ELSEVIER

Journal of Geometry and Physics 43 (2002) 133–145

JOURNAL OF GEOMETRY AND PHYSICS

www.elsevier.com/locate/jgp

Differential equations and conformal geometry

Simonetta Frittelli [a], Niky Kamran [b,*], Ezra T. Newman [c]

[a] *Department of Physics, Duquesne University, Pittsburgh, PA 15282, USA*
[b] *Department of Mathematics and Statistics, McGill University, 805 Sherbrooke Street West, Montreal, Que., Canada H3A 2K6*
[c] *Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260, USA*

Received 7 June 2001; received in revised form 20 October 2001

Abstract

It has recently been proved [3] that the solution spaces of certain classes of differential equations whose local solutions are parametrized by three or four arbitrary constants can be endowed with conformal Lorentzian metrics in a natural way. We shall prove that these conformal structures are preserved when the differential equations are transformed by a contact transformation.

MSC: 52G34; 53B30

Subj. Class.: Differential geometry; General relativity

Keywords: Differential equations; Conformal geometry; Lorentzian geometry; Contact transformations

1. Introduction

It has recently been shown [3] that the solution spaces of certain classes of differential equations whose local solutions are parametrized by arbitrary real constants are naturally endowed with conformal Lorentzian structures.

In the three-dimensional case, this result is classical and can be viewed as a corollary of Chern's solution [2] of the local equivalence problem for third-order ODEs

$$\frac{d^3u}{ds^3} = F\left(s, u, \frac{du}{ds}, \frac{d^2u}{ds^2}\right), \tag{1}$$

under the Lie pseudogroup of contact transformations (see also [1]).

* Corresponding author.
E-mail address: nkamran@scylla.math.mcgill.ca (N. Kamran).

0393-0440/02/$ – see front matter
PII: S0393-0440(01)00088-2

In four-dimensional case, the starting point is given by an overdetermined system of second-order partial differential equations of the form

$$\frac{\partial^2 u}{\partial s^2} = U\left(s, t, u, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial s \partial t}\right), \qquad \frac{\partial^2 u}{\partial t^2} = V\left(s, t, u, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial s \partial t}\right), \tag{2}$$

where the functions U and V are chosen in such a way that the solutions depend smoothly on four arbitrary constants.

A basic issue is to decide whether to use the pair of real variables s and t as our independent variables or to combine them into a pair of conjugate complex coordinates s and s^*. While the analysis can be carried out with either choice, it turns out that using the pair of real variables is most natural when seeking a metric of split signature $(1, 1, -1, -1)$ on the solution space, while the use of the complex conjugate pair is better adapted in the Lorentzian case $(1, -1, -1, -1)$. We will use the complex conjugate pair and stress that there is no implication of holomorphicity in this choice. It will thus be convenient to re-formulate this overdetermined system as a single complex partial differential equation of the form

$$\frac{\partial^2 u}{\partial s^2} = S\left(s, s^*, u, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial s^*}, \frac{\partial^2 u}{\partial s \partial s^*}\right), \tag{3}$$

where s is a complex, s^* denotes the complex conjugate of s and S a complex-valued function which is determined by U and V. It was shown in [3] that, locally, every four-dimensional Lorentzian metric can be realized in a natural way as a metric on the solution space of (3), and that further assumptions on S give rise to all the local solutions of the Einstein equations.

Our goal in this paper is to further clarify the relationship between the contact geometry of the differential equations (3) and (1) and the conformal geometry of their solution spaces. More precisely, we will show that the action induced by the Lie pseudogroup of contact transformations will preserve the conformal classes of the underlying Lorentzian metrics. Our proof is based on the equivalence between the classical envelope construction which is used to solve the eikonal equation and Lie's description of contact transformations in terms of characteristic functions. It is thus different in spirit from the proof given in Chern's paper [2] in the three-dimensional case.

2. Contact geometry of a third-order ODE

To the third-order ODE

$$\frac{\mathrm{d}^3 u}{\mathrm{d}s^3} = F\left(s, u, \frac{\mathrm{d}u}{\mathrm{d}s}, \frac{\mathrm{d}^2 u}{\mathrm{d}s^2}\right), \tag{4}$$

we associate the completely integrable Pfaffian system $\mathcal{I}_F$ on $J^2(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}^4$ generated by the 1-forms

$$\theta^1 = \mathrm{d}u - u'\,\mathrm{d}s, \qquad \theta^2 = \mathrm{d}u' - u''\,\mathrm{d}s, \qquad \theta^3 = \mathrm{d}u'' - F(s, u, u', u'')\,\mathrm{d}s, \tag{5}$$

where (s, u, u', u'') denote local jet coordinates on $J^2(\mathbb{R}, \mathbb{R})$ in which the contact Pfaffian system is generated by the 1-forms θ^1, θ^2. The local solutions of (4) correspond to integral curves $c : \mathbb{R} \to J^2(\mathbb{R}, \mathbb{R})$ satisfying the independence condition $c^*\,\mathrm{d}s \not\equiv 0$.

