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Symmetries of first-order stochastic ordinary differential equations revisited

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SUMMARY

Symmetries of stochastic ordinary differential equations (SODEs) are analysed. This work focuses on maintaining the properties of the Weiner processes after the application of infinitesimal transformations. The determining equations (DEs) for first-order SODEs are derived in an Itô calculus context. These DEs are non-stochastic. This article reconciles earlier works in this area. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Two years after the seminal work by Gaeta and Quintero [1] which brought to the fore the relations between the symmetries of the Fokker–Planck (FP) equation and its corresponding Itô stochastic (ordinary) differential equation (SDE), a paper by Wafo Soh and Mahomed [2] explained how to derive these Lie point symmetries without referring to the corresponding FP equation and without using these symmetries to transform the Itô SDE into a different one as had been done in [3]. This novelty in methodology was able to incorporate higher-order SDEs like the governing equation for the response of a mass–spring oscillator to a white noise random excitation. Ünal [4] observed that the determining equations (DEs) he obtained for finding symmetries of first-order SDEs were not in agreement with the version of [2], as it precluded an extra condition given in his derivation. This paper is aimed at reconciling these two seminal works.

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In the first section, we present the basic mathematical background needed in order to transform our spatial, temporal and Wiener variables according to the usual Lie symmetry approach. The following section then derives the DEs that are required for solving the symmetries. We closely follow the methodology of [2] in this regard. However, we extend the derivation of [2] further and arrive at an alternative form of the same DEs that were constructed by Ünal [4]. This route leads to the same extra condition that was found in [4]. It also yields another important condition on the temporal symmetry variable τ, which ensures that the transformed Wiener differential still behaves like a standard Wiener process. We thus, in the third section, review the steps given in [4]; deriving these DEs and comparing them with those found in the previous section mentioned above. We conclude with the same example used in [4] to provide evidence that we have reconciled the works of [2, 4]. We, in fact, show that Ünal's extra condition is a direct consequence of our extension using the properties of the Wiener process.

2. PRELIMINARIES

In order to work with SDEs, we first have to familiarize ourselves with how we associate events ω belonging to a sample space Ω with a probability measure $\mathbb{P}$. We apply the probability measure specifically to a system of subsets of Ω, which we denote by $\mathscr{F}$. This σ-algebra $\mathscr{F}$ contains the complement and countable union of any of its arbitrary members, which we call open sets (refer to [5] for summarized definitions concerning measure theory). We then form a *natural filtration* by forming an indexed family of σ-algebras $\mathscr{F}_t$, where t is a time index, to which the sample paths of our processes are *adapted* (see [6]). The probability space $(\Omega, \mathscr{F}, \mathbb{P})$ that we have introduced allows us to proceed with the introduction of the randomness which drives the SDE, namely the Wiener process. The Wiener process is a family of random variables indexed, for our purposes, by time t, which belongs to the interval I, which can be taken to be the positive real line. This process is a mathematical tool used for formalizing the physical phenomena of Brownian motion; its sample paths, which are obtained by focusing on a fixed realization of particular event $\omega \in \Omega$ and following their families of random variables through time, are almost surely continuous, and are almost surely nowhere differentiable in the usual sense. (There are many books that explain these concepts, e.g. [7, 8].) We represent it as a function $W(t, \omega)$ which performs the following:

$$(t, \omega) \in I \times \Omega \longrightarrow W(t, \omega) \in \mathbb{R}$$

The ω in the argument of our function is an arbitrary event and is thus suppressed throughout the paper. This process $W(t)$ also has the following characteristics:

- At time zero with probability 1, $W(0) = 0$.
- For any strictly increasing sequence of indexed times $\{t_i\} \subset I$, the random variables $W(t_{i+1}) - W(t_i)$ are independent.
- For times $s < t$, $W(t) - W(s)$ is normally distributed with a zero mean and a variance of $t - s$.
- The covariance between two scalar processes at different times $\mathbb{E}(W(s)W(t))$ is just the minimum between the two different times $\min(t, s)$.

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The conditions used in deriving the DEs in [2, 4] are based upon what is known as the *Itô's multiplication table*—simple mneumonics based on *Itô Isometry*, see [9]:

$$\begin{array}{c|ccc} & \mathrm{d}W_l(t) & \mathrm{d}W_m(t) & \mathrm{d}t \\ \hline \mathrm{d}W_l(t) & \mathrm{d}t & 0 & 0 \\ \mathrm{d}W_m(t) & 0 & \mathrm{d}t & 0 \\ \mathrm{d}t & 0 & 0 & 0 \end{array} \tag{1}$$

Here, $\mathrm{d}W_l(t)$ and $\mathrm{d}W_m(t)$ are two independent standard Wiener processes, where $l, m = 1, \ldots, N$. From this, we begin to realize that the Newton–Leibnitz chain rule in differential form, that we need to use in order to apply invariance arguments to our spatial, temporal and Wiener variables, has to be adjusted. The justification for this lies in the quadratic variation of the Wiener process, i.e. $(\mathrm{d}W(t))^2$ has mean value of $\mathrm{d}t$, which is finite. This leads to the following theorem.

