generalizations to the case of G-manifolds, where G is a Lie group acting on a manifold M, and also to the case of manifolds with boundary. We do not consider these generalizations but refer the reader to Thurston's works.

As was mentioned early, the geometry of three-dimensional hyperbolic manifolds is considerably distinct from the geometry of hyperbolic manifolds of higher dimension ($\dim M \geq 3$). For example, in contrast to the general case, the set $\mathcal{H}$ of all three-dimensional hyperbolic manifolds can be transformed into a topological space.

Definition 5.2.1. Manifolds M_i *converge* to M if the following conditions hold. There exist points $x_i \in M$, $x \in M$, such that for any $\delta > 0$ and $s > 0$ and for $i > i_0(\delta, s)$, there exist mappings f_i of metric balls $B(x_i, s) \subset M_i$ into the manifold M such that $f(x_i) = x$ and $f(B(x_i, s)) \supset B(x, s-\delta)$, and for all $y, z \in B(x_i, s)$, we have the inequalities $(1-\delta)\rho(y,z) \leq \rho(f_i(y), f_i(z)) \leq (1+\delta)\rho(y,z)$.

Thurston has proved the following two important theorems on the structure of topology in the set $\mathcal{H}$.

Theorem 5.2.3 ([193]). *If three-dimensional manifolds M_i converge to M and* $c = \sup \mathrm{vol}(M_i) < \infty$*, then M is a hyperbolic manifold and* $\mathrm{vol}(M_i) \to \mathrm{vol}(M)$ *for $i \to \infty$.*

To state the next result, we need a new concept. We consider the decomposition $M = M_{(0,\varepsilon]} \cup M_{[\varepsilon,\infty)}$, where $M_{(0,\varepsilon]}$ is the set of points $x \in M$ that are vertices of geodesic loops which are noncontractible in M and have length $\leq \varepsilon$ and $M_{[\varepsilon,\infty)} = \overline{M \setminus M_{(0,\varepsilon]}}$. In Fig. 41, we schematically depict a two-dimensional analog of this decomposition. The Margulis lemma 2.3.16 implies the existence of a constant C such that for $0 < \varepsilon < \frac{1}{2}C$, the set $M_{(0,\varepsilon]}$ consists of finitely many components of standard form: these are either an "annulus," a tubular neighborhood of a closed geodesic, or a horospheric "horn," a manifold $T^2 \times [0, \infty)$ with metric $ds^2 = e^{-t} ds_0^2 + dt^2$, where ds_0^2 is the metric of the flat torus.

Now can present the second fundamental result of Thurston.

Theorem 5.2.4 ([193]). *For any three-dimensional hyperbolic manifold M with $\mathrm{vol}(M) < \infty$ that has k horns and for any number $0 \le l < k$, there exists a sequence of hyperbolic manifolds M_i with l horns that converge to M, and, moreover, $\mathrm{vol}(M_i) < \mathrm{vol}(M)$.*

From these results of Thurston, we can obtain information about the structure of the function $V : \mathcal{H} \to \mathbb{R}^+$ whose value is the volume $\mathrm{vol}(M)$ of the three-dimensional manifold $M \in \mathcal{H}$. The set $V(\mathcal{H})$ of all values of volumes in $\mathbb{R}^+$ forms a closed linearly ordered subset in $\mathbb{R}^+$. Moreover, for any $x \in \mathbb{R}^+$, $V^{-1}(x)$ contains finitely many manifolds. A similar statement also holds for the simplicial volume $\|M\|$.

5.3. Fill-in radius of a Riemannian manifold. We can estimate the volume of a Riemannian manifold using a new metric invariant defined by Gromov. Let M be a closed Riemannian n-dimensional manifold, $\rho(x, y)$ be a metric on M, and $L^\infty(M)$ be the space of all bounded functions on M equipped with the sup norm $\|f\|$. The function $d_x(y) = \rho(x, y)$, $x, y \in M$, belongs to the space $L^\infty(M)$. The canonical embedding $M \to L^\infty(M)$, $x \to d_x$ yields an isometric embedding. Let $U_\varepsilon(M) \subset L^\infty(M)$ be an ε-neighborhood of the space M in $L^\infty(M)$, and let $\alpha_\varepsilon : H_n(M, \mathbb{Z}_2) \to H_n(U_\varepsilon(M), \mathbb{Z}_2)$ be a homomorphism induced by the embedding. In [77], Gromov introduced a new metric invariant of a Riemannian manifold, its fill-in radius.

Definition 5.3.1. The *fill-in radius* $\mathrm{frad}(M)$ of a Riemannian manifold M is the minimum number $\varepsilon > 0$ such that $\alpha_\varepsilon([M]) = 0$, where $[M]$ is the fundamental class of the manifold M and $[M] \in H_n(M, \mathbb{Z}_2)$.

For an arbitrary Riemannian manifold M, the fill-in radius has, e.g., the following properties: for any Riemannian manifold M, we have the inequality $\mathrm{frad}(M) \le \frac{1}{3} d(M)$; if M_1 and M_2 are Riemannian manifolds, then $\mathrm{frad}(M_1 \times M_2) = \min(\mathrm{frad}\, M_1, \mathrm{frad}\, M_2)$. For a detailed discussion and additional properties of the fill-in radius $\mathrm{frad}(M)$ of a Riemannian manifold M, see [77]. We present two results concerning the exact value of the fill-in radius. Their proofs can be found in [103].

Theorem 5.3.1. (a) *The fill-in radius of the projective space $\mathbb{R}P^n$ with metric of constant curvature 1 is equal to $\mathrm{frad}\, \mathbb{R}P^n = \frac{1}{3} d(\mathbb{R}P^n) = \pi/6$.*

(b) *The fill-in radius of the sphere S^n with metric of constant curvature 1 equals half of the spherical distance between the vertices of the inscribed regular $(n+1)$-simplex, i.e., $\mathrm{frad}\, S^n = \frac{1}{2} \arccos(-\frac{1}{n+1})$.*

For estimating the volume of a Riemannian manifold in terms of its fill-in radius, we have the following theorem, proved in [77].

Theorem 5.3.2. *Let M be a closed Riemannian manifold of dimension n. Then $\mathrm{frad}\, M \le C_n (\mathrm{vol}\, M)^{1/n}$ for a certain universal constant C_n such that $0 < C_n < n(n+1) \sqrt[n]{n!}$.*

Chapter 4

RIEMANNIAN MANIFOLDS WITH ADDITIONAL STRUCTURE AND THEIR APPLICATIONS

1. Symmetric Spaces

1.1. Main constructions. Riemannian spaces (M, g_{ij}) with curvature tensor R_{abcd} of a symmetric connection compatible with g_{ij} are covariantly constant, i.e., $\nabla_s R_{abcd} = 0$, where ∇_s is the covariant derivative, are of great interest. The relation $\nabla_s R_{abcd} = 0$ implies that the manifold M is homogeneous whenever it is simply connected, i.e., $\pi_1(M) = \{e\}$. Nonsimply connected manifolds whose curvature tensor R_{abcd} is covariantly constant can be obtained by factorizing by a discrete motion group. In this case, it can happen that a discrete group Γ does not commute with its own motion group of the manifold M, and then the quotient space M/Γ is not homogeneous. Such spaces are called *locally symmetric*. Usually, one initially introduces the homogeneity condition into the definition of symmetric space, and then the definition given above is obtained from this (see [94]).

Definition 1.1.1. A Riemannian manifold M is called a *symmetric space* if, for any point $p \in M$, there exists a mapping $\sigma_p : M \to M$ such that the following conditions hold:

(a) $\sigma_p \neq \mathrm{id}$ is an isometry of the manifold M;
(b) $\sigma_p^2 = \mathrm{id}$;
(c) $\sigma_p(p) = p$;
(d) if $\gamma(t)$ is a geodesic such that $\gamma(0) = p$, then $\sigma_p(\gamma(t)) = \gamma(-t)$.

The existence of the mapping σ_p implies the homogeneity of the manifold M. To prove this, we connect two points $x, y \in M$ by a broken geodesic line (see Fig. 42). Then the mapping $g = \sigma_{y_k} \circ \ldots \circ \sigma_{y_1}$ maps the point x into y; here σ_{y_j} is the reflection with respect to the middle of the jth link of the broken geodesic line. If we want to get the connected component of the identity of the automorphism group, then in the case of an odd k, we need one more symmetry, the symmetry with respect to the point x.

Symmetric Riemannian spaces were defined by E. Cartan in the paper "On one remarkable class of Riemannian spaces." This class of spaces was also considered by A. P. Shirokov.

If, in Definition 1.1.1, we require the isometry σ_p to be defined only in a certain neighborhood of the point p, then we obtain the definition of a locally symmetric space. We present the simplest example of a locally symmetric space that is not globally symmetric.

Example. We consider the space $\mathbb{R}^3$ equipped with the metric $ds^2 = dx^2 + dy^2 + dz^2$. This space is globally symmetric. Let $\alpha \in (0, 2\pi)$ be a fixed number. We define the action of the group $\mathbb{Z}$ on $\mathbb{R}^3$ as follows. We fix a generator $a \in \mathbb{Z}$ of the group $\mathbb{Z}$. By definition, it acts on $\mathbb{R}^3$ by

$$\begin{cases} x' = x\cos\alpha + y\sin\alpha, \\ y' = -x\sin\alpha + y\cos\alpha, \\ z' = z + 1. \end{cases}$$

Identifying equivalent points, we obtain the desired example of a three-dimensional locally symmetric space M^3. Obviously, M^3 is locally symmetric. This space is not homogeneous with respect to the motion group, and, therefore, it is not symmetric, since points located on the axis Oz have a specific property that there exists a closed geodesic of length 1. Other points do not have this property (see Fig. 43). The geodesic aa' is not closed, and the closed geodesic aa'' is of length greater than 1 (for the notation, see Fig. 43).

There exists a remarkable connection of Riemannian globally symmetric spaces and Lie group theory (see [94]). Let M be a Riemannian globally symmetric space, and let G be the maximal connected motion group. If the Riemannian space M is of class C^2, then the motion group of this manifold is a Lie group (see [139]). This statement holds without the assumption that M is a symmetric space. The group G acts transitively on the symmetric space M. Let $x_0 \in M$, and let H be the stationary subgroup of the point x_0: $H = \{g \in G \mid gx_0 = x_0\}$.

Proposition 1.1.1. *Let M be a Riemannian symmetric space, G be its maximal connected motion group, and H be the stationary subgroup. Then in the group G, there exists an involutive automorphism φ whose subgroup of fixed points contains H and has the same connected component of the identity as H: $(G_\varphi)_0 \subset H \subset G_\varphi$, where $G_\varphi = \{g \in G \mid \varphi(g) = g\}$ and $(G_\varphi)_0$ is the connected component of the identity.*

In a certain sense, this proposition admits a converse statement (see [113]).

Theorem 1.1.1. *Let M be a homogeneous Riemannian space with motion group G and stationary subgroup H. If the group G admits an involutive automorphism φ whose fixed subgroup contains H and has*

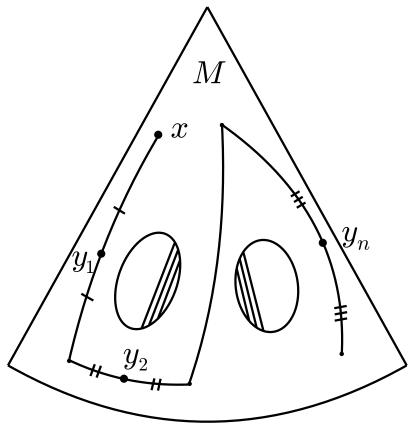


Fig. 42

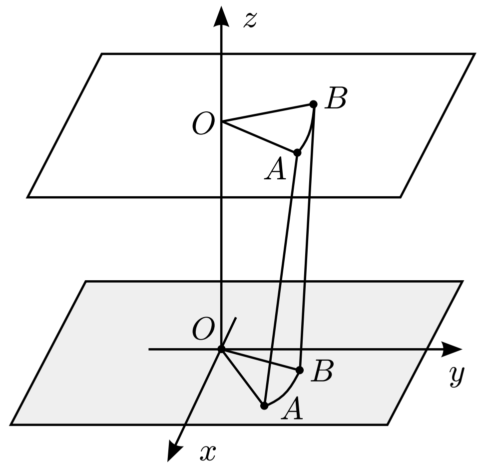


Fig. 43

the same connected component of the identity as H: $(G_\varphi)_0 \subset H \subset G_\varphi$, *then the space* M *is a Riemannian globally symmetric space.*

In Chap. 1, we take this statement as the definition. In terms of the involutive automorphism of the motion group, we can give a description of geodesics in symmetric spaces.

Proposition 1.1.2. *Let* M *be a simply connected Riemannian space with motion group* G *and stationary subgroup* H. *As we know,* G *admits an involutive automorphism* φ *such that* $(G_\varphi)_0 \subset H \subset G_\varphi$. *These geodesics on* M *coincide with trajectories of one-parameter subgroups whose tangent lines belong to the subspace* $\mathfrak{R} = \{Y \in \mathfrak{G} \mid d\varphi(Y) = -Y\}$; *here* $\mathfrak{G}$ *is the Lie algebra of the Lie group* G.

1.2. Cartan models. The following two main theorems of the theory of symmetric spaces hold (see, e.g., [59, 94]).

Theorem 1.2.1. *Let* M *be a complete simply connected Riemannian manifold such that* $\nabla_s R^i_{j,pq} = 0$ *and* $S^i_{pq} = 0$. *Then* M *is a globally symmetric space*; *here* $R^i_{j,pq}$ *is the curvature tensor and* S^i_{pq} *is the torsion tensor.*

We note that the statement converse to this theorem is obvious. Therefore, we obtain the characterization of symmetric spaces given in the beginning of this section, which historically is the first such characterization.

Theorem 1.2.2. *Any globally symmetric Riemannian space* M *is realized as a totally geodesic surface in an appropriate Lie group.*

In this theorem, the Lie group is considered as the space of affine connection Γ^i_{jk}. There are the following three connections on an arbitrary Lie group: the left connection ${}^{\mathrm{l}}\Gamma^i_{jk}$ defined by the parallelization via left translations, the right connection ${}^{\mathrm{r}}\Gamma^i_{jk}$ defined by right translations, and the neutral connection $\Gamma^i_{jk} = \frac{1}{2}({}^{\mathrm{l}}\Gamma^i_{jk} + {}^{\mathrm{r}}\Gamma^i_{jk})$. With respect to the connection Γ^i_{jk}, the Lie group is a symmetric space.

The Lie algebra $\mathfrak{G}$ of the motion group G of a symmetric space M falls into a direct sum $\mathfrak{G} = \mathfrak{Y} + \mathfrak{R}$, where $\mathfrak{Y} = \{X \in \mathfrak{G} \mid d\sigma(X) = X\}$, $\mathfrak{R} = \{X \in \mathfrak{G} \mid d\sigma(X) = -X\}$, and σ is the involutive automorphism of the motion group described above. The commutativity of elements from the subspaces $\mathfrak{Y}$ and $\mathfrak{R}$ is described by the relations $[\mathfrak{Y}, \mathfrak{Y}] \subset \mathfrak{Y}$, $[\mathfrak{Y}, \mathfrak{R}] \subset \mathfrak{R}$, and $[\mathfrak{R}, \mathfrak{R}] \subset \mathfrak{Y}$. It turns out that the decomposition of the Lie algebra generates a certain "decomposition" of the corresponding Lie group. In the Lie group

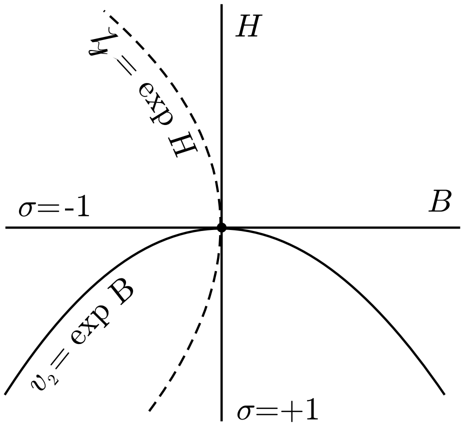


Fig. 44

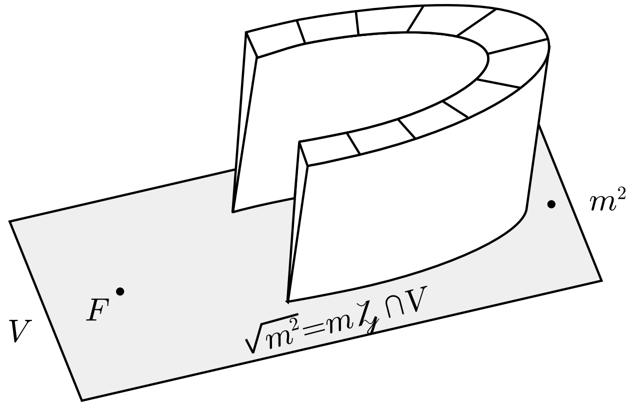


Fig. 45

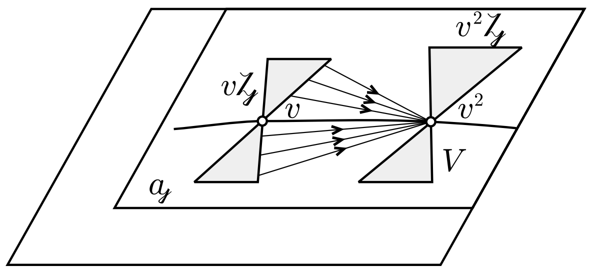


Fig. 46

G, we have a subgroup H, the stationary subgroup of the space M. One asks: is it possible to embed the symmetric space itself in the group G as a "homogeneous surface"? If M is not merely homogeneous, then its embedding in the group is not possible in general. However, for symmetric space, the case is more favorable. It turns out that in this case, it can be realized in the form of a certain "homogeneous surface" in the simply connected group G. In the group G, we consider a simply connected component V_1 of the subset of elements $g \in G$ such that $\sigma(g) = g^{-1}$. We denote by V_2 the subset in G that is swept by geodesics γ of the group G passing through the identity of the group such that $\dot{\gamma}(0) \in \mathfrak{R}$ and $\gamma(0) = e$. Finally, let $V_3 \subset G$ be the subset in the group G that is the image of the group G under its mapping into itself via the mapping $p(g) = g\sigma(g^{-1})$ (see Fig. 44).

Theorem 1.2.3. *The sets V_1, V_2, and V_3 coincide in the group G. This subset is a smooth manifold in the group G that is diffeomorphic to the symmetric space $M = G/H$. Moreover, this submanifold is totally geodesic, i.e., any geodesic of the group G that is tangent to the submanifold M lies entirely in this submanifold. The continuous mapping p is a principal bundle $p : G \to M$ with fiber H.*

The proof of this theorem can be found in [59]. The mapping $M \to G$ of the symmetric space M into the group G constructed above is called the *Cartan model* of the symmetric space G/H. One should not think that this embedding is a section of the bundle $p : G \to M$. The following statement describes the intersection of cosets by the subgroup H with the submanifold $M \subset G$.

Proposition 1.2.1. *Each coset gH has nonzero intersection with the submanifold M. Let mH be an arbitrary coset (where $m \in M$). Then the relation $mH \cap M = \{\sqrt{m^2}\} \cap M$ holds (see* Fig. 45*).*

The action of the projection $p: G \to M$ is schematically shown in Fig. 46.

1.3. Classification problems. There is a natural action $\operatorname{ad}_{\mathfrak{Y}}$ of a subalgebra $\mathfrak{Y}$ on the space tangent to a symmetric space G/H at a point $O = H \in G/H$.

Definition 1.3.1. A symmetric space $M = G/H$ is said to be *irreducible* if the adjoint representation $\operatorname{ad}_{\mathfrak{Y}}$ of the subalgebra $\mathfrak{Y}$ on the tangent space T_0M is irreducible, i.e., does not have proper nonzero invariant subspaces.

Theorem 1.3.1 (E. Cartan)**.** *A compact simply connected Riemannian symmetric space M falls into a direct product $M = M_1 \times \ldots \times M_s$ of irreducible compact symmetric spaces.*

There is a complete classification of compact irreducible symmetric spaces (E. Cartan, [37]). They fall into the following two series: series I and series II. Spaces of series I are not Lie groups, and a complete list of them can be found, e.g., in [94]. For example, they are $\mathrm{SU}(n)/\mathrm{SO}(n)$, $\mathrm{SU}(2n)/\mathrm{Sp}(n)$, $\mathrm{SO}(2n)/\mathrm{U}(n)$, $\mathrm{SU}(n)/\mathrm{SU}(k)\times\mathrm{SU}(n-k)$, $\mathrm{SO}(n)/\mathrm{SO}(k)\times\mathrm{SO}(n-k)$, $\mathrm{Sp}(n)/\mathrm{Sp}(k)\times\mathrm{Sp}(n-k)$, and $\mathrm{Sp}(n)/\mathrm{U}(n)$. Compact Riemannian irreducible symmetric spaces of series II are compact connected simple Lie groups equipped with a two-sided invariant metric. Each Lie group is naturally transformed into a symmetric space. Indeed, the isometry group of a group G is naturally isomorphic to the direct product $G \times G$. The action of the group $G \times G$ on the group G is generated by left and right translations: $(g_1, g_2) : g \to g_1 g g_2^{-1}$. The stationary subgroup H coincides with the diagonal Δ of the direct product $G \times G$, i.e., the subgroup H consists of elements of the form (g, g), $g \in G$. Therefore, the group G is represented in the form of the symmetric space $G \times G/G$, where the symmetry g_x is given by the relation $g_x(h) = xh^{-1}x$. Along with the two series of spaces with a compact motion group indicated above, there is one more class of symmetric spaces with a noncompact symmetry group (see [94]). For Lie groups, various characteristics related to the curvature tensor are calculated in an especially simple way. This is explained by the existence of explicit formulas for the Riemannian tensor.

Definition 1.3.2. Let G be a Lie group. A metric g_{ij} on G is said to be *bi-invariant* if left translations $L_a(x) = ax$ and right translations $R_a(x) = xa$ are isometries with respect to the metric g_{ij} for all $a \in G$.

In the case of a bi-invariant Riemannian metric, all transformations of the form $i(x) = R_{x^{-1}} L_x$ and $s(x) = x^{-1}$ are also isometries. Bi-invariant Riemannian metrics exist on all compact Lie groups (see [94]).

Theorem 1.3.2 ([94])**.** *Let G be a Lie group with a bi-invariant Riemannian metric $(X, Y) = g_{ij} X^i Y^j$, and let $R(X, Y)Z$ be the Riemannian tensor. If X, Y, and Z are arbitrary vector fields on the Lie group G (i.e., $X, Y, Z \in \mathfrak{G}$), then the following formulas hold*:

$$(1) \quad R(X, Y)Z = \frac{1}{2}[[X, Y]Z];$$

$$(2) \quad (R(X, Y)Z, W) = \frac{1}{4}([X, Y], [Z, W]).$$

Since the fields X, Y, and Z can be considered as elements of the Lie algebra $\mathfrak{G}$, these formulas are in fact written at the identity of the Lie group G. Theorem 1.3.2 implies that in the case of the Lie group G, the curvature in any two-dimensional direction has the form

$$(R(X, Y)X, Y) = \frac{1}{4}([X, Y], [X, Y]) = \frac{1}{4}\|[X, Y]\|^2;$$

therefore, it is always nonnegative and vanishes iff the vectors X and Y commute in the Lie algebra $\mathfrak{G}$.

In the case of a bi-invariant metric, geodesics on a Lie group admit a simple description in group-theoretic terms. More precisely, the following statement holds.

Theorem 1.3.3 ([94])**.** *Let g_{ij} be a bi-invariant Riemannian metric on a Lie group G. Then each geodesic is obtained by a left translation from a certain one-parametric subgroup and vice versa.*

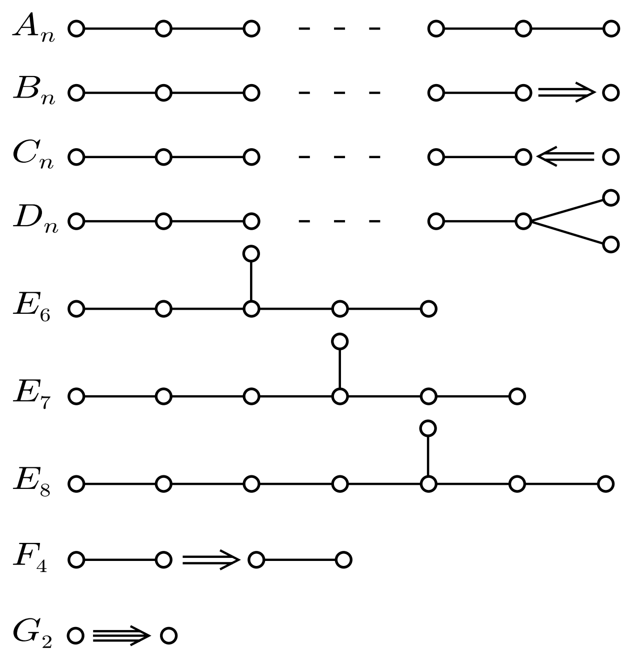


Fig. 47

We now consider briefly the classification of Riemannian compact irreducible symmetric spaces of series II. It turns out that there are four infinite series and five isolated exceptional groups. All of them are enumerated by certain graphs on the plane. The complete list of graphs describing simple compact Lie groups is presented in Fig. 47. The construction of graphs of simple compact Lie groups can be found, e.g., in [30, 59, 158].

The classification of complex simple Lie groups coinciding with the classification of compact real simple groups was obtained by W. Killing in the paper "Structure of continuous finite transformation groups"; gaps of the Killing proof were removed by E. Cartan. The classification of real noncompact simple Lie groups was given by E. Cartan in the paper "Real simple finite continuous groups."

1.4. Conjugate points of compact symmetric spaces. Let M be a compact Riemannian manifold, and let $p \in M$. We denote by $Q_p(M)$ the set of all first conjugate points in M for p (see Sec. 2.4 of Chap. 1, where the definition of the set $Q_p(M)$ is given, and also see Chap. 2 for the general concept of conjugate points).