We shall work locally by restricting the domain of $F(s, u, u', u'')$ to an open neighborhood U of $J^2(\mathbb{R}, \mathbb{R})$ where F is C^∞ and where the Cauchy problem for (4) admits a unique C^∞ solution depending in a C^∞ fashion on Cauchy data given in U. It follows from this assumption that the set M_3 of local solutions of (4) is endowed with the structure of a three-dimensional C^∞ manifold. We will denote the local coordinates in M_3 by $(x^a) = (x^1, x^2, x^3)$ and refer to M_3 as the *solution space* of (4). The ODE (4) thus gives rise to a local fibration $\rho_F : J^2(\mathbb{R}, \mathbb{R}) \to M_3$, where

$$\ker \rho_{F_*} = \mathcal{I}_F^\perp = \left\{ \frac{\mathrm{D}}{\mathrm{D}s} \right\}, \tag{6}$$

and

$$\frac{\mathrm{D}}{\mathrm{D}s} = \frac{\partial}{\partial s} + u' \frac{\partial}{\partial u} + u'' \frac{\partial}{\partial u'} + F(s, u, u', u'') \frac{\partial}{\partial u''}. \tag{7}$$

By working locally in M_3, we obtain a C^∞ map $z : M_3 \times \mathbb{R} \to \mathbb{R}$, $u = z(x^1, x^2, x^3, s)$, such that for fixed x_0 in M_3 with local coordinates (x_0^1, x_0^2, x_0^3), the induced map $z_{x_0} : \mathbb{R} \to \mathbb{R}$, $s \to u = z(x_0^1, x_0^2, x_0^3, s)$ is a solution of (4), that is

$$(j^2 z_{x_0})^* \mathcal{I}_F = 0. \tag{8}$$

Consider now on $M_3 \times \mathbb{R}$ the three 1-forms given by

$$\beta^1 = (\partial_a z)\,\mathrm{d}x^a, \qquad \beta^2 = (\partial_a z_s)\,\mathrm{d}x^a, \qquad \beta^3 = (\partial_a z_{ss})\,\mathrm{d}x^a. \tag{9}$$

It follows from the preceding discussion that there exists a local diffeomorphism $\varsigma : J^2(\mathbb{R}, \mathbb{R}) \to M_3 \times \mathbb{R}$ which fibers over the identity map id: $\mathbb{R} \to \mathbb{R}$ through the source map $\alpha : J^2(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$, $\alpha(s, u, u', u'') = s$, and the projection pr: $M_3 \times \mathbb{R} \to \mathbb{R}$, $\mathrm{pr}(x^1, x^2, x^3, s) = s$,

$$\begin{array}{ccc} J^2(\mathbb{R}, \mathbb{R}) & \xrightarrow{\ \varsigma\ } & M_3 \times \mathbb{R} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle pr} \\ \mathbb{R} & \xrightarrow[id]{} & \mathbb{R} \end{array}$$

and which pulls back the completely integrable Pfaffian system $\mathcal{J}_F = \{\beta^1, \beta^2, \beta^3\}$ on $M_3 \times \mathbb{R}$ to $\mathcal{I}_F$,

$$\varsigma^* \mathcal{J}_F = \mathcal{I}_F. \tag{10}$$

We shall consider the ODEs (4) from the point of view of the Lie pseudogroup of contact transformations of $J^2(\mathbb{R}, \mathbb{R})$. We will say that the ODE (4) and the third-order ODE

$$\frac{\mathrm{d}^3 \bar{u}}{\mathrm{d}\bar{s}^3} = \bar{F}\left(\bar{s}, \bar{u}, \frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{s}}, \frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}\bar{s}^2} \right) \tag{11}$$

are locally equivalent if there exists a contact transformation $f: J^2(\mathbb{R},\mathbb{R}) \to J^2(\mathbb{R},\mathbb{R})$, $(s,u,u',u'') \to (\bar{s},\bar{u},\bar{u}',\bar{u}'')$ such that

$$f^*\mathcal{I}_{\bar{F}} = \mathcal{I}_F, \tag{12}$$

that is a local diffeomorphism $f: J^2(\mathbb{R},\mathbb{R}) \to J^2(\mathbb{R},\mathbb{R})$ such that

$$f^*\begin{pmatrix}\bar{\theta}^1\\ \bar{\theta}^2\\ \bar{\theta}^3\end{pmatrix} = \begin{pmatrix}a_{11} & 0 & 0\\ a_{21} & a_{22} & 0\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}\theta^1\\ \theta^2\\ \theta^3\end{pmatrix}, \tag{13}$$

where a_{ij} are C^∞ functions on U satisfying $\prod_{i=1}^{3} a_{ii} \not\equiv 0$. Note that the matrix appearing in the right-hand side of (13) is triangular as a direct consequence of Bäcklund's theorem on contact transformations, [5].