Theorem 2.1 (*Itô's Formula, Øksendal* [9])
If $\mathbf{X}(t)$, an N-dimensional vector, is an Itô process,

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}\,\mathrm{d}t + \mathbf{G}\,\mathrm{d}\mathbf{W}(t) \tag{2}$$

where $\mathbf{f}(t, \mathbf{X}(t))$ and $\mathbf{G}(t, \mathbf{X}(t))$ are an N-dimensional drift vector coefficient and diffusion matrix coefficient of dimension $N \times M$, respectively; then for an arbitrary application $\mathbf{F}: I \times \mathbb{R}^N \rightarrow \mathbb{R}^M$, which is twice differentiable in the spatial coordinates, $\mathbf{F}(t, .) \in \mathscr{C}^2(\mathbb{R}^N, \mathbb{R}^M)$ and only differentiable with respect to time once, $\mathbf{F}(., \mathbf{x}) \in \mathscr{C}^1(I, \mathbb{R}^M)$ for all $(s, \mathbf{y}) \in I \times \mathbb{R}^{\mathcal{N}}$, an Itô process $\mathbf{F}(t, \mathbf{X}(t))$ exists and is written in component form as

$$\mathrm{d}F_j(t, \mathbf{X}(t)) = \left.\frac{\partial F_j(t, \mathbf{x})}{\partial t}\right|_{(t,\mathbf{X}(t))} \mathrm{d}t + \left.\frac{\partial F_j(t, \mathbf{x})}{\partial x_i}\right|_{(t,\mathbf{X}(t))} \mathrm{d}X_i(t) + \frac{1}{2}\left.\frac{\partial^2 F_j(t, \mathbf{x})}{\partial x_i \partial x_m}\right|_{(t,\mathbf{X}(t))} \mathrm{d}X_i(t)\,\mathrm{d}X_m(t) \quad \text{for } j = 1, \ldots, N$$

The evaluation of each of the partial derivatives on the right-hand side is made at $(t, \mathbf{X}(t))$, which we give as

$$\mathrm{d}F_j(t, \mathbf{X}(t)) = \frac{\partial F_j}{\partial t}\mathrm{d}t + \frac{\partial F_j}{\partial x_i}\mathrm{d}X_i(t) + \frac{1}{2}\frac{\partial^2 F_j}{\partial x_i \partial x_m}\mathrm{d}X_i(t)\,\mathrm{d}X_m(t) \tag{3}$$

It should be kept in mind that though $\mathbf{X}(t)$ is indexed by time, it is by its random nature independent of time. The repeated index summation convention is assumed throughout this work. The terms $\mathrm{d}\mathbf{X}_i(t)$ and $\mathrm{d}\mathbf{X}_i(t)\,\mathrm{d}\mathbf{X}_m(t)$ are evaluated using (2) and the Itô multiplication table to obtain

$$\mathrm{d}F_j(t, \mathbf{X}(t)) = \Gamma(F_j)(t, \mathbf{X}(t))\,\mathrm{d}t + Y(F_j)^l(t, \mathbf{X}(t))\,\mathrm{d}W_l(t) \tag{4}$$

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where

$$\Gamma(F_j)(t,\mathbf{X}(t)) = \frac{\partial F_j}{\partial t} + f_i \frac{\partial F_j}{\partial x_i} + \frac{1}{2}\sum_{k=1}^{M} G_i^k G_m^k \frac{\partial^2 F_j}{\partial x_i \partial x_m} \tag{5}$$

$$Y(F_j)^l(t,\mathbf{X}(t)) = \frac{\partial F_j}{\partial x_i} G_i^l \quad \text{for each } l=1,\ldots,M \tag{6}$$

For the existence and uniqueness of a temporally continuous solution, besides the assumption that $\mathbf{X}(t)$ belongs to $\mathscr{L}^2$ for an interval $[0, T]$, we also assume that the instantaneous mean and diffusion coefficients of (2) are Lipschitz continuous (see [8, Chapter 7]). We give an example to illustrate how Itô's theorem could be applied to find the integral of a function of the Wiener process. From this example, one notices how the Newtonian calculus differs from the Itô calculus.

Example 2.1
The Wiener process $\mathbf{W}^2(t)$ is an Itô process. We apply Itô's formula (3) to $\mathbf{W}^3(t)$ to find the integral of the process $\mathbf{W}^2(t)$. We, therefore, obtain

$$\begin{aligned} \mathrm{d}(\mathbf{W}^3(t)) &= 3\mathbf{W}^2(t)\,\mathrm{d}\mathbf{W}(t) + \tfrac{1}{2}6\mathbf{W}(t)(\mathrm{d}\mathbf{W}(t))^2 \\ &= 3\mathbf{W}^2(t)\,\mathrm{d}\mathbf{W}(t) + 3\,\mathbf{W}(t)\,\mathrm{d}t \end{aligned} \tag{7}$$

Thus, by integrating, we arrive at

$$\mathbf{W}^3(T) - \mathbf{W}^3(0) = 3\int_0^T \mathbf{W}^2(t)\,\mathrm{d}\mathbf{W}(t) + \int_0^T 3\mathbf{W}(t)\,\mathrm{d}t$$

Dividing throughout by 3 and rearranging terms now simplifies to

$$\int_0^T \mathbf{W}^2(t)\,\mathrm{d}\mathbf{W}(t) = \frac{1}{3}\mathbf{W}(T)^3 - \int_0^T \mathbf{W}(t)\,\mathrm{d}t \tag{8}$$

One easily identifies the extra term $-\int_0^T \mathbf{W}(t)\,\mathrm{d}t$, as Itô's correction term. This adjusts the answer we would have obtained had we used basic Newtonian calculus methods. Since the calculus governing Wiener processes is not as straightforward as Newtonian calculus, in this case that the transformation of a Wiener process into another Wiener process would have to be contended with. This brings us to the following theorem.

