Progress in Mathematics

From Geometry to Quantum Mechanics

In Honor of Hideki Omori

Yoshiaki Maeda
Peter Michor
Takushiro Ochiai
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Editors

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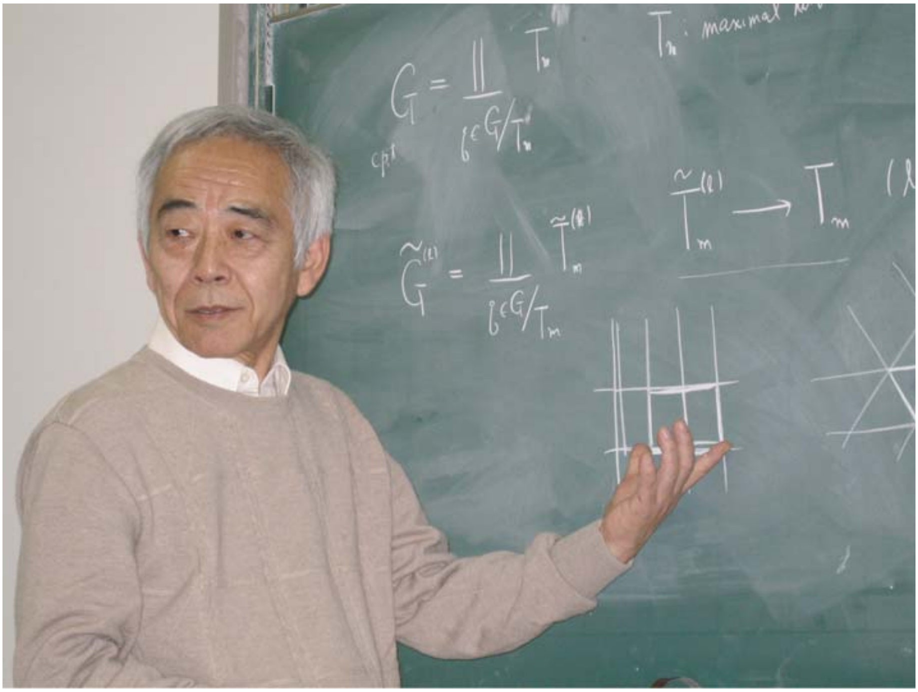

Hideki Omori, 2006

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Preface

Hideki Omori is widely recognized as one of the world's most creative and original mathematicians. This volume is dedicated to Hideki Omori on the occasion of his retirement from Tokyo University of Science. His retirement was also celebrated in April 2004 with an influential conference at the Morito Hall of Tokyo University of Science.

Hideki Omori was born in Nishionmiya, Hyogo prefecture, in 1938 and was an undergraduate and graduate student at Tokyo University, where he was awarded his Ph.D degree in 1966 on the study of transformation groups on manifolds [3], which became one of his major research interests. He started his first research position at Tokyo Metropolitan University. In 1980, he moved to Okayama University, and then became a professor of Tokyo University of Science in 1982, where he continues to work today.

Hideki Omori was invited to many of the top international research institutions, including the Institute for Advanced Studies at Princeton in 1967, the Mathematics Institute at the University of Warwick in 1970, and Bonn University in 1972. Omori received the Geometry Prize of the Mathematical Society of Japan in 1996 for his outstanding contributions to the theory of infinite-dimensional Lie groups.

Professor Omori's contributions are deep and cover a wide range of topics as illustrated by the numerous papers and books in his list of publications. His major research interests cover three topics: Riemannian geometry, the theory of infinite-dimensional Lie groups, and quantization problems. He worked on isometric immersions of Riemannian manifolds, where he developed a maximum principle for nonlinear PDEs [4]. This maximum principle has been widely applied to various problems in geometry as indicated in Chen–Xin [1]. Hideki Omori's lasting contribution to mathematics was the creation of the theory of infinite-dimensional Lie groups. His approach to this theory was founded in the investigation of concrete examples of groups of diffeomorphisms with added geometric data such as differential structures, symplectic structures, contact structures, etc. Through this concrete investigation, Omori produced a theory of infinite-dimensional Lie groups going beyond the categories of Hilbert and Banach spaces to the category of inductive limits of Hilbert and Banach spaces. In particular, the notion and naming of ILH (or ILB) Lie groups is due to Omori [O2]. Furthermore, he extended his theory of infinite-dimensional Lie groups to the category of Fréchet spaces in order to analyze the group of invertible zeroth order Fourier integral operators on a closed manifold. In this joint work with Kobayashi, Maeda, and Yoshioka, the notion of a regular Fréchet Lie group was formulated. Omori developed and unified these ideas in his book [6] on generalized Lie groups.

Beginning in 1999, Omori focused on the problem of deformation quantization, which he continues to study to this day. He organized a project team, called OMMY after the initials of the project members: Omori, Maeda, Miyazaki and Yoshioka. Their first work showed the existence of deformation quantization for any symplectic manifold. This result was produced more or less simultaneously by three different approaches, due to Lecomte–DeWilde, Fedosov and Omori–Maeda–Yoshioka. The approach of the Omori team was to realize deformation quantization as the algebra of a "noncommutative manifold." After this initial success, the OMMY team has continued to develop their research beyond formal deformation quantization to the convergence problem for deformation quantization, which may lead to new geometric problems and insights.

Hideki Omori is not only an excellent researcher, but also a dedicated educator who has nurtured several excellent mathematicians. Omori has a very charming sense of humor that even makes its way into his papers from time to time. He has a friendly personality and likes to talk mathematics even with non-specialists. His mathematical ideas have directly influenced several researchers. In particular, he offered original ideas appearing in the work of Shiohama and Sugimoto [2], his colleague and student, respectively, on pinching problems. During Omori's visit to the University of Warwick, he developed a great interest in the work of K. D. Elworthy on stochastic analysis, and they enjoyed many discussions on this topic. It is fair to say that Omori was the first person to introduce Elworthy's work on stochastic analysis in Japan. Throughout their careers, Elworthy has remained one of Omori's best research friends.

In conclusion, Hideki Omori is a pioneer in Japan in the field of global analysis focusing on mathematical physics. Omori is well known not only for his brilliant papers and books, but also for his general philosophy of physics. He always remembers the long history of fruitful interactions between physics and mathematics, going back to Newton's classical dynamics and differentiation, and Einstein's general relativity and Riemannian geometry. From this point of view, Omori thinks the next fruitful interaction will be a geometrical description of quantum mechanics. He will no doubt be an active participant in the development of his idea of "quantum geometry."

The intended audience for this volume includes active researchers in the broad areas of differential geometry, global analysis, and quantization problems, as well as aspiring graduate students, and mathematicians who wish to learn both current topics in these areas and directions for future research.

We finally wish to thank Ann Kostant for expert editorial guidance throughout the publication of this volume. We also thank all the authors for their contributions as well as their helpful guidance and advice. The referees are also thanked for their valuable comments and suggestions.

References

1. Q. Chen, Y. L. Xin, A generalized maximum principle and its applications in geometry. *Amer. J. Math.*, **114** (1992), 355–366. *Comm. Pure Appl. Math.*, **28** (1975), 333–354.
2. M. Sugimoto, K. Shiohama, On the differentiable pinching problem. *Math. Ann.*, **195** (1971), 1–16.
3. H. Omori, A study of transformation groups on manifolds, *J. Math. Soc. Japan*, **19** (1967), 32–45.
4. H. Omori, Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan*, **19** (1967), 205–214.
5. H. Omori, Infinite dimensional Lie transformation groups, *Lec. Note Math.*, **427**, 1974, Springer.
6. H. Omori, Infinite dimensional Lie groups, *A.M.S. Translation Monograph*, **158**, 1997, AMS.

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Curriculum Vitae

Hideki Omori

Born 1938. 12. 3

BA : University of Tokyo, 1961
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PhD : University of Tokyo, 1966

1963–1966 Research Assistant, Tokyo Metropolitan University
1966–1967 Lecturer, Tokyo Metropolitan University
1967–1980 Associate Professor, Tokyo Metropolitan University
1980–1982 Professor, Okayama University
1982–2004 Professor, Tokyo Unversity of Science

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1970–1971 Visiting Professor, The University of Warwick
1972–1973 Visiting Professor, Bonn University
1975–1975 Visiting Professor, Northwestern University

List of Publications

[1] H. Omori, Homomorphic images of Lie groups, *J. Math. Soc. Japan*, **18** (1966), 97–117.
[2] H. Omori, Some examples of topological groups, *J. Math. Soc. Japan*, **18** (1966), 147–153.
[3] H. Omori, A transformation group whose orbits are homeomorphic to a circle of a point, *J. Fac. Sci. Univ. Tokyo*, **13** (1966), 147–153.
[4] H. Omori, A study of transformation groups on manifolds, *J. Math. Soc. Japan*, **19** (1967), 32–45.
[5] H. Omori, Isometric immersions of Riemannian manifolds, *J. Math. Soc. Japan*, **19** (1967), 205–214.
[6] H. Omori, A class of riemannian metrics on a manifold, *J. Diff. Geom.*, **2** (1968), 233–252.
[7] H. Omori, On the group of diffeomorphisms on a compact manifold, *Global Analysis, Proc. Sympos. Pure Math.*

[8] H. Omori, P. de la Harpe, Opération de groupes de Lie banachiques sur les variétés différentielle de dimension finie, *C.R. Ser. A-B.*, **273** (1971), A395–A397.

[9] H. Omori, On regularity of connections, *Differential Geometry, Kinokuniya Press, Tokyo*, (1972), 385–399.

[10] H. Omori, Local structures of group of diffeomorphisms, *J. Math. Soc. Japan*, **24** (1972), 60–88.

[11] H. Omori, On smooth extension theorems, *J. Math. Soc. Japan*, **24** (1972), 405–432.

[12] H. Omori, P. de la Harpe, About interactions between Banach–Lie groups and finite dimensional manifolds, *J. Math. Kyoto Univ*, **12** (1972), 543–570.

[13] H. Omori, Groups of diffeomorphisms and their subgroups, *Trans. A.M.S.* **179** (1973), 85–432.

[14] H. Omori, Infinite dimensional Lie transformation groups, Lecture Note in Mathematics, **427**, Springer, 1974.

[15] A. Koriyama, Y. Maeda, H. Omori, Lie algebra of vector fields on expansive sets, *Japan. J. Math. (N.S.)*, **3** (1977), 57–80.

[16] A. Koriyama, Y.Maeda, H. Omori, Lie algebra of vector fields, *Trans. Amer. Math. Soc.* **226** (1977), 89–117.

[17] H. Omori, Theory of infinite-dimensional Lie groups (in Japanese). *Kinokuniya Book Store Co., Ltd., Tokyo*, 1978

[18] H. Omori, On Banach Lie groups acting on finite dimensional manifolds, *Tohoku Math. J.* **30** (1978), 223–250.

[19] H. Omori, On the volume elements on an expansive set, *Tokyo J. Math.* **1** (1978), 21–39.

[20] H. Omori, Theory of infinite-dimensional Lie groups, (in Japanese) *Sugaku* **31** (1979), 144–158.

[21] H. Omori, A mehtod of classifying expansive sigularities, *J. Differential. Geom.* **15** (1980), 493–512.

[22] H. Omori, Y. Maeda, A. Yoshioka, On regular Fréchet–Lie groups I, Some differential geometrical expressions of Fourier integral operators on a Riemannian manifold, *Tokyo J. Math.* **3** (1980), 353–390.

[23] H. Omori, A remark on nonenlargeable Lie algebras, *J. Math. Soc. Japan*, **33** (1981), 707–710.

[24] H. Omori, Y. Maeda, A. Yoshioka, On regular Fréchet–Lie groups II, Composition rules of Fourier integral operators on a Riemannian manifold, *Tokyo J. Math.* **4** (1981), 221–253.

[25] H. Omori, Y. Maeda, A. Yoshioka, O. Kobayashi, On regular Fréchet–Lie groups III, A second cohomology class related to the Lie algebra of pseudo-differential operators of order one, *Tokyo J. Math.* **4** (1981), 255–277.

[26] H. Omori, Y. Maeda, A. Yoshioka, O. Kobayashi, On regular Fréchet–Lie groups IV, Definitions and fundamental theorems, *Tokyo J. Math.* **5** (1982), 365–398.

[27] H. Omori, Construction problems of riemannian manifolds, *Spectra of riemannian manifolds, Proc. France-Japan seminar, Kyoto, 1981*, (1982), 79–90.

[28] D. Fujiwara, H. Omori, An example of globally hypo-elliptic operator, *Hokkaido Math. J.* **12** (1983), 293–297.

[29] H. Omori, Y. Maeda, A. Yoshioka, O. Kobayashi, On regular Fréchet–Lie groups V, Several basic properties, *Tokyo J. Math.* **6** (1983), 39–64.

[30] H. Omori, Y. Maeda, A. Yoshioka, O. Kobayashi, On regular Fréchet–Lie groups VI, Infinite dimensional Lie groups which appear in general relativity, *Tokyo J. Math.* **6** (1983), 217–246.

[31] H. Omori, Second cohomology groups related to ψDOs on a compact manifold. *Proceedings of the 1981 Shanghai symposium on differential geometry and differential equations, Shanghai/Hefei, 1981*, (1984), 239–240.

[32] A. Yoshioka, Y. Maeda, H. Omori, O. Kobayashi, On regular Fréchet–Lie groups VII, The group generated by pseudo-differential operators of negative order, *Tokyo J. Math.* **7** (1984), 315–336.

[33] Y. Maeda, H. Omori, O. Kobayashi, A. Yoshioka, On regular Fréchet–Lie groups VIII, Primodial operators and Fourier integral operators, *Tokyo J. Math.* **8** (1985), 1–47.

[34] O. Kobayashi, A. Yoshioka, Y. Maeda, H. Omori, The theory of infinite dimensional Lie groups and its applications, *Acta Appl. Math.* **3** (1985), 71–105.

[35] H. Omori, On global hypoellipticity of horizontal Laplacians on compact principal bundles, *Hokkaido Math. J.* **20** (1991), 185–194.

[36] H. Omori, Y. Maeda, A. Yoshioka, Weyl manifolds and deformation quantization, *Advances in Math.* **85** (1991), 224–255.

[37] H. Omori, Y. Maeda, A. Yoshioka, Global calculus on Weyl manifolds, *Japanese J. Math.* **17** (1991), 57–82.

[38] H. Omori, Y. Maeda, A. Yoshioka, Existence of a closed star-product, *Lett. Math. Phys.* **26** (1992), 285–294.

[39] H. Omori, Y. Maeda, A. Yoshioka, Deformation quantization of Poisson algebras, *Proc. J. Acad. Ser.A. Math. Sci.* **68** (1992), 97–118

[40] H. Omori, Y. Maeda, A. Yoshioka, The uniqueness of star-products on $P_n(C)$, *Differential Geometry, Shanghai, 1991*, (1992), 170–176.

[41] H. Omori, Y. Maeda, A. Yoshioka, Non-commutative complex projective space, *Progress in differential geometry*, Advanced Studies in Pure Math. **22**, (1993), 133–152.

[42] T. Masuda, H. Omori, Algebra of quantum groups as quantized Poisson algebras, *Geometry and its applications, Yokohama, 1991*, (1993), 109–120.

[43] H. Omori, Y. Maeda, A. Yoshioka, A construction of a deformation quantization of a Poisson algebra, *Geometry and its applications. Yokohama, 1991*, (1993), 201–218.

[44] H. Omori, Y. Maeda, A. Yoshioka, Poincaré–Birkhoff–Witt theorem for infinite dimensional Lie algebras, *J. Math. Soc. Japan*, **46** (1994), 25–50.

[45] T. Masuda, H. Omori, The noncommutative algebra of the quantum group $SU_q(2)$ as a quantized Poisson manifold, *Symplectic geometry and quantization*, *Contemp. Math.* **179**, (1994) 161–172.

[46] H. Omori, Y. Maeda, A. Yoshioka, Deformation quantizations of Poisson algebras, *Symplectic geometry and quantization*, *Contemp. Math.* **179**, (1994), 213–240.

[47] H. Omori, Y. Maeda, A. Yoshioka, Deformation quantization of Poisson algebras.(Japanese) Nilpotent geometry and analysis (in Japanese), *RIMS Kokyuroku*, **875** (1994), 47–56.

[48] H. Omori, Berezin representation of a quantized version of the group of volume-preserving transformations. (Japanese) Geometric methods in asymptotic analysis (in Japanese), *RIMS Kokyuroku*, **1014** (1997), 76–90.

[49] H. Omori, N. Miyazaki, A. Yoshioka, Y. Maeda, Noncommutative 3-sphere: A model of noncommutative contact algebras, *Quantum groups and quantum spaces, Warsaw, 1995*, Banach Center Publ., **40** (1997), 329–334.

[50] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Noncommutative contact algebras, *Deformation theory and symplectic geometry, Ascona, 1996*, Math. Phys. Studies, **20** (1997), 333–338.

[51] H. Omori, Infinite dimensional Lie groups, Translations of Mathematical Monographs, **158**. *American Mathematical Society, Providence, RI*, 1997.

[52] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Groups of quantum volume preserving diffeomorphisms and their Berezin representation, *Analysis on infinite-dimensional Lie groups and algebras, Marseille, 1997*, (1998), 337–354.

[53] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Deformation quantization of the Poisson algebra of Laurent polynomials, *Lett. Math. Pysics*, **46** (1998), 171–180.

[54] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Noncommutative 3-sphere: A model of noncommutative contact algebras, *J. Math. Soc. Japan*, **50** (1998), 915–943.

[55] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Poincaŕe–Cartan class and deformation quantization of Kähler manifolds, *Commun. Math. Phys.* **194** (1998), 207–230.

[56] H. Omori, The noncommutative world: a geometric description. (in Japanese) *Sugaku* **50** (1998), 12–28.

[57] H. Omori, Introduction to noncommutative differential geometry, *Lobachevskii J. Math.* **4** (1999), 13–46.

[58] H. Omori, T. Kobayashi, On global hypoellipticity on compact manifolds, *Hokkaido Math. J*, **28** (1999), 613–633.

[59] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Anomalous quadratic exponentials in the star-products, *Lie groups, geometric structures and differential equations—one hundred years after Sophus Lie* (in Japanese), *RIMS Kokyuroku*, **1150** (2000), 141–165.

[60] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Deformation quantization of Fréchet–Poisson algebras: Convergence of the Moyal product, *in Conférence Moshé Flato 1999, Quantizations, Deformations, and Symmetries*, Math. Phys. Studies **22**, Vol. II, (2000), 233–246.

[61] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, An example of convergent star product, *Dynamical systems and differential geometry* (in Japanese), *RIMS Kokyuroku*, **1180** (2000), 141–165.

[62] H. Omori, Noncommutative world, and its geometrical picture, *A.M.S. translation of Sugaku expositions* **13** (2000), 143–171.

[63] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Singular system of exponential functions, *Noncommutative differential geometry and its application to physics*, Math. Phys. Studies **23** (2001), 169–186,

[64] H. Omori, T. Kobayashi, Singular star-exponential functions, *SUT J. Math.* **37** (2001), 137–152.

[65] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Convergent star products on Fréchet linear Poisson algebras of Heisenberg type, *Global differential geometry: the mathematical legacy of Alfred Gray, Bilbao, 2000*, Contemp. Math., **288** (2001), 391–395.

[66] H. Omori, Associativity breaks down in deformation quntization, *Lie groups, geometric structures and differential equations—one hundred years after Sophus Lie*, Advanced Studies in Pure Math., **37** (2002), 287–315.

[67] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Star exponential functions for quadratic forms and polar elements, *Quantization, Poisson brackets and beyond, Manchester, 2001*, Contemp. Math., **315** (2002), 25–38.

[68] H. Omori, One must break symmetry in order to keep associativity, *Geometry and analysis on finite- and infinite-dimensional Lie groups, Bedlewo, 2000*, Banach Center Publi., **55** (2002), 153-163.

[69] H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, Strange phenomena related to ordering problems in quantizations, *Jour. Lie Theory* **13** (2003), 481–510.

[70] Y. Maeda, N. Miyazaki, H. Omori, A. Yoshioka, Star exponential functions as two-valued elements, *The breadth of symplectic and Poisson geometry*, Progr. Math., **232** (2005), 483–492.

From Geometry to
Quantum Mechanics

Part I

Global Analysis and Infinite-Dimensional Lie Groups

Aspects of Stochastic Global Analysis

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Dedicated to Hideki Omori

Summary. This is a survey of some topics where global and stochastic analysis play a role. An introduction to analysis on Banach spaces with Gaussian measure leads to an analysis of the geometry of stochastic differential equations, stochastic flows, and their associated connections, with reference to some related topological vanishing theorems. Following that, there is a description of the construction of Sobolev calculi over path and loop spaces with diffusion measures, and also of a possible L^2 de Rham and Hodge-Kodaira theory on path spaces. Diffeomorphism groups and diffusion measures on their path spaces are central to much of the discussion. Knowledge of stochastic analysis is not assumed.

AMS Subject Classification: Primary 58B20; 58J65; Secondary 53C17; 53C05; 53C21; 58D20; 58D05; 58A14; 60H07; 60H10; 53C17; 58B15.

Key words: Path space, diffeomorphism group, Hodge–Kodaira theory, infinite dimensions, universal connection, stochastic differential equations, Malliavin calculus, Gaussian measures, differential forms, Weitzenbock formula, sub-Riemannian.

1 Introduction

Stochastic and global analysis come together in several distinct ways. One is from the fact that the basic objects of finite dimensional stochastic analysis naturally live on manifolds and often induce Riemannian or sub-Riemannian structures on those manifolds, so they have their own intrinsic geometry. Another is that stochastic analysis is expected to be a major tool in infinite dimensional analysis because of the singularity of the operators which arise there; a fairly prevalent assumption has been that in this situation stochastic methods are more likely to be successful than direct attempts to extend PDE techniques to infinite dimensional situations. (Ironically that situation has been reversed in recent work on the stochastic 3D Navier–Stokes equation, [DPD03].) Stimulated particularly by the approach of Bismut to index theorems, [Bis84], and by other ideas from topology, representation theory, and theoretical physics, this has been extended to attempts to use stochastic analysis in the construction of infinite dimensional geometric structures, for example on loop spaces of Riemannian manifolds. As examples see [AMT04], and [Léa05]. In any case global analysis was firmly embedded in stochastic analysis with the advent of Malliavin calculus, a theory of Sobolev spaces and calculus on the space of continuous paths on $\mathbb{R}^n$, as described briefly below, and especially its relationships with diffusion operators and processes on finite dimensional manifolds.

In this introductory selection of topics, both of these aspects of the intersection are touched on. After a brief introduction to analysis on spaces with Gaussian measure there is a discussion of the geometry of stochastic differential equations, stochastic flows, and their associated connections, with reference to some related topological vanishing theorems. Following that, there is a discussion of the construction of Sobolev calculi over path and loop groups with diffusion measures, and also of de Rham and Hodge–Kodaira theory on path spaces. The first part can be considered as an updating of [Elw92], though that was written for stochastic analysts. A more detailed introductory survey on geometric stochastic analysis 1950–2000 is in [Elw00]. The section here on analysis on path spaces is very brief, with a more detailed introduction to appear in [ELb], and a survey for specialists in [Aid00]. Many important topics which have been developed since 2000 have not been mentioned. These include, in particular, the extensions of Nevanlinna theory by Atsuji, [Ats02], stochastic analysis on metric spaces [Stu02] and geometry of mass transport and couplings [vRS05], geometric analysis on configuration spaces, [Dal04], and on infinite products of compact groups, [ADK00], and Brownian motion on Jordan curves and representations of the Virasoro algebra [AMT04].

In this exposition the diffeomorphism group takes its central role: I was introduced to it by Hideki Omori in 1967 and I am most grateful for that and for the continuing enjoyment of our subsequent mathematical and social contacts.

2 Convolution semi-groups and Brownian motions

Consider a Polish group G. Our principle examples will be $G = \mathbb{R}^m$ or more generally a separable real Banach space, and $G = \mathrm{Diff}(M)$ the group of smooth diffeomorphisms of a smooth connected finite dimensional manifold M with the C^∞-compact open topology, and group structure given by composition; see [Bax84]. By a *convolution semigroup of probability measures* on G we mean a family of Borel measures $\{\mu_t\}_{t\geqslant 0}$ on G such that:

(i) $\mu_t(G) = 1$
(ii) $\mu_t \star \mu_s = \mu_{s+t}$

where $\star$ denotes convolution, i.e., the image of the product measure $\mu_t \otimes \mu_s$ on $G \times G$ by the multiplication $G \times G \to G$.

The standard example on $\mathbb{R}^m$ is given by the standard Gaussian family $\{\gamma_t^m\}_t$ whose values on a Borel set A are given by:

$$\gamma_t^m(A) = (2\pi t)^{-m/2} \int_A e^{-|x|^2/2t} dx.$$

More generally when G is a finite dimensional Lie group with right invariant metric we could set $\mu_t = p_t(id, x)dx$, the fundamental solution of the heat equation on G from the identity element. In these examples we also have *symmetry* and *continuity*, i.e.,

(iii) $\mu_t(A^{-1}) = \mu_t(A)$ for all Borel sets A where $A^{-1} = \{g^{-1} : g \in A\}$.
(iv) $(1/t)\mu_t(G - U) \to 0$ as $t \to 0$ for all neighbourhoods U of the identity element.

Given a convolution semigroup satisfying (i), (ii), and (iv) there is an associated Markov process on G; that is, a family of measurable maps

$$z_t : \Omega \to G, t \geqslant 0,$$

defined on some probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ such that:

(a) $z_0(\omega) = id$ for all $\omega \in \Omega$
(b) $t \mapsto z_t(\omega)$ is continuous for all $\omega \in \Omega$
(c) for each Borel set A in G and times $0 \leqslant s \leqslant t < \infty$

$$\mathbb{P}\{\omega \in \Omega : z_t(\omega)z_s(\omega)^{-1} \in A\} = \mu_{t-s}(A).$$

In particular we can take Ω to be the space of continuous maps of the positive reals into G which start at the identity element, and then take $z_t(\omega) = \omega(t)$. This is the *canonical process*. In any case the process satisfies:

(A) (independent increments on the left) If $0 \leqslant s < t \leqslant u < v$, then $z_t z_s^{-1}$ and $z_v z_u^{-1}$ are independent.
(B) (time homogeneity) For $0 \leqslant s \leqslant t$ and a Borel set A, we have $\mathbb{P}\{\omega : z_t(\omega)z_s(\omega)^{-1} \in A\}$ depends only on $t - s$.

For proofs in this generality see [Bax84]. Baxendale calls such processes *Brownian motions* on G, though such terminology may be restricted to the case where the symmetry condition (iii) holds with the general case referred to as *Brownian motions with a drift*. In the symmetric case we will call the measure $\mathbb{P}$ on the path space of G the *Wiener measure*. However it will often be more convenient to restrict our processes to run for only a finite time,T, say. Our canonical probability space will then be the space $C_{id}([0, T]; G)$ of continuous paths in G starting at the identity and running for time T. In the example above where $G = \mathbb{R}^m$ we obtain the standard, classical Brownian motion and classical Wiener measure on $C_0([0, T]; \mathbb{R}^m)$.

There are also corresponding semi-groups. For this we refer to the following lemma of Baxendale:

Lemma 2.1 ([Bax84]) *Let B be a Banach space and $G \times B \to B$ a continuous action of G by linear maps on B. Set $P_t b = \int (gb) d\mu_t(g)$. This integral exists and $\{P_t\}_{t \geqslant 0}$ forms a strongly continuous semi-group of bounded linear operators on B satisfying $\|P_t\| \leqslant ce^{dt}$ for some constants c and d.*

In our example with $G = \mathbb{R}^m$ we can take B to be the space of bounded continuous real-valued functions on $\mathbb{R}^m$, or those vanishing at infinity, or L^2 functions etc., with the action given by $(x, f) \mapsto f(\cdot + x)$. The resulting semi-group is then just the usual heat semi-group with generator $-\frac{1}{2}\triangle$ where we use the sign convention that Laplacians are non-negative. From convolution semi-groups on Diff M we will similarly obtain semi-groups acting on differential forms and other tensors on M as well as the semi-group $\{\mathbf{P}_t\}_t$ acting on functions on Diff(M): see below. Note that if our convolution semi-group satisfies (i), (ii), and (iv) so does the family $\{\mu_{rt}\}_{t\geqslant 0}$ for each $r > 0$. We therefore get a family of probability measures $\{\mathbb{P}_r\}_{r\geqslant 0}$ on $C_{id}([0, T]; G)$ with $\mathbb{P} = \mathbb{P}_1$, which will also form a convolution semi-group.

2.1 Gaussian measures on Banach spaces

Take G to be a separable (real) Banach space E. If E is finite dimensional, a probability measure γ on E is said to be (centred) *Gaussian* if its Fourier transform $\widehat{\gamma}(l) := \int_E e^{il(x)} d\gamma(x) = exp(-\frac{1}{2}B(l, l))$ for all l in E^*, the dual space of E, for some positive semi-definite bilinear form B on E^*. General Gaussians are just translates of these. When E is infinite dimensional γ is said to be Gaussian if its push forward $l_*\gamma$ is Gaussian on $\mathbb{R}$ for each $l \in E^*$. The Levy–Khinchin representation gives a decomposition of any convolution semigroup on E, e.g., see [Lin86], from this, (even just the one-dimensional version), we see that each measure μ_t of a convolution family on E satisfying (iv) is Gaussian.

Gaussian measures have a rich structure. If γ is a centred Gaussian measure on E, by a result given in a general form in [DFLC71] but going back to Kuelbs, Sato, and Stefan, there is a separable Hilbert space H, $\langle, \rangle_H$ and an injective bounded linear map $i : H \to E$ such that $\widehat{\gamma}(l) = \exp(-\frac{1}{2}\|j(l)\|_H^2)$ for all $l \in E^*$ where $j : E^* \to H$ is the adjoint of i. If γ is strictly positive (i.e., the measure of any non-empty open subset of E is positive) it is said to be *non-degenerate* and then i has dense image. Any triple $\{i, H, E\}$ which arises this way is called an *abstract Wiener space* following L. Gross, e.g., in [Gro67]. If $\gamma = \mu_1$ for a convolution semi-group, then $\mu_t = \gamma_t$ for each t where γ_t has Fourier transform $\widehat{\gamma_t}(l) = \exp(-\frac{1}{2}t\|j(l)\|_H^2)$ for $l \in E^*$.

Among the important properties of abstract Wiener spaces and their measures are:

- the image $i[H]$ in E has γ-measure zero,
- translation by an element v of E preserves sets of measure zero if and only if v lies in the image of H,
- if $T : E \to K$ is a continuous linear map into a Hilbert space K then the composition $T \circ i$ is Hilbert–Schmidt,
- if $s \neq t$, then γ_t and γ_s are orthogonal, in the sense that there is a set which has full measure for one and measure zero for the other.

Gross showed that to do analysis, and in particular potential theory, using these measures, it was natural to differentiate only in the H-directions, and to consider the H-derivative of a suitable function $f : E \to K$ of E into a separable Hilbert space K, e.g., a Fréchet differentiable function, as a map of E into the space of Hilbert–Schmidt maps of H into K:

$$d_H f : E \to \mathcal{L}_2(H; K).$$

He generalised an integration by parts theorem, for classical Wiener space, of Cameron and Martin to this context. Malliavin calculus took this much further; going to the closure $\overline{d_H}$ of the H-derivative as an operator between L^p spaces of functions on E and showing that a wide class of functions defined only up to sets of measure zero on classical Wiener space (for example as solutions of stochastic differential equations) actually lie in the domain of the closure of the H-derivative, and so can be considered to have H-derivatives lying in L^2. The closability of the H-derivative can be deduced from the integration by parts theorem.

In its simplest form the integration by parts formula is as follows: Let $f : E \to \mathbb{R}$ be Fréchet differentiable with bounded derivative and let $h \in H$. Then

$$\int_E (d_H f)_x(h) d\gamma(x) = -\int_E f(x) div(h)(x) d\gamma(x)$$

where $div(h) : E \to \mathbb{R}$ is $-\mathcal{W}(h)$ where $\mathcal{W}(h) = \lim_{L^2} l_n$ for $\{l_n\}_n$ a sequence in E^* such that $j(l_n) \to h$ in H.

If E is finite dimensional, $\mathcal{W}(h)(x)$ is just $\langle h, x\rangle$. For classical Wiener space it is often written $\int_0^T \langle \frac{dh}{dt}, dx\rangle$ and known as the Paley–Wiener integral. Unless $\frac{dh}{dt}$ is of bounded variation, or has some similar smoothness property, it will have no classical meaning since almost all paths x will not have bounded variation. It is the simplest example of a 'stochastic integral'. In general it is not continuous in $x \in C_0([0, T]; \mathbb{R}^m)$. However it is in the domain of $\overline{d_H}$ with $\overline{d_H}(\mathcal{W}(h))_x(k) = \langle h, k\rangle_H$ for all $x \in E$ and $k \in H$.

More generally we have a divergence operator acting on a class of H-vector fields, i.e., maps $V : E \to H$. Let $\mathbb{D}^{p,1}$ be the domain of $\overline{d_H}$ acting from $L^p(E; \mathbb{R})$ to $L^p(E; H^*)$ with its graph norm. Then

$$\int_E (d_H f)_x(V(x)) d\gamma(x) = -\int_E f(x)\,\mathrm{div}(V)(x) d\gamma(x)$$

for $f \in \mathbb{D}^{2,1}$ if V is in the domain of div in L^2. In the classical Wiener space case an H-vector field is a map $V : C_0([0, T]; \mathbb{R}^m) \to L^{2,1}([0, T] : \mathbb{R}^m)$ and so we have $\frac{\partial V(\sigma)}{\partial t} \in L^2([0, T]; \mathbb{R}^m)$, for $\sigma \in C_0([0, T]; \mathbb{R}^m)$. This can be considered as a stochastic process in $\mathbb{R}^m$ with probability space the classical Wiener space with its Wiener measure. If this process is *adapted* or *non-anticipating*, (which essentially means that for each $t \in [0, T]$, $\frac{\partial V(\sigma)}{\partial t}$ depends only on the path σ up to time t), and is square integrable with respect to the Wiener measure, then V is in the domain of the divergence and its divergence turns out to be minus the Ito integral, written

$$\mathrm{div}(V) = -\int_0^T \frac{\partial V(\sigma)}{\partial t} d\sigma(t).$$

This is the stochastic integral which is the basic object of stochastic calculus (and so, of course to its applications, for example to finance). It has the important isometry property that

$$\left\| \int_0^T \frac{\partial V(\sigma)}{\partial t} d\sigma(t) \right\|_{L^2}^2 = \int_{C_0([0,T];\mathbb{R}^m)} \int_0^T \left\| \frac{\partial V(\sigma)}{\partial t} \right\|^2 dt d\gamma(\sigma).$$

In the non-adapted case it is, now by definition, the *Skorohod integral*, or *Ramer–Skorohod integral*, although it may involve differentiation as is standard in finite dimensions.

Corresponding to $\overline{d_H}$ there is the gradient operator, acting on L^2 as $\nabla : \mathbb{D}^{2,1} \to L^2(E; H)$. The relevant version of the Laplacian is the 'Ornstein–Uhlenbeck' operator $\mathcal{L}$ given by $\mathcal{L} = \overline{d_H}^* \overline{d_H} = -\operatorname{div} \nabla$. With its natural domain this is self-adjoint. Its spectrum in L^2 consists of eigenvalues of infinite multiplicity, apart from the ground state. The eigenspace decomposition it induces is Wiener's homogeneous chaos decomposition, at least in the classical Wiener space case, or in field theoretic language the Fock space decomposition with $\mathcal{L}$ the number operator. When $E = H = \mathbb{R}^n$ the operator $\mathcal{L}$ is given by

$$\mathcal{L}(f)(x) = \triangle(f)(x) + \langle \nabla(f)(x), x \rangle$$

for $\triangle$ the usual Laplacian on $\mathbb{R}^n$ (with the sign convention that it is a *positive* operator).

The H-derivative also gives closed operators $\overline{d_H} : \operatorname{Dom}(\overline{d_H}) \subset L^p(E; G) \to L^p(E; \mathcal{L}_2(H; G))$ for $1 \leqslant p < \infty$ where the Hilbert space of Hilbert–Schmidt operators, $\mathcal{L}_2(H; G)$, is sometimes identified with the completed tensor product $G \bigotimes_2 H$. This leads to the definitions of higher derivatives and Sobolev spaces. An L^2–de Rham theory of differential forms was described by Shigekawa, [Shi86], in this context. It was based on H-forms, i.e., maps $\varphi : E \to \bigwedge^k H^*$ for k-forms, where $\bigwedge^k H^*$ refers to the Hilbert space completion of the k-th exterior power of H^* with itself. He defined an L^2-Hodge–Kodaira Laplacian, gave a Hodge decomposition and proved vanishing of L^2 harmonic forms with consequent triviality of the de Rham cohomology. In finite dimensions these Laplacians could be considered as Bismut–Witten Laplacians for the Gaussian measure in question.

2.2 Brownian motions on diffeomorphism groups

For convolution semi-groups on a finite dimensional Lie group G there is an analogous Levy–Khinchin description to that described above. It is due to Hunt [Hun56]. In particular given the continuity condition (iv) above, the semi-group $\{\mathbf{P}_t\}_{t \geqslant 0}$ induced on functions on G has generator a second-order right-invariant semi-elliptic differential operator with no zero-order term (a right invariant *diffusion operator*) on the group.

For diffeomorphism groups of compact manifolds Baxendale gave an analogue of this result of Hunt. Given a convolution semi-group of probability measures, satisfying (iii) and (iv), on $\operatorname{Diff}(M)$ for M compact, he showed that there is a Gaussian measure, γ say, on the tangent space $T_{id}\operatorname{Diff}(M)$ at the identity, i.e., the space of smooth vector fields on M, with an induced convolution semi-group of Gaussian measures and Brownian motion $\{W_t\}_t$ on $T_{id}\operatorname{Diff}(M)$ such that the Brownian motion on $\operatorname{Diff}(M)$ can be taken to be the solution, starting at the identity, of the right invariant *stochastic differential equation*

$$d\xi_t = TR_{\xi_t} \circ dW_t$$

where R_g denotes right translation by the group element g and TR_g its derivative acting on tangent vectors. Such a (Stratonovich) stochastic differential equation gives the solution$\{\xi_t\}_t$ as a non-anticipating function of $\{W_t\}_t$ (taking W_- to be the canonical Brownian motion). The solution can be obtained by taking piecewise linear approximations $\{W_t^n\}_t$ to each path W_-, solving the ordinary differential equations

$$\frac{d\xi_t^n}{dt} = TR_{\xi_t^n}\frac{dW_t^n}{dt}$$

with $\xi_0 = id$ to obtain measurable maps

$$\xi_-^n : C_0([0,T]; T_{id}\mathrm{Diff}(M)) \to C_{id}([0,T]; \mathrm{Diff}(M)),$$

$n = 1, 2, \ldots$. These will converge in measure, (and so a subsequence almost surely), to the required solution of the stochastic differential equation. The point of this procedure being that typical Brownian paths are too irregular for our stochastic differential equation to have classical meaning. The solution will be measurable but not continuous in W_-. However one of the main points of the Malliavin calculus is that for such equations, at each time t, it is possible to define the H-derivative of the solution.

As in the case of finite dimensional Lie groups the situation is also determined by a diffusion operator $\mathcal{B}$, say, acting on functions $f : \mathrm{Diff}(M) \to \mathbb{R}$. This is given by the sum of Lie derivatives

$$\mathcal{B}(f) = 1/2 \sum_j \mathcal{L}_{\widetilde{X^j}} \mathcal{L}_{\widetilde{X^j}}$$

where $\{X^j\}_j$ is an orthonormal base for the Hilbert space H_γ of vector fields determined by the Gaussian measure γ and $\widetilde{X^j}$ denotes the corresponding right invariant vector field on $\mathrm{Diff}(M)$: for $\theta \in \mathrm{Diff}(M)$ we have $\widetilde{X^j}(\theta) : M \to TM$ given by $\widetilde{X^j}(\theta)(y) = X^j(\theta(y))$.

If we fix a point $x_0 \in M$, there is the *one-point motion* $\{\xi_t(x_0) : 0 \leqslant t \leqslant T\}$. This almost-surely defined function of $W_- \in C_0([0,T]; T_{id}\mathrm{Diff}(M))$ solves the stochastic differential equation on M:

$$dx_t = ev_{x_t} \circ dW_t$$

where ev_{x_t} denotes evaluation at x_t, which could equally be written as

$$dx_t = \sum_j X^j(x_t) \circ dW_t^j$$

where W_t^j now is the jth component of the Brownian motion W_- (or to be more precise it is $\mathcal{W}(s \mapsto (s \wedge t)X^j)$, for $\mathcal{W}$ the Paley–Wiener map described above), which is a Brownian motion on $\mathbb{R}$.

In case the symmetry condition (iii) on the convolution semi-group does not hold, the only difference is the appearance of a vector field A, say, on M, whose right translate $\widetilde{A}$ needs to be added to our expression for $\mathcal{B}$ as a first-order operator, and which has to be added on to the stochastic differential equations. The stochastic differential equation for the one-point motion is then:

$$dx_t = ev_{x_t} \circ dW_t + A(x_t)dt \tag{1}$$

which has the same interpretation via approximations as that described for the first equation on Diff(M), or can be interpreted in terms of stochastic integrals as described below. In any case the Brownian motion $\{\xi_t : 0 \leqslant t \leqslant T\}$ is the solution flow of the S.D.E. for the one-point motion.

The semi-group $\{P_t\}_t$ of operators on functions on M that $\{\mu_t\}_t$ determines has generator the diffusion operator $\mathcal{A}$ for

$$\mathcal{A} = 1/2 \sum_j \mathcal{L}_{X^j}\mathcal{L}_{X^j} + \mathcal{L}_A. \tag{2}$$

Thus for bounded measurable $f : M \to \mathbb{R}$ if we set $f_t = P_t(f) = \int f \circ \xi_t$, then f_t solves the equation $\frac{\partial f_t}{\partial t} = \mathcal{A}(f_t)$ at least if f is smooth, or more generally if $\mathcal{A}$ is elliptic. In fact the standard definition of a solution to 1 is that for any C^2 function $f : M \to \mathbb{R}$ we have for all relevant t:

$$f(x_t) = f(x_0) + \int_0^t (df)_{x_s} dW_s(x_s) + \int_0^t \mathcal{A}(f)(x_s)ds$$

where the first integral is an Ito stochastic integral, described above as minus the divergence of $W \mapsto \int_0^{\cdot}(df)x_s(W_-)ds$ considered as an H-vector field $E \to L_0^{2,1}([0,T]; H_\gamma)$, where E is the closure of H_γ in the space $C_0([0,T]; T_{id}\mathrm{Diff}\, M)$, i.e., the support of γ. In summary the main result of [Bax84] can be expressed as:

Theorem 2.2 (Baxendale) *Every Brownian motion on the diffeomorphism group of a compact manifold is the solution flow of a stochastic differential equation driven by, possibly infinitely many, Brownian motions on $\mathbb{R}$. The flow is determined by the expression of the generator $\mathcal{A}$ in Hörmander form, or more precisely by the Hilbert space H of vector fields which has the vector fields X^j as orthonormal basis, together with the "drift" A.*

It is important to appreciate that there are in general many ways to write a diffusion generator such as $\mathcal{A}$ in Hörmander form, even using only finitely many vector fields. These different ways correspond to flows which may have very different behaviour, [CCE86]. We shall look below a bit more deeply at the extra structure a Hörmander form decomposition involves. When M is Riemannian and $\mathcal{A} = -1/2\triangle$, we can obtain a Hörmander form decomposition via Nash's isometric embedding theorem. For this take such an embedding $\alpha : M \to \mathbb{R}^m$ say, write α in components $(\alpha^1, \ldots, \alpha^m)$ and set $X^j = \mathrm{grad}(\alpha^j)$. The corresponding S.D.E. equation 1, with $A = 0$, has solutions which have $-1/2\triangle$ as generator in the sense described above. This means they are *Brownian motions on M* by definition of a Brownian motion on a Riemannian manifold. Such an S.D.E. is called a *gradient Brownian S.D.E.* For a compact Riemannian symmetric space the symmetric space structure can be used to give a Hörmander form decomposition or equivalently an SDE for its Brownian motion, e.g., see [ELL99]. The flow will then consist almost surely of isometries and, equivalently, the measures μ_t will be supported on the subgroup of Diff(M) consisting of isometries.