We shall restrict our attention to ODEs (4) satisfying the contact-invariant condition

$$W_F := F_u - aF_{u''} + \frac{\mathrm{D}a}{\mathrm{D}s} - ab = 0, \tag{14}$$

where a and b are defined by

$$2a = -F_{u'} - \frac{2}{9}F_{u''}^2 + \frac{1}{3}\frac{\mathrm{D}F_{u''}}{\mathrm{D}s}, \qquad b = -\frac{1}{3}F_{u''}. \tag{15}$$

The function W_F, known as the *Wünschmann invariant* [2] of (4), is a relative invariant of the contact geometry of (4), in the sense that if (4) and (11) are locally contact equivalent, then

$$f^*W_{\bar{F}} = \lambda W_F \tag{16}$$

for some non-vanishing C^∞ multiplier λ. Alternatively, the Wünschmann invariant W_F can be viewed as a section of a certain natural line bundle naturally associated to (4).

3. The conformal Lorentzian structure on the solution space

Our purpose in this section is to exhibit a correspondence which associates to the contact orbit of each ODE (4) satisfying the contact-invariant condition $W_F = 0$ a local conformal Lorentzian structure on its solution space M. This correspondence is mentioned briefly by Chern in [2] as a byproduct of his solution of the equivalence problem for (4) under the Lie pseudogroup of contact transformations. The approach we have adopted here is a bit different in the sense that it is based on the characterization of the conformal class of a Lorentzian metric by its characteristic surfaces. Our main reason for treating the three-dimensional case first is that it serves as guide for the four-dimensional case, which is treated in the next section of our paper.

We start on $J^2(\mathbb{R},\mathbb{R})$ with the quadratic differential form h given by

$$h_F = \eta^1 \otimes \eta^3 + \eta^3 \otimes \eta^1 - \eta^2 \otimes \eta^2, \tag{17}$$

where

$$\eta^1 = \varsigma^* \beta^1, \qquad \eta^2 = \varsigma^* \beta^2, \qquad \eta^3 = \varsigma^* \beta^3 + a\varsigma^* \beta^1 + b\varsigma^* \beta^2, \tag{18}$$

where $\varsigma : J^2(\mathbb{R}, \mathbb{R}) \to M_3 \times \mathbb{R}$ was defined in (10) and a, b were given in (15).

Lemma 1. *Consider a third-order ODE (4) with vanishing Wünschmann invariant,*

$$W_F = 0.$$

Then, we have [3]

$$\mathcal{L}_{\mathrm{D/D}s} h_F = \tfrac{2}{3} F_{u''} h_F.$$

We now let g_F denote the quadratic differential form defined on $M_3 \times \mathbb{R}$ by

$$g_F = (\varsigma^{-1})^* h_F. \tag{19}$$

It follows from the preceding lemma that g_F induces on M_3 a conformal Lorentzian structure, which we shall denote by $[g_F]$. We can thus write a representative for $[g_F]$ in the form

$$g_F = \omega^1 \otimes \omega^3 + \omega^3 \otimes \omega^1 - \omega^2 \otimes \omega^2, \tag{20}$$

where

$$\omega^1 = \beta^1, \qquad \omega^2 = \beta^2, \qquad \omega^3 = \beta^3 + [(\varsigma^{-1})^* a]\beta^1 + [(\varsigma^{-1})^* b]\beta^2. \tag{21}$$

Before stating the main result of this section, we remark that any contact transformation $f : J^2(\mathbb{R}, \mathbb{R}) \to J^2(\mathbb{R}, \mathbb{R})$ relating (4)–(11) will map local solutions to local solutions and will therefore induce a local diffeomorphism $\tilde{f} : M_3 \to \bar{M}_3$ between the solution spaces of these ODEs. We shall choose adapted charts in M_3 and $\bar{M}_3$ in which the local diffeomorphism $\tilde{f}$ is represented by the identity.

Theorem 1. *Let (4) be a third-order ODE with vanishing Wünschmann invariant and let (11) be a third-order ODE locally equivalent to (4) under a contact transformation $f : J^2(\mathbb{R}, \mathbb{R}) \to J^2(\mathbb{R}, \mathbb{R})$. Then the local diffeomorphism of solution spaces $\tilde{f} : M_3 \to \bar{M}_3$ induced by f preserves the corresponding conformal Lorentzian structures, that is*

$$\tilde{f}^*[g_{\bar{F}}] = [g_F].$$

In particular, any three-parameter family of solutions $u = z(x^1, x^2, x^3, s)$ of the ODE (4) is a complete integral of the eikonal equation

$$g_F^{ab}\, \partial_a z\, \partial_b z = 0 \tag{22}$$

for $[g_F]$, and, conversely, any complete integral of the eikonal equation (22) gives rise to a solution of a third-order ODE which is contact equivalent to (4).

Proof. We shall give a proof in the case of contact transformations which are not the prolongation of point transformations, and leave the case of prolonged point transformations as an exercise to the reader.