Theorem 2.2 (*Random time change for Itô integrals, Øksendal* [*10*])
Let $c(t,\omega)$ be the measurable time change rate, which is related to our time change scalar stochastic process $\beta(t,\omega)$, by the following equation:

$$\beta(t,\omega) = \int^t c(s,\omega)\,\mathrm{d}s \tag{9}$$

and $\alpha(t,\omega)$ be a scalar stochastic process satisfying

- $\alpha(0,\omega) = 0$.
- $\mathrm{d}\alpha(t,\omega)/\mathrm{d}t = 1/c(\alpha(t),\omega) \geqslant 0$, for almost all positive time and almost all $\omega \in \Omega$.
- $\beta(t,\omega)$ and $\alpha(t,\omega)$ are left and right inverses of each other, respectively, $\alpha(\beta(t,\omega),\omega) = \beta(\alpha(t,\omega),\omega) = t$ for all $(t,\omega) \in I \times \Omega$.

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Then, under the (random) time change $\bar{t}=\beta(t,\omega)$, the Wiener process $\mathbf{W}(\alpha(t),\omega)$ is mapped to another Wiener process $\overline{\mathbf{W}}(t,\omega)$ according to the relation

$$\sqrt{\frac{\mathrm{d}\alpha(t)}{\mathrm{d}t}}\,\mathrm{d}\overline{\mathbf{W}}(t)=\mathrm{d}\mathbf{W}(\alpha(t)) \tag{10}$$

where we have suppressed ω in the expression above. This can then expressed as

$$\mathrm{d}\overline{\mathbf{W}}(\beta(t))=\sqrt{c(t)}\,\mathrm{d}\mathbf{W}(t) \tag{11}$$

by using the inverse relation between $\alpha(t)$ and $\beta(t)$ in conjunction with (9).

3. DERIVATION OF THE DETERMINING EQUATIONS

Consider an Itô process

$$\mathrm{d}\mathbf{X}(t)=\mathbf{f}(t,\mathbf{X}(t))\,\mathrm{d}t+\mathbf{G}(t,\mathbf{X}(t))\,\mathrm{d}\mathbf{W}(t) \tag{12}$$

where $\mathbf{f}(t,\mathbf{x})$ is a vector of N dimension, which is the same as the dimension of the process $\mathbf{X}(t)$ and $\mathbf{G}(t,\mathbf{x})$ is an $N\times M$-matrix. These functions are evaluated at $\mathbf{X}(t)$ in the system of Itô processes above. The Lie Point Theorem symmetry approach for ODEs requires spatial and temporal infinitesimals $\xi(t,x)$ and $\tau(t,x)$ in its analysis. In the SODE framework, these entities are functionally based on the spatial stochastic process, $\mathbf{X}(t)$, and using Itô's formula (4), we have that the jth spatial infinitesimal, for $j=1,\ldots,N$, and temporal infinitesimal are themselves solutions to Itô processes given in component form, respectively, as

$$\mathrm{d}\xi_j(t,\mathbf{X}(t))=\Gamma(\xi_j)(t,\mathbf{X}(t))\,\mathrm{d}t+Y(\xi_j)^l(t,\mathbf{X}(t))\,\mathrm{d}W_l(t) \tag{13}$$

and

$$\mathrm{d}\tau(t,\mathbf{X}(t))=\Gamma(\tau)(t,\mathbf{X}(t))\,\mathrm{d}t+Y(\tau)^l(t,\mathbf{X}(t))\,\mathrm{d}W_l(t) \tag{14}$$

where $\Gamma(\xi_j)$, $Y(\xi_j)^l$, $\Gamma(\tau)$, and $Y(\tau)^l$ are the drift and diffusion coefficients of our spatial and temporal infinitesimals, respectively, and defined using (5) and (6). The Lie Point Theorem (see [2]), as in [2], uses DEs to furnish symmetries that would enable the transformation of a solution of the equation to another. These DEs are in fact $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$ equations derived from form invariant ODE point transformation analysis. The resultant higher-order equations of this form invariant analysis are functionally dependent on the solution of these equations. We perform a similar point transformation of (12)'s spatial, temporal and the Wiener variables

$$\begin{aligned}\overline{X}_j(t)&=\mathrm{e}^{\varepsilon H}(X_j(t))\\&=\int^t\Gamma(\mathrm{e}^{\varepsilon H}(X_j(s)))\,\mathrm{d}s+\int^t Y(\mathrm{e}^{\varepsilon H}(X_j(s)))\,\mathrm{d}W(s)\end{aligned} \tag{15}$$

$$\begin{aligned}\bar{t}&=\mathrm{e}^{\varepsilon H}(t)\\&=\int^t\Gamma(\mathrm{e}^{\varepsilon H}(s))\,\mathrm{d}s+\int^t Y(\mathrm{e}^{\varepsilon H}(s))\,\mathrm{d}W(s)\end{aligned} \tag{16}$$