We consider a compact Lie group G and its involutive automorphism $\theta : G \to G$. We denote by $G_\theta = \{g \in G \mid \theta(g) = g\}$ the set of fixed points of the automorphism θ. Let $K \subset G$ be a closed subgroup of G such that $(G_\theta)_0 \subset K \subset G_\theta$, where H_0 is the connected component of the identity in H. In a natural way, the bi-invariant metric on G induces a G-invariant Riemannian metric on the homogeneous space $M = G/K$. With respect to this metric, the space M is a Riemannian symmetric space. Any compact symmetric space is obtained from the construction presented above. We now assume that M is an irreducible space. The Lie algebra of the Lie group G is denoted by $\mathfrak{G}$, and the Lie algebra K is denote by $\mathfrak{K}$. The involutive automorphism θ of the Lie group G induces an involutive automorphism of the Lie algebra $\mathfrak{G}$, which is also denoted by θ. Since K lies between G_θ and the connected component of the identity of the subgroup G_θ, we have $\mathfrak{K} = \{X \in \mathfrak{G} \mid \theta(X) = X\}$. We set $\mathfrak{M} = \{X \in \mathfrak{G} \mid \theta(X) = -X\}$. Since θ is an involutive automorphism, the decomposition $\mathfrak{G} = \mathfrak{K} \oplus \mathfrak{M}$ holds. We take the maximal Abelian subspace $\mathfrak{N}$ in $\mathfrak{M}$ and the maximal Abelian subalgebra $\mathfrak{I}$ in $\mathfrak{G}$ containing $\mathfrak{N}$. Then the complexification $\mathfrak{I}^{\mathbb{C}}$ of the subalgebra $\mathfrak{I}$ is a Cartan subalgebra in the complexification $\mathfrak{G}^{\mathbb{C}}$ of the Lie algebra $\mathfrak{G}$. The bi-invariant metric g_{ij} on G induces an Ad_G-invariant inner product $\langle X, Y\rangle$ on $\mathfrak{G}$. For $a \in \mathfrak{J}$, we set $\mathfrak{G}_a = \{X \in \mathfrak{G}^{\mathbb{C}} \mid [X, H] = \sqrt{-1}\langle a, H\rangle X$ for all $H \in \mathfrak{J}\}$. An element $a \in \mathfrak{J} \setminus O$ is called a *root* if $\mathfrak{G}_a \neq 0$. Let Δ be the set of all roots. Then we have a decomposition into the direct sum: $\mathfrak{G}^{\mathbb{C}} = \mathfrak{J}^{\mathbb{C}} + \sum\limits_{\alpha\in\Delta} \mathfrak{G}_\alpha$. For an arbitrary $\gamma \in \mathfrak{N}$, we define the subspace $\mathfrak{G}_\gamma \subset \mathfrak{G}^{\mathbb{C}}$ by $\mathfrak{G}_\gamma = \{X \in \mathfrak{G}^{\mathbb{C}} \mid [H, X] = \sqrt{-1}\langle \gamma, H\rangle X$ for all $H \in \mathfrak{N}\}$ and set $\Sigma = \{\gamma \in \mathfrak{N} \setminus O \mid \mathfrak{G}_\gamma \neq 0\}$. Let $H \to \bar{H}$ be the orthogonal projection of $\mathfrak{R}$ on $\mathfrak{N}$. Then $\Sigma = \{\bar{\alpha} \mid \alpha \in \Delta$ and $\bar{\alpha} \neq 0\}$. We choose a lexicographic order $>$ on $\mathfrak{R}$ and $\mathfrak{N}$ in such a way that $\alpha \in \Delta$ and $\bar{\alpha} > 0$ imply $\alpha > 0$. We denote by Δ_+ and Σ_+ the set of all positive roots in Δ and Σ, respectively. We introduce the subspaces $\mathfrak{R}_\gamma = \mathfrak{R} \cap (\mathfrak{G}_\gamma + \mathfrak{G}_{-\gamma})$ and $\mathfrak{M}_\gamma = \mathfrak{M} \cap (\mathfrak{G}_\gamma + \mathfrak{G}_{-\gamma})$ for $\gamma \in \Sigma_+$, $\mathfrak{R}_0 = \{X \in \mathfrak{R} \mid [H, X] = 0$ for all $H \in \mathfrak{N}\}$.

The root decomposition has the following properties (the proof can be found in [94]):

(a) all summands in the direct sums $\mathfrak{R} = \mathfrak{R}_0 + \sum\limits_{\gamma\in\Delta_+} \mathfrak{R}_\gamma$ and $\mathfrak{M} = \mathfrak{N} + \sum\limits_{\gamma\in\Sigma_+} \mathfrak{M}_\gamma$ are pairwise orthogonal;

(b) one can choose elements $S_\alpha \in \mathfrak{R}$ and $T_\alpha \in \mathfrak{M}$ for all $\alpha \in \Delta$, where $\bar{\alpha} \neq 0$, such that the following conditions hold. For all $\gamma \in \Sigma_+$, the sets $\{S_\alpha \mid \alpha \in \Delta_+, \bar{\alpha} = \gamma\}$ and $\{T_\alpha \mid \alpha \in \Delta_+, \bar{\alpha} = \gamma\}$ form an orthonormal basis for $\mathfrak{R}_\gamma$ and $\mathfrak{M}_\gamma$, respectively. For each $\alpha \in \Delta_+$ such that $\bar{\alpha} = \gamma \in \Sigma_+$ and for any $H \in \mathfrak{N}$, we have

$$[H, S_\alpha] = \langle \gamma, H\rangle T_\alpha, \qquad [H, T_\alpha] = -\langle \gamma, H\rangle S_\alpha,$$
$$\mathrm{Ad}_{\exp H} S_\alpha = \cos\langle \gamma, H\rangle S_\alpha + \sin\langle \gamma, H\rangle T_\alpha,$$
$$\mathrm{Ad}_{\exp H} T_\alpha = -\sin\langle \gamma, H\rangle S_\alpha + \cos\langle \gamma, H\rangle T_\alpha.$$

The rank of a symmetric space M is the maximal dimension of flat totally geodesic submanifolds M. For example, the spheres S^n and various projective spaces $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, and $\mathbb{C}\mathrm{a}P^2$ are symmetric spaces of rank one.

If a symmetric space M is irreducible, then the simple root system of the space M contains r roots $\gamma_1, \dots, \gamma_r$, where r is the rank of the space M. In this case, the root system Σ is irreducible, and there exists a unique maximum root δ in Σ. We set $S = \{H \in \mathfrak{N} \mid \langle \delta, H\rangle = \pi,\ \langle \gamma_i, H\rangle \geq 0,\ i = 1, \dots, r\}$. The set of first conjugate points for the space M is described by the following theorem.

Theorem 1.4.1. *The set of first conjugate points $Q_0(M)$ of a symmetric irreducible space M with respect to a point $O \in M$ coincides with the set*

$$Q_0(M) = \bigcup_{k\in K} k\,\mathrm{Exp}(S).$$

1.5. Rigidity of symmetric spaces. In Sec. 1.5 of Chap. 3, we have indicated the rigidity theorem for hyperbolic spaces. An analog of Theorem 1.5.4 was proved by Mostow for symmetric spaces (see [137]). To state this generalization, we need new concepts. Let M_1 and M_2 be complete connected locally symmetric spaces, and let $\bar{M}_1$, and $\bar{M}_2$ be their universal coverings, so that $M_i = \bar{M}_i/\Gamma_i$, $i = 1, 2$, where Γ_i is a discrete isometry group acting on $\bar{M}_i$. Let $\bar{M}_1 = N_0 \times N_1 \times \ldots \times N_t$ and $\bar{M}_2 = N'_0 \times N'_1 \times \ldots \times N'_{t'}$ be their decompositions into Euclidean irreducible factors (see [113]).

Definition 1.5.1. Two locally symmetric spaces M_1 and M_2 are said to be *isometric with accuracy up to a renormalization* if $t = t'$ and there exist positive numbers $\lambda_1, \dots, \lambda_t$ such that M_1 is isometric to the manifold $(N'_0 \times \lambda_1 N'_1 \times \ldots \times \lambda_t N'_t)/\Gamma_2$, where the action of the group Γ_2 is naturally defined.

The following theorem is due to Mostow.

Theorem 1.5.1 ([137])**.** *Let M_1 and M_2 be complete connected compact locally symmetric spaces. Let the local decompositions of the manifolds M_1 and M_2 into the direct metric decompositions into Euclidean and irreducible spaces not contain one-dimensional and two-dimensional factors (in particular, N_0 and N'_0 are zero-dimensional). If, moreover, the fundamental groups $\pi_1(M_1)$ and $\pi_1(M_2)$ are isomorphic, then the spaces M_1 and M_2 are isometric with accuracy up to a normalization.*

2. Riemannian Geometry on Homogeneous Spaces

2.1. Geometry on G/H. Let G be a compact Lie group, H be its connected closed subgroup, and $\mathfrak{G}$ and $\mathfrak{Y}$ be the Lie algebras of the Lie groups G and H, respectively. We denote by $N(H)$ the normalizer of the subgroup H in G and by $\mathfrak{N}$ its Lie algebra. We consider decompositions $\mathfrak{G} = \mathfrak{N} + \mathfrak{L}$ and $\mathfrak{N} = \mathfrak{Y} + \mathfrak{R}$ of the Lie algebras $\mathfrak{G}$ and $\mathfrak{N}$, respectively, such that $\mathrm{Ad}_{N(H)}\,\mathfrak{L} \subset \mathfrak{L}$ and $\mathrm{Ad}_H\,\mathfrak{R} \subset \mathfrak{R}$. We define $\mathfrak{J} = \mathfrak{R} + \mathfrak{L}$. Then we have the inclusions $[\mathfrak{Y}, \mathfrak{R}] = 0$, $[\mathfrak{R}, \mathfrak{R}] \in \mathfrak{R}$, and $\mathrm{Ad}_H\,\mathfrak{J} \subset \mathfrak{J}$. We can identify the space $\mathfrak{J}$ with the tangent space to the homogeneous space G/H at the origin $H \in G/H$ and $\mathfrak{R}$ with the Lie algebra $K = N(H)/H$. In addition, we have $\mathfrak{R} = \{\xi \in \mathfrak{J} \mid \mathrm{Ad}_h\,\xi = \xi,\ h \in H\}$. On the homogeneous space G/H, we can construct a canonical movable frame. For this purpose, according to each element $v \in \mathfrak{G}$, we construct the corresponding vector field Z_v on the space G/H, the so-called fundamental vector field generated by v. By definition, $Z_v(y) = \left.\dfrac{d}{dt}\right|_{t=0} ye^{tv}$. Clearly, $(R_a)_* Z_v = Z_{\mathrm{Ad}_{a^{-1}} v}$. This implies $[Z_v, Z_w] = Z_{(v,w)}$. Let ε_i be a basis of the Lie algebra $\mathfrak{G}$ compatible with the decomposition $\mathfrak{G} = \mathfrak{Y} + \mathfrak{R} + \mathfrak{L}$. We denote by $\varepsilon_{\alpha'}$ a part of the basis lying in $\mathfrak{Y}$ and by $\varepsilon_\alpha \in \mathfrak{J}$ a part of the basis lying in $\mathfrak{J} = \mathfrak{R} + \mathfrak{L}$. The fundamental vector fields on G/H corresponding to ε_i are denoted by e_i. Then $[e_i, e_j] = c_{ij}^k e_k$, where c_{ij}^k is the structural tensor of the Lie algebra $\mathfrak{G}$. At the point $O = [e]$, all vector fields $e_{\alpha'}$ corresponding to the stationary subgroup H vanish, and $e_\alpha(0)$ forms a basis of the space $T_0(G/H)$. e_α is called the *canonical movable frame* of the homogeneous space G/H.

By Theorem 5.1.2 of Chap. 1, we have a G-invariant Riemannian metric $g = (g_{ij})$ on the space G/H. Its restriction to the space $T_0(G/H) = \mathfrak{J}$ is an Ad_H-invariant metric. Conversely, each Ad_H-invariant inner product on $\mathfrak{J}$ defines a G-invariant Riemannian metric on the space G/H. The fundamental vector fields Z_v, $v \in \mathfrak{G}$, are Killing fields for the invariant metric g. Therefore, the space G/H admits the movable frame consisting of Killing vectors.

We calculate the curvature of homogeneous spaces in explicit form. Let ∇ be the covariant derivative corresponding to a connection compatible with a Riemannian metric g_{ij}, i.e., $X(Y,Z) = (\nabla_X Y, Z) + (Y, \nabla_X Z)$ and $\nabla_X Y - \nabla_Y X - [X,Y] = 0$. We decompose $\nabla_X Y$ into the symmetric and antisymmetric parts: $\nabla_X Y = S(X,Y) + A(X,Y)$; here $S(X,Y) = S(Y,X)$ and $A(X,Y) = -A(Y,X)$. For arbitrary vector fields X and Y, we have $A(X,Y) = \frac{1}{2}[X,Y]$. If X, Y, and Z are Killing vector fields, then $(S(X,Y), Z) = -\frac{1}{2}\{([Z,X],Y) + (X,[Z,Y])\}$. For the curvature tensor $(R(X,Y)Z,W)$, we have

$$\begin{aligned}(R(X,Y)Z,W) = &-\frac{1}{4}\{([[X,Y],Z],W) - ([[X,Y],W],Z)\} - \frac{1}{2}\{([[Z,W],X],Y) - ([[Z,W],Y],X)\}\\ &+ \frac{1}{4}\{([X,Z],[Y,W]) + 2([X,Y],[Z,W]) - ([Y,Z],[X,W])\}\\ &+ (S(X,Z),S(Y,W)) - (S(Y,Z),S(X,W)),\end{aligned}$$

where X, Y, Z, and W are arbitrary Killing vector fields. We now can express the curvature of a homogeneous space G/H using the language of the structural tensor $c_{i,\beta}^\delta$. Let $c_{i\beta,\gamma} = g_{\gamma\delta} c_{i\beta}^\delta$. Then $S_{\alpha,\beta\gamma} = -\frac{1}{2}(c_{\gamma\alpha,\beta} + c_{\gamma\beta,\alpha})$ (we note that, in the general case, the structural constants $c_{i\alpha,\beta}$ are not antisymmetric with respect to the last two subscripts, since the metric is not bi-invariant in general). For the Christoffel symbols $\Gamma_{\alpha\beta,\gamma} = (\nabla_\alpha e_\beta, e_\gamma)$, we have the following explicit expression: $\Gamma_{\alpha\beta,\gamma} = \frac{1}{2}(c_{\alpha\beta,\gamma} - c_{\gamma\alpha,\beta} + c_{\beta\gamma,\alpha})$. Further, for $R_{\alpha\beta\gamma\delta} = (R(e_\alpha, e_\beta)e_\gamma, e_\delta)$, we have

$$\begin{aligned}R_{\alpha\beta\gamma\delta} = &-\frac{1}{4}\{c_{\alpha\beta}^i c_{i\gamma,\delta} - c_{\alpha\beta}^i c_{i\delta,\gamma} + c_{\gamma\delta}^i c_{i\alpha,\beta} - c_{\gamma\delta}^i c_{i\beta,\alpha}\} + \frac{1}{4}\{c_{\alpha\gamma}^k c_{\beta\delta,k} + 2c_{\alpha\beta}^k c_{\gamma\delta,k} - c_{\beta\gamma}^k c_{\alpha\delta,k}\}\\ &+ \frac{1}{4}g^{k\lambda}\{(c_{k\alpha,\gamma} + c_{k\gamma,\alpha})(c_{\lambda\beta,\delta} + c_{\lambda\delta,\beta})(c_{k\beta,\gamma} + c_{k\gamma,\beta})(c_{\lambda\alpha,\delta} + c_{\lambda\delta,\alpha})\}.\end{aligned}$$

The Ricci tensor $R_{\beta\gamma} = g^{\alpha\delta} R_{\alpha\beta\gamma\delta}$ is given by

$$R_{\beta\gamma} = \frac{1}{4}c_{\alpha k,\beta} c_{\alpha k,\gamma} - \frac{1}{2}c_{\beta\alpha,k} c_{\gamma\alpha,k} - \frac{1}{2}c_{\beta\alpha,k} c_{\gamma k,\alpha} - \frac{1}{2}c_{\beta\alpha}^{k'} c_{\gamma k'}^\alpha - \frac{1}{2}c_{\gamma\alpha}^{k'} c_{\beta k'}^\alpha - \frac{1}{2}(c_{k\beta,\gamma} + c_{k\gamma,\beta})c_{k\alpha}^\alpha.$$

In all these expressions, if a certain index occurs twice on the same level, then one sums up with respect to this index after the compression with the metric tensor; for example,

$$c_{\alpha k,\beta} c_{\alpha k,\gamma} = g^{\alpha\delta} g^{ki} g_{\beta\varepsilon} g_{\gamma\xi} c^{\varepsilon}_{\alpha k} c^{\xi}_{\delta i}.$$

Finally, for the scalar curvature $R = g^{\beta\gamma} R_{\beta\gamma}$, we have

$$R = -\frac{1}{4} c_{\alpha\beta,\gamma} c_{\alpha\beta,\gamma} - \frac{1}{2} c_{\alpha\beta,\gamma} c_{\alpha\gamma,\beta} - c^{k'}_{\beta\alpha} c^{\alpha}_{\beta k'} - c^{\alpha}_{k\alpha} c^{\beta}_{k\beta}.$$

In terms of structural constants, we can give a reductivity criterion of a homogeneous space G/H. The space G/H is reductive if the tensor $c_{\alpha\beta,\gamma}$ is anti-symmetric with respect to the last two subscripts. In the case of a reductive space, $\nabla_X Y = \frac{1}{2}[X,Y]$, $R_{\beta\gamma} = \frac{1}{4} c^{\alpha}_{\beta k} c^{k}_{\gamma\alpha} + \frac{1}{2} k_{\beta\gamma}$, and $R = \frac{1}{4} c^{\alpha}_{\beta k} c^{k}_{\beta\alpha} + \frac{1}{2} g^{\beta\gamma} k_{\beta\gamma}$, where $k_{ij} = -c^{m}_{il} c^{l}_{jm}$ is the Killing metric. In particular, if G/H is a symmetric space, i.e., $c_{\alpha\beta,\gamma} = 0$, then $R_{\beta\gamma} = \frac{1}{2} k_{\beta\gamma}$ and $R = \frac{1}{2} g^{\beta\gamma} k_{\beta\gamma}$. If $H = \{e\}$ (the case of a group) and $g_{\alpha\beta} = k_{\alpha\beta}$, then $R_{\beta\gamma} = \frac{1}{4} k_{\beta\gamma}$ and $R = \frac{1}{4} \dim \mathfrak{G}$.

2.2. The Ambrose–Singer theorem. A simply connected complete Riemannian manifold is a symmetric space iff the covariant derivative of the curvature tensor field and the torsion tensor vanishes (see Theorem 1.2.1). Ambrose and Singer generalized this classical result and characterized the homogeneity of a Riemannian manifold by using certain conditions for the covariant derivative of the curvature tensor and a certain additional tensor of type $(1,2)$, which is trivial in the symmetric case. An elegant proof of this theorem was obtained by Costant (see [115]).

Theorem 2.2.1. (a) *Let (M, g_{ij}) be a homogeneous Riemannian manifold (i.e., a Riemannian manifold admitting a transitive isometry group). Then there exists a tensor field D of type $(1,2)$ on M such that for any vector $X \in T(M)$, we have*

(i) *$D(X) = D(X, \cdot)$ is a skew-symmetric endomorphism;*

(ii) *$\nabla_X R = D(X) \cdot R$;*

(iii) *$\nabla_X D = D(X) \cdot D$, where ∇ and R stand for the connection compatible with a Riemannian metric and the curvature tensor field on the manifold (M, g_{ij}), respectively.*

(b) *Let (M, g_{ij}) be a simply connected complete Riemannian manifold. Let there exist a tensor field D of type $(1,2)$ on it that satisfies conditions (i)–(iii). Then (M, g_{ij}) is a homogeneous manifold.*

2.3. Isometry groups of left-invariant metrics. We consider a connected compact Lie group G. Denote by L_g (resp. R_g) left (resp. right) translations by an element $g \in G$, where $L_g(x) = gx$ (resp. $R_g(x) = xg$). Let g_{ij} be a Riemannian metric on the Lie group G, and let $I(G, g_{ij})$ be the isometry group of the metric g_{ij}. The metric g_{ij} is said to be *left-invariant* if it is invariant with respect to all left translations on the group G, i.e., $L(G) \subset I(G, g_{ij})$ ($L(G)$ is the group of all left translations of the group G). In a similar way, one defines a *right-invariant* metric on G. A Riemannian metric on the Lie group G is said to be *bi-invariant* if L_x and R_x are isometries with respect to g_{ij} for all $x \in G$. We denote by $I_0(G, g_{ij})$ the connected component of the identity of the group $I(G, g_{ij})$. Let G be a connected semisimple Lie group, and let g_{ij} be a bi-invariant Riemannian metric on the Lie group G. Then $I_0(G, g_{ij}) = L(G)R(G)$, where $R(G)$ is the group of right translations on G. For the proof, see [94]. We can generalize this statement to the case of left-invariant metrics on a Lie group. For the isometry groups of left-invariant metrics on Lie groups, we refer the reader to [151].

Theorem 2.3.1. *Let G be a connected simple compact Lie group, and let g_{ij} be a left-invariant Riemannian metric on the Lie group G. Then $I_0(G, g_{ij}) \subset L(G)R(G)$, i.e., for any element $f \in I_0(G, g_{ij})$, there exist elements $x, y \in G$ such that $f = L_x \circ R_y$.*

In specific examples, this theorem yields rich-in-content information about the structure of the isometry group of left-invariant metrics.

We now consider infinitesimal isometries of left-invariant metrics, i.e., Killing fields. Let g_{ij} be a left-invariant Riemannian metric on a connected Lie group G. Then the Lie algebra of the Lie group $I_0(G, g_{ij})$ can be naturally identified with the Lie algebra of all Killing vector fields, and the Lie algebra of the group $L(G)$ (or $R(G)$) can be identified with the Lie algebra of all right- (resp. left-) invariant vector fields on the Lie group G (see [113]).

Theorem 2.3.2. *Let G be a compact connected simple Lie group, and let g_{ij} be a left-invariant Riemannian metric on G. Then:*

(1) *a right-invariant vector field on G is the Killing vector field corresponding to g_{ij};*
(2) *the Killing vector field on the group G corresponding to g_{ij} is the sum of left-invariant and right-invariant vector fields on G;*
(3) *the Lie algebra of all right-invariant vector fields on G is an ideal in the Lie algebra of all Killing fields on G corresponding to the metric g_{ij}.*

Example. On the Lie group $\mathrm{SO}(n)$, we define a left-invariant metric, which is important for applications; to assign it, we need to define a linear operator $\varphi : \mathrm{so}(n) \to \mathrm{so}(n)$ in the Lie algebra $\mathrm{so}(n)$. We describe a slightly more general construction of such operators for an arbitrary compact Lie algebra $\mathfrak{G}_u$. Each complex semisimple Lie algebra $\mathfrak{G}$ admits a compact real form $\mathfrak{G}_u$. We recall that $\mathfrak{G}_u = \{E_\alpha + E_{-\alpha}, i(E_\alpha - E_{-\alpha}), iH_\alpha\}$, where E_α are root vectors with respect to the Cartan subalgebra $\mathfrak{T} \subset \mathfrak{G}$ and $H_\alpha \in \mathfrak{T}$ are vectors representing these roots (see [94,101]). Let $a, b \in i\mathfrak{T}_0$ (where $\mathfrak{T}_0$ is the real subspace in the Cartan subalgebra spanned by all roots $H_\alpha \in \mathfrak{T}$) be generic elements. Clearly,

$$\mathrm{ad}_a(E_\alpha + E_{-\alpha}) = \alpha(a')(i(E_\alpha - E_{-\alpha})),$$

$$\mathrm{ad}_a(i(E_\alpha - E_{-\alpha})) = -\alpha(a')(E_\alpha + E_{-\alpha}),$$

where $a = ia'$ and $a' \in \mathfrak{T}_0$. We define the operator $\varphi = \varphi_{abD} : \mathfrak{G}_u \to \mathfrak{G}_u$ by

$$\varphi(X) = \varphi(X' + t) = \varphi_{a,b}(X') + D(t) + \mathrm{ad}_a^{-1} \mathrm{ad}_b X' + D(t),$$

where $X = X' + t$ is a unique decomposition of X into those components for which $t \in i\mathfrak{T}_0$, $X' \perp i\mathfrak{T}_0$. In each compact form $\mathfrak{G}_u$, we consider the subalgebra $\mathfrak{G}_n$, which is called a normal compact subalgebra and is spanned by the vectors $E_\alpha + E_{-\alpha}$, where α runs over the set of roots of the Lie algebra $\mathfrak{G}$ with respect to the Cartan subalgebra. Since all these vectors are eigenvectors of the operators φ, we obtain a normal series when these vectors are restricted to the subalgebra $\mathfrak{G}_n$. These operators merely coincide with $\varphi_{a,b} : G_u \to G_u$, $\varphi(X) = \mathrm{ad}_a^{-1} \mathrm{ad}_b(X)$, $X \in \mathfrak{G}_n$. The construction of the operators $\varphi_{a,b,D}$ presented here was given by A. S. Mishchenko and A. T. Fomenko. The Lie algebra $\mathrm{so}(n)$ can be realized as a normal compact subalgebra in $\mathrm{u}(n)$. The classical metric of a "rigid body" (see [10]) is obtained if we set

$$a = \begin{pmatrix} i\lambda_1 & & 0 \\ & \ddots & \\ 0 & & i\lambda_n \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} i\lambda_1^2 & & 0 \\ & \ddots & \\ 0 & & i\lambda_n^2 \end{pmatrix}.$$

Then $\varphi_{ab}(X) = XI + IX$, where $I = -ia$. We assign the metric on $\mathrm{so}(n)$ according to the operator φ_{ab} in the standard way: $(X, Y) = \langle X, \varphi_{ab}(Y)\rangle$, where $\langle X, Y\rangle$ is the Killing form of the algebra $\mathfrak{G}_n$. In our case, $(X, Y) = \mathrm{tr}(XIY^t + YIX^t)$. We describe the connected component of the identity of the isometry group of this metric. Since $\mathrm{SO}(n)$ is a connected compact simple Lie group, we have $I_0(\mathrm{SO}(n)) = L(\mathrm{SO}(n))H$, where H is the stationary subgroup of the Lie group $I_0(\mathrm{SO}(n))$ that leaves the point $e \in \mathrm{SO}(n)$ fixed. Moreover, $H \subset \mathrm{Int}(G)$ (i.e., any element $h \in H$ is represented in the form $h = L_{x^{-1}}R_x$) and H is a connected compact subgroup (see [151]). Therefore, to describe $I_0(\mathrm{SO}(n))$, we need to find the group of right translations preserving the metric and take the maximal connected subgroup B in it. Then $H = \{L_{b^{-1}}R_b \mid b \in B\}$. We find H in our example. A simple calculation shows that if φ is a generic operator (all λ_i are pairwise distinct), then $I_0(\mathrm{SO}(n)) \cong L(\mathrm{SO}(n)) \cong \mathrm{SO}(n)$. In particular, for $G = \mathrm{SO}(3)$, in the case of a generic operator, we have $I_0(\mathrm{SO}(3)) \cong \mathrm{SO}(3)$. In a similar way, we can examine, for example, the case where $\lambda_1 = \lambda_2$; under this assumption, we have

$$I_0(\mathrm{SO}(3)) \cong L(\mathrm{SO}(3)) \times \mathrm{SO}(2) \cong \mathrm{SO}(3) \times S^1.$$

2.4. Left-invariant Einstein metrics on Lie groups. A left-invariant metric on a Lie group G is completely determined by an inner product on the Lie algebra $\mathfrak{G}$. We will consider metrics of a fixed volume. Let $g_{ij}^{(0)}$ be a fixed metric, $X_1, \dots, X_n$ be an orthonormal basis, and g_{ij} be one more metric with the same volume element and with an orthonormal basis $Y_1, \dots, Y_n$. If $Y_j = \sum\limits_{k=1}^{n} a_{kj} X_k$, then the matrix $\|a_{ij}\|$ belongs to the group $\mathrm{SL}(n, \mathbb{R})$. Therefore, we obtain a one-to-one correspondence between bases having a fixed volume element in $\mathfrak{G}$ and the group $\mathrm{SL}(n, \mathbb{R})$. Any basis in $\mathfrak{G}$ defines an inner product in which it is orthonormal. Two bases $(Y_1, \dots, Y_n)$ and $(Z_1, \dots, Z_n)$ define the same inner product iff $(Z_1, \dots, Z_n) = (Y_1, \dots, Y_n)u$, where $u \in \mathrm{SO}(n)$. Therefore, the set of inner products is in a one-to-one correspondence with the homogeneous space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$. It is not necessary that distinct inner products generate nonisometric left-invariant metrics on a Lie group. If two inner products $g_{ij}^{(1)}$ and $g_{ij}^{(2)}$ yield isometric Riemannian manifolds after left translations, then there exists an isometry $\varphi : G \to G$ such that $\varphi^* g_{ij}^{(1)} = g_{ij}^{(2)}$. Since $L_{\varphi(e)^{-1}} \circ \varphi$ is also a diffeomorphism, $L_{\varphi(e)^{-1}} \circ \varphi(e) = e$, and

$$(L_{\varphi(e)^{-1}} \circ \varphi) * g_{ij}^{(1)} = \varphi^* L^*_{\varphi(e)^{-1}} g_{ij}^{(1)} = \varphi^* g_{ij}^{(1)} = g_{ij}^{(2)},$$

we can assume that $\varphi(e) = e$. Since the metrics are of the same volume, we have $\det \varphi_e = 1$. There arises the problem of describing all volume-preserving diffeomorphisms for which $\varphi(e) = e$ and $\varphi^* g_{ij}$ is a left-invariant metric for any left-invariant metric g_{ij}.