A Hörmander form decomposition of a diffusion operator $\mathcal{A}$ on M also determines operators on differential forms and general tensor fields (in fact on sections of arbitrary natural vector bundles) using the standard interpretation of the Lie derivative of such sections. It turns out that this operator is the generator of the semi-group of operators on sections induced by the corresponding convolution semi-group $\{\mu_t\}_t$ of measures on Diff(M), see [ELJL] with special cases in [ELL99], [ER96], and [Elw92]. Consequently, for example on differential forms, a solution to the equation

$$\frac{\partial \phi_t}{\partial t} = 1/2 \sum_j \mathcal{L}_{X^j} \mathcal{L}_{X^j} \phi_t + \mathcal{L}_A \phi_t \tag{3}$$

is given by

$$\phi_t = \int_{C_{id}([0,T];\mathrm{Diff}(M))} \xi_t^*(\phi_0) d\mathbb{P}(\xi) \tag{4}$$

for a suitably smooth initial differential form ϕ_0.

In [ELL99] it is observed that for $A = 0$ these operators on forms also can be written as $1/2(\partial d + d\partial)$ where d is the usual exterior derivative and $\partial = \sum \mathcal{L}_{X^j} \iota_{X^j}$ for ι_{X^j} the interior product.

Note, for example by the path integral formula, equation (4), that under these non-standard heat flows of forms, if an initial form ϕ_0 is closed, then so is ϕ_t and the de Rham cohomology class is preserved, [ELL99]. Thus decay properties of the semi-groups on forms will be reflected in vanishing of the relevant de Rham cohomology. Such decay is implied by suitable decay of the norm of the derivative $T\xi_t$ of the flow. This relates to stability of the flow in the sense of having negative Lyapunov exponents, but it is the stronger moment exponents, e.g.,

$$\mu_M^q(p) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in M} \int_{C_{id}([0,T];\mathrm{Diff}(M))} \| \wedge^q T_x \xi_t \|^p d\mathbb{P}(\xi)$$

which are needed, and stability in this sense leads to vanishing of homotopy and/ or integral homology of our compact manifold M by considering the action of the flow on integral currents representing homology classes [ER96]. When applied to the gradient Brownian flow of a compact submanifold in Euclidean space these yield such topological vanishing results given positivity of an expression in the sectional and mean curvatures of the manifold, regaining results in [LS73]. However the approach via stochastic flows in [ER96] does not require the strict positivity needed in [LS73]. The property that Brownian motion instantly explores every part of the manifold allows the use of forms of "spectral positivity" or "stochastic positivity", see below and [ELR93], can be used. The following vanishing result for the fundamental group of certain *non-compact* submanifolds of $\mathbb{R}^n$ is due to Xue-Mei Li. Analogous results for higher homotopy, or integral homology, groups for non-compact manifolds seem to be lacking.

Theorem 2.3 ([Li95]) *Let M be a complete Riemannian manifold isometrically immersed in a Euclidean space with second fundamental form α satisfying $\|\alpha\|^2 \leqslant const.(1+\log[1+d(x)])$, $x \in M$ where d is the Riemannian distance from a fixed point of M. Denote its mean curvature by H and let $\underline{Ric}(x)$ be the smallest eigenvalue of the Ricci curvature at the point x. Suppose $\underline{Ric} - \|\alpha\|^2/2 + \frac{n}{2}|H|^2$ is positive, or more generally spectrally positive. Then $\pi_1(M) = 0$.*

3 Reproducing kernel Hilbert spaces, connections, and stochastic flows

3.1 Reproducing kernels and semi-connections on the diffeomorphism bundle

We have seen how a stochastic flow on M or equivalently a convolution semi-group of probability measures on Diff(M) determines a Gaussian measure γ with Hilbert space H_γ of smooth vector fields on M.

From now on we assume that the principal symbol of the generator $\mathcal{A}$ of the one-point motion on M,

$$\sigma_{\mathcal{A}} : T^*M \to TM$$

has constant rank and so has image in a sub-bundle E, say of TM. The symbol then induces an inner product on each fibre E_x, giving E a Riemannian metric.

Then our Hilbert space H_γ consists of sections of E and is ample for E in the sense that at any point x of M its evaluations span E_x. It determines, and is determined by, a smooth *reproducing kernel* $k_\gamma(x, y) : E_x^* \to E_y$ [Bax76] defined by

$$k_\gamma(x, -) = (ev_x)^* : E_x^* \to H_\gamma$$

where ev_x denotes evaluation at x. Using the metric to identify E_x^* with E_x we obtain $k_\gamma^\sharp(x, y) : E_x \to E_y$. The defining properties of such a reproducing kernel are that

(i) $k^\sharp(x, y) = k^\sharp(y, x)^*$;
(ii) $k^\sharp(x, x) = \text{identity} : E_x \to E_x$;
(iii) for any finite set $x_1, \dots, x_q$ of elements of M we have

$$\sum_{i,j=1}^{q} \langle k^\sharp(x_i, x_j)u_i, u_j\rangle \geqslant 0$$

for all $\{u_j\}_{j=1}^q$ with $u_j \in E_{x_j}$.

Let π : Diff$(M) \to M$ be the evaluation map at the point x_0 of M. We will consider it as a principal bundle with group the subgroup Diff$_{x_0}(M)$ consisting of those diffeomorphisms which fix x_0. We are being indecisive about the precise differentiability class of these diffeomorphisms and related vector fields, and the differential structure we are using on these infinite dimensional spaces: see [Mic91] for a direct approach for C^∞ diffeomorphisms, via the Frólicher–Kriegl calculus, otherwise we can work with Hilbert manifolds of mappings in sufficiently high Sobolev classes as [EM70], [Elw82]. The latter approach has the advantage that stochastic differential equations on Hilbert manifolds are well behaved, but it is necessary to be aware of the drops in regularity of compositions. More details will be found in [Elw]. In any case the tangent space $T_\theta \mathrm{Diff}(M)$ to $\mathrm{Diff}(M)$ at a diffeomorphism θ can be identified with the relevant space of maps $V : M \to TM$ lying over θ.

A reproducing kernel k on sections of E as above determines a horizontal lift map

$$h_\theta : E_{\pi_\theta} \to T_\theta \mathrm{Diff}(M)$$

for each $\theta \in \mathrm{Diff}(M)$. This is the linear map given by

$$h_\theta(u)(x) = k^\sharp(\theta(x_0), \theta(x))u$$

for $u \in E_{\theta(x_0)}$ and $x \in M$. Set H_θ equal to the image of h_θ, the *horizontal subspace at* θ. This is equivariant under the action of $\mathrm{Diff}_{x_0}(M)$ and would correspond to a connection if $E = TM$, i.e., when $\mathcal{A}$ is elliptic. In general we call it a *semi-connection over* E. It gives a horizontal lift $\tilde{\sigma} : [0, T] \to \mathrm{Diff}(M)$ for any piecewise C^1 curve $\sigma : [0, T] \to M$ with derivative $\dot{\sigma}(t) \in E_{\sigma}(t)$ for all $t \in [0, T]$. For example, [ELJL04], [ELJL], if $\sigma(0) = x_0$ the lift $\tilde{\sigma}$ starting at the identity diffeomorphism is just the solution flow of the dynamical system on M given by

$$\dot{z}(t) = k^\sharp(\sigma(t), z(t))\dot{\sigma}(t).$$

'Semi-connections' are also known as 'partial connections' or 'E-connections', [Ge92]. [Gro96].

Given a metric on E the map, Ξ say, from reproducing kernels satisfying (i), (ii), (iii) above, to semi-connections on $\mathrm{Diff}(M)$ is easily seen to be injective [ELJL]. In theory therefore all the properties of the flow, e.g., its stability properties, should be obtainable from this semi-connection.

Our bundle $\pi : \mathrm{Diff}(M) \to M$ can be considered as a universal natural bundle over M and a semi-connection on it induces one on each natural bundle over M. For example for the tangent bundle TM, given our curve σ above, the parallel translation $\widehat{//}_t : T_{x_0}M \to T_{\sigma(t)}M$ along σ is simply given by the derivative of the horizontal lift $\tilde{\sigma}$, i.e., $\widehat{//}_t = T_{x_0}\tilde{\sigma}(t)$. The corresponding covariant derivative operator (in this case differentiating a vector field in the direction of an element of E to obtain a tangent vector) will be denoted by $\widehat{\nabla}$, or $\widehat{\nabla}^\gamma$ if it comes initially from our Gaussian measure γ and we wish to emphasise that fact.

3.2 The adjoint connection

The kernel $k^\sharp$ also determines a connection on E which is given by its covariant derivative $\check{\nabla}$ defined by:

$$\check{\nabla}_v U = d\{k(-, x)(U(-))\}(v)$$

for $v \in T_xM$ and U a differentiable section of E. In other words it is the projection on E of the trivial connection on the trivial H- bundle over M by the evaluation map at x_0. This is a metric connection and all metric connections on E can be obtained by a suitable Hilbert space H of sections of E, in fact by a finite dimensional H, see [ELL99]. The latter fact is a consequence of Narasimhan and Ramanan's construction of universal connections, [NR61]; see [Qui88] for a direct proof. This connection was discussed in detail in [ELL99] together with its relationships to stochastic flows, and called the LW-connection for the flow. It had appeared in a very different form in the elliptic case in [LW84], see also [AMV96].

There is a correspondence, $\Gamma \leftrightarrow \Gamma'$ between connections on E and certain semi-connections on TM over E, [Dri92], [ELL99], given in terms of the co-variant derivatives by :

$$\nabla'_u V = \nabla_v U + [U, V](x).$$

We say ∇ and ∇' are *adjoints*. (When $E = TM$ this relationship is shown to be one of the complete list of natural automorphisms of the space of connections given in section 25 of [KMS93].) It is shown in [ELJL04], [ELJL], that $\widehat{\nabla}^\gamma$ and $\check{\nabla}^\gamma$ are adjoints.

3.3 The space of Hörmander form decompositions of $\mathcal{A}$

Now fix an infinite dimensional separable Hilbert space $\underline{\mathrm{H}}$. For our fixed diffusion operator $\mathcal{A}$ on M with constant rank symbol and associated Riemannian sub-bundle E of TM, let $SDE(E)$ denote the space of all smooth vector bundle maps

$$X : M \times \underline{H} \to E$$

which are surjective and induce the given metric on E. Let q be the dimension of the fibres of E and let $\mathbf{G}$ be its gauge group i.e., the space of all metric preserving vector bundle automorphisms of E over the identity of M. For a fixed orthonormal basis $\underline{e_1}, \underline{e_2}, \ldots$ of $\underline{H}$ there is a bijection between $SDE(E)$ and the set of Hörmander form representations as in equation (2) obtained by taking $X^i = X(-)(\underline{e_i})$ and then choosing the vector field A so that the equation (2) is satisfied. There is an obvious right action of $\mathbf{G}$ on $SDE(E)$ leading to a principal $\mathbf{G}$-bundle

$$\pi_1 : SDE(E) \to SDE(E)/\mathbf{G}.$$

Local sections can be obtained on noting the injection

$$\kappa_0 : SDE(E)/\mathbf{G} \to \mathrm{Map}[M; G(q, \underline{H})]$$

into the space of maps of M into the Grassmannian of q-dimensional subspaces in $\underline{H}$, where κ_0 sends X to the map $x \mapsto [\ker X(x)]^\perp$.

Let $V(q, \underline{H})$ denote the space of orthonormal q-frames in $\underline{H}$ and $p : V(q, \underline{H}) \to G(q, \underline{H})$ the projection, a universal $O(q)$-bundle. Let $\mathrm{Map}_E[M; G(q, \underline{H})]$ be the subspace of $\mathrm{Map}[M; G(q, \underline{H})]$ consisting of maps which classify E, i.e., those maps f for which $f^*(p)$ is equivalent to $O(E)$, the orthonormal frame bundle of E. This is the image of κ_0. Indeed if $\mathrm{Map}_{O(q)}[O(E); V(q, \underline{H})]$ denotes the space of $O(q)$-equivariant maps $\widetilde{f} : O(E) \to V(q, \underline{H})$, the left action of $\mathbf{G}$ on $O(E)$ induces a right action on $\mathrm{Map}_{O(q)}[O(E); V(q, \underline{H})]$ leading to a principle $\mathbf{G}$-bundle

$$\pi_2 : \mathrm{Map}_{O(q)}[O(E); V(q, \underline{H})] \to \mathrm{Map}_E[M; G(q, \underline{H})]$$

and this is equivalent to the bundle π_1 by the map

$$\Phi : SDE(E) \to \mathrm{Map}_{O(q)}[O(E); V(q, \underline{H})]$$

given by $\Phi(X)(u) = (Y_x u(e_1), \ldots, Y_x u(e_q))$ for u an orthonormal frame at a point x of M, with $e_1, \ldots, e_q$ the standard base of $\mathbb{R}^q$ and $Y_x : E_x \to \underline{H}$ the adjoint of $X(x)$, c.f. Chapter 6 of [ELL99]. According to [AB83] the bundle π_2 is a universal bundle for G, and so therefore must be π_1.

From our earlier discussions we have some related spaces and maps. One is the space $SC_E(\mathrm{Diff}\, M)$ the space of semi-connections over E on our bundle $p : \mathrm{Diff}(M) \to M$, considered as the space of $\mathrm{Diff}_{x_0}(M)$-equivariant horizontal lift maps $h : p^*(E) \to T\mathrm{Diff}(M)$. Another is the space of reproducing kernels of Hilbert spaces of sections of E satisfying (i), (ii) and (iii) of Section 3.1. This can also be considered as the space of stochastic flows which have $\mathcal{A}$ as one-point generator and will be denoted by $\mathrm{Flow}_{\mathcal{A}}$. This has a right-action of $\mathbf{G}$ given in terms of reproducing kernels by $(k^\sharp.g)(x, y) = g(y)^{-1} k^\sharp(x, y) g(x)$. There is also the usual space $\mathbf{C}_E$ of metric connections on E with its right $\mathbf{G}$-action.

We can summarise the situation by the following diagram, [Elw]:

$$\begin{array}{ccccc}
\mathrm{Map}_{O(q)}[O(E); V(q, \underline{H})] & \xrightarrow{\;\Phi^{-1}\;} & SDE(E) & \xrightarrow{\;\mathcal{H}\;} & \mathrm{Flow}_{\mathcal{A}} \\
\Big\downarrow{\scriptstyle NR} & & & & \Big\downarrow{\scriptstyle \Xi} \\
\mathbf{C}_E & \xrightarrow{\;\text{adjoint}\;} & SC_E(TM) & \longleftarrow & SC_E(\mathrm{Diff}\, M).
\end{array}$$

Here the vertical maps are $\mathbf{G}$-equivariant; the map NR refers to the pull-back of Narasimhan and Ramanan's universal connection and so is surjective and $\mathbf{G}$-equivariant. The diagram shows how the use of Narasimhan and Ramanan's construction to give a connection on a metric sub-bundle of a tangent bundle TM gives a semi-connection on the diffeomorphism bundle and so on all natural bundles over M.

4 Heat semi-groups on differential forms

4.1 Spectral and stochastic positivity

As mentioned above there are various weakenings of the the notion of positivity as applied, in particular, to the sort of curvature terms which arise in Bochner type vanishing results. For a measurable function $\rho : M \to \mathbb{R}$ and a diffusion operator $\mathcal{A}$, as before, we say:

1) The function ρ is $\mathcal{A}$-*stochastically positive* if

$$\limsup_{t\to\infty}\frac{1}{t}\log\int_{C_{x_0}([0,\infty);M)} e^{(-\frac{1}{2}\int_0^t \rho(\sigma(s))ds)}d\mu^{\mathcal{A}}_{x_0}(\sigma)=0$$

for each $x_0 \in M$, where the measure $\mu^{\mathcal{A}}_{x_0}$ is the diffusion measure corresponding to $\mathcal{A}$ and could be obtained as a push-forward measure by the evaluation map at x_0 of our measure $\mathbb{P}$ on paths (now defined for all time) on $\mathrm{Diff}(M)$, or by solving an SDE such as equation (1). We are not assuming M compact in this section but for simplicity we will assume that the solutions to equation (1) exist for all time or equivalently that the semi-group on functions on M, with generator $\mathcal{A}$, satisfies $P_t(1) = 1$ for all $t \geqslant 0$.

2) For M Riemannian, the function ρ is *spectrally positive* if there exists $c > 0$ such that $\int_M (\triangle(f)(x) + \rho(x)f(x))f(x)dx \geqslant c\|f\|^2_{L_2}$ for all smooth compactly supported functions f on M.

When M is Riemannian and $\mathcal{A} = -\frac{1}{2}\triangle$ we just refer to 'stochastic positivity' and if also M is compact this is equivalent to spectral positivity. Also $\mathcal{A}$-stochastic positivity of ρ implies the corresponding property for its lifts to any covering of M, see [ER91], [ELR93], [Li95], [ELR98] and also [Li02] for a similar condition. The fact that stochastic positivity lifts to covers made it an especially effective condition to apply and led to results beyond the scope of the usual Bochner methods, as pointed out in Ruberman's Appendix in [ER91].

4.2 Refined path integrals for the semi-groups on forms and generalised Weitzenbock curvatures

As remarked, for the compact case, near the end of Section 2.2, a convolution semi-group on Diff M determines semi-groups$\{P^k_t : t \geqslant 0\}$ on spaces of differential forms via the natural action of diffeomorphisms on forms, and these semi-groups map closed forms to closed forms in the same de Rham cohomology class. There are analogous results in many non-compact situations, especially covering spaces, but some care is required: see [Li94], [ELL99] and we will only discuss the compact case here.

The path integral giving $P^k_t\phi$ is given by equation (4). However it can be simplified by integrating first over the fibres of the map

$$C_{id}([0,T];\mathrm{Diff}(M)) \to C_{x_0}([0,T];M),$$

in probabilistic terms "conditioning on the one-point motion", or "filtering out the redundant noise", [EY93], [ELL99]. We are then left with the more intrinsic path integral representation

$$P^k_t\phi(V_0) = \int_{C_{x_0}([0,T];M)} \phi_0(V_t(\sigma))d\mu^{\mathcal{A}}_{x_0}(\sigma)$$

where, for almost all paths σ we have defined the vector field $\{V_t(\sigma) : 0 \leqslant t \leqslant T\}$ along σ, by the covariant differential equation:

$$\frac{\widehat{D}}{dt}V_t = -\frac{1}{2}\breve{\mathcal{R}}^k(V_t) + d\wedge^k(\breve{\nabla}_{(-)}A)V_t \tag{5}$$

where the hat and breve refer respectively to the semi-connection over E determined by our flow, and its adjoint connection on E, with the linear operator $\breve{\mathcal{R}}^k : \wedge^k TM \to \wedge^k TM$ a *generalised Weitzenbock curvature* obtained from the curvature of the connection on E by the same use of annihilation and creation operators as in the classical Levi-Civita case. For example for $k = 1$ it is just the Ricci-curvature

$$\breve{Ricci}^{\sharp} : TM \to E$$

given by

$$\breve{Ricci}^{\sharp}(v) = \text{trace}_E \breve{R}(v, -)-$$

for $v \in E$.

If the drift A does not take values in E this differential equation needs special interpretation, [ELL99].

4.3 Generalised Bochner type theorems

For the case of Riemannian manifolds with $\mathcal{A} = -\frac{1}{2}\triangle$, so $E = TM$, and if our flow is chosen to give the Levi-Civita connection, for example by using a gradient Brownian SDE, then the semi-groups on forms are seen to be the standard heat semi-groups, cf. [Kus88], [Elw92] and Bochner type vanishing theorems result from the refined path integral formula as discussed in [ER91] for example, but going back to the work of Malliavin and his co-workers, as examples: [Mal74], [Mér79]. The extension of these to more general connections and operators is not at present among the usual preoccupations of geometers and we will state only one simple result. However clearly many of the usual theorems will have more versions in this sort of generality:

Theorem 4.1 *Suppose M is compact and $k \in 1, 2, \dots, \dim(M) - 1$. If TM admits a metric with a metric connection $\breve{\nabla}$ such that its adjoint connection $\widehat{\nabla}$ is adapted to some metric, $\langle\,,\rangle'$ say, on TM, and its generalised Weitzenbock curvature, $\breve{R}^k$, in particular its Ricci curvature if $k = 1$, is such that $\inf\{\langle \breve{R}^k(V), V\rangle' : V \in \wedge^k TM, |V|' = 1\}$ is positive, then the cohomology group $H^k(M; \mathbb{R})$ vanishes.*

Positivity can be replaced by $\mathcal{A}$-stochastic positivity where $\mathcal{A}f = \frac{1}{2}\text{trace}_E \breve{\nabla}_-(df)$. For a proof, and a version with M non-compact, see the proof of Proposition 3.3.13 in [ELL99].

5 Analysis on path spaces

5.1 Bismut tangent spaces and associated Sobolev calculus

Consider the path space $C_{x_0} = C_{x_0}([0, T]; M)$ furnished with a diffusion measure $\mu_{x_0}^{\mathcal{A}}$, for example with *Brownian motion measure* μ_{x_0}, taking $\mathcal{A} = -\frac{1}{2}\triangle$. As with Gaussian measures on Banach spaces, to do analysis in this situation it seems that differentiation should be restricted to a special set of directions, giving an analogue of H-differentiation. To do this for Brownian motion measure, the standard procedure, going back at least to [JL91], is to use the Levy-Civita connection (of the Riemannian structure determined by the Laplace–Beltrami operator $\triangle$) and define Hilbert spaces $\mathcal{H}_\sigma$ for almost all paths σ, by

$$\mathcal{H}_\sigma = \{v \in TC_{x_0} : [t \mapsto (//_t)^{-1}v(t)] \in L_0^{2,1}([0,T]; T_{x_0}M\} \tag{6}$$

where $//_t$ refers to parallel translation along σ. Since almost all σ are non-differentiable the parallel translation is made using stochastic differential equations, hence the fact that it is defined only for almost all σ. Following the integration by parts formula in [Dri92] the Sobolev calculus was defined, as described in Section 2.1, giving closed operators $\bar{d}_H$ and $\bar{\nabla}_H$ from their domains $\mathbb{D}^{p,1}$ in $L^p(M,\mathbb{R})$ to L^p sections of the "Bismut tangent bundle" $\mathcal{H} = \bigcup_\sigma \mathcal{H}_\sigma$ and its dual bundle $\mathcal{H}^*$ respectively. Again we have a "Laplacian", $(\bar{d}_H)^*\bar{d}_H$, acting on functions. However in general little is known about it apart from the important result of S. Fang that it has a spectral gap and the refinements of that to logarithmic Sobolev and related inequalities; see [ELL99] for versions valid for more general diffusion measures.

In [Dri92], Driver showed Bismut tangent spaces using more general, but 'torsion skew-symmetric', metric connections could be used. This was extended by Elworthy, LeJan, and Li to cover a wide class of diffusion measures, with operator $\mathcal{A}$ possibly degenerate but having symbol of constant rank, and so having an associated sub-bundle E of TM with induced metric as in Section 3.1. In this situation a metric connection $\breve{\nabla}$ on E is chosen. The space $\mathcal{H}_\sigma$ of admissible directions at the path σ is defined by a modification of equation (6). Essentially it consists of those tangent vectors v at σ to TC_{x_0} for which the covariant derivative along σ, $\frac{\widehat{D}v}{dt}$, using the adjoint semi-connection, exists for almost all $t \in [0,T]$, takes values in E, and has $\int_0^T |\frac{\widehat{D}v}{dt}|^2 dt < \infty$, see [ELL99].

By the Narasimhan and Ramanan construction we can find a finite dimensional SDE

$$dx_t = X(x_t) \circ dB_t + A(x_t)dt \tag{7}$$

where $X : M \times \mathbb{R}^m \to TM$ is a smooth vector bundle map, with image E, which induces the connection $\breve{\nabla}$ (and so has flow inducing $\widehat{\nabla}$). Here A is a smooth vector field chosen so that the equation corre sponds to a Hörmander form decomposition of $\mathcal{A}$ or equivalently so that the solutions of the equation form an $\mathcal{A}$-diffusion. The Brownian motion $\{B_t : 0 \leqslant t \leqslant T\}$ will be taken to be the canonical process $B_t(\omega) = \omega(t)$ defined on classical Wiener space $C_0([0,T];\mathbb{R}^m)$. The solution map

$$\mathcal{I} : C_0([0,T];\mathbb{R}^m) \to C_{x_0} = C_{x_0}([0,T];M)$$

given by $\mathcal{I}(\omega)(t) = x_t(\omega)$ for $\{x_t : 0 \leqslant t \leqslant T\}$ the solution to equation (7) starting at the point x_0, is called the *Ito map* of the SDE. It sends the Wiener measure on $C_0([0,T];\mathbb{R}^m)$ to the measure $\mu_{x_0}^{\mathcal{A}}$. Moreover it has an H-derivative

$T_\omega\mathcal{I} : L_0^{2,1}([0,T] : \mathbb{R}^m) \to T_{x.(\omega)}C_{x_0}([0,T]; M)$ which is continuous, linear and defined for almost all ω. It is given by Bismut's formula:

$$T\mathcal{I}(h)(t) = T\xi_t \int_0^t T\xi_s^{-1} X(x_s)(\dot{h}(s))ds \tag{8}$$

for $h \in L_0^{2,1}([0,T]; \mathbb{R}^m)$ and where $\{\xi_t : 0 \leqslant t \leqslant T\}$ is the solution flow of our SDE. (In fact the integration by parts formula, and log-Sobolev formula in this context can be derived from "mother" formulae for paths on the diffeomorphism group, [ELL99].)

It can be shown that with such a careful choice of SDE, composition with the Ito map pulls back functions in the L^p domains of the H-derivative operator on the path space of M to elements in $\mathbb{D}^{p,1}$, at least when the semi-connection $\widehat{\nabla}$ is compatible with some metric on TM. For this see [EL05] where some fundamental, but still open, problems in this direction are discussed. A key point is that although $T_\omega\mathcal{I}$ does not map $h \in L_0^{2,1}([0,T]; \mathbb{R}^m)$ to $\mathcal{H}_{x.}$ we can 'filter out the redundant noise' by conditioning as at the beginning of Section 4.2. The result is a map $\overline{T\mathcal{I}}_\sigma : L_0^{2,1}([0,T]; \mathbb{R}^m) \to \mathcal{H}_\sigma$ given by:

$$\overline{T\mathcal{I}}_\sigma(h)_t = W_t^1 \int_0^t (W_s^1)^{-1} X(\sigma(s))(\dot{h}_s)ds \tag{9}$$

where for $r = 1, 2, \ldots, n$, the map $W_t^r : \wedge^r T_{x_0}M \to \wedge^r T_{\sigma(t)}M$ is the evolution determined by equation (5). This map, $\overline{T\mathcal{I}}_\sigma$, maps onto the Bismut tangent space and it is convenient to use the inner product it induces on those spaces, [ELL99]. It is also convenient to use the connection it induces on the 'Bismut tangent bundle' by projection, to define higher order derivatives, [EL05]. This connection is conjugate, by the operator $\frac{\widehat{D}}{dt}V_t + \frac{1}{2}\breve{\mathcal{R}}^1(V_t) - \breve{\nabla}_{(-)}A)V_t$, to the pointwise metric connection, [Eli67], induced on the bundle of L^2-paths in E which lie over C_{x_0}. It appeared in the work of Cruzeiro and Fang, e.g., in [CF95] (in the Brownian motion measure situation) called the *damped Markovian* connection following an 'undamped version' described earlier by Cruzeiro and Malliavin.

5.2 L^2-de Rham and Hodge–Kodaira theory

From the discussion above and the results of Shigekawa in the flat case, [Shi86], it would be natural to look for a differential form theory of "H-forms" these being sections of the dual 'bundle' to the completed exterior powers of the Bismut tangent bundle. However in the presence of curvature this fails at the definition of the exterior derivative of an H-one-form ϕ. The standard definition would give

$$d\phi(V^1 \wedge V^2) = d(\phi(V^2))(V^1) - d(\phi(V^1))(V^1)) - \phi([V^1, V^2]) \tag{10}$$

for H-vector fields (i.e., sections of the Bismut tangent bundle), V^1, V^2. However in general the Lie bracket of H-vector fields is not an H-vector field so the final term in (10) does not make sense (at least not classically). One approach, by Léandre, was to interpret this last term as a stochastic integral. This leads to rather complicated analysis but he was able to develop a de Rham theory, [Léa96]. Much earlier there had been an approach by Jones and Léandre using stochastic Chen forms, [JL91]. However Hodge–Kodaira theory, and a more standard form of L^2-cohomology did not appear in this work.

A different approach, by Elworthy and Li, was to modify the space of H-forms. This was done by using the conditional expectation

$$\overline{\wedge^r T\mathcal{I}}\sigma : \wedge^r L_0^{2,1}([0,T]:\mathbb{R}^m) \to \wedge^r T_\sigma C_{x_0}$$

of $\wedge^r T\mathcal{I}$, 'filtering out the redundant noise' or 'integrating over the fibres of $\mathcal{I}$,' [EL00]. Weitzenbock curvature terms come in rather as the first term, the Ricci curvature, did in equation (9), through (5). This led to a closed exterior derivative on H-one-forms and H-two-forms, and an L^2 Hodge–Kodaira decomposition in these cases [EL00], [ELa]. The situation for higher forms is unclear, and the algebra involved appears complicated, but there is some positive evidence in [EL03]. Even in dimensions 1 and 2 it is not known if the corresponding L^2 cohomology is trivial. The question of whether any reasonable L^2-cohomology for such a contractible space should be expected to be trivial, or if defined on loop spaces whether it should agree with the standard de Rham cohomology, stimulated work on L^2 de Rham cohomology for finite dimensional Riemannian manifolds with measures which have a smooth density decaying at infinity, (or growing rapidly), see [Bue99], [BP02], [GW04].

5.3 Geometric analysis on loops

The case of based loops is rather easier to deal with than free loops. There is a natural measure on the space of based on a Riemannian manifold M, the so called Brownian bridge measure. This corresponds to Brownian motion conditioned to return at time T, say, to its starting point. The conditioning is achieved by adding a time dependent vector field which is singular at time T, to the SDE, or equivalently to the generator $\mathcal{A}$, [Dri97]. This is obtained from the gradient of the heat kernel of M, and estimates on that play a vital role in the consequent analysis. For free loops an averaged version of this is used [Lea97]. There is also a *heat kernel* measure which is used, especially for loops on Lie groups, [Dri97], [AD00].

A beautiful and important result by Eberle, [Ebe02], showed that the spectrum of the natural Laplacian on these spaces does not have a gap at 0 if there is a closed geodesic on the underlying compact manifold M with a suitable neighbourhood of constant negative curvature.

For based loops and free loops on a compact Lie group with bi-invariant metric the (right invariant say) flat connection can be used to define Bismut tangent spaces and the absence of curvature allows the construction of a full L^2 de Rham and Hodge–Kodaira theory [FF97]. The work of Léandre and of Jones and Léandre referred to above included loop spaces, giving the topological real cohomology groups. More recently Léandre has been advocating the use of diffeologies, with a stochastic version for loop spaces, again leading to the usual cohomology groups, [Léa01].

Another approach to analysing based loop spaces has been to consider them as submanifolds of the space of paths on the the tangent space to M at the base point by means of the stochastic development map

$$\mathcal{D} : C_0([0, T]; T_{x_0}M) \to C_{x_0}([0, T]; M).$$

Since this map is obtained by stochastic differential equations, [Elw82], [IW89], [Elw00], it is defined only up to sets of measure zero and is not continuous, although it is smooth in the sense of Malliavin calculus. Quasi-sure analysis, [Mal97] has to be invoked to choose a nice version for which the inverse image of the based loops has at least the rudiments of the structure of a submanifold of $C_0([0, T]; T_{x_0}M)$. Even so as a space it is only defined up to 'slim' sets and there has not been a proof that its homotopy type is well determined and equal to that of the loop space itself. For a de Rham theory in this context see [Kus91].

References

[AB83] M. F. Atiyah and R. Bott. The Yang–Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[AD00] S. Aida and B. K. Driver. Equivalence of heat kernel measure and pinned Wiener measure on loop groups. *C.R. Acad. Sci. Paris Sér. I Math.*, 331(9):709–712, 2000.

[ADK00] S. Albeverio, A. Daletskii, and Y. Kondratiev. De Rham complex over product manifolds: Dirichlet forms and stochastic dynamics. In *Mathematical physics and stochastic analysis (Lisbon, 1998)*, pages 37–53. World Sci. Publishing, River Edge, NJ, 2000.

[Aid00] S. Aida. Stochastic analysis on loop spaces [translation of Sūgaku **50** (1998), no. 3, 265–281; MR1652019 (99i:58155)]. *Sugaku Expositions*, 13(2):197–214, 2000.

[AMT04] H. Airault, P. Malliavin, and A. Thalmaier. Canonical Brownian motion on the space of univalent functions and resolution of Beltrami equations by a continuity method along stochastic flows. *J. Math. Pures Appl. (9)*, 83(8):955–1018, 2004.

[AMV96] L. Accardi, A. Mohari, and C. V. Volterra. On the structure of classical and quantum flows. *J. Funct. Anal.*, 135(2):421–455, 1996.

[Ats02] A. Atsuji. Brownian motion and harmonic maps: value distribution theory for regular maps. *Sūgaku*, 54(3):235–248, 2002.

[Bax76] P. Baxendale. Gaussian measures on function spaces. *Amer. J. Math.*, 98(4):891–952, 1976.

[Bax84] P. Baxendale. Brownian motions in the diffeomorphism groups I. *Compositio Math.*, 53:19–50, 1984.

[Bis84] J.-M. Bismut. The Atiyah–Singer theorems: a probabilistic approach: I and II. *J. Funct. Anal.*, 57:56–99&329–348, 1984.

[BP02] E. Bueler and I. Prokhorenkov. Hodge theory and cohomology with compact supports. *Soochow J. Math.*, 28(1):33–55, 2002.

[Bue99] E. L. Bueler. The heat kernel weighted Hodge Laplacian on noncompact manifolds. *Trans. Amer. Math. Soc.*, 351(2):683–713, 1999.

[CCE86] A. P. Carverhill, M. J. Chappell, and K. D. Elworthy. Characteristic exponents for stochastic flows. In *Stochastic processes—mathematics and physics (Bielefeld, 1984)*, volume 1158 of *Lecture Notes in Math.*, pages 52–80. Springer, Berlin, 1986.

[CF95] A. B. Cruzeiro and S. Fang. Une inégalité l^2 pour des intégrales stochastiques anticipatives sur une variété riemannienne. *C. R. Acad. Sci. Paris, Série I*, 321:1245–1250, 1995.

[Dal04] A. Daletskii. Poisson configuration spaces, von Neumann algebras, and harmonic forms. *J. Nonlinear Math. Phys.*, 11(suppl.):179–184, 2004.

[DFLC71] R. M. Dudley, Jacob Feldman, and L. Le Cam. On seminorms and probabilities, and abstract Wiener spaces. *Ann. of Math. (2)*, 93:390–408, 1971.

[DPD03] Giuseppe Da Prato and Arnaud Debussche. Ergodicity for the 3D stochastic Navier–Stokes equations. *J. Math. Pures Appl. (9)*, 82(8):877–947, 2003.

[Dri92] B. K. Driver. A Cameron–Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold. *J. Functional Analysis*, 100:272–377, 1992.

[Dri97] Bruce K. Driver. Integration by parts and quasi-invariance for heat kernel measures on loop groups. *J. Funct. Anal.*, 149(2):470–547, 1997.

[Ebe02] Andreas Eberle. Absence of spectral gaps on a class of loop spaces. *J. Math. Pures Appl. (9)*, 81(10):915–955, 2002.

[ELa] K. D. Elworthy and Xue-Mei Li. An L^2 theory for 2-forms on path spaces I & II. In preparation.

[ELb] K. D. Elworthy and Xue-Mei Li. Geometric stochastic analysis on path spaces. In *Proceedings of the International Congress of Mathematicians, Madrid, 2006*. Vol III, pages 575–594. European Mathematical Society, Zurich, 2006.

[EL00] K. D. Elworthy and Xue-Mei Li. Special Itô maps and an L^2 Hodge theory for one forms on path spaces. In *Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999)*, pages 145–162. Amer. Math. Soc., 2000.

[EL03] K. D. Elworthy and Xue-Mei Li. Some families of q-vector fields on path spaces. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 6(suppl.):1–27, 2003.

[EL05] K. D. Elworthy and Xue-Mei Li. Ito maps and analysis on path spaces. Warwick Preprint, also www.xuemei.org, 2005.

[Eli67] H. Eliasson. Geometry of manifolds of maps. *J. Diff. Geom.*, 1:169–194, 1967.

[ELJL] K. D. Elworthy, Yves Le Jan, and Xue-Mei Li. A geometric approach to filtering of diffusions. In preparation.

[ELJL04] K. D. Elworthy, Yves Le Jan, and Xue-Mei Li. Equivariant diffusions on principal bundles. In *Stochastic analysis and related topics in Kyoto*, volume 41 of *Adv. Stud. Pure Math.*, pages 31–47. Math. Soc. Japan, Tokyo, 2004.

[ELL99] K. D. Elworthy, Y. LeJan, and X.-M. Li. *On the geometry of diffusion operators and stochastic flows, Lecture Notes in Mathematics 1720*. Springer, 1999.

[ELR93] K. D. Elworthy, X.-M. Li, and Steven Rosenberg. Curvature and topology: spectral positivity. In *Methods and applications of global analysis*, Novoe Global. Anal., pages 45–60, 156. Voronezh. Univ. Press, Voronezh, 1993.

[ELR98] K. D. Elworthy, X.-M. Li, and S. Rosenberg. Bounded and L^2 harmonic forms on universal covers. *Geom. Funct. Anal.*, 8(2):283–303, 1998.

[Elw] K. D. Elworthy. The space of stochastic differential equations. In *Stochastic analysis and applications—A symposium in honour of Kiyosi Itô*. Proceedings of Abel Symposium, 2005. Springer-Verlag. To appear.

[Elw82] K. D. Elworthy. *Stochastic Differential Equations on Manifolds, London Mathematical Society Lecture Notes Series 70*. Cambridge University Press, 1982.

[Elw92] K. D. Elworthy. Stochastic flows on Riemannian manifolds. In Diff*usion processes and related problems in analysis, Vol. II (Charlotte, NC, 1990)*, volume 27 of *Progr. Probab.*, pages 37–72. Birkhäuser Boston, Boston, MA, 1992.

[Elw00] K. D. Elworthy. Geometric aspects of stochastic analysis. In *Development of mathematics 1950–2000*, pages 437–484. Birkhäuser, Basel, 2000.

[EM70] D. G. Ebin and J. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. Math.*, pages 102–163, 1970.

[ER91] K. D. Elworthy and Steven Rosenberg. Manifolds with wells of negative curvature. *Invent. Math.*, 103(3):471–495, 1991.

[ER96] K. D. Elworthy and S. Rosenberg. Homotopy and homology vanishing theorems and the stability of stochastic flows. *Geom. Funct. Anal.*, 6(1):51–78, 1996.

[EY93] K. D. Elworthy and M. Yor. Conditional expectations for derivatives of certain stochastic flows. In J. Azéma, P.A. Meyer, and M. Yor, editors, *Sem. de Prob. XXVII. Lecture Notes in Mathematics 1557*, pages 159–172. Springer-Verlag, 1993.

[FF97] S. Fang and J. Franchi. De Rham–Hodge–Kodaira operator on loop groups. *J. Funct. Anal.*, 148(2):391–407, 1997.

[Ge92] Z. Ge. Betti numbers, characteristic classes and sub-riemannian geometry. *Illinois J. of Mathematics*, 36:372–403, 1992.

[Gro67] Leonard Gross. Potential theory on Hilbert space. *J. Functional Analysis*, 1:123–181, 1967.

[Gro96] M. Gromov. Carnot–Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.

[GW04] F.-Z. Gong and F.-Y. Wang. On Gromov's theorem and L^2-Hodge decomposition. *Int. J. Math. Math. Sci.*, (1-4):25–44, 2004.

[Hun56] G. A. Hunt. Semigroups of measures on lie groups. *Trans. Amer. Math. Soc.*, 81:264–293, 1956.

[IW89] N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes, second edition*. North-Holland, 1989.

[JL91] J. D. S. Jones and R. Léandre. L^p-Chen forms on loop spaces. In *Stochastic analysis (Durham, 1990)*, volume 167 of *London Math. Soc. Lecture Note Ser.*, pages 103–162. Cambridge Univ. Press, Cambridge, 1991.

[KMS93] I. Kolar, P. W. Michor, and J. Slovak. *Natural operations in differential geometry*. Springer-Verlag, Berlin, 1993.

[Kus88] S. Kusuoka. Degree theorem in certain Wiener Riemannian manifolds. In *Stochastic analysis (Paris, 1987)*, volume 1322 of *Lecture Notes in Math.*, pages 93–108. Springer, Berlin, 1988.

[Kus91] S. Kusuoka. de Rham cohomology of Wiener–Riemannian manifolds. In *Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990)*, pages 1075–1082, Tokyo, 1991. Math. Soc. Japan.

[Léa96] R. Léandre. Cohomologie de Bismut–Nualart–Pardoux et cohomologie de Hochschild entière. In *Séminaire de Probabilités, XXX*, volume 1626 of *Lecture Notes in Math.*, pages 68–99. Springer, Berlin, 1996.

[Lea97] R. Léandre. Invariant Sobolev calculus on the free loop space. *Acta Appl. Math.*, 46(3):267–350, 1997.

[Léa01] R. Léandre. Stochastic cohomology of Chen–Souriau and line bundle over the Brownian bridge. *Probab. Theory Related Fields*, 120(2):168–182, 2001.

[Léa05] R. Léandre. Brownian pants and Deligne cohomology. *J. Math. Phys.*, 46(3):330–353, 20, 2005.

[Li94] X.-M. Li. Stochastic differential equations on noncompact manifolds: moment stability and its topological consequences. *Probab. Theory Related Fields*, 100(4):417–428, 1994.

[Li95] X.-M. Li. On extensions of Myers' theorem. *Bull. London Math. Soc.*, 27(4):392–396, 1995.

[Li02] P. Li. Differential geometry via harmonic functions. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, pages 293–302, Beijing, 2002. Higher Ed. Press.

[Lin86] W. Linde. *Probability in Banach spaces—stable and infinitely divisible distributions*. A Wiley-Interscience Publication. John Wiley & Sons Ltd., Chichester, 1986.

[LS73] H. Blaine Lawson, Jr. and James Simons. On stable currents and their application to global problems in real and complex geometry. *Ann. of Math. (2)*, 98:427–450, 1973.

[LW84] Y. LeJan and S. Watanabe. Stochastic flows of diffeomorphisms. In *Stochastic analysis (Katata/Kyoto, 1982), North-Holland Math. Library, 32,*, pages 307–332. North-Holland, Amsterdam, 1984.

[Mal74] P. Malliavin. Formules de la moyenne, calcul de perturbations et théorèmes d'annulation pour les formes harmoniques. *J. Functional Analysis*, 17:274–291, 1974.

[Mal97] P. Malliavin. *Stochastic analysis, Grundlehren der Mathematischen Wissenschaften, 313*. Springer-Verlag, 1997.

[Mér79] A. Méritet. Théorème d'annulation pour la cohomologie absolue d'une variété riemannienne à bord. *Bull. Sci. Math. (2)*, 103(4):379–400, 1979.

[Mic91] P. W. Michor. *Gauge theory for fiber bundles*, volume 19 of *Monographs and Textbooks in Physical Science. Lecture Notes*. Bibliopolis, Naples, 1991.

[NR61] M. S. Narasimhan and S. Ramanan. Existence of universal connections. *American J. Math.*, 83:563–572, 1961.

[Qui88] D. Quillen. Superconnections; character forms and the Cayley transform. *Topology*, 27(2):211–238, 1988.

[Shi86] I. Shigekawa. de Rham–Hodge–Kodaira's decomposition on an abstract Wiener space. *J. Math. Kyoto Univ.*, 26(2):191–202, 1986.