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and

$$\mathrm{d}\overline{W}_l(\bar{t}) = \sqrt{\frac{\mathrm{d}(\mathrm{e}^{\varepsilon H}(t))}{\mathrm{d}t}}\,\mathrm{d}W_l(t) \quad \text{for each } l = 1, \ldots, M \tag{17}$$

using the random time change formula and Itô's formula, where H is the symmetry generator

$$H = \tau(t, \mathbf{x})\frac{\partial}{\partial t} + \xi_j(t, \mathbf{x})\frac{\partial}{\partial x_j} \tag{18}$$

with the spatial and temporal infinitesimals $\xi(t, \mathbf{x})$ and $\tau(t, \mathbf{x})$, respectively. The point transformation of the drift and diffusion coefficients is given by

$$f_j(\bar{t}, \bar{\mathbf{x}}) = \mathrm{e}^{\varepsilon H}(f_j(t, \mathbf{x})) \tag{19}$$

and

$$g_i^k(\bar{t}, \bar{\mathbf{x}}) = \mathrm{e}^{\varepsilon H}(g_i^k(t, \mathbf{x})) \tag{20}$$

for each $i, j = 1, \ldots, N$ and $k = 1, \ldots, N$. The transformations (15)–(17), (19) and (20) are used in conjunction with Itô's formula to form an invariant version of the original SODE (12):

$$\mathrm{d}\overline{\mathbf{X}}(\bar{t}) = \mathbf{f}(\bar{t}, \overline{\mathbf{X}}(\bar{t}))\,\mathrm{d}\bar{t} + \mathbf{G}(\bar{t}, \overline{\mathbf{X}}(\bar{t}))\,\mathrm{d}\overline{\mathbf{W}}(\bar{t}) \tag{21}$$

The transformed standard Wiener process, $\mathrm{d}\overline{\mathbf{W}}(\bar{t})$, should be invariant in terms of the existence of an instantaneous mean and variance which implies that the following should still hold, viz:

$$\mathbb{P}[|\mathrm{d}\overline{W}_l(\bar{t})| > \varepsilon | \mathbf{W}(t) = \mathbf{w}, \mathbf{X}(t) = \mathbf{x}] = 0 \quad \text{for all } \varepsilon > 0 \tag{22}$$

$$\mathbb{E}[\mathrm{d}\overline{W}_l(\bar{t}) | \mathbf{W}(t) = \mathbf{w}, \mathbf{X}(t) = \mathbf{x}] = 0 \tag{23}$$

$$\mathbb{E}[\mathrm{d}\overline{W}_l(\bar{t})\,\mathrm{d}\overline{W}_m(\bar{t}) | \mathbf{W}(t) = \mathbf{w}, \mathbf{X}(t) = \mathbf{x}] = \mathrm{d}\bar{t}\,\delta_l^m \tag{24}$$

This implies that if we expand (23) by using (14) in conjunction with (17), we can show that the diffusion coefficient of temporal infinitesimal, $\tau(t, \mathbf{X}(t))$, is zero, i.e.

$$Y(\tau)^l(t, \mathbf{X}) = 0 \quad \text{for each } l = 1, \ldots, M \tag{25}$$

which is exactly the condition that Ünal [4] derived using a form invariant argument on the Itô multiplication table. As a result (16) and (17) become

$$\bar{t} = \int^t \Gamma(\mathrm{e}^{\varepsilon H}(s))(s, \mathbf{X}(s))\,\mathrm{d}s \tag{26}$$

and

$$\mathrm{d}\overline{W}_l(\bar{t}) = \sqrt{\Gamma(\mathrm{e}^{\varepsilon H}(t))}\,\mathrm{d}W_l(t) \quad \text{for each } l = 1, \ldots, M \tag{27}$$

respectively, where the temporal infinitesimal instantaneous drift, $\Gamma(\mathrm{e}^{\varepsilon H}(s))$, can be viewed as the time change rate, $c(s)$ in (9). Since the temporal instantaneous drift is measurable as a result of

Itô's formula, the random time change formula still holds for this application. Expanding the drift term $\mathbf{f}(\bar{t}, \overline{\mathbf{X}}(\bar{t}))\,\mathrm{d}\bar{t}$ on the right-hand side of (21) with simple algebra gives

$$\Bigg\{\mathbf{f}(t,\mathbf{X}(t)) + \varepsilon(\Gamma(H(t)) + H)\mathbf{f}(t,\mathbf{X}(t)) + \sum_{k=2}^{\infty}\frac{\varepsilon^k}{k!}\Bigg((\Gamma(H(t)) + H)^k\mathbf{f}(t,\mathbf{X}(t)) + \sum_{j=0}^{k-2}\binom{k}{k-j}\mathbf{f}(t,\mathbf{X}(t))H^j(t)(\Gamma(H^{k-j}(t)) - [\Gamma(H(t))]^{k-j})\Bigg)\Bigg\}\,\mathrm{d}t \tag{28}$$