A Lie group G is said to be *unimodular* if $\mathrm{Tr}(\mathrm{ad}_X) = 0$ for any element X of its Lie algebra, where $\mathrm{ad}_X(Y) = [X, Y], Y \in \mathfrak{G}$.

If a Lie group G is unimodular, then for any $[a] \in \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$, each element in $\mathrm{Int}(G)[a]$ leads to isometric left-invariant Riemannian metrics.

We now consider the scalar curvature function of a left-invariant metric on a Lie group. Since G is a Riemannian homogeneous space with respect to the left-invariant metric on the group G, the scalar curvature of the left-invariant metric is a constant function. For distinct metrics, the scalar curvature function R assumes distinct values in general.

Definition 2.4.1. The *scalar curvature function* of a left-invariant metric is the function

$$R : \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n) \to \mathbb{R}$$

such that with each class $[a]$, it associates the value of the scalar curvature of the left-invariant metric that is generated by this class.

If G is a unimodular group, then the scalar curvature function is invariant with respect to the action of the group $\mathrm{Int}(G)$ on the space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$.

A metric g_{ij} is called an *Einstein metric* if its metric tensor is proportional to the Ricci tensor. Einstein metrics can be obtained as critical points of certain functionals defined on the set of all Riemannian metrics. Let M be a compact oriented manifold. Nagano proved that all Einstein metrics are exactly critical points of the functional

$$I(g) = \int\limits_M R_g \, dv,$$

where R_g is the scalar curvature and dv is a fixed volume element. Riemannian metrics are considered with the same volume element. If $M = G$ is a Lie group with a left-invariant metric, then the restriction of the functional $I(g)$ to the set of all left-invariant metrics yields

$$I = \int\limits_G R_g \, dv = R_g \operatorname{vol}(G).$$

Therefore, there naturally arises the problem on the correspondence between the set of all Einstein metrics and the set of all critical points of the scalar curvature function.

Theorem 2.4.1. *Let G be a Lie group, $\mathfrak{G}$ be its Lie algebra, and G be unimodular. Then left-invariant Einstein metrics are exactly critical points of the scalar curvature function.*

For matrices that are not unimodular, this statement is not true in the general case. As an example, it suffices to take

$$\mathfrak{G} = \left\{ \begin{pmatrix} x_1 \dots x_n \\ 0 \end{pmatrix}, \quad x \in \mathbb{R}, \ 1 \le i \le n \right\}, \qquad X_i = \begin{pmatrix} \delta_{1i} \dots \delta_{ni} \\ 0 \end{pmatrix}.$$

This Lie algebra is the Lie algebra of the Lie group

$$G = \left\{ \begin{pmatrix} x_1 \dots x_n \\ \hline 0 \mid E_{n-1} \end{pmatrix}, \quad x_1 > 0 \right\}.$$

The formulas presented above easily imply $R_{pq} = 0$, $p \ne q$, $R_{11} = -n/2$, and $R_{kk} = -n/2$, $k = 2, \dots, n$, i.e., $R_{jk} = -\dfrac{n}{2}\delta_{jk}$, and we obtain an Einstein metric. For this metric to be a critical point, it is necessary that

$$a_{jk} = \left(\frac{1}{2} \sum (c^j_{pk} + c^k_{pj}) T_r c_p - T_r c_j T_r c_k \right) = \lambda \delta_{jk},$$

where $c_i = \|c^k_{ij}\|$, c^k_{ij} being the structural constants of the Lie algebra $\mathfrak{G}$ in the basis $X_1, X_2, \dots, X_n$. We have $a_{11} = -n^2$ and $a_{kk} = n/2$, $k = 2, \dots, n$. Therefore, this metric is not a critical point of the scalar curvature function.

We us now consider the case of semisimple Lie groups in more detail. Let G be a compact semisimple Lie group. Then its Killing form $\varphi(X) \operatorname{tr} \operatorname{ad}^2_X$ defined on its Lie algebra is nondegenerate and negative definite on $\mathfrak{G}$. Let $\theta : \mathfrak{G} \to \mathfrak{G}$ be a nontrivial involutive automorphism of the semisimple Lie algebra $\mathfrak{G}$. Then the subalgebra of fixed points $\mathfrak{R} = \{X \mid \theta(X) = X\}$ is a compactly embedded subalgebra in $\mathfrak{G}$. If $\mathfrak{M} = \{X \in \mathfrak{G} \mid \theta(X) = -X\}$, then $\mathfrak{G} = \mathfrak{R} \oplus \mathfrak{M}$ is a direct sum that is orthogonal with respect to the Killing metric. The form φ is negative definite on $\mathfrak{R}$ and is positive definite or negative definite on $\mathfrak{M}$ depending on whether the Lie algebra $\mathfrak{G}$ is of compact or noncompact type. Let $n = \dim \mathfrak{G}$, and let $r = \dim \mathfrak{R}$. We consider the matrix

$$B = \left(\begin{array}{c|c} \frac{1}{r} E_r & 0 \\ \hline 0 & -\frac{1}{n-r} E_{n-r} \end{array} \right),$$

where E_s is the identity matrix of size $s \times s$ and $\operatorname{Tr} B = 0$. We examine the behavior of $R(e^{tB})$.

Theorem 2.4.2. *We have*

$$R(e^{tB}) = \frac{\lambda}{8}(3r - n)e^{\frac{2t}{r}} - \frac{1}{2}\varepsilon\lambda(n - r)e^{-\frac{2t}{n-r}} - \frac{1}{8}\lambda(n - r)e^{-2t(\frac{2}{n-r} + \frac{1}{r})},$$

where $\varepsilon = +1$ in the case of noncompact type and $\varepsilon = -1$ in the case of compact type.

The behavior of the function $R(e^{tB})$ is depicted in Figs. 48–51.

A simple calculation shows that the function $R(e^{tB})$ has the critical point $t_0 = \dfrac{r(n-r)}{2n \ln \left[\dfrac{n+1}{3n-r} \right]}$, $t_0 > 0$, in the case where $\mathfrak{G}$ is compact and $\mathfrak{R}$ is non-Abelian.

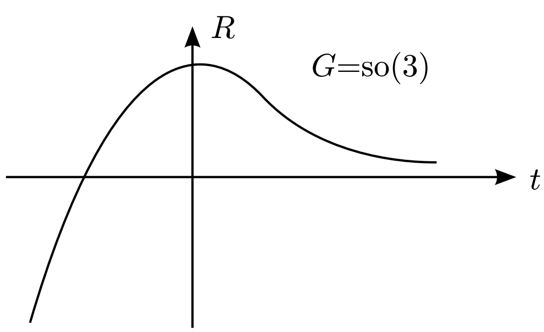


Fig. 48

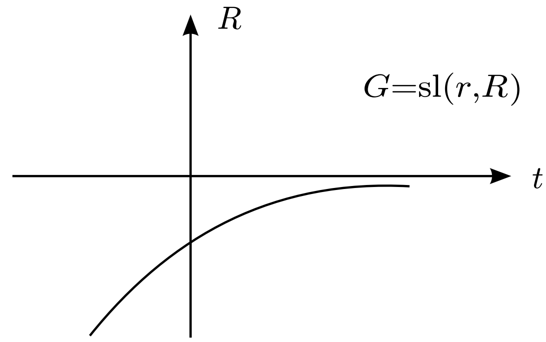


Fig. 49

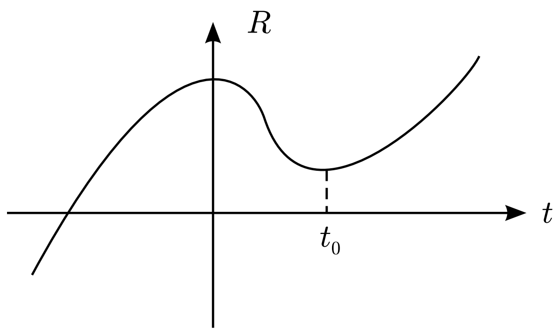


Fig. 50

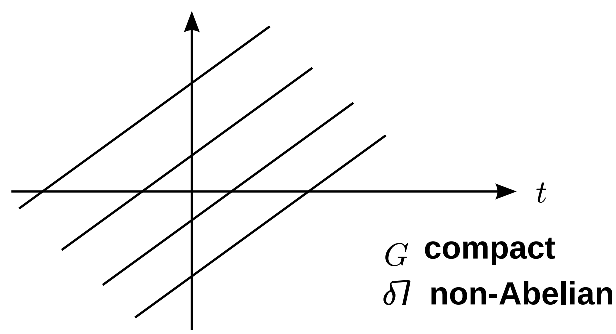


Fig. 51

The following statement gives an answer to the question of when the metric corresponding to t_0 is an Einstein metric.

Theorem 2.4.3. *Let* $e^{t_0 B}$ *be a critical point of the functional* $I(g)$, *i.e., the corresponding metric is Einstein. Then the Killing form of the subalgebra* $\mathfrak{R}$ *is proportional to the Killing form of the algebra* $\mathfrak{G}$ *that is restricted to* $\mathfrak{R}\times\mathfrak{R}$ (*the proportionality coefficient is a positive constant*); *this condition is necessary and sufficient. The necessary condition is that* $\mathfrak{R}$ *be a compact and semisimple algebra, and the sufficient condition is that* $\mathfrak{R}$ *be a compact and simple algebra.*

We now return to arbitrary Lie groups. There exists a complete classification of left-invariant Einstein metrics on four-dimensional Lie groups owing to Jensen. Let G be a four-dimensional Lie group equipped with a left-invariant Riemannian metric. Then G is an Einstein space iff its Lie algebra $\mathfrak{G}$ is one of the following solvable Lie groups with an inner product such that the basis X_1, X_2, X_3, X_4 becomes orthonormal with respect to it. For distinct t, we obtain nonisomorphic Lie algebras.

1.

$$[X_1, X_2] = 0, \qquad [X_2, X_3] = 0,$$
$$[X_1, X_3] = X_4, \qquad [X_2, X_4] = 0,$$
$$[X_1, X_4] = -X_3, \qquad [X_3, X_4] = 0.$$

In this case, we obtain a flat manifold.

2.

$$[X_1, X_2] = X_2 - tX_3, \qquad [X_2, X_3] = 2X_4,$$

$$[X_1, X_3] = tX_2 + X_3, \qquad [X_2, X_4] = 0,$$
$$[X_1, X_4] = 2X_4, \qquad [X_3, X_4] = 0, \quad 0 \le t < \infty.$$

The corresponding Riemannian space is isometric to a hyperbolic space of constant sectional curvature k, $-1 \ge k \ge -4$.

3.

$$[X_1, X_2] = X_2, \qquad [X_2, X_3] = 0,$$
$$[X_1, X_3] = X_3 - tX_4, \qquad [X_2, X_4] = 0,$$
$$[X_1, X_4] = tX_3 + X_4, \qquad [X_3, X_4] = 0, \quad 0 \le t < \infty.$$

The corresponding Riemannian space is isometric to a hyperbolic space of constant sectional curvature $k = -1$.

4.

$$[X_1, X_2] = 0, \qquad [X_2, X_3] = 0,$$
$$[X_1, X_3] = X_3, \qquad [X_2, X_4] = X_4,$$
$$[X_1, X_4] = 0, \qquad [X_3, X_4] = 0.$$

This Lie algebra is the direct sum of two copies of a two-dimensional Lie algebra. The corresponding Riemannian space is the product of a two-dimensional solvable group of curvature $k = -1$ by itself.

3. Geometry of Kählerian Manifolds

3.1. Main concepts and definitions. Let M^{2n} be a complex manifold (see [44, 48, 94, 113, 211]) on which an Hermitian inner product (ξ, η) is given. We consider the form $\omega(\xi, \eta) = \operatorname{Im}(\xi, \eta)$. Obviously, ω is a skew-symmetric nondegenerate 2-form. Generally speaking, this form is not closed on an arbitrary complex manifold equipped with an Hermitian metric.

Definition 3.1.1. A complex manifold equipped with an Hermitian metric is said to be *Kählerian* if the imaginary part ω of the inner product (ξ, η) is a closed differential form: $d\omega = 0$.

The complex projective space $\mathbb{C}P^n$ is one of the classical examples of a Kählerian manifold. There is a natural holomorphic mapping $\pi : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}P^n$. On $\mathbb{C}^{n+1} \setminus 0$, we consider the covariant 2-tensor

$$\hat{F} = \frac{4R}{\left(\sum\limits_{k=0}^{n} z_k \bar{z}_k\right)^2} \left\{ \left(\sum_{k=0}^{n} z_k \bar{z}_k\right) \left(\sum_{k=0}^{n} dz_k \otimes d\bar{z}_k\right) - \left(\sum_{k=0}^{n} \bar{z}_k dz_k\right) \otimes \left(\sum_{k=0}^{n} z_k \otimes d\bar{z}_k\right) \right\},$$

where $z_0, z_1, \ldots, z_n$ are standard coordinates in $\mathbb{C}^{n+1}$ and $R = \text{const}$.

Proposition 3.1.1. *On $\mathbb{C}P^n$, there exists a Kählerian metric F such that $\pi^* F = \hat{F}$, where $\hat{F}$ is defined by the above formula.*

This statement is implied by the following four obvious properties of the tensor $\hat{F}$:

(a) the restriction of $\hat{F}$ to a fiber of the mapping $\pi : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}P^n$ vanishes;
(b) the tensor $\hat{F}$ is invariant with respect to the natural action of the group $\mathbb{C}^* = \mathbb{C} \setminus 0$ on $\mathbb{C}^{n+1} \setminus 0$: $z(z_0, z_1, \ldots, z_n) = (zz_0, zz_1, \ldots .zz_n)$, $z \in \mathbb{C}^*$;
(c) the restriction of $\hat{F}$ to the orthogonal complement to the fiber with respect to a flat metric on $\mathbb{C}^{n+1}$ is positive definite;
(d) the differential 3-form $d(\operatorname{Im} F)$ on $\mathbb{C}P^n$ is invariant with respect to the mapping induced by unitary transformations A of the space $\mathbb{C}^{n+1}$, $A \in U(n+1)$.

The metric F on $\mathbb{C}P^n$ constructed above is called the *Fubini–Study metric* (for more details, see, e.g., [10]). As a consequence of this construction, we obtain that any complex analytic submanifold of $\mathbb{C}P^n$ is Kählerian.

A one-dimensional complex manifold is called a *Riemann surface*. A simple example of a one-dimensional manifold that is not reduced to one chart is the *Riemann sphere* (or extended complex plane) $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Any Riemann surface is obviously a Kählerian manifold. We indicate certain important properties of Riemann surfaces. Let Y be a Riemann surface. Then there exist a simply connected Riemann surface X and a covering $p : X \to Y$. We denote by Γ the group of this covering ($\Gamma \cong \pi_1(Y)$). This is a discrete group of analytic automorphisms of the surface X, and $Y \cong X/\Gamma$. Therefore, for describe Riemann surfaces, we need to solve the following two problems:

(a) to classify simply connected Riemann surfaces;
(b) to classify discrete automorphism groups of simply connected Riemann surfaces.

The first problem was solved by Poincaré and Koebe in 1907.

Theorem 3.1.1. *Any simply connected Riemann surface is isomorphic to one of the following surfaces*: $\mathbb{C}$, $\hat{\mathbb{C}}$, *and* U, *where* $\mathbb{C}$ *is the field of complex numbers,* $\hat{\mathbb{C}}$ *is the Riemann sphere, and* U *is the upper half-plane.*

This theorem is a sufficiently deep analytic property.

Complete automorphism groups of simply connected surfaces are described by the following theorem.

Theorem 3.1.2. (a) *The automorphism group* $\mathrm{Aut}(\hat{\mathbb{C}})$ *consists of all linear-fractional transformations* $W = \dfrac{az+b}{cz+d}$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *is an arbitrary nonsingular matrix. These transformations can be normalized in such a way that* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(r, \mathbb{C})$.

(b) *The automorphism group* $\mathrm{Aut}(\mathbb{C})$ *consists of all linear transformations* $W = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$.

(c) *The automorphism group* $\mathrm{Aut}(U)$ *consists of linear-fractional transformations* $W = \dfrac{az+b}{cz+d}$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(r, \mathbb{R})$.

We note that any automorphism of the sphere $\hat{\mathbb{C}}$ has a fixed point. Discrete automorphism groups of the plane $\mathbb{C}$ consist of parallel translations. All possible case are represented in the following table.

Γ	$\{\mathrm{id}\}$	$\{\mathrm{id}\}$	$z \mapsto z + 2\pi i n$, $n \in \mathbb{Z}$	$z \mapsto z + \omega_1 n_1 + \omega_2 n_2$, $n_1, n_2 \in \mathbb{Z}$
Y	$\hat{\mathbb{C}}$	$\mathbb{C}$	$\mathbb{C}^* = \mathbb{C} \setminus 0$	torus
X	$\hat{\mathbb{C}}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

In the case of a torus, the complex structure depends on ω_1 and ω_2, and the ratio ω_1/ω_2 is an invariant of the structure. We obtain a continuum of nonisomorphic tori (from the topological viewpoint, all tori are the same, and as Riemann surfaces, they are different. For the metric classification of tori, see Sec. 1.4 of Chap. 3.

Any Riemann surface admits a conformal geometry, i.e., we can measure angles in it. The tangent space is a one-dimensional complex vector space. Therefore, any two directions are obtained from one another via multiplication by $e^{i\varphi}$, and φ is the angle between these directions. Any holomorphic mapping with a nonzero derivative preserves angles. Any $\mathbb{R}$-smooth mapping preserving angles is said to be *conformal*. Any conformal orientation-preserving mapping is holomorphic. The conformal geometry on simply connected Riemann surface is related to a rich-in-content geometry on the classical spaces $\mathbb{R}^2$, S^2, and H^2.

Theorem 3.1.3. (a) *The space* $\hat{\mathbb{C}}$ *is conformally isomorphic to the standard sphere* S^2.
(b) *The space* $\mathbb{C}$ *is conformally isomorphic to the Euclidean plane* $\mathbb{R}^2$.
(c) *The space* U *is conformally isomorphic to the Lobachevskii plane* H^2.

We now reveal the connection between the motion groups and the automorphism groups of Riemann surfaces. We denote by I_+ the group of orientation-preserving isometries of a Riemann surface.

Theorem 3.1.4. (a) *The group* $I_+(\hat{\mathbb{C}})$ *consists of transformations of the form* $W = \dfrac{az+b}{cz+d}$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(r)$.
(b) *The group* $I_+(\mathbb{C})$ *consists of transformations of the form* $W = az + b$, *where* $|a| = 1$, $a, b \in \mathbb{C}$.
(c) *The group* $I_+(U)$ *coincides with the whole automorphism group* $\mathrm{Aut}(U)$, *i.e., it consists of all transformations of the form* $W = \dfrac{az+b}{cz+d}$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(r, \mathbb{R})$.

Therefore, we have more analytic automorphisms. However, in all three cases, discrete automorphism groups consist of motions.

Proposition 3.1.2. *Any discrete group of analytic automorphisms of a simply connected Riemann surface* X *is conjugate to the motion group in the group* $\mathrm{Aut}(X)$.

There are finitely many automorphisms of a compact Riemann surface.

Theorem 3.1.5. *There are no more than* $84(g-1)$ *automorphisms of a compact Riemann surface* Y *of genus* $g > 1$.

This bound is attained for infinitely many but not for all g.

The condition of existence of the Kählerian structure on a manifold M^n imposes strong topological conditions on M^n. We present the simplest obstructions of such a type. If a compact manifold admits a Kählerian structure, even-dimensional Betti numbers $b_{2k}(M^n)$ of the manifold M^n are positive. To observe that $b_{2q}(M^n) > 0$, it suffices to verify that the closed $2q$-form ω^q is not exact. Indeed, if $\omega^q = d\psi$ for a certain form ψ, then

$$\int_M \underbrace{\omega \wedge \ldots \wedge \omega}_{n} = \int_M d(\psi \wedge \underbrace{\omega \wedge \ldots \wedge \omega}_{n-q}) = 0,$$

where ω is the form $\mathrm{Im}\, ds^2$ associated with the Kählerian metric ds^2. The form $\omega^n/n!$ coincides with the volume form of the manifold M^n; therefore, the above relation is not possible.

Further examples of Kählerian manifolds will be presented below.

The Kählerian condition admits the following geometric interpretation. We say that a metric ds^2 on a manifold M has a *contact of order* k with the Euclidean metric on $\mathbb{C}^n$ if, in a neighborhood of each point $z_0 \in M$, we can find holomorphic local coordinates $z^1, \ldots, z^n$ in which

$$ds^2 = \sum_{i,j=1}^{n} (\delta_{ij} + g_{ij}) dz^i \otimes dz^j,$$

where the function g_{ij} has a zero of order k at the point z_0.

The following characterization of Kählerian manifolds holds. A manifold M with metric ds^2 is Kählerian iff it has a contact of the second order with the Euclidean metric everywhere.

Kählerian manifolds can be equivalently defined in a slightly different way. A Riemannian manifold M of even dimension is said to be Kählerian if on M, we have a field J of tensor of type $(1, 1)$ such that

$J^2 = E$ (J a linear operator on each tangent plane) and

$$\text{(a)}\quad (JX, JY) = (X, Y)$$
$$\text{(b)}\quad \nabla_X(J) = 0$$

for arbitrary vector fields X and Y on the manifold M. If we have a Kählerian manifold in the sense of Definition 3.1.1, then it suffices to take as J the linear operator of multiplication by the imaginary unit $\sqrt{-1}$. We recall that a tensor field J of type $(1, 1)$ such that $J^2 = E$ is called an *almost complex structure* on the manifold M. The curvature tensor on Kählerian manifolds satisfies the additional identity $R_{JX,JY} = R_{X,Y}$.

Definition 3.1.2. A two-dimensional subspace in the tangent space T_xM spanned by the vectors x and Jx ($x \neq 0$) is called a *holomorphic section* at $x \in M$. The *holomorphic sectional* or *bisectional* curvature H of a Kählerian manifold M is the usual sectional curvature restricted to holomorphic sections. Therefore, we can assume that H is a real-valued function of a tangent vector $X \in T_pM$, $p \in M$.

The following analog of the Schur theorem holds for Kählerian manifolds (see Sec. 1.1 of Chap. 3).