[Stu02] K.-T. Sturm. Nonlinear martingale theory for processes with values in metric spaces of nonpositive curvature. *Ann. Probab.*, 30(3):1195–1222, 2002.

[vRS05] M.-K. von Renesse and Karl-Theodor Sturm. Transport inequalities, gradient estimates, entropy, and Ricci curvature. *Comm. Pure Appl. Math.*, 58(7):923–940, 2005.

A Lie Group Structure for Automorphisms of a Contact Weyl Manifold

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Dedicated to Professor Hideki Omori

Summary. In the present article, we are concerned with the automorphisms of a contact Weyl manifold, and we introduce an infinite-dimensional Lie group structure for the automorphism group.

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Key words: Infinite-dimensional Lie group, contact Weyl manifold, star product, deformation quantization.

1 Introduction

The concept of Lie group has a long history. It originated from Sophus Lie who initiated the systematic investigation of group germs of continuous transformations. As can be seen in the introduction of the monograph by H. Omori [32], S. Lie seemed to be motivated by the following:

- To construct a theory for differential equations similar to Galois theory.
- To investigate groups such as continuous transformations that leave various geometrical structures invariant.

It is well known that the theory of Lie groups has expanded in two directions:

(A) The theory of finite dimensional Lie groups and Lie algebras.
(B) The theory expanded to include Banach–Lie groups and transformations that leave various geometrical structures invariant.

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There are a large number of works from the standpoint of (A). With respect to (B), there are also numerous works which are concerned with Banach–Lie groups and their geometrical and topological properties (cf. [42]). However, it was already known in [31] that a Banach–Lie group acting effectively on a finite dimensional smooth manifold is necessarily finite dimensional. So there is no way to model the diffeomorphism group on a Banach space as a manifold. Under the situation above, at the end of the 1960s, Omori initiated the theory of infinite-dimensional Lie groups, called "ILB-Lie groups", beyond Banach–Lie groups, taking ILB-chains as model spaces in order to treat the diffeomorphism group on a manifold (see [32] for the precise definition). Shortly after these foundations were laid, Omori et al. [39] introduced the definition of Lie group modeled on a Fréchet space equipped with a certain property called "regularity" by relaxing the conditions of an ILB-Lie group. Roughly speaking, regularity means that the smooth curves in the Lie algebra integrate to smooth curves in the Lie group in a smooth way (see also [26], [32] and [40]). Using this notion, they studied subgroups of a diffeomorphism group, and the group of invertible Fourier integral operators with suitable amplitude functions on a manifold. For technical reasons, they assumed that the base manifold is compact (cf. [25], [39], [1], [2] and [3]). Beyond a compact base manifold, in order to treat the diffeomorphism group on a noncompact manifold, we need a more general category of Lie groups, i.e., infinite-dimensional Lie groups modeled on locally convex spaces which are Mackey complete (see §2. See also [10] and [19]).

In this article, we are concerned with the group [1] $\mathrm{Aut}(M,*)$ of all modified contact Weyl diffeomorphisms on a contact Weyl manifold over a symplectic manifold (M,ω), where a contact Weyl manifold is a geometric realization of the star product introduced by A. Yoshioka in [50]. In this context, a modified contact Weyl diffeomorphism is regarded as an automorphism on a contact Weyl manifold. As to the group $\mathrm{Aut}(M,*)$, we have the following.

Theorem 1.1 *1) Set*

$$\underline{\mathrm{Aut}}(M,*) = \{\Phi \in \mathrm{Aut}(M,*) \mid \Phi \textit{ induces the base identity map.}\}.$$

Then $\underline{\mathrm{Aut}}(M,*)$ is a Lie group modeled on a Mackey complete locally convex space.

2) Any element $\Psi \in \mathrm{Aut}(M,*)$ induces a symplectic diffeomorphism on the base manifold and there exists a group homomorphism p from $\mathrm{Aut}(M,*)$ into $\mathrm{Diff}(M,\omega)$, where $\mathrm{Diff}(M,\omega)$ is the regular Lie group of all symplectic diffeomorphisms on the symplectic manifold (M,ω).

3) Assume that there exists a map (not necessarily a Lie group homomorphism) j from $\mathrm{Diff}(M,\omega)$ into $\mathrm{Aut}(M,*)$ satisfying $p \circ j = \mathrm{identity}$. Then $\mathrm{Aut}(M,*)$ is a Lie group modeled on a Mackey complete locally convex space [2].

[1] See Definition 5.1 for the precise definition.

[2] If the base manifold is compact, the model spaces of $\mathrm{Aut}(M,*)$ and $\underline{\mathrm{Aut}}(M,*)$ are Fréchet spaces.

4) *Under the same assumption above,*

$$1 \to \underline{\mathrm{Aut}}(M,*) \to \mathrm{Aut}(M,*) \to \mathrm{Diff}(M,\omega) \to 1$$

is a short exact sequence of Lie groups. Moreover $\underline{\mathrm{Aut}}(M,*)$ *and* $\mathrm{Aut}(M,*)$ *are regular Lie groups.*

Remark that from the point of view of differential geometry, a contact Weyl manifold might be seen as a "*prequantum bundle*" over a symplectic manifold (M,ω) where the symplectic structure ω is not necessarily *integral*, and a modified contact diffeomorphism can be regarded as a quantum symplectic diffeomorphism over a "prequantum bundle".

As is well known, the theory of infinite-dimensional Lie algebras including Kac–Moody algebras has made rapid and remarkable progress for the past two decades involving completely integrable systems (Sato's theory), loop groups, conformal field theory and quantum groups. However, it would be difficult for me to review this entire fruitful field. A definitive treatment of the infinite-dimensional Lie algebras is found in Kac [17], Tanisaki [45] and Wakimoto [46].

Since the purpose of this article is to give an exhibitory review of relations between contact Weyl manifolds and deformation quantization, and automorphisms on a contact Weyl manifold, please consult [29], [11], [37], [38] and [50] for the detailed proofs omitted in the present article.

2 Infinite-dimensional Lie groups

In this section we give a survey of regular Lie groups. For this purpose, we first recall Mackey completeness; see the excellent monographs [16], [19] for details.

Definition 2.1 A locally convex space E is called Mackey complete (MC for short) if one of the following equivalent conditions is satisfied:

1) For any smooth curve c in E there is a smooth curve C in E with $C' = c$.
2) If $c : \mathbf{R} \to E$ is a curve such that $l \circ c : \mathbf{R} \to \mathbf{R}$ is smooth for all $\ell \in E^*$, then c is smooth.
3) Local completeness: that is, for every absolutely convex closed bounded[3] subset B, E_B is complete, where E_B is a normed space linearly generated by B with a norm $p_B(v) = \inf\{\lambda > 0 | v \in \lambda B\}$.
4) Mackey completeness: that is, any Mackey–Cauchy net converges in E.
5) Sequential Mackey completeness: that is, any Mackey–Cauchy sequence converges in E.

A net $\{x_\gamma\}_{\gamma\in\Gamma}$ is called Mackey–Cauchy if there exists a bounded set B and a net $\{\mu_{\gamma,\gamma'}\}_{(\gamma,\gamma')\in\Gamma\times\Gamma}$ in $\mathbf{R}$ converging to 0, such that $x_\gamma - x_{\gamma'} \in \mu_{\gamma,\gamma'}B = \{\mu_{\gamma,\gamma'} \cdot x \mid x \in B\}$.

[3] A subset B is called bounded if it is absorbed by every 0-neighborhood in E, i.e., for every 0-neighborhood $\mathcal{U}$, there exists a positive number p such that $[0, p] \cdot B \subset \mathcal{U}$.

Next we recall the fundamentals relating to infinite-dimensional differential geometry.

1) Infinite-dimensional manifolds are defined on Mackey complete locally convex spaces in much the same way as ordinary manifolds are defined on finite-dimensional spaces. In this article, a manifold equipped with a smooth group operation is referred to as a Lie group. Remark that in the category of infinite-dimensional Lie groups, the existence of exponential maps is not ensured in general, and even if an exponential map exists, the local surjectivity of it does not hold (cf. Definition 2.2).
2) A *kinematic tangent vector* (a tangent vector for short) with a foot point x of an infinite-dimensional manifold X modeled on a Mackey complete locally convex space F is a pair (x, X) with $X \in F$. Let $T_x F = F$ be the space of all tangent vectors with foot point x. It consists of all derivatives $c'(0)$ at 0 of a smooth curve $c : \mathbf{R} \to F$ with $c(0) = x$. Remark that operational tangent vectors viewed as derivations and kinematic tangent vectors via curves differ in general. A kinematic vector field is a smooth section of a kinematic vector bundle $TM \to M$.
3) We set $\Omega^k(M) = C^\infty(L_{\mathrm{skew}}(TM \times \cdots \times TM, M \times \mathbf{R}))$ and call it the space of *kinematic differential forms*, where "skew" denotes "skew-symmetric". Remark that the space of kinematic differential forms turns out to be the right ones for calculus on manifolds; especially for them the theorem of de Rham is proved.

Next we recall the precise definition of regularity (cf. [26], [32], [39] and [40]):

Definition 2.2 A Lie group G modeled on a Mackey complete locally convex space $\mathfrak{G}$ is called a regular Lie group if one of the following equivalent conditions is satisfied:

1) For each $X \in C^\infty(\mathbf{R}, \mathfrak{G})$, there exists $g \in C^\infty(\mathbf{R}, G)$ satisfying

$$g(0) = e, \quad \frac{\partial}{\partial t} g(t) = R_{g(t)}(X(t)), \tag{1}$$

2) For each $X \in C^\infty(\mathbf{R}, \mathfrak{G})$, there exists $g \in C^\infty(\mathbf{R}, G)$ satisfying

$$g(0) = e, \quad \frac{\partial}{\partial t} g(t) = L_{g(t)}(X(t)), \tag{2}$$

where $R(X)$ (resp. $L(X)$) is the right (resp. left) invariant vector field defined by the right (resp. left)-translation of a tangent vector X at e.

The following lemma is useful (cf. [19], [26], [39] and [40]):

Lemma 2.3 *Assume that*

$$1 \to N \to G \to H \to 1 \tag{3}$$

is a short exact sequence of Lie groups with a local smooth section[4] *j from a neighborhood $U \subset H$ of 1_H into G, and N and H are regular. Then G is also regular.*

[4] Remark that this does not give global splitting of the short exact sequence.

To end this section, we remark that the fundamental properties of a principal regular Lie group bundle (P, G) over M are (these are the usual properties for principal finite-dimensional Lie group bundles):

1) The parallel transformation is well defined.
2) The horizontal distribution $\mathcal{H}$ of a flat connection is integrable, i.e., there exists an integral submanifold for $\mathcal{H}$ at each point.

3 Deformation quantization

The concept of quantization has a long history. Mathematically it originated from H. Weyl [47], who introduced a map from classical observables (functions on the phase space) to quantum observables (operators on Hilbert space). The inverse map was constructed by E. Wigner by interpreting functions (classical observables) as symbols of operators. It is known that the exponent of the bidifferential operator (Poisson bivector) coincides with the product formula of (Weyl type) symbol calculus developed by L. Hörmander who established the theory of pseudo-differential operators and used them to study partial differential equations (cf. [20] and [30]).

In the 1970s, supported by the mathematical developments above, Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer [6] considered quantization as a deformation of the usual commutative product of classical observables into a noncommutative associative product which is parametrized by the Planck constant $\hbar$ and satisfies the correspondence principle. Nowadays deformation quantization, or more precisely, the star product becomes an important notion. In fact, it plays an important role to give passes from Poisson algebras of classical observables to noncommutative associative algebras of quantum observables. In the approach above, the space of quantum observables and star product is defined in the following way(cf. [6]):

Definition 3.1 A star product of Poisson manifold (M, π) is a product $*$ on the space $C^\infty(M)[[\hbar]]$ of formal power series of parameter $\hbar$ with coefficients in $C^\infty(M)$, defined by

$$f *_\hbar g = fg + \hbar\pi_1(f, g) + \cdots + \hbar^n\pi_n(f, g) + \cdots, \quad \forall f, g \in C^\infty(M)[[\hbar]]$$

satisfying
(a) $*$ is associative,
(b) $\pi_1(f, g) = \frac{1}{2\sqrt{-1}}\{f, g\}$,
(c) each π_n $(n \geq 1)$ is a $\mathbf{C}[[\hbar]]$-bilinear and bidifferential operator,
where $\{,\}$ is the Poisson bracket defined by the Poisson structure π.

A deformed algebra (resp. a deformed algebra structure) is called a *star algebra* (resp. a *star product*). Note that on a symplectic vector space $\mathbf{R}^{2n}$, there exists the "canonical" deformation quantization, the so-called Moyal product:

$$f * g = f \exp\left[\frac{\nu}{2}\overleftarrow{\partial_x} \wedge \overrightarrow{\partial_y}\right] g,$$

where $f,\ g$ are smooth functions of a Darboux coordinate (x, y) on $\mathbf{R}^{2n}$ and $\nu = i\hbar$.

The existence and classification problems of star products have been solved by succesive steps from special classes of symplectic manifolds to general Poisson manifolds. Because of its physical origin and motivation, the problems of deformation quantization was first considered for symplectic manifolds, however, the problem of deformation quantization is naturally formulated for the Poisson manifolds as well. For example, Etingof and Kazhdan proved every Poisson–Lie group can be quantized in the sense above, and investigated quantum groups as deformation quantization of Poisson–Lie groups. After their works, for a while, there were no specific developments for existence problems of deformation quantization on any Poisson manifold. The situation drastically changed when M. Kontsevich [10] proved his celebrated formality theorem. As a collorary, he showed that deformation quantization exists on any Poisson manifold. (cf. [8], [12], [10], [11], [37], [44] and [50]).

4 Contact Weyl manifold over a symplectic manifold

As mentioned in the introduction, for a symplectic manifold, the notion of a Weyl manifold was introduced in [37]. Later, Yoshioka [50] proposed the notion of a contact Weyl manifold as a bridge joining the theory of Weyl manifold (Omori–Maeda–Yoshioka quantization) and Fedosov quantization. In order to recall the construction of a contact Weyl manifold, we have to give precise definitions of fundamental algebras.

Definition 4.1 1) An associative algebra W is called a Weyl algebra if W is formally generated by $\nu, Z^1, \dots, Z^n, Z^{n+1}, \dots, Z^{2n}$ satisfying the following commutation relations:

$$[Z^i, Z^j] = \nu\Lambda^{ij}, \quad [\nu, Z^i] = 0, \tag{4}$$

where $\Lambda = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}$, and the product of this algebra is denoted by $*$. This algebra has the canonical involution $^-$ such that

$$\overline{a * b} = \bar{b} * \bar{a}, \quad \bar{\nu} = -\nu, \quad \bar{Z^i} = Z^i. \tag{5}$$

We also define the degree d by $d(\nu^l Z^\alpha) = 2l + |\alpha|$.

2) A Lie algebra C is called a contact Weyl algebra if $C = \tau\mathbf{C} \oplus W$ with an additional generator τ satisfying the following relations:

$$[\tau, \nu] = 2\nu^2, \quad [\tau, Z^i] = \nu Z^i, \tag{6}$$

and $^-$ is naturally extended by $\bar{\tau} = \tau$.

Remark that the relation (4) is nothing but the commutation relation of the Moyal product, and called the *canonical commutation relation*.

Definition 4.2 1) A $\mathbf{C}[[\nu]]$-linear isomorphism Φ from W onto W is called a ν-automorphism of Weyl algebra W if

(a) $\Phi(\nu) = \nu$,
(b) $\Phi(a * b) = \Phi(a) * \Phi(b)$,
(c) $\Phi(\bar{a}) = \overline{\Phi(a)}$.

2) A $\mathbf{C}[[\nu]]$-linear isomorphism Ψ from C onto C is called a ν-automorphism of contact Weyl algebra C if
(a) Ψ is an algebra isomorphism,
(b) $\Psi|_W$ is a ν-automorphism of Weyl algebra.

In order to explain the construction of contact Weyl manifolds, it is useful to recall how to construct the prequantum line bundle, which plays a crucial role in the theory of Souriau–Kostant (geometric) quantization[49]. This bundle is constructed in the following way: Let ω be an integral symplectic structure, then we have $d(\theta_\alpha) = (\delta\omega)_\alpha$, $d(f_{\alpha\beta}) = (\delta\theta)_{\alpha\beta}$, $c_{\alpha\beta\gamma} = (\delta f)_{\alpha\beta\gamma}$ where $f_{\alpha\beta}$ (resp. θ_α) is a local function (resp. a local 1-form) defined on an open set $U_\alpha \cap U_\beta$ (resp. U_α), $\mathcal{U} = \{U_\alpha\}$ is a good covering of a symplectic manifold (M, ω), d is the deRham exterior differential operator, and δ is the Čech coboundary operator. Setting $h_{\alpha\beta} = \exp[2\pi i f_{\alpha\beta}]$, we see that

$$\theta_\alpha - \theta_\beta = \frac{1}{2\pi i} d \log h_{\alpha\beta}. \tag{7}$$

This equation ensures the exsistence of a line bundle defined by

$$L = \coprod (U_\alpha \times \mathbf{C}) / \overset{h_{\alpha\beta}}{\sim}, \qquad \nabla_\xi (\phi_\alpha 1_\alpha) = (\xi\phi_\alpha + 2\pi i \theta_\alpha(\xi)\phi_\alpha) 1_\alpha. \tag{8}$$

This gives the desired bundle with a connection whose curvature equals ω.

Inspired by the idea above, Yoshioka proposed the notion of contact Weyl manifolds and obtained the fundamental results (cf. [50]). To state the precise definition of contact Weyl manifolds and theorems related to them, we first recall the definitions of *Weyl continuation* and *locally modified contact Weyl diffeomorphism*:

Definition 4.3 Set $(X^1, \ldots, X^n, Y^1, \ldots, Y^n) := (Z^1, \ldots, Z^n, Z^{n+1}, \ldots, Z^{2n})$ (see Definition 4.1). Consider the trivial contact Weyl algebra bundle $C_U := U \times C$ over a local Darboux chart $(U; (x, y))$. A section

$$f^\# := f(x + X, y + Y) = \sum_{\alpha\beta} \frac{1}{\alpha!\beta!} \partial_x^\alpha \partial_y^\beta f(x, y) X^\alpha Y^\beta \in \Gamma(C_U)$$

determined by a local smooth function $f \in C^\infty(U)$ is called a Weyl function, and $\# : f \mapsto f^\#$ is referred to as Weyl continuation. We denote by $\mathcal{F}_U$ the set of all Weyl functions on U.

A bundle map $\Phi : C_U \to C_U$ is referred to as a locally modified contact Weyl diffeomorphism if it is a fiberwise ν-automorphism of the contact Weyl algebra and its pull-back preserves the set of all Weyl functions $\mathcal{F}_U$.

Definition 4.4 Let $\pi : C_M \to M$ be a locally trivial bundle with a fiber being isomorphic to the contact Weyl algebra over a symplectic manifold M. Take an atlas

$\{(V_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ of M such that $\varphi_\alpha : V_\alpha \to U_\alpha \subset \mathbb{R}^{2n}$ gives a local Darboux coordinate for every $\alpha \in A$. Denote by $\Psi_\alpha : C_{V_\alpha} \to C_{U_\alpha}$ a local trivialization and by $\Psi_{\alpha\beta} = \Psi_\beta \Psi_\alpha^{-1} : C_{U_{\alpha\beta}} \to C_{U_{\beta\alpha}}$ the glueing map, where $C_{U_\alpha} := \pi^{-1}(U_\alpha)$, $U_{\alpha\beta} := \varphi_\alpha(V_\alpha \cap V_\beta)$, $U_{\beta\alpha} := \varphi_\beta(V_\alpha \cap V_\beta)$, $C_{U_{\alpha\beta}} := \Psi_\alpha(C_{V_\alpha}|_{V_\alpha \cap V_\beta})$, etc. Under the notation above,

$$\big(\pi : C_M \to M, \{\Phi_\alpha : C_{V_\alpha} \to C_{U_\alpha}\}_{\alpha\in A}\big) \tag{9}$$

is called a contact Weyl manifold if the glueing maps $\Psi_{\alpha\beta}$ are modified contact Weyl diffeomorphisms.

Theorem 4.5 *Let (M, ω) be an arbitrary (not necessarily integral) symplectic manifold. For any closed form $\Omega_M(\nu^2) = \omega + \omega_2\nu^2 + \omega_4\nu^4 + \cdots$, where $\nu = \sqrt{-1}\hbar$ is a formal parameter, there exists a contact Weyl manifold C_M with a connection ∇^Q whose curvature equals $\Omega_M(\nu^2)$, and the restriction of ∇^Q to W_M is flat, where W_M is the Weyl algebra bundle associated to M equipped with the canonical fiber-wise product $\hat{*}$.*

This bundle C_M is called a contact Weyl manifold equipped with a *quantum connection* ∇^Q. Yoshioka [50] also proved that the connection $\nabla|_{W_M}$ is essentially the same as the Fedosov connection [12]. It is known (cf. [50] and [11]) that

Theorem 4.6 *There is a bijection between the space of the isomorphism classes of a contact Weyl manifold and $[\omega] + \nu^2 H^2_{dR}(M)[[\nu^2]]$, which assigns a class $[\Omega_M(\nu^2)] = [\omega + \omega_2\nu^2 + \cdots]$ to a contact Weyl manifold $(C_M \to M, \{\Psi_\alpha\})$.*

The flatness of $\nabla^Q|_{W_M}$ ensures the existence of a linear isomorphism # between $C^\infty(M)[[\nu]]$ and $\mathcal{F}_M$ the space of all parallel sections with respect to the quantum connection restricted to W_M. An element of $\mathcal{F}_M$ is called a *Weyl function*. Using this map #, we can recapture a star product in the following way:

$$f * g = \#^{-1}\big(\#(f)\hat{*}\#(g)\big). \tag{10}$$

Furthermore, the following theorem is known (cf. [11]):

Theorem 4.7 *There is a bijection between the space of the equivalence classes of star products and $[\omega] + \nu^2 H^2_{dR}(M)[[\nu^2]]$.*

5 A Lie group structure of Aut($M, *$)

With the preliminaries in the previous section, we give a precise definition of $\mathrm{Aut}(M, *)$:

Definition 5.1

$$\mathrm{Aut}(M, *) = \{\Psi : C_M \to C_M \mid \text{fiber-wise } \nu\text{-automorphism}, \Psi^*(\mathcal{F}_M) = \mathcal{F}_M\} \tag{11}$$

$$\underline{\mathrm{Aut}}(M,*) = \{\Psi \in \mathrm{Aut}(M,*) \mid \Psi \text{ induces the base identity map}\}. \tag{12}$$

An element of $\mathrm{Aut}(M,*)$ is called a modified contact Weyl diffeomorphism (an MCWD for short).

To illustrate automorphisms of a contact Weyl manifold, we consider the automorphisms of a contact Weyl algebra. For any real symplectic matrix $A \in Sp(n,\mathbf{R})$, set a ν-automorphism of C by $\hat{A}Z^i = \sum a^i_j Z^j$ and $\hat{A}\nu = \nu$. Then we easily have $\hat{A}([a,b]) = [\hat{A}a, \hat{A}b]$. Conversely, combining the Baker–Campbell–Hausdorff formula with the Poincaré lemma, we have the following.

Proposition 5.2 ([50]) *If Ψ is a ν-automorphism of contact Weyl algebra, there exists uniquely*

$$A \in Sp(n,\mathbf{R}),$$
$$F \in \left\{ a = \sum_{2\ell+|\alpha|\geq 3, |\alpha|>0} a_{\ell\alpha}\nu^\ell Z^\alpha \right\},$$
$$c(\nu^2) = \sum_{i=0}^{\infty} c_{2i}\nu^{2i} \in \mathbf{R}[[\nu^2]],$$

such that $\Phi = \hat{A} \circ e^{ad(\frac{1}{\nu}(c(\nu^2)+F))}$, where $\hat{A}Z^i = \sum a^i_j Z^j$ and $\hat{A}\nu = \nu$.

This ν-automorphism can be seen as a "linear" example appearing in the simplest model of contact Weyl manifolds.

Next we study the basic properties of a modified contact Weyl diffeomorphism. Set $\tilde{\tau}_U = \tau + \sum z^i \omega_{ij} Z^j$ where $U \subset \mathbf{R}^{2n}$ is an open subset and $\omega_{ij} dz^i \wedge dz^j$ stands for the symplectic structure. Then for any modified contact Weyl diffeomorphism, we may set $\Psi|^*_{C_U}(\tilde{\tau}_U) = a\tilde{\tau}_U + F$, where $a \in C^\infty(U)$, $F \in \Gamma(W_U)$, where W_U is a trivial bundle $W_U = U \times W$. Furthermore the following proposition is known (cf. Lemma 2.21 in [50]).

Proposition 5.3 *Let U be an open set in $\mathbf{R}^{2n}$, Ψ a modified contact Weyl diffeomorphism and ϕ the induced map on the base manifold. Then the pull-back of $\tilde{\tau}_{\phi(U)}$ by Ψ can be written as*

$$\Psi^* \tilde{\tau}_{\phi(U)} = \tilde{\tau}_U + f^\# + a(\nu^2), \tag{13}$$

for some Weyl functions $f^\# := \#(f) \in \mathcal{F}_U$ with $\bar{f^\#} = f^\#$ and $a(\nu^2) \in C^\infty(U)[[\nu^2]]$.

Definition 5.4 A modified contact Weyl diffeomorphism Ψ is called a contact Weyl diffeomorphism (CWD, for short) if

$$\Psi^* \tilde{\tau}_{U'} = \tilde{\tau}_U + f^\#. \tag{14}$$

For a contact Weyl diffeomorphism, we obtain the following (see Corollary 2.5 in [11] and Proposition 2.24 in [50]).

Proposition 5.5 *1) Suppose that* $\Psi : C_U \to C_U$ *is a contact Weyl diffeomorphism which induces the identity map on the base space. Then, there exists uniquely a Weyl function* $f^{\#}(\nu^2)$ *of the form*

$$f^{\#} = f_0 + \nu^2 f_+^{\#}(\nu^2) \quad (f_0 \in \mathbf{R},\ f_+(\nu^2) \in C^\infty(U)[[\nu^2]]), \tag{15}$$

such that $\Psi = e^{ad\frac{1}{\nu}\{f_0+\nu^2 f_+^{\#}(\nu^2)\}}$.

2) If Ψ *induces the identity map on* W_U*, then there exists a unique element* $c(\nu^2) \in \mathbf{R}[[\nu^2]]$ *with* $\overline{c(\nu^2)} = c(\nu^2)$*, such that*[5] $\Psi = e^{ad\frac{1}{\nu}c(\nu^2)}$.

Combining Propositions 5.3 and 5.5, we have the following.

Proposition 5.6 *For any modified contact Weyl diffeomorphism* $\Psi : C_U \to C_U$ *which induces the identity map on the base space, there exists a Weyl function* $f^{\#}(\nu^2)$ *of the form*

$$f^{\#}(\nu^2) = f_0 + \nu^2 f_+^{\#}(\nu^2) \quad (f_0 \in \mathbf{R},\ f_+(\nu^2) \in C^\infty(U)[[\nu^2]]), \tag{16}$$

and smooth function $g(\nu^2) \in C^\infty(U)[[\nu^2]]$ *such that* $\Psi = e^{ad(\frac{1}{\nu}\{g(\nu^2)+f^{\#}(\nu^2)\})}$.

Remark Please compare this result with Proposition 5.2.

Furthermore, we have

Proposition 5.7 *Let* Ψ_{U_α} *(resp.* Ψ_{U_β}*) be a modified contact Weyl diffeomorphism on* C_{U_α} *(resp.* C_{U_β}*) inducing the identity map on the base manifold. Suppose that*

$$\Psi_{U_\alpha}|_{C_{U_{\alpha\beta}}} = \Psi_{U_\beta}|_{C_{U_{\beta\alpha}}},$$

where $U_{\alpha\beta} := \varphi_\alpha(V_\alpha \cap V_\beta),\ U_{\beta\alpha} := \varphi_\beta(V_\alpha \cap V_\beta),\ C_{U_{\alpha\beta}} := \Psi_\alpha(C_{V_\alpha}|_{V_\alpha \cap V_\beta})$ *etc.*[6] *Then*

$$\Psi_{\alpha\beta}^{-1,*}\big((g_\alpha(\nu^2) + \nu^2 f_\alpha^{\#}(\nu^2))|_{U_{\alpha\beta}}\big) = (g_\beta(\nu^2) + \nu^2 f_\beta^{\#}(\nu^2))|_{U_{\alpha\beta}}. \tag{17}$$

Thus, patching $\{g_U + \nu^2 f_U^{\#}\}$ *together we can make a global function* $g + \nu^2 f^{\#} \in C^\infty(M)[[\nu^2]] + \nu^2 C^\infty(M)^{\#}[[\nu^2]]$*. Hence there is a bijection between* $\underline{\mathrm{Aut}}(M, *)$ *and* $C^\infty(M)[[\nu^2]] + \nu^2 C^\infty(M)^{\#}[[\nu^2]]$.

The propositions mentioned above imply that the space

$$\mathfrak{C}_c(M) = C_c^\infty(M)[[\nu^2]] + \nu^2 C_c^\infty(M)^{\#}[[\nu^2]]$$

is a candidate for the model space of $\underline{\mathrm{Aut}}(M, *)$. In fact, the Baker–Campbell–Hausdorff formula shows the smoothness of group operations. Therefore we have the following:

[5] Note that this does not induce the identity on the whole of C_U. In [50], a notion of modified contact Weyl diffeomorphism is introduced to make $\{C_{U_\alpha}, \Psi_{\alpha\beta}\}$ a contact Weyl algebra bundle by adapting the glueing maps to satisfy the cocycle condition and patching them together.

[6] See also Definition 4.4.

Theorem 5.8 $\underline{\mathrm{Aut}}(M,*)$ *is a Lie group modeled on* $\mathfrak{C}_c(M)$.

We can now state

Lemma 5.9 *Suppose that* $\star$ *is a quasi-multiplicative,*[7] *associative product on a Mackey complete locally convex space*[8] $(E,\ \{||\cdot||_\rho\}_\rho)$*; that is, there exists a positive number* C_ρ *such that*

$$||f \star g||_\rho \leq C_\rho ||f||_\rho \cdot ||g||_\rho. \tag{18}$$

Then, $\sum_{n=0}^{\infty} \frac{f\star\cdots\star f}{n!}$ *converges. Set* $e_\star^f = \sum_{n=0}^{\infty} \frac{f\star\cdots\star f}{n!}$. *Then we have*

$$||e_\star^f||_\rho \leq \sum \frac{C_\rho^{n-1}||f||_\rho^n}{n!}. \tag{19}$$

We also have

Lemma 5.10 *The space*

$$\mathfrak{C}_c(M) = \mathrm{ind} \lim_{K:compact} \left(C_K^\infty(M)[[\nu^2]] + \nu^2 C_K^\infty(M)^\#[[\nu^2]] \right)$$

is Mackey complete and quasi-multiplicative, where $C_K^\infty(M)$ *is the space equipped with the standard locally convex topology.*

Proof. The first assertion is followed by [19]. Since the proof of second assertion is bit long and a messy one, we does not give it here. See [29] or [27].

Combining Lemma 5.10 with Lemma 5.9, we can show the exsistence of solution for the equation (1) when $G = \underline{\mathrm{Aut}}(M,*)$ in Definition 2.2 (cf. [27] and [29]). Then we see that smooth curves in the Lie algebra integrate to smooth curves in the Lie group in a smooth way. Thus we have

Theorem 5.11 $\underline{\mathrm{Aut}}(M,*)$ *is a regular Lie group modeled on* $\mathfrak{C}_c(M)$.

As will be seen in the next proposition, general modified contact Weyl diffeomorphims are closely related to symplectic diffeomorphisms.

Proposition 5.12 *For any modified contact Weyl diffeomorphism* Ψ*, it induces a symplectic diffeomorphism on the base symplectic manifold. Moreover, there exists a group homomorphism* p *from* $\mathrm{Aut}(M,*)$ *into* $\mathrm{Diff}(M,\omega)$.

---

[7] The assumption (18) can be replaced by $||f \star g||_\rho \leq C_\rho ||f||_\rho \cdot ||g||_{\hat{\rho}}$.

[8] Here $\{||\cdot||_\rho\}_\rho$ denotes a family of seminorms which gives a locally convex topology.

Conversely, we consider the following problem:

Problem *For any* globally *defined symplectic diffeomorphism* $\phi : M \to M$*, does there exist a* globally *defined modified contact Weyl diffeomorphism* (*referred to as a MCW-lift*) $\hat{\phi}$ *which induces* ϕ?

Although the author does not know the proof, he believes that the problem has an affirmative answer. Instead of the problem above, we consider the existence of a *local* CWD-lift of a locally symplectic diffeomorphism. Although the following argument seems well known for specialists, we review it for readers' convenience.

Assume that

$$\big(U, z = (z^1, \dots, z^{2n})\big), \quad \big(\phi(U), z' = (z'^1, \dots, z'^{2n})\big)$$

are star-shaped Darboux charts. Then $\phi|_U$ is expressed as

$$\big(z'^1, \dots, z'^{2n}\big) = \big(\phi^1(z), \dots, \phi^{2n}(z)\big)$$

and satisfies

$$\{\phi^i, \phi^j\} = \{\phi^{i+n}, \phi^{j+n}\} = 0, \ \ \{\phi^i, \phi^{n+j}\} = -\delta^{ij} \qquad (1 \le i, j \le n),$$

because ϕ is a symplectic diffeomorphism defined on U. The Weyl continuations $\phi^{i\#}$ $(i = 1, \dots, 2n)$ only satisfy

$$\begin{aligned}
[\phi^{i\#}, \phi^{j\#}] &= \nu^3 a^{i,j\#}_{(3)} + \cdots + \nu^{2l+1} a^{i,j\#}_{(2l+1)} + \cdots, \\
[\phi^{i\#}, \phi^{n+j\#}] &= -\nu\delta^{ij} + \nu^3 a^{i,n+j\#}_{(3)} + \cdots + \nu^{2l+1} a^{i,n+j\#}_{(2l+1)} + \cdots, \\
[\phi^{n+i\#}, \phi^{n+j\#}] &= \nu^3 a^{n+i,n+j\#}_{(3)} + \cdots + \nu^{2l+1} a^{n+i,n+j\#}_{(2l+1)} + \cdots.
\end{aligned} \tag{20}$$

However the Jacobi identity holds:

$$[\phi^{s\#}[\phi^{t\#}, \phi^{u\#}]] + \text{c.p.} = 0, \tag{21}$$

where "c.p." means "cyclic permutation". This gives

$$\{z'^s, a^{t,u}_{(3)}\} + \text{c.p.} = \{\phi^s, a^{t,u}_{(3)}\} + \text{c.p.} = 0 \qquad (1 \le s, t, u \le 2n). \tag{22}$$

Set

$$\begin{aligned}
\omega'(z') = \frac{1}{2} \sum_{1 \le i,j \le n} \Big[& a^{(3)}_{n+i,n+j}(z') dx'^i \wedge dx'^j \\
& -2a^{(3)}_{n+i,j}(z') dx'^i \wedge dy'^j + a^{(3)}_{i,j}(z') dy'^i \wedge dy'^j \Big] \quad (z' \in U').
\end{aligned} \tag{23}$$

A direct computation shows that (22) is equivalent to $d\omega' = 0$. As in the proof of Lemma 3.4 in [38], the closedness of ω' above ensures the existence of elements $b'_j \in C^\infty(\phi(U))[[\nu]]$, $(j = 1, \dots, 2n)$ such that replacing $\phi^{s\#}$ by

$$\phi^s_{(1)} = \begin{cases} \phi^j(z) + \nu^2 b'_{j+n}(\phi(z)), \ s = j \\ \phi^{j+n}(z) - \nu^2 b'_j(\phi(z)), \ s = j+n \end{cases} \qquad (1 \le j \le n), \tag{24}$$

shows that ν^3-components of (20) vanish. Repeating the argument above for the ν^5-, ν^7- components gives

$$\phi_{(\infty)} = (\phi^1_{(\infty)}, \dots, \phi^{2n}_{(\infty)}),$$

where

$$\phi^i_{(\infty)} = \phi^i(z) + \sum_{p\ge 1} \nu^{2p} g^i_p(z) \tag{25}$$

such that

$$[\phi^{i\#}_{(\infty)}, \phi^{j\#}_{(\infty)}] = [\phi^{n+i\#}_{(\infty)}, \phi^{n+j\#}_{(\infty)}] = 0, \quad [\phi^{i\#}_{(\infty)}, \phi^{n+j\#}_{(\infty)}] = -\nu\delta^{ij}, \ (i, j = 1, \dots, n).$$

Thus, by Lemma 3.2 in [37], there exists a local Weyl diffeomorphism Φ_U which induces the base map ϕ_U. We next extend Φ_U to a local contact Weyl diffeomorphism Ψ_U. Set

$$\Psi^*_U(a) = \begin{cases} \Phi^*_U(a), \ \ (a \in \mathcal{F}_U), \\ \tilde{\tau}_U + H, \ (a = \tilde{\tau}_{\phi(U)}) \end{cases} \tag{26}$$

where $H = \sum_m \nu^m h^\#_m$ is an unknown term. Ψ_U is a contact Weyl diffeomorphism if it satisfies the following equation w.r.t. H,

$$[\Psi^*_U(\tilde{\tau}_{\phi(U)}), \Psi^*_U(z'^{i\#})] = \Psi^*_U[\tilde{\tau}_{\phi(U)}, z'^{i\#}]. \tag{27}$$

As to the equation, we easily have

$$\text{R.H.S. of (27)} = \Psi^*_U(\nu z'^{i\#}) \overset{def}{=} \nu(\phi^\#_i + B^\#(\nu)), \tag{28}$$

where $B(\nu) = \sum_{l\ge 1} \nu^{2l} g_l$. On the other hand, we also obtain

$$\text{L.H.S. of (27)} \overset{(2.18)\ \text{in [50]}}{=} \nu\Big(\sum_l z^l \frac{\partial z'^i}{\partial z^l}\Big)^\# + \Big[\sum_m \nu^m h_m(z'^i \circ \phi) + \sum_p \nu^{2p} g_p\Big]^\# + \Big(2\nu^2 \partial_\nu B + \nu(EB)\Big)^\# \tag{29}$$

where $E = \nu \sum_{l=1}^{2n} z^l \partial_{z^l}$. As in the proof of Theorem 3.6 in [37], comparing the components w.r.t. ν^1-, ν^2-,ν^3-,... of both sides splits the equation w.r.t. H above into infinitely many equations. Since the component of ν is

$$\{h_0, z'^i \circ \phi\} = (z'^i \circ \phi) - \sum z^l \left(\frac{\partial z'^i}{\partial z^l}\right), \tag{30}$$

we can find the solution h_0 for this equation, and then we can solve the infinitely many equations recursively.[9] Summing up the above, we have

[9]Thanks to star-shapeness of U, we can fix b'_s and H canonically.

Proposition 5.13 *Take a star-shaped Darboux chart U. For any symplectic diffeomorphism $\phi : U \to \phi(U)$, there canonically exists a contact Weyl diffeomorphism (CW-lift) $\hat{\phi}$ which induces ϕ.*

Then we have

Corollary 5.14 *Assume that a symplectic manifold M is covered by a star-shaped Darboux chart. Then for any symplectic diffeomorphism $\phi : M \to M$, there canonically exists a contact Weyl diffeomorphism (CW-lift) $\hat{\phi}$ which induces ϕ.*

We also have

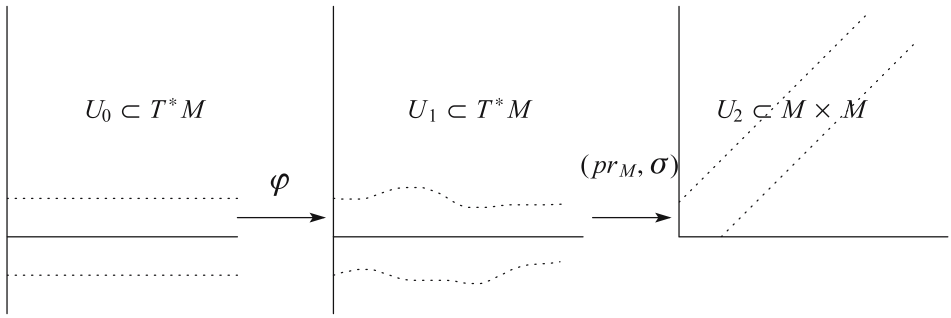
Proposition 5.15 *Assume that there exists a map* [10] *j from* $\mathrm{Diff}(M, \omega)$ *into* $\mathrm{Aut}(M, *)$ *satisfying* $p \circ j = \mathrm{identity}$. *Then we have a bijection:*

$$\mathrm{Aut}(M, *) \cong \mathrm{Diff}(M, \omega) \times \underline{\mathrm{Aut}}(M, *). \tag{31}$$

Proof. As mentioned in Proposition 5.12, any element $\Psi \in \mathrm{Aut}(M, *)$ induces a symplectic diffeomorphism $\phi = p(\Psi)$ on the base manifold. Set $\hat{\phi} = j(\phi)$ and $\Phi = \hat{\phi}^{-1} \circ \Psi$. By the assumption, Φ induces the base identity map. According to Proposition 5.6, we see $\Phi = \exp[ad(\frac{1}{\nu}(g(\nu^2) + \nu^2 f^{\#}(\nu^2)))]$.

As seen in the proposition above, in order to determine the model space of $\mathrm{Aut}(M, *)$, we have to determine the model space of $\mathrm{Diff}(M, \omega)$. Take a diffeomorphism (pr_M, σ) from an open neighborhood U_0 of the zero section in T^*M onto an open neighborhood U_2 of the diagonal set of $M \times M$, such that $\sigma(0\text{-section}|_x) = x$. Let ω_0 be the canonical symplectic structure of T^*M, and $\omega_1 := (pr_M, \sigma)^*(\omega \oplus \omega^-)$, where the reversed symplectic structure of ω is denoted by ω^-. Since ω_0 and ω_1 vanish when restricted to the zero section, by virtue of Moser's technique (cf. [5]), there exists a diffeomorphism $\varphi : U_0 \to U_1$ between two open neighborhoods U_0 and U_1 of the zero section in T^*M which is the identity on the zero section and satisfies $\varphi^*\omega_1 = \omega_0$. Thus we obtain that

$$\eta = (pr_M, \sigma) \circ \varphi : (U_0, \omega_0) \overset{\varphi}{\longleftrightarrow} (U_1, \omega_1) \overset{(pr_M, \sigma)}{\longleftrightarrow} (U_2, \omega \oplus \omega^-). \tag{32}$$



[10]The map j is not a Lie group homomorphism in general.

We also see that

$$\begin{aligned}&\{\eta^{-1}(x, f(x)) \mid x \in M\} \text{ is a closed form } (\in \Omega^1_c(T^*M))\\ &\quad\Leftrightarrow \{\eta^{-1}(x, f(x)) \mid x \in M\} \text{ is a Lagrangian submanifold of } (T^*M, \omega_0)\\ &\quad\Leftrightarrow \text{ the graph is a Lagrangian submanifold of } (M \times M, \omega \oplus \omega^-)\\ &\quad\Leftrightarrow 0 = (Id_M, f)^*(pr_1^*(\omega) - pr_2^*(\omega)) = Id_M^*\omega - f^*\omega\\ &\quad\Leftrightarrow f \in \mathrm{Diff}_c(M, \omega).\end{aligned}$$

Let $\mathcal{U}$ be an open neighborhood of Id_M consisting of all $f \in \mathrm{Diff}(M)$ with compact support satisfying $(Id_M, f)(M) \subset U_2$ and $pr_M : \eta^{-1}(\{(x, f(x))|x \in M\}) \to M$ is still a diffeomorphism. For $f \in \mathcal{U}$, the map $(Id_M, f) : M \to graph(f) \subset M \times M$ is the natural diffeomorphism onto the graph of f. According to (32), we can define the smooth chart of $\mathrm{Diff}(M)$ which is centered at the identity in the following way:

$$\mathrm{Diff}_c(M) \supset \mathcal{U} \overset{\Psi}{\to} \Psi(\mathcal{U}) \subset \Omega^1_c(M), \quad \Psi(f) = \eta^{-1}(Id_M, f)\ ;\ M \to T^*M.$$

Since $\Omega^1_c(T^*M)$ is Mackey complete (cf. [19]), $\mathcal{U} \cap \mathrm{Diff}(M, \omega)$ gives a submanifold chart for $\mathrm{Diff}(M, \omega)$ at Id_M. Moreover, conditions of Definition 2.2 can be shown by the standard argument for an ordinary differential equation under a certain identification of T^*M with TM. Therefore, we have the following.