First recall [5] that a contact transformation which is not a prolonged point transformation is determined in terms of a generating function $S(s, u, \bar{s}, \bar{u})$ by solving the following implicit relations:

$$S(s, u, \bar{s}, \bar{u}) = 0, \qquad S_s + u' S_u = 0, \qquad S_{\bar{s}} + \bar{u}' S_{\bar{u}'} = 0 \tag{23}$$

for $\bar{s}, \bar{u}, \bar{u}'$ as functions of s, u, u', respectively. (We are of course assuming that S satisfies the solvability conditions required by the implicit function theorem.) With no loss of generality, we take S to be of the form

$$S(s, u, \bar{s}, \bar{u}) = \bar{u} - \bar{V}(s, u, \bar{s}), \tag{24}$$

and write the contact transformation generated by S in the form

$$\bar{u} = \bar{V}(s, u, I(s, u, u')), \qquad \bar{s} = I(s, u, u'), \qquad \bar{u}' = \bar{V}_{\bar{s}}(s, u, I(s, u, u')), \tag{25}$$

where I is determined by solving

$$\bar{V}_s + u' \bar{V}_u = 0 \tag{26}$$

for $\bar{s}$ in terms of s, u, u'. The contact orbit of (1) is thus obtained by applying the transformation (25) to the 1-forms (5).

Next, note that, from (20), it follows that for each value of s, the 1-form ω^1 is null for the conformal class $[g_F]$, so that any three-parameter family of solutions $u = z(x^1, x^2, x^3, s)$ of (4) gives rise to a one-parameter family of solutions of the eikonal equation (22). In other words, the solutions of (4) are complete integrals of the eikonal equation.

We now want to invoke the envelope construction to take one complete integral of the eikonal equation into another such solution. To this effect, we must first pull back (25) and (26) to $M_3 \times \mathbb{R}$ by means of the local diffeomorphism $\varsigma^{-1} : M_3 \times \mathbb{R} \to J^2(\mathbb{R}, \mathbb{R})$.

We now consider the function $\bar{z}(x^1, x^2, x^3, \bar{s})$ defined by

$$\bar{z} = \bar{V}(s, z(x^1, x^2, x^3, s), \bar{s}), \tag{27}$$

where s is defined implicitly as a function of x^1, x^2, x^3 and $\bar{s}$ by the envelope condition

$$\bar{V}_u u' + \bar{V}_s = 0. \tag{28}$$

Note that although (28) has the same form as (26), it now lives on $M_3 \times \mathbb{R}$, and thus involves the variables x^1, x^2, x^3, and s.

It is important to note that since the function $z(x^1, x^2, x^3, s)$ solves the eikonal equation (22), the function $\bar{z}(x^1, x^2, x^3, \bar{s})$ will also solve it. This proves the second part of the statement of our theorem.

From (27) and (28), it follows that

$$\bar{z}' = \frac{\mathrm{d}\bar{V}}{\mathrm{d}s} \frac{\mathrm{d}s}{\mathrm{d}\bar{s}}. \tag{29}$$

But we have

$$\frac{\mathrm{d}\bar{V}}{\mathrm{d}s} = \frac{\partial \bar{V}}{\partial s} + \frac{\partial \bar{V}}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}s} + \frac{\partial \bar{V}}{\partial \bar{s}}\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} \tag{30}$$

Now, the sum of the first two terms on the right-hand side of the above equation is zero by virtue of (26), so that we have

$$\bar{z}' = \frac{\partial \bar{V}}{\partial \bar{s}}\frac{\mathrm{d}\bar{s}}{\mathrm{d}s} = \frac{\partial \bar{V}}{\partial \bar{s}}. \tag{31}$$

This shows that the map $\tilde{f} : M_3 \to \bar{M}_3$ induced by a contact transformation $f : J^2(\mathbb{R}, \mathbb{R}) \to J^2(\mathbb{R}, \mathbb{R})$ will map the envelope of a 1-parameter family of null surfaces for g_F to another, and will therefore preserve the conformal class of g_F.

Stated more informally, what we have shown is that the contact equivalence class of a third-order ODE satisfying $W_F = 0$ is characterized by the conformal equivalence class of a three-dimensional Lorentzian metric. Furthermore, we have shown that the three-parameter set of solutions of each ODE in a given class form a one-parameter family of solutions of the eikonal equation for that Lorentzian metric. □

We conclude this section by illustrating the proof of our theorem in the simplest case of the differential equation

$$\frac{\mathrm{d}^3 u}{\mathrm{d}s^3} = 0, \tag{32}$$

which will give rise to three-dimensional conformal Minkowski space. We will thus recover Lie's classical correspondence between circle geometry in the Euclidean plane and conformal Minkowskian geometry [4].

We will change the notation slightly and rewrite (32) as

$$\frac{\mathrm{d}^3 \hat{u}}{\mathrm{d}p^3} = 0, \tag{33}$$

whose general solution may be written as

$$\hat{u} = (1 + p^2)t + 2px + (1 - p^2)y, \tag{34}$$

where the parameters (t, x, y) are constants of integration, which will serve as local coordinates on the solution space. It is straightforward to check from the eikonal equation (22) that the corresponding conformal structure on the solution space is Minkowskian,

$$g = \Omega^2 \,\mathrm{diag}(1, -1, -1), \tag{35}$$

and that for any fixed value of p, the level surfaces of (34) are null planes for (35).