Theorem 3.1.6. *Let M be a Kählerian manifold with* $\dim M \geq 4$. *If the holomorphic sectional curvature H of the manifold M is pointwise constant, then it is constant.*

This theorem is a particular case of the construction presented in Sec. 1.1 of Chap. 3.

In conclusion, we describe certain important properties of the Fubini–Study metric on the projective space $\mathbb{C}P^n$. The sectional curvature of this metric has the maximum value $4R$ and the minimum value R. More precisely, the curvature tensor $(R(w, x)y, z)$ of the Fubini–Study metric equals

$$R\{\langle w, y\rangle\langle x, z\rangle - \langle w, z\rangle\langle x, y\rangle + \langle Jw, y\rangle\langle Jx, z\rangle - \langle Jw, z\rangle\langle Jx, y\rangle + 2\langle Jw, x\rangle\langle Jy, z\rangle\}.$$

Therefore, the holomorphic sectional curvature is identically equal to $4R$. Because of this observation, the projective space $\mathbb{C}P^n$ plays the same role among Kählerian manifolds as the ordinary sphere among real Riemannian manifolds. The general formula for the sectional curvature of the projective space $\mathbb{C}P^n$ has the form

$$K_{xy} = R\left[1 + \frac{3\langle Jx, y\rangle^2}{|x|^2|y|^2 - \langle x, y\rangle^2}\right].$$

This implies the assertion on the maximum and minimum of the sectional curvature K_{xy}. Let $v(x)$ be the volume of a geodesic ball of radius r. Then $v(x) = \dfrac{1}{n!}\left(\dfrac{\pi}{R}\right)^n \sin^{2n}(r\sqrt{R})$. In particular, the volume of the projective space $\mathbb{C}P^n(R)$ is equal to $\dfrac{1}{n!}\left(\dfrac{\pi}{R}\right)^n$. The volume of nonsingular projective manifolds is easily calculated through their algebraic characteristics. More precisely, the following assertion holds. Let $S \subset \mathbb{C}P^n$ be a nonsingular r-dimensional projective manifold. Then $\mathrm{vol}_{2r}(S) = \deg S\,\mathrm{vol}_{2r}(L)$, where vol_{2r} is the $2r$-dimensional volume with respect to the Fubini–Study metric ($R = 1$) and L is any r-dimensional linear subspace of the projective space $\mathbb{C}P^n$.

3.2. Hodge manifolds. We now focus on a specific class of Kählerian manifolds defined by a certain topological condition. If M is a compact manifold, then the closed differential form φ on M is said to be *integral* if its cohomology class $[\varphi] \in H^*(M, \mathbb{C})$ belongs to the image of the natural mapping $H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{C})$ induced by an embedding of the group $\mathbb{Z}$ into the field $\mathbb{C}$. Let h be a Kählerian metric on a complex manifold, and let ω be the associated form: $\omega = \mathrm{Im}\, h$.

Definition 3.2.1. If ω is an integral differential form, then it is called a Hodge form on M, and ω is also called a *Hodge metric*. A Kählerian manifold is called a *Hodge manifold* if it admits a Hodge metric.

Hodge manifolds play an important role because of the following fundamental Kodaira theorem.

Theorem 3.2.1. *A compact complex manifold is a Hodge manifold iff it is a projective algebraic variety.*

There are many examples of Hodge manifolds; moreover, for some of them, it is not obvious in advance that they are projective algebraic varieties. Also, we mention the Chow theorem, which asserts that projective manifolds are algebraic varieties, i.e., they are defined by zeros of homogeneous polynomials. We now present examples of Hodge manifolds. Any complex submanifold of a Hodge manifold is itself a Hodge manifold. Let X be a compact complex manifold such that it is a covering of a Hodge manifold Y, i.e., there exists a holomorphic mapping $\pi : X \to Y$ such that $\pi^{-1}(p)$ is discrete for all $p \in Y$ and π is locally biholomorphic at each point $x \in X$. Then X is a Hodge manifold. Therefore, basic operations on Riemannian manifolds preserve the class of Hodge manifolds. If X is a connected compact Riemannian surface, then X is a Hodge manifold. Let D be a bounded domain in $\mathbb{C}^n$, and let Γ be a proper discontinuous subgroup of the group $\mathrm{Aut}(D)$ of biholomorphic self-mappings of D acting without fixed points such that $M = D/\Gamma$ is a compact set. Then M is a Hodge manifold. We consider the case of tori in more detail. Let $\omega_1, \dots, \omega_{2n}$ be $2n$ vectors of the space $\mathbb{C}^n$ that are linearly independent over $\mathbb{R}$, and let Γ be the lattice consisting of all integer linear combinations of the vectors $\omega_1, \dots, \omega_{2n}$. The lattice Γ naturally acts on the space $\mathbb{C}^n$ by translations: $z \mapsto z + \gamma$ if $\gamma \in \Gamma$. The quotient space $\mathbb{C}^n/\Gamma$ is a complex manifold, and its universal covering is $\mathbb{C}^n$. The manifold $M = \mathbb{C}/\Gamma$ is called the *complex torus*. It is homeomorphic to the product of $2n$ circles $S^1 \times \ldots \times S^1$. A Kählerian metric h on $\mathbb{C}^n$ is invariant with respect to the action of the group Γ. Because of this invariance, there exists an Hermitian metric $\tilde{h}$ on M such that if $\pi : \mathbb{C}^n \to \mathbb{C}^n/\Gamma$ is a natural holomorphic projection, then $\pi^*(\tilde{h}) = h$. The metric $\tilde{h}$ is Kählerian. The complex torus $M = \mathbb{C}^n/\Gamma$ is called an *Abelian variety* if it is a projective algebraic variety, i.e., admits an embedding in the projective space. Since $\mathbb{C}^n$ is a universal covering of the torus M, we can identify $H_1(M, \mathbb{Z}) = \Gamma$. Let $\omega_1, \dots, \omega_n$ be vectors of the lattice that form an integer basis of Γ; it is also a basis of the real vector space $\mathbb{R}^{2n} = \mathbb{C}^n$. We denote by $x_1, \dots, x_n$ the dual real coordinates on $\mathbb{R}^{2n}$ and by $dx_1, \dots, dx_n$ the corresponding 1-forms on M. We set $dx_i = \sum\limits_{\alpha} \pi_{i\alpha} dz_\alpha + \sum\limits_{\alpha} \bar{\pi}_{i\alpha} d\bar{z}_\alpha$ and denote by $\Pi = \|\pi_{i\alpha}\|$ the matrix with entries $\pi_{i\alpha}$.

Theorem 3.2.2 (Riemann)**.** *The torus M is an Abelian variety iff there exists an integer skew-symmetric matrix Q for which $\Pi^t Q \Pi = 0$ and $-\sqrt{-1}\Pi^t Q \bar{\Pi} > 0$; here the sign $>$ stands for the positive definiteness.*

Usually the Riemann conditions are written in terms of the dual matrix of basis change. Let $\omega_1, \dots, \omega_n$ be an integer basis in Γ, and let $e_1, \dots, e_n$ be a basis of the complex space $\mathbb{C}^n$. We define the period matrix for $\Gamma \subset \mathbb{C}^n$ as an $n \times 2n$-matrix $\Omega = (\omega_{\alpha i})$ for which $\omega_i = \sum\limits_{\alpha} \omega_{\alpha i} e_\alpha$. Then $dz_\alpha = \sum\limits_{i} \omega_{\alpha i} dx^i$ and $d\bar{z}_\alpha = \sum\limits_{\alpha} \bar{\omega}_{\alpha i} dx_i$.

Theorem 3.2.3. *The torus M is an Abelian variety iff there exists an integer skew-symmetric matrix Q such that $\Omega Q^{-1} \Omega^t = 0$ and $-\sqrt{-1}\Omega Q^{-1} \Omega^t > 0$.*

Definition 3.2.2. A matrix Ω for which the conditions of Theorem 3.2.3 hold is called a *Riemannian matrix*.

Not every torus is an Abelian variety. An example of a complex torus that does not satisfy the Riemann conditions is given by the period matrix

$$\Omega = \begin{pmatrix} 1 & 0 & \sqrt{-2} & \sqrt{-5} \\ 0 & 1 & \sqrt{-3} & \sqrt{-7} \end{pmatrix}.$$

We present an important example of Abelian varieties that arise in the Riemann surface theory. We consider a compact Riemann surface S of genus g (from the topological viewpoint, it is the sphere with g handles) (see Fig. 52). Let $\delta_1, \dots, \delta_{2g}$ be a basis in the homology group $H_1(S, \mathbb{Z})$ (see Fig. 52), and let $\alpha_1, \dots, \alpha_g$ be a basis of one-dimensional differentials on S. The *Jacobian of the surface S* is the quotient

Fig. 52

manifold $J(S) = \mathbb{C}^g/\{\lambda_1, \dots, \lambda_g\}$, where λ_i are vectors of the form

$$\lambda_i = \left(\int\limits_{\delta_i} \alpha_1, \dots, \int\limits_{\delta_i} \alpha_g \right)^t .$$

There exist bases $\delta_1, \dots, \delta_{2g}$ and $\alpha_1, \dots, \alpha_g$ for which

$$\int\limits_{\delta_i} \omega_\alpha = \delta_{i\alpha}, \qquad 1 \le i, \alpha \le g.$$

Then the period matrix Ω has the form $\Omega = (E, Z)$. Let $Z = X + \sqrt{-1}Y$. In this case, Z is symmetric and $Y > 0$. As a result, the torus $J(S)$ is an Abelian variety. We can give an intrinsic definition of the Jacobian $J(S)$ of the surface S (in this connection, see [44]). The theory of invariant Kählerian structures is presented in [211] in more detail.

3.3. Chern forms of Kählerian manifolds. Let M^n be a Riemannian manifold, and let $E_1, \dots, E_n$ be a local orthonormal basis. Then we can define real curvature forms $\Omega_{ij}(X, Y)$, $1 \le i, j \le n$, on M by $\Omega_{ij}(X, Y) = \langle R_{XY} E_i, E_j \rangle$, where R_{XY} is $\nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$ as usual. The matrix $\Omega = \|\Omega_{ij}\|$ is skew-symmetric, and we can calculate the Pfaffian $\mathrm{Pf}(\Omega)$. If M is an oriented Riemannian manifold of dimension $2n$, then the form $\chi = (2\pi)^{-n}\,\mathrm{Pf}(\Omega)$ is called the *Euler class* of the manifold M. In fact, we have seen it in the Gauss–Bonnet formula.

We now consider a Kählerian manifold M and fix a holomorphic basis $\{e_1, Je_1, \dots, e_n, Je_n\} = \{e_1, e_{1^*}, \dots, e_n, e_{n^*}\}$.

Definition 3.3.1. *Complex curvature forms* are differential forms Ξ_{ij} defined by $\Xi_{ij}(v, w) = \Omega_{ij}(v, w) - \sqrt{-1}\Omega_{ij^*}(v, w)$. In contrast to the anti-symmetric matrix $\|\Omega_{ij}\|$, the matrix $\|\Xi_{ij}\|$ is anti-Hermitian.

Definition 3.3.2. Let $\det \|\delta_{ij} - \dfrac{1}{2\pi\sqrt{-1}}\Xi_{ij}\| = 1 + C_1 + \dots + C_n = C$. The form C is called the *full Chern form* of the manifold M, and C_i is called the ith Chern form of the manifold M.

It is easily verified that C does not depend on the choice of the orthonormal basis $e_1, e_{1^*}, \dots, e_n, e_{n^*}$. Under an isomorphism of the de Rham cohomology space onto topological cohomologies, the full form C transforms into the full Chern class of the manifold M by the de Rham theorem. It is easy to see that $2\pi C(v,w) = \sum\limits_{i=1}^{n} \Omega_{ii^*}(v,w)$. The first form C_1 plays the role of the Ricci curvature, since $R_{ij}x^iy^j = 2\pi C_1(x, Jy)$.

Let

$$\Omega_{i_1 \dots i_{2k}} = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} \varepsilon(\sigma) \Omega_{i_{\sigma(1)} i_{\sigma(2)}} \wedge \dots \wedge \Omega_{i_{\sigma(2k-1)} i_{\sigma(2k)}}.$$

The forms C_k are explicitly expressed through $\Omega_{i_1 \dots i_k}$ by the formula

$$(2\pi)^k C_k = \sum_{i_1 < \dots < i_k} \Omega_{i_1 i_{1^*} \dots i_k i_{k^*}}.$$

This implies that the form C_n of degree $n = \dim_{\mathbb{C}} M$ coincides with the Euler form $(2\pi)^n C_n = \mathrm{Pf}(\Omega)$. Therefore, the Chern form is a generalization of the Euler form.

We construct analogs of the Euler characteristic for compact Kählerian manifolds ($n = \dim_{\mathbb{C}} M$). Since the form C_n coincides with the Euler form, the Euler characteristic $\chi(M)$ is equal to the integral $\chi(M) = \int\limits_M C_n$.

Definition 3.3.3. Let $(i_1, \dots, i_k)$ be a sequence of integers such that $i_1 + 2i_2 + \dots + ki_k = 2n$. Then the *Chern number* $C_{i_1 i_2 \dots i_k}(M)$ defined by

$$C_{i_1 i_2 \dots i_k}(M) = \int\limits_M C_{i_1} \wedge \dots \wedge C_{i_k}$$

corresponds to this sequence.

As an example, we consider the complex projective space. The full Chern form C of the projective space $\mathbb{C}P^n(R)$ is equal to $C = \left(1 + \frac{R}{\pi}\omega\right)^{n+1}$. This implies that the ith Chern form is given by $C_i = C_{n+1}^i \left(\frac{R\omega}{\pi}\right)^i$. Here ω is the imaginary part of the Fubini–Study metric. Further,

$$C_{i_1 \dots i_k}(\mathbb{C}P^n(R)) = C_{n+1}^{i_1} \dots C_{n+1}^{i_k}.$$

We note that the Chern numbers of the projective space $\mathbb{C}P^n(R)$ are always integers. In fact, this statement is true for an arbitrary Kählerian manifold M. These numbers are a useful generalization of the Euler characteristic of the manifold M; they give a lot of information about the topology of the manifold M. As an example of assertions of such a type, we present a theorem owing to Michelson (see [182]). Let M be a compact simply connected Kählerian manifold with $C_1(M) = 0$. Then $\mathrm{td}(M) = 0$ or 2^k for a certain k. More precisely, M is the product $M = M_1 \times \dots \times M_k$ of simply connected Kählerian manifolds M_i with zero Chern class, and, moreover, $\mathrm{td}(M_i) = 0$, if the dimension of M_i is odd and $\mathrm{td}(M_i) = 2$ if the dimension of M_i is even. Here we use the *Todda genus* $\mathrm{td}(M)$, which is constructed by a multiplicative sequence corresponding to the power series $Q(x) = x/(1 - e^{-x})$ (see [95]).

3.4. Kählerian manifolds and the Wirtinger inequality. Let M be a Kählerian manifold equipped with an almost complex structure J and a metric $\langle X, Y \rangle$. We say that a submanifold $P \subset M$ is a Kählerian submanifold if for any vector $X \in T_aP$, one has $JX \in T_aP$. Clearly, the restriction of J to P assigns an almost complex structure, called the induced almost complex structure. The following statement holds. Let P be a Kählerian submanifold of a Kählerian manifold M. Then the induced Riemannian metric and the induced almost complex structure assign the structure of a Kählerian manifold on P. Moreover, the mean curvature vector of the manifold P vanishes identically, i.e., P is a minimal submanifold in M.

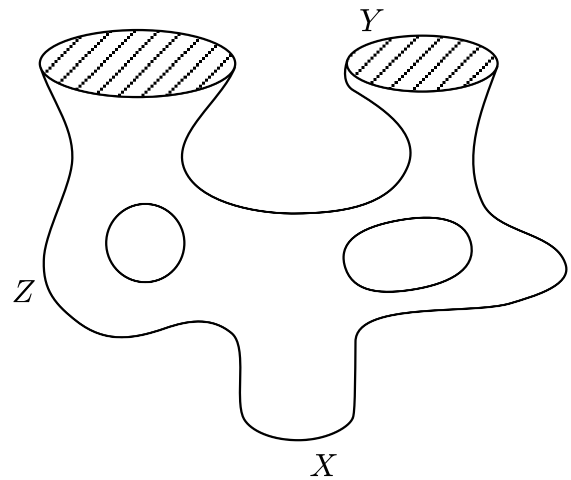


Fig. 53

We recall the definition of the mean curvature vector. Let M be a submanifold of a Riemannian manifold W, and let ∇ be the covariant derivative with respect to a connection that is compatible with the Riemannian metric on the manifold W. Let $x, y \in T_m(M)$. We include the vector y into a smooth vector field Y on W that is tangent to the submanifold M. We define the *second fundamental form* $B(x, y)$ of the submanifold M. By definition, we set $B(x, y) = (\nabla_x Y)^N$, i.e., we differentiate the field Y in the direction of the field x and project the result obtained on the normal subspace. We consider the second fundamental form represented as the form B on the tangent space $T_m(M)$, $m \in M$, with values in the normal space $N_m(M)$. Since an inner product is defined on $T_m(M)$, we can consider the trace of the form B, which is a certain vector at each point m that belongs to $N_m(M)$. This section H of the normal bundle NM is called the *mean curvature* of the embedded submanifold $M \subset W$. If $e_1, \dots, e_k$ is a certain orthonormal basis of the space $T_m M$, then $H = \sum\limits_{i=1}^{k} B(e_i, e_i) \in N_m(M)$. A submanifold $M \subset W$ is said to be *locally minimal* if its mean curvature H vanishes identically (at all points $m \in M$).

The following fundamental fact holds (see [60] and Fig. 53).

Theorem 3.4.1 (Wirtinger). *If S is a complex submanifold of dimension d in a manifold M, then*

$$\operatorname{vol}(S) = \frac{1}{d!} \int\limits_S \omega^d = \int\limits_S \underbrace{\omega \wedge \ldots \wedge \omega}_{d}.$$

This theorem shows the fundamental distinction between Riemannian geometry and Hermitian differential geometry. The volume of a complex submanifold S of a complex manifold M is expressed as the integral over S of the differential form globally defined on M. This is no longer true in the real case. For example, for a smooth arc $(x(t), y(t))$ in $\mathbb{R}^2$, the length element is given by $(x'(t)^2 + y'(t)^2)^{1/2} dt$, which is not the inverse image of any differential form on the space $\mathbb{R}^2$. Moreover, on a Kählerian manifold, we have a canonical orientation defined by the volume form $\underbrace{\omega \wedge \ldots \wedge \omega}_{d}$. Using the Wirtinger theorem, we easily obtain the following Wirtinger inequality. Let P be a Kählerian submanifold in M having a finite volume, and let P' be any submanifold that is homologic to P. Then $\operatorname{vol}(P) \le \operatorname{vol}(P')$, i.e., P is a minimal submanifold.

3.5. Manifolds of negative bisectional curvature. We only have certain results this problem. The classification of such manifolds remains an open problem. Certain important results on the structure of such manifolds are obtained in dimension two. We mention some of them. Let M be a Kählerian surface of negative curvature, and let $M = \tilde{M}/\Gamma$, where $\tilde{M}$ is the universal covering over M and Γ is a discrete subgroup in the component of the identity in the automorphism group $\mathrm{Aut}(\tilde{M})$ acting freely on $\tilde{M}$. Then $\tilde{M}$ is biholomorphically isomorphic to the unit disk in $\mathbb{C}^2$. If we omit the condition that the surface M is Kählerian, then we obtain the following assertion. If $M = \tilde{M}/\Gamma$ is a compact complex surface such that $\tilde{M}$ is a bounded domain in $\mathbb{C}^2$, $\partial\tilde{M}$ is a three-dimensional topological manifold, and $\Gamma \subset \mathrm{Aut}_0(\tilde{M})$ is a discrete subgroup acting freely on $\tilde{M}$, then $\tilde{M}$ is a bounded symmetric domain in the space $\mathbb{C}^2$.

We now impose a weaker condition for the biholomorphic curvature of a surface to be negative. We therefore consider a compact complex surface $M = \tilde{M}/\Gamma$, where $\tilde{M}$ is a universal covering of M, $\tilde{M}$ is a bounded domain in $\mathbb{C}^2$, the group $\Gamma \subset \mathrm{Aut}_0(\tilde{M})$ acts freely on $\tilde{M}$, and Γ is not isomorphic to the fundamental group of a real surface. Then $\tilde{M}$ is a bounded symmetric domain. It is useful to compare these results with the Poincaré–Koebe theorem from Sec. 3.1. For the proofs of these statements, see [214].

Theorem 3.5.1 ([217]). *The polydisk Δ_n $(n > 1)$ does not admit a complete Kählerian metric of bisectional curvature H satisfying the inequality $-c^2 \le H \le -d^2$.*

3.6. The Calabi problem. In Calabi's paper [36], there are several conjectures on the existence of certain classes of metrics. One of his conjectures is as follows: if M is a Kählerian manifold with $C_1(M) = 0$, then there exists a Kählerian metric of nonzero Ricci curvature. A confirmation of this conjecture was given by Yau in [219]. A detailed solution can be found in [29].

There arises a natural question: is it true that the class $C_1(M)$ can be represented by the Ricci form of a certain metric? An affirmative answer is given by the following theorem.

Theorem 3.6.1. *On a compact Kählerian manifold, each form representing the class $C_1(M)$ is the Ricci form of a certain Kählerian metric.*

We can improve this theorem. With each positive cohomology class, one associates one and only one metric. In particular, if $\dim H_2(M, \mathbb{R}) = 1$, then a solution of the problem of realization of the class $C_1(M)$ by the Ricci form is unique with accuracy up to a homothetic transformation of a metric. For the proof of the above statements, see [11].

For Kählerian manifolds with negative class $C_1(M)$, in order to obtain uniqueness of the solution, it is necessary to restrict the class of metrics to the class of Einstein metrics.

Theorem 3.6.2 ([11]). *A compact Kählerian manifold with negative first Chern class admits a Kählerian–Einstein metric, and all such metrics are proportional.*

In concluding of this subsection, we present one useful extremal property of Kählerian–Einstein metrics. In Sec. 4.8 of Chap. 3, we have defined the functional $\nu(g)$ on the space of all Riemannian metrics on a given manifold M. For this functional, we have the following estimate in the Kählerian case.

Theorem 3.6.3. *Let* $\dim M = 4$, *and let g be a Kählerian metric of a certain complex structure on the manifold M. Then*

$$\nu(g) \ge 24\pi^2|\tau| + \frac{16}{3}\pi^2 \min\{2\chi - 6\tau, 2\chi + 3\tau\},$$

where τ and χ are the signature and the Euler characteristic of the manifold, respectively. The equality holds iff M is a Kählerian–Einstein metric.

As a consequence of this statement, we obtain that for any Kählerian metric g on the product $S^2 \times S^2$, the inequality $\nu(g) \ge \frac{128}{3}\pi^2$ holds, since $\tau(S^2 \times S^2) = 0$ and $\chi(S^2 \times S^2) = 4$.

3.7. Kählerian manifolds of positive curvature. Let M be a Kählerian manifold of dimension n, $\sigma = \{X, Y\}$ be a two-dimensional subspace in T_xM, and $K(\sigma)$ be the sectional curvature in direction σ. We denote $|g(X, JY)|$ by $\cos\alpha(\sigma)$ (this is the angle between the planes σ and $J(\sigma)$). We define the *Kählerian sectional curvature* by $K^*(\sigma) = 4K(\sigma)/(1 + 3\cos^2\alpha(\sigma))$. This definition is natural since the sectional curvature of a Kählerian manifold of constant holomorphic curvature 1 equals $\frac{1}{4}(1 + 3\cos^2\alpha(\sigma))$. In the theory of manifolds of positive curvature, results whose form is similar to those in Sec. 2.1 of Chap. 3 were obtained. A Kählerian manifold M is said to be *δ-clamped* if it is δ-clamped as a Riemannian manifold, and M is said to be *Kählerian δ-clamped* if, after the corresponding normalization of the metrics, the inequality $\delta \le K^*(\sigma) \le 1$ holds for all planes σ. If this inequality holds for all planes invariant with respect to J, then M is said to be *holomorphically δ-clamped*.

The connections between various concepts of clamping are reflected in the following theorem.

Theorem 3.7.1. *A complete Kählerian manifold M of complex dimension n is holomorphically isometric to the complex projective space $\mathbb{C}P^n$ equipped with the Fubini–Study metric iff one of the following three conditions holds*:

(a) *the manifold M is $\frac{1}{4}$-clamped*;
(b) *M is holomorphically 1-clamped*;
(c) *M is Kählerian 1-clamped.*

The clamping condition imposes various restrictions on the topology as in the case of real Riemannian manifolds. For example, a complete holomorphically δ-clamped Kählerian manifold with $\delta > 0$ is compact and simply connected. In fact, for a manifold to be simply connected, we need a weaker condition than the positivity of the curvature. We have the following complex analog of the Meyers theorem.

Theorem 3.7.2. *A compact Kählerian manifold M with positive Ricci tensor is simply connected.*

With accuracy up to a homotopy equivalence, the projective space $\mathbb{C}P^n$ can be characterized by using the language of δ-clamping. In the following two theorems, this is done for two other types of clamping.

Theorem 3.7.3. *A compact n-dimensional Kählerian manifold of Kählerian clamping $\delta > 9/16$ has the same homotopy type as the complex projective space $\mathbb{C}P^n$.*

Theorem 3.7.4. *A compact n-dimensional Kählerian manifold M of holomorphic δ-clamping has the same homotopy type as the complex projective space $\mathbb{C}P^n$ for $\delta > 4/5$.*

To obtain an isometry of a given Kählerian manifold to the projective complex space, along with the condition of δ-clamping we need to impose additional conditions. As an example of such a type, we present the following theorem.