Theorem 5.16 ([19], [32]) *Let (M, ω) be a finite-dimensional symplectic manifold. Then the group* $\mathrm{Diff}(M, \omega)$ *of symplectic diffeomorphisms is a regular Lie group and a closed submanifold of the regular Lie group* $\mathrm{Diff}(M)$ *of diffeomorphisms. The Lie algebra of* $\mathrm{Diff}(M, \omega)$ *is a Mackey complete locally convex space $\mathfrak{X}_c(M, \omega)$ of symplectic vector fields with compact supports.*

Combining the Baker–Campbell–Hausdorff formula with Theorem 5.11 and Proposition 5.15, we have the following:

Lemma 5.17 *The following maps are smooth:*

(i) $\mathrm{Diff}(M, \omega) \times \underline{\mathrm{Aut}}(M, *) \to \underline{\mathrm{Aut}}(M, *)$; $(\phi, \Psi) \mapsto \hat{\phi}^{-1} \circ \Psi \circ \hat{\phi}$,
(ii) $\mathrm{Diff}(M, \omega) \times \mathrm{Diff}(M, \omega) \to \underline{\mathrm{Aut}}(M, *)$; $(\phi, \psi) \mapsto \widehat{(\phi \circ \psi)}^{-1} \circ \hat{\phi} \circ \hat{\psi}$,
(iii) $\mathrm{Diff}(M, \omega) \to \underline{\mathrm{Aut}}(M, *)$; $\phi \mapsto \hat{\phi} \circ \widehat{\phi^{-1}}$.

According to Theorem 5.11 and Proposition 5.15, $\mathfrak{X}_c(M, \omega) \times \mathfrak{C}_c(M)$ is a model space, which is a Mackey complete locally convex space. Let $\Psi_i = \hat{\psi}_i \circ e^{ad(\frac{1}{\nu}H_i(\nu^2))}$, where $H_i(\nu^2) = g_i(\nu^2) + \nu^2 f_i^\#(\nu^2)$ $(i = 1, 2)$. Then the multiplication is written in the following way:

$$\begin{aligned}\Psi_1 \circ \Psi_2 &= \hat{\psi}_1 \circ e^{ad(\frac{1}{\nu}H_1(\nu^2))} \circ \hat{\psi}_2 \circ e^{ad(\frac{1}{\nu}H_2(\nu^2))}\\ &= \widehat{\psi_1 \circ \psi_2} \circ \left\{ \widehat{(\psi_1 \circ \psi_2)}^{-1} \circ \hat{\psi}_1 \circ \hat{\psi}_2 \right\}\end{aligned}$$

$$\circ \left\{ \hat{\psi_2}^{-1} \circ e^{ad(\frac{1}{\nu}H_1(\nu^2))} \circ \hat{\psi_2} \right\} \circ e^{ad(\frac{1}{\nu}H_2(\nu^2))}. \tag{33}$$

According to (i) and (ii) of Lemma 5.17, (33) is written as

$$\left(\widehat{\psi_1 \circ \psi_2} \right) \circ e^{ad(\frac{1}{\nu}H(\psi_1, \psi_2, H_1(\nu^2), H_2(\nu^2)))},$$

and we see the smoothness of

$$(\psi_1, \psi_2, H_1(\nu^2), H_2(\nu^2)) \mapsto H(\psi_1, \psi_2, H_1(\nu^2), H_2(\nu^2)).$$

In a similar way, we can verify the smoothness of the inverse operation. Summing up the above, we have

Theorem 5.18 *Under the assumption of Proposition 5.15,* $\mathrm{Aut}(M, *)$ *is a Lie group modeled on a Mackey complete locally convex space* $\mathfrak{X}_c(M, \omega) \times \mathfrak{C}_c(M)$.

Furthermore, combining Definition 5.1 of Aut(∗) with Proposition 5.12 gives a short exact sequence

$$1 \to \underline{\mathrm{Aut}}(M, *) \to \mathrm{Aut}(M, *) \to \mathrm{Diff}(M, \omega) \to 1.$$

As mentioned in Theorem 5.16, the group of all symplectic diffeomorphisms $\mathrm{Diff}(M, \omega)$ is a regular Lie group modeled on a Mackey complete locally convex space $\mathfrak{X}_c(M, \omega)$. Therefore, combining Theorem 5.11 with Lemma 2.3, that $\mathrm{Aut}(M, *)$ is regular. Thus, we obtain the following.

Theorem 5.19 *Suppose the assumption of Proposition 5.15.*

1) $1 \to \underline{\mathrm{Aut}}(M, *) \to \mathrm{Aut}(M, *) \to \mathrm{Diff}(M, \omega) \to 1$ *is a short exact sequence of Lie groups.*
2) $\underline{\mathrm{Aut}}(M, *)$ *and* $\mathrm{Aut}(M, *)$ *are regular Lie groups.*

This completes the proof of Theorem 1.1.

6 Concluding remarks

In the previous section, we proved that $\mathrm{Aut}(M, *)$ has a regular Lie group structure under suitable assumptions. In this section, we note the advantages of the smooth structure of $\mathrm{Aut}(M, *)$. It is known that, in order to analyse properties of the group $\mathrm{Aut}(M, *)$, there are several tools such as *Floer theory* and the *residue trace* of Wodzicki [48] and Guillemin [13] (see also [7]). In the present section, we focus on the former one and then our goal is to suggest a problem with respect to the relation between $\mathrm{Aut}(M, *)$ and symplectic Floer and quantum homology.

First we give a brief review of the fundamentals for the Floer theory. There exist analytical difficulties to be overcome and certain conditions to be assumed in order to give the definition. However, we will not mention them again. Roughly speaking, Floer homology theory can be seen as Morse homology theory for a certain functional with a suitable index. In fact, we use the symplectic action[11] a_H on a loop space with the $\dim_M - \mu_{\text{Conley–Zhender}}$ instead of the Morse function f with the Morse index μ_{Morse} used in the finite-dimensional Morse theory.

On the other hand, quantum homology $QH_*(M, \omega) := \oplus_k \oplus_{i+j=k} H_i(M, \mathbf{Z}_2) \otimes \Lambda_j$ under suitable assumptions, where $\Lambda = \oplus \Lambda_j$ is called the Novikov ring of (M, ω), see [43] for details.

Next we explain the advantages of the smooth structure of Aut(M, $*$) and Diff(M, ω). According to the argument developed in [28], with certain assumptions, we can find a secondary characteristic 1-form μ which gives the nontrivial cohomology class[12]

$$[\mu] \in H^1_{dR}(\text{Aut}(M, *)),$$

and then we can explicitly find nontrivial elements in $\pi_1(\text{Aut}(M, *))$ under a suitable condition of the base manifold[13] M.

As to Floer theory, it is known that there exists a map $\mathfrak{S}$ (called the Seidel map [43]) from $\pi_1(\text{Diff}(M, \omega))$ into the Floer homology group $HF_*(M, \omega)$. Thus composing $\mathfrak{S}$ with

$$\mathfrak{W}\mathfrak{F}_* : \pi_1(\text{Aut}(M, *)) \to \pi_1(\text{Diff}(M, \omega)),$$

we obtain a map

$$\mathfrak{S} \circ \mathfrak{W}\mathfrak{F}_* : \pi_1(\text{Aut}(M, *)) \to HF_*(M, \omega).$$

Furthermore Piunikhin–Salamon–Schwarz [41] showed the existence of an isomorphism between symplectic Floer homology $HF_*(M, \omega)$ and quantum homology $QH_*(M, \omega)$. Hence under the above situation, it is natural to ask that

Problem *Do the images of nontrivial cycles of* Aut($*$) *by* $\mathfrak{S} \circ \mathfrak{W}\mathfrak{F}_*$ *give invertible elements in a quantum homology ring* $QH_*(M, \omega)$*?*

Remark that invertible elements in quantum homology seem to play a crucial role in particle physics (cf. [41] and [43]). We will be concerned with this problem in a forthcoming paper.

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[11] Here, the symplectic action integral is defined by $a_H : \mathcal{L}M \to \mathbf{R}/\mathbf{Z}$ defined by

$$a_H(x) = -\int_{\{z \in \mathbf{C}: |z| \le 1\}} u^* \omega - \int_0^1 H_t(x(t)) dt$$

for $x \in \mathcal{L}M$ with a suitable periodic hamiltonian function H and a suitable smooth map $u : \{z \in \mathbf{C} : |z| \le 1\} \to M$ such that $u(e^{e\pi i t}) = x(t)$.

[12] For construction of differential forms, we need the Lie group structure.

[13] For example, when $M = D_{pq}(\mathbf{C})$, i.e., bounded symmetric domain, we can find closed paths Φ_λ of Aut$(M, *)$, where $\lambda = (\lambda_1, \ldots, \lambda_p, \lambda_{p+1}, \ldots, \lambda_{p+q})$, $\lambda_i \in \mathbb{Z}$ such that $\langle \mu, \Phi_\lambda \rangle = q \sum_{i=1}^{p} \lambda_i - p \sum_{i=p+1}^{p+q} \lambda_i$.

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References

1. Adams, M., Ratiu, T. and Schmid, R. *The Lie group structure of diffeomorphism groups and invertible Fourier integral operators with applications,* Infinite-dimensional Groups with applications, eds. V. Kac, (1985), 1–69, Springer .
2. Adams, M., Ratiu, T. and Schmid, R. *A Lie group structure for Pseudodifferential Operators*, Math. Ann., 273 (1986), 529–551.
3. Adams, M., Ratiu, T. and Schmid, R. *A Lie group structure for Fourier integral Operators*, Math. Ann., 276 (1986), 19–41.
4. Arnol'd, V. I. *On a characteristic class entering in quantization conditions*, Func. Anal. Appl. 1 (1967), 1–13.
5. Banyaga, A. *The structure of classical diffeomorphism groups,* Mathematics and its applications 400 (1997), Kluwer Academic Publishers.
6. Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D. *Deformation theory and quantization I,* Ann. of Phys. 111 (1978), 61–110.
7. Deligne, P. *Déformations de l'Algébre des Fonctions d'une variété Symplectique: Comparison enter Fedosov et De Wilde, Lecomte,* Selecta Math. N.S.1 (1995), 667–697.
8. De Wilde, M. and Lecomte, P. B. *Existence of star-products and formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds,* Lett. Math. Phys. 7 (1983), 487–496.
9. Dito,G and Sternheimer,D. *Deformation Quantization: Genesis, Developments and Metamorphoses* math.QA/0201168.
10. Eighhorn, J. and Schmid, R. *Lie groups of Fourier integral operators on open manifolds,* Comm. Ana. Geom. 9 (2001), no.5, 983–1040.
11. Etingof, P. and Kazhdan, D. *Quantization of Lie bialgebras, I*, Selecta Math., New Series, 2 (1996), 1–41.
12. Fedosov, B. V. *A simple geometrical construction of deformation quantization,* Jour. Diff. Geom. 40 (1994), 213–238.
13. Guillemin, V. W. *A new proof of Weyl's formula on the asymptotic distribution of eigenvalues*, Adv. Math. 55, (1985), 131–160.
14. Gutt, S. and Rawnsley, J. *Equivalence of star products on a symplectic manifold; an introduction of Deligne's Cech cohomology classes,* Jour. Geom. Phys. 29 (1999), 347–392.
15. Hamilton, R. *The inverse function theorem of Nash and Morse*, Bull. A.M.S., 7 (1982), 65-225.
16. Jarchow, H. *Locally convex spaces,* (1981), Teubner.
17. Kac, V. *Infinite-dimensional Lie algebras,* (1990), Cambridge, University Press.
18. Kontsevich, M. *Deformation quantization of Poisson manifolds,* Lett. Math. Phys. 66, no.3 (2003), 157–216.
19. Kriegl, A. and Michor, P. *The convenient setting of Global Analysis,* SURV. 53, (1997), Amer. Math. Soc.
20. Kumano-go, H.*Pseudodifferential Operators*, MIT, (1982).
21. Leray, J. *Analyse lagrangienne et mechanique quantique*, Seminaire du College de France 1976–1977; R.C.P.25, (1978), Strasbourg.

22. Leslie, J. A. *Some Frobenius theorems in global analysis*, Jour. Diff. Geom., 2 (1968), 279–297.
23. Melrose, R. *Star products and local line bundles,* Annales de l'Institut Fourier, 54 (2004), 1581–1600.
24. Maeda, Y. and Kajiura, H. *Introduction to deformation quantization*, Lectures in Math. Sci., The Univ. of Tokyo, 20 (2002), Yurinsya.
25. Michor, P. *Manifolds of smooth maps, II, III,* Cahiers topo. et Géom. Diff. XIX-1 (1978), 47–78, XX-3, (1979), 63–86, XXI-3,(1980), 325–337.
26. Milnor, J. *Remarks on infinite-dimensional Lie groups,* Proc. Summer School on Quantum Gravity, ed B. Dewitt, (1983), Les Houches.
27. Miyazaki, N. *On regular Fréchet–Lie group of invertible inhomogeneous Fourier integral operators on* $\mathbf{R}^{\mathbf{n}}$*,* Tokyo Jour. Math., 19, No.1 (1996), 1–38.
28. Miyazaki, N. *A remark on the Maslov form on the group generated by invertible Fourier integral operators*, Lett. Math. Phys. 42 (1997), 35–42.
29. Miyazaki, N. *Automorphisms of the Weyl manifold,* in preparation.
30. Moyal, J.E.*Quantum mechanics as Statistical Theory,* Proc. Cambridge Phil. Soc., 45 (1949), 99–124.
31. Omori, H. and de la Harp, P. *About interactions between Banach–Lie groups and finite-dimensional manifolds*, Jour. Math. Kyoto Univ. 12-3 (1972), 543–570.
32. Omori, H. *Infinite-dimensional Lie groups*, MMONO 158 (1995), Amer. Math. Soc.
33. Omori, H. *Physics in Mathematics: Toward Geometrical Quantum Theory*, (2004), University of Tokyo Press.
34. Omori, H., Maeda, Y., Miyazaki, N. and Yoshioka, A. *Poincaré–Cartan class and deformation quantization of Kähler manifolds,* Commun. Math. Phys. 194 (1998), 207–230.
35. Omori, H., Maeda,Y., Miyazaki, N. and Yoshioka, A., *Strange phenomena related to ordering problems in quantizations*, Jour. Lie Theory vol. 13, no 2 (2003), 481–510.
36. Omori, H., Maeda, Y., Miyazaki, N. and Yoshioka, A. *Star exponential functions as two-valued elements*, Progr. Math. 232 (2005), 483–492, Birkhäuser.
37. Omori, H., Maeda, Y. and Yoshioka, A. *Deformation quantization and Weyl manifolds,* Adv. Math. 85 (1991), 224–255.
38. Omori, H., Maeda, Y. and Yoshioka, A. *Global calculus on Weyl manifolds,* Japan. Jour. Math. 17 (2), (1991), 57–82.
39. Omori, H., Maeda, Y., Yoshioka, A. and Kobayashi, O. *On regular Fréchet–Lie groups IV,* Tokyo Jour. Math. 5 (1982), 365–398.
40. Omori, H., Maeda, Y., Yoshioka, A. and Kobayashi, O. *The theory of infinite-dimensional Lie groups and its applications,* Acta Appl. Math. 3 (1985), 71–105.
41. Piunikhin, S., Salamon, D. and Schwarz, M. *Symplectic Floer–Donaldson theory and quantum cohomology,* in "Contact and symplectic geometry" (C.B.Thomas,ed.) (1996), 171–200, Cambridge Univ. Press.
42. Pressley, A. and Segal, G. *Loop groups,* (1988), Clarendon Press, Oxford.
43. Seidel, P. π_1 *of symplectic automorphism groups and invertibles in quantum homology rings,* GAFA. 7 (1997), 1046–1095.
44. Sternheimer, D. *Deformation quantization twenty years after,* AIP Conf. Proc. 453 (1998), 107–145, (q-alg/9809056).
45. Tanisaki, T. *Lie algebras and Quantum groups*, (2002), Kyoritsu.
46. Wakimoto, M. *Infinite-dimensional Lie algebras,* (1999), Iwanami.
47. Weyl, H. *Gruppentheorie und Quantenmechanik,* Hirzel, Leibzig, (1928).
48. Wodzicki, M. *Noncommutative residue, I.* 320–399, Lecture Notes in Math. (1987), Springer.

49. Woodhouse, N. *Geometric quantization,* Clarendon Press, (1980), Oxford.
50. Yoshioka, A. *Contact Weyl manifold over a symplectic manifold,* in "Lie groups, Geometric structures and Differential equations", Adv. St. Pure Math. 37(2002), 459–493.

Part II

Riemannian Geometry

Projective Structures of a Curve in a Conformal Space

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Summary. In this paper we will show that a Möbius structure or a conformal structure of a manifold induces a projective structure of a regular curve on the manifold, and that for a regular curve on the sphere, the curve has no self-intersection if the projective developing map of the curve is injective.

AMS Subject Classification: 53A30.

Key words: Möbius structure, conformal structure, projective structure, Schwarzian derivative.

1 Introduction

In the paper [1], the author and M. Wada introduced two kinds of Schwarzian derivative of a regular curve $x\colon I \to (M, g)$ in a Riemannian n-manifold. One is defined as

$$s_g x = (\nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x})\dot{x}^{-1} - \frac{3}{2}((\nabla_{\dot{x}}\dot{x})\dot{x}^{-1})^2 - \frac{R_g}{2n(n-1)}\dot{x}^2,$$

where multiplications are understood to be the Clifford multiplications with respect to the metric g, and R_g is the scalar curvature of the metric g. The other is defined as

$$\tilde{s}_g x = (\nabla_{\dot{x}} \nabla_{\dot{x}} \dot{x})\dot{x}^{-1} - \frac{3}{2}((\nabla_{\dot{x}}\dot{x})\dot{x}^{-1})^2 - \frac{1}{n-2}\dot{x}(L_g \cdot \dot{x}),$$

where

$$L_g = \mathrm{Ric}_g - \frac{R_g}{2(n-1)}g,$$

and Ric_g is the Ricci curvature. We call $s_g x$ and $\tilde{s}_g x$ the *Schwarzian* and the *conformal Schwarzian* of the curve respectively. These two Schwarzian derivatives coincide if g is an Einstein metric.

In this paper we will show that the Möbius (resp. conformal) structure of (M, g) induces a projective structure of the curve x through $s_g x$ (resp. $\tilde{s}_g x$), and we rephrase Theorem 1.3 of [1] as follows:

Theorem *Let I be an interval and $x\colon I \to (S^n, g_0)$ be a regular curve of the Euclidean sphere. If the projective developing map*

$$\mathrm{dev}_x\colon I \to \mathbf{R}\mathrm{P}^1$$

is injective, then $x\colon I \to S^n$ is injective.

2 Projective structures of a curve

We recall some basic properties of the Schwarzian derivatives of a regular curve ([1]). First we note that our Schwarzians $s_g x$ and $\tilde{s}_g x$ have decompositions into their 0-parts and 2-parts:

$$s_g x(t) = s_g x^{(0)}(t) + s_g x^{(2)}(t) \in \mathbf{R} \oplus \Lambda^2 T_{x(t)} M,$$
$$\tilde{s}_g x(t) = \tilde{s}_g x^{(0)}(t) + \tilde{s}_g x^{(2)}(t) \in \mathbf{R} \oplus \Lambda^2 T_{x(t)} M.$$

Lemma 2.1 ([1]) *For a regular curve $x = x(t)$ on a Riemannian manifold (M, g), we have*

(i) $s_g x^{(0)} = 2|\dot{x}|^2 \left(\dfrac{\frac{d^2}{ds^2}\sqrt{|\dot{x}|}}{\sqrt{|\dot{x}|}} + \dfrac{1}{4}\left(\kappa^2 + \dfrac{R_g}{n(n-1)}\right)\right),$

(ii) $\tilde{s}_g x^{(0)} = s_g x^{(0)} - \dfrac{1}{n-2}\mathrm{Ric}_g^\circ(\dot{x}, \dot{x}),$

where $\cdot = d/dt$, $d/ds = (1/|\dot{x}|)d/dt$ is the derivation with respect to an arclength parameter s, κ is the geodesic curvature of the curve x, and $\mathrm{Ric}_g^\circ = \mathrm{Ric}_g - (R_g/n)g$.

Lemma 2.2 ([1]) *Suppose $\hat{g} = e^{2\varphi} g$. Then,*

(i) $s_{\hat{g}} x^{(0)} = s_g x^{(0)} + P_\varphi(\dot{x}, \dot{x}),$
(ii) $\tilde{s}_{\hat{g}} x^{(0)} = \tilde{s}_g x^{(0)},$

where $P_\varphi = -e^\varphi \nabla^2 e^{-\varphi} + \frac{1}{n} e^\varphi (\Delta e^{-\varphi}) g$.

Thus if g and $\hat{g}$ are Möbius equivalent, then $s_{\hat{g}} x^{(0)} = s_g x^{(0)}$ (cf. [1], [2], [3]), and if g and $\hat{g}$ are conformal and $n = \dim M \geq 3$, then $\tilde{s}_{\hat{g}} x^{(0)} = \tilde{s}_g x^{(0)}$.

We are interested in parametrization of the curve $x\colon I \to M$. Let $U \subset I$ be an open set and $u\colon U \to \mathbf{R}$ be a new local parameter. We assume $\dot{u} = du/dt > 0$, and put

$$f = \sqrt{\frac{dt}{du}},$$

that is, $u = \int dt/f^2$. Put $\hat{x} := x \circ u^{-1}$, that is, $\hat{x}(u(t)) = x(t)$. Then we have

Lemma 2.3

(i) $s_g\hat{x}^{(0)} = 2f^3(\ddot{f} + \frac{1}{2}(s_g x^{(0)})f)$.
(ii) $\tilde{s}_g\hat{x}^{(0)} = 2f^3(\ddot{f} + \frac{1}{2}(\tilde{s}_g x^{(0)})f)$.

Proof. By a straightforward calculation, we have

$$\frac{d^2}{ds^2}\sqrt{\left|\frac{d}{du}\hat{x}\right|} = |\dot{x}|^{-\frac{3}{2}}\ddot{f} + (\frac{d^2}{ds^2}\sqrt{|\dot{x}|})f.$$

This, together with Lemma 2.1, yields the desired equalities. □

From the above lemma we see that for any point $t \in I$, there is a neighborhood $U \subset I$ of t where we have a local parameter $u\colon U \to \mathbf{R}$ such that either

$$s_g(x \circ u^{-1})^{(0)} = 0 \tag{2.1}$$

or

$$\tilde{s}_g(x \circ u^{-1})^{(0)} = 0 \tag{2.2}$$

holds.

Proposition 2.4 *Let $u\colon U \to \mathbf{R}$ and $v\colon V \to \mathbf{R}$ be local parameters of the curve $x\colon I \to M$ such that either*

$$s_g(x \circ u^{-1})^{(0)} = s_g(x \circ v^{-1})^{(0)} = 0$$

or

$$\tilde{s}_g(x \circ u^{-1})^{(0)} = \tilde{s}_g(x \circ v^{-1})^{(0)} = 0.$$

Then,

$$u = \frac{av + b}{cv + d}$$

for some $a, b, c, d \in \mathbf{R}$.

Proof. Since the argument is local, we may assume $t = v$. Then it follows from Lemma 2.3 that $\ddot{f} = 0$, i.e., $((du/dv)^{-1/2})\ddot{} = 0$, which implies

$$\frac{\dddot{u}}{\dot{u}} - \frac{3}{2}\left(\frac{\ddot{u}}{\dot{u}}\right)^2 = 0.$$

Hence u is a linear fractional function in $t = v$. □

In this way we have an open covering $\{U_\lambda\}$ of I and maps $u_\lambda\colon U_\lambda \to \mathbf{R}$ which satisfy either (2.1) or (2.2), and then Proposition 2.4 says that the coordinate transformations $u_\lambda \circ u_\mu^{-1}$ are 1-dimensional projective transformations. Namely, two projective structures are defined on the interval I. It follows from Lemma 2.2 that the projective structure defined through $s_g x$ with (2.1) (resp. $\tilde{s}_g x$ with (2.2)) depends only on the Möbius (resp. conformal) structure of (M, g). These two projective structures through $s_g x$ and $\tilde{s}_g x$ may be essentially different. For example, let $(M, g) = \mathbf{R} \times S^{n-1}(1)$, and $x\colon \mathbf{R} \to M; x(t) = (t, p_0)$ for some $p_0 \in S^{n-1}(1)$. Then it is easy to see that the projective structures on $\mathbf{R}$ through $s_g x$ and $\tilde{s}_g x$ are different because the projective developing map, $\mathrm{dev}_x : \mathbf{R} \to \mathbf{R}\mathrm{P}^1 = \mathbf{R} \cup \{\infty\}$ for the former one, is not injective but the one for the latter is injective.

It is natural to consider a projective developing map $\mathrm{dev}_x : I \to \mathbf{R}\mathrm{P}^1$ in order to see the projective structure of I. Here a *projective developing map* means simply a projective map. Once a projective structure is defined on the interval I, a developing map $\mathrm{dev}_x : I \to \mathbf{R}\mathrm{P}^1$ is defined, since I is simply connected. Moreover the developing map is uniquely determined up to a projective transformation of $\mathbf{R}\mathrm{P}^1$. The following propositions give some conditions on the injectivity of the projective developing maps.

Proposition 2.5 *The following are equivalent:*

(i) *There is a global parameter* $u\colon I \to \mathbf{R}$, *such that* $s_g(x \circ u^{-1}) = 0$.
(ii) *There is a global parameter* $v\colon I \to \mathbf{R}$, *such that* $s_g(x \circ v^{-1}) \leq 0$.
(iii) *The developing map* $\mathrm{dev}_x : I \to \mathbf{R}\mathrm{P}^1$ *with respect to the projective structure defined through* $s_g x$ *with* (2.1) *is injective.*

Proposition 2.5′ *The following are equivalent:*

(i) *There is a global parameter* $u\colon I \to \mathbf{R}$, *such that* $\tilde{s}_g(x \circ u^{-1}) = 0$.
(ii) *There is a global parameter* $v\colon I \to \mathbf{R}$, *such that* $\tilde{s}_g(x \circ v^{-1}) \leq 0$.
(iii) *The developing map* $\mathrm{dev}_x : I \to \mathbf{R}\mathrm{P}^1$ *with respect to the projective structure defined through* $\tilde{s}_g x$ *with* (2.2) *is injective.*

Proof. The proofs of Propositions 2.5 and 2.5′ are completely similar. and we will prove Proposition 2.5. It is obvious that the conditions (i) and (iii) are equivalent, and it is trivial that (i) implies (ii). So we have only to show that (ii) implies (i).

We may assume $s_g x^{(0)}(t) \leq 0$ for any $t \in I$. Consider an ordinary differential equation

$$\ddot{f} + \frac{1}{2}(s_g x^{(0)}) f = 0. \tag{2.3}$$

Since this is a linear ordinary differential equation, we have a solution $f\colon I \to \mathbf{R}$ which satisfies (2.3) with initial conditions $f(t_0) = 1$ and $\dot{f}(t_0) = 0$ for some fixed $t_0 \in I$. Then $\ddot{f} \geq 0$ whenever $f > 0$ since $s_g x^{(0)} \leq 0$. Hence we have $f \geq 1$ on I. Thus $u = \int dt/f^2\colon I \to \mathbf{R}$ is a parameter of x for which $s_g(x \circ u^{-1})^{(0)} = 0$ because of Lemma 2.3 (i). □

The Theorem in our Introduction is readily proved from Propositions 2.5 and Theorem 1.3 of [1]. The theorem in [1] asserts that *a regular curve x of the Euclidean n-sphere satisfying $s_g x^{(0)} \leq 0$ is injective.*

References

1. Kobayashi, O. and Wada, M.: Circular geometry and the Schwarzian, *Far East J. Math. Sci.* Special Volume(2000), 335–363
2. Osgood, B. and Stowe, D.: The Schwarzian derivative and conformal mapping of Riemannian manifolds, *Duke Math. J.* **67** (1992), 57–99.
3. Yano, K.: Concircular Geometry I. Concircular Transformations, *Proc. Imp. Acad. Japan* **16** (1940), 195–200.

Deformations of Surfaces Preserving Conformal or Similarity Invariants

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Dedicated to professor Hideki Omori

Summary. We study Möbius applicable surfaces, i.e., conformally immersed surfaces in Möbius 3-space which admit deformations preserving the Möbius metric. We show new characterizations of Willmore surfaces, Bonnet surfaces and harmonic inverse mean curvature surfaces in terms of Möbius or similarity invariants.

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Key words: Deformation of surfaces, Möbius geometry, similarity geometry.

Introduction

In [11], Burstall, Pedit and Pinkall gave a fundamental theorem of surface theory in Möbius 3-space in modern formulation. Surfaces in Möbius 3-space are determined by conformal Hopf differential and Schwarzian derivative up to conformal transformations. Isothermic surfaces are characterized as surfaces in Möbius 3-space which admit deformations preserving the conformal Hopf differential.

On every surface in Möbius 3-space, a (possibly singular) conformally invariant Riemannian metric is introduced. This metric is called the *Möbius metric* of the surface. The Gaussian curvature of the Möbius metric is called the Möbius curvature.

Here we point out that the preservation of Möbius metric is weaker than that of conformal Hopf differential.

Constant mean curvature surfaces (abbreviated as CMC surfaces) in the space forms are typical examples of isothermic surfaces. Bonnet showed that every constant mean curvature surface admits a one-parameter family of isometric deformations preserving the mean curvature. A surface which admits such a family of deformations is called a *Bonnet surface*. Both the isothermic surfaces and Bonnet surfaces are regarded as *geometric* generalizations of constant mean curvature surfaces.

On the other hand, from the viewpoint of integrable system theory, Bobenko introduced the notion of surface with harmonic inverse mean curvature (HIMC surface, in short) in Euclidean 3-space $\mathbb{R}^3$. The first named author extended the notion of HIMC surface in $\mathbb{R}^3$ to that of 3-dimensional space forms [19]. HIMC surfaces have deformation families (associated family) which preserve the conformal structure of the surface and the harmonicity of the reciprocal mean curvature. Moreover, there exist local bijective conformal correspondences between HIMC surfaces in different space forms.

It should be remarked that while every Bonnet surface is isothermic, HIMC surfaces are not necessarily isothermic. In fact, the associated family of Bonnet surfaces or HIMC surfaces preserves the Möbius metrics, while the conformal Hopf differential of HIMC surfaces are not preserved in the associated family.

These observations motivate us to study surfaces in Möbius 3-space (or space forms) which admit deformations preserving the Möbius metric. We call such surfaces *Möbius applicable surfaces*.

In this paper we study Möbius applicable surfaces.

First, we shall show the following new characterization of Willmore surfaces.

Theorem 1.5 *A surface in Möbius 3-space is Willmore if and only if it is a Möbius applicable surface whose deformation family preserves the Schwarzian derivative.*

Next, we shall characterize both Bonnet surfaces and HIMC surfaces in the class of Möbius applicable surfaces in terms of similarity invariants:

Theorem 2.4 *A surface in Euclidean 3-space is a Bonnet surface or a HIMC surface if and only if it is a Möbius applicable surface with specific deformation family in which the ratio of principal curvatures is preserved.*

Furthermore we shall give the following characterization of flat Bonnet surfaces:

Theorem 2.6 *A Bonnet surface of non-constant mean curvature in Euclidean 3-space is flat if and only if its ratio of principal curvatures or Möbius curvature is constant.*

Our characterization results imply that "Bonnet" and "HIMC" are similarity notions. Thus these classes of surfaces fit naturally into similarity geometry.

We emphasize that similarity geometry provide us non-trivial differential geometry of integrable surfaces. In fact, the Burgers hierarchy is derived as a deformation of plane curves in similarity geometry.

1 Deformation of surfaces preserving conformal invariants

1.1 Generalities of surface theory in conformal geometry

Let $\mathbb{R}^3$ be the Euclidean 3-space. The group Conf(3) of all conformal diffeomorphisms are generated by isometries, dilations and inversions. The conformal compactification

$\mathcal{M}^3$ of $\mathbb{R}^3$ is called the *Möbius 3-space*. By definition, $\mathcal{M}^3$ is the 3-sphere equipped with the canonical flat conformal structure.

In this paper, we use the projective lightcone model of the Möbius 3-space introduced by Darboux.

Let $\mathbb{R}^5_1$, be the *Minkowski 5-space* with canonical Lorentz scalar product:

$$\langle \xi, \eta \rangle = -\xi_0\eta_0 + \xi_1\eta_1 + \xi_1\eta_2 + \xi_1\eta_3 + \xi_4\eta_4.$$

We denote the natural basis of $\mathbb{R}^5_1$ by $\{e_0, e_1, \dots, e_4\}$. The unit timelike vector e_0 time-orients $\mathbb{R}^5_1$. The linear isometry group of $\mathbb{R}^5_1$ is denoted by $\mathrm{O}_1(5)$ and called the *Lorentz group* [27]. The *lightcone* $\mathcal{L}$ of $\mathbb{R}^5_1$ is

$$\mathcal{L} = \{v \in \mathbb{R}^5_1 \mid \langle v, v \rangle = 0,\ v \neq 0\}.$$

The lightcone has two connected components

$$\mathcal{L}_\pm := \{v \in \mathcal{L} \mid \pm \langle e_0, v \rangle < 0\}.$$

These connected components $\mathcal{L}_+$ and $\mathcal{L}_-$ are called the *future lightcone* and *past lightcone*, respectively.

For $v \in \mathcal{L}$ and $r \in \mathbb{R}^\times$, clearly, $rv \in \mathcal{L}$. Thus $\mathbb{R}^\times$ acts freely on $\mathcal{L}$. The quotient $\mathbb{P}(\mathcal{L})$ of $\mathcal{L}$ by the action of $\mathbb{R}^\times$ is called the *projective lightcone*.

The projective lightcone has a conformal structure with respect to which it is conformally equivalent to the unit sphere S^3 with constant curvature 1 metric.

In fact, let us take a unit timelike vector t_0 and set

$$S_{t_0} := \{v \in \mathbb{P}(\mathcal{L}) \mid \langle t_0, v \rangle = -1\}.$$

For $v \in S_{t_0}$, express v as $v = v^\perp + t_0$ so that $v^\perp \perp t_0$. Then

$$0 = \langle v, v \rangle = \langle v^\perp, v^\perp \rangle + \langle t_0, t_0 \rangle = \langle v^\perp, v^\perp \rangle - 1.$$

This implies that the projection $v \mapsto v^\perp$ is an isometry from $S_{t_0} \subset \mathbb{P}(\mathcal{L})$ onto the unit 3-sphere S^3 in the Euclidean 4-space $\mathbb{R}^4 = (\mathbb{R}t_0)^\perp$. This identification induces the following identification:

$$\mathcal{M}^3 \to \mathbb{P}(\mathcal{L});\ v \longmapsto [1 : v]$$

between the Möbius 3-space and the projective lightcone.

More generally, all space forms are realized as conic sections of $\mathcal{L}$. In fact, for a non-zero vector v_0, the section S_{v_0} inherits a Riemannian metric of constant curvature $-\langle v_0, v_0 \rangle$.

Definition 1.1 A diffeomorphism of $\mathcal{M}^3$ is said to be a *Möbius transformation* if it preserves 2-spheres. The Lie group Möb(3) of Möbius transformations is called the *Möbius group*.

Any conformal diffeomorphism of $\mathcal{M}^3$ is a Möbius transformation.
The following result is due to Liouville:

Proposition 1.2 *Let $\phi : U \to V$ be a conformal diffeomorphism between two connected open subsets of $\mathcal{M}^3$. Then there exists a unique Möbius transformation Φ of $\mathcal{M}^3$ such that $\phi = \Phi|_U$.*

The linear action of Lorentz group $\mathrm{O}_1(5)$ on $\mathbb{R}^5_1$ preserves $\mathcal{L}$ and descends to an action on $\mathbb{P}(\mathcal{L})$. For a unit timelike vector t_0 and $T \in \mathrm{O}_1(5)$, T restricts the action to giving an isometry $S_{t_0} \to S_{Tt_0}$ so that the induced transformation on $\mathbb{P}(\mathcal{L})$ is a conformal diffeomorphism. These facts together with Liouville's theorem imply that the sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{O}_1(5) \to \mathrm{M\ddot{o}b}(3) \to 0$$

is exact. Hence $\mathrm{M\ddot{o}b}(3) \cong \mathrm{O}^+_1(5)$, where $\mathrm{O}^+_1(5)$ is the subgroup of $\mathrm{O}_1(5)$ that preserves $\mathcal{L}_\pm$. (See [9, Theorem 1.2, 1.3].)

The *de Sitter* 4*-space*

$$S^4_1 = \{v \in \mathbb{R}^5_1 \mid \langle v, v\rangle = 1\}$$

parametrizes the space of all oriented conformal 2-spheres in $\mathcal{M}^3$. In fact, take a unit spacelike vector $v \in S^4_1$ and denote by V the 1-dimensional linear subspace spanned by v. Then $\mathbb{P}(\mathcal{L} \cap V^\perp)$ is a conformal 2-sphere in $\mathcal{M}^3$. Conversely any conformal 2-sphere can be represented in this form. Via this correspondence, the space of all conformal 2-spheres is identified with $S^4_1/\mathbb{Z}_2$. Viewed as a surface $S_{v_0} \cap V^\perp$ of the conic section S_{v_0}, this conformal 2-sphere has the mean curvature vector $\mathbb{H}_v$,

$$\mathbb{H}_v = -v_0^\perp - \langle v_0^\perp, v_0^\perp\rangle v$$

at v, where v_0 is decomposed as $v_0 = v_0^T + v_0^\perp$ according to the orthogonal direct sum $\mathbb{R}^5_1 = V \oplus V^\perp$.

Let $F : M \to \mathcal{M}^3 = \mathbb{P}(\mathcal{L})$ be a conformal immersion of a Riemann surface into the Möbius 3-space. The *central sphere congruence* (or *mean curvature sphere*) of F is a map $S : M \to S^4_1$ which assigns to each point $p \in M$, the unique oriented 2-sphere $S(p)$ tangent to F at $F(p)$ which has the same orientation to M and the same mean curvature vector $\mathbb{H}_{S(p)} = \mathbb{H}_p$ at $F(p)$ as F. The pull-back $\mathrm{I}_\mathcal{M} := \langle dS, dS\rangle$ of the metric of S^4_1 by the central sphere congruence gives a (possibly singular) metric on M and called the *Möbius metric* of (M, F). The Möbius metric is singular at umbilics. The area functional $\mathcal{A}_\mathcal{M}$ of $(M, \mathrm{I}_\mathcal{M})$ is called the *Möbius area* of (M, F). A conformally immersed surface (M, F) is said to be a *Willmore surface* if it is a critical point of the Möbius area functional.

1.2 The integrability condition

Let $F : M \to \mathcal{M}^3$ be a conformal immersion of a Riemann surface. A *lift* of F is a map $\psi : M \to \mathcal{L}_+$ into the future lightcone such that $\mathbb{R}\psi(p) = F(p)$ for any

$p \in M$. For instance, $\phi := (1, F) : M \to S_{e_0} \subset \mathcal{L}_+$ is a lift of F. This lift is called the *Euclidean lift* of F. Now let ϕ be the Euclidean lift of F. Then for any positive function μ on M, $\phi\mu$ is still a lift of F. Direct computation shows that

$$\langle d(\phi\mu), d(\phi\mu)\rangle_1 = \mu^2 \langle dF, dF\rangle_1,$$

where $\langle\cdot,\cdot\rangle_1$ is the constant curvature 1 metric of $\mathcal{M}^3$. Take a local complex coordinate z. Then the *normalized lift* ψ with respect to z is defined by the relation:

$$\langle d\psi, d\psi\rangle = dz d\bar{z}.$$

This lift is Möbius invariant. For another local complex coordinate $\tilde{z}$, the normalized lift $\tilde{\psi}$ with respect to $\tilde{z}$ is computed as $\tilde{\psi} = \psi|\tilde{z}_z|$.

The normalized lift ψ satisfies the following inhomogeneous Hill equation:

$$\psi_{zz} + \frac{c}{2}\psi = \kappa.$$

Under the coordinate change $z \mapsto \tilde{z}$, the coefficients c and κ are changed as

$$\begin{aligned} \widetilde{\kappa}\frac{d\tilde{z}^2}{|d\tilde{z}|} &= \kappa\frac{dz^2}{|dz|}, \\ \tilde{c}d\tilde{z}^2 &= (c - S_z(\tilde{z}))dz^2, \end{aligned} \tag{1}$$

where $S_z(\tilde{z})$ is the *Schwarzian derivative* of $\tilde{z}$ with respect to z. Here we recall that the Schwarzian derivative $S_z(f)$ of a meromorphic function f on M is defined by

$$S_z(f) := \left(\frac{f_{zz}}{f_z}\right)_z - \frac{1}{2}\left(\frac{f_{zz}}{f_z}\right)^2.$$

Moreover two meromorphic functions f and g are *Möbius equivalent*, i.e., related by a linear fractional transformation:

$$g = \frac{af+b}{cf+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2\mathbb{C}$$

if and only if their Schwarzian derivatives $S_z(f) = S_z(g)$ agree.

Now we denote by L the 1-*density bundle* of M:

$$L := (K \otimes_{\mathbb{C}} \overline{K})^{-1/2}, \ K \text{ is the canonical bundle of } M.$$

The transformation law (1) implies that $\kappa\, dz^2/|dz|$ is a section of $L\,K^2$, i.e., an L-valued quadratic differential on M. This section is called the *conformal Hopf differential* of (M, F). The differential cdz^2 is called the *Schwarzian* of (M, F). The coefficient function c is also called the Schwarzian.

Note that the conformal Hopf differential vanishes identically if and only if M is totally umbilical.

The integrability condition for a conformal immersion $F : M \to \mathcal{M}^3$ is given in terms of κ and c as follows:

$$\begin{cases} \frac{1}{2}c_{\bar{z}} = 3\bar{\kappa}_z\kappa + \bar{\kappa}\kappa_z, \\ \mathrm{Im}\left(\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa\right) = 0. \end{cases} \tag{2}$$

These equations are called the *conformal Gauss equation* and the *conformal Codazzi equation*, respectively.

The Möbius metric $\mathrm{I}_{\mathcal{M}}$ is represented by

$$\mathrm{I}_{\mathcal{M}} = 4|\kappa|^2 dz d\bar{z}. \tag{3}$$

The Euler–Lagrange equation for the Möbius area functional $\mathcal{A}_{\mathcal{M}}$ is called the *Willmore surface equation* and given in terms of κ and c as follows if [11, p. 51]:

$$\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa = 0. \tag{4}$$

1.3 Deformation of surfaces preserving the Schwarzian derivative or the conformal Hopf differential

Generally speaking, the conformal Hopf differential alone determines surfaces in $\mathcal{M}^3$. However, there are the only exceptional surfaces–*isothermic surfaces* [10]. Isothermic surfaces are defined as surfaces in $\mathcal{M}^3$ conformally parametrized by their curvature lines away from umbilics. Away from umbilics, there are holomorphic coordinates in which the conformal Hopf differential is real valued. Such holomorphic coordinates (and their associated real coordinates) are called *isothermic coordinates*.

Now let (M, F) be an isothermic surface parametrized by an isothermic coordinate z. Then under the deformation:

$$c \longrightarrow c_r := c + r, \quad r \in \mathbb{R},$$

the conformal Gauss–Codazzi equations

$$c_{\bar{z}} = 4(\kappa^2)_z, \quad \mathrm{Re}\left(\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa\right) = 0$$

are invariant. Hence, as in the case of CMC surfaces, one obtains a 1-parameter family $\{F_r\}$ of deformations through $F = F_0$ preserving the conformal Hopf differential κ. Since all c_r are distinct, the surfaces $\{F_r\}$ are non-congruent to each other. The family $\{F_r\}$ is referred to as the *associated family* of an isothermic surface (M, F). The correspondence $F \mapsto F_r$ is called the *T-transformation* by Bianchi [3]. The T-transformation was independently introduced by Calapso [12] and also called the *Calapso transformation*.