We now apply a suitable contact transformation the differential equation (33) to map it into the equation

$$\frac{\mathrm{d}^3 \bar{u}}{\mathrm{d}\bar{s}^3} = 3\frac{(\mathrm{d}\bar{u}/\mathrm{d}\bar{s})(\mathrm{d}^2\bar{u}/\mathrm{d}\bar{s}^2)^2}{1 + (\mathrm{d}\bar{u}/\mathrm{d}\bar{s})^2}, \tag{36}$$

whose solutions, given implicitly by

$$(\bar{u} - y)^2 + (\bar{s} - x)^2 - t^2 = 0, \tag{37}$$

are the light cones with apex at $(\bar{u}, \bar{s}, 0)$. This contact transformation can be conveniently expressed as the composition of three relatively simple transformations. We first apply the fiber-preserving point transformation

$$u^* = \frac{\hat{u}}{1+p^2}, \qquad \cos s = \frac{2p}{1+p^2}, \tag{38}$$

to transform (33) into

$$\frac{\mathrm{d}^3 u^\dagger}{\mathrm{d}s^3} = -\frac{\mathrm{d}u^\dagger}{\mathrm{d}s}, \tag{39}$$

with general solution given by

$$u^\dagger = t + x\cos s + y\sin s, \tag{40}$$

where (t, x, y) are the same constants of integration as before. The level surfaces of $u^\dagger$ are null planes. Next, we perform the fiber-preserving point transformation on (39) given by

$$u^\dagger = u\sin s \tag{41}$$

to obtain

$$\frac{\mathrm{d}^3 u}{\mathrm{d}s^3} = -3\frac{\mathrm{d}^2 u}{\mathrm{d}s^2}\cot s + 2\frac{\mathrm{d}u}{\mathrm{d}s}, \tag{42}$$

whose general solution is given by

$$u = x\cot s + y + \frac{t}{\sin s}, \tag{43}$$

with the level surfaces of u being null planes. Finally, we apply to (42) the contact transformation with generating function $H(s, u, \bar{s}, \bar{u})$ given by

$$H(s, u, \bar{s}, \bar{u}) = (\bar{u} - u)\sin s + \bar{s}\cos s, \tag{44}$$

which yields (36).

We now show that the ODE (36) and the light cones (37) can be constructed by forming envelopes of planes (43). From (44), we have

$$\bar{u} = \bar{V}(s, u, \bar{s}) = u - \bar{s}\cot s, \tag{45}$$

or, in view of (43)

$$\bar{u} = x\cot s + y + \frac{t}{\sin s} - \bar{s}\cot s. \tag{46}$$

To apply the envelope construction, we set to zero the derivative of (46) with respect to s, so that

$$\cos s = \frac{\bar{s} - x}{t}. \tag{47}$$

When (47) is substituted into (46), we obtain the family of light cones (37), and therefore the ODE (36).

4. Contact geometry of a pair of second-order PDEs

Our purpose in this section is to show how the main theorem of the preceding section can be extended to the case of four-dimensional Lorentzian metrics. We will only give details in the instances where there are notable differences with the three-dimensional case.

We first point out that it is not our intention at this stage to carry out a complete analysis of the conformal geometry of the Lorentzian metric in terms of differential equations. We will only concern ourselves with the relevant problem of establishing a correspondence between conformal geometry of four-dimensional Lorentzian metrics and the contact geometry of certain differential equations. These differential equations will be overdetermined systems of two second-order PDEs for one function of two variables, whose solutions depend smoothly on four arbitrary constants. These constants will serve as local coordinates on the four-dimensional solution space of our PDE system.

Recall from (3) that the differential equations which serve as the starting point of our construction are overdetermined systems of PDEs of the form

$$\frac{\partial^2 u}{\partial s^2} = S\left(s, s^*, u, \frac{\partial u}{\partial s}, \frac{\partial u}{\partial s^*}, \frac{\partial^2 u}{\partial s \partial s^*}\right), \tag{48}$$

where s is complex-valued, s^* denotes the complex conjugate of s and S is complex-valued.