In order to use the Lie Point Theorem in the SODE context, we require that all terms of order higher than $\mathcal{O}(\varepsilon)$ be functionally dependent on terms of order $\mathcal{O}(1)$ and $\mathcal{O}(\varepsilon)$. As a result of this dependency, higher-order terms can be ignored completely and justifies the methods of [2, 4]. This dependency, however, forces the following condition:

$$\mathrm{e}^{\varepsilon\Gamma(H(t))}(t,\mathbf{X}(t)) = \Gamma(\mathrm{e}^{\varepsilon H}(t)(t,\mathbf{X}(t))) \tag{29}$$

and the resultant relationship, by separation of coefficients of ε, between the drift components of the left- and right-hand side of (21) can be expressed as

$$\Gamma(H^k(\mathbf{x}))(t,\mathbf{X}(t)) = (\Gamma(H(t)) + H)^k f(t,\mathbf{X}(t)) \tag{30}$$

for $k = 1, 2, 3, \ldots$. Thus for $k = 1$, we have our first DE

$$\Gamma(H(\mathbf{x})) = (\Gamma(H(t)) + H)\, f(t,\mathbf{X}(t)) \tag{31}$$

which partially solves for the spatial and temporal infinitesimals. By using the DE (31) in (30) for the remaining higher-order equations, a direct functional dependency between the two is established by the following:

$$\Gamma(H^k(\mathbf{x})) = (\Gamma(H(t)) + H)^{k-1}\Gamma(H(\mathbf{x})) \quad \text{for } k = 2, 3, 4, \ldots \tag{32}$$

Before deriving the remaining DE, we first note that (27) can be expressed as

$$\mathrm{d}\overline{W}_l(\bar{t}) = \mathrm{e}^{\varepsilon\Gamma(H(t))/2}\,\mathrm{d}W_l(t) \quad \text{for each } l = 1, \ldots, M \tag{33}$$

as a result of (29). If we expand the diffusion component $\mathbf{G}(\bar{t}, \overline{\mathbf{X}}(\bar{t}))\,\mathrm{d}\overline{\mathbf{W}}(\bar{t})$ of (21) and then compare these components on both sides of (21) by separation of coefficients of ε, we obtain the following:

$$Y^l(H(\mathbf{x}))(t,\mathbf{X}(t)) = \left(\frac{\Gamma(H(t))}{2} + H\right)G^l(t,\mathbf{X}(t)) \tag{34}$$

$$Y^l(H^k(\mathbf{x}))(t,\mathbf{X}(t)) = \left(\frac{\Gamma(H(t))}{2} + H\right)^{k-1} Y^l(H(\mathbf{x})) \quad \text{for } k = 2, 3, 4, \ldots \tag{35}$$

for each $l = 1, \ldots, M$, where (34) is the last DE required to solve for the infinitesimals. The functional dependency of higher-ordered equations on zero- and first-order ones is satisfied in (35). All that remains to be shown is that the DEs are unique to their SODEs from which

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they are derived. If we are given the DEs (31) and (34), the canonical symmetry that is immediately applicable is the time scaling symmetry $H=\partial/\partial t$. From this, we observe that the drift and diffusion coefficients have to be functions of the spatial variable only in order to satisfy (31) and (34). Thus, the SODE associated with this particular symmetry is given by

$$\mathrm{d}\mathbf{X}(t)=\mathbf{f}(\mathbf{X}(t))\,\mathrm{d}t+\mathbf{G}(\mathbf{X}(t))\,\mathrm{d}\mathbf{W}(t) \tag{36}$$

Thus, we have shown the following theorem which was partially proved in Wafo Soh and Mahomed [2].

Theorem 3.1 (*Lie Point Theorem for SODE*)
The Itô SODE

$$\mathrm{d}\mathbf{X}(t)=\mathbf{f}(t,\mathbf{X}(t))\,\mathrm{d}t+\mathbf{G}(t,\mathbf{X}(t))\,\mathrm{d}\mathbf{W}(t) \tag{37}$$

has the following DEs and conditions that have to hold in order to transform a solution of (37) to that of another solution using Lie point symmetry methods:

$$\Gamma(H(x))(t,\mathbf{X}(t))=(\Gamma(H(t))+H)f(t,\mathbf{X}(t)) \tag{38}$$

$$Y^l(H(x))(t,\mathbf{X}(t))=\left(\frac{\Gamma(H(t))}{2}+H\right)G^l(t,\mathbf{X}(t)) \tag{39}$$

$$\mathrm{e}^{\varepsilon\Gamma(H(t))}(t,\mathbf{X}(t))=\Gamma(\mathrm{e}^{\varepsilon H}(t))(t,\mathbf{X}(t)) \tag{40}$$

and

$$Y(\tau)^l(t,\mathbf{X}(t))=0 \quad \text{for each } l=1,\ldots,M \tag{41}$$

To establish a comparison between these results and those of [2], we resort to the definition of the first prolongation of an infinitesimal generator for non-stochastic ODEs:

$$H^{[1]}=H+\xi_j^{[1]}\frac{\partial}{\partial\dot{x}_j} \tag{42}$$

where

$$\dot{x}_j=\frac{\mathrm{d}x_j}{\mathrm{d}t} \tag{43}$$

$$=D_t x_j \tag{44}$$

$$\xi_j^{[1]}=D_t(\xi_j)-\dot{x}_j D_t(\tau) \tag{45}$$

$$=\frac{\partial\xi_j}{\partial t}+\dot{x}_i\frac{\partial\xi_j}{\partial x_i}-\dot{x}_j\left(\frac{\partial\tau}{\partial t}-\dot{x}_i\frac{\partial\tau}{\partial x_i}\right) \tag{46}$$

with the total time derivative D_t given as

$$D_t=\frac{\partial}{\partial t}+\dot{x}_i\frac{\partial}{\partial x_i}+\ddot{x}_i\frac{\partial}{\partial\dot{x}_i}+\cdots \tag{47}$$