The existence of deformations preserving the conformal Hopf differential characterizes isothermic surfaces as follows:

Theorem 1.3 ([11]) *A surface in $\mathcal{M}^3$ is isothermic if and only if it has deformations preserving the conformal Hopf differential.*

Corollary 1.4 ([11]) *Let $F_1, F_2 : M \to \mathcal{M}^3$ be two non-congruent surfaces with the same conformal Hopf differential. Then both F_1 and F_2 belong to the same associated family of an isothermic surface.*

On the other hand, for deformations preserving Möbius metric and Schwarzian, we have the following *new* characterization of Willmore surfaces.

Theorem 1.5 *A surface in $\mathcal{M}^3$ is Willmore if and only if it has Möbius-isometric deformations preserving the Schwarzian derivative.*

Proof. Let F be a surface in $\mathcal{M}^3$ with the Schwarzian derivative c and the conformal Hopf differential κ. If F has deformation preserving the Möbius metric $\mathrm{I}_{\mathcal{M}}$ and c, there exists an S^1-valued function λ such that $\lambda\kappa$ and c satisfy the conformal Gauss equation. Combining this with the conformal Gauss equation for F, we have

$$3\bar{\lambda}_z\lambda + \bar{\lambda}\lambda_z = 0, \tag{5}$$

which implies that $\lambda^3\bar{\lambda}$ is holomorphic and hence λ is an S^1-valued constant. Since $\lambda\kappa$ and c satisfy the conformal Codazzi equation, combining this with the conformal Codazzi equation for F, we have

$$\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa = 0, \tag{6}$$

which implies that F is Willmore. □

Remark 1.6 É. Cartan formulated a general theory of deformation of submanifolds in homogeneous spaces. The classical deformation problems (also called *applicability* of submanifolds in classical literatures) in Euclidean, projective and conformal geometry are covered by Cartan's framework [13]–[14].

According to Griffiths [6] and Jensen [24], two immersions $F_1,\ F_2 : M \to G/K$ of a manifold into a homogeneous manifold are said to be *kth order deformation* of each other if there exists a smooth map $g : M \to G$ such that, for every $p \in M$, the Taylor expansions about p of F_2 and $g(p)F_1$ agree through kth order terms. An immersion $F : M \to G/K$ is said to be *deformable of order k* if it admits a non-trivial kth order deformation.

Musso [25] showed that a conformal immersion of a Riemann surface M into the Möbius 3-space is 2nd-order deformable if and only if it is isothermic.

Remark 1.7 (Special isothermic surfaces) Among isothermic surfaces in $\mathbb{R}^3$, Darboux [18] distinguished the class of *special isothermic surfaces*. An isothermic surface $F : M \to \mathbb{R}^3$ with first and second fundamental forms,

$$\mathrm{I} = e^{\omega}(dx^2 + dy^2), \quad \mathbb{I} = e^{\omega}(k_1 dx^2 + k_2 dy^2),$$

is called *special* of type (A, B, C, D) if its mean curvature H satisfies the equation:

$$4e^{\omega}|\nabla H|^2 + m^2 + 2Am + 2BH + 2C\ell + D = 0,$$

where $\ell = 2e^{\omega}\sqrt{H^2 - K}$, $m = -H\ell$ and A, B, C, D are real constants. Constant mean curvature surfaces are particular examples of a special isothermic surface. Special isothermic surfaces with $B = 0$ are conformally invariant. Moreover, Bianchi [2] and Calapso [12] showed that an umbilic free isothermic surface in $\mathcal{M}^3$ is special with $B = 0$ if and only if it is conformally equivalent to a constant mean curvature surface in space forms. For modern treatment of special isothermic surfaces and their Darboux transformations, we refer to [26]. In [1], Bernstein constructed non-special, non-canal isothermic tori in $\mathcal{M}^3$ with spherical curvature lines.

Let $F : M \to \mathcal{M}^3$ be a conformal immersion. Then F is said to be a *constrained Willmore surface* if it is a critical point of the Möbius area functional under (compactly supported) conformal variations.

Proposition 1.8 ([6]) *A compact surface $F : M \to \mathcal{M}^3$ is constrained Willmore if and only if there exists a holomorphic quadratic differential $q dz^2$ such that*

$$\kappa_{\bar{z}\bar{z}} + \frac{1}{2}\bar{c}\kappa = \mathrm{Re}\,(\bar{q}\kappa). \tag{7}$$

The constrained Willmore surface equation (7) has the following deformation:

$$\kappa \longrightarrow \kappa_\lambda := \lambda\kappa,\ c \longrightarrow c_\lambda := c + (\lambda^2 - 1)q,\ q \longrightarrow q_\lambda := \lambda q,$$

for $\lambda \in S^1$.

Hence we obtain a one-parametric conformal deformation family $\{F_\lambda\}$ of a constrained Willmore surface (M, F). This family is referred to as the *associated family* of F.

Obviously, for Willmore surfaces ($q = 0$), the associated family preserves the Schwarzian.

The following characterization of a constrained Willmore surface equation can be verified in a way similar to the proof of Theorem 1.5:

Proposition 1.9 *A surface $F : M \to \mathcal{M}^3$ has a deformation of the form*

$$\kappa \mapsto \lambda\kappa,\ c \mapsto c + r$$

for some S^1-valued function λ and a holomorphic quadratic differential $r dz^2$ if and only if M satisfies (7).

Remark 1.10 A classical result by Thomsen says that a surface is isothermic Willmore if and only if it is minimal in a space form ([8], [23, Theorem 3.6.7], [29]). Constant mean curvature surfaces in space forms are isothermic and constrained Willmore. Richter [28] showed that in the case of immersed tori in $\mathcal{M}^3$, all isothermic constrained Willmore tori are constant mean curvature tori in some space forms. In contrast to Thomsen's result, the assumption "tori" is essential for Richter's result. In fact, Burstall constructed isothermic constrained Willmore cylinders which are not realized as constant mean curvature surfaces in any space forms. See [6].

2 Deformation of surfaces preserving similarity invariants

As we saw in the preceding section, preservation of conformal Hopf differentials is a strong restriction in the study of deformation of surfaces. Clearly, preservation of the Möbius metric is weaker than that of the conformal Hopf differential. In this section we study deformation of surfaces preserving the Möbius metric.

2.1 Möbius invariants via metrical language

First, we discuss relations between metrical invariants and Möbius invariants.

Let $F : M \to \mathbb{R}^3$ be a conformal immersion of a Riemann surface into the Euclidean 3-space. Denote by I the first fundamental form (induced metric) of M. The Levi-Civita connections D of $\mathbb{R}^3$ and ∇ of M are related by the *Gauss equation*:

$$D_X F_* Y = F_*(\nabla_X Y) + \mathrm{I\!I}(X, Y)n.$$

Here n is the unit normal vector field. The symmetric tensor field $\mathrm{I\!I}$ is the *second fundamental form* derived from n.

The trace free part of the second fundamental form is given by $\mathrm{I\!I} - H\mathrm{I}$, where H is the mean curvature function. Define a function h by $h := \sqrt{H^2 - K}$. This function h is called the *Calapso potential*.

Then one can check that the normal vector field n/h and the symmetric tensor field h^2 I are invariant under the conformal change of the ambient Euclidean metric. Moreover the trace free symmetric tensor field

$$\mathrm{I\!I}_{\mathcal{M}} := h(\mathrm{I\!I} - H\mathrm{I})$$

is also conformally invariant. It is easy to see that h^2 I coincides with the *Möbius metric* $\mathrm{I}_{\mathcal{M}}$ of (M, F). The pair $(\mathrm{I}_{\mathcal{M}}, \mathrm{I\!I}_{\mathcal{M}})$ is called *Fubini's conformally invariant fundamental forms*. The Gaussian curvature $K_{\mathcal{M}}$ of $(M, \mathrm{I}_{\mathcal{M}})$ is called the *Möbius curvature* of (M, F). The Möbius area functional $\mathcal{A}_{\mathcal{M}}$ of $(M, \mathrm{I}_{\mathcal{M}})$ is computed as

$$\mathcal{A}_{\mathcal{M}} = \int_M (H^2 - K) dA_{\mathrm{I}}.$$

Now let us take a local complex coordinate z and express the first fundamental form as $\mathrm{I} = e^{\omega} dz d\bar{z}$. The (metrical) *Hopf differential* is defined by

$$Q^{\#} := Q dz^2, \ Q = \langle F_{zz}, n \rangle.$$

Then the conformal Hopf differential and the metric one are related by the formula:

$$\kappa = Q e^{-\omega/2}. \tag{2.1}$$

The Schwarzian derivative is represented as

$$c = \omega_{zz} - \frac{1}{2}(\omega_z)^2 + 2HQ.$$

2.2 Similarity geometry

The similarity geometry is a subgeometry of Möbius geometry whose symmetry group is the *similarity transformation group*:

$$\mathrm{Sim}(3) = \mathrm{CO}(3) \ltimes \mathbb{R}^3,$$

where CO(3) is the linear conformal group

$$\mathrm{CO}(3) = \{A \in \mathrm{GL}_3\mathbb{R} \mid {}^{\exists}c \in \mathbb{R};\ {}^{t}AA = cE\ \}.$$

Let $F : M \to \mathbb{R}^3$ be an immersed surface with unit normal n as before.

Under the similarity transformation of $\mathbb{R}^3$, Levi-Civita connections D and ∇ are invariant. Hence the vector-valued second fundamental form $\mathbb{I}n$ is similarity invariant. The shape operator $S = -dn$ itself is not similarity invariant, but the ratio of principal curvatures are invariant. It is easy to see that the constancy of the ratio of principal curvatures is equivalent to the constancy of K/H^2. The function K/H^2 is similarity invariant. The principal directions are yet another similarity invariant.

2.3 Deformation of surfaces preserving the Möbius metric and the ratio of principal curvatures

Let $F : M \to \mathbb{R}^3$ be a surface in Euclidean 3-space. Then the Gauss–Codazzi equations of (M, F) are given by

$$\begin{cases} \omega_{z\bar{z}} + \frac{1}{2}H^2 e^{\omega} - 2|Q|^2 e^{-\omega} = 0, \\ Q_{\bar{z}} = \frac{1}{2}H_z e^{\omega}. \end{cases} \tag{2.2}$$

The Gauss–Codazzi equations imply the following fundamental fact due to Bonnet.

Proposition 2.1 ([7]) *Every non-totally umbilical constant mean curvature surface admits a one-parameter isometric deformation preserving the mean curvature.*

Here we exhibit two examples of surfaces which admit deformations preserving the Möbius metric.

Example 2.2 (Bonnet surfaces) Let $F : M \to \mathbb{R}^3$ be a Bonnet surface. Namely (M, F) admits a non-trivial isometric deformation $F \mapsto F_\lambda$ preserving the mean curvature. The deformation family $\{F_\lambda\}$ is called the *associated family* of (M, F).

Since all the members F_λ have the same metric and mean curvature, they have the same Möbius metric. Note that the conformal Hopf differential is not preserved under the deformation.

Example 2.3 (HIMC surfaces) A surface $F : M \to \mathbb{R}^3$ is said to be a *surface with harmonic inverse mean curvature* (*HIMC surface*, in short) if its inverse mean curvature function $1/H$ is a harmonic function on M [4]. Since $1/H$ is harmonic, H can be expressed as $1/H = h + \bar{h}$ for some holomorphic function h. The associated family $\{F_\lambda\}$ of a HIMC surface F is given by the following metric data $(\mathrm{I}_\lambda, H_\lambda, Q_\lambda)$:

$$\mathrm{I}_\lambda = e^{\omega_\lambda} dz d\bar{z}, \quad e^{\omega_\lambda} = \frac{e^{\omega}}{(1 - 2\sqrt{-1}\,\bar{h}t)^2 (1 + 2\sqrt{-1}ht)^2},$$

$$\frac{1}{H_\lambda} = h_\lambda + \overline{h_\lambda}, \quad h_\lambda = \frac{h}{1 + 2\sqrt{-1}ht},$$

$$Q_\lambda = \frac{Q}{(1 + 2\sqrt{-1}ht)^2}, \quad \lambda = \frac{1 - 2\sqrt{-1}\bar{h}t}{1 + 2\sqrt{-1}ht}, \quad t \in \mathbb{R}.$$

From these, we have

$$(H_\lambda^2 - K_\lambda) = (1 - 2\sqrt{-1}\bar{h}t)(1 + 2\sqrt{-1}ht)(H^2 - K),$$

Hence

$$(H_\lambda^2 - K_\lambda) e^{\omega_\lambda} = (H^2 - K) e^{\omega}.$$

Thus the Möbius metric is preserved under the deformation $F \mapsto F_\lambda$. On the other hand, the conformal Hopf differential is not preserved under the deformation. In fact, the conformal Hopf differential of F_λ is

$$\kappa_\lambda := Q_\lambda e^{-\frac{\omega_\lambda}{2}} = \kappa\, \frac{1 - 2\sqrt{-1}\bar{h}t}{1 + 2\sqrt{-1}ht}.$$

Clearly κ_λ is not preserved under the deformation.

While Bonnet surfaces are isothermic, HIMC surfaces are not necessarily so. The dual surfaces of Bonnet surfaces are isothermic HIMC surfaces. Since the associated families of Bonnet's surfaces or isothermic HIMC surfaces do not preserve the conformal Hopf differential, these families differ from the T-transformation families. Note that T-transformations are only well defined up to Möbius transformations [9, section 2.2.3].

Now we prove the following theorem which characterizes Bonnet surfaces and HIMC surfaces in the class of surfaces which posses Möbius metric preserving deformations. We call such surfaces *Möbius applicable surfaces*.

Theorem 2.4 *Let F be a surface in $\mathbb{R}^3$ which has deformation preserving the Möbius metric and the ratio of principal curvatures. Then the deformation is given by*

$$e^{\omega} \to |\lambda|^2 e^{\omega}, \ H \to \frac{1}{|\lambda|} H, \ Q \to \lambda Q, \tag{2.3}$$

where λ is a function with $|\lambda| = |f|$ for some holomorphic function f. Moreover if $|\lambda| = 1$ (respectively λ is holomorphic), then F is a Bonnet surface (respectively a HIMC surface).

Proof. Note that the quantities $|Q|^2e^{-\omega}$ and $e^{-\omega}/H^2$ are invariant under the deformation, which implies that the deformation is given as above for some function λ (see (2.1)). From the Gauss equation we have

$$(\log|\lambda|^2)_{z\bar{z}} = 0,$$

which implies that $|\lambda| = |f|$ for some holomorphic function f.

If $|\lambda| = 1$, the deformation is nothing but the isometric deformation preserving the mean curvature. Hence F is a Bonnet surface.

If λ is holomorphic, putting $(H')^2 = H^2/|\lambda|^2$ and differentiating it by z, we have

$$2H'H'_z = -\frac{\bar{\lambda}\lambda_z}{|\lambda|^4}H^2 + \frac{2}{|\lambda|^2}HH_z.$$

Note that $Q \neq 0$ since F is umbilic-free. Combining the Codazzi equations for F and the surface obtained by deformation, we have

$$H'_z = \frac{1}{\bar{\lambda}}H_z. \tag{2.4}$$

Hence we have

$$H' = -\frac{\lambda_z}{2\lambda^2}\frac{H^2}{H_z} + \frac{1}{\lambda}H.$$

Differentiating it by $\bar{z}$ and using (2.4) again, we have

$$H_{z\bar{z}} - \frac{2|H_z|^2}{H} = 0,$$

which implies that F is a HIMC surface. □

2.4 Flat Bonnet surfaces

Let M be a Bonnet surface in $\mathbb{R}^3$. Then away from umbilics, there exists an isothermic coordinate z such that the Gauss–Codazzi equations of M reduce to the following third-order ordinary differential equation (*Hazzidakis equation* [22]):

$$\left\{\left(\frac{H_{ss}}{H_s}\right)_s - H_s\right\}R^2 = 2 - \frac{H^2}{H_s}, \quad H_s < 0, \tag{2.5}$$

where $s = z + \bar{z}$ and the coefficient function $R(s)$ is one of the following functions [5, p. 30]:

$$R_A(s) = \frac{\sin(2s)}{2}, \quad R_B(s) = \frac{\sinh(2s)}{2}, \quad R_C(s) = s.$$

The modulus $|Q|$ of the metrical Hopf differential Qdz^2 is given by

$$|Q(z,\bar{z})| = \frac{1}{R(s)^2}. \tag{2.6}$$

A Bonnet surface is said to be of *type* A, B or C, respectively, if away from critical points of the mean curvature, it is determined by a solution to a Hazzidakis equation with coefficient R_A, R_B or R_C ([5, Definition 3.2.1], [15]).

Proposition 2.5 ([5], [20]) *Flat Bonnet surfaces in* $\mathbb{R}^3$ *are of* C*-type.*

Flat Bonnet surfaces are characterized as follows in terms of conformal (Möbius) or similarity invariants.

Theorem 2.6 *A Bonnet surface in* $\mathbb{R}^3$ *with non-constant mean curvature is flat if the Möbius curvature or the ratio of the principal curvatures is constant.*

Proof. First we consider Bonnet surfaces with constant ratio of principal curvatures. By the assumption the function K/H^2 is constant. Computing K/H^2 by using (2.5) and (2.6), one can deduce that $K = 0$ if K/H^2 is constant.

Next, the Möbius curvature $K_{\mathcal{M}}$ is computed as

$$K_{\mathcal{M}} = \frac{1}{H_s}(\log H_s)_{ss}$$

by using the Hazzidakis equation (2.5).

If $K_{\mathcal{M}}$ is constant, a direct computation shows that the solution of (2.5) is

$$H = -\frac{2}{K_{\mathcal{M}}}\frac{1}{s}$$

with $K_{\mathcal{M}} < 0$. Hence the surface is flat. □

Appendices

A.1 Curves in similarity geometry

Let us consider plane curve geometry in the 2-dimensional similarity geometry ($\mathbb{R}^2$, Sim(2)). Here Sim(2) denotes the similarity transformation group of $\mathbb{R}^2$.

Let $\gamma(s)$ be a regular curve on $\mathbb{R}^2$ parametrized by the Euclidean arclength σ. Then the Sim(2)-invariant parameter s is the *angle function* $s = \int^{\sigma} \kappa_E(s)d\sigma$, where κ_E is the Euclidean curvature function. The Sim(2)-invariant curvature κ_S is given by $\kappa_S = (\kappa_E)_\sigma/\kappa_E^2$. Obviously, every circle is a curve of similarity curvature 0. The Sim(2)-invariant frame field $\mathcal{F} = (T, N)$ is given by

$$T = \gamma_s, \;\; N = T_s + \kappa T.$$

The Frenet–Serret equation of $\mathcal{F}$ is

$$\mathcal{F}^{-1}\frac{d\mathcal{F}}{ds} = \begin{pmatrix} -\kappa_S & -1 \\ 1 & -\kappa_S \end{pmatrix}.$$

Now let us consider plane curves of nonzero constant similarity curvature.

Put $\kappa_S = c_1$ (constant). Then we have $1/\kappa_E = (-c_1)\sigma + c_2$, namely γ is a curve whose inverse Euclidean curvature $1/\kappa_E$ is a linear function of the Euclidean arclength parameter. Thus γ is a log-spiral (if $c_1 \neq 0$) or a circle ($c_1 = 0, c_2 \neq 0$).

These curves provide fundamental examples of Bonnet surfaces as well as HIMC surfaces. In fact, let γ be a plane curve of constant similarity curvature. Then a cylinder over γ is a flat Bonnet surface in $\mathbb{R}^3$ as well as a flat HIMC surface in $\mathbb{R}^3$. Generally, the Hazzidakis equation of Bonnet or isothermic HIMC surfaces reduces to Painlevé equations of type III, V or VI. The solutions to a log-spiral cylinder are elementary function solutions to these Painlevé equations. (see [5], [20]).

A.2 Time evolutions

Let us consider the time evolution of a plane curve $\gamma(s)$ in similarity geometry.

Denote by $\gamma(s;t)$ the time evolution which preserves the similarity arclength parameter s;

$$\frac{\partial}{\partial t}\gamma(s;t) = gN + fT.$$

Then the similarity curvature $u = \kappa_S$ obeys the following partial differential equation:

$$u_t = f_{sss} - 2uf_{ss} - (3u_s - u^2 - 1)f_s - (u_{ss} - 2uu_s)f + au_s, \quad a \in \mathbb{R}.$$

In particular, if we choose $f = -1, a = 0$, then the time evolution of κ obeys the *Burgers equation*:

$$u_t = u_{ss} - 2uu_s.$$

More generally, the Burgers hierarchy is induced by the above time evolution, see [16, pp. 17–18]. Space curves in similarity geometry and their time evolution, we refer to [17].

References

1. H. Bernstein, Non-special, non-canal isothermic tori with spherical lines of curvature, Trans. Amer. Math. Soc. **353** (2000), no. 6, 2245–2274.
2. L. Bianchi, Ricerche sulle superficie isoterme e sulle deformazione delle quadriche, Annali di Mat. **11** (1905), 93–157.
3. L. Bianchi, Complementi alle ricerche sulle superficie isoterme, Annali di Mat. **12** (1905), 20–54.
4. A. I. Bobenko, Surfaces in terms of 2 by 2 matrices. Old and new integrable cases, in: *Harmonic maps and integrable systems*, 83–127, Aspects Math., E23, Vieweg, Braunschweig, 1994.
5. A. I. Bobenko and U. Eitner, *Painlevé Equations in the Differential Geometry of Surfaces*, Lecture Notes in Math. **1753** (2000), Springer Verlag.
6. C. Bohle, G. P. Peters and U. Pinkall, Constrained Willmore surfaces, preprint, math.DG/0411479.
7. O. Bonnet, Mémoire sur la théorie des surfaces applicables sur une surface donnee, J. l'École Pol. Paris **42** (1867) Cahier 72–92.
8. W. Blaschke, *Vorlesungen über Differentialgeometrie* III: Differentialgeometrie der Kreise und Kugeln. Grundlehren **29**, Springer Verlag, Berlin, 1929.

9. F. E. Burstall, Isothermic surfaces: conformal geometry, Clifford algebras and integrable systems, in: *Integrable Systems, Geometry and Topology*, pp. 1–82, AMS/IP Stud. Adv. Math. **36**, Amer. Math. Soc., Providence, RI, 2006.
10. F. E. Burstall and D. M. J. Calderbank, Submanifold geometry in generalized flag manifolds, Rend. Circ. Mat. Palermo (2) Suppl. **72** (2004) 13-41.
11. F. Burstall, F. Pedit and U. Pinkall, Schwarzian derivatives and flows of surfaces, in: *Differential Geometry and Integrable Systems* (Tokyo, 2000), pp. 39–61, Contemp. Math. **308**, Amer. Math. Soc., Providence, RI, 2002.
12. P. Calapso, Sulle transformazioni delle superficie isoterme, Annali di Mat. **24** (1915), 11–48.
13. É. Cartan, Sur le problème général de la déformation, C. R. Congrés Strasbourg (1920), 397–406. Reprinted as: Oeuvres Complètes III 1, pp. 539–548.
14. É. Cartan, Sur la déformation projective des surfaces, Ann. Scient. Éc. Norm. Sup. (3) **37** (1920), 259–356, Reprinted as: Oeuvres Complètes III 1, pp. 441–538.
15. É. Cartan, Sur les couples de surfaces applicables avec conservation des courbures principales, Bull. Sci.Math. **66** 1942 55–85 Reprinted as: Oeuvres Complètes III 1, pp.1591–1620.
16. K. S. Chou and C. Z. Qu, Integrable equations arising from motions of plane curves, Physica D **162** (2002), 9–33.
17. K. S. Chou and C. Z. Qu, Motions of curves in similarity geometries and Burgers-mKdV hierarchies, Chaos, Solitons and Fractals **19** (2004), 47–53.
18. G. Darboux, Sur une classe des surfaces isothermiques liées à la deformations dea surfaces du second degré, C. R. Acad. Sci. Paris **128** (1899), 1483–1487.
19. A. Fujioka, Surfaces with harmonic inverse mean curvature in space forms, Proc. Amer. Math. Soc. **127** (1999), no. 10, 3021–3025.
20. A. Fujioka and J. Inoguchi, Bonnet surfaces with constant curvature, Results Math. **33** (1998), no. 3-4, 288–293.
21. P. A. Griffiths, On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, Duke Math. J. **41** (1974), 775–814.
22. J. N. Hazzidakis, Biegung mit Erhaltung der Hauptkrümmungsradien, J. reine Angew. Math. **117** (1897), 42–56.
23. U. Hertrich-Jeromin, *Introduction to Möbius Differential Geometry*, London Math. Soc. Lecture Note Series **300**, Cambridge Univ. Press, 2003.
24. G. R. Jensen, Deformation of submanifolds of homogeneous spaces, J. Diff. Geom. **16** (1981), 213–246.
25. E. Musso, Deformazione superficie nello spazio di Möbius, Rend. Ist. Mat. Univ. Trieste **27** (1995), 25–45.
26. E. Musso and L. Nicolodi, Special isothermic surfaces and solitons. in: *Global Differential Geometry: The Mathematical Legacy of Alfred Gray* (Bilbao, 2000), pp. 129–148, Contemp. Math., **288**, Amer. Math. Soc., Providence, RI, 2001.
27. B. O'Neill, *Semi-Riemannian Geometry with Application to Relativity*, Pure and Applied Math., vol. 130, Academic Press, Orlando, 1983.
28. J. Richter, *Conformal maps of a Riemann surface into the space of quaternions*, Ph. D. Thesis, TU-Berlin, 1997.
29. G. Thomsen, Über konforme Geometrie I: Grundlagen der konformen flächentheorie, Hamb. Math. Abh. **3** (1923), 31–56.

Global Structures of Compact Conformally Flat Semi-Symmetric Spaces of Dimension 3 and of Non-Constant Curvature

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Summary. Let (M, g) be a compact connected locally conformally flat semi-symmetric space of dimension 3 and with principal Ricci curvatures $\rho_1 = \rho_2 \neq \rho_3 = 0$. Then M is a Seifert fibre space. Moreover, in case the holonomy group is discrete, M is commensurable to a Kleinian manifold. If the holonomy group is indiscrete, $(M, \overline{g})$ is a hyperbolic surface bundle over a circle and (M, g) has negative scalar curvature. Here $\overline{g}$ denotes a metric induced from the flat conformal structure.

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1 Introduction

The aim of this note is to investigate global structures of compact connected conformally flat semi-symmetric spaces of dimension 3 and of non-constant curvature, using the method of W. Thurston's geometric structures in [19]. We determine such spaces completely.

A semi-symmetric space is a smooth Riemannian manifold (M, g) with the curvature tensor R satisfying the identity $R(X, Y) \cdot R = 0$ for all vector fields $X,\ Y$ on M, where $R(X, Y)$ acts as a derivation on R. The condition implies that, at each point p, R_p is the same as the curvature tensor of a symmetric space (which may change with the point).

A motivation of the present study is the following. In [2] p. 179, there is a problem asking if there exist compact semi-symmetric spaces of dimension $n \geqslant 3$ which are locally irreducible and not locally symmetric. This is the compact version of Nomizu's conjecture in [12]. Most investigations of compact Riemannian manifolds would belong to geometry in the large. As a matter of fact, complete Riemannian manifolds have

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no cones. So, we consider, as a geometry in the large, compact semi-symmetric spaces except for cases of constant sectional curvature. In this note, we restrict ourselves to cases of dimension 3 that are (locally) conformally flat.

The first work on global structures of compact conformally flat manifolds goes back to Kuiper's around 1949, see [9] and [10]. Since Thurston's lectures on the geometry and topology of 3-manifolds, flat conformal structures on compact manifolds have been studied extensively by many authors, especially, in the field of topology, cf. [1], [3], [6] and [11]. In the study of conformally flat n-manifolds, developing maps and homomorphisms from fundamental groups into Möbius groups of S^n form the most important invariants. The image of the fundamental group under the homomorphism is called the *holonomy group*. Limit sets of holonomy groups give rise to distinctions on global structures of compact conformally flat manifolds. Hence our work in this note concerns the study of limit sets of holonomy groups.

Semi-symmetric spaces have been investigated by many authors since E. Cartan's work on symmetric spaces in the middle 1940s. In 1982, Z. I. Szabó gave the full local classification of semi-symmetric spaces. He proved that a semi-symmetric space is locally a de Rham product of irreducible semi-symmetric spaces. However, he did not give explicit expressions for the metric of such spaces. So, in 1996 O. Kowalski studied the class of foliated semi-symmetric spaces in dimension 3. He solved the partial differential equations to give explicit descriptions of spaces in the class, and he classified them. The notion 'semi-symmetricity' has already been generalized as 'pseudo-symmetricity'. Most of those researches belong to the local geometry.

Recently, G. Calvaruso classified in [20] the class of conformally flat semi-symmetric spaces. He proved that a conformally flat semi-symmetric space M (of dimension $n > 2$) is either locally symmetric or it is locally irreducible and isometric to a semi-symmetric real cone.

Let M be a connected, locally conformally flat, semi-symmetric space of dimension 3 and with principal Ricci curvatures $\rho_1 = \rho_2 \neq \rho_3 = 0$. When M is complete, we see in Section 2 that the universal covering of M is the Riemannian direct product of a 2-dimensional space of constant curvature and a line $\mathbb{R}$. In case M is, further, compact, then we see that M turns out to be a Seifert fibre space (Theorem 2.6). By looking into holonomy groups, we consider the cases that the holonomy group is discrete or indiscrete, separately. We obtain the following theorems:

Theorem 5.3 *Let M be a compact, locally conformally flat, semi-symmetric space of dimension 3 and with the principal Ricci curvatures $\rho_1 = \rho_2 \neq \rho_3 = 0$. Suppose that the holonomy group is discrete. Then M is commensurable to a Kleinian manifold.*

Theorem 6.6 *Let (M, g) be a compact, locally conformally flat, semi-symmetric space of dimension 3 and with the principal Ricci curvatures $\rho_1 = \rho_2 \neq \rho_3 = 0$. Suppose that the holonomy group is indiscrete. Then the developing map is a homeomorphism onto $S^3 \setminus S^1$ and $(M, \overline{g})$ is a hyperbolic surface bundle over S^1. Here $\overline{g}$ denotes a metric induced from the flat conformal structure.*

Corollary 6.7 *Let (M, g) be as in the above Theorem 6.6. Suppose that the holonomy group is indiscrete. Then (M, g) has negative scalar curvature.*

In this note we always assume that manifolds are smooth, connected and without boundary. Also we assume that dimensions of manifolds are greater than or equal to 3 unless mentioned otherwise.

The contents of the paper is as follows: In Section 2 we give preliminaries for semi-symmetric spaces. In Section 3, we recall the definition of geometric structure and some basic notions. In Section 4, we introduce limit sets of holonomy groups. The proofs of the above theorems are in Sections 5 and 6.

For the sake of completeness, brief proofs of some known results are included.

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2 Preliminaries

Let (M, g) be a three-dimensional semi-symmetric space. Let Ric be the Ricci form of (M, g) and Q the field of symmetric endomorphisms satisfying $\mathrm{Ric}(X, Y) = g(QX, Y)$ for vector fields X and Y on M. Since M is of dimension 3, the curvature tensor R of (M, g) is given by

$$R(X, Y) = QX \wedge Y + X \wedge QY - \frac{\mathrm{trace}\, Q}{2} X \wedge Y \tag{1}$$

for all vector fields X and Y. At each point of M we may choose an orthonormal basis $\{e_1, e_2, e_3\}$ such that $Qe_i = \lambda_i e_i$ for $i = 1, 2, 3$. Then we have $g(R(e_i, e_j)e_k, e_h) = 0$ whenever at least three of the indices i, j, k and h are distinct. Hence we can see that one of the following three cases occurs:

$$\begin{aligned} &\lambda_1 = \lambda_2 = \lambda_3 = \lambda, \quad \lambda \neq 0; \\ &\lambda_1 = \lambda_2 = \lambda, \quad \lambda_3 = 0, \quad \lambda \neq 0; \\ &\lambda_1 = \lambda_2 = \lambda_3 = 0. \end{aligned}$$

It is known that, if the rank of the Ricci form is 3 at least at one point of M, then (M, g) is a space of constant curvature, cf [15]. And, if $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then M is flat. Next we shall assume that the rank of the Ricci form is 2 at any point of M. Namely, we consider the case that the principal Ricci curvatures $\lambda_1 = \lambda_2 = \lambda \neq 0$ and $\lambda_3 = 0$ everywhere. We may assume that M is orientable, by taking the orientable double covering space of M if necessary.

In [8], O. Kowalski proved the following

Proposition 2.1 *Let (M, g) be a 3-dimensional semi-symmetric space with the principal Ricci curvatures $\lambda_1 = \lambda_2 = \lambda \neq 0$ and $\lambda_3 = 0$ everywhere. Then, in a normal coordinate neighborhood U of any point p, there exists a local coordinate system $(U; x, y, t)$ such that*

$$g = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$$

where

$$\omega^1 = f_1(x, y, t)dx, \quad \omega^2 = f_2(x, y, t)dy + q(x, y, t)dx, \quad \omega^3 = dt + h(x, y)dx$$

and $f_1 f_2 \neq 0$. *Furthermore, it follows that the equations* $\omega^1 = \omega^2 = 0$ *determine the principal directions of zero Ricci curvature and the corresponding integral curves in* (U, g) *are geodesics; and the variable* t *measures the arc-length along any geodesic of this family.*

Let $(U; x, y, t)$ be the local coordinate system as in Proposition 2.1 and $\{E_1, E_2, E_3\}$ the local orthonormal frame dual to the coframe $\{\omega^1, \omega^2, \omega^3\}$. Then E_i, $i = 1, 2, 3$, are vector fields of eigenvectors of the Ricci operator Q corresponding to the eigenvalues λ_i, respectively. The Levi-Civita connection ∇ of (M, g) is given by

$$\nabla_{E_1}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix} = \begin{pmatrix}0 & \frac{-f_1{}'_y}{f_1 f_2} & -a\\ \frac{f_1{}'_y}{f_1 f_2} & 0 & -c\\ a & c & 0\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix},$$

$$\nabla_{E_2}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix} = \begin{pmatrix}0 & -\alpha & -b\\ \alpha & 0 & -e\\ b & e & 0\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix},$$

$$\nabla_{E_3}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix} = \begin{pmatrix}0 & -b & 0\\ b & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}E_1\\E_2\\E_3\end{pmatrix},$$

where

$$a = \frac{f_1{}'_t}{f_1}, \quad b = \frac{1}{2 f_1 f_2}(h'_y + f_2 q'_t - q f_2{}'_t), \quad c = b - \frac{h'_y}{f_1 f_2}, \quad e = \frac{f_2{}'_t}{f_2},$$

and

$$\alpha = \frac{-1}{f_1 f_2}(f_2{}'_x - q'_y - h f_2{}'_t).$$

Notice that the last identity implies that the integral curves of the vector field E_3 are geodesics. With respect to the basis $\{E_1, E_2, E_3\}$, we have

$$R(E_1, E_2) = \lambda E_1 \wedge E_2, \quad R(E_1, E_3) = 0, \quad R(E_2, E_3) = 0. \tag{2}$$

If (M, g) is locally conformally flat, the following identity holds:

$$(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{4}(X(\text{trace } Q)Y - Y(\text{trace } Q)X) \tag{3}$$

for all vector fields X, Y on M.

Taking X, Y in (3) from $\{E_1, E_2, E_3\}$ and using $QE_i = \lambda_i E_i$, we have

$$E_1\lambda = 0, \quad E_2\lambda = 0, \quad h'_y = 0, \quad b = 0, \quad c = 0 \tag{4}$$

and

$$E_3\lambda + 2\lambda a = 0, \quad E_3\lambda + 2\lambda e = 0. \tag{5}$$

By $\lambda \neq 0$, it follows that $a = e$. Because of $R(E_1, E_3)E_3 = 0$ and (5), we have

$$E_3 a + a^2 = 0. \tag{6}$$

Hence we have $a = 1/(t + c)$ for some constant c. Thus, we have $\lambda = c'/(t + c)^2$, where c and c' are some constants. If we, furthermore, assume that M is complete, then the integral curve of E_3 is infinitely extendible and $\lambda(t)$ must be defined for any t along the integral curve of E_3. But, if $a \neq 0$, $\frac{1}{\lambda}$ will be 0 for $t = -c$, which is a contradiction. Thus, $a = e = 0$ and $f_i = f_i(x, y)$ for $i = 1, 2$. Also $q = q(x, y)$. In the case when $a = 0$, it follows that λ is constant. Since $h'_y = 0$, applying a similar argument as in the proof of Theorem 7.10 in [8], we may assume that $h = 0$. Thus, summarizing the above argument, we obtain

Proposition 2.2 *Let M be a 3-dimensional complete, locally conformally flat, semi-symmetric space with the principal Ricci curvatures $\lambda_1 = \lambda_2 = \lambda \neq 0$ and $\lambda_3 = 0$ everywhere and $\widetilde{M}$ its universal covering space. Then $\widetilde{M}$ is a Riemannian product space of a 2-dimensional space of constant curvature and a line $\mathbb{R}$.*

Applying G. Calvaruso's work, Proposition 4.3 in [20], we see that M in the above Proposition 2.2 is locally symmetric.

Corollary 2.3 *Let M be a compact locally conformally flat 3-dimensional semi-symmetric space with the principal Ricci curvatures $\lambda_1 = \lambda_2 = \lambda \neq 0$ and $\lambda_3 = 0$ everywhere. Then the fundamental group $\pi_1(M)$ is infinite.*

Now, we shall define a Seifert fibre space.

Definition 2.4 A 3-manifold M is called a *Seifert fibre space* if it has a decomposition into disjoint circles, called fibres, such that each circle has a neighborhood in M which is a union of fibres and is isomorphic to a fibred solid torus or a Klein bottle.

For compact 3-manifolds, a manifold is a Seifert fibre space if and only if it is foliated by circles.

Let M be a compact, locally conformally flat, semi-symmetric space of dimension 3 and with the principal Ricci curvatures $\lambda_1 = \lambda_2 = \lambda \neq 0$ and $\lambda_3 = 0$ everywhere. Let $\widetilde{M}$ be the universal covering space of M. Then $\widetilde{M}$ is $N(k) \times \mathbb{R}$ (the Riemannian direct product), where $N(k)$ is a 2-dimensional manifold of constant curvature k. The group of isometries of $N(k) \times \mathbb{R}$, denoted by $I(N(k) \times \mathbb{R})$, can be identified with $I(N(k)) \times I(\mathbb{R})$ and the factors are regarded as subgroups naturally. Let G be the discrete subgroup of $I(N(k) \times \mathbb{R})$ which acts freely and has quotient M. If we set $K := G \cap I(\mathbb{R})$, then K is discrete and must be $\{1\}, \mathbb{Z}, \mathbb{Z}_2$ or $\mathbb{Z}_2 \ltimes \mathbb{Z}$. As G acts freely on $N(k) \times \mathbb{R}$, it is torsion free. So K cannot be $\mathbb{Z}_2$. Let $\phi : G \to I(N(k))$ be the projection, and Γ the image of ϕ. Then, as K is normal in G and is the kernel of ϕ, we have the exact sequence

$$1 \to K \to G \to \Gamma \to 1.$$

If K is $\mathbb{Z}$ or $\mathbb{Z}_2 \ltimes \mathbb{Z}$, each line $\{x\} \times \mathbb{R}$ covers a circle in $(N(k) \times \mathbb{R})/K$. Thus we have

Proposition 2.5 *Let M be a compact, locally conformally flat, semi-symmetric space of dimension 3 and with the principal Ricci curvatures $\lambda_1 = \lambda_2 = \lambda \neq 0$ and $\lambda_3 = 0$ everywhere. Let G be a discrete subgroup of $I(N(k) \times \mathbb{R})$ which acts freely and has quotient M. If $K = G \cap I(\mathbb{R})$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_2 \ltimes \mathbb{Z}$, then the natural foliation of $N(k) \times \mathbb{R}$ by lines descends to a Seifert bundle structure on M.*

When M is compact, K cannot be a finite group. Consequently, we have

Theorem 2.6 *Let M be a compact, locally conformally flat, semi-symmetric space of dimension 3 and with the principal Ricci curvatures $\lambda_1 = \lambda_2 = \lambda \neq 0$ and $\lambda_3 = 0$ everywhere. Then M is a Seifert fibre space. Furthermore, let G be the discrete subgroup of $I(N(k) \times \mathbb{R})$ which acts freely and has quotient M. Then $G \cap I(\mathbb{R})$ is $\mathbb{Z}$ or $\mathbb{Z}_2 \ltimes \mathbb{Z}$.*

3 Geometric structures

We say two metrics g and g_1 on a manifold M are *(pointwise) conformal* if $g = f(x)g_1$ for some positive smooth function f on M. A smooth map $\psi : (M, g) \to (M_1, g_1)$ is called a *conformal map* if g is (pointwise) conformal to $\psi^* g_1$.

We shall recall the definition of a geometric structure originally defined by W. Thurston. We refer to, say [19], [1] or [3] for more details.

Definition 3.1 Let X be a real analytic manifold of dimension $n \geqslant 3$ and G a Lie group acting on X faithfully, analytically and transitively. Let M be a paracompact smooth manifold, $\{U_\lambda\}_{\lambda \in \Lambda}$ a collection of open sets of M and $\varphi_\lambda : U_\lambda \to X$ an open embedding into X . Then $\{(U_\lambda, \varphi_\lambda)\}_{\lambda \in \Lambda}$ is called an (X, G)-*atlas* if it satisfies the following conditions:

1) $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open covering of M, and $U_\lambda \cap U_\mu$ is connected if it is non-empty;
2) If $U_\lambda \cap U_\mu \neq \emptyset$, then there exists ψ of G such that $\varphi_\mu \circ \varphi_\lambda^{-1}|_{\varphi_\lambda(U_\lambda \cap U_\mu)}$ is the restriction of ψ.

A maximal (X, G)-atlas is called an (X, G)-*structure* on M. If M has an (X, G)-structure, we say M is *modeled* on the pair (X, G), or M is an (X, G)-*manifold*.

Two (X, G)-atlases on M are called *equivalent* if their union is an (X, G)-atlas.

When $X = S^n = \mathbb{R}^n \cup \{\infty\} =: \widehat{\mathbb{R}^n}$ is the one-point compactification of $\mathbb{R}^n$ and $G = \mathcal{M}(S^n)$ is the group of Möbius transformations of S^n, we call an (X, G)-manifold M a *(locally) conformally flat* manifold, and the corresponding (X, G)-structure on M a *flat conformal structure*.

If $\{(U_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ is a flat conformal structure on M, there is the induced Riemannian metric $\varphi_\lambda^* g_s$ on U_λ, where g_s is the standard metric on S^n. As $U_\lambda \cap U_\mu$ is connected, there exists a unique $\psi \in \mathcal{M}(S^n)$ such that $\varphi_\mu = \psi \circ \varphi_\lambda$ by the Liouville theorem. Since ψ is conformal with respect to g_s, two metrics $\varphi_\lambda^* g_s$ and $\varphi_\mu^* g_s$ are (pointwise) conformal on $U_\lambda \cap U_\mu$. Therefore, this metric pieces together to give a Riemannian metric $\overline{g}$ on M: $\overline{g} = \sum_\lambda t_\lambda \phi_\lambda^* g_s$, where $\{t_\lambda\}$ is a locally finite partition of unity subordinating to the open covering $\{U_\lambda\}$ of M. It follows that a flat conformal structure on a manifold M corresponds to a conformal class of Riemannian metrics on M bijectively.

Let $p : N \to M$ be a covering map. If $\mathcal{U} = \{(U_\lambda, \phi_\lambda)\}_{\lambda\in\Lambda}$ is a flat conformal structure on M, we call a flat conformal structure on N containing $\{(V_\lambda, \phi_\lambda \circ p)\}_\lambda$ the *lift* of $\{(U_\lambda, \phi_\lambda)\}_{\lambda\in\Lambda}$ by p, where V_λ is a connected component of $p^{-1}(U_\lambda)$. In particular, when p is a homeomorphism, $p^*\mathcal{U}$ and $\mathcal{U}$ are said to be *isomorphic*.