Theorem 3.7.5. *A compact Kählerian manifold of dimension n and constant sectional curvature is holomorphically isometric to the projective space $\mathbb{C}P^n$ equipped with the Fubini–Study metric.*

4. Pseudo-Riemannian Manifolds

4.1. Pseudo-Euclidean spaces. In Sec. 1 of Chap. 1, we have presented the definition of a pseudo-Riemannian manifold (M, g_{ij}). If we restrict the metric tensor g_{ij} to the tangent space T_xM, then we obtain the so-called pseudo-Euclidean space on it, i.e., a linear space on which a nondegenerate symmetric inner product is given. We consider the simplest example of a pseudo-Euclidean space. We associate the space $\mathbb{R}^n$ with Cartesian coordinates $x^1, \dots, x^n$ and endow it with an additional structure by assigning the metric $-\sum\limits_{i=1}^{s}(dx^i)^2 + \sum\limits_{j=s+1}^{n}(dx^j)^2$. The symmetric indefinite nondegenerate inner product

$$(a, b)_s = -\sum_{i=1}^{s} a^i b^i + \sum_{j=s+1}^{n} a^j b^j,$$

where a, b are vectors from $\mathbb{R}^n$, corresponds to this metric. A linear space equipped with such a metric is said to be *pseudo-Euclidean of index s* and is denoted by $\mathbb{R}^n_s$. As in $\mathbb{R}^n_0$, the length of a vector a in $\mathbb{R}^n_s$ is defined by $|a|_s = \sqrt{(a, a)_s}$. Since the form $(X, Y)_s$ is not positive definite, the set of all vectors a in $\mathbb{R}^n_s$ emanating, e.g., from the origin, is divided into the following three disjoint subsets:

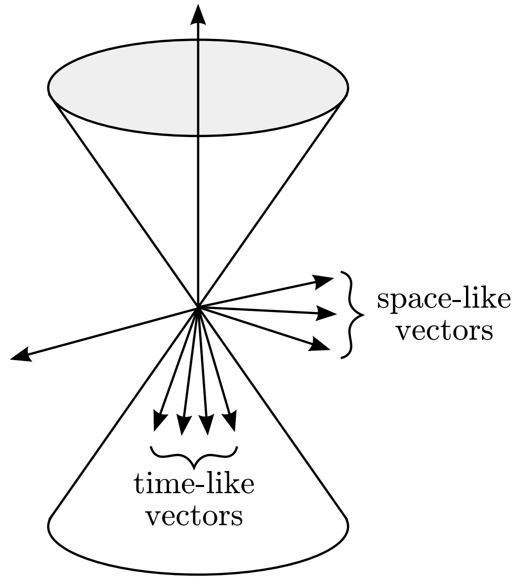


Fig. 54

(1) *time-like* vectors for which $(a,a)_s < 0$,
(2) *light or isotropic* vectors for which $(a,a)_s = 0$,
(3) *space-like vectors* for which $(a,a)_s > 0$ (see Fig. 54).

These vectors have imaginary, zero, or real length, respectively. The concepts indicated above originate from special relativity theory in which the pseudo-Euclidean space $\mathbb{R}^4_s$ of index 1 is called the *Minkowski space* (see [48, 117, 161]). The existence of three types of vectors in $\mathbb{R}^n_s$, which are considerably different in their properties, defines a geometry more rich in content as compared with Euclidean geometry in a certain sense. At each point of $\mathbb{R}^n_s$, the set of isotropic vectors emanating from it form the cone given by the equation

$$-\sum_{i=1}^{s}(a^i)^2 + \sum_{j=s+1}^{n}(a^j)^2 = 0;$$

this cone is said to be *isotropic* or *light*. In the case of the Minkowski space, a light ray emanating from the origin goes along one of the rulings of the isotropic cone if as the coordinate a^1, we choose the parameter ct, where c is the speed of light and t is time.

Pseudo-Euclidean geometry is closely related to Riemannian geometry; this can be illustrated by examining $\mathbb{R}^3_1$ and the Lobachevskii plane. The pseudo-sphere S^{n-1}_s of index s in the space $\mathbb{R}^n_s$ is the set of points that are distant from a fixed point (for example, the origin) by a fixed distance ρ. In this case, ρ can be a real number, a purely imaginary number, or zero. The pseudo-sphere of zero radius coincides with the isotropic cone. We consider the example where $n = 2$ and $s = 1$. The isotropic cone with vertex at the origin consists of two lines $x = \pm y$, where x and y are Cartesian coordinates on the plane. Pseudo-circles of real radius are hyperbolas $-x^2 + y^2 = \alpha^2$, where α is a real number. Pseudo-circles of imaginary radius are hyperbolas $-x^2 + y^2 = -\alpha^2$. Let $n = 3$, and let $s = 1$. The isotropic cone, i.e., the pseudo-sphere of zero radius, is the usual second-order cone $-x^2 + y^2 + z^2 = 0$. Pseudo-spheres of real radius are hyperboloids of one sheet $-x^2 + y^2 + z^2 = \alpha^2$. Pseudo-spheres of imaginary radius are two-sheeted hyperboloids $-x^2+y^2+z^2 = -\alpha^2$. We consider a metric on the pseudo-sphere of purely imaginary radius that is induced by the ambient pseudo-Euclidean metric. For this purpose, it is convenient to use the stereographic projection as in the case of the ordinary sphere (see Fig. 55).

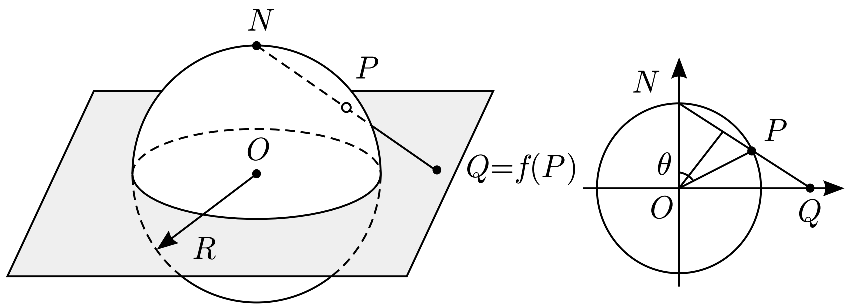


Fig. 55

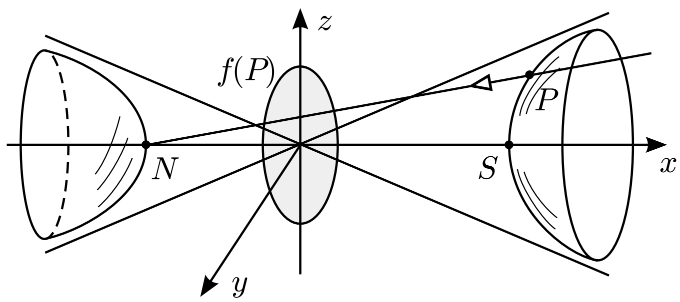


Fig. 56

As the center of the pseudo-sphere, we take the origin, as the north pole N, the point with the Cartesian coordinates $(-\alpha, 0, 0)$, and as the south pole S, the point with the coordinates $(\alpha, 0, 0)$. As the plane on which the pseudo-sphere is projected, we take the plane YOZ. We consider a variable point P on the right sheet of the two-sheeted hyperboloid and join it with the north pole N by a segment. This segment intersects the plane YOZ at a certain point, which is denoted by $f(P)$ and is called the image of the point P under the stereographic projection $f : S_1^2 \to \mathbb{R}^2$ (see Fig. 56). In Fig. 57, we show the section of the pseudo-sphere of imaginary radius by the plane XOZ. The image of the right sheet of the hyperboloid covers the interior of the disk $y^2 + z^2 < \alpha^2$ of radius α in the plane YOZ. The image of the left sheet of the hyperboloid covers the complement to the closure of this disk. The isotropic cone is projected on the boundary of the disk, i.e., on the circle $y^2 + z^2 = \alpha^2$. Let u^1, u^2 be Cartesian coordinates on the disk D^2 of radius α in the plane YOZ. Then it is easy to state the relation between the coordinates x, y, z of the point $P \in S_1^2$ and the coordinates u^1, u^2 of the point $f(P)$ on the disk D^2. Namely,

$$x = \alpha \frac{|u|^2 + \alpha^2}{\alpha^2 - |u|^2}, \quad y = \frac{2\alpha^2 u^1}{\alpha^2 - |u|^2}, \quad z = \frac{2\alpha^2 u^2}{\alpha^2 - |u|^2},$$

where $u = (u^1, u^2)$ and $|u|^2 = (u^1)^2 + (u^2)^2$. It is easily verified that the coordinates on the disk D^2 assign a regular coordinate system on the right sheet of the hyperboloid, i.e., the stereographic projection $f : S_1^2 \to \mathbb{R}^2$ described above assigns a regular coordinate system on the pseudo-sphere of imaginary radius (for the left sheet, the arguments are similar). It is easy to compute an explicit form of the metric induced on the pseudo-sphere by its embedding in the space $\mathbb{R}_1^3$. In coordinates u^1, u^2, this metric becomes $4\frac{(du^1)^2 + (du^2)^2}{(\alpha^2 - |u|^2)^2}$. In particular, the induced metric turns out to be Riemannian (i.e., positive definite), although the ambient metric is indefinite. The open disk of radius α with the metric indicated above is called the Poincaré model of the Lobachevskii geometry (see [48, 53]). In polar coordinates (for $\alpha = 1$),

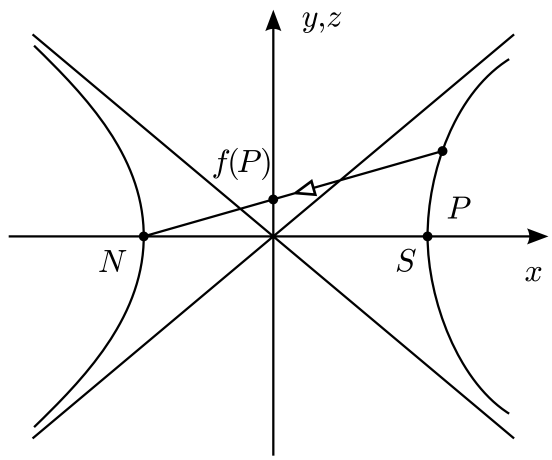


Fig. 57

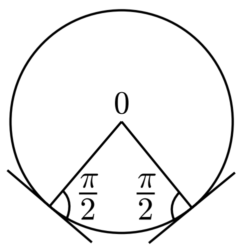


Fig. 58

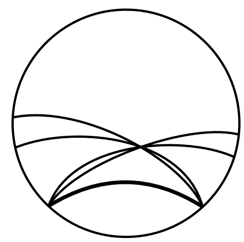


Fig. 59

the Lobachevskii metric becomes $4\dfrac{dr^2 + r^2 d\varphi^2}{(1 - r^2)^2}$. In some cases, it is convenient to use this metric in the pseudo-spherical coordinates that are defined similarly to the spherical coordinates but replacing the usual trigonometric functions by hyperbolic functions. Replacing the coordinates r, φ by new coordinates χ, φ according to the formulas $r = \coth \dfrac{\chi}{2}$ and $\varphi = \varphi$, we obtain $d\chi^2 + \sinh^2 \chi d\varphi^2$.

Sometimes, it is convenient to represent the properties of the Lobachevskii metric in terms of "points" and "lines" on the Poincaré model. We consider a plane section of a pseudo-sphere of imaginary radius, i.e., the two-sheeted hyperboloid, by planes in $\mathbb{R}^3_1$ passing through the origin, i.e., the center of the pseudo-sphere. We consider the images of these plane sections under the stereographic projection $f : S^2_1 \to \mathbb{R}^2$. It turns out that each line of intersection of the pseudo-sphere with a plane of the form $ax+by+cz = 0$ passes to an arc of the circle on the Poincaré model under the mapping f, and, moreover, this arc intersects the circle $y^2 + z^2 = \alpha^2$ by a right angle (see Fig. 58). Therefore, the properties of the Lobachevskii plane (geometry) can be studied considering the open disk of radius α as the set of points of Lobachevskii geometry; moreover, as "lines," one should take arcs of circles intersecting the boundary of the disk $y^2 + z^2 = \alpha^2$ (it is called the absolute) by a right angle. In particular, "lines" are all diameters of the disk (circles of infinite radius). We see from Fig. 59 in which form the "postulate on parallel lines" holds in Lobachevskii geometry. There are an infinite set of lines passing through each point located outside a line that do not intersect it, i.e., lines parallel to it. If we let the parameter α tend to infinity, then in any finite domain on the Poincaré model, the Lobachevskii geometry "tends" to the Euclidean geometry, since the arcs of circles become straightened and transform into Euclidean lines.

4.2. Einstein manifolds. A pseudo-Riemannian (and also Riemannian) manifold (M^n, g_{ij}) is called an *Einstein space* if it satisfies the relation $R_{ij} = \rho(x) g_{ij}(x)$, where R_{ij} is the Ricci tensor of the space M^n. Compressing this relation with the tensor g^{ij} defined by $g^{ij} g_{jk} = \delta^i_k$, we obtain $\rho = \frac{R(x)}{n}$, where $R(x) = g^{\alpha\beta} R_{\alpha\beta}$ is the scalar curvature of the space M. Therefore, in an Einstein space, we have the equation

$$R_{ij}(x) = \frac{R(x)}{n} g_{ij}(x),$$

which is called the *Einstein equation.*

As an example of an Einstein space, we consider the four-dimensional space $\mathbb{R}^4$ with coordinates t, r, θ, φ and metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2).$$

The geometry of this space is called the *Scwarzschild geometry.* The constant M is called the mass of the field source. As one more example, we present the so-called *Kerr–Newman black hole.* In this case, the metric has the form

$$ds^2 = -\left(1 - \frac{2Mr - G^2}{\Sigma}\right) dt^2 - \frac{(2Mr - \theta^2)2a\sin^2\theta}{\Sigma} dt d\varphi$$
$$+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{(2Mr - \theta^2)a^2 \sin\theta}{\Sigma}\right) \sin^2\theta d\varphi^2,$$

and in the coordinates t, φ, r, θ, where $a^2 + \theta^2 \leq M^2$, M is called the mass, a is the motion momentum per mass unit, and θ the charge, $\Delta = r^2 - 2Mr + a^2 + \theta^2$, $\Sigma = r^2 + a^2 \cos^2\theta$. A large number of examples of Einstein space can be found in the work [154] of A. Z. Petrov. Also, an extensive bibliography can be found therein.

We begin our study of general properties of Einstein manifolds with examples of manifolds on which it is not possible, in principle, to assign an Einstein metric. On the product $S^1 \times S^3$ and on the connected sum $n\mathbb{C}P^2 = \mathbb{C}P^2 \# \cdots \# \mathbb{C}P^2$ (n copies of the space $\mathbb{C}P^2$), there is no Einstein metric for $n \geq 4$ (see [96]).

The topological restrictions that are imposed on manifolds equipped with an Einstein metric are presented in the following Hitchin theorem.

Theorem 4.2.1 ([96]). *Let M be a compact four-dimensional Einstein manifold of signature τ and Euler characteristic χ. Then we have the inequality $|\tau| \leq \frac{2}{3}\chi$. Moreover, if the equality holds in it, then M is either a flat manifold, or a $K3$ surface ($\pi_1(M) = e$), or an Enriques surface ($\pi_1(M) = \mathbb{Z}_2$), or a quotient Enriques surface with respect to a free antiholomorphic involution ($\pi_1(M) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$).*

We recall that a $K3$ surface is a complex surface with first nonzero Betti number $b_1 = 0$ and first nonzero Chern number $c_1 = 0$; an Enriques surface is a complex surface with $b_1 = 0$ and $2c_1 = 0$.

The following examples of four-dimensional manifolds admitting an Einstein metric are known:

(a) flat Riemannian manifolds;
(b) the complex symmetric spaces S^4, $S^2 \times S^2$, and $\mathbb{C}P^2$;
(c) manifolds whose universal covering is the corresponding noncompact symmetric space (see [24]).

If we impose additional geometric restrictions on the space M, then the estimate of Theorem 2.2.1 can be improved. As an example of such a type, we mention the following inequality.

Theorem 4.2.2 ([97]). *Let M be a compact four-dimensional Einstein manifold with nonnegative* (*or positive*) *sectional curvature. Then $|\tau| \leq (\frac{2}{3})^{3/2}\chi$. Since $(\frac{2}{3})^{3/2}$ is irrational, the equality is possible here only in the case of flat manifolds M.*

This result of Hitchin was recently improved by Kobayashi; moreover, he used the invariant $\nu(g)$ considered in Sec. 4.8 of Chap. 3.

Theorem 4.2.3. *Let M be a compact four-dimensional manifold. If M admits an Einstein metric, then the inequality $\nu(M) \le 16\pi^2\chi$ holds. If M admits an Einstein metric of nonnegative sectional curvature, then $\nu(M) \le \frac{64}{5}\pi^2\chi$.*

As a consequence, we obtain that if a compact orientable four-dimensional manifold M admits an Einstein metric of nonnegative sectional curvature, then $|\tau| \le \dfrac{8\chi}{15}$ and the equality holds here iff M has a flat metric.

4.3. Local immersions of pseudo-Riemannian manifolds. Similar to Riemannian manifolds, there arises the problem on the embeddability of an arbitrary pseudo-Riemannian manifold into a certain pseudo-Euclidean space. Let $M^n(p,q)$ be a pseudo-Riemannian manifold endowed with a metric of the form

$$dx_1^2 + \ldots + dx_p^2 - dy_1^2 - \ldots - dy_q^2, \quad p + q = n.$$

Theorem 4.3.1 ([62])**.** *Any pseudo-Riemannian manifold $M^n(p,q)$ with analytic metric admits an analytic isometric local immersion into the space $\mathbb{R}^m_s$, where $m = \frac{1}{2}n(n+1)$ and $m-s$ and s are arbitrarily given numbers satisfying the conditions $m - s \ge p$ and $s \ge q$.*

Let k_0 be the least nonnegative integer such that $M^n(p,q)$ admits a local immersion in the space $\mathbb{R}^{n+k_0}(p, q+k_0)$. For each k, $0 \le k \le k_0$, we define the kth immersion class of $M^n(p,q)$ as the least number N_k such that $M^n(p,q)$ admits a local isometric immersion in the space $\mathbb{R}^{n+N_k}(q + q_k, q + k)$, where $a_k + k = N_k$. The *immersion class* of $M^n(p,q)$ is $\min N_k$ for $0 \le k \le k_0$.

According to Theorem 4.3.1, $N_k \le \frac{1}{2}n(n+1)$ for all k. The main problem consists of finding, for a given manifold $M^n(p,q)$, the number N_k. To determine whether the relation $N_k = 0$ holds or not, it suffices to verify whether the Riemannian curvature tensor vanishes identically. In [171, 192], there is a sufficiently ambiguous criterion for determination of whether the immersion class equals 1 or not. We present an example of the simplest necessary conditions for validity of the relation $N_k = 1$.

Theorem 4.3.2. *If the Ricci tensor of the manifold $M^n(p,q)$ vanishes, then $N_k \ne 1$.*

4.4. Geodesics on a pseudo-Riemannian manifold. A curve $x^k = x^k(s)$ satisfying the differential equations

$$\frac{d^2x^2}{ds^2} + \Gamma^s_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0, \qquad \Gamma^m_{kl} = \frac{1}{2}\left(-\frac{\partial g_{kl}}{\partial x^m} + \frac{\partial g_{ml}}{\partial x^k} + \frac{\partial g_{km}}{\partial x^l}\right)$$

is called a geodesic as above, and the parameter s is said to be natural. The quantity $\Phi = g_{jk}\dot{x}^j\dot{x}^k$ is constant along a geodesic, and the following three cases are possible: if $\Phi > 0$, then the geodesic is said to be *space-like*, if $\Phi = 0$, then this geodesic is said to be *zero*, and if $\Phi < 0$, then the geodesic is said to be *time-like*. Since Φ is a quadratic function, this classification does not depend on the choice of a natural parameter. The parameter s can be chosen such that $\Phi = 1$ in the first case and $\Phi = -1$ in the second case; in these cases s is respectively called the distance and the proper time along the geodesic considered. Geodesics play an important role in general relativity theory.

A *nonremovable singularity* (*singularity*) of a pseudo-Riemannian manifold is a certain point (or a set points) at (or on) which a certain invariant of curvature (i.e., a certain scalar value constructed according to the functions g_{ij} and their derivatives of various order) vanishes. It is not possible to remove singularities of such a type by a change of coordinates, since the scalar value indicated above, being an invariant, tends to infinity in any coordinate system when it approaches this set of points.

A pseudo-Riemannian manifold is said to be *geodesically complete* if a geodesic emanating from any point in the initial direction given by any tangent vector either can be prolonged in the manifold for arbitrary large values of the natural parameter or meets a nonremovable singularity for a certain finite value of it.

The general problem of expanding Einstein manifolds is far from solution. Almost nothing is known on the existence or uniqueness of geodesically complete expansions. It is possible to construct simple examples in which a geodesically complete expansion does not exist or is not unique.

4.5. Dirac operator on pseudo-Riemannian spaces. The Dirac operator of a pseudo-Riemannian manifold is a generalization of the classical Dirac equation of relativistic quantum mechanics. The definition of the Dirac operator is given in Sec. 4.2 of Chap. 3. The properties of the Dirac operator of compact Riemannian manifolds listed there are not extended to pseudo-Riemannian manifolds. We indicate the following self-adjointness property of the Dirac operator. Denote by D^* the operator formally adjoint to D.

Theorem 4.5.1 (Baum). *Let (M^n_k, g_{ij}) be an oriented spinor manifold, $\xi^k \subset TM$ be a time-like k-dimensional subbundle, and $g(\xi)$ be a Riemannian metric on M that is naturally defined by g and ξ. If the Riemannian manifold $(M, g(\xi))$ is complete, then the operators $\operatorname{Re}(D) = \frac{1}{2}(D+D^*)$ and $\operatorname{Im} D = \frac{i}{2}(D^*-D)$ are essentially self-adjoint in the Hilbert space of all square-integrable spinor fields.*

This results generalizes the essential self-adjointness of the free Hamilton operator of the classical Dirac operator of relativistic quantum mechanics.

On compact pseudo-Riemannian manifolds, there exist (real) complex eigenvalues and eigenvalues of infinite multiplicity.

Example. We consider the sphere $S^3 = \mathrm{SU}(r)$ with the following properties of left-invariant metrics $g_{(\lambda,\mu)}$, $\lambda, \mu \in \mathbb{R}^+$, of index 2. Let

$$X_1 = \frac{\sqrt{2}}{4}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad X_2 = \frac{\sqrt{2}}{4}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad X_3 = \frac{\sqrt{2}}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

be matrices orthogonal with respect to $g_{(\lambda,\mu)}$; we set

$$g_{(\lambda,\mu)}(X_1, X_1) = g_{(\lambda,\mu)}(X_2, X_2) = -\lambda^2, \quad g_{(\mu,\lambda)}(X_3, X_3) = \mu^2.$$

The eigenvalues of the operator D on $(S^3, g_{\lambda\mu})$ are described as follows:

$$\alpha_m = -\frac{\sqrt{2}}{8}\frac{\mu}{\lambda^2} + \frac{\sqrt{2}}{4\mu}(m+1)$$

have the multiplicity $2(m+1)$, and

$$\alpha^{\pm}_{r,l} = -\frac{\sqrt{2}}{8}\frac{\mu}{\lambda^2} \pm \frac{\sqrt{2}}{4\mu}\sqrt{(r-l)^2 - 4rl\frac{\mu^2}{\lambda^2}}$$

have the multiplicities $r+l$, where $m = 0, 1, 2, \dots, r$ and $l = 1, 2, \dots$.

Example. Let T^3 be the three-dimensional torus, and let g_λ, where $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{R}^+)^3$, stand for the left-invariant metric defined by $g_\lambda(X_l, X_j) = \varkappa(j)\delta_{lj}\lambda_j^2$, $\varkappa(j) = -1$ for $j = 1$ and $\varkappa(j) = 1$ for $j = 2, 3$. For $a = (a_1, a_2, a_3) \in \mathbb{Z}_2^3 = H^1(T^3, \mathbb{Z}_2)$, we denote by D_a the Dirac operator of the manifold (T^3, g_λ) with respect to the spinor structure corresponding to a. Then D has the eigenvalues

$$\alpha^{\pm}(m) = \pm\frac{\pi}{2}\sqrt{-\left(\frac{4m_1+1-a_1}{\lambda_1}\right)^2 + \left(\frac{4m_2+1-a_2}{\lambda_2}\right)^2 + \left(\frac{4m_3+1-a_3}{\lambda_3}\right)^2},$$

$m = (m_1, m_2, m_3) \in \mathbb{Z}^3$. In particular, the dimension of harmonic spinors depends on the metric and the chosen spin-structure. It can be equal to zero, a finite number, or infinity for the same metric.

5. Classical Mechanics from the Viewpoint of Riemannian Geometry

5.1. Kinematic line element. In mechanics, one usually describes the motion of a mechanical system by using a set of parameters that assigns the position of the system and velocities of its parts. The set of all positions of a mechanical system is called the *configuration space*. As a rule, it is a smooth manifold. The dimension of the configuration space is called the number of degrees of freedom. The set of all positions of a mechanical system, together with their velocities, is called the *state space*. The state space is identified with the tangent bundle TM of the configuration space M. Local coordinate systems $(q^1, \dots, q^n)$ on the manifold M^n are usually called *generalized coordinates* of this mechanical system (see [190]).

Example. The configuration space of the "spherical" mathematical pendulum is the two-dimensional sphere S^2. As generalized coordinates, we can take the usual spherical coordinates.

If a mechanical system moves in a certain way, then its generalized coordinates are functions of time. Therefore, the motion is determined by equations of the form $q^i = q^i(t)$. The corresponding curve in the configuration space is called a *trajectory* of this mechanical system. The object $\dot{q}^i$, which is a vector with respect to coordinate transformations, is called the *generalized velocity*. The *kinetic energy* T of a system, being written in generalized coordinates q^i, is a positive-definite form with respect to $\dot{q}^i$ with coefficients depending on $\dot{q}^i$: $T = \frac{1}{2} a_{ik}(q^1, \dots, q^n)\dot{q}^i\dot{q}^k$. The collection of functions a_{ik} forms a tensor field of type (0,2) on the configuration space M.

Definition 5.1.1. The Riemannian metric $ds^2 = a_{ij} dq^i dq^j$ on the configuration space M of a given mechanical system is called the *kinematic linear element*.