We will be interested in the case in which the Pfaffian system naturally associated to (48) is completely integrable, so that the local solutions of (48) will depend on arbitrary constants. We thus consider $J^2(\mathbb{R}^2, \mathbb{R})$ with local jet coordinates $(s, s^*, u, u_s, u_{s^*}, u_{ss}, u_{ss^*}, u_{s^*s^*})$ in which the contact Pfaffian system is generated by the 1-forms

$$\theta_u = \mathrm{d}u - u_s\,\mathrm{d}s - u_{s^*}\,\mathrm{d}s^*, \tag{49}$$

$$\theta_{u_s} = \mathrm{d}u_s - u_{ss}\,\mathrm{d}s - u_{ss^*}\,\mathrm{d}s^*, \tag{50}$$

$$\theta_{u_{s^*}} = \mathrm{d}u_{s^*} - u_{ss^*}\,\mathrm{d}s - u_{s^*s^*}\,\mathrm{d}s^*. \tag{51}$$

To (48) is naturally associated the locus L_S in $J^2(\mathbb{R}^2, \mathbb{R})$, defined by the equations

$$u_{ss} = S(s, s^*, u, u_s, u_{s^*}, u_{ss^*}), \qquad u_{s^*s^*} = S^*(s, s^*, u, u_s, u_{s^*}, u_{ss^*}). \tag{52}$$

We shall work locally and assume that this locus is a six-dimensional C^∞ submanifold of $J^2(\mathbb{R}^2, \mathbb{R})$, with local coordinates given by $(s, s^*, u, u_s, u_{s^*}, u_{ss^*})$. Furthermore, we assume that the signature condition

$$1 - S_{u_{ss^*}} S^*_{u_{ss^*}} > 0, \tag{53}$$

and the integrability condition

$$\frac{\mathrm{D}^2 S^*}{\mathrm{D}s^2} = \frac{\mathrm{D}^2 S}{\mathrm{D}s^{*2}}, \tag{54}$$

where

$$\frac{\mathrm{D}}{\mathrm{D}s} = \frac{\partial}{\partial s} + u_s\frac{\partial}{\partial u} + S\frac{\partial}{\partial u_s} + u_{ss^*}\frac{\partial}{\partial u_{s^*}} + Q\frac{\partial}{\partial u_{ss^*}}, \tag{55}$$

$$\frac{\mathrm{D}}{\mathrm{D}s^*}=\frac{\partial}{\partial s^*}+u_{s^*}\frac{\partial}{\partial u}+u_{ss^*}\frac{\partial}{\partial u_s}+S^*\frac{\partial}{\partial u_{s^*}}+Q^*\frac{\partial}{\partial u_{ss^*}}, \tag{56}$$

and where Q is defined by

$$Q=u_{sss^*}=(1-S_{u_{ss^*}}S^*_{u_{ss^*}})^{-1}\left[\frac{\partial S}{\partial s}+u_s\frac{\partial S}{\partial u}+S\frac{\partial S}{\partial u_s}+u_{ss^*}\frac{\partial S}{\partial u_{s^*}}\right.$$
$$\left.+S_{u_{ss^*}}\left(\frac{\partial S^*}{\partial s^*}+u_{s^*}\frac{\partial S^*}{\partial u}+u_{ss^*}\frac{\partial S^*}{\partial u_s}+S^*\frac{\partial S^*}{\partial u_{s^*}}\right)\right], \tag{57}$$

are satisfied at every point of L_S. The signature condition (53) can be shown to ensure the Lorentzian character of the signature of the metric on the solution space.

To (48), we associate on L_S the rank 4 Pfaffian system $\mathcal{I}_S$ generated by the 1-forms

$$\theta^1=\mathrm{d}u-u_s\,\mathrm{d}s-u_{s^*}\,\mathrm{d}s^*, \tag{58}$$

$$\theta^2=\mathrm{d}u_s-S(s,s^*,u_s,u_{s^*},u_{ss^*})\,\mathrm{d}s-u_{ss^*}\,\mathrm{d}s^*, \tag{59}$$

$$\theta^3=\mathrm{d}u_{s^*}-u_{ss^*}\,\mathrm{d}s-S^*(s,s^*,u_s,u_{s^*},u_{ss^*})\,\mathrm{d}s^*, \tag{60}$$

$$\theta^4=\mathrm{d}u_{ss^*}-Q(s,s^*,u_s,u_{s^*},u_{ss^*})\,\mathrm{d}s-Q^*(s,s^*,u_s,u_{s^*},u_{ss^*})\,\mathrm{d}s^*. \tag{61}$$

The local solutions of (48) are in one-to-one correspondence with the two-dimensional local integral manifolds $c:\mathbb{R}^2\to L_S$ of $\mathcal{I}_S$ satisfying the independence condition

$$c^*(\mathrm{d}s\wedge\mathrm{d}s^*)\not\equiv 0 \tag{62}$$

It is now easy to show using the Frobenius theorem that the local solutions of our overdetermined system are parametrized by four arbitrary real constants.

Lemma 2. *Consider an overdetermined system (48) satisfying the rank condition (53) and the integrability condition (54). Then the corresponding Pfaffian system $\mathcal{I}_S=\{\theta^1,\theta^2,\theta^3,\theta^4\}$ on the 6-manifold L_S is completely integrable.*

It follows from this lemma that the set M_4 of local solutions of the system (48), where S satisfies (53), is a four-dimensional C^∞ manifold M_4. The local coordinates on M_4 will be denoted by (x^1,x^2,x^3,x^4), and we have now a C^∞ map $z:M_4\times\mathbb{R}^2\to\mathbb{R}$, $u=z(x^1,x^2,x^3,x^4,s,s^*)$ such that for fixed x_0 in M_4 with local coordinates $(x_0^1,x_0^2,x_0^3,x_0^4)$, the induced map $z_{x_0}:\mathbb{R}\to\mathbb{R}$, $s\to u=z(x_0^1,x_0^2,x_0^3,x_0^4,s,s^*)$ satisfies

$$(j^2 z_{x_0})^*\mathcal{I}_S=0 \tag{63}$$

for every $x_0\in M_4$.