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Applying the first prolongation on $(\dot{x}_j - f_j)$ at $\dot{\mathbf{x}} = \mathbf{f}$ can be represented as

$$H^{[1]}(\dot{x}_j - f_j)|_{\dot{\mathbf{x}}=\mathbf{f}} = \xi_j^{[1]} - H(f_j) \tag{48}$$

Using (46) we find that (48) in conjunction with the second-order derivative terms of the instantaneous spatial and temporal drifts constitute the whole of (31) and we can express this as

$$\left(H^{[1]}(\dot{x}_j - f_j)|_{\dot{\mathbf{x}}=\mathbf{f}} + \frac{1}{2}\sum_{k=1}^{M} G_i^k G_p^k \left(\frac{\partial^2 \xi_j}{\partial x_i \partial x_p} - f_j \frac{\partial^2 \tau}{\partial x_i \partial x_p} \right) \right)(t, \mathbf{X}(t)) = 0 \tag{49}$$

If we now consider (34), there is no *white noise* term, $\mathrm{d}W_l(t)/\mathrm{d}t$, as was found in the previous attempt by Wafo Soh and Mahomed [2] since $Y(\tau)^l = 0$.

4. ÜNAL's EXTRA CONDITION

Ünal [4] commented that the Itô multiplication table for the transformed variables must be applicable, i.e.

$$\mathrm{d}\overline{W}_l(\bar{t})\,\mathrm{d}\overline{W}_m(\bar{t}) = \delta_l^m\,\mathrm{d}\bar{t} \tag{50}$$

$$\mathrm{d}\overline{W}_i(\bar{t})\,\mathrm{d}\bar{t} = 0 \tag{51}$$

$$\mathrm{d}\bar{t}\,\mathrm{d}\bar{t} = 0 \tag{52}$$

for each i, l and $m = 1, \ldots, M$ and derived his DEs from this standpoint. Recently, Srihirun *et al.* [11] stated that no strict proof had been obtained in the past to verify that the transformed Wiener processes using the random time change formula would still satisfy the properties of a Wiener process. All that the random time change formula requires for it to be applicable to SODEs is the measurability of the rate of time change, which Itô's formula preserves. The spatial process $\mathbf{X}(t)$ is measurable at the onset, so all functions of this stochastic process will be measurable too. The strict proof has been obtained in [9, 10]; the consequences of these properties on the symmetry infinitesimals were investigated in [4]. Using the results (26) and (33), we find

$$\mathrm{d}\overline{W}_l(\bar{t})\,\mathrm{d}\overline{W}_m(\bar{t}) = \mathrm{e}^{\varepsilon\Gamma(\tau)/2+\varepsilon\Gamma(\tau)/2}\,\mathrm{d}W_l(t)\,\mathrm{d}W_m(t)\delta_l^m = \delta_l^m\,\mathrm{e}^{\varepsilon\Gamma(\tau)}\,\mathrm{d}t = \delta_l^m\,\overline{\mathrm{d}t} \tag{53}$$

$$\mathrm{d}\overline{W}_l(\bar{t})\,\mathrm{d}\bar{t} = \mathrm{e}^{(3/2)\varepsilon\Gamma(\tau)}\,\mathrm{d}W(t)\,\mathrm{d}t = 0 \tag{54}$$

and

$$\mathrm{d}\bar{t}\,\mathrm{d}\bar{t} = 0 \quad \text{are automatically satisfied} \tag{55}$$

for each i, l and $m = 1, \ldots, M$. Thus, our application of the Lie Point Theorem for SODE is consistent with the criteria set by Ünal [4]. We use the same example as Ünal [4] to show that the symmetries, which we arrive at using (31) and (34), are the same as those found in [4].

Example 4.1
Let $\mathbf{X}(t)$ be an Itô process

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}\,\mathrm{d}t + \mathbf{G}\,\mathrm{d}W(t) \tag{56}$$

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where $\mathbf{f}$ is the vector

$$\begin{pmatrix} -\frac{1}{2}X_1(t) \\ -\frac{1}{2}X_2(t) \end{pmatrix} \tag{57}$$

and $\mathbf{G}$ the vector

$$\begin{pmatrix} -X_2(t) \\ X_1(t) \end{pmatrix} \tag{58}$$

Thus from Wafo Soh and Mahomed's [2] corrected version of the DEs (49) and (34), we have for $j=1$:

$$H^{[1]}\left(\dot{x}_1+\frac{1}{2}x_1\right)\bigg|_{\dot{\mathbf{x}}=\mathbf{f}}+\frac{1}{2}G_i^1G_p^1\left(\frac{\partial^2\xi_1}{\partial x_i\partial x_p}+\frac{1}{2}x_1\frac{\partial^2\tau}{\partial x_i\partial x_p}\right)=0 \tag{59}$$