By the Liouville theorem, the conformal transformations of S^n are determined locally and are given by Möbius transformations of S^n. Therefore, by a standard monodromy argument, a simply connected conformally flat manifold with dimension $\geqslant 3$ has a conformal immersion into S^n which is unique up to composition with a Möbius transformation of S^n. We call such an immersion the *developing map*.

For a general (X,G)-manifold M, we can determine the developing map from the universal cover $\widetilde{M}$. We call it the developing map of M also.

Let M be a locally conformally flat manifold of dimension $n \geqslant 3$, $p : \widetilde{M} \to M$ the universal covering and $D : \widetilde{M} \to S^n$ the developing map. The fundamental group $\pi_1(M)$ with base point $p(x_o)$ is identified, via x_o, with the group of deck transformations on $\widetilde{M}$. If $\gamma \in \pi_1(M)$, the relation $D \circ \gamma = \xi \circ D$ holds for some $\xi \in \mathcal{M}(S^n)$ by the uniqueness of the developing map up to composition with a conformal transformation of S^n. Hence we have a representation $\rho : \pi_1(M) \to \mathcal{M}(S^n)$, which is called the *holonomy representation* of the flat conformal structure. The image $\rho(\pi_1(M))$ is called the *holonomy group*.

Let M be locally conformally flat and g a Riemannian metric on M. We say g is *compatible* with the flat conformal structure if for each λ the map $\varphi_\lambda : (U_\lambda, g) \to S^n$ is a conformal map. The following result was established in [7], cf. [13].

Proposition 3.2 *If M is a compact, locally conformally flat manifold, then M admits a compatible metric whose scalar curvature does not change sign. The sign is uniquely determined by the conformal structure: M admits a compatible metric of (1) positive, (2) negative, or (3) identically zero scalar curvature.*

Lastly, we shall recall the following theorem, called the holonomy theorem in the literature. For a compact manifold M with a flat conformal structure, let us denote by $FC^r(M)$ the set of C^r-developing maps of M and $\mathrm{Diff}^r_H(M)$ the group of C^r-diffeomorphisms of M which are homotopic to the identity.

Theorem 3.3 (see [19], [3]). *Let M be a compact, locally conformally flat manifold of dimension n ($\geqslant 3$) and $D : \widetilde{M} \to S^n$ the C^r-developing map. Then* *there is a neighborhood* $V \subset FC^r(M)$ *of* D *homeomorphic to* $V_1 \times V_2$*, where* $V_1 \subset \mathrm{Hom}(\pi_1(M), \mathcal{M}(S^n))$ *is a neighborhood of the holonomy representation* ρ *of* D *and* V_2 *is a neighborhood of the identity in* $\mathrm{Diff}^r_H(M)$.

4 Limit sets

In this section we define limit sets of subgroups of the Möbius group $\mathcal{M}(S^n)$ and briefly discuss their properties. We refer to, say [1] for details.

Let Γ be any (may not be discrete nor finitely generated) subgroup of $\mathcal{M}(S^n)$. A subset $A \subset S^n$ is said to be Γ-*invariant* if $\gamma(A) = A$ for any $\gamma \in \Gamma$. We denote by Ω_Γ the set of points $x \in S^n$ such that there exists a neighborhood U of x with $\gamma U \cap U = \emptyset$ except for finitely many $\gamma \in \Gamma$. The set Ω_Γ , called the *domain of discontinuity* of Γ, is the maximal Γ-invariant open subset of S^n on which Γ acts discontinuously. We say that Γ is a *Kleinian group* if $\Omega_\Gamma \neq \emptyset$. A Kleinian group is discrete in $\mathcal{M}(S^n)$. If a Kleinian group Γ acts freely on a Γ-invariant domain $\Omega \subset S^n$, we call the quotient manifold Ω / Γ a *Kleinian manifold*.

Definition 4.1 Let Γ be any subgroup of $\mathcal{M}(S^n)$. The set of accumulation points in S^n of the orbit $\Gamma(a)$ of some (and hence any) point $a \in D^{n+1}$ is called the *limit set* of Γ. Here D^{n+1} denotes the unit disk bounded by the unit sphere S^n.

We denote by $L(\Gamma)$ the limit set of Γ, which is a closed and Γ-invariant subset of S^n. A subgroup Γ is precompact if and only if $L(\Gamma) = \emptyset$.

Remark 4.2 There are several other ways to define limit sets of subgroups of $\mathcal{M}(S^n)$. However, it is known that for the holonomy group of a compact, locally conformally flat manifold they are identical.

For an arbitrary subgroup $\Gamma \subset \mathcal{M}(S^n)$ the cardinality of $L(\Gamma)$ is $0, 1, 2$ or infinite. If it is 1 or 2, then every point of $L(\Gamma)$ is a common fixed point of elements of Γ. Whereas, if it is infinite, then $L(\Gamma)$ is the unique minimal set (i.e., $L(\Gamma)$ is contained in any non-empty closed Γ-invariant subset of S^n), and any two points x, y of $L(\Gamma)$ are dual relative to Γ. Hence, for two distinct points x, y of $L(\Gamma)$, we can find a loxodromic element of Γ whose fixed points are close to x, y, respectively. Here we say two points x, y of $L(\Gamma)$ are *dual relative to* Γ if there exists a sequence $\{\gamma_k\} \subset \Gamma$ such that $\gamma_k(a) \to x$ and $\gamma_k^{-1}(a) \to y$ as $k \to \infty$.

Let Γ be a discrete subgroup of $\mathcal{M}(S^n)$ and Ω a Γ-invariant open subset of S^n for which $S^n \setminus \Omega$ is neither empty nor a singleton. Then Γ acts on Ω discontinuously.

Now we shall discuss holonomy groups.

Let M be a compact, locally conformally flat manifold. Let $D : \widetilde{M} \to S^n$ be the developing map, $\rho : \pi_1(M) \to \mathcal{M}(S^n)$ the holonomy homomorphism and $H := \rho(\pi_1(M))$ the holonomy group. Note that if the developing map is a covering map onto its image, then $D(\widetilde{M}) \cap L(H) = \emptyset$.

Proposition 4.3 *Let M be a compact, locally conformally flat manifold of dimension $n \geqslant 3$. Suppose that the developing map $D : \widetilde{M} \to S^n$ is a covering map onto its image. If the cardinality of the limit set $L(H)$ is less than or equal to 2, then the holonomy group $H := \rho(\pi_1(M))$ is discrete.*

Proof. By $|L|$ we denote the cardinality of the set L.

If $|L(H)|$ =0, then the developing map D is a homeomorphism and hence the holonomy group H is a discrete group.

If $|L(H)| = 1$, we have $D(\widetilde{M}) \subseteq S^n \setminus \{x\}$. We choose coordinates on $S^n \setminus \{x\}$ so that $S^n \setminus \{x\}$ is identified with $\mathbb{R}^n$ and x corresponds to the point ∞. In these coordinates a Möbius transformation of S^n which leaves the point ∞ fixed is a similarity map. It follows that the developing map is a homeomorphism and the holonomy group is a discrete group.

Let $|L(H)| = 2$. If $S^n \setminus D(\widetilde{M}) = \{x, x'\}$, then H has a subgroup of index 2 which leaves the point x fixed. Thus the situation reduces to the preceeding one. □

Corollary 4.4 *Let M be as in Proposition 4.3. If $H = \rho(\pi_1(M))$ is indiscrete, then $|L(H)|$ is infinite.*

Remark 4.5 Let M be a compact, locally conformally flat manifold of dimension 3. The developing map $D : \widetilde{M} \to S^3$ is an immersion. Suppose that M has infinite fundamental group. Then $D(\widetilde{M}) \neq S^3$ if and only if the developing map is a covering map (cf. [6]).

Proposition 4.6 *Let M be a compact, locally conformally flat, semi-symmetric manifold of dimension 3 and with the principal Ricci curvatures $\lambda_1 = \lambda_2 \neq \lambda_3 = 0$ everywhere. Then the developing map is a covering map.*

Proof. If $D(\widetilde{M}) = S^3$, the developing map D is a homeomorphism. Because the fundamental group of M is infinite by Corollary 2.3, we have a contradiction. □

5 Discrete holonomy groups

Let M be a compact, locally conformally flat manifold of dimension $n \geqslant 3$ and $\overline{g}$ a Riemannian metric induced from the flat conformal structure. Let $H = \rho(\pi_1(M))$ be the holonomy group. In this section we consider the case that H is discrete.

If the developing map $D : \widetilde{M} \to S^n$ is surjective onto S^n, then it is a homeomorphism. The holonomy group is discrete and $(M, \overline{g})$ is a spherical space form (i.e., a complete Riemannian manifold of constant positive curvature).

If $\partial D(\widetilde{M}) = \{\text{a point}\}$, then $(M, \overline{g})$ is a Euclidean space form. This is due to Fried [4] and Matsumoto [11]. In this case the holonomy group is discrete and is isomorphic to $\pi_1(M)$.

A closed similarity manifold M is said to be a *Hopf manifold* if the developing map D is a homeomorphism onto $S^n \setminus \{\text{two points}\}$ and the holonomy group is a subgroup of the group of similarities of $\mathbb{R}^n$. A Hopf manifold has a finite covering which is homeomorphic to $S^{n-1} \times S^1$, and the holonomy group is discrete.

In the last two cases, if M has infinite fundamental group, the developing map D is a covering map. In particular, $\partial D(\widetilde{M}) = L(H)$.

Next, we shall study the general case under the following two conditions:

(H1) The developing map $D : \widetilde{M} \to S^n$ is a covering map onto its image.

(H2) The holonomy group $H := \rho(\pi_1(M))$ is discrete.

Suppose that the conditions (H1) and (H2) hold for M. Let Γ be a torsion free subgroup of the holonomy group H with finite index. Then Γ acts on $D(\widetilde{M})$ freely. The existence of such a Γ is due to Selberg ([16]). Since the action of Γ is conformal on a locally conformally flat manifold $D(\widetilde{M})$, the Kleinian manifold $D(\widetilde{M})/\Gamma$ admits a flat conformal structure. The developing map D induces the covering map

$$\overline{D} : \widetilde{M}/\rho^{-1}(\Gamma) \to D(\widetilde{M})/\Gamma.$$

Since $\rho^{-1}(\Gamma)$ is of finite index in $\pi_1(M)$, the covering $q : \widetilde{M}/\rho^{-1}(\Gamma) \to M$ is a finite covering. Therefore $\widetilde{M}/\rho^{-1}(\Gamma)$ is compact and the map $\overline{D}$ is also a finite covering.

Here we introduce a terminology and summarize the above observation using it.

Definition 5.1 Two locally conformally flat manifolds are said to be *commensurable* if they have isomorphic finite coverings.

Proposition 5.2 *Let M be a compact, conformally flat manifold of dimension $n \geqslant 3$. Suppose that two conditions (H1) and (H2) hold for M. Then M is commensurable to a Kleinian manifold.*

Thus Proposition 5.2, together with Proposition 4.6, yields

Theorem 5.3. *Let M be a compact, locally conformally flat semi-symmetric space of dimension 3 and with the principal Ricci curvatures $\lambda_1 = \lambda_2 \neq \lambda_3 = 0$ everywhere. Suppose that M satisfies the condition (H2). Then M is commensurable to a Kleinian manifold.*

6 Indiscrete holonomy groups

Let M be a compact, locally conformally flat manifold of dimension $n \geqslant 3$. In this section we shall study the case that the holonomy group is indiscrete. Recall that the limit set $L(H)$ of the holonomy group is infinite in this case, cf. Corollary 4.4. We consider the conditions:

(H1) The developing map $D : \widetilde{M} \to S^n$ is a covering map onto its image.

(H3) The holonomy group $H = \rho(\pi_1(M))$ is indiscrete.

Let $\overline{H}_0$ be the closure of the identity component of H. Note that $\overline{H}_0 \neq \{1\}$ by (H3). Since $\overline{H}_0$ is a normal subgroup of $\overline{H}$, the closure of H, it follows that the limit set $L(\overline{H}_0)$ is invariant under the action of H. We have $L(H) = L(\overline{H}_0)$.

Lemma 6.1. *Fix $x \in L(H)$ and let $K := \overline{H}_0(x)$ be the orbit of x. Then K is dense in $L(H)$.*

Proof. Let $y \in L(H)$ and V a neighborhood of y in $\overline{D}^{n+1}$. Let $z \in L(H)$ be dual to y. Since $L(H)$ is an infinite set, there is $\psi \in \overline{H}_0$ such that $\psi(x) \neq z$. If U is a neighborhood of z in $\overline{D}^{n+1}$ with $\psi(x) \notin U$, then there exists $\phi \in \overline{H}_0$ such that $\phi(\overline{D}^{n+1} \setminus U) \subset V$. In particular $\phi\psi(x) \in V$. □

First, we shall study the case that $\overline{H}_0$ is non-compact.

Proposition 6.2. *Let M be a compact, locally conformally flat manifold of dimension $n \geqslant 3$. Suppose that the conditions (H1) and (H3) hold for M. If $\overline{H}_0$ is non-compact, then the limit set $L(H)$ of the holonomy group H is $\widehat{\mathbb{R}}^{n-2}$.*

Proof. We consider the coordinates of $\widehat{\mathbb{R}}^n$. We note that the orbit $\overline{H}_0(x)$ of $x \in L(H)$ under the action of $\overline{H}_0$ is dense in $L(H)$ by Lemma 6.1. Moreover, since $\overline{H}_0$ is non-compact, there is no fixed point of $\overline{H}_0$ in $L(H)$. In fact, if there were a fixed point of $\overline{H}_0$ in $L(H)$, there must be at least three fixed points of $\overline{H}_0$ in $L(H)$. Hence there is a fixed point in D^{n+1}, a contradiction.

Fix any point $x \in L(H)$ and let $K = \overline{H}_0(x)$ be the orbit of x. Then K is a smoothly injective immersed submanifold in $\widehat{\mathbb{R}}^n$. We may assume that the origin o lies in K. Since $L(H)$ is infinite, there is a loxodromic transformation, cf. Section 4. We call it f. We may assume the fixed points of f are 0 and ∞. Then f can be written as

$$f(x) = \lambda P(x), \qquad \text{where } \lambda > 1,\ P \in O(n).$$

Since K is invariant under f, we have $K = \widehat{\mathbb{R}}^k$ for some positive integer $k (\leqslant n)$. Thus $L(H) = \widehat{\mathbb{R}}^k$. By (H1) we have $k \neq n$. Finally we obtain $k = n-2$, since otherwise the developing map is a homeomorphism onto a connected component of $\widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^k$ and H must be discrete, contradicting (H3). □

Next we shall study the case that $\overline{H}_0$ is compact.

Proposition 6.3 *Let M be a compact, locally conformally flat manifold of dimension $n \geqslant 3$. Suppose that the conditions (H1) and (H3) hold for M. If $\overline{H}_0$ is compact, then the limit set $L(H)$ is S^{n-2}.*

Proof. The coordinates of S^n are convenient in this case. We assert that if $\overline{H}_0$ is compact, the fixed point set of $\overline{H}_0$ in S^n is S^k for some k, $0 \leqslant k \leqslant n$. Assuming this assertion, we prove Proposition 6.3 as follows: We have $k \neq n$ since $\overline{H}_0$ is non-trivial. Suppose $k = n-1$. Since $\overline{H}_0$ is connected, $\overline{H}_0 \subseteq SO(n+1)$ Thus $H \subseteq O(n+1)$ and M turns out to be modeled on the pair $(S^n, O(n+1))$ with finite fundamental group since M is compact, a contradiction. If $k < n-2$, then $S^n \setminus S^k$ is simply connected. Since the developing map D is a covering map, D must be a homeomorphism and H must be a discrete group, a contradiction. We have $k = n-2$.

We shall prove the above assertion. Since $\overline{H}_0$ is compact, there is a fixed point of $\overline{H}_0$ in D^{n+1}. We may assume that it is the origin o. If o is the only fixed point of $\overline{H}_0$. Then o is also a fixed point of H since $\overline{H}_0$ is a normal subgroup of $\overline{H}$. As $\overline{H}_0$ is connected, we have $\overline{H}_0 \subseteq SO(n+1)$. Thus $H \subseteq O(n+1)$ and we have a contradiction. Therefore at least one of the fixed points of $\overline{H}_0$ lies in S^n . Let us denote by S^k the fixed point set of $\overline{H}_0$ in S^n. By the H-invariance of S^k and the minimality of the limit set we have $L(H) \subseteq S^k$. To prove the inverse inclusion, suppose that there were a point $x \in S^k \setminus L(H)$. Let V be an open neighborhood in S^n of x such that $V \cap L(H) = \emptyset$. We may assume $k \leqslant n-2$. Then $D^{-1}(V \setminus S^k)$ is connected and hence $D^{-1}(V)$ is connected. So D must be a homeomorphism and H discrete, a contradiction. □

With the above results we shall study global structures of compact, locally conformally flat semi-symmetric spaces. We need another preparation.

Let M be a compact, locally conformally flat space of dimension 3. Suppose that the conditions (H1) and (H3) hold for M. Then it follows from Propositions 6.2 and 6.3 that the limit set of the holonomy group has no interior points.

Let H^{n-1} be the upper half space model of the hyperbolic space, i.e.,

$$H^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{n-1} > 0, x_n = 0\}$$

with the Riemannian metric $ds_H^2 = (dx_1^2 + \cdots + dx_{n-1}^2)/x_{n-1}^2$, and $H^{n-1} \times \mathbb{R}$ the Riemannian product space with the metric $ds_H^2 + dt^2$, where dt^2 is the metric on $\mathbb{R}$. The isometry group $I(H^{n-1})$ is identified with $\mathcal{M}(\widehat{\mathbb{R}}^{n-2})$. Furthermore, we can identify $\mathcal{M}(\widehat{\mathbb{R}}^{n-2})$ with the group

$$\{f \in \mathcal{M}(\widehat{\mathbb{R}}^n) : f(H^{n-1}) = H^{n-1}\}.$$

Theorem 6.4. *Let M be a compact, locally conformally flat manifold of dimension $n (\geqslant 3)$. Suppose that the conditions (H1) and (H3) hold for M. Then $(M, \overline{g})$ is a hyperbolic manifold bundle over the circle, where $\overline{g}$ is a Riemannian metric induced from the $(H^{n-1} \times \mathbb{R}, \mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R})$-structure.*

To prove the theorem we need the following lemma.

Lemma 6.5 *Let M be a compact, locally conformally flat manifold of dimension n $(\geqslant 3)$. Let $D : \widetilde{M} \to \widehat{\mathbb{R}}^n$ be the developing map and $\rho : \pi_1(M) \to \mathcal{M}(\widehat{\mathbb{R}}^n)$ the holonomy homomorphism. Suppose that $D(\widetilde{M}) = \widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^{n-2}$.*

Then the pair (D, ρ) can be lifted to the pair $(\overline{D}, \overline{\rho})$, where

$$\overline{D} : \widetilde{M} \to H^{n-1} \times \mathbb{R},$$

$$\overline{\rho} : \pi_1(M) \to \mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R}.$$

Proof of Lemma 6.5. Let us denote by $R_\theta \in \mathcal{M}(\widehat{\mathbb{R}}^n)$ the rotation by angle θ around $\widehat{\mathbb{R}}^{n-2}$. We define $p : H^{n-1} \times \mathbb{R} \to \widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^{n-2}$ by $p(x, t) = R_{2\pi t}(x)$. Then p is a universal covering map. Let

$$\mathcal{S} = \{f \in \mathcal{M}(\widehat{\mathbb{R}}^n) : f(\widehat{\mathbb{R}}^{n-2}) = \widehat{\mathbb{R}}^{n-2}\}.$$

If $f \in \mathcal{S}$, then f maps H^{n-1} to a half plane bounded by $\widehat{\mathbb{R}}^{n-2}$. Namely, such an f is determined by $f|_{\widehat{\mathbb{R}}^{n-2}}$ and the image $f(H^{n-1})$. So, it commutes with R_θ. We have the surjective homomorphism

$$\xi : \mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R} \to \mathcal{S}$$

defined by $\xi(f, t) = R_{2\pi t} \circ f$. Taking the diagonal action of $\mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R}$ on $H^{n-1} \times \mathbb{R}$, we have an equivariant map

$$(p, \xi) : (H^{n-1} \times \mathbb{R}, \mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R}) \to (\widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^{n-2}, \mathcal{S}).$$

Thus we have the lift $(\overline{D}, \overline{\rho})$ of the pair (D, ρ) as desired. □

Proof of Theorem 6.4 Let M be as in Theorem 6.4. We consider the coordinates $\widehat{\mathbb{R}}^n$. Then the image of $\widetilde{M}$ by the developing map is $\widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^{n-2}$. We have the pair (D, ρ) such that $D : \widetilde{M} \to H^{n-1} \times \mathbb{R}$ and $\rho : \pi_1(M) \to \mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R}$ by Lemma 6.5. Theorem 3.3 (the holonomy theorem) allows us a small perturbation of a pair (D, ρ): Namely, let us denote by p_i the projection map from $\mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R}$ to the i-th factor. Define a map $\rho' : \pi_1(M) \to \mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R}$ so that $p_1 \circ \rho' = p_1 \circ \rho$ and $p_2 \circ \rho'(\pi_1(M)) \subseteq \mathbb{Q}$. Set $\rho'' = \xi \circ \rho'$, where $\xi : \mathcal{M}(\widehat{\mathbb{R}}^{n-2}) \times \mathbb{R} \to \mathcal{S}$ is the map defined in the proof of Lemma 6.5 above. Then there is a smooth immersion $D' : \widetilde{M} \to \widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^{n-2}$ such that $D'(\gamma x) = \rho''(\gamma) \circ D'(x)$ for $\gamma \in \pi_1(M)$ and $x \in \widetilde{M}$. Moreover, the limit set of the pair (D', ρ'') is $\widehat{\mathbb{R}}^{n-2}$, because ρ' was perturbed only in the $\mathbb{R}$-direction. The immersion D' is a covering map onto $\widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^{n-2}$.

Again by Lemma 6.5, the map D' lifts to a homeomorphism

$$\overline{D'} : \widetilde{M} \to H^{n-1} \times \mathbb{R}.$$

Since $p_2 \circ \rho'(\pi_1(M)) \subseteq \mathbb{Q}$, it follows that $p_2 \circ \rho'(\pi_1(M))$ is infinite cyclic with a generator, say θ. Let Γ be the kernel of $p_2 \circ \rho'$. Then we have an exact sequence

$$1 \to \Gamma \to \pi_1(M) \to \theta\mathbb{Z} \to 1$$

and a bundle structure of M with fibre $H^{n-1}/p_1 \circ \rho'(\Gamma)$ over $\mathbb{R}/\theta\mathbb{Z} \cong S^1$.

We introduce the Riemannian metric on $\widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^{n-2}$ so that the universal covering map $p : (H^{n-1} \times \mathbb{R}, ds_H^2 + dt^2) \to \widehat{\mathbb{R}}^n \setminus \widehat{\mathbb{R}}^{n-2}$, defined in the proof of Lemma 6.5, is (locally) isometric. □

Let (M, g) be a compact, locally conformally flat, semi-symmetric space of dimension 3 and with the principal Ricci curvatures $\lambda_1 = \lambda_2 \neq \lambda_3 = 0$ everywhere. Then the developing map is a covering map by Proposition 4.6. Therefore we have

Theorem 6.6 *Let (M, g) be a compact, locally conformally flat, semi-symmetric space of dimension 3 and with the principal Ricci curvatures $\lambda_1 = \lambda_2 \neq \lambda_3 = 0$ everywhere. Suppose that the holonomy group is indiscrete. Then the developing map is a homeomorphism onto $S^3 \setminus S^1$ and $(M, \overline{g})$ is a hyperbolic surface bundle over S^1.*

Owing to Proposition 3.2, we obtain

Corollary 6.7 *Let M be as in above Theorem 6.6. Suppose that the holonomy group is indiscrete. Then (M, g) has negative scalar curvature.*

References

1. N. Apanasov: *Discrete Groups in Space and Uniformization Problems*, Kluwer Academic Publishers, 1991.
2. E. Boeckx, O. Kowalski, L. Vanheck: *Riemannian Manifolds of Conullity two*, World Science, Singapore 1996.
3. R. D. Canary, D. B. A. Epstein, P. Green: Notes on notes of Thurston, London Math. Soc. Lecture Notes 111(O.B.A. Epstein editor), Cambridge University Press, 1984.
4. D. Fried: Closed similarity manifolds, Comm. Math. Helv., 55(1980), 576–582.
5. M. Goto: When is a diffeomorphism of a hyperbolic space isotopic to the identity? Differential Geometry and its applications Proc. Conf., Opava(Czech Republic), August 27–31, 2001 Silesian University, Opava, 2001, 23–26.
6. N. A. Gusevskii, M. È. Kapovich.: Conformal structures on three-dimensional manifolds, Soviet Math.Dokl. 34(1987), no.2, 314–318.
7. J. Kazdan, F. Warner: Scalar curvature and conformal deformation of Riemannian structure, J. Differential Geometry 10(1975), 113–134.
8. O. Kowalski: An explicit clasification of 3-dimensional Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, Czech.Math.J. 46(121)(1996), 427–474.
9. N. H. Kuiper: On conformally flat spaces in the large, Ann. Math., 50(1949), 916–924.
10. N. H. Kuiper: On compact conformally Euclidean spaces of dimension > 2, Ann. Math., 52(1950), 478–490.
11. S. Matsumoto: Foundations of flat conformal structure, Adv. Studies in Pure Math. 20(1992), 167–261.
12. K. Nomizu: On hypersurfaces satisfying a certain condition on the curvature tensor, Tohoku Math. J. 20(1968), 46–59.
13. Schoen,R., Yau,S.-T.: Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math.,92(1988), 47–71.
14. P. Scott: The geometry of 3-manifolds, Bull. London Math. Soc 15(1983), 401–487.
15. K. Sekigawa, H. Takagi: On conformally flat spaces satisfying a certain condition on the Ricci tensor, Tohoku Math. J. 23(1971), 1–11.
16. A. Selberg: On discontinuous groups in higher dimensional symmetric spaces, in "Contribution to Function Theory, Bombay", Tata Institute, 1960, 147–164.
17. Z. I. Szabó: Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, I. The local version, J.D.G.,17(1982), 531–582.
18. Z. I. Szabó: Structure theorems on Riemannian spaces satisfying $R(X, Y)\cdot R = 0$, II. Global version, Geometriae Dedicata 19(1985), 65–108.
19. W. P. Thurston: The geometry and topology of 3-manifolds, Princeton University, 1979.

20. G. Calvaruso: Conformally flat semi-symmetric spaces, Arch. Math. (Brno), Tomus 41 (2005), 27–36.
21. E. Boeckx: Einstein-like semi-symmetric spaces, Arch. Math. (Brno), Tomus 29(1993), 235–240.

Differential Geometry of Analytic Surfaces with Singularities

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Dedicated to my teacher, Professor Hideki Omori

Summary. First the fundamental results of analytic curves with singularities which correspond to those in the classical theory of curves in Euclidean space are described. And then the geometry of surfaces with singularities are studied.

AMS Subject Classification: 53A04, 53A05, 53A40.

Key words: Analytic, curve, surface, singularity.

Introduction

In the previous paper [S] we studied elementary properties of analytic curves with singularities and obtained two fundamental theorems. In the present paper we first review those and, next, study elementary concepts and formulae on analytic surfaces around singularities.

1 Curves at singularities

Let C be an analytic curve with a singular point x_0 in the Euclidean 3-space $\mathbb{E}^3$. We may take a local parameter t defined on a small open interval L containing $t = 0$ and an analytic mapping $x(t)$ of L into C with $x(0) = x_0$ and $(dx/dt)(0) = 0$ ([S], §1). Let us define a function $h(t)$ by

$$h(t) = \|dx/dt\|^2,$$

where $\| \cdot \|$ is the Euclidean norm in $\mathbb{E}^3$. It is analytic on L and vanishes only at $t = 0$. It coincides with $(ds/dt)^2$ at any non-zero point where s is the arc-length parameter.

We define two functions on L which are said to be the curvature and the torsion of C by

$$K(t) = (h \cdot \|dx^2/dt^2\|^2 - (dx/dt, d^2x/dt^2)^2)/h^3,$$
$$\tau(t) = \det(dx/dt, d^2x/dt^2, d^3x/dt^3)/(h^3K),$$

respectively, where $(\,,\,)$ is the Euclidean inner product of $\mathbb{E}^3$ and $\det(\cdot)$ denotes the determinant of a matrix. They are analytic on $L - \{0\}$, and coincide with the square of the curvature and the torsion in the classical meaning at non-singular points in $x(L)$, respectively.

We define a frame $\{e_i(t)\}$ $(i = 1, 2, 3)$ on C as follows:

$$e_1 = dx/dt, \quad e_2 = d^2x/dt^2 - (dh/dt)e_1/(2h), \quad e_3 = e_1 \times e_2.$$

We can easily show that e_i is analytic in t. The relation between the classical Frenet frame $\{\mathbf{e}_i\}$ and the above is stated as follows.

$$e_1 = h^{1/2}\mathbf{e}_1, \quad e_2 = hK^{1/2}\mathbf{e}_2, \quad e_3 = (h^3K)^{1/2}\mathbf{e}_3.$$

Then

$$\begin{cases} \|e_1\|^2 = h, \quad \|e_2\|^2 = h^2K, \quad \|e_3\|^2 = h^3K, \\ (e_i, e_j) = 0, \quad \text{for} \quad i \neq j. \end{cases} \tag{$*$}$$

An elementary calculation shows that;

$$de/dt = \begin{pmatrix} (\log h)'/2 & 1 & 0 \\ -hK & (\log h^2K)'/2 & \tau \\ 0 & -h\tau & (\log h^3K)'/2 \end{pmatrix} e$$

where $e = {}^t({}^te_1, {}^te_2, {}^te_3)$. This corresponds to the classical Frenet–Serret equation and is uniquely determined by h, K and τ. It is said to be the *fundamental equation of C with respect to t.*

Theorem (The first fundamental theorem). *Let C be an analytic curve in $\mathbb{E}^3$. Let x_0 be a singular point of C and let t be a local parameter of C at x_0. Then there exists an orthogonal frame $\{e_i(t)\}$ which satisfies $(*)$ and the fundamental equation. In particular, the directions of e_i $(i = 1, 2, 3)$ are independent of the choice of t.*

Observing the power series expansions of h, K and τ by virtue of a canonical local parameter ([S], §1) where L is small enough, we can summarize those properties as follows:

$$\begin{cases} h(t) = \sum_{k=2(\lambda-1)}^{\infty} h_k t^k, \ h_{2(\lambda-1)} = \lambda^2 \text{ and } h > 0 \text{ on } L - \{0\}, \\ K(t) = \sum_{k=2\mu}^{\infty} K_k t^k, \ K_{2\mu} > 0, K > 0 \text{ on } L - \{0\}, \\ \tau(t) = \sum_{k=\nu}^{\infty} \tau_k t^k, \tau_\nu \neq 0, \ \lambda \geq 2, \ \lambda + \mu > 0, \ \lambda + \nu > 0. \end{cases} \tag{$**$}$$

We note that K and τ may have poles at $t = 0$.

Finally we have obtained:

Theorem (The second fundamental theorem). Let $h(t)$, $K(t)$ and $\tau(t)$ be functions which satisfy the conditions $(**)$. Then there exists an analytic curve $C : x = x(t)$ which admits an orthogonal frame $\{e_i\}$ satisfying $(*)$ and the fundamental equation with given K and τ as its curvature and torsion, respectively. Such a curve is uniquely determined up to a motion of $\mathbb{E}^3$.

2 Surfaces around singularities

Let S be an analytic surface defined by

$$\mathbb{X} = \mathbb{X}(u, v) = (x(u, v), y(u, v), z(u, v)),$$

in $\mathbb{E}^3$, where x, y and z are analytic functions.

We know that

Definition 1 At $\mathbb{X}(u, v)$, S has a singularity (or, S is singular) if and only if

$$r(u, v) = \operatorname{rank} \begin{bmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{bmatrix} < 2.$$

Thus the set $\mathcal{S}(S)$ of singularities of S has dimension ≤ 1 and, $r(u, v) = 0$ or 1 there.

Example 1 $\mathbb{X} = (uv, u^3, v^3)$ i.e., $x^3 - yz = 0, \mathcal{S}(S) = \{(u, v) = (0, 0)\}$ and $r(0, 0) = 0$.

Example 2 (Whitney umbrella without handle). $\mathbb{X} =(uv, u^2, v)$, $\mathcal{S}(S) = \{(0, 0)\}$ and $r(0, 0) = 1$.

Example 3 (Cone). $\mathbb{X} = (u \cos v, u \sin v, u))$. Then $r(\mathcal{S}(S)) = r(0, 0) = 1$.

Let us define E, F and G as usual by

$$E = \langle \mathbb{X}_u, \mathbb{X}_u \rangle, \quad F = \langle \mathbb{X}_u, \mathbb{X}_v \rangle, \quad G = \langle \mathbb{X}_v, \mathbb{X}_v \rangle,$$

where $\langle \cdot, \cdot \rangle$ means the inner product in $\mathbb{E}^3$. Then the following is obvious.

Lemma 1 $\mathbb{X}(u, v) \in \mathcal{S}(S)$ *if and only if* $g = g(u, v) = EG - F^2 = 0$.

We set

Definition 2 $\mathbb{N} = \mathbb{X}_u \times \mathbb{X}_v$, *where* $\times$ *is the exterior product in* $\mathbb{E}^3$.

Since $\langle \mathbb{N}, \mathbb{X}_u \rangle = \langle \mathbb{N}, \mathbb{X}_v \rangle = 0$, the following hold;

$$\mathcal{L} = -\langle \mathbb{X}_u, \mathbb{N}_u \rangle = \langle \mathbb{N}, \mathbb{X}_{uu} \rangle = \det(\mathbb{X}_{uu}, \mathbb{X}_u, \mathbb{X}_v),$$
$$\mathcal{M} = -\langle \mathbb{X}_u, \mathbb{N}_v \rangle = \langle \mathbb{N}, \mathbb{X}_{uv} \rangle = \det(\mathbb{X}_{uv}, \mathbb{X}_u, \mathbb{X}_v),$$
$$\mathcal{N} = -\langle \mathbb{X}_v, \mathbb{N}_v \rangle = \langle \mathbb{N}, \mathbb{X}_{vv} \rangle = \det(\mathbb{X}_{vv}, \mathbb{X}_u, \mathbb{X}_v),$$

where det is the determinant of a matrix. Then the following relations hold between the above and the quantities defined by Gauss, i.e., the unit normal vector $\mathbf{n}$, the second fundamental quantities L, M and N;

$$\mathbb{N} = \sqrt{g}\,\mathbf{n}, \quad \mathcal{L} = \sqrt{g}\,L, \quad \mathcal{M} = \sqrt{g}\,M, \quad \mathcal{N} = \sqrt{g}N.$$

We note that these are analytic.

Lemma 2 *If* $\mathbb{X} = (u, v) \in \mathcal{S}(S)$, $\mathbb{N} = 0$, $\mathcal{L} = \mathcal{M} = \mathcal{N} = 0$, *where* 0 *is the zero vector.*

Lemma 3 (Gauss' equation) *The Gaussian curvature is*

$$K = \frac{1}{g^2}(\mathcal{L}\mathcal{N} - \mathcal{M}^2) = -\frac{1}{g^2}R_{1212}$$

where R_{abcd} *is the so-called Riemann–Christoffel tensor.*

Remark By the definition R_{1212} is analytic.

Now, we rewrite u, v; $\mathbb{X}_u, \mathbb{X}_v$; E, F, G and $\mathcal{L}, \mathcal{M}, \mathcal{N}$ by u^a; $\mathbb{X}_a$; g_{ab} and $\mathcal{L}_{ab}$, respectively. Then Gauss' and Weingarten's formulae are stated as follows;

$$\frac{\partial \mathbb{X}_a}{\partial u^b} = \Gamma^c_{ab}\,\mathbb{X}_c + \frac{1}{g}\mathcal{L}_{ab}\,\mathbb{N} \tag{\#}$$

$$\frac{\partial \mathbb{N}}{\partial u^a} = g^{bc}\mathcal{L}_{ca}\mathbb{X}_b + \frac{1}{2}\frac{\partial}{\partial u^a}(\log\ g)\mathbb{N} \tag{b}$$

where Γ^c_{ab} is the Christoffel symbol. These equations constitute a system of total differential equations with singularities $\{g = 0\}$. The integrability conditions are Lemma 3 and

$$\mathcal{L}_{ab;c} - \mathcal{L}_{ac;b} = \frac{1}{2}\left(\frac{\partial}{\partial u^c}(\log\ g)\,\mathcal{L}_{ab} - \frac{\partial}{\partial u^b}(\log\ g)\,\mathcal{L}_{ac}\right) \tag{bb}$$

where ; is the covariant differential. It is the so-called Mainardi–Codazzi's equation.

Next we consider the converse. Let analytic g_{ab} and $\mathcal{L}_{ab}$ be given. Suppose they satisfy (#) and (b) with integrability conditions Lemma 3 and (bb). Then;

Conjecture *There always exists an analytic solution of (#) and (b), i.e., an analytic surface with arbitrary* g_{ab} *and* $\mathcal{L}_{ab}$ *as the first and second fundamental quantities, and with singularities* $\{g = 0\}$.

At the present time the best result applicable to this problem is due to Takano and Yoshida ([T] and [TY]).

Theorem *Let* $g = \prod_a h^a$, *where* $h^a = \sum_b c^a_b u^b$ *is linear in* u^b. *Then the above conjecture is true.*

References

[S] T. Sasai, *Geometry of analytic space curves with singularities and regular singularities of differential equations*, Funkcial. Ekvac., **30**(1989), 283–303.

[T] T. Takano, *A reducion theorem for a linear Pfaffian system with regular singular points*, Arch. Math., **31** (1978), 310–316.

[TY] T. Takano and M. Yoshida, *On a linear system of Pfaffian equations with regular singular points*, Funkcial. Ekvac., **19** (1976), 175–189.

Part III

Symplectic Geometry and Poisson Geometry

The Integration Problem for Complex Lie Algebroids*

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Summary. A complex Lie algebroid is a complex vector bundle over a smooth (real) manifold M with a bracket on sections and an anchor to the complexified tangent bundle of M which satisfy the usual Lie algebroid axioms. A proposal is made here to integrate analytic complex Lie algebroids by using analytic continuation to a complexification of M and integration to a holomorphic groupoid. A collection of diverse examples reveal that the holomorphic stacks presented by these groupoids tend to coincide with known objects associated to structures in complex geometry. This suggests that the object integrating a complex Lie algebroid should be a holomorphic stack.

AMS Subject Classification: H5805, 32Q99.

Key words: Lie algebroid, groupoid, stack, involutive structure, complex manifold, pseudoconvex domain.

1 Introduction

It is a pleasure to dedicate this paper to Professor Hideki Omori. His work over many years, introducing ILH manifolds [30], Weyl manifolds [32], and blurred Lie groups [31] has broadened the notion of what constitutes a “space.” The problem of “integrating” complex vector fields on real manifolds seems to lead to yet another kind of space, which is investigated in this paper.

Recall that a *Lie algebroid* over a smooth manifold M is a real vector bundle E over M with a Lie algebra structure (over $\mathbb{R}$) on its sections and a bundle map ρ (called the *anchor*) from E to the tangent bundle TM, satisfying the Leibniz rule

$$[a, fb] = f[a, b] + (\rho(a)f)b$$

for sections a and b and smooth functions $f : M \to \mathbb{R}$. Sections of a Lie algebroid may be thought of as “virtual” vector fields, which are mapped to ordinary vector fields by the anchor.

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There is an analogous definition for complex manifolds, in which E is a holomorphic vector bundle over M, and the Lie algebra structure is defined on the sheaf of local sections. Such objects are called complex Lie algebroids by Chemla [6], but they will be called in this paper *holomorphic Lie algebroids* to distinguish them from the "hybrid" objects defined in [5] as follows.

Definition 1.1 A complex Lie algebroid (CLA) over a smooth (real) manifold M is a complex vector bundle E over M with a Lie algebra structure (over $\mathbb{C}$) on its space $\mathcal{E}$ of sections and a bundle map ρ (called the anchor) from E to the complexified tangent bundle $T_{\mathbb{C}}M$, satisfying the Leibniz rule

$$[a, fb] = f[a, b] + (\rho(a)f)b$$

for sections a and b in $\mathcal{E}$ and smooth functions $f : M \to \mathbb{C}$.

The unmodified term "Lie algebroid" will always mean "real Lie algebroid."

Every Lie algebroid may be realized as the bundle whose sections are the left invariant vector fields on a *local* Lie groupoid Γ. The *integration problem* of determining when Γ can be taken to be a global groupoid was completely solved in [8], but, for a complex Lie algebroid, it is not even clear what the corresponding local object should be. The main purpose of the present paper is to propose a candidate for this object.

Any CLA E whose anchor is injective may be identified with the involutive subbundle $\rho(E) \subseteq T_{\mathbb{C}}M$. Such subbundles have been studied extensively under the name of "involutive structures" or "formally integrable structures," for instance by Treves [39]. An important issue in these studies has been to establish the existence (or nonexistence) of "enough integrals," i.e., smooth functions which are annihilated by all the sections of E. In the general C^∞ case, the question is very subtle and leads to deep problems and results in linear PDE theory. When E is analytic,[1] though, one can sometimes proceed in a fairly straightforward way by complexifying M and extending E by holomorphic continuation to an involutive holomorphic tangent subbundle of the complexification, where it defines a holomorphic foliation. The leaf space of this foliation is then a complex manifold whose holomorphic functions restrict to M to give integrals of E. (A succinct example of this may be found at the end of [35]; see Section 3.2 below.)

The leaf space described above may be thought of as the "integration" of the involutive subbundle E; this suggests a similar approach to analytic CLAs whose anchors may not be injective. Any analytic CLA E over M may be holomorphically continued to a holomorphic Lie algebroid E' over a complexification $M_{\mathbb{C}}$; E' may then be integrated to a (possibly local) holomorphic groupoid G. Since G will generally have nontrivial isotropy, one must take this into account by considering not just the orbit space of G, but the "holomorphic stack" associated to G.

Some intuition behind the complexification approach to integration comes from the following picture in the real case. If G is a Lie group, there is a long tradition of

[1] In this paper, "analytic" will always mean "real analytic", and "holomorphic" will be used for "complex analytic."

thinking of its Lie algebra elements as tiny arrows pointing from the identity of G to "infinitesimally nearby" elements. If G is now a Lie groupoid over a manifold M, M may be identified with the identity elements of G, and an element a of the Lie algebroid E of G may be thought of as an arrow from the base $x \in M$ of a to a groupoid element with its source at x and its target at an infinitesimally nearby $y \in G$. The tangent vector $\rho(a)$ is then viewed as a tiny arrow in M pointing from x to y.

Now suppose that E is a complex Lie algebroid over M. Then $\rho(a)$ is a complex tangent vector. To visualize it, one may still think of the tail of the tiny arrow as being at a, but the imaginary part of the vector will force the head to lie somewhere "out there" in a complex manifold $M_{\mathbb{C}}$ containing M as a totally real submanifold. To invert (and compose) such groupoid elements requires that their sources as well as targets be allowed to lie in this complexification $M_{\mathbb{C}}$. Thus, the integration should be a groupoid over the complexification.

What exactly is this complexification? Haefliger [18], Shutrick [37], and Whitney and Bruhat [44] all showed that every analytic manifold M may be embedded as an analytic, totally real submanifold of a complex manifold $M_{\mathbb{C}}$. Any two such complexifications are canonically isomorphic near M. Consequently, the identity map extends uniquely near M to an antiholomorphic involution of $M_{\mathbb{C}}$ ("complex conjugation") having M as its fixed point set. Finally, Grauert [16] showed that the complexification may be taken to have a pseudoconvex boundary and therefore be a Stein manifold. $M_{\mathbb{C}}$ is then called a *Grauert tube*.

Of course, constructing the complexification requires that the Lie algebroid have a real analytic structure. For the underlying smooth manifold M, such a structure exists and is unique up to isomorphism [43], though the isomorphism between two such structures is far from canonical. Extending the analyticity to E is an issue which must be deal with in each example.