The generalized *acceleration* f^2 is the covariant derivative in t of the velocity vector $\dot{q}^i$. We have the following explicit expression for f^2:

$$f^2 = \ddot{q}^i + \Gamma^2_{st}\dot{q}^s\dot{q}^t,$$

where Γ^i_{jk} is an affine connection compatible with the kinematic linear element. It is easy to see that the components $f_r = a_{rs} f^s$ have the form

$$f_r = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r}.$$

The *force field* on the configuration space is usually the differential form ω of degree 1. Its integral over a certain curve on the configuration space M is equal to the work that is done under the motion of the system along this curve. We recall that the evolution of a mechanical system is represented in this language by the motion of a point along a curve in the configuration space. If the force field is conservative, then $\omega = -dU$, where U is the *potential energy*. We consider the vector field X_ω canonically corresponding to the 1-form ω. This field is uniquely defined by the relation $2\langle X, Y \rangle = \omega(Y)$, which should hold for any vector field Y on M. In this notation, the *Newton law* of motion is written as $\nabla_{\dot{\gamma}}(\dot{\gamma}) = X_\omega$, where $\gamma(t)$ is a trajectory on M, $\dot{\gamma}$ is the vector field of velocities of the trajectory γ, and ∇ is the covariant derivative defined by the Riemannian connection. In particular, a trajectory of a free motion of a mechanical system (i.e., in the case where the force field vanishes) is a geodesic, and, moreover, time along the trajectory is proportional to the length of this geodesic, i.e., to the natural parameter. The Riemannian manifold described above is geodesically complete if after any shock, the motion of the system is unboundedly long. In this case, we can direct the mechanical system from any initial position x with unit kinetic energy so that under the free motion, it attains any configuration y (i.e., a point of the manifold M) given in advance at time equal to the distance between these points x and y in the manifold M.

Therefore, if the form ω has coordinates Q_r, then the equation of motion of our mechanical system is

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^r} - \frac{\partial T}{\partial q^r} = Q_r.$$

5.2. Linear action element. One can associate mechanics with Riemannian geometry in an essentially different way. In the case of conservative forces, on trajectories of a mechanical system, we have the energy conservation law $T + U = h$, where h is the total mechanical energy, T is the kinetic energy, and U is the potential energy. In this subsection, we consider only those trajectories for which the complete energy is equal to a constant h.

Definition 5.2.1. The Riemannian metric $ds^2 = 2(h - U)a_{rs}dq^r dq^s$ is called the linear action element.

Theorem 5.2.1. *Trajectories of a mechanical system with a given total energy h are geodesics of the configuration space if, as a metric in this space, we take the linear action element.*

The proof of this theorem can be found in any course on theoretical mechanics.

5.3. Brachistochrones. Trajectories such that the time of motion along them is stationary are called brachistochrones. They are trajectories on which the integral

$$t = \int_A^B \sqrt{\frac{a_{mn}\dot{q}^m \dot{q}^n}{2(h - U)}} ds$$

assumes a stationary value.

Theorem 5.3.1. *Brachistochrones of a given system coincide with geodesics in the configuration space if the linear element is defined by* $ds^2 = \dfrac{a_{mn}dq^m dq^n}{(h - U)}$.

5.4. Geometry of the configuration space of a rigid body rotating around a fixed point. A rigid body rotating around a fixed point is the most well known and important mechanical system (see [10, 190]). The space of positions of this system is the three-dimensional projective space $\mathbb{R}P^3$ (see [10]). The kinetic energy of a rigid body assigns a Riemannian metric on this space. The class of Riemannian manifolds thus obtained admits a geometric description in terms of the Ricci curvature.

Theorem 5.4.1 ([206]). *A Riemannian manifold is the configuration space of a rigid body rotating around a fixed point iff the roots of the characteristic equation of the Ricci tensor are constant and negative and the principal directions form a geodesic net.*

Let r_1, r_2, and r_3 be roots of the characteristic equation of the Ricci tensor. Then the principal inertia moments I_i, $1 \le i \le 3$, of a body for which this space is the configuration space are found from the relations

$$I_i = -\frac{2r_i}{r_i^2 + r_1 r_2 + r_1 r_3 + r_2 r_3}.$$

6. Yang–Mills Connections

6.1. Geometry and physics of Yang–Mills fields. The Maxwell equations describe an electromagnetic interaction. The Yang–Mills equations describe a class of interactions between particles, which are called weak interactions, and arise, for example, in certain process of radioactive decay. The Yang–Mills equations are nonlinear generalization of the Maxwell equations. A natural geometric language for description of these fields is given by fiber bundle theory. With a Yang–Mills field, one associates the concept of connection in a principal bundle over a Riemannian manifold.

One says that a smooth manifold M is a *principal fiber manifold* if

(1) a Lie group G acts smoothly on M, and, moreover, it has no fixed points on M;
(2) the quotient space $B = M/G$ is a smooth manifold, and, moreover, the canonical projection $p : M \to B$ is a smooth mapping;

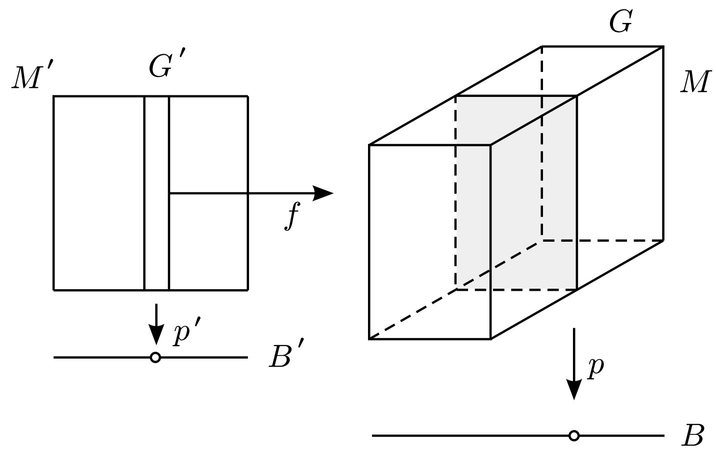


Fig. 60

(3) the bundle $p : M \to B$ is locally trivial, i.e., for any point $b \in B$, there exists a neighborhood U such that its full inverse image $p^{-1}(U)$ is homeomorphic to the direct product $U \times G$, and, moreover, the mapping $x \to (p(x), h(x)) \in U \times G$, where $x \in p^{-1}(U)$, is a diffeomorphism and $h(g)x)) = g(h(x))$ for any $g \in G$ (see [22, 48, 113, 120, 178, 189]).

The manifold B is called the *base* of the bundle and G the *fiber*, and at the same time, the *structural group*. On each fiber manifold, we define *fundamental vector fields*. Since G acts on M, there exists a homeomorphism α of the Lie algebra $\mathfrak{G}$ of the Lie group G into the Lie algebra $V(M)$ on smooth vector fields on M that is defined by

$$\alpha(X)_p = \left.\frac{d}{dt}\right|_{t=0} (\exp tX)p.$$

The image $\alpha(X)$ of an element $X \in G$ is called the fundamental vector field on the manifold M corresponding to X. Since G has no fixed points, α is an isomorphism of the Lie algebra $\mathfrak{G}$ onto the subalgebra $\alpha(\mathfrak{G})$ in $V(M)$. We consider two principal bundles $p : M \xrightarrow{G} B$ and $p' : M' \xrightarrow{G'} B$ having the same base B but, in general, distinct structural groups G and G'.

Definition 6.1.1. A smooth mapping $f : M' \to M$ is called a *homomorphism* of principal bundles p' and p if there is a fixed homomorphism f of the group G' into the group G such that $f(g'(x')) = f(g')(f(x'))$ for any $x' \in M'$ and $g \in G'$ (see Fig. 60) and the self-mapping of f of the base B induced by the homomorphism $f : G' \to G$ is smooth. If f is a one-to-one mapping, then it is called an isomorphism of principal bundles. If G' is a subgroup of G, then the principal bundle homomorphism $f : M' \to M$ corresponding to an embedding of G' in G is called an *embedding of the bundle* $p' : M' \to B$ in the bundle $p : M \to B$ if the mapping f induced by it is the identical transformation of the base B.

Definition 6.1.2. Let $p : M \xrightarrow{G} B$ be a principal bundle. One says that its structural group G is reducible to a Lie subgroup G if there exists a principal bundle $p' : M' \xrightarrow{G'} B$ with the same base B and the structural group G' such that it admits an embedding in the bundle $p : M \xrightarrow{G} B$. Then M', together with the mapping f, is called a reduced fiber manifold.

A principal bundle can be given by using the so-called gluing functions. Let the base B be covered by open domains U_i whose full inverse images (under the projection p) are equipped with the structural group of a direct product by using diffeomorphisms $\varphi_i : U_i \times F \to p^{-1}(U_i)$ such that $p(\varphi_i(b, e)) = e$ for $b \in U_i$ and $e \in F$, where F is a fiber of the mapping $p : F = p^{-1}(x)$. The transformations $\lambda_{ij} = \varphi_j^{-1}\varphi_i : U_{ij} \times F \to U_{ij} \times F$, where $U_{ij} = U_i \cap U_j$, are called *gluing functions* (or *transition functions*) of the bundle.

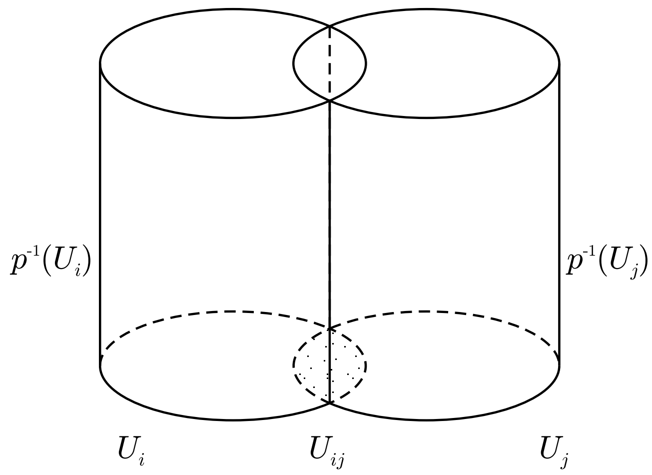


Fig. 61

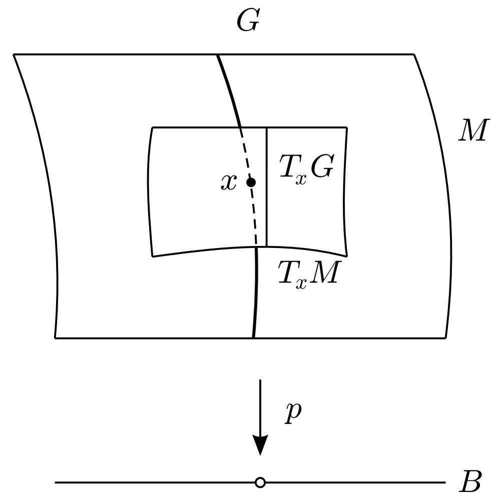


Fig. 62

They can be written as follows: $\tilde{\lambda}_{ij}(b, e) = (b, \omega_{ij}(b)e)$. It is required that for any i, j, and $b \in B$ the transformation $\omega_{ij}(b) : F \to F$ be an element of the group G (which was already fixed). Therefore, the gluing functions ω_{ij} define smooth mappings of the domain U_{ij} into the group G, i.e., $\omega_{ij} : U_{ij} \to G$, where $b \to \omega_{ij}(b)$ (see Fig. 61). It follows from the definition that $\omega_{ij} = \omega_{ji}^{-1}$ and $\omega_{ij}\omega_{jk}\omega_{ki} = \omega_{ii} = 1$. Here 1 stands for the identity transformation. The latter relation holds on the intersection $U_{ijk} = U_i \cap U_j \cap U_k$ of three domains in B. The domains U_i are called coordinate neighborhoods of the bundle. The geometrical sense of the reducibility of principal bundles is explained by the following statement. The structural group G of the principal fiber manifold $p : M \xrightarrow{G} B$ is reducible to a subgroup G' iff there exists a covering of the base B by coordinate neighborhoods U_i such that the transition functions ω_{ij} assume their values in the subgroup $G' \subset G$.

We define the concept of an associated fiber manifold. Let there be a principal bundle $p : M \xrightarrow{G} B$ and a smooth manifold F on which the group G acts on the right, i.e., $(g, h) \to hg$, where $g \in G$, $h \in F$. We consider the direct product $M \times F$; on it, we can define a left action of the group G as follows: $(x, h) \to (x, h)g = (g(x), hg^{-1})$, where $(x, h) \in M_xF$, $g \in G$. We consider the quotient space $E = (M \times F)/G$; let $p : M \to B$ be the initial projection. We construct a mapping that transforms a point of E corresponding to the class $(x, h) \in M \times F$ into the point $p(x) \in B$. Obviously, this mapping induces a projection λ of the space E on the base B. The fiber of this projection over a point $b \in B$ is the set of points of the space E that correspond to the class (x, h), where x is any point of M for which $p(x) = b$ and h runs over the whole fiber F. The manifold E is endowed with the structure of a smooth manifold, and, moreover, the projection $\lambda : E \xrightarrow{F} B$ turns out to be smooth. Therefore, we have constructed the bundle $\lambda : E \xrightarrow{F} B$ with the structural group G, which is called the bundle associated with the initial fiber space $p : M \xrightarrow{G} B$. Roughly speaking, we "change the fiber": instead of the group G, there arises the space F on which the right action of the Lie group G is defined.

We consider an arbitrary principal fiber manifold M with base B and structural group G. Let T_mM be the tangent space to the manifold M at the point $m \in M$, and let T_mG be the subspace T_mM tangent to the fiber passing through the point m (see Fig. 62).

Definition 6.1.3. A *connection* Γ in the fiber manifold M is a correspondence that to each point $x \in M$ sets in correspondence a certain tangent subspace Q_x in T_xM for which the following conditions hold:

(1) T_xM is the direct sum of the subspaces Q_x and T_xG;

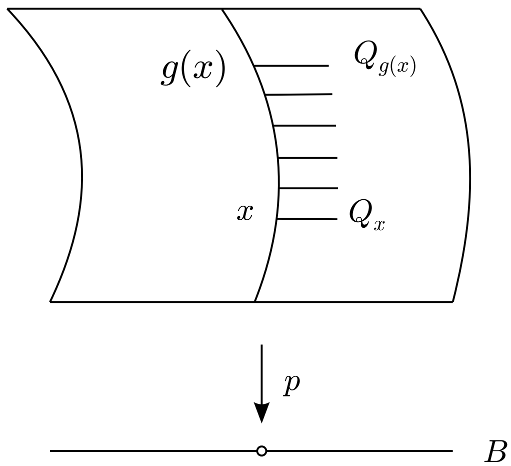


Fig. 63

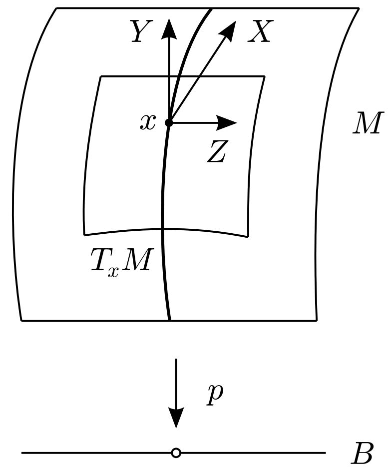


Fig. 64

(2) for any $x \in M$ and $g \in G$, the subspace $Q_{g(x)}$ is the image of the subspace Q_x under the mapping induced by the left translation $L_g(a) = ga$ (see Fig. 63);
(3) the subspace Q_x smoothly depends on the point x.

Let X be an arbitrary vector field on M. Then at each point $m \in M$, there arises a unique decomposition of the vector X_m into two components $X_m = Y_m + Z_m$, where $Y_m \in T_xG$ and $Z_x \in Q_x$ (see Fig. 64). The vector field Y is called the *vertical component* of the field X, and the vector field Z is called the *horizontal component* of the field X. For a given connection, the subspace Q_x is usually called the horizontal subspace.

For each connection Γ in a principal fiber manifold, we can define a certain 1-form ω that assumes its values in the Lie algebra $\mathfrak{G}$ of the group G. Let $x \in M$ be an arbitrary point, and let $T_xM = Q_x + T_xG$ be the decomposition of the tangent space generated by the connection Γ. Since the fiber G passing through the point x is homeomorphic to the group G and the horizontal subspaces $\{Q_{g(x)}\}$ pass to one another under the action of the group G by left translations on itself, the tangent space T_xG coincides with the set of tangent vectors of the form $\alpha(h)_x$, where $h \in \mathfrak{G}$ and $\alpha(h)$ is the fundamental vector field corresponding to the element $h \in \mathfrak{G}$. The *connection form* ω_x at each point x is the linear mapping defined on the set of all vectors tangent to M at the point x that to a vector $X \in T_xM$ sets in correspondence the vector $\omega_x(vX) \in T_xG$, which is the vertical component of the field X at the point x, and at the same time, corresponds to an element of the Lie algebra $\mathfrak{G}$. In particular, if X is tangent to the fiber G at a point x, then the value of the connection form ω_x at it is the element of the Lie algebra $\mathfrak{G}$ corresponding to X under the identification α indicated above. If X lies in the horizontal plane Q_x, then $\omega_x(X) = 0$. The connection form ω satisfies the following conditions:

(1) $\omega(\alpha(h)) = h$ for any element $h \in \mathfrak{G}$;
(2) for any element $g \in G$, the left translation Lg transforms the form ω into the form $\mathrm{Ad}_g(\omega)$, where $\mathrm{Ad}_g : \mathfrak{G} \to \mathfrak{G}$ is the adjoint representation of the Lie group G on its Lie algebra $\mathfrak{G}$.

Proposition 6.1.1. *If a* 1*-form* ω *with values in the Lie algebra* $\mathfrak{G}$ *of the structural group* G *is given on a principal fiber manifold* M *and it satisfies conditions* (1) *and* (2) *indicated above, then it is possible to construct a connection* Γ *for which this form is its connection form.*

We consider the projection $p : M \to B$; clearly, at each point $x \in M$, the differential of this projection assigns a linear isomorphism between the spaces Q_x and $T_{p(x)}B$. This allows us to define the concept of a lift of a vector field from the base to the space M.

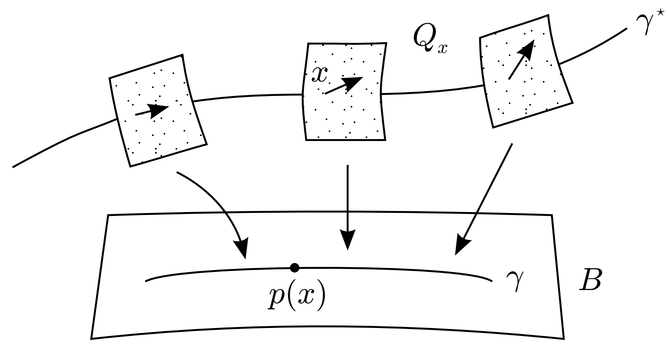


Fig. 65

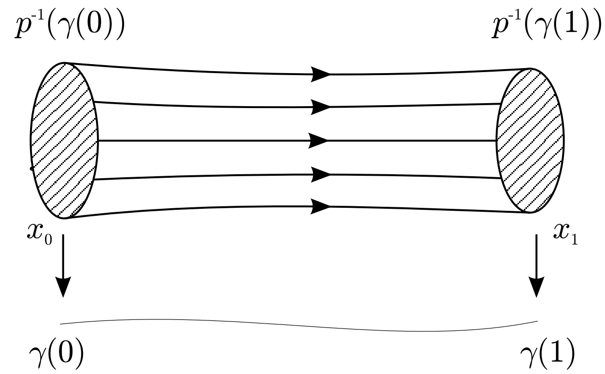


Fig. 66

Definition 6.1.4. Let v be a vector field on the base B. Its *lift* is a unique horizontal vector field v^* on M covering the field v under the projection $p : M \to B$.

The existence and uniqueness of a lift is obviously implied by the above remark on the isomorphism between Q_x and $T_{p(x)}B$ under $(dp)_x$.

In Chap. 1, we have considered an affine connection, which allows one to assign the operation of parallel translation along paths on a smooth manifold. Similarly, the operation of parallel translation is naturally related to each connection Γ in a principal bundle; we now describe this operation. Let τ be a certain curve in a principal fiber manifold M. We say that this curve is *horizontal* if all its tangent vectors are horizontal, i.e., $\dot{\tau}(t) \in Q_{\tau(t)}$, where $Q_{\tau(t)}$ is a horizontal at the point $\tau(t) \in M$. If $\gamma(t)$ is an arbitrary piecewise smooth curve in the base B, then its lift $\gamma^*(t)$ is a horizontal curve $\gamma^*(t)$ in the manifold M such that $p\gamma^*(t) = \gamma(t)$ for all t (see Fig. 65).

Proposition 6.1.2. *Let $\gamma(t)$, $0 \leq t \leq 1$, be an arbitrary piecewise-smooth curve in the base B of a principal fiber manifold $p : M \to B$, and let x_0 be a point such that $p(x_0) = \gamma(0)$. Then there exists a unique lift $\gamma^*(t)$ of the curve $\gamma(t)$ with the initial point $x_0 \in M$.*

The existence and uniqueness of the lift of curves allows us to define an important concept of parallel translation generated by a given connection. Let $\gamma(t)$, $0 \leq t \leq 1$, be a piecewise-smooth curve in the base B, and let x_0 be an arbitrary point of the fiber $p^{-1}(\gamma(0))$ that projected into the point $\gamma(0)$. According to Proposition 6.1.2, there exists a unique lift γ^* of the curve γ emanating from the point x_0 (see Fig. 66). The endpoint $\gamma^*(1)$ of the curve γ^* is projected into the endpoint $\gamma(1)$ of the curve $\gamma(t)$. This point $\gamma^*(1)$ is uniquely defined if the point x_0 is fixed. We define the mapping f_γ of the fiber $p^{-1}(\gamma(0))$ onto the fiber $p^{-1}(\gamma(1))$ by associating with each point x_0 of the fiber $p^1(\gamma(0))$ the point $\gamma^*(1)$ corresponding to it. The mapping f_γ is a diffeomorphism of the fiber $p^{-1}(\gamma(0))$ onto the fiber $p^{-1}(\gamma(1))$.

We consider a principal bundle $p : M \to B$ with a connection Γ; let ω be the corresponding connection form.

Definition 6.1.5. Let α be an arbitrary differential form of order k on the manifold M assuming the values in an arbitrary vector space. The *covariant differential* $D\alpha$ of the form α is the form given by $(D\alpha)(X_1, \dots, X_{k+1}) = (d\alpha)(hX_1, \dots, hX_{k+1})$ for any vector fields $X_1, \dots, X_{k+1}$ on the manifold M, where d stands for the usual exterior differential, and h stands for the horizontal components of vector fields.

Definition 6.1.6. The form $\Omega = D\omega$ is called the *curvature form* of a given connection Γ on the principal bundle $p : M \to B$.

Therefore, the curvature form is a 2-form assuming the values in the Lie algebra $\mathfrak{G}$ of the Lie group G.

We show that electromagnetic phenomena can be described in the framework of connection theory. We first recall that the *Maxwell equations* in their traditional form in the unit system with $c = \mu_0 = \varepsilon_0 = 1$ have the form

$$\begin{aligned} \operatorname{rot} B - \frac{\partial}{\partial t} E &= 4\pi J, \\ \operatorname{rot} E + \frac{\partial}{\partial t} B &= 0, \\ \operatorname{div} B &= 0, \\ \operatorname{div} E &= 4\pi\rho. \end{aligned}$$

In this case, we naturally used the operators rot and div in the usual three-dimensional flat space. We recall how to write these equations in terms of the metric and exterior derivative. For this purpose, we introduce the *Faraday* 2-*form* F with the components

$$(F_{ij}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_z \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

The second and third Maxwell equations become $F_{[\mu\nu,\gamma]} = 0$, i.e., $dF = 0$, $(x,\gamma) = \dfrac{\partial x}{\partial x^\gamma}$. We introduce the following pseudo-Euclidean metric (of special relativity theory) in the space $\mathbb{R}^4$:

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we can raise the subscripts of the form F and obtain the antisymmetric tensor $F^{ij} = g^{ik} g^{jl} F_{kl}$, i.e.,

$$(F_{ij}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix}.$$

Moreover, the remaining Maxwell equations become $F^{\mu\nu}_{,\nu} = 4\pi J^\mu$, where $(J^t = \rho, J^x, J^y, J^z) = (J^\mu)$ is the four-dimensional current vector. Let $\omega = dt \wedge dx \wedge dy \wedge dz$ be the volume form in $\mathbb{R}^4(t,x,y,z)$. We define the 2-form $*F$ as the compression $*F = \frac{1}{2}\omega(F)$, i.e., $(*F)_{\mu\nu} = \frac{1}{2}\omega_{\alpha\beta\mu\nu} F^{\alpha\beta}$. This is the so-called *dual form* to F. We define the 3-form $*J$ as the compression $*J = \omega(J)$, i.e., $(*J)_{\mu\nu\gamma} = \omega_{\mu\nu\gamma} J^\gamma$. Then the equation $F^{\mu\nu}_{,\nu} = 4\pi J^\mu$ is equivalent to $d(*F) = 4\pi(*J)$. Therefore, in our new notation, the Maxwell equations become $dF = 0$, $d(*F) = 4\pi(*J)$. Now they are represented in coordinate-free form and, therefore, in such a form they can be written on any manifold with metric (Riemannian or pseudo-Riemannian). The metric is needed in order to obtain $*F$ from F. The *operation* $*$ is defined on an arbitrary Riemannian manifold. To this end, we define the tensor $\varepsilon_{i_1\ldots i_n}$ according to the rule $\varepsilon_{i_1 i_2\ldots i_n} \neq 0$ only if among the subscripts $i_1, \ldots, i_n$, we now have equal ones; then

$$\varepsilon_{i_1\ldots i_n} = \begin{cases} +1, & \operatorname{sgn}(i_1\ldots i_n) = +1, \\ -1, & \operatorname{sgn}(i_1\ldots i_n) = -1. \end{cases}$$

If $T_{i_1\ldots i_k}$ is a skew-symmetric tensor of type $(0,k)$, then $*T$ denotes the skew-symmetric tensor of type $(0, n-k)$ given by

$$(*T)_{i_{k+1},\ldots,i_n} = \frac{1}{k!}\sqrt{\det(g_{ij})}\,\varepsilon_{i_1\ldots i_n} T^{i_1\ldots i_k},$$

where $T^{i_1\ldots i_k} = g^{i_1 j_1}\ldots g^{i_k j_k} T_{j_1\ldots j_k}$.