We proceed in analogy with the three-dimensional case and consider on $M_4\times\mathbb{R}^2$ the 1-forms given by

$$\beta^0=(\partial_a z)\,\mathrm{d}x^a,\qquad \beta^+=(\partial_a z_s)\,\mathrm{d}x^a,\qquad \beta^-=(\partial_a z_{s^*})\,\mathrm{d}x^a,$$
$$\beta^1=(\partial_a z_{ss^*})\,\mathrm{d}x^a. \tag{64}$$

We now have a local diffeomorphism $\varsigma:L_S\to M_4\times\mathbb{R}^2$ which pulls back the completely integrable Pfaffian system generated by $\beta^0,\beta^+,\beta^-,\beta^1$ to $\mathcal{I}_S$.

In order to have a well-defined conformal Lorentzian structure on M_4, we will have to restrict our attention to a subclass of the class of overdetermined systems (48) for which a certain invariant analogous to the Wünschmann invariant is identically zero. This invariant, which we will denote by M_S, is defined in [3] and is given by

$$M_S = \frac{1}{3}(\mathrm{D}_s S) - S_{u_{ss^*}} S_{u_s} - S_{u_{s^*}} + S_{u_{ss^*}} \frac{g^{1+}}{g^{01}}, \tag{65}$$

where

$$\frac{g^{1+}}{g^{01}} = -\frac{1}{2\Delta}((\mathrm{D}_s S)_{u_{ss^*}} - S_{u_s} - S_{u_{s^*}} S^*_{u_{ss^*}}) + \frac{1}{4\Delta} S_{u_{ss^*}} ((\mathrm{D}_s S^*)_{u_{ss^*}} - S^*_{u_{s^*}} - S^*_{u_s} S_{u_{ss^*}}), \tag{66}$$

and where

$$\Delta = 1 - \tfrac{1}{4} S_{u_{ss^*}} S^*_{u_{ss^*}}. \tag{67}$$

The condition

$$M_S = 0 \tag{68}$$

is invariant under contact transformations of $J^2(\mathbb{R}^2, \mathbb{R})$ preserving L_S.

We now consider the 1-forms

$$\omega^0 = \varsigma^* \beta^0, \tag{69}$$

$$\omega^+ = \alpha(\varsigma^* \beta^+ + b \varsigma^* \beta^-), \tag{70}$$

$$\omega^- = \alpha(\varsigma^* \beta^- + b^* \varsigma^* \beta^+), \tag{71}$$

$$\omega^1 = \varsigma^* \beta^1 + a \varsigma^* \beta^+ + a^* \varsigma^* \beta^- + c\, \varsigma^* \beta^0, \tag{72}$$

where the coefficients α, a, b and c are defined in the following way, [3]

$$b = \frac{\sqrt{1 - S^*_{u_{ss^*}} S_{u_{ss^*}}} - 1}{S^*_{u_{ss^*}}}, \tag{73}$$

$$\alpha^2 = \frac{\left(\sqrt{1 - S^*_{u_{ss^*}} S_{u_{ss^*}}} + 1\right)}{2(1 - S^*_{u_{ss^*}} S_{u_{ss^*}})} = \frac{(1 + bb^*)}{(1 - bb^*)^2} \tag{74}$$

$$\begin{aligned} a = {} & (1 - S_{u_{ss^*}} S^*_{u_{ss^*}})^{-1} \left(1 - \frac{1}{4} \left(\frac{\mathrm{D}S^*}{\mathrm{D}s}\right)_{u_{ss^*}} S^*_{u_{ss^*}}\right)^{-1} \\ & \times \left\{ \frac{1}{2} \left[S^*_{u_{s^*}} + S^*_{u_s} S_{u_{ss^*}} - \left(\frac{\mathrm{D}S^*}{\mathrm{D}s}\right)_{u_{ss^*}} \right] \left(1 + \frac{1}{2} S^*_{u_{ss^*}} S_{u_{ss^*}}\right) \right. \\ & \left. - \frac{3}{4} S^*_{u_{ss^*}} \left[S_{u_s} + S_{u_{s^*}} S^*_{u_{ss^*}} - \left(\frac{\mathrm{D}S}{\mathrm{D}s^*}\right)_{u_{ss^*}} \right] \right\} \end{aligned} \tag{75}$$