$$-\xi_2-G_i^1\left(\frac{\partial\xi_1}{\partial x_i}\right)-\frac{1}{2}x_2\left(\frac{\partial\tau}{\partial t}+f_i\frac{\partial\tau}{\partial x_i}+\frac{1}{2}G_i^1G_p^1\frac{\partial^2\tau}{\partial x_i\partial x_p}\right)=0 \tag{60}$$

and for $j=2$:

$$H^{[1]}\left(\dot{x}_2+\frac{1}{2}x_2\right)\bigg|_{\dot{\mathbf{x}}=\mathbf{f}}+\frac{1}{2}G_i^1G_p^1\left(\frac{\partial^2\xi_2}{\partial x_i\partial x_p}+\frac{1}{2}x_2\frac{\partial^2\tau}{\partial x_i\partial x_p}\right)=0 \tag{61}$$

$$\xi_1-G_i^1\left(\frac{\partial\xi_2}{\partial x_i}\right)+\frac{1}{2}x_1\left(\frac{\partial\tau}{\partial t}+f_i\frac{\partial\tau}{\partial x_i}+\frac{1}{2}G_i^1G_p^1\frac{\partial^2\tau}{\partial x_i\partial x_p}\right)=0 \tag{62}$$

The prolongations of the spatial infinitesimals are given for j equal to 1 and 2, respectively, as

$$\begin{aligned}\xi_1^{[1]}&=\frac{\partial\xi_1}{\partial t}+\dot{x}_i\frac{\partial\xi_1}{\partial x_i}-\dot{x}_1\left(\frac{\partial\tau}{\partial t}-\dot{x}_i\frac{\partial\tau}{\partial x_i}\right)\\&=\frac{\partial\xi_1}{\partial t}+f_i\frac{\partial\xi_1}{\partial x_i}+\frac{1}{2}x_1\left(\frac{\partial\tau}{\partial t}-f_i\frac{\partial\tau}{\partial x_i}\right)\end{aligned} \tag{63}$$

$$\begin{aligned}\xi_2^{[1]}&=\frac{\partial\xi_2}{\partial t}+\dot{x}_i\frac{\partial\xi_2}{\partial x_i}-\dot{x}_2\left(\frac{\partial\tau}{\partial t}-\dot{x}_i\frac{\partial\tau}{\partial x_i}\right)\\&=\frac{\partial\xi_1}{\partial t}+f_i\frac{\partial\xi_1}{\partial x_i}+\frac{1}{2}x_2\left(\frac{\partial\tau}{\partial t}-f_i\frac{\partial\tau}{\partial x_i}\right)\end{aligned} \tag{64}$$

as we are evaluating at $\dot{x}_i=f_i$ in both cases of j. Substituting the above into the refurbished DEs of Wafo Soh and Mahomed [2], i.e. Equations (49) and (34), we find the following once we have

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multiplied Equations (59) and (61) by a factor of 2:

$$-\xi_2 + x_2 \frac{\partial \xi_2}{\partial x^2} - x_1^2 \frac{\partial^2 \xi_2}{\partial x_2^2} + x_1 \frac{\partial \xi_2}{\partial x^1} + 2x_1 x_2 \frac{\partial^2 \xi_2}{\partial x^1 \partial x^2} - x_2^2 \frac{\partial^2 \xi_2}{\partial x_1^2} = 0 \tag{65}$$

$$\xi_1 - x_1 \frac{\partial \xi_2}{\partial x^2} + x_2 \frac{\partial \xi_2}{\partial x^1} = 0 \tag{66}$$

$$-\xi_1 + x_2 \frac{\partial \xi_1}{\partial x^2} - x_1^2 \frac{\partial^2 \xi_1}{\partial x_2^2} + x_1 \frac{\partial \xi_1}{\partial x^1} + 2x_1 x_2 \frac{\partial^2 \xi_1}{\partial x^1 \partial x^2} - x_2^2 \frac{\partial^2 \xi_1}{\partial x_1^2} = 0 \tag{67}$$

$$\xi_2 - x_1 \frac{\partial \xi_1}{\partial x^2} + x_2 \frac{\partial \xi_1}{\partial x^1} = 0 \tag{68}$$

The final DE now required is the extra condition (25) that reconciles both papers, viz

$$-x_1 \frac{\partial \tau}{\partial x^2} + x_2 \frac{\partial \tau}{\partial x^1} = 0 \tag{69}$$

where the evaluation at $(t, \mathbf{X}(t))$ has not taken place. Solving these deterministic equations gives

$$\tau(t, \mathbf{X}(t)) = C_0 F_0 \left(\frac{X(t)_2^2 + X(t)_1^2}{2} \right) \tag{70}$$

$$\xi_1(t, \mathbf{X}(t)) = C_1 F_1 \left(\frac{X(t)_2^2 + X(t)_1^2}{2} \right) X(t)_1 + C_2 F_2 \left(\frac{X(t)_2^2 + X(t)_1^2}{2} \right) X(t)_2 \tag{71}$$

and

$$\xi_2(t, \mathbf{X}(t)) = C_1 F_1 \left(\frac{X(t)_2^2 + X(t)_1^2}{2} \right) X(t)_2 - C_2 F_2 \left(\frac{X(t)_2^2 + X(t)_1^2}{2} \right) X(t)_1 \tag{72}$$