We note that if in nature we have magnetic monopoles, then the Maxwell equations take the symmetrical form $dF = 4\pi(*J_m)$, $d(*F) = 4\pi(*J_e)$, where J_m and J_e are the corresponding currents.

Since F is a closed 2-form, there exists a 1-form A such that $F = dA$ in a certain neighborhood of each point. Using the metric, with this 1-form, we can associate a vector called the *potential vector*. The form A is not uniquely defined: $A' = A + df$ for any function f yields the same F. Such a change is called a *gauge transformation*. We note that if there exist magnetic monopoles, then A in this case can be defined only on simply connected domains containing no magnetic monopoles.

The above formalism can be rewritten in terms of bundles and connections in them. We consider a trivial U(1)-bundle over the pseudo-Riemannian manifold $\mathbb{R}^4_1 = \mathbb{R}^4$ with metric (g_{ij}). We consider the potential A of the electromagnetic field as a connection in this bundle. As was shown above, such a connection is given by the form with values in the Lie algebra u(1) of the Lie group U(1). In our case, $\mathrm{u}(1) \cong i\mathbb{R}$, and the form is equal to iA. Gauge transformations are smooth mappings $g : \mathbb{R}^4 \to \mathrm{U}(1)$ that can be represented in the form $g(x) = \exp(-if(x))$. Then the connection is transformed according to the law $A^g = A + df$. The curvature of the connection iA coincides with $\omega = dA$. On the space of all connections, we consider the functional

$$L = \frac{1}{2}\int \|\omega\|^2.$$

It is easy to see that the condition $\delta L = 0$, where δL is the first variation of the functional L, yields the Maxwell equations for the field ω.

Weak interactions can be described similarly with the replacement of the group U(1) by a more complicated group. For the first time, this was observed by Yang and Mills; they wrote the corresponding Lagrangian for the potential, assuming its values in the Lie algebra su(2) of the Lie group SU(2): $g : \mathbb{R}^4_1 \to \mathrm{SU}(2)$, which is transformed according to the formula

$$a^g = gag^{-1} + gdg^{-1}$$

of transformation of a connection in the trivial SU(2)-bundle over $\mathbb{R}^4_1$. The curvature form is $\Omega = da + \frac{1}{2}[a, a]$. Equations describing the corresponding interaction are obtained as equations for extremals of the functional

$$\frac{1}{2}\int \|\Omega\|.$$

At present, in physics, one uses the connections in SU(2)-, SU(3)-, and SU(4)-principal bundles (see [186]).

From the geometric viewpoint, we have the following. We consider an arbitrary principal bundle E with structural Lie group G (compact Lie group) over an arbitrary smooth manifold. The *potential* is an arbitrary connection ∇ in this bundle. The field associated with the potentials are the curvatures R^∇ of the connection ∇. The group of gauge transformations G_E is the automorphism group of the bundle E. With the group G_E, one associates the Lie algebra $\mathfrak{G}_E \cong E \underset{\mathrm{ad}}{\times} \mathfrak{G}$, where $\mathfrak{G}$ is the Lie algebra of the Lie group G; we have $G_E \cong E \underset{\mathrm{Ad}}{\times} G$. If M is a Riemannian manifold, then on the space $C(E)$ of all connections in the bundle E, we define the *Yang–Mills functional*

$$\mathrm{YM}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2,$$

where $\|Z\|$ is the norm, which is naturally defined by using the Riemannian metric on M and the G-invariant norm in $\mathfrak{G}_E$. The group of gauge transformations G_E acts on the space $C(E)$ in a natural way. The Yang–Mills functional is invariant with respect to this action.

Definition 6.1.7. A connection $\nabla \in C(E)$ with a finite action is called a *Yang–Mills potential*, and R^∇ is called a Yang–Mills field if ∇ is a critical point of the Yang–Mills functional, i.e., if the first variation of this functional vanishes at the point ∇.

6.2. Four-dimensional case. An important property of the Yang–Mills functional in dimension four consists of its conformal invariance, i.e., $\mathrm{YM}(\nabla)$ depends only on the conformal class of metric of this manifold (for a detailed discussion of this concept, see Sec. 4.8 of Chap. 3; an example of another important functional that is also conformally invariant was considered there). Over an arbitrary compact orientable manifold M^4, any G-bundle E has characteristic numbers. If G is a simple group, then there exists only one characteristic number, namely, the Pontryagin number $p_1(E)$ (it is called a *topological charge* by physicists). We denote its absolute value by $k = |p_1(E)|$. For a connection ∇ on E, it can be expressed through the curvature tensor:

$$4\pi^2 p_1(E) = \int\limits_M \langle R^\nabla, R^\nabla \rangle,$$

where $\langle X, Y \rangle$ is the inner product on $\mathfrak{G}_E$.

The space of 2-forms Ω^2 on an oriented four-dimensional manifold M^4 admits the decomposition $\Omega^2 = \Omega^+ \oplus \Omega^-$ into eigensubspaces $\Omega^\pm$ of the involution $*$; the space Ω^+ corresponds to the eigenvalue $+1$, and Ω^- corresponds to the eigenvalue -1. This decomposition generates the corresponding decomposition of the curvature tensor: $R^\nabla = R^\nabla_+ + R^\nabla_-$, where $*R^\nabla_\pm = \pm R^\nabla_\pm$ and

$$4\pi p_1(E) = \int\limits_M \{\|R^\nabla_+\|^2 - \|R^\nabla_-\|^2\}, \qquad \mathrm{YM}(\nabla) = \int\limits_M \{\|R^\nabla_+\|^2 + \|R^\nabla_-\|^2\}.$$

Therefore, $4\pi k^2 \leq \mathrm{YM}(\nabla)$ for any connection ∇ on E. This bound is attained iff $R^\nabla_\varepsilon \equiv 0$, where $\varepsilon k = -p_1(E)$.

Definition 6.2.1. Fields for which R^∇_- (resp. R^∇_+) vanishes are said to be *self-dual* (resp. *anti-self-dual*). In the case S^4 and $G = \mathrm{SU}(2)$ such connections are called *instantons*.

Atiyah, Hitchin, and Singer proved a remarkable fact that the instanton space is finite-dimensional, and its dimension equals $8k - 3$.

Definition 6.2.2. One says that a Yang–Mills connection is stable if the second variation of the functional $\mathrm{YM}(\nabla)$ at the point ∇ is nonnegative (a local minimum is always stable).

Simons showed that on the sphere S^n, there is no stable Yang–Mills fields for $n \geq 5$. On the other hand, stable critical points of the functional $\mathrm{YM}(\nabla)$ are related to the self-duality or anti-self-duality.

Theorem 6.2.1. *Any stable Yang–Mills field over the sphere S^4 with the group* $\mathrm{SU}(2)$*,* $\mathrm{SU}(3)$*, or* $\mathrm{U}(2)$ *is self-dual or anti-self-dual.*

A similar assertion holds for the Lie group $\mathrm{SO}(4)$.

As the following theorem shows, sometimes the stability implies that the Yang–Mills functional attains its minimum value.

Theorem 6.2.2. *Let M^4 be a compact oriented homogeneous four-dimensional Riemannian manifold. Then any stable Yang–Mills field on M^4 with the group* $\mathrm{SU}(2)$*,* $\mathrm{SU}(3)$*,* $\mathrm{U}(2)$*, or an Abelian group is an absolute minimum.*

In some cases, critical points of the Yang–Mills functional are isolated. As an example, we consider the case of the sphere S^n. To state the corresponding results, we need a new concept. Let a principal bundle $p : M \to B$ be trivial, i.e., the direct product $M = B \times G$. Then a section N_g of the bundle $p : B \times G \to B$, consisting of points of the form (q, g), where q runs over all points of the base and the element g is fixed, passes through any point $(b, g) \in M$. It is clear that N_g is a submanifold in the direct product homeomorphic to the base (see Fig. 67). A connection in M is said to be *flat* if at each point $x = (b, g)$, as the horizontal plane Q_x, we take the tangent space to the submanifold N_g. Clearly, the curvature form of such a connection is identically equal to zero.

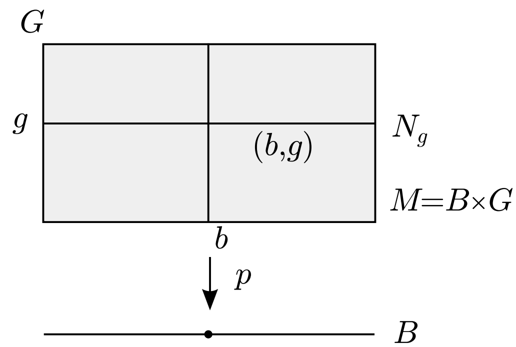


Fig. 67

A connection in a principal bundle $p : M \xrightarrow{G} B$ is said to be *locally flat* if for each point $b \in B$, there exists a neighborhood U such that the connection induced by the ambient connection in $p^{-1}(U) \cong U \times G$ is flat in the direct product $U \times G$. The following important property, characterizing locally flat connections through their curvature tensor, holds. A connection in a principal bundle $p : M \to B$ is locally flat iff the curvature form of this connection is identically equal to zero.

In the following estimates, we use the norm on $\mathfrak{G}_E$ defined by $\|A\|^2 = \frac{1}{2}\operatorname{tr}(A^t A)$. We need to consider the following three cases separately: $n \geq 5$, $n = 4$, and $n = 3$. In the following three theorems, it is shown in which sense critical points of the functional $\mathrm{YM}(\nabla)$ are isolated on the sphere S^n.

Theorem 6.2.3 ($n \geq 5$)**.** *Any Yang–Mills connection ∇ over the standard sphere S^n of dimension n, $n \geq 5$, such that $\|R^\nabla\|^2 \leq \frac{1}{2}C_n^2$ is locally flat, i.e., $R^\nabla \equiv 0$.*

Theorem 6.2.4 ($n = 4$)**.** *Let R^∇ be a Yang–Mills field on a bundle E over the sphere S^4 satisfying the pointwise condition $\|R^\nabla\|^2 \leq 3$. Then either E is a flat bundle or $E = E_0 + S$, where E_0 is a flat bundle and S is one of two four-dimensional spinor bundles with canonical connection. Further, if R^∇ satisfies the condition $\|R_+^\nabla\|^2 < 3$ (or $\|R_-^\nabla\|^2 < 3$) pointwise, then $R_+^\nabla \equiv 0$ (resp. $R_-^\nabla \equiv 0$).*

Theorem 6.2.5 ($n = 3$)**.** *Let R^∇ be a Yang–Mills field on a bundle E over the sphere S^3 that satisfies the condition $\|R^\nabla\|^2 \leq 3/2$ pointwise. Then either E is a flat bundle or $E = E_0 + S$, where E_0 is a flat bundle and S is a four-dimensional spinor bundle with canonical Riemannian connection.*

The proof of these assertions can be found in [27].

In many respects, the problem of finding critical points of the functional $\mathrm{YM}(\nabla)$ on four-dimensional manifolds is similar to the problem of studying harmonic mappings of two-dimensional manifolds (see, e.g., the paper [205] of Uhlenbeck).

6.3. Moduli spaces of self-dual Yang–Mills connections. As a rule, all self-dual connections form a "good" space. To explain what we mean, certain additional considerations are necessary. Let $\mathfrak{G}_E = E \underset{\mathrm{Ad}_G}{\times} \mathfrak{G}$ be the vector bundle associated with a principal bundle E by the adjoint representation. The operator $*$ acts on sections of the bundle $\mathfrak{G}_E \otimes \Lambda^p$, where Λ^p is the space of forms of degree p on a manifold M. We have

$$\mathfrak{G}_E \otimes \Lambda^2 = (\mathfrak{G}_E \otimes P_+\Lambda^2) \oplus (\mathfrak{G}_E \otimes P_-\Lambda^2),$$

where $P_\pm = \frac{1}{2}(1 \pm *)$. The curvature tensor R_A of the connection A is a section of the bundle $\mathfrak{G}_E \otimes \Lambda^2$. If $P_- R_A = 0$, then the connection A is said to be *self-dual*, and if $P_+ R_A = 0$, the connection A is said to be *anti-self-dual*. The equation $P_- R_A = 0$ is called the duality condition. Solutions of this equation are also called *instantons*. Each instanton is a Yang–Mills connection. The curvature tensor assigns the mapping $R : \Lambda^2 \to \Lambda^2$. We consider the following decomposition of R into irreducible components with respect to the representation of the group SO(4) (see [28]):

$$R = u_+^{ij} x_+^i \otimes x_+^j + \omega_-^{ij} x_-^i \otimes x_-^j + B^{ij} x_+^i \otimes x_-^j + B_+^{ij} x_-^i \otimes x_+^j + \frac{S}{6}(x_+^j \otimes x_+^j + x_-^j \otimes x_-^j),$$

where $(x_\pm^i)$, $i = 1, 2, 3$, is a local orthonormal basis for $\pm\Lambda^2$, respectively, and the function S coincides with the scalar curvature on M. We call ω_-^{ij} the *self-dual part* of the curvature tensor. We denote by $H^K(M)$ the de Rham cohomology space of the complex

$$0 \to \Gamma(\Lambda^0) \xrightarrow{d} \Gamma(\Lambda^1) \xrightarrow{d} \Gamma(\Lambda^2) \xrightarrow{d} \Gamma(\Lambda^3) \xrightarrow{d} \Gamma(\Lambda^4) \to 0,$$

where d is the exterior derivative and $\Gamma(\xi)$ denotes the space of all sections of the bundle ξ.

We first present the following existence theorem.

Theorem 6.3.1 ([191])**.** *Let M be a compact orientable Riemannian manifold of dimension* 4*. Let $P_- H^2(M) = 0$, and let G be a compact semisimple Lie group. Then there exists a principal G-bundle over the manifold M that admits smooth irreducible self-dual connections.*

We consider a compact semisimple Lie group G. With accuracy up to an isomorphism, principal G-bundles over M are in a one-to-one correspondence with homotopy classes of mappings of the manifold M into the classifying space BG of the Lie group G, and, moreover, there is a surjective mapping $\varphi : [M, BG] \to \mathbb{Z}^p \to 0$, where p is the number of nontrivial ideals in the Lie algebra $\mathfrak{G}$ of the Lie group G. Moreover, there is a canonical mapping (see [191])

$$\eta : [M, BG] \to H^2(M\pi_1(G)),$$

where $[X, Y]$ is the set of homotopy classes of mappings of the space X into Y.

Theorem 6.3.2 ([191])**.** *Let M be a compact orientable four-dimensional Riemannian manifold satisfying the condition $P_- H^2(M) = 0$, G be a compact semisimple Lie group, and $E \to M$ be a principal G-bundle such that all its Pontryagin classes $p_j^k(\mathfrak{G}_E)$, $k = 1, \dots, p$, are nonnegative. Moreover, we assume that the image of the bundle E in $H^2(M, \pi_1(G))$ under the mapping η is trivial. Then*:

(1) *the space $C(E)$ contains a smooth self-dual connection*;
(2) *if the principal G-bundle over S^4 with unit Pontryagin classes admits an irreducible self-dual connection, then in the space $C(E)$, there also exists such a connection*;
(3) *if M is a real-analytic manifold, then there exists a real analytic principal G-bundle E' isomorphic to E for which assertions* (1) *and* (2) *hold for real analytic connections.*

We consider the group $\operatorname{Aut} E = \Gamma(E \underset{\mathrm{Ad}_G}{\times} G)$. It naturally acts on the space $C(E)$. Consider the orbit space with respect to the action of the group $\operatorname{Aut} E$ on the spaces of connections $C(E)$. The set of irreducible self-dual connections in $C(E)$ modulo this action is called the *moduli space of self-dual connections* in $C(E)$. Atiyah, Hitchin, and Singer proved that if the self-dual part of the curvature tensor of the manifold M vanishes, then the moduli space is a finite-dimensional manifold.

Theorem 6.3.3 ([191])**.** *Let the conditions of the preceding theorem hold. Let $E \to M$ be a principal G-bundle, and, moreover, let G be a compact semisimple Lie group. Consider the connection A from assertion* (2) *of the previous theorem. Then, in a neighborhood of the connection A in the space $C(E)/\operatorname{Aut} E$, the moduli space of irreducible self-dual connections is a smooth manifold of dimension $p_1(\mathfrak{G}_E) = \frac{1}{2}(\dim G)(\chi - \tau)$, where $p_1(\mathfrak{G}_E) = \sum\limits_{j=1}^{l} p_j^1(\mathfrak{G}_E)$ is the sum of Pontryagin classes of the vector bundle $\mathfrak{G}_E = E \underset{\mathrm{Ad}_G}{\times} \mathfrak{G}$, χ is the Euler characteristic of the manifold M, and τ is the signature of the manifold M.*

The previous theorem is of local character. If we impose additional restrictions, a global statement can be obtained.

Theorem 6.3.4 ([191]). *In addition to the conditions of the previous theorem, we assume that the Riemannian metric on M is conformally equivalent to the Riemannian metric g' on M for which $S'^2 - 3\omega'_- > 0$, where $S'(x)$ is the scalar curvature of the metric g' and $\omega'_-(x) = \sup_{\xi \in S^2 \subset \mathbb{R}^3} \omega'^{ij} \xi^i \xi^j$, ω' is the self-dual part of the curvature tensor of the metric g'. Then the moduli space of irreducible self-dual connections is a Hausdorff manifold as a whole, and its dimension is calculated by the formula presented above.*

The theory of self-dual connections found surprising applications in topology. For these applications, see [46, 47].

REFERENCES

1. R. Abraham, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis, and Applications,* Addison-Wesley, Reading, Massachusetts (1983).
2. J. Adem, "Relations on iterated reduced powers," *Proc. Natl. Acad. Sci. USA,* **39**, 636–638 (1953).
3. S. I. Alber, "Topology of functional manifolds and the calculus of variations," *Usp. Mat. Nauk,* **25**, No. 4, 57–123 (1970).
4. A. D. Aleksandrov, *Intrinsic Geometry of Convex Surfaces* [in Russian], OGIZ, Moscow–Leningrad (1948).
5. A. D. Aleksandrov, *Convex Polyhedrons* [in Russian], Gostekhisdat, Moscow (1950).
6. C. B. Allendorfer and A. Weil, "The Gauss–Bonnet theorem for Riemannian polyhedra," *Trans. Amer. Math. Soc.,* **53**, 101–129 (1943).
7. S. Aloff and N. R. Wallach, "An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures," *Bull. Amer. Math. Soc.*, **81**, 93–97 (1985).
8. D. V. Anosov, "Some homotopies in the space of closed curves," *Izv. Akad. Nauk SSSR, Ser. Mat.,* **44**, No. 6, 1219–1254 (1980).
9. D. V. Anosov, "Geodesic flows on closed Riemannian manifolds of negative curvature," *Tr. Mat. Inst. Akad. Nauk SSSR,* Vol. 90 (1967).
10. V. A. Arnol'd, *Mathematical Methods of Classical Mechanics* [in Russian], Nauka, Moscow (1974).
11. T. Aubin, *Nonlinear Analysis on Manifolds. Monge–Ampere Equations*, Springer-Verlag, New York–Heidelberg–Berlin (1984).
12. I. Ya. Bakel'man, A. D. Verner, and B. E. Kantor, *Introduction to Differential Geometry in the Large* [in Russian], Nauka, Moscow (1973).
13. W. Ballmann, G. Thorbergsson, and W. Ziller, "Some existence theorems for closed geodesics," *Comment. Math. Helv.,* **58**, No. 3, 416–432.
14. S. Bando, "On the classification of three dimensional compact Kähler manifolds of nonnegative bisectional curvature," *J. Diff. Geom.,* **19**, No. 2, 283–297 (1984).
15. V. Bangert and N. Hingston, "Closed geodesics on manifolds with infinite Abelian fundamental groups," *J. Diff. Geom.,* **19**, No. 2, 277–282 (1984).
16. J. Beem and P. Ehrlich, *Global Lorentzian Geometry,* Marcel Dekker, New York–Basel (1981).
17. M. Berger, "Les varieties Riemanniannes 1/4-pincees," *Ann. Scuola Norm. Sup. Pisa, Ser. III*, **14**, 161–170 (1960).
18. M. Berger, "Les varieties Rimanniannes a courbure positive," *Bull. Soc. Math. Belg.,* **10**, 89–104 (1958).
19. M. Berger, P. Gauduchon, and E. Mazet, *Le spectre d'une variete Riemannenienne, Lect. Notes Math.,* Vol. 194 (1971).

20. M. Berger, "An extension theorem of Rauch's metric comparison theorem and some applications," *Illinois J. Math.,* **6**, No. 4, 700–712 (1962).
21. A. L. Besse, *Manifolds All of Whose Geodesics Are Closed,* Springer-Verlag, Berlin–Heidelberg–New York (1978).
22. R. L. Bishop and R. J. Crittenden, *Geometry of Manifolds,* Academic Press, New York–London (1964).
23. R. L. Bishop and B. O'Neil, "Manifolds of negative curvature," *Trans. Amer. Math. Soc.,* **145**, 1–49 (1969).
24. A. Borel, "Compact Clifford–Klein forms of symmetric spaces," *Topology,* **2**, No. 2, 111–122 (1963).
25. Yu. F. Borisov, "Manifolds with boundary of bounded curvature," *Dokl. Akad. Nauk SSSR*, **74**, 877–880 (1950).
26. R. Bott, "On manifolds all of whose geodesics are closed," *Ann. Math.,* **60**, 375–382 (1954).
27. J. P. Bourguignon and H. B. Lawson, Jr., "Stability and isolation phenomena for Yang–Mills fields," *Commun. Math. Phys.,* **79**, 189–230 (1980).
28. J. P. Bourguignon and H. Karcher, "Curvature operators: Pinching estimates and geometric examples," *Ann. Sci. Ecole Norm. Super. Ser. 4,* **11**, No. 1, 71–92 (1978).
29. J. P. Bourguignon, "Premiere classe de Chern at courbure de Ricci: Preve de le conjecture de Calabi," *Asterisque,* Vol. 58 (1978).
30. T. Bröcker and T. von Dieck, "Representations of compact Lie groups," *Grad. Texts Math.,* **98**, (1985).
31. R. Brooks, "The fundamental group and the spectrum of the Laplacian," *Comment. Math. Helv.,* **56**, 581–598 (1981).
32. R. Brooks, "Amenability and the spectrum of the Laplacian," *Bull. Amer. Math. Soc.,* **6**, No. 1, 87–89 (1982).
33. C. Burstin, "Ein Beitragzum Problem der Einbettung der Riemannschen Raume in Euklidischen Raume," *Mat. Sb.,* **38**, No. 3/4, 74–85 (1931).
34. H. Busemann, *The Geometry of Geodesics,* Academic Press, New York (1955).
35. S. V. Buyalo, "Volume and the fundamental group of manifolds of nonpositive curvature," *Mat. Sb.*, **122**, No. 2, 142–156 (1983).
36. E. Calabi, "On Kähler manifolds with vanishing canonical class," In: *Algebraic Geometry and Topology. Symp. in Honor of S. Lefschetz,* Princeton Univ. Press, Princeton (1957), pp. 78–89.
37. E. Cartan, "Les groupes d'holonomie des espaces generalizes," *Acta Math.,* **48**, 11–42 (1926).
38. E. Cartan, "Sur la possibilite de plonger un espace Riemannien donne dans un espace euclidien," *Ann. Soc. Polon. Math.,* **6**, 1–7 (1927).
39. L. S. Charlap and A. T. Vasquez, "Compact flat Riemannian manifolds," *Amer. J. Math.,* **95**, 471–494 (1973).
40. J. Cheeger, "Some examples of manifolds of nonnegative curvature," *J. Diff. Geom.,* **8**, No. 4, 623–629 (1973).
41. J. Cheeger, "On the structure on complete manifolds of nonnegative curvature," *Ann. Math., Ser. 2,* **96**, No. 3, 413–443 (1972).
42. J. Cheeger, "Finiteness theorems for Riemannian manifolds," *Amer. J. Math.,* **92**, No. 1, 61–74 (1970).
43. J. Cheeger and D. Ebin, "Comparison theorems in Riemannian geometry," *North Holland Math. Library,* No. 9 (1975).
44. S. S. Chern, *Complex Manifolds, Textos de Mat.,* No. 5, Univ. do Recife (1959).
45. A. Dold, *Lectures on Algebraic Topology.* Springer-Verlag, Berlin–Heidelberg–New York (1972).
46. S. K. Donaldson, "Instantons and geometric invariant theory," *Commun. Math. Phys.,* **93**, 453–460 (1984).