In fact, examples are at the heart of this paper. Except for some brief final remarks, the many observations and questions about CLAs which arise naturally by extension from the real theory and from complex geometry will be left for future work. Concepts such as cohomology, connections, modular classes, Kähler structure, and quantization are discussed by Block [3] and in work with Cannas [5] and with Leichtnam and Tang [24].

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2 Complexifications of real Lie algebroids

A complex Lie algebroid over a point is just a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. It seems natural to take as integration of $\mathfrak{g}$ a holomorphic Lie group G with this Lie algebra. In particular, if $\mathfrak{g}$ is the complexification of a real Lie algebra $\mathfrak{g}_{\mathbb{R}}$, then G is a complexification of a real Lie group $G_{\mathbb{R}}$.

Next, given any real Lie algebroid $E_{\mathbb{R}}$, its complexification E becomes a complex Lie algebroid when the bracket and anchor are extended by complex (bi)linearity. If $E_{\mathbb{R}}$ is integrated to a (possibly local) Lie groupoid $G_{\mathbb{R}}$, then a natural candidate for G would be a complexification of $G_{\mathbb{R}}$. For this complexification to exist, $G_{\mathbb{R}}$ must admit an analytic structure, and, when this structure does exist, it is rarely unique (though it may be unique up to isomorphism). Examples follow.

2.1 Zero Lie algebroids

Let $E_{\mathbb{R}}$ be the zero Lie algebroid over M. An analytic structure on $E_{\mathbb{R}}$ is just an analytic structure on M, which exists but is unique only up to noncanonical isomorphism. Now the unique source-connected Lie groupoid integrating $E_{\mathbb{R}}$ is the manifold M itself, which always admits a complexification $M_{\mathbb{C}}$. This complex manifold is far from unique, but its germ along M is unique up to natural (holomorphic) isomorphism, given the analytic structure on M. One could say that the choice of analytic structure on M is part of the integration of this zero complex Lie algebroid.

This example suggests that the object integrating M should be the germ along M of a complexification of M. Getting rid of all the choices, including that of the analytic structure, requires that the complexification $M_{\mathbb{C}}$ be shrunk even further, to a formal neighborhood of M in $M_{\mathbb{C}}$. Both of these possibilities will be considered in many of the examples which follow.

Remark 2.1 One could define the germ as an object for which the underlying topological space is M, but with a structure sheaf given by germs along M of holomorphic functions on $M_{\mathbb{C}}$. But these are just the analytic functions on M. For the formal neighborhood, the structure sheaf becomes simply the infinite jets of smooth complex-valued functions.

2.2 Tangent bundles

Let $E = T_{\mathbb{C}}M$ be the full complexified tangent bundle. Once again, an analytic structure on $E_{\mathbb{R}} = TM$ is tantamount to an analytic structure on M, which leads to many complexifications $M_{\mathbb{C}}$, as above. A source-connected Lie groupoid integrating TM is the pair groupoid $M \times M$, while the source-simply connected groupoid is the fundamental groupoid $\pi(M)$. The pair groupoid $M_{\mathbb{C}} \times M_{\mathbb{C}}$ is then a complexification of $M \times M$ and may be taken as an integration of the complex Lie algebroid $T_{\mathbb{C}}M$. On the other hand, $\pi(M)$ could be complexified to $\pi(M_{\mathbb{C}})$; however, the result is sensitive, even after restriction to M, to the choice of $M_{\mathbb{C}}$. If $M_{\mathbb{C}}$ is taken to be a small neighborhood of M, the restriction to M is just $\pi(M)$.

2.2.1 Interlude: The integration as a stack

Some of the dependence on the choice of $M_{\mathbb{C}}$ disappears when two groupoids are declared to be “the same” when they are Morita equivalent. The groupoid is then seen as a presentation of a differential stack (see Behrend [2] and Tseng and Zhu [40]) or, more precisely, a holomorphic stack. Since a transitive groupoid is equivalent to any of its isotropy groups, the stack represented by a pair groupoid $M \times M$ is just a point. The only difference between this and $M_{\mathbb{C}} \times M_{\mathbb{C}}$ is that the latter represents a "holomorphic point." Depending on the choice of groupoid, this point as a stack might carry isotropy equal to the fundamental group of M or even of one its complexifications.

2.3 Action groupoids

Any (right) action of a Lie algebra $\mathfrak{k}$ on M induces an action, or transformation, groupoid structure on the trivial vector bundle $E_{\mathbb{R}} = M \times \mathfrak{k}$. The complexified bundle $E = M \times \mathfrak{k}_{\mathbb{C}}$ becomes a complex Lie algebroid whose anchor maps the constant sections of E to a finite dimensional Lie algebra of complex vector fields on M.

When the $\mathfrak{k}$ action comes from a (left) action of a Lie group K, $E_{\mathbb{R}}$ integrates to the transformation groupoid $K \times M$; in fact, Dazord [9] showed that $E_{\mathbb{R}}$ is always integrable to a global groupoid $G_{\mathbb{R}}$ which encodes the (possibly local) integration of the $\mathfrak{k}$ action.

Passing from $E_{\mathbb{R}}$ to E complicates issues significantly. First, complexifying G requires an analytic structure on it, which amounts to an analytic structure on M for which the $\mathfrak{k}$ action is analytic. But this can fail to exist even when $\mathfrak{k} = \mathbb{R}$, in other words, when the action is simply given by a vector field. For instance, if the vector field vanishes to infinite order at a point p of M, but not on a neighborhood of p, it can never be made analytic, so complexification of the action groupoid $G_{\mathbb{R}}$ and hence integration of E become impossible except on the formal level.

In addition, it is conceivable that some smooth action groupoids may be made analytic in essentially different ways, even though, according to Kutzschebauch [21], this cannot happen for proper actions by groups with finitely many connected components. Perhaps there is a smooth action which admits several quite different complexifications.

When M and the $\mathfrak{k}$ action are analytic, the vector fields generating the action extend to holomorphic vector fields on a complexification $M_{\mathbb{C}}$, leading to a holomorphic Lie algebroid structure on $M_{\mathbb{C}} \times k_{\mathbb{C}}$. This integrates to a holomorphic Lie groupoid G, the "local transformation groupoid" of the complexified $K_{\mathbb{C}}$ action.

Note the slightly different strategy here—the Lie algebroid is first extended to the complexification and then integrated, rather than the other way around. This strategy will be used extensively below.

Example 2.2 Let $\mathfrak{k} = \mathbb{R}$ act on $M = \mathbb{R}$ via the vector field $x\,\frac{\partial}{\partial x}$. When $\mathfrak{k}$ is considered as the Lie algebra of the multiplicative group $\mathbb{R}^+$, the resulting action groupoid is $\mathbb{R}^+ \times \mathbb{R}$, with the first component acting on the second by multiplication. The orbits of this groupoid are the two open half lines and the origin.

A natural complexification of $\mathbb{R}^+ \times \mathbb{R}$ is the action groupoid $\mathbb{C}^\times \times \mathbb{C}$, whose orbits are the origin in $\mathbb{C}$ and its complement $\mathbb{C}^\times$. When this groupoid is restricted to the original manifold $\mathbb{R}$, the two half lines now belong to the same orbit, even if the complexification $\mathbb{C}$ is replaced by a small neighborhood of the real axis. (In this case, the complexified groupoid would no longer be an action groupoid, but it would still have just the two orbits.) As a stack, the complexified groupoid represents a space with two points, one of which is an ordinary holomorphic point. The second point is in the closure of the first and has isotropy group $\mathbb{C}^{\times}$.

After restriction of the groupoid to the germ of $\mathbb{C}$ around M, or to the formal neighborhood, the notion of "orbit" is harder to pin down, since the groupoid does not directly define an equivalence relation.

A somewhat different result is obtained if the algebroid is first extended and then integrated. The extended complex Lie algebroid is $\mathbb{C} \times \mathbb{C}$; for its natural integration, the group is the simply connected cover $\mathbb{C}$ of $\mathbb{C}^{\times}$. The action groupoid is now $\mathbb{C} \times \mathbb{C}$ with the action $w \cdot z = e^{w} z$, for which the orbits are the same as before, but the isotropy group of nonzero z (including real z) is now $2\pi i \mathbb{Z}$.

Remark 2.3 A similar but slightly more complicated example is given by the vector field on the phase plane $M = \mathbb{R}^2$ which describes a classical mechanical system near a local maximum of the potential function. The complexication of the action groupoid $\mathbb{R} \times \mathbb{R}^2$ includes groupoid elements connecting states on opposite sides of the potential maximum which cannot be connected by real classical trajectories. These groupoid elements are not without physical interest, though, since they may be interpreted as representing quantum tunneling.

2.4 Foliations

An analytic foliation $E_{\mathbb{R}} \subset TM$ extends to a holomorphic foliation of $M_{\mathbb{C}}$, and, if $M_{\mathbb{C}}$ is small enough, the leaf stack of the latter is just a straightforward complexification of the (analytic) leaf stack of the former. In particular, if the former is a manifold, so is the latter.

But there are many foliations which admit no compatible analytic structure. Take for example the Reeb [33] foliation (or for that matter, according to Haefliger [18], any foliation) on S^3. The leaf space of the Reeb foliation consists of two circles and a special point whose only open neighborhood is the entire space. The isotropy group of the holonomy groupoid is trivial for the leaves on the circles and $\mathbb{Z}^2$ for the special leaf.

To complexify the Lie algebroid by complexifying the foliation groupoid, one might look instead at the equivalent groupoid given by restriction to a cross section to the leaves. This cross section can be taken as a copy of $\mathbb{R}$ on which $\mathbb{Z}^2$ acts, fixing the origin, with one of the two generators acting by 1-sided contractions on the left half line and the other by contractions on the right. Complexifying the action of the generators gives maps on $\mathbb{C}$ which have essential singularities at the origin, and there seems to be no way to make a holomorphic stack out of this data.

3 Involutive structures

A complex Lie algebroid E over M with injective anchor may be identified with the image of its anchor, which is an involutive subbundle of $T_{\mathbb{C}}M$. Following Treves [39], these subbundles will be called here *involutive structures*. An analytic structure on E is just an analytic structure on M for which E admits local bases of analytic complex vector fields.

Let E be an analytic subbundle of $T_{\mathbb{C}}M$, then, and $M_{\mathbb{C}}$ a complexification of M. Identifying $T_{\mathbb{C}}M$ with the restriction to M of $TM_{\mathbb{C}}$, one may extend the local bases of analytic sections of E to local holomorphic sections of $TM_{\mathbb{C}}$. For $M_{\mathbb{C}}$ sufficiently small, local bases again determine a holomorphic subbundle E' of $TM_{\mathbb{C}}$. Holomorphic continuation of identities implies that E' is itself involutive; by the holomorphic Frobenius theorem, it determines a holomorphic foliation of $M_{\mathbb{C}}$. The holonomy groupoid of this foliation determines a holomorphic stack which may be considered as the integration of the complex Lie algebroid E.

The rest of this section is devoted to examples of involutive structures viewed as CLAs.

3.1 Complex structures

An almost complex structure on M is an endomorphism $J : TM \to TM$ such that $-J^2$ is the identity. $T_{\mathbb{C}}M$ is the direct sum of the $-i$ and $+i$ eigenspaces of the complexified operator $J_{\mathbb{C}}$. These conjugate complex subbundles, denoted by $T_J^{0,1}M$ and $T_J^{1,0}M$ respectively, are involutive if and only if J is integrable in the sense that the Nijenhuis tensor N_J vanishes. The eigenspace $T_J^{0,1}M$ is then a CLA which, like J itself, is called a *complex structure*. It is a standard fact that every subbundle $E \subset T_{\mathbb{C}}M$ such that $E \oplus \overline{E} = T_{\mathbb{C}}M$ is $T_J^{0,1}M$ for some almost complex structure J.

Theorems of Eckmann–Frölicher [11] and Ehresmann [12] (analytic case)[2] and Newlander–Nirenberg [29] (smooth case) tell us that any complex structure on M is locally isomorphic to the standard one on $\mathbb{R}^{2n} = \mathbb{C}^n$; i.e., it gives a reduction of the atlas of smooth charts on M to a subatlas with holomorphic transition functions, making M into a complex manifold. Let us pretend for a moment, though, that we do not know those theorems and look directly at the integration of an analytic complex structure as a holomorphic stack. (The result of this exercise will turn out to be the original 1951 proof!)

According to the discussion above, complexification gives a foliation E' of a suitably small $M_{\mathbb{C}}$ whose leaves, by the condition $E \oplus \overline{E} = T_{\mathbb{C}}M$, have tangent spaces along M which are complementary to the real subbundle TM. As a result, shrinking $M_{\mathbb{C}}$ again can insure that each leaf is a holomorphic ball intersecting M exactly once, transversely, so that the leaf space of this foliation may be identified with M. This leaf space being a complex manifold, M itself inherits the structure of a complex manifold. Holomorphic local coordinates on M result from sliding open sets in M along the foliation E' to identify them with open sets in holomorphic transversals, e.g., leaves of the holomorphic foliation $\overline{E'}$ which extends $\overline{E}$.

The holomorphic stack in this case may be identified with M as a complex manifold, presented by the holonomy groupoid of the foliation E'. An alternate presentation is the etale groupoid obtained by restricting the holonomy groupoid to the union

[2]The cited authors also attribute the result to de Rham.

of enough transversals to cover M under projection along E'. The latter groupoid is just the equivalence relation associated to a covering of M by holomorphic charts.

When E is given simply as a smooth complex structure, the only recourse is to invoke the Newlander–Nirenberg theorem. This has the consequence that M has an analytic structure in which E is analytic, so the previous situation is obtained.

Remark 3.1 The analytic structure on M which makes a complex structure E analytic is unique, since it must be the one attached to the holomorphic structure determined by E. The situation is therefore different from that for the complex Lie algebroid $T_{\mathbb{C}}M$ and the zero Lie algebroid, whose integration depends on an arbitrary choice of analytic structure compatible with the given smooth structure.

3.2 CR structures

A step beyond the complex structures within the class of involutive systems are the general *CR structures*. These are subbundles E of $T_{\mathbb{C}}M$ such that E and $\overline{E}$ intersect only in the zero section, but $E \oplus \overline{E}$ is not necessarily all of $T_{\mathbb{C}}M$.[3]

Any "generic" real submanifold M in a complex manifold X inherits a CR structure, namely the intersection $G_{M,X} = T_{\mathbb{C}}M \cap T_J^{0,1}X$. To be precise, the submanifold is called generic when $G_{M,X}$ has constant dimension; note that real hypersurfaces are always generic in this sense. $G_{M,X} \oplus \overline{G_{M,X}}$ is the complexification of the maximal complex subbundle $F_{M,X}$ of TM. A natural geometric problem, which has led to fundamental developments in linear PDE theory, is whether a given CR manifold can be realized either locally or globally as a submanifold in some complex manifold, and in particular in $\mathbb{C}^n$. For analytic CR structures, the integration method of this paper solves this problem. What follows below essentially reproduces an argument of Andreotti and Fredricks [1], or more precisely, that in the review by Rossi [35] of that paper.

Let E' be the integrable holomorphic subbundle of $TM_{\mathbb{C}}$ which extends E. The corresponding foliation will be called the *CR foliation*. If M has (real) dimension $2n+r$ and E has complex dimension n, then $M_{\mathbb{C}}$ has complex dimension $2n + r$, and the leaves of the CR foliation have complex dimension n; each of them meets M in a point, with no common tangent vectors (since E contains no real vectors). It follows that $M_{\mathbb{C}}$ can be chosen so that the leaves are simply connected; the stack defined by the foliation groupoid is then simply a complex manifold N of complex dimension $n + r$ containing M as a real hypersurface of real codimension r. When $r = 0$, $N = M$, and M is a complex manifold; when $n = 0$ (zero Lie algebroid), $N = M_{\mathbb{C}}$. (Andreotti and Fredricks [1] call N a complexification of M for any n; thus, the complexification of a complex manifold is the manifold itself.)

3.3 The Mizohata structure

The next example shows that the natural map from M to a stack which integrates a complex Lie algebroid $E \to M$ may not be injective.

[3] Some authors use the term "CR structure" only when $E \oplus \overline{E}$ is of codimension 1 in $T_{\mathbb{C}}M$.

As in Example I.10.1 of Treves [39], the *Mizohata structure* over $M = \mathbb{R}^2$ is defined to be the involutive system E spanned by the complex vector field

$$i\partial/\partial t - t\,\partial/\partial x.$$

It is a complex structure except along the x-axis, where it is the complexification of the real subspace spanned by $\partial/\partial t$. The holomorphic continuation of E over $\mathbb{C}^2$ is spanned by the same vector field in which (x, t) are taken as complex variables, and the leaves of the corresponding foliation E' are the levels of the invariant function $\zeta = x - it^2/2$. These levels, which can be described as graphs $x = it^2/2 + \zeta$ with the parameter t running through $\mathbb{C}$, are contractible, so the stack defined by the foliation groupoid is isomorphic to $\mathbb{C}$ with ζ as its complex coordinate. The natural map from M to this stack folds $\mathbb{R}^2$ along the x-axis, and the image is the (closed) lower half plane.

The situation becomes more complicated rather than simpler if the complexification is shrunk to a neighborhood of $\mathbb{R}^2$ in $\mathbb{C}^2$, for instance that defined by the bounds $|\Im t| < \epsilon$ and $|\Im z| < \epsilon$ on the imaginary parts. In this case, some of the level manifolds of ζ split into two components, so that the corresponding part of the leaf space (the complement of a strip near the origin in the lower half plane) bifurcates into two branches.[4] The common closure of these branches is a family of leaves depending on one (real) parameter, so we can describe the integration of the Mizohata structure (or the "complexification", in the language used in CR geometry) as the non-Hausdorff complex manifold which is the union of an open strip along the real axis in the complex ζ-plane with two copies of the rest of the lower half plane. The map from M to this stack now separates points except those in a strip along the x axis, which is folded as before.

Integrals of the involutive structure on M must be even in t near the x axis; since they are holomorphic away from the x axis, they must be even everywhere. In this case, there are integrals of E which are not the pullback of holomorphic functions on the stack. (See Example III.2.1 in Treves [39].)

It is not clear what kind of geometric object is obtained in the limit as the complexification shrinks down to M, or for the formal complexification.

A test problem for any global theory of integration is to describe the integration of involutive structures on smooth surfaces which have Mizohata-type singularities along a collection of simple closed curves and which are complex structures elsewhere.

3.4 Eastwood–Graham and LeBrun–Mason structures

In the next example, due to Eastwood and Graham [10], the map from M to the stack integrating a complex Lie algebroid has nondiscrete fibres.

Consider $\mathbb{C}^2$ with coordinates $z = x + iy$ and $w = s + it$ and the involutive structure spanned by $\partial/\partial x + i\,\partial/\partial y$ and $\partial/\partial t - (x + iy)\,\partial/\partial s$, or, in complex notation, $\partial/\partial \bar{z}$ and $\partial/\partial t - z\,\partial/\partial s$. When $y \neq 0$, this is a complex structure, while when $y = 0$, it contains the real subspace spanned by $\partial/\partial t - x\,\partial/\partial s$. The integrals for this structure are generated by $z = x + iy$ and $\zeta = s + zt$. On the complexification $\mathbb{C}^2_{\mathbb{C}} = \mathbb{C}^4, x, y, s,$

[4] There is no bifurcation in the upper half plane.

and t may have complex values, and then the map $(z, \zeta) : \mathbb{C}^2_{\mathbb{C}} \to \mathbb{C}^2$ is a submersion whose fibres are the leaves of the extended foliation; thus, the leaf space (and hence the stack which integrates the structure) may be identified with the complex (z, ζ) plane.

What is singular here is the map ϕ from the original $\mathbb{C}^2 = \mathbb{R}^4$ to this stack. When the variables (x, y, s, t) are real, ϕ is a local diffeomorphism, except on the hypersurface $y = 0$, where each of the orbits of the vector field $\partial/\partial t - x\, \partial/\partial s$ is mapped to a constant. The image of this hypersurface is the subset of the (z, ζ) plane on which the variables are both real, and, as is clearly described by Eastwood and Graham [10], the map ϕ realizes the (real) blow-up of $\mathbb{R}^2$ in $\mathbb{C}^2$.

A similar involutive structure was constructed by Lebrun and Mason [23] on the projectivized complexified tangent bundle of a surface with affine connection; the singular curves in their example are the geodesics.

4 Boundary Lie algebroids

This section exhibits CLAs which are neither involutive systems nor the complexification of real Lie algebroids. The example is taken from work with Leichtnam and Tang [24] on Kähler geometry and deformation quantization in the setting of CLAs. The description of the integration of these CLAs is not complete.

Let X be a complex manifold of (complex) dimension $n + 1$ with boundary M, and let $\mathcal{E}_{M,X}$ be the space of complex vector fields on X whose values along M lie in the induced CR structure $G_{M,X}$. $\mathcal{E}_{M,X}$ is a module over $C^\infty(X)$ and is closed under bracket. The following lemma shows that it may be identified with the space of sections of a complex Lie algebroid $E_{M,X}$.

Lemma 4.1 *$\mathcal{E}_{M,X}$ is a locally free $C^\infty(X)$-module.*

Proof. Away from the boundary, $\mathcal{E}_{M,X}$ is the space of sections of $T_{\mathbb{C}}M$, hence locally free. Near a boundary point, choose a defining function ψ, i.e., a function which vanishes on the boundary and has no critical points there. Next, choose a local basis $\overline{v}_1, \dots, \overline{v}_n$ of $G_{M,X}$ and extend it to a linearly independent set of sections of $T^{0,1}X$, still denoted by $\overline{v}_j$, defined in an open subset of X, to be shrunk as necessary. Let v_j be the complex conjugate of $\overline{v}_j$. These vectors all annihilate ψ on M; there is no obstruction to having them annihilate ψ everywhere. Next, choose a local section $\overline{v}_0$ of $T^{0,1}X$ such that $\overline{v}_0 \cdot \psi = 1$, and let v_0 be its conjugate. This gives a local basis $(v, \overline{v})$ for the complex vector fields. Such a vector field belongs to $\mathcal{E}_{M,X}$ if and only if, when it is expanded with respect to this basis, the coefficients of $\overline{v}_0$ and all the v_j vanish along M. Since this means that all these coefficients are divisible by ψ with smooth quotient, setting $u'_0 = \psi \overline{v}_0$, $u'_j = \overline{v}_j$ for $j = 1, \dots, n$, and $u_j = \psi v_j$ for $j = 0, \dots, n$ produces a local basis (u, u') for $\mathcal{E}_{M,X}$. □

To integrate the boundary Lie algebroid $E_{M,X}$, assuming analyticity as usual, one may begin by extending X slightly beyond M, so that M becomes an embedded hypersurface. In the complexification $X_{\mathbb{C}}$, M extends to a submanifold $M_{\mathbb{C}}$ of complex codimension 1. The CR structure on M extends (see Section 3.2) to the tangent bundle E' of the CR foliation on $M_{\mathbb{C}}$. The holomorphic continuation of $E_{M,X}$ is then the holomorphic Lie algebroid whose local sections are the vector fields on $X_{\mathbb{C}}$ whose restrictions to $M_{\mathbb{C}}$ have their values in E'.

What is the groupoid of this Lie algebroid over $X_{\mathbb{C}}$? Over the complement of $M_{\mathbb{C}}$, the Lie algebroid is the tangent bundle, so the groupoid could be taken to be the pair groupoid. Since $M_{\mathbb{C}}$ has complex codimension 1, though, its complement generally has a nontrivial fundamental group, and the fundamental groupoid or one of its nontrivial quotients might be appropriate as well. The choice depends in part on compatibility with the choice made on $M_{\mathbb{C}}$ itself.

Over $M_{\mathbb{C}}$, the image of the anchor of the extended Lie algebroid is the tangent bundle E' to the CR foliation, but now, unlike in the pure CR situation, there is nontrivial isotropy. To describe this isotropy, note that, at each point x of $M_{\mathbb{C}}$, there is a flag $E'_x \subset T_x M_{\mathbb{C}} \subset T_x X_{\mathbb{C}}$. The isotropy algebra may be identified with the endomorphisms of the normal space $T_x X_{\mathbb{C}}/E'_x$ which vanish on $T_x M_{\mathbb{C}}$. Given two points x and y in $M_{\mathbb{C}}$, there are morphisms in the integrating groupoid from x to y if and only if x and y lie in the same leaf of the CR foliation. Each such morphism is then a linear map $T_x X_{\mathbb{C}}/E'_x \rightarrow T_y X_{\mathbb{C}}/E'_y$ whose restriction $T_x M_{\mathbb{C}}/E'_x \rightarrow T_y M_{\mathbb{C}}/E'_y$ coincides with the linearized holonomy map along any path in the leaf. (Assume that the complexification is small enough so that the leaves are simply connected.) In particular, when $x = y$, the isotropy group consists of the automorphisms of $T_x X_{\mathbb{C}}/E'_x$ which fix $T_x M_{\mathbb{C}}/E'_x$. (Compare the author's discussion in Section 6 of [42], where the Lie algebroid and its integrating groupoid are studied for the vector fields tangent to the boundary of a real manifold, as well as the treatment by Mazzeo [25] of vector fields tangent to the fibres of a submersion on the boundary. Finally, a slightly different, class of vector fields on a manifold with fibred boundary is used by Mazzeo and Melrose [26].)

When x lies on the real hypersurface M, the space above admits an explicit description in terms of the CR geometry. Over M, $TX_{\mathbb{C}}$ restricts to $T_{\mathbb{C}}X$, $TM_{\mathbb{C}}$ is just $T_{\mathbb{C}}M$, and E' is the CR structure $G_{M,X} = T_{\mathbb{C}}M \cap T_J^{0,1}X$. Thus, the isotropy of the integrating groupoid consists of the automorphisms of $T_{\mathbb{C}}X/T_{\mathbb{C}}M \cap T_J^{0,1}X$ which fix its codimension 1 subspace $T_{\mathbb{C}}M/T_{\mathbb{C}}M \cap T_J^{0,1}X$. These automorphisms act on the complexified normal bundle $T_{\mathbb{C}}X/T_{\mathbb{C}}M$, and those which act trivially on the normal bundle are "shears" which may be identified with the additive group of linear maps from that normal bundle to $T_{\mathbb{C}}M/T_{\mathbb{C}}M \cap T_J^{0,1}X$. The choice of a defining function trivializes the normal bundle, so the isotropy is an extension of the automorphism (or "dilation") group of the normal bundle by the abelian group $T_{\mathbb{C}}M/T_{\mathbb{C}}M \cap T_J^{0,1}X$.

The preceding description of the integrating groupoid is not complete, since it lacks an explanation of how the piece over the interior and the piece over the boundary fit together. In particular, if one were to use the fundamental groupoid on the interior, as described above, it may be necessary to use a covering of the automorphisms of the line bundle on the boundary.

5 Generalized complex structures

In the rapidly developing subject of *generalized geometry*, originated by Hitchin [19], the tangent bundle TM of a manifold with its Lie algebroid structure is replaced by the *generalized tangent bundle* $\mathcal{T}M$, which is the direct sum $TM \oplus T^*M$ equipped with the Courant algebroid structure consisting of the bracket

$$[\![(\xi_1, \theta_1), (\xi_2, \theta_2)]\!] = \left([\xi_1, \xi_2], \mathcal{L}_{\xi_1}\theta_2 - \mathcal{L}_{\xi_2}\theta_1 - \frac{1}{2}d(i_{\xi_1}\theta_2 - i_{\xi_2}\theta_1)\right),$$

the anchor $\mathcal{T}M \to TM$ which projects to the first summand, and the symmetric bilinear form

$$\langle(\xi_1, \theta_1), (\xi_2, \theta_2)\rangle = \frac{1}{2}(i_{\xi_1}\theta_2 + i_{\xi_2}\theta_1).$$

Like the tangent bundle, $\mathcal{T}M$ may be complexified to the "complex Courant algebroid" $\mathcal{T}_{\mathbb{C}}M$. It is not a complex Lie algebroid, but it contains many CLAs, in particular the *complex Dirac structures*, i.e., the (complex) subbundles E which are maximal isotropic for the symmetric form and whose sections are closed under the bracket. For instance, if $A \subseteq T_{\mathbb{C}}M$ is an involutive system and $A^{\perp} \subseteq T^*_{\mathbb{C}}M$ is its annihilator, then $A \oplus A^{\perp}$ is a complex Dirac structure.

Of special interest among the complex Dirac structures are those for which $E \oplus \overline{E} = \mathcal{T}M$. These are called *generalized complex structures* and are the $-i$ eigenspaces of (the complexifications of) integrable almost complex structures $\mathcal{J} : \mathcal{T}M \to \mathcal{T}M$; the integrability condition here is that the Nijenhuis torsion is zero, the usual bracket of vector fields in the definition of the torsion being replaced by the Courant bracket.

In particular given a complex structure $J : TM \to TM$, with associated CLA $T_J^{0,1}M$, the direct sum with its annihilator is the generalized complex structure $\mathcal{T}_J^{0,1}M = T_J^{0,1}M \oplus T_J^{1,0^*}M$. The image of the anchor is the involutive system $T_J^{0,1}M$, but $\mathcal{T}_J^{0,1}M$ itself is not an involutive system, since the kernel of its anchor is the nontrivial bundle $T_J^{1,0^*}M$. Also, $\mathcal{T}_J^{0,1}M$ is not isomorphic to the complexification of a real Lie algebroid, since the image of its anchor is not invariant under complex conjugation.

Another kind of example arises from symplectic structures on M, viewed as bundle maps $\omega : TM \to T^*M$. Here, the generalized complex structure E_ω is defined to be the graph of the complex 2-form $i\omega$. This time, the anchor is bijective, so, as a Lie algebroid, E_ω is isomorphic to $T_{\mathbb{C}}M$.

What is the integration, in the sense of this paper, of a generalized complex structure? First, let J be a complex structure on M, $\mathcal{T}_J^{0,1}M = T_J^{0,1}M \oplus T_J^{1,0^*}M$ the corresponding generalized structure. Complexifying M and J as in Section 3.1 gives a foliation on $M_{\mathbb{C}}$. The groupoid which integrates the holomorphic continuation of $\mathcal{T}_J^{0,1}M$ is the semidirect product groupoid obtained from the action of the holonomy groupoid of the foliation (via the "Bott connection") on its conormal bundle. (This is just the holomorphic version of a construction by Bursztyn, Crainic, Zhu, and the author [4].)

This action groupoid is equivalent to the holomorphic leaf space M carrying the cotangent bundle $T_J^{1,0*}M$ of additive groups as its isotropy. The corresponding stack is the bundle over M whose fibres are the "universal classifying stacks" of the cotangent spaces.

Next let ω be a symplectic structure on M. Since the generalized complex structure E_ω is isomorphic to $T_{\mathbb{C}}M$, its integration must be that of $T_{\mathbb{C}}M$, i.e., the holomorphic point, perhaps carrying the fundamental group of M as isotropy. To see what has become of ω, it is best to look again at (real and complex) Dirac structures.

As a subbundle of $\mathcal{T}M$, a Dirac structure E carries a natural skewsymmetric bilinear form, the restriction of

$$B(\xi_1, \theta_1), (\xi_2, \theta_2)) = (1/2)(i_{\xi_1}\theta_2 - i_{\xi_2}\theta_1).$$

It is shown in [4] that this form gives rise to a multiplicative closed 2-form on a groupoid integrating E, producing a *presymplectic groupoid*. Applying this construction to the holomorphic extension of any complex Dirac structure E shows that its integration as a CLA is a holomorphic symplectic groupoid over $M_{\mathbb{C}}$. In particular, for E_ω or any other complex Poisson structure, it is a holomorphic symplectic groupoid. For E_J, or any other direct sum of an involutive structure with its annihilator, the restriction of B is zero, and hence so is the presymplectic structure on the integrating groupoid.

6 Further topics and questions

A notion of integration for complex Lie algebroids has been proposed in this paper. There are many interesting questions about other extensions of Lie algebroid theory to the complex case, including the relation between these extensions and the integration construction proposed here. Some examples conclude this paper.

6.1 Integrability

Does the integrability criterion of Crainic and Fernandes [8] apply in the holomorphic case? What are the conditions on an analytic CLA which determine whether its holomorphic continuation is integrable? What can one do in the nonanalytic case?

6.2 Cohomology

A "van Est" theorem of Crainic [7] describes the relation between the cohomology of a Lie algebroid and that of its integrating groupoids. The definition of cohomology extends in a straightforward to CLAs (for instance, it gives the Dolbeault cohomology in the case of a complex structure). Is there a van Est theorem in this case, too?

6.3 Bisections

One consequence of the integration of a Lie algebroid E is that the submanifolds of an integrating groupoid which are sections for the source and target maps form a group whose Lie algebra in some formal sense is the space of sections of E. Is there a similar construction for the case of a complex Lie algebroid? Some hints might come from the constructions by Neretin [27] and Segal [36] (also see Yuriev [45]) of a semigroup which in some sense integrates the complexified Lie algebra of vector fields on a circle. Conversely, a general construction for CLAs could provide complexifications for the diffeomorphism groups of other manifolds.

6.4 Quantization

Once a Lie algebroid E has been integrated, the groupoid algebra of an integrating groupoid may be considered, following Landsman and Ramazan [22], as a deformation quantization of the Poisson structure on the dual bundle E^*, or as a completion of Rinehart's [34] universal enveloping algebra of E. Is there a corresponding application for the integration of a CLA?

On the other hand, given a complex Poisson structure Π on M, it defines a CLA structure on the complexified cotangent bundle. Integration of this structure should give a holomorphic symplectic groupoid which should be somehow related to the deformation quantization of (M, π). On the formal level (without integration), it is possible [24] to extend the methods of Karabegov [20] and Nest and Tsygan [28] to construct deformation quantizations of certain boundary Lie algebroids as in Section 4 above.

6.5 Connections and representations

If E is a CLA over M and V is a complex vector bundle V, an E-connection on V is a map $a \mapsto \nabla_a$ from the sections of E to the $\mathbb{C}$-endomorphisms of the sections of V which satisfies the conditions $\nabla_{fa} u = f \nabla_a u$ and $\nabla_a g u = g \nabla_a u + (\rho(a)g)u$. The connection is flat and is also called a representation of E on V if the map $a \mapsto \nabla_a$ is a Lie algebra homomorphism.

For instance, if E is a complex structure, a representation of E on V is a holomorphic structure on V. More generally, representations of CR structures correspond to CR vector bundles, as in the work of Webster [41]. After complex extension, an analytic representation of an analytic CR structure becomes a flat connection along the leaves of the CR foliation, which leads to a holomorphic vector bundle on the complexification.

If E is the generalized complex structure associated to a complex structure on M, a representation on V is a holomorphic structure on V together with a holomorphic action of the (abelian) cotangent spaces of M as endomorphisms of the fibres of V.

6.6 The modular class

The modular class of a Lie algebroid, introducted by Evens, Lu, and the author [14] is the obstruction to the existence of an "invariant measure." Its definition extends directly to the case of CLAs. For a complex structure, the modular class is the obstruction to the existence of a Calabi–Yau structure.

References

1. Andreotti, A., and Fredricks, G.A., Embeddability of real analytic Cauchy–Riemann manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **6** (1979), 285–304.
2. Behrend, K. Cohomology of stacks, Intersection theory and moduli, *ICTP Lect. Notes* **XIX**, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, 249–294 (electronic).
3. Block, J., Duality and equivalence of module categories in noncommutative geometry, preprint math.QA/0509284.
4. Bursztyn, H., Crainic, M., Weinstein, A., and Zhu, C., Integration of twisted Dirac brackets, *Duke Math. J.* **123** (2004), 549–607.
5. Cannas da Silva, A., and Weinstein, A., *Geometric Models for Noncommutative Algebras*, Berkeley Math. Lecture Notes, Amer. Math. Soc., Providence, 1999.
6. Chemla, S., A duality property for complex Lie algebroids, *Math. Z.* **232** (1999), 367–388.
7. Crainic, M., Differentiable and algebroid cohomology, van Est isomorphisms, and characteristic classes, *Comment. Math. Helv* **78** (2003), 681–721.
8. Crainic, M., and Fernandes, R., Integrability of Lie brackets, *Ann. Math.* **157** (2003), 575–620.
9. Dazord, P., Groupoïde d'holonomie et géométrie globale, *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), 77–80.
10. Eastwood, M., and Graham, C.R., The involutive structure on the blow-up of $\mathbb{R}^n$ in $\mathbb{C}^n$, *Comm. Anal. Geom.* **7** (1999), 609–622.
11. Eckmann, B., and Frölicher, A., Sur l'intégrabilité des structures presque complexes, *C. R. Acad. Sci. Paris* **232** (1951), 2284–2286.
12. Ehresmann, C., Sur les variétés presque complexes, *Proceedings of the International Congress of Mathematicians, Cambridge, Mass., 1950, vol. 2*, Amer. Math. Soc., Providence, R. I., **1952**, 412–419.
13. Epstein, C.L., A relative index on the space of embeddable CR-structures, I and II, *Ann. of Math.* **147** (1998), 1–59 and 61-91.
14. Evens, S., Lu, J.-H., and Weinstein, A., Transverse measures, the modular class, and a cohomology pairing for Lie algebroids, *Quart. J. Math* **50** (1999), 417–436.
15. Fedosov, B., A simple geometrical construction of deformation quantization, *J. Diff. Geom.* **40** (1994), 213–238.
16. Grauert, H., On Levi's problem and the imbedding of real-analytic manifolds, *Ann. of Math.* **68** (1958), 460–472.
17. Gualtieri, M., Generalized complex geometry, Oxford University D.Phil. thesis, 2003, math.DG/0401221.
18. Haefliger, A., Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes, *Comment. Math. Helv.* **32** (1958), 248–329.
19. Hitchin, N., Generalized Calabi–Yau manifolds, *Q. J. Math.* **54** (2003), 281–308.

20. Karabegov, A., Deformation quantizations with separation of variables on a Kähler manifold, *Comm. Math. Phys.* **180** (1996), 745–755
21. Kutzschebauch, F., On the uniqueness of the analyticity of a proper G-action, *Manuscripta Math.* **90** (1996), 17–22.
22. Landsman, N. P., and Ramazan, B., Quantization of Poisson algebras associated to Lie algebroids, Groupoids in analysis, geometry, and physics (Boulder, CO, 1999), *Contemp. Math.* **282**, Amer. Math. Soc., Providence, RI, (2001) 159–192.
23. Lebrun, C., and Mason, L.J., Zoll manifolds and complex surfaces, *J. Diff. Geom.* **61** (2002), 453–535.
24. Leichtnam, E., Tang, X, and Weinstein, A., Poisson geometry near a strictly pseudo-convex boundary and deformation quantization, *J. Eur. Math. Soc.* (to appear). Preprint math.DG/0603350.
25. Mazzeo, R., Elliptic theory of differential edge operators. I, *Comm. Partial Differential Equations* **16** (1991), 1615–1664.
26. Mazzeo, R., and Melrose, R., Pseudodifferential operators on manifolds with fibred boundaries, in *Mikio Sato: a great Japanese mathematician of the twentieth century, Asian J. Math.* **2** (1998), 833–866.
27. Neretin, Yu. A., On a complex semigroup containing the group of diffeomorphisms of the circle, *Funct. Anal. Appl.* **21** (1987), 160–161.
28. Nest, R., and Tsygan, B., Deformations of symplectic Lie algebroids, deformations of holomorphic symplectic structures, and index theorems, *Asian J. Math.* **5** (2001), 599–635.
29. Newlander, A., and Nirenberg, L., Complex analytic coordinates in almost complex manifolds. *Ann. of Math.* **65** (1957), 391–404.
30. Omori, H., On the group of diffeomorphisms on a compact manifold, *Global Analysis (Proc. Sympos. Pure Math., Vol. XV, Berkeley, Calif., 1968)*, Amer. Math. Soc., Providence, R.I. (1970),OA 167–183.
31. Omori, H., Maeda, Y., Miyazaki, N., and Yoshioka, A., Strange phenomena related to ordering problems in quantizations, *J. Lie Theory* **13** (2003), 479–508.
32. Omori, H., Maeda, Y., and Yoshioka, A., Weyl manifolds and deformation quantization, *Advances in Math.* **85** (1991), 224–255.
33. Reeb, G., Sur certaines propriétés topologiques des variétés feuilletées, Actualités Sci. Ind., no. 1183, *Publ. Inst. Math. Univ. Strasbourg* **11**, Hermann & Cie., Paris (1952), 5–89, 155–156.
34. Rinehart, G.S., Differential forms on general commutative algebras, *Trans. Amer. Math. Soc.* **108** (1963), 195-222.
35. Rossi, H., review of [1], *MathSciNet* review MR0541450 (1980).
36. Segal, G., The definition of conformal field theory, *Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser.* **308**, Cambridge Univ. Press, Cambridge, 2004, 421–577.
37. Shutrick, H.B., Complex extensions, *Quart. J. Math. Oxford Ser. (2)* **9** (1958) 189–201.
38. Tanaka, N., A Differential Geometric Study on Strongly Pseudo-convex Manifolds, Tokyo, Kinokuniya Book-store Co., 1975.
39. Treves, F., *Hypo-Analytic Structures*, Princeton University Press, Princeton, 1992.
40. Tseng, H.-H., and Zhu, C., Integrating Lie algebroids via stacks, *Compositio Math.* **142** (2006), 251–270.
41. Webster, S.M., The integrability problem for CR vector bundles, Several Complex Variables and Complex Geometry, Part 3 (Santa Cruz, CA, 1989), *Proc. Sympos. Pure Math.*, **52** Part 3, Providence, Amer. Math. Soc., 1991, 355–368.
42. Weinstein, A., Groupoids: unifying internal and external symmetry, *Notices A.M.S.* **43** (1996), 744–752, reprinted in *Contemp. Math.* **282**, (2001), 1–19.

43. Whitney, H., Differentiable manifolds, *Annals of Math.* **37** (1936), 645–680.
44. Whitney, H, and Bruhat, F., Quelques propriétés fondamentales des ensembles analytiques-réels, *Comment. Math. Helv.* **33** (1959) 132–160.
45. Yuriev, D.V., Infinite-dimensional geometry of the universal deformation of the complex disk, *Russian J. Math. Phys.* **2** (1994), 111–121.

Reduction, Induction and Ricci Flat Symplectic Connections

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We are pleased to dedicate this paper to Hideki Omori on the occasion of his 65th birthday

Summary. In this paper we present a construction of Ricci-flat connections through an induction procedure. Given a symplectic manifold (M, ω) of dimension $2n$, we define induction as a way to construct a symplectic manifold (P, μ) of dimension $2n + 2$. Given any symplectic manifold (M, ω) of dimension $2n$ and given a symplectic connection ∇ on (M, ω), we define induction as a way to construct a symplectic manifold (P, μ) of dimension $2n+2$ and an induced connection ∇^P which is a Ricci-flat symplectic connection on (P, μ).

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Introduction

A symplectic connection on a symplectic manifold (M, ω) is a torsionless linear connection ∇ on M for which the symplectic 2-form ω is parallel. A symplectic connection exists on any symplectic manifold and the space of such connections is an affine space modelled on the space of symmetric 3-tensorfields on M.

In what follows, the dimension $2n$ of the manifold M is assumed to be ≥ 4 unless explicitly stated. The curvature tensor R^∇ of a symplectic connection ∇ decomposes [5] under the action of the symplectic group into two irreducible components, $R^\nabla = E^\nabla + W^\nabla$. The E^∇ component is defined only in terms of the Ricci-tensor r^∇ of ∇. All traces of the W^∇ component vanish.

Two particular types of symplectic connections thus arise:

— symplectic connections for which $W^\nabla = 0$; we call them Ricci-type symplectic connections;

— symplectic connections for which $E^\nabla = 0$; they are called Ricci-flat since $E^\nabla = 0 \Leftrightarrow r^\nabla = 0$.

When studying [1] local and global models for Ricci-type symplectic connections, (or more generally [2] so-called special symplectic connections), Lorenz Schwachhöfer and the present authors were led to consider examples of the following construction:

- start with a symplectic manifold (M, ω) of dimension $2n$;
- build a (cooriented) contact manifold (N, α) of dimension $2n+1$ and a submersion $\pi : N \to M$ such that $d\alpha = \pi^*\omega$;
- define on the manifold $P = N \times \mathbb{R}$ a natural symplectic structure μ.