which are the same results that Ünal [4] had found. The condition (Lie point SODE condition) is satisfied, since $\Gamma(\tau) = 0$ and $H(\tau) = \tau\,\Gamma(\tau) = 0$. To demonstrate that a solution of one SODE is transformed to that of another, we choose a simple example where $F_1((X(t)_2^2 + X(t)_1^2)/2) = F_2((X(t)_2^2 + X(t)_1^2)/2) = 1$. Thus, we have the following resulting symmetry generators:

$$H_0 = F_0 \left(\frac{x_2^2 + x_1^2}{2} \right) \frac{\partial}{\partial t} \tag{73}$$

$$H_1 = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \tag{74}$$

and

$$H_2 = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \tag{75}$$

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The point transformations associated with (73) are

$$\overline{x}_1(\overline{t}) = x_1 \tag{76}$$

$$\overline{x}_2(\overline{t}) = x_2 \tag{77}$$

and

$$\overline{t} = t + F_0\left(\frac{x_2^2 + x_1^2}{2}\right)\varepsilon \tag{78}$$

The point transformations associated with (74) are

$$\overline{x}_1(\overline{t}) = x_1 \mathrm{e}^{\varepsilon} \tag{79}$$

$$\overline{x}_2(\overline{t}) = x_2 \mathrm{e}^{\varepsilon} \tag{80}$$

and

$$\overline{t} = t \tag{81}$$

The point transformation associated with (75) are

$$\overline{x}_1(\overline{t}) = x_1 \cos(\varepsilon) + x_2 \sin(\varepsilon) \tag{82}$$

$$\overline{x}_2(\overline{t}) = -x_1 \sin(\varepsilon) + x_2 \cos(\varepsilon) \tag{83}$$

and

$$\overline{t} = t \tag{84}$$

the point transformations associated with (73) and (74) trivially verify form invariance when Itô's formula is applied. This is especially for H_0, where the temporal infinitesimal is zero under both the Γ and Y^1 operators. Applying Itô's formula to (82) and (83) gives the following:

$$\begin{aligned}
\mathrm{d}\overline{X}_1(\overline{t}) &= \mathrm{d}X_1(t)\cos(\varepsilon) + \mathrm{d}X_2(t)\sin(\varepsilon)\\
&= \left(\frac{-X_1(t)}{2}\cos(\varepsilon) + \frac{-X_2(t)}{2}\sin(\varepsilon)\right)\mathrm{d}\overline{t} + (-X_2(t)\cos(\varepsilon) - X_1(t)\sin(\varepsilon))\,\mathrm{d}\overline{W}(\overline{t})\\
&= (\mathrm{e}^{\varepsilon H_2}(f_1(X_1(t))))\,\mathrm{d}\overline{t} + (\mathrm{e}^{\varepsilon H_2}(G_1(X_2(t))))\,\mathrm{d}\overline{W}(\overline{t}) && (85)\\
&= f_1(\overline{t}, \overline{\mathbf{X}}(\overline{t}))\,\mathrm{d}\overline{t} + G_1(\overline{t}, \overline{\mathbf{X}}(\overline{t}))\,\mathrm{d}\overline{W}(\overline{t}) && (86)\\
\mathrm{d}\overline{X}_2(\overline{t}) &= -\mathrm{d}X_1(t)\sin(\varepsilon) + \mathrm{d}X_2(t)\cos(\varepsilon)\\
&= \left(\frac{-X_2(t)}{2}\cos(\varepsilon) + \frac{X_1(t)}{2}\sin(\varepsilon)\right)\mathrm{d}\overline{t} + (X_1(t)\cos(\varepsilon) + \varepsilon X_2(t)\sin(\varepsilon))\,\mathrm{d}\overline{W}(\overline{t})\\
&= (\mathrm{e}^{\varepsilon H_2}(f_2(X_2(t))))\,\mathrm{d}\overline{t} + (\mathrm{e}^{\varepsilon H_2}(G_2(X_1(t))))\,\mathrm{d}\overline{W}(\overline{t}) && (87)\\
&= f_2(\overline{t}, \overline{\mathbf{X}}(\overline{t}))\,\mathrm{d}\overline{t} + G_2(\overline{t}, \overline{\mathbf{X}}(\overline{t}))\,\mathrm{d}\overline{W}(\overline{t}) && (88)
\end{aligned}$$

which demonstrates form invariance.

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5. CONCLUDING REMARKS

It has been shown that by taking special care that the transformed Wiener variable is still a standard Wiener process, overlooked in the pioneering work [2], for the Itô process

$$\mathrm{d}\mathbf{X}(t) = \mathbf{f}\,\mathrm{d}t + \mathbf{G}\,\mathrm{d}\mathbf{W}(t)$$

leads to the same results as that of [4] meaning that no recourse to the Itô's multiplication table for the transformed variables is necessary to find the extra condition (25).

This work allows us to investigate the symmetries of stochastic ordinary differential equations (SODEs) without recourse to the FP equation; precluding the assumption that the symmetry H of the SDE had to be projectable, i.e. $\tau = \tau(t)$. This work has successfully reconciled the works of Wafo Soh and Mahomed [2] and Ünal [4]. We have also found a new condition that allows us to use Lie point symmetry in the SODE context.

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