47. S. K. Donaldson, “An application of gauge theory to four-dimensional topology,” *J. Diff. Geom.,* **18**, 279–315 (1983).
48. B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern Geometry* [in Russian], Nauka, Moscow (1979).
49. B. A. Dubrovin, S. P. Novikov, and A. T. Fomenko, *Modern Geometry. Homology Theory* [in Russian], Nauka, Moscow (1984).
50. B. A. Dubrovin, I. M. Krichever, and S. P. Novikov, “Integrable systems,” In: *Progress in Science and Technology, Series on Contemporary Problems in Mathematics, Fundamental Directions. Integrable Systems* [in Russian], Vol. 4, All-Union Institute for Scientific and Technical Information (VINITI), Akad. Nauk SSSR, Moscow (1985), pp. 179–284.
51. P. Eberlein, “Euclidean de Rham factor of a lattice of nonpositive curvature,” *J. Diff. Geom.,* **18**, 209–220 (1983).
52. J. Eells and L. Lemaire, “A report on harmonic maps,” *Bull. London Math. Soc.,* **10**, No. 1, 1–68 (1978).
53. N. V. Efimov, *Higher Geometry* [in Russian], Nauka, Moscow (1971).
54. I. P. Egorov, “Riemannian spaces of the second lacunarity,” *Dokl. Akad. Nauk SSSR*, **3**, No. 2, 276–279 (1956).
55. L. P. Eisenhart, *Riemannian Geometry,* Princeton Univ. Press, Princeton (1926).
56. J. H. Eschenburg, “New examples of manifolds with strictly positive curvature,” *Invent. Math.,* **66**, No. 3, 469–480 (1982).
57. F. T. Farrell and W. C. Hsiang, “Topological characterization of flat and almost flat manifolds $M^n\,(n \neq 3, 4)$,” *Amer. J. Math.,* **105**, No. 3, 641–672 (1983).
58. W. J. Floyd, “Group completions and limit sets of Kleinian groups,” *Invent. Math.,* **57,** No. 3, 205–218 (1980).
59. A. T. Fomenko, *Differential Geometry and Topology. Additional Chapters* [in Russian], Moscow Univ., Moscow (1983).
60. A. T. Fomenko, *Variational Methods in Topology* [in Russian], Nauka, Moscow (1982).
61. A. T. Fomenko, “Higher-dimensional variational methods in topology of extremals,” *Usp. Mat. Nauk*, **36**, No. 6, 105–135 (1981).
62. A. Friedman, “Local isometry embedding of Riemannian manifolds with indefinite metrics,” *J. Math. Mech.,* **10,** No. 4, 625–649 (1961).
63. H. Gluck, F. Warner, and C. T. Yang, “Division algebras fibrations of spheres by great spheres and the topological determination of space by the cross behaviour of its geodesics,” *Duke Math. J.,* **50,** No. 4, 1041–1076 (1983).
64. C. Godbillon, “Geometrie differentielle et mechanique analitique,” In: *Collection Methodes,* Hermann, Paris (1969).
65. R. E. Gompf, “Three exotic $\mathbb{R}^3$’s and other anomalies,” *J. Diff. Geom.,* **18**, 317–328 (1983).
66. A. Gray, “A generalization of F. Schur’s theorem,” *J. Math. Soc. Jpn.,* **21,** No. 3, 454–457 (1969).
67. A. Gray and L. Vanhecke, “The volume of a tube in a Riemannian manifold,” *Rend. Semin. Math. Univ. Politech. Torino,* **39,** No. 3, 1–50 (1981).
68. F. P. Greenleaf, *Invariant Means of Topological Groups and Their Applications,* Van Nostrand–Reinhold (1969).
69. W. Greub, S. Halperin, and R. Vanstone, *Connections, Curvature, and Cohomology,* Vol. 1, Academic Press, New York–London (1972); Vol. 2, Academic Press, New York–London (1973).
70. R. I. Grigorchuk, “To the Milnor problem on the group growth,” *Dokl. Akad. Nauk SSSR,* **271**, No. 1, 30–33 (1983).
71. D. Gromoll, W. Klingenberg, and W. Meyer, *Riemannsche Geometrie in Grossen,* Springer-Verlag, Berlin–Heidelberg–New York (1968).

72. D. Gromoll and W. Meyer, "An exotic sphere with nonnegative sectional curvature," *Ann. Math.,* **100,** 401–406 (1974).
73. D. Gromoll and J. Wolf, "Some relations between the metric structure and the algebraic structure of the fundamental groups in manifolds of nonpositive curvature," *Bull. Amer. Math. Soc.,* **77,** No. 4, 545–552 (1971).
74. M. Gromov, "Manifolds of negative curvature," *J. Diff. Geom.,* **13,** No. 2, 223–230 (1978).
75. M. Gromov, "Curvature, diameter and Betti numbers," *Comment. Math. Helv.,* **56,** No. 2, 179–194 (1982).
76. M. Gromov, "Volume and bounded cohomology," *Publ. Math. IHES,* **56,** 2–99 (1982).
77. M. Gromov, "Filling Riemannian manifolds," *J. Diff. Geom.,* **18,** No. 1, 1–147 (1983).
78. M. L. Gromov, "Topological methods for constructing solutions to differential equations and inequalities," In: *Int. Congr. Math., Nizza, 1970, Rep. Sov. Math.* [in Russian], Nauka, Moscow (1972), pp. 72–76.
79. M. L. Gromov, "Isometric embeddings and immersions," *Dokl. Akad. Nauk SSSR,* **192**, No. 5, 1206–1209 (1970).
80. M. Gromov and H. B. Lawson, Jr., "The classification of simply connected manifolds of positive scalar curvature," *Ann. Math.,* **111**, 423–434 (1980).
81. M. Gromov and H. B. Lawson, Jr., "Blaine positive scalar curvature and the Dirac operator on complete Riemannian manifolds," *Publ. Math. IHES,* **58**, 295–408 (1983).
82. M. L. Gromov and V. A. Rokhlin, "Embeddings and immersions in Riemannian Geometry," *Usp. Mat. Nauk,* **25**, No. 5, 3–62 (1970).
83. K. Grove and H. Karcher, "On pinched manifolds with fundamental group $\mathbb{Z}_2$," *Compos. Math.,* **27**, No. 1, 49–61 (1973).
84. K. Grove, H. Karcher, and E. Ruh, "Jacobi fields and Finsler metrics on compact Lie groups with applications to differential pinching problems," *Math. Ann.,* **211**, No. 1, 7–21 (1974).
85. K. Grove and K. Shiohama, *A generalized sphere theorem,* Københavns Universitet, Matematisk Institut, Preprint No. 23 (1975).
86. V. Guillemin and S. Sternberg, *Symplectic Technique in Physics,* Cambridge University Press (1984).
87. N. A. Gusevskii, "On the fundamental group of manifolds of negative curvature," *Dokl. Akad. Nauk SSSR,* No. 4, 777–781 (1983).
88. I. S. Gutsul, "On one group of compact three-dimensional manifolds of constant negative curvature," *Dokl. Akad. Nauk SSSR,* **248**, No. 2, 283–286 (1979).
89. I. S. Gutsul, "On compact three-dimensional manifolds of constant negative curvature," *Tr. Mat. Inst. Akad. Nauk SSSR*, **152**, 89–96 (1980).
90. G. S. Hall, "Sectional curvature and the determination of the metric in space-time," *Gen. Relativ. Grav.,* **16**, No. 1, 79–88 (1984).
91. R. S. Hamilton, "Three-manifolds with positive Ricci curvature," *J. Diff. Geom.,* **17**, No. 2, 255–306 (1982).
92. S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time,* Cambridge University Press (1973).
93. E. Heintze, "On homogeneous manifolds of negative curvature," *Math. Ann.,* **211**, No. 1, 23–34 (1974).
94. S. Helgason, *Differential Geometry and Symmetric Spaces,* Academic Press, New York–London (1962).
95. F. Hirzebruch, *Topological Methods in Algebraic Geometry,* Springer-Verlag, New York (1960).
96. N. Hitchin, "Compact four-dimensional Einstein manifolds," *J. Diff. Geom.,* **9**, No. 3, 435–441 (1974).
97. N. Hitchin, "The space of harmonic spinors," *Adv. Math.,* **14**, 1–55 (1974).

98. H. Hopf and W. Rinov, "Über der Begriff der vollstandigeb differentialgeomentrischen Flache," *Comment. Math. Helv.,* **3**, 209–225 (1931).
99. H. C. Im Hof and E. Ruh, "An equivariant pinching theorem," *Comment. Math. Helv.,* **50**, No. 3, 389–402 (1975).
100. Y. Itokawa, "The topology of certain Riemannian manifolds with positive Ricci curvature," *J. Diff. Geom.,* **18**, 151–155 (1983).
101. N. Jacobson, *Lie Algebras,* Interscience Publishers, New York–London (1962).
102. M. Janet, "Sur la possibilite de plonger un espace Riemannien donne dans espace Euclidien," *Ann. Soc. Polon. Math.,* **5** (1926).
103. M. Katz, "The filling radius of two-point homogeneous spaces," *J. Diff. Geom.,* **18**, 505–511 (1983).
104. J. L. Kazdan and F. W. Warner, "Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvatures," *Ann. Math.,* **101**, No. 2, 317-331 (1975).
105. J. L. Kazdan and F. W. Warner, "A direct approach to the determination of Gaussian and scalar curvature functions," *Invent. Math.,* **28**, No. 3, 227–230 (1975).
106. W. Klingenberg, *Riemannian Geometry,* Walter de Gruyter, Berlin–New York (1982).
107. W. Klingenberg, *Lectures on Closed Geodesics,* Springer-Verlag, Berlin–Heidelberg–New York (1978).
108. W. Klingenberg, "Contribution to Riemannian geometry in the large," *Ann. Math.,* **69**, No. 3, 654–666 (1959).
109. W. Klingenberg, "Über Riemannishe Mannigfaltigkeiten mit positive Krümmung," *Comment. Math. Helv.,* **35**, No. 1, 47–54 (1961).
110. W. Klingenberg, "Über Riemannishe Manigfaltigkeiten mit nach oben beschränkter Krümmung," *Ann. Math.,* **60**, 49–59 (1962).
111. H. Knörrer, "Geodesics on the ellipsoid," *Invent. Math.,* **59**, No. 2, 119–144 (1980).
112. S. Kobayashi, *Transformation Groups in Differential Geometry,* Springer-Verlag (1972).
113. S. Kobayashi and N. Nomizu, *Foundation of Differential Geometry,* Interscience Publishers, New York–London–Sydney, Vol. 1 (1963), Vol. 2 (1969).
114. V. N. Kolokol'tsov, "Geodesic flows on two-dimensional manifolds with additional first integral polynomial in velocities," *Izv. Akad. Nauk SSSR, Ser. Mat.*, No. 5, 994–1010 (1983).
115. O. Kowalski, "Generalized symmetric spaces," *Lect. Notes Math.,* Vol. 805 (1980).
116. R. S. Kulkarni, "Curvature and metrics," *Ann Math.,* **91**, No. 2, 311–331 (1970).
117. L. D. Landau and E. M. Lifshits, *Field Theory* [in Russian], Nauka, Moscow (1973).
118. H. B. Lawson, Jr. and J. Simons, "On stable currents and their application to global problems in real and complex geometry," *Ann. Math.,* **98**, No. 3, 427–450 (1973).
119. H. B. Lawson, Jr. and S. T. Yau, "Compact manifolds of nonpositive curvature," *J. Diff. Geom.,* **7**, No. 1, 211–228 (1972).
120. A. Lichnerowich, *Theorie Globale des Connexions et des Groups d'Holonomie,* Rome Fd. Gremones (1955).
121. A. Lichnerowich, "Spineurs harmoniques," *C. R. Acad. Sci. Paris,* **257**, 7–9 (1963).
122. L. A. Lyusternik and L. G. Shnirel'man, "Topological methods in variational problems and their applications to the differential geometry of surfaces," *Usp. Mat. Nauk*, **2**, No. 2, 166–217 (1947).
123. L. A. Lyusternkik and L. G. Shnirelman, "Sur le problem de geodesiques fermees sur les surfaces de genre 0," *C. R. Acad. Sci., Paris,* **189**, 269–271 (1927).
124. V. S. Makarov and I. S. Gutsul, "On noncompact three-dimensional manifolds of constant negative curvature having a finite measure," *Tr. Mat. Inst. Akad. Nauk SSSR*, **152**, 165–169 (1980).
125. S. V. Manakov, V. E. Zakharov, S. P. Novikov, and L. P. Pitaevskii, *Soliton Theory. The Inverse Problem Method* [in Russian], Nauka, Moscow (1980).
126. O. V. Manturov, "Homogeneous Riemannian spaces with irreducible rotation group," *Tr. Sem. Vekt. Tenz. Anal.,* No. 13, 68–145 (1966).

127. G. A, Margulis, “Isometry of closed manifolds with the same fundamental group,” *Dokl. Akad. Nauk SSSR,* **192**, No. 4, 736–737 (1970).
128. J. Milnor, “A note on curvature and fundamental groups,” *J. Diff. Geom.,* **2**, No. 1, 1–7 (1968).
129. J. Milnor, *Morse Theory,* Princeton University Press, Princeton, New Jersey (1963).
130. J. Milnor, “On the existence of connection with curvature zero,” *Comment. Math. Helv.,* **32**, No. 2, 15–233 (1958).
131. J. Milnor, “Remarks concerning spin manifolds,” In: *Differential and Combinatorial Topology,* Princeton University Press, Princeton, (1965), pp. 55–62.
132. J. Milnor, “On fundamental groups of complete affinely flat manifolds,” *Adv. Math.,* **25**, 178–187 (1977).
133. J. Milnor, “Curvature of left invariant metrics on Lie groups,” *Adv. Math.,* **21**, No. 3, 293–329 (1976).
134. A. S. Mishchenko and A. T. Fomenko, *A Course on Differential Geometry and Topology* [in Russian], Moscow Univ., Moscow (1980).
135. S. A. Molchanov, “Diffusive processes and Riemanian geometry,” *Usp. Mat. Nauk*, **30**, No. 1, 3–59 (1975).
136. H. M. Morse, “The calculus of variations in the large,” *Amer. Math. Soc., Colloq. Publ.,* **18** (1934).
137. G. D. Mostow, *Strong Rigidity of Locally Symmetric Spaces,* Ser. Ann. Math. Stud., No. 78, Princeton, (1973), pp. 1–195.
138. S. Meyers, “Riemannian manifolds with positive mean curvature,” *Duke Math. J.,* **8**, 401–404 (1941).
139. S. Meyers and N. E. Steenrod, “The group of isometries of a Riemann manifold,” *Ann. Math.,* **40**, 400–416 (1939).
140. T. Nagayoshi and Y. Tsukamoto, “On positively curved Riemannian manifolds with bounded volume,” *Tôhoku Math. J.,* **25**, No. 2, 213–218 (1973).
141. H. Nakagawa and K. Shiohama, “On Riemannian manifolds with certain cut loci,” *Tôhoku Math. J.,* **22**, 14–23 (1970).
142. H. Nakagawa and K. Shiohama, “On Riemannian manifolds with certain cut loci, II,” *Tôhoku Math. J.,* **22**, 357–361 (1970).
143. J. Nash, “C^1-isometric immersions,” *Matematika*, **1**, No. 2, 3–16 (1951).
144. J. Nash, “The imbedding problem for Riemannian manifolds,” *Ann. Math.,* **63**, No. 1, 20–63 (1956).
145. V. V. Nikulin and I. R. Shafarevuch, *Geometry and Groups* [in Russian], Nauka, Moscow (1983).
146. K. Nomizu, *Lie Groups and Differential Geometry,* Publ. Math. Soc., Japan (1956).
147. K. Nomizu and H. Ozeki, “The existence of complete Riemannian metrics,” *Proc. Amer. Math. Soc.,* **12**, 889–891 (1961).
148. K. Nomizu, “Conformal transformations of compact Riemannian manifolds,” *Illinois J. Math.,* **6**, No. 2, 292–295 (1962).
149. A. P. Norden, *Affine Connection Spaces* [in Russian], Nauka, Moscow (1976).
150. S. P. Novikov, “Hamiltonian formalism and a multivalued analog of the Morse theory,” *Usp. Mat. Nauk*, **37**, No. 5, 3–49 (1982).
151. T. Ochiai and T. Takahashi, “The group of isometries of a left-invariant Riemannian metric of a Lie group,” *Math. Ann.,* **223**, No. 1, 91–96 (1976).
152. H. Omori, “A class of Riemannian metrics on a manifolds,” *J. Diff. Geom.,* **2**, 233–252 (1968).
153. Ya. B. Pesin, “Geodesic flows with hyperbolic behavior of trajectories and related objects,” *Usp. Mat. Nauk,* **36**, No. 4, 3–52 (1981).
154. A. Z. Petrov, *New Methods in General Relativity* [in Russian], Nauka, Moscow (1966).
155. A. V. Pogorelov, *Differential Geometry* [in Russian], Nauka, Moscow (1974).
156. H. Poincaré, “Sur les lignes geodesiques des surfaces convexe,” *Trans. Amer. Math. Soc.,* **5**, 237–274 (1905).

157. L. S. Pontryagin, "Certain topological invariants of closed Riemannian manifolds," *Izv. Akad. Nauk SSSR, Ser. Mat.,* **13**, 125–162 (1949).
158. L. S. Pontryagin, *Continuous Groups* [in Russian], Nauka, Moscow (1973).
159. M. M. Postnikov, *Introduction to the Morse Theory* [in Russian], Nauka, Moscow (1971).
160. E. G. Poznyak, "Isometric embeddings of two-dimensional Riemannian metrics in Euclidean spaces," *Usp. Mat. Nauk,* **28**, No. 5, 47–76 (1973).
161. P. K. Rashevskii, *Riemannian Geometry and Tensor Calculus* [in Russian], Nauka, Moscow (1967).
162. P. K. Rashevskii, *A Course on Differential Geometry* [in Russian], Moscow (1956).
163. P. K. Rashevskii, "On geometry of homogeneous spaces," *Dokl. Akad. Nauk SSSR*, **80**, 169–171 (1951).
164. H. E. Rauch, *Geodesics and Curvature in Differential Geometry in the Large,* Yeshiva University, New York (1959).
165. H. E. Rauch, "A contribution to Riemannian geometry in the large," *Ann. Math.,* **54**, 38–55 (1951).
166. Yu. G. Reshetnyak, "Isothermal coordinates in manifolds of bounded curvature, I," *Sib. Mat. Zh.,* **1**, No. 1, 88–116 (1960).
167. Yu. G. Reshetnyak, "Isothermal coordinates in manifolds of bounded curvature, II", *Sib. Mat. Zh.*, **1**, No. 1, 248–276 (1960).
168. Yu. G. Reshetnyak, "To the theory of spaces of curvature not greater than k," *Mat. Sb.,* **52**, No. 3, 789–798 (1960).
169. G. de Rham, "Sur espace de Riemann," *Comment. Math. Helv.,* **26**, 328–344 (1952).
170. E. R. Rozendorn, "Weakly regular surfaces of negative curvature," *Usp. Mat. Nauk*, **21**, No. 5, 59–116 (1966).
171. N. A. Rozenson, "On Riemannian spaces of class 1," *Izv. Akad. Nauk SSSR*, **7**, No. 6, 253–284 (1943).
172. S. Salamon, "Topics in four-dimensional Riemannian geometry," *Lect. Notes Math.,* Vol. 1022 (1983), pp. 33–124.
173. L. Schläfli, "Nota alla memorici del sig Beltrami sugli spazii si curvatura constante," *Ann. Math., Ser. 2,* **5**, 170–193 (1871–1873).
174. R. Schoen and S. T. Yau, "On the structure of manifolds with positive scalar curvature," *Manuscr. Math.,* **28**, No. 1–3, 159–183 (1979).
175. R. Schoen and S. T. Yau, "Positivity of the total mass of a general space-time," *Phys. Rev.,* **43,** 1457–1459 (1979).
176. R. Schoen and S. T. Yau, "On the proof of the positive mass conjecture in general relativity," *Commun. Math. Phys.,* **65**, 45–76 (1979).
177. B. F. Schutz, *Geometrical Methods of Mathematical Physics,* Cambridge University Press (1980), pp. 250.
178. J. T. Schwartz, *Differential Geometry and Topology,* New York (1968).
179. P. Scott, "The geometry of 3-manifolds," *Bull. London Math. Soc.,* **15,** No. 5, 401–487 (1983).
180. L. I. Sedov, *Mechanics of Continuous Media* [in Russian], Nauka, Moscow (1976).
181. Seminar Arthur Besse, *Geometrie Riemannianne,* Textes Mathematiques 3. Cedic/Fernand Nathan Paris (1981).
182. Seminar on Differential Geometry, S. T. Yau, Ed., *Ann. Math. Stud.,* Vol. 102, Princeton Univ. Press (1982).
183. V. A. Sharafutdinov, "Relative Euler class and the Gauss–Bonnet theorem," *Sib. Mat. Zh.*, **14**, No. 6, 1321–1335 (1973).
184. V. A. Sharafutdinov, "Complete open manifolds of nonnegative curvature," *Sib. Mat. Zh.*, **15**, No. 1, 177–191 (1974).
185. J. L. Sing, "On the geometry of dynamics," *Phil. Trans. Roy. Soc. London, Ser. A* (1926).

186. A. A. Slavnov and L. D. Faddeev, *Introduction to the Quantum Gauge Field Theory* [in Russian], Nauka, Moscow (1978).
187. A. S. Solodovnikov, "Projective transformations of Riemannian spaces," *Usp. Mat. Nauk,* **11**, No. 4, 45–116 (1956).
188. S. Sternberg, *Lectures on Differential Geometry,* Prentice–Hall, New Jersey (1964).
189. R. Sulanke and P. Wintgen, *Differentialgerometrie und Faserbundel,* Veb Deutscher Verlag der Wissenschaften, Berlin (1972).
190. Ya. V. Tatarinov, *Lectures on Classical Dynamics* [in Russian], Moscow Univ., Moscow (1984).
191. C. H. Taubes, "Self-dual Yang–Mills connections on non-self-dual 4-manifolds," *J. Diff. Geom.,* **17**, No. 1, 139–170 (1982).
192. T. Y. Thomas, "Riemannian spaces of class one and their characterization," *Acta Math.,* **67**, No. 3–4, 169–211 (1936).
193. W. Thurston, *The Geometry and Topology of* 3*-Manifolds,* Princeton Univ. Press, Princeton (1978), pp. 311.
194. V. A. Toponogov, "Convexity property of Riemannian spaces of positive curvature," *Dokl. Akad. Nauk SSSR,* **115**, No. 4, 674–676 (1957).
195. V. A. Toponogov, "Riemannian spaces of curvature bounded from below by a positive number," *Dokl. Akad. Nauk SSSR*, **120**, No. 4, 719–721 (1958).
196. V. A. Toponogov, "Riemannian spaces of bounded from below curvature," *Usp. Mat. Nauk*, **14**, No. 1, 87–130 (1959).
197. V. A. Toponogov, "Dependence between the curvature and the topological structure of Riemannian spaces," *Dokl. Akad. Nauk SSSR,* **133**, No. 5, 1031–1033 (1960).
198. V. A. Toponogov, "Metric structure of Riemannian spaces of nonnegative curvature containing straight lines," *Sib. Mat. Zh.,* **5**, No. 6, 1358–1369 (1964).
199. V. A. Toponogov, "Certain extremal problems of Riemannian geometry," *Sib. Mat. Zh.,* **8**, No. 5, 1079–1103 (1967).
200. F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators,* Plenum Press, New York–London (1982).
201. V. V. Trofimov, "Completely integrable geodesic flows of left-invariant metrics on Lie groups connected with commutative graded algebra with Poincaré duality," *Dokl. Akad. Nauk SSSR,* **263**, No. 4, 812–816 (1982).
202. V. V. Trofimov and A. T. Fomenko, "Liouville integrability of Hamiltonian systems on Lie algebras," *Usp. Mat. Nauk*, **39**, No. 2, 3–56 (1984).
203. Yu. Yu. Trokhimchuk, Yu. B. Zelinskii, and V. V. Sharko, "On certain results in topology of manifolds, multivalued mapping theory, and the Morse theory," *Tr. Mat. Inst. Akad. Nauk USSR*, **154**, 222–230 (1983).
204. Y. Tsukamoto, "Über gewisse Riemannische mannigfaltigkeiten mit positiver Krümmung," *Nagoya Math. J.,* **52**, 35–38 (1973).
205. K. K. Uhlenbeck, "Connections with bounds on curvature," *Commun. Math Phys.,* **83**, 31–42 (1982).
206. V. Vagner, "Geometry of the configuration space of a rigid body rotating along a fixed point," *Uch. Zap. Saratov Univ.,* **1**, No. 2, 34–59 (1938).
207. F. M. Valiev, "Exact bounds of sectional curvatures of homogeneous Riemannian metrics on Wollach spaces," *Sib. Mat. Zh.*, **20**, No. 2, 248–262 (1979).
208. M. Vigué-Poirrier and D. Sullivan, "The homology theory of the closed geodesic problem," *J. Diff. Geom.,* **11**, No. 4, 633–644 (1976).
209. N. R. Wallach, "Compact homogeneous Riemannian manifolds with strictly positive curvature," *Ann. Math.,* **96**, No. 2, 277–295 (1972).
210. H. C. Wang, "Topics in totally discontinuous groups," In: *Symmetric Spaces,* Vol. 4, Marcel Dekker, (1972), pp. 460-485.

211. A. Weil, *Introduction à l'Étude des Variétés Kählériennes,* Hermann, Paris (1958).
212. H. Weyl, "Über die Bestimmung einer geschlossen konvexen Flache durch ihr Linienelement," *Vierteljahresschrift der Natur forschenden gesellschaft in Zürich,* 40–72 (1916).
213. J. A. Wolf, *Spaces of Constant Curvature,* Univ. California, Berkeley, California (1972).
214. B. Wong, "The uniformization of compact Kähler surfaces of negative curvature," *J. Diff. Geom.,* **16**, 407–420 (1981).
215. H. Yamabe, "On the deformation of Riemannian structure on compact manifolds," *Osaka Math. J.,* **12**, 21–37 (1960).
216. N. N. Yanenko, "To the embedding theory of higher-dimensional Riemanian spaces in Euclidean spaces," *Tr. Mosk. Mat. Obshch.,* **3**, 89–180 (1054).
217. P. Yang, "On Kähler manifolds with negative bisectional curvature," *Duke Math. J.,* **43**, 871–874 (1967).
218. K. Yano and S. Bochner, *Curvature and Betti Numbers,* Princeton Univ. Press, Princeton, New Jersey (1953).
219. S. T. Yau, "On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampere equation, I," *Commun. Pure Appl. Math.,* **31**, 339–411 (1978).
220. S. T. Yau, "On the fundamental group of compact manifolds of nonpositive curvature," *Ann. Math.,* **93**, 579-585 (1971).
221. K. Yosida, *Functional Analysis,* Springer-Verlag, Berlin–Göttingen–Heidelberg (1965).
222. W. Ziller, "Geschlossene geodätische auf global symmetrischen und homogenen Räumen," *Bonn. Math. Schr.,* No. 85 (1976).
223. O. Zoll, "Über Flächen mit Scharen geschlossener geodatischer Linien," *Math. Ann.,* **57**, 108–133 (1903).