$$c=-\tfrac{1}{2}G-(a-a^*b^*)(a^*-ab)(1+bb^*)^{-1}, \tag{76}$$

and

$$\begin{aligned}
&G\left(1+\frac{1}{2}S_{u_{ss^*}}S^*_{u_{ss^*}}\right)\\
&\quad=\left(\frac{\mathrm{D}S}{\mathrm{D}s^*}\right)_{u_s}+\left(\frac{\mathrm{D}S}{\mathrm{D}s^*}\right)_{u_s^*}S^*_{u_{ss^*}}+\left[\left(\frac{\mathrm{D}S}{\mathrm{D}s^*}\right)_{u_s^*}\right]^*+\left[\left(\frac{\mathrm{D}S}{\mathrm{D}s^*}\right)_{u_s}\right]^*S_{u_{ss^*}}\\
&\qquad-\frac{1}{2}\left(\frac{\mathrm{D}^2S}{\mathrm{D}s^{*2}}\right)_{u_{ss^*}}+\frac{1}{2}(S^*_{u_s}S_{u_s}S_{u_{ss^*}}+S_{u_s}S^*_{u_s^*}+S^*_{u_s^*}S_{u_s^*}S^*_{u_{ss^*}}+S_{u_s^*}S^*_{u_s}\\
&\qquad-S^*_{u_{ss^*}}S_u-S_{u_{ss^*}}S^*_u)-\frac{1}{2}\left(S_{u_s}S^*_{u_{ss^*}}+S_{u_{ss^*}}S^*_{u_s}+2\left[\left(\frac{\mathrm{D}S}{\mathrm{D}s^*}\right)_{u_{ss^*}}\right]^*\right)\frac{g^{1+}}{g^{01}}\\
&\qquad-\frac{1}{2}\left(S_{u_{ss^*}}S^*_{u_s^*}+S_{u_s^*}S^*_{u_{ss^*}}+2\left(\frac{\mathrm{D}S}{\mathrm{D}s^*}\right)_{u_{ss^*}}\right)\frac{g^{1-}}{g^{01}},
\end{aligned} \tag{77}$$

where g^{1-}/g^{01} denotes the complex conjugate of the quantity g^{1+}/g^{01} defined in (66).

Using these 1-forms, we define a quadratic differential form h_S on the 6-manifold L_S by

$$h_S=\omega^0\otimes\omega^1+\omega^1\otimes\omega^0-\omega^+\otimes\omega^--\omega^-\otimes\omega^+. \tag{78}$$

We have

Lemma 3. *Consider an overdetermined system (48), with vanishing generalized Wünschmann invariant*

$$M_S=0.$$

Then, we have [3]

$$\mathcal{L}_{\mathrm{D}/\mathrm{D}s}h_S=\Lambda h_S$$

for some multiplier Λ.

We now let g_S denote the quadratic differential form defined on $M_4\times\mathbb{R}^2$ by

$$g_S=(\varsigma^{-1})^*h_S. \tag{79}$$

It follows from the preceding lemma that g_S induces on M_4 a conformal Lorentzian structure, which we shall denote by $[g_S]$.

We can now state our main result in the four-dimensional case:

Theorem 2. *Let (48) be an overdetermined PDE system with vanishing generalized Wünschmann invariant,* $M_S=0$, *and consider a PDE system locally equivalent to (4) under a contact transformation* $f: J^2(\mathbb{R}^2,\mathbb{R})\to J^2(\mathbb{R}^2,\mathbb{R})$. *Then the local diffeomorphism of solution spaces* $\tilde{f}: M_4\to\bar{M}_4$ *induced by* f *preserves the corresponding conformal Lorentzian structures, that is*

$$\tilde{f}^*[\bar{g}_{\bar{S}}]=[g_S].$$

The proof is similar to the one given in the three-dimensional case. We remark that in the four-dimensional case, a complete integral of the eikonal equation for $[g_S]$ will depend on two arbitrary real constants.

Acknowledgements

We thank Robert Bryant and Peter Vassiliou for their interest in this work, and for some helpful discussions. This research was supported by NSERC grant no. OGP0105490 and by the NSF under grant nos. PHY 97-22049 and PHY 98-03301.

References

[1] E. Cartan, La geometria de las equaciones diferenciales de tercer orden, Rev. Mater. Hispano-Am. 4 (1941) 1–31. Reprinted in: E. Cartan, Oeuvres Complètes, Partie III-2, Editions du CNRS, 1984, pp. 1535–1566.
[2] S.-S. Chern, The geometry of a differential equation $y''' = F(x, y, y', y'')$, Sci. Rep. Tsing Hua Univ. 4 (1940) 97–111. Reprinted in: S.-S. Chern, Selected Papers, Springer, Berlin, 1978.
[3] S. Frittelli, C. Kozameh, E.T. Newman, Differential geometry from differential equations, Commun. Math. Phys., gr-qc/0012058, in press.
[4] T. Hawkins, Emergence of the Theory of Lie Groups: An Essay in the History of Mathematics, Springer, Berlin, 2000, pp. 1869–1926.
[5] P.J. Olver, Equivalence, Invariants and Symmetry, Cambridge University Press, Cambridge, 1995.