It was observed [1] that if (M, ω) admits a symplectic connection of Ricci type, one could "lift" this connection to P and the lifted connection is symplectic (relative to μ) and flat.

The aim of this paper is to generalize this result. More precisely we formalize a notion of induction for symplectic manifolds. Starting from a symplectic manifold (M, ω), we define a contact quadruple (M, N, α, π), where N, α and π are as above, and we build the corresponding $2n + 2$-dimensional symplectic manifold (P, μ). We prove the following:

Theorem 4.1 *Let (M, ω) be a symplectic manifold which is the first element of a contact quadruple (M, N, α, π). Let ∇ be an arbitrary symplectic connection on (M, ω). Then one can lift ∇ to a symplectic connection on (P, μ) which is Ricci-flat.*

This theorem has various applications. In particular one has

Theorem 5.3 *Let (P, μ) be a symplectic manifold admitting a conformal vector field S which is complete, a symplectic vector field E which commutes with S and assume that, for any $x \in P$, $\mu_x(S, E) > 0$. Assume the reduction of $\Sigma = \{x \in P \mid \mu_x(S, E) = 1\}$ by the flow of E has a manifold structure M with $\pi : \Sigma \to M$ a surjective submersion. Then (P, μ) admits a Ricci-flat connection.*

The paper is organized as follows. In Section 1 we study sufficient conditions for a symplectic manifold (M, ω) to be the first element of a contact quadruple and we give examples of such quadruples. Section 2 is devoted to the lift of hamiltonian (resp. conformal) vector fields from (M, ω) to the induced symplectic manifold (P, μ) constructed via a contact quadruple. We show that if (M, ω) is conformal homogeneous, so is (P, μ). Section 3 describes the structure of conformal homogeneous symplectic manifolds; this part is certainly known but as we had no immediate reference we decided to include it. Section 4 gives some constructions of lifts of symplectic connections of (M, ω) to symplectic connections on the induced symplectic manifold (P, μ) constructed via a contact quadruple. We also prove Theorem 4.1. In Section 5 we give conditions for a symplectic manifold (P, μ) to be obtained by induction from a contact quadruple (M, N, α, π). We give also a proof of Theorem 5.3.

1 Induction and contact quadruples

Definition 1.1 A *contact quadruple* is a quadruple (M, N, α, π) where M is a $2n$-dimensional smooth manifold, N is a smooth $(2n+1)$-dimensional manifold, α is a cooriented contact structure on N (i.e., α is a 1-form on N such that $\alpha \wedge (d\alpha)^n$ is nowhere vanishing), $\pi : N \to M$ is a smooth submersion and $d\alpha = \pi^*\omega$ where ω is a symplectic 2-form on M.

Definition 1.2 Given a contact quadruple (M, N, α, π) the *induced symplectic manifold* is the $(2n+2)$-dimensional manifold

$$P := N \times \mathbb{R}$$

endowed with the (exact) symplectic structure

$$\mu := 2e^{2s}\, ds \wedge p_1^*\alpha + e^{2s}\, dp_1^*\alpha = d(e^{2s}\; p_1^*\alpha)$$

where s denotes the variable along $\mathbb{R}$ and $p_1 : P \to N$ the projection on the first factor.

Remark 1.3 The word induction has been used by various authors in symplectic geometry, with different meanings. In [4], Guillemin and Sternberg consider a construction which is a symplectic analogue of the induced representation construction. Induction in the sense of building a $(2n+2)$-dimensional symplectic manifold from a symplectic manifold of dimension $2n$ is considered by Kostant in [4] (see further Example 2).

Remark 1.4

- The vector field $S := \partial_s$ on P is such that $i(S)\mu = 2\mathrm{e}^{2s}(p_1^*\alpha)$; hence $L_S\mu = 2\mu$ and S is a conformal vector field.
- The Reeb vector field Z on N (i.e., the vector field Z on N such that $i(Z)d\alpha = 0$ and $i(Z)\alpha = 1$) lifts to a vector field E on P such that: $p_{1*}E = Z$ and $ds(E) = 0$. Since $i(E)\mu = -d(\mathrm{e}^{2s})$, E is a Hamiltonian vector field on (P, μ). Furthermore

$$[E, S] = 0,$$
$$\mu(E, S) = -2\mathrm{e}^{2s}.$$

- Observe also that if $\Sigma = \{\, y \in P \mid s(y) = 0 \,\}$, the reduction of (P, μ) relative to the constraint manifold Σ (which is isomorphic to N) is precisely (M, ω).
- For $y \in P$ define $H_y(\subset T_yP) => E, S <^{\perp_\mu}$. Then H_y is symplectic and $(\pi \circ p_1)_{*y}$ defines a linear isomorphism between H_y and $T_{\pi p_1(y)}M$. Vector fields on M thus admit "horizontal" lifts to P.

We shall now make some remarks on the existence of a contact quadruple, the first term of which corresponds to a given symplectic manifold (M, ω).

Lemma 1.5 *Let (M, ω) be a smooth symplectic manifold of dimension $2n$ and let N be a smooth $(2n+1)$-dimensional manifold admitting a smooth surjective submersion π on M. Let $\mathcal{H}$ be a smooth $2n$-dimensional distribution on N such that $\pi_{*x} : \mathcal{H}_x \to T_{\pi(x)}M$ is a linear isomorphism (remark that such a distribution may always be constructed by choosing a smooth Riemannian metric g on N and setting $\mathcal{H}_x = (\ker \pi_{*x})^{\perp}$). Then either there exists a smooth nowhere vanishing 1-form α and a smooth vector field Z such that $\forall x \in N$ we have (i) $\ker \alpha_x = \mathcal{H}_x$ (ii) $Z_x \in \ker \pi_{*x}$ (iii) $\alpha_x(Z_x) = 1$ or the same is true for a double cover of N.*

Proof. Choose an auxiliary Riemannian metric g on M and consider $N' = \{Z \in TN \mid Z \in \ker \pi_* \text{ and } g(Z, Z) = 1\}$. If N' has two components, one can choose a global vector field $Z \in \ker \pi_*$ on N and define a smooth 1-form α with $\ker \alpha = \mathcal{H}$ and $\alpha(Z) = 1$. If N' is connected, N' is a double cover of N ($p : N' \to N : Z_x \to x$) and we can choose coherently $Z' \in T_Z N'$ by the rule that its projection on $T_x N$ is precisely Z. □

This says that if we have a pair (M, N) with a surjective submersion $\pi : N \to M$ we can always assume (by passing eventually to a double cover of N) that there exists a nowhere vanishing vector field $Z \in \ker \pi_*$ and a nowhere vanishing 1-form α such that $\alpha(Z) = 1$ and $\ker \alpha$ projects isomorphically on the tangent space to M. The vector field Z is determined up to a non-zero multiplicative factor by the submersion π; on the other hand, having chosen Z, the 1-form α can be modified by the addition of an arbitrary 1-form β vanishing on Z. If $\tilde{\alpha} = \alpha + \beta$ is another choice, the 2-form $d\tilde{\alpha}$ is the pullback of a 2-form on M iff $i(Z)d\tilde{\alpha} = 0$; i.e., iff:

$$\text{(i) } L_Z \beta = -L_Z \alpha, \qquad \text{(ii) } \beta(Z) = 0.$$

This can always be solved locally. We shall assume this can be solved globally.

Lemma 1.6 *Let (M, ω) be a smooth symplectic manifold of dimension $2n$ and let N be a smooth $(2n+1)$-dimensional manifold admitting a smooth surjective submersion π on M. Let Z be a smooth nowhere vanishing vector field on N belonging to $\ker \pi_*$. Let α be a 1-form such that $\alpha(Z) = 1$. If $L_Z \alpha = \mu \alpha$, for a certain $\mu \in C^\infty(N)$, then $\mu = 0$ and $d\alpha$ is the pullback of a closed 2-form ν on M. Furthermore if X (resp. Y) is a vector field on M and $\bar{X}$ (resp. $\bar{Y}$) is the vector field on N such that (i) $\pi_* \bar{X} = X$ (resp. $\pi_* \bar{Y} = Y$) (ii) $\alpha(\bar{X}) = \alpha(\bar{Y}) = 0$, then:*

$$[\bar{X}, \bar{Y}] - \overline{[X, Y]} = -\pi^*(\nu(X, Y))Z,$$
$$[Z, \bar{X}] = 0.$$

Proof. We have $\pi_*[Z, \bar{X}] = 0, [Z, \bar{X}] = \alpha([Z, \bar{X}])Z = -(L_Z\alpha)(\bar{X})Z = 0$. Since $(L_Z\alpha)(Z) = d\alpha(Z, Z) = 0$, μ vanishes. Also:

$$i(Z)d\alpha = \mathcal{L}_Z \alpha = 0,$$

so $d\alpha$ is the pullback of a closed 2-form ν on M. Finally:

$$\pi_*[\bar{X}, \bar{Y}] = \pi_*\overline{[X, Y]},$$
$$[\bar{X}, \bar{Y}] = \overline{[X, Y]} + \alpha([\bar{X}, \bar{Y}])Z = \overline{[X, Y]} - d\alpha(\bar{X}, \bar{Y})Z.$$

□

Corollary 1.7 *If $\nu = \omega$, the manifold (N, α) is a contact manifold and Z is the corresponding Reeb vector.*

We shall now give examples of contact quadruples for given symplectic manifolds.

Example 1 Let $(M, \omega = d\lambda)$ be an exact symplectic manifold. Define $N = M \times \mathbb{R}$, $\pi = p_1$ (= projection of the first factor), $\alpha = dt + p_1^*\lambda$; then (N, α) is a contact manifold and (M, N, α, π) is a contact quadruple.

The associated induced manifold is $P = N \times \mathbb{R} = M \times \mathbb{R}^2$; with coordinates (t, s) on $\mathbb{R}^2$ and obvious identification

$$\mu = \mathrm{e}^{2s}\,[\,d\lambda + 2ds \wedge (dt + \lambda)\,].$$

Example 2 Let (M, ω) be a quantizable symplectic manifold; this means that there is a complex line bundle $L \xrightarrow{p} M$ with hermitian structure h and a connection ∇ on L preserving h whose curvature is proportional to $i\omega$.

Define $N := \{\xi \in L \mid h(\xi, \xi) = 1\} \subset L$ to be the unit circle sub-bundle. It is a principal $U(1)$ bundle and L is the associated bundle $L = N \times_{U(1)} \mathbb{C}$. The connection 1-form on N (representing ∇) is $u(1) = i\mathbb{R}$-valued and will be denoted α'; its curvature is $d\alpha' = ik\omega$. Define $\alpha := \frac{1}{ik}\alpha'$ and $\pi := p|_N : N \to M$ the surjective submersion. Then (M, N, α, π) is a contact quadruple.

The associated induced manifold P is in bijection with $L_0 = L\backslash$ zero section; indeed, consider

$$\Psi : L_0 \to P = N \times \mathbb{R} : \xi \to \left(\frac{\xi}{h(\xi, \xi)^{1/2}}, k \ln h(\xi, \xi)^{1/2} \right).$$

Clearly L_0 is a $\mathbb{C}^*$ principal bundle on M; denote by $\check{\alpha}$ the $\mathbb{C}^*$-valued 1-form on L_0 representing ∇; if $j_1 : N \to L_0$ is the natural injection and similarly $j_2 : i\mathbb{R} \to \mathbb{C}$ the obvious injection, we have

$$j_1^*\check{\alpha} = j_2 \circ \alpha'.$$

Then

$$\begin{aligned}\left((\Psi^{-1})^*\check{\alpha}\right)_{(\xi_0, s)}(X_{\xi_0} + a\partial_s) &= \check{\alpha}_{\xi_0 \mathrm{e}^{s/k}}(\Psi_*^{-1}(X_{\xi_0} + a\partial_s)) \\ &= \check{\alpha}_{\xi_0}(R_{\mathrm{e}^{-s/k}\,*} \circ \Psi_*^{-1}(X_{\xi_0} + a\partial_s)) \\ &= \check{\alpha}_{\xi_0}(X_{\xi_0} + a\partial_s) = j_2\,\alpha'(X_{\xi_0}) + \frac{a}{k}\end{aligned}$$

i.e.,

$$\Psi^{-1*}\check{\alpha} = p_1^* j_2\,\alpha' + \frac{ds}{k}.$$

On the other hand the 1-form $e^{2s} p_1^* \alpha = \frac{1}{ik} e^{2s} p_1^* j_2^* \alpha'$; this shows how the symplectic form $\mu = d(e^{2s}\ p_1^* \alpha)$ on P is related to the connection form on L_0 [$\mu = d(\frac{e^{2s}}{ik} \Psi^{-1*} \check{\alpha})$]. Such examples have been studied by Kostant in [4].

Example 3 Let (M, ω) be a connected homogeneous symplectic manifold; i.e., $M = G/H$ where G is a Lie group which we may assume connected and simply connected and where H is the stabilizer in G of a point $x_0 \in M$. If $p : G \to M : g \to gx_0$, $p^*\omega$ is a left invariant closed 2-form on G and $\Omega = (p^*\omega)_e$, (e = neutral element of G) is a Chevalley 2-cocycle on $\mathfrak{g}$ (= Lie Algebra of G) with values in $\mathbb{R}$ (for the trivial representation). Notice that Ω vanishes as soon as one of its arguments is in $\hbar$ (= Lie algebra of H). Let $\mathfrak{g}_1 = \mathfrak{g} \oplus \mathbb{R}$ be the central extension of $\mathfrak{g}$ defined by Ω; i.e.,

$$[(X, a), (Y, b)] = ([X, Y], \Omega(X, Y)).$$

Let $\hbar'$ be the subalgebra of $\mathfrak{g}_1$, isomorphic to $\hbar$, defined by $\hbar' := \{ (X, 0) \mid X \in \hbar \}$. Let G_1 be the connected and simply connected group of algebra $\mathfrak{g}$, and let H' be the connected subgroup of G_1 with Lie algebra $\hbar'$. *Assume H' is closed*. Then G_1/H' admits a natural structure of smooth manifold; define $N := G_1/H'$. Let $p_1 : G_1 \to G$ be the homomorphism whose differential is the projection $\mathfrak{g}_1 \to \mathfrak{g}$ on the first factor; clearly $p_1(H') \subset H$. Define $\pi : N = G_1/H' \to M = G/H : g_1H' \mapsto p_1(g_1)H$; it is a surjective submersion.

We shall now construct the contact form α on N: $p_1^* \circ p^*\omega$ is a left invariant closed 2-form on G_1 vanishing on the fibers of $p \circ p_1 : G_1 \to M$. Its value Ω_1 at the neutral element e_1 of G_1 is a Chevalley 2-cocycle of $\mathfrak{g}_1$ with values in $\mathbb{R}$. Define the 1-cochain $\alpha_1 : \mathfrak{g}_1 \to \mathbb{R} : (X, a) \to -a$. Then

$$\begin{aligned}\Omega_1((X, a), (Y, b)) &= (p^*\omega)_e(X, Y) = \Omega(X, Y) = -\alpha_1([(X, a), (Y, b)]) \\ &= \delta\alpha_1((X, a), (Y, b)),\end{aligned}$$

i.e., $\Omega_1 = \delta\alpha_1$ is a coboundary. Let $\tilde{\alpha}_1$ be the left invariant 1-form on G_1 corresponding to α_1. Let $q : G_1 \to G_1/H' = N$ be the natural projection. We shall show that there exists a 1-form α on N so that $q^*\alpha = \tilde{\alpha}_1$. For any $U \in \mathfrak{g}_1$ denote by $\tilde{U}$ the corresponding left invariant vector field on G_1. For any $X \in \hbar'$ we have

$$\begin{aligned}&i(\tilde{X})\tilde{\alpha}_1 = \alpha_1(X) = 0, \\ &(L_{\tilde{X}}\tilde{\alpha}_1)(\widetilde{(Y, b)}) = -\tilde{\alpha}_1([\tilde{X}, \widetilde{(Y, b)}]) = -\alpha_1([X, (Y, b)]) = \Omega(X, Y) = 0,\end{aligned}$$

so that indeed $\tilde{\alpha}_1$ is the pullback by q of a 1-form α on $N = G_1/H'$. Furthermore $d\alpha = \pi^*\omega$ because both are G_1 invariant 2-forms on N and:

$$\begin{aligned}(d\alpha)_{q(e_1)}((X, a)^{*N}, (Y, b)^{*N}) &= (q^*d\alpha)_{e_1}(\widetilde{(X, a)}, \widetilde{(Y, b)}) \\ &= (d\tilde{\alpha}_1)_{e_1}(\widetilde{(X, a)}, \widetilde{(Y, b)}) \\ &= \Omega(X, Y), \\ &= \omega_{x_0}(X^{*M}, Y^{*M}) \\ &= (\pi^*\omega)_{q(e_1)}((X, a)^{*N}, (Y, b)^{*N})\end{aligned}$$

where we denote by U^{*N} the fundamental vector field on N associated to $U \in \mathfrak{g}_1$. Thus

Lemma 1.8 *Let $(M = G/H, \omega)$ be a homogeneous symplectic manifold; let Ω be the value at the neutral element of G of the pullback of ω to G. This is a Chevalley 2-cocycle of the Lie algebra $\mathfrak{g}$ of G. If $\mathfrak{g}_1 = \mathfrak{g} \oplus \mathbb{R}$ is the central extension of $\mathfrak{g}$ defined by this 2-cocycle and G_1 is the corresponding connected and simply connected group, let H' be the connected subgroup of G_1 with algebra $\hbar' = \{ (X, 0) \mid X \in \hbar \} \cong \hbar$. Assume H' is a closed subgroup of G_1. Then $N = G_1/H'$ admits a natural submersion π on M and has a contact structure α such that $d\alpha = \pi^*\omega$. Hence $(G/H, G_1/H', \alpha, \pi)$ is a contact quadruple.*

Remark 1.9 The center of G_1 is connected and simply connected, hence the central subgroup $\exp t(0, 1)$ is isomorphic to $\mathbb{R}$. The subgroup $p_1^{-1}(H)$ is a closed Lie subgroup of G_1 whose connected component is $p_1^{-1}(H_0)$ (H_0 = connected component of H). The universal cover $\widetilde{p_1^{-1}(H_0)}$ of $p_1^{-1}(H_0)$) is the direct product of $\tilde{H}_0$ (= universal cover of H_0) by $\mathbb{R}$. If $\nu : \widetilde{p_1^{-1}(H_0)} \to p_1^{-1}(H_0)$ is the covering homomorphism, the subgroup H' we are interested in is $H' = \nu(\tilde{H}_0)$. Clearly if $\pi_1(H_0) \sim \ker \nu$ is finite , H' is closed and the construction proceeds.

2 Lift of hamiltonian vector fields and of conformal vector fields

Let (M, ω) be a symplectic manifold of dimension $2n$ and let (P, μ) be the induced symplectic manifold of dimension $2n + 2$ constructed via the contact quadruple (M, N, α, π). Let X be a hamiltonian vector field on M; i.e.,

$$\mathcal{L}_X\omega = 0, \qquad i(X)\omega = df_X.$$

Consider the horizontal lift $\bar{X}$ of X to N defined by

$$\alpha(\bar{X}) = 0, \qquad \pi_*(\bar{X}) = X,$$

and the lift $\bar{\bar{X}}$ of $\bar{X}$ to P defined by

$$p_{1*}\bar{\bar{X}} = \bar{X}, \qquad ds(\bar{\bar{X}}) = 0.$$

Let Z be the Reeb vector field on (N, α) and let E be its lift to P defined by

$$p_{1*}E = Z, \qquad ds(E) = 0.$$

Definition 2.1 Define the lift $\tilde{X}$ of a hamiltonian vector field X on (M, ω) as the vector field on P defined by:

$$\tilde{X} = \bar{\bar{X}} - (p_1^*\pi^* f_X) \cdot E =: \bar{\bar{X}} - \widetilde{f_X} E.$$

Lemma 2.2 *The vector field $\tilde{X}$ is a hamiltonian vector field on (P,μ). Furthermore if $\mathfrak{g}$ is a Lie algebra of vector fields X on M having a strongly hamiltonian action, then the set of vector fields $\tilde{X}$ on P form an algebra isomorphic to $\mathfrak{g}$ and its action on (P,μ) is strongly hamiltonian.*

Proof. $i(\tilde{X})\mu = i(\bar{\bar{X}} - \widetilde{f_X}E)(e^{2s}(p_1^*\pi^*\omega + 2ds \wedge p_1^*\alpha)) = e^{2s}(d\tilde{f}_X + 2ds\tilde{f}_X) = d(e^{2s}\tilde{f}_X)$ which shows that $\tilde{X}$ is hamiltonian and that the hamiltonian function is $f_{\tilde{X}} = e^{2s}\tilde{f}_X$. Also if $X, Y \in \mathfrak{g}$:

$$\begin{aligned}
[\tilde{X},\tilde{Y}] &= [\bar{\bar{X}} - \widetilde{f_X}E, \bar{\bar{Y}} - \widetilde{f_Y}E] \\
&= \overline{\overline{[X,Y]}} - (\pi\circ p_1)^*\omega(X,Y)E - \widetilde{(Xf_Y)}E + Y\widetilde{f_X}E \\
&= \overline{\overline{[X,Y]}} - (\pi\circ p_1)^* f_{[X,Y]}E \\
&= \widetilde{[X,Y]}
\end{aligned}$$

and

$$\{f_{\tilde{X}}, f_{\tilde{Y}}\} = (\bar{\bar{X}} - \widetilde{f_X}E)(e^{2s}\widetilde{f_Y}) = e^{2s}\widetilde{Xf_Y} = e^{2s}\tilde{f}_{[X,Y]} = f_{\widetilde{[X,Y]}}.$$

□

If C is a conformal vector field on (M,ω) we may assume

$$L_C\omega = \omega, \qquad di(C)\omega = \omega.$$

By analogy of what we just did, define the lift $\tilde{C}_1$ of C to (P,μ) by:

$$ds(\tilde{C}_1) = 0, \qquad p_{1*}\tilde{C}_1 = \bar{C} + bZ$$

(i.e., $\pi_* p_{1*}\tilde{C}_1 = C$ and $\tilde{C}_1 = \bar{\bar{C}} + p_1^* bE$). Then

$$\begin{aligned}
L_{\tilde{C}_1}\mu &= di(\tilde{C}_1)e^{2s}(p_1^*\pi^*\omega + 2ds\wedge p_1^*\alpha) = d[e^{2s}(p_1^*\pi^* i(C)\omega - 2p_1^* b\, ds)] \\
&= e^{2s}[p_1^*\pi^*\omega + 2ds \wedge p_1^*\pi^* i(C)\omega - 2p_1^* db \wedge ds] \\
&= e^{2s}[p_1^*\pi^*\omega + 2ds \wedge (p_1^*\pi^*(i(C)\omega) + p_1^* db)].
\end{aligned}$$

Thus $\tilde{C}_1$ is a conformal vector field provided:

$$p_1^*\pi^* i(C)\omega + p_1^* db = p_1^*\alpha.$$

Or equivalently

$$\alpha - \pi^* i(C)\omega = db.$$

The left-hand side is a closed 1-form. If this form is exact we are able to lift C to a conformal vector field $\tilde{C}_1$ on P. Notice that the rate of variation of b along the flow of the Reeb vector field is prescribed:

$$Zb = 1.$$

A variation of this construction reads as follows. Let

$$\tilde{C}_2 = \bar{\bar{C}} + aE + l\partial_s.$$

Then:

$$\begin{aligned} L_{\tilde{C}_2}\mu &= d(i(\bar{\bar{C}} + aE + l\partial_s))\mathrm{e}^{2s}(p_1^*\pi^*\omega + 2ds \wedge p_1^*\alpha) \\ &= d\left(\mathrm{e}^{2s}(p_1^*\pi^*(i(C)\omega) - 2ads + 2lp_1^*\alpha)\right) \\ &= \mathrm{e}^{2s}[p_1^*\pi^*\omega - 2da \wedge ds + 2lp_1^*\pi^*\omega + 2ds \wedge p_1^*\pi^*i(C)\omega + 2lds \wedge p_1^*\alpha \\ &\qquad +2dl \wedge p_1^*\alpha] \\ &= \mathrm{e}^{2s}[(1+2l)p_1^*\pi^*\omega + 2ds \wedge (da + p_1^*\pi^*i(C)\omega + 2lp_1^*\alpha) + 2dl \wedge p_1^*\alpha]. \end{aligned}$$

If we choose $l = -1/2$,

$$L_{\tilde{C}_2}\mu = 2\mathrm{e}^{2s}ds \wedge (p_1^*\pi^*i(C)\omega - p_1^*\alpha + da).$$

Thus $\tilde{C}_2$ is a symplectic vector field on (P, μ) if the closed 1-form $\pi^*i(C)\omega - \alpha$ is exact. If this is the case the lift $\tilde{C}_2$ is hamiltonian and

$$f_{\tilde{C}_2} = -a\mathrm{e}^{2s}.$$

Lemma 2.3 *If C is a conformal vector field on (M, ω), it admits a lift $\tilde{C}_1$ (resp. $\tilde{C}_2$) to (P, μ) which is conformal (resp. hamiltonian) if the closed 1-form $\pi^*i(C)\omega - \alpha$ is exact.*

Let $\mathfrak{g}$ be an algebra of conformal vector fields on (M, ω). Let $X \in \mathfrak{g}$ be such that $L_{X^*}\omega = \omega$ (where $X_x^* = \frac{d}{dt}\exp -tX.x|_0$; $x \in M$). Then $\mathfrak{g} = \mathbb{R}X \oplus \mathfrak{g}_1$, where the vector fields associated to the elements of $\mathfrak{g}_1$, are symplectic. We shall assume here that they are hamiltonian; i.e., $\forall\, Y \in \mathfrak{g}_1$, $i(Y^*)\omega = df_Y$. Consider the lifts of these vector fields to (P, μ).

$$\begin{aligned} [\tilde{X}_1^*, \tilde{Y}^*] &= [\bar{\bar{X}}^* + p_1^*bE, \bar{\bar{Y}}^* - \tilde{f}_Y E] \\ &= [\bar{\bar{X}}^*, \bar{\bar{Y}}^*] - p_1^*\pi^*(Xf_Y)E - p_1^*(\bar{Y}^*b)E + \tilde{f}_Y p_1^*(Zb)E \\ &= \overline{\overline{[X, Y]}}^* \\ &\quad +[-p_1^*\pi^*\omega(X,Y) - p_1^*\pi^*\omega(Y,X) + p_1^*\pi^*\omega(X,Y) + \tilde{f}_Y]E \\ &= \overline{\overline{[X, Y]}}^* + p_1^*\pi^*(\omega(X,Y) + f_Y)E; \\ i([X^*, Y^*])\omega &= -L_{Y^*}i(X^*)\omega = -(i(Y^*)d + di(Y^*))i(X^*)\omega \\ &= -i(Y^*)\omega - d\omega(X,Y) = -d(\omega(X,Y) + f_Y). \end{aligned}$$

Hence

$$[\tilde{X}_1^*, \tilde{Y}^*] = [\widetilde{X^*, Y^*}].$$

A similar calculation shows that

$$[\tilde{X}_2^*, \tilde{Y}^*] = \widetilde{[X^*, Y^*]}.$$

Notice as before that $L_E\mu = 0$ and $L_{\partial_s}\mu = -2\mu$.

Proposition 2.4 *Let (M, ω) be the first term of a contact quadruple (M, N, α, π) and let (P, μ) be the associated induced symplectic manifold. Then*

(i) *If G is a connected Lie group acting in a strongly hamiltonian way on (M, ω), this action lifts to a strongly hamiltonian action of $\tilde{G}$ (= universal cover of G) on (P, μ).*
(ii) *If X is a conformal vector field on (M, ω) it admits a conformal (resp. symplectic) lift to (P, μ) if the closed 1-form $\pi^*(i(X)\omega) - \alpha$ is exact. The symplectic lift is in fact hamiltonian.*
(iii) *The vector field E on P is hamiltonian and the vector field ∂_s is conformal.*

Corollary 2.5 *If (M, ω) admits a transitive hamiltonian action, (P, μ) admits a transitive conformal action. If (M, ω) admits a transitive conformal (hamiltonian) action, then so does (P, μ).*

The stability of the class of conformally homogeneous spaces under this construction leads us to the study of these spaces.

3 Conformally homogeneous symplectic manifolds

Definition 3.1 Let (M, ω) be a smooth connected $2n \geq 4$-dimensional symplectic manifold. A connected Lie group G is said to *act conformally on* (M, ω) if

(i) $\forall g \in G, g^*\omega = c(g)\omega$.
(ii) There exists at least one $g \in G$ such that $c(g) \neq 1$.

As ω is closed, $c(g) \in \mathbb{R}$; also $c : G \to \mathbb{R}$ is a character of G. Let $G_1 = \ker c$; it is a closed, normal, codimension 1 subgroup of G. Let $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) be the Lie algebra of G (resp. G_1). Then there exists $0 \neq X \in \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{R}X \qquad \text{and} \qquad c_*(X) = 1.$$

The 1-parametric group $\exp tX$ is such that

$$(\exp tX)^*\omega = e^t\omega$$

and this group $\exp tX$ is thus isomorphic to $\mathbb{R}$. Hence the group G_1 is connected and if G is simply connected so is G_1. If X^* is the fundamental vector field on M associated to X, remark that $L_{X^*}\omega = -\omega$ since $X^*_x = \frac{d}{dt} \exp -tX \cdot x|_0$.

Definition 3.2 A symplectic manifold (M, ω) of dimension $2n \geq 4$ is called *conformal homogeneous* if there exists a Lie group G acting conformally and transitively on (M, ω).

We assume M and G connected. Then $\tilde{G}$ (= the universal cover of G) is the semi-direct product of $\tilde{G}_1$ (= the universal cover of G_1) by $\mathbb{R}$. By transitivity the orbits of G_1 are of dimension $\geq 2n-1$. So there are two cases:

(i) The maximum of the dimension of the G_1 orbits is $(2n-1)$.
(ii) G_1 admits an open orbit.

Case (i) By transitivity the dimension of all G_1 orbits is $(2n-1)$. If we write as above $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathbb{R}X$, the vector field X^* is everywhere transversal to the G_1 orbits. In particular it is everywhere $\neq 0$. Since $\mathfrak{g}_1$ is an ideal in $\mathfrak{g}$, the group $\exp tX$ permutes the G_1 orbits. Clearly if θ_1 is a G_1 orbit, $\cup_{t\in\mathbb{R}} \exp tX \cdot \theta_1 = M$. The restriction $\omega|_{T_x\theta_1}$ has rank $(2n-1)$. Let Z_x span the radical of $\omega|_{T_x\theta_1}$ and let $\alpha := -i(X^*)\omega \neq 0$ (so $d\alpha = \omega$). As $\alpha_x(Z_x) \neq 0$, we normalize Z_x so that $\alpha_x(Z_x) = 1$. Then

$$T_xM = \mathbb{R}X^* \oplus T_x\theta_1 = \mathbb{R}X^* \oplus (\mathbb{R}Z_x \oplus \ker \underline{\alpha}_x)$$

if $\underline{\alpha}_x = \alpha_x|_{T_x\theta_1}$. If $j : \theta_1 \to M$ denotes the canonical injection,

$$\underline{\alpha} \wedge (d\underline{\alpha})^{n-1} = j^*(\alpha \wedge (\omega)^{n-1}) \neq 0.$$

Thus the orbit θ_1 is a contact manifold, and Z is the Reeb vector field. Notice that

$$\begin{aligned}
(L_{X^*}\alpha)(X^*) &= X^*\alpha(X^*) = 0,\\
(L_{X^*}\alpha)(Y^*) &= X^*\alpha(Y^*) - \alpha([X^*, Y^*])\\
&= -X^*\omega(X^*, Y^*) + \omega(X^*, [X^*, Y^*])\\
&= -(L_{X^*}\omega)(X^*, Y^*) = \omega(X^*, Y^*) = -\alpha(Y^*)
\end{aligned}$$

for any $Y \in \mathfrak{g}_1$. Hence

$$L_{X^*}\alpha = -\alpha.$$

This says that the various orbits of G_1 have "conformally" equivalent contact structure; i.e.,

$$\underline{\alpha}_{\exp tX\cdot x}(\exp tX_* \cdot Y^*) = e^t \underline{\alpha}_x(Y^*) \qquad Y \in \mathcal{G}_1.$$

Furthermore

$$\omega([X^*, Z], Y^*) = X^*\omega(Z, Y^*) - (L_{X^*}\omega)(Z, Y^*) - \omega(Z, [X^*, Y^*]) = 0$$

as $[X^*, Y^*]$ is tangent to the orbit. This says that $[X^*, Z]$ is proportional to Z; also

$$\alpha([X^*, Z]) = X^*\alpha(Z) - (L_{X^*}\alpha)(Z) = \alpha(Z) = 1,$$

hence $[X^*, Z] = Z$ and thus

$$(\exp tX)_* Z_x = e^t Z_{\exp tX\cdot x}.$$

Finally

$$\alpha([Y^*, Z]) = -\omega(X^*, [Y^*, Z])$$

$$= -Y^*\omega(X^*, Z) + L_{Y^*}\omega(X^*, Z) + \omega([Y^*, X^*], Z) = 0,$$
$$\omega([Y^*, Z], Y'^*) = Y^*\omega(Z, Y'^*) - L_{Y^*}\omega(Z, Y'^*) - \omega(Z, [Y^*, Y'^*]) = 0.$$

Hence $[Y^*, Z]$ must be proportional to Z and thus

$$[Y^*, Z] = 0$$

which says that the Reeb vector is G_1 stable.

Case (ii) G_1 admits an open orbit. We shall assume that this orbit coincides with M. Thus (M, ω) is a G_1 homogeneous sympletic manifold and ω is exact.

$$\omega = d\eta \quad \text{where} \quad \eta := -i(X^*)\omega.$$

Assume that the action of G_1 is strongly hamiltonian; i.e., $\forall Y \in \mathfrak{g}_1$

$$i(Y^*)\omega = df_Y,$$
$$\{f_Y, f_{Y'}\} = -\omega(Y^*, Y'^*) = f_{[Y,Y']}$$

where U^* denotes the fundamental vector field associated to $U \in \mathfrak{g}_1$ on θ_1. Then

$$L_{Y^*}\eta = -L_{Y^*}i(X^*)\omega = -(L_{Y^*}i(X^*) - i(X^*)L_{Y^*})\omega = -i([Y^*, X^*])\omega$$
$$= df_{[X,Y]} = df_{DY}$$

if $D = \mathrm{ad}X|_{\mathfrak{g}_1}$. We also have$L_{X^*}\eta = -\eta$.

By Kostant's theorem we may identify M (up to a covering) with a coadjoint orbit θ_1 of G_1. Let $\xi \in \theta_1$, let $\pi : G_1 \to \theta_1 : g_1 \to g_1 \cdot \xi = \mathrm{Ad}^* g_1 \xi$ and let H_1 be the stabilizer of ξ in G_1. It is no restriction to assume $X^*_\xi = 0$ (since one can replace X by $X + Y$ for any $Y \in \mathfrak{g}_1$ and any tangent vector at ξ can be written in the form Y^*_ξ). Assuming G (hence G_1) to be connected and simply connected, the derivation D exponentiates to a 1-parametric automorphism group of $\mathfrak{g}_1$ given by e^{tD} and these "exponentiate" to a 1-parametric automorphism group of G_1 which will be denoted $a(t)$. The product law in $G = G_1 \cdot \mathbb{R}$ reads:

$$(g_1, t_1)(g_2, t_2) = (g_1 a(t_1) g_2, t_1 + t_2).$$

As $X^*_\xi = 0$ we have:

$$(1, t) \cdot \xi = \xi,$$
$$(1, t)(g_1, 0) \cdot \xi = (a(t)g_1, t)\xi = (a(t)g_1, 0)(1, t) \cdot \xi$$
$$= (a(t)g_1, 0) \cdot \xi = (a(t)g_1 \circ g_1^{-1}, 0)(g_1, 0) \cdot \xi.$$

In particular if $g_1 \in H_1$ (= stabilizer of ξ in G_1) $a(t)g_1 \in H_1$; hence if $Y \in \hbar_1$ (= Lie algebra of H_1), $[Y, X] \in \hbar_1$. Furthermore

$$(L_{X^*}\omega)(Y_1^*, Y_2^*) = -\omega(Y_1^*, Y_2^*)$$
$$= X^*\omega(Y_1^*, Y_2^*) - \omega([X^*, Y_1^*], Y_2^*) - \omega(Y_1^*, [X^*, Y_2^*]).$$

The above relation at ξ reads:

$$\omega_\xi(Y_1^*, Y_2^*) = \omega_\xi([X, Y_1^*], Y_2^*) + \omega_\xi(Y_1^*, [X, Y_2^*]).$$

But on θ_1, ω is the Kostant–Souriau symplectic form; hence

$$\langle \xi, [Y_1, Y_2] \rangle = \langle \xi, D[Y_1, Y_2] \rangle,$$
$$\langle \xi - \xi \circ D, [Y_1, Y_2] \rangle = 0.$$

That is $\xi - \xi D$ vanishes identically on the derived algebra $\mathfrak{g}_1'$.

Conversely suppose we are given an algebra $\mathfrak{g}_1$, an element $\xi \in \mathfrak{g}_1^*$ and a derivation D of $\mathfrak{g}_1$ such that

$$\xi - \xi \circ D \qquad \text{vanishes on } \mathfrak{g}_1'.$$

Then, if, as above, H_1 denotes the stabilizer of ξ in G_1 and $\hbar_1$ its Lie algebra, one observes that $Y \in \hbar_1$ implies $DY \in \hbar_1$. On the orbit $\theta_1 = G_1 \cdot \xi = G_1/H_1$ define the vector field $\hat{X}$ at $\tilde{\xi} = g_1 \cdot \xi$ by:

$$\hat{X}_{\tilde{\xi}} = \frac{d}{dt} a(-t) g_1 \cdot g_1^{-1} \cdot \tilde{\xi}|_{t=0}.$$

This can be expressed in a nicer way as:

$$\langle \hat{X}_{\tilde{\xi}=g_1\xi}, Z \rangle = \frac{d}{dt} \langle a_{-t}(g_1) g_1^{-1} g_1 \xi, Z \rangle|_0 = \frac{d}{dt} \langle a_{-t}(g_1)\xi, Z \rangle|_0$$

for $Z \in \mathfrak{g}_1$,

$$\begin{aligned} \operatorname{Ad} a_{-t}(g_1^{-1}) Z &= \frac{d}{ds} a_{-t}(g_1^{-1}) \mathrm{e}^{sZ} a_{-t}(g_1)|_0 = \frac{d}{ds} a_{-t}(g_1^{-1} a_t \mathrm{e}^{sZ} g_1)|_0 \\ &= \frac{d}{ds} a_{-t}(g_1^{-1} \mathrm{e}^{s\mathrm{e}^{tD}Z} g_1)|_0 = a_{-t*} \operatorname{Ad} g_1^{-1} \mathrm{e}^{tD} Z, \end{aligned}$$

$$\frac{d}{dt} \operatorname{Ad} a_{-t}(g_1^{-1}) Z|_0 = -D \circ \operatorname{Ad} g_1^{-1} Z + \operatorname{Ad} g_1^{-1} DZ,$$

i.e.,

$$\hat{X}_{\tilde{\xi}=g\xi} = -\xi \circ D \circ \operatorname{Ad} g_1^{-1} + \xi \circ \operatorname{Ad} g_1^{-1} \circ D.$$

Observe that this expression has a meaning; indeed if we assume that $g \in H_1$ (= stabilizer of ξ)

$$\xi \circ \operatorname{Ad} g^{-1} = \xi.$$

Also if $Y \in \hbar_1$, $\frac{d}{ds} \langle \xi \circ D \circ \operatorname{Ad} \mathrm{e}^{-sY} \rangle|_s = -\langle \xi \circ D \circ adY \circ \operatorname{Ad} \mathrm{e}^{-sY}, Z \rangle = 0$ so that $\langle \xi \circ D \circ \operatorname{Ad} \mathrm{e}^{-sY}, Z \rangle = \langle \xi \circ D, Z \rangle$. Thus $\hat{X}_\xi = 0$ and, if $h \in H_1$:

$$\begin{aligned} \hat{X}_{\tilde{\xi}=g\cdot\xi=g\cdot h\cdot\xi} &= \xi \circ D \circ \operatorname{Ad} h^{-1} \circ \operatorname{Ad} y^{-1} + \xi \circ \operatorname{Ad} h^{-1} \circ \operatorname{Ad} g^{-1} D \\ &= \xi \circ D \circ \operatorname{Ad} g^{-1} + \xi \circ \operatorname{Ad} g^{-1} D. \end{aligned}$$

Furthermore if $Y \in \mathfrak{g}_1$:

$$[Y^*, \hat{X}]_{\xi} = (L_{Y^*}\hat{X})_{\xi} = \frac{d}{dt}(\varphi^{Y^*}_{-t*}\hat{X}_{\varphi^{Y^*}_t\xi})|_0 = -(DY)^*_{\xi}.$$

Hence, if $Y_1, Y_2 \in \mathfrak{g}_1$:

$$\begin{aligned}(L_{\hat{X}}\omega)_\xi(Y_1^*, Y_2^*) &= \hat{X}_\xi\omega(Y_1^*, Y_2^*) - \omega((DY_1)^*, Y_2^*) - \omega(Y_1^*, (DY_2)^*)\\ &= \langle -\xi, D[Y_1, Y_2]\rangle = -\langle \xi, [Y_1, Y_2]\rangle = -\omega_\xi(Y_1^*, Y_2^*)\end{aligned}$$

and similarly at any other point, so that $\hat{X}$ is a conformal vector field ($L_{\hat{X}}\omega = -\omega$). We conclude by

Proposition 3.3 *Let (M, ω)be a smooth connected $2n(\geq 4)$-dimensional symplectic manifold which is conformal homogeneous and let G denote the connected component of the conformal group. Then*

(i) *G admits a codimension 1 closed, connected, invariant subgroup G_1 which acts symplectically on M and $G/G_1 = \mathbb{R}$.*
(ii) *If the maximum dimension of the G_1 orbits is $(2n-1)$, M is a union of $(2n-1)$-dimensional G_1 orbits; each of these orbits is a contact manifold.*
(iii) *If G_1 acts transitively on M in a strongly hamiltonian way, M is a covering of a G_1 orbit θ in $\mathfrak{g}_1^*$ (= dual of the Lie algebra $\mathfrak{g}_1$ of G_1). Furthermore if $\xi \in \theta$, there exists a derivation D of $\mathfrak{g}_1$ such that*

$$\xi - \xi \circ D$$

vanishes on the derived algebra. Conversely if we are given an element $\xi \in \mathfrak{g}_1^*$ and a derivation such that $\xi - \xi \circ D$ vanishes on the derived algebra, the orbit θ has the structure of a conformal homogeneous symplectic manifold.

4 Induced connections

We consider the situation where we have a smooth symplectic manifold (M, ω) of dim $2n$, a contact quadruple (M, N, α, π) and the corresponding induced symplectic manifold (P, μ).

Let as before Z be the Reeb vector field on the contact manifold (N, α) (i.e., $i(Z)d\alpha = 0$ and $\alpha(Z) = 1$). At each point $x \in N$, $\mathrm{Ker}\,(\pi_{*x}) = \mathbb{R}Z$ and $L_Z\alpha = 0$. Recall that $P = N \times \mathbb{R}$ and $\mu = 2e^{2s}\, ds \wedge p_1^*\alpha + e^{2s}\, dp_1^*\alpha$ where s is the variable along $\mathbb{R}$ and $p_1 : P \to N$ the projection on the first factor.

Let ∇ be a smooth symplectic connection on (M, ω). We shall now define a connection ∇^P on P induced by ∇.

Let us first recall some notation: Denote by p the projection $p = \pi \circ p_1 : P \to M$. If X is a vector field on M, $\bar{\bar{X}}$ is the vector field on P such that

$$\text{(i)}\ \ p_*\bar{\bar{X}} = X, \qquad \text{(ii)}\ \ (p_1^*\alpha)(\bar{\bar{X}}) = 0, \qquad \text{(iii)}\ \ ds(\bar{\bar{X}}) = 0.$$

We denote by E the vector field on P such that

$$\text{(i)}\ \ p_{1*}E = Z, \qquad \text{(ii)}\ \ ds(E) = 0.$$

Clearly the values at any point of P of the vector fields $\bar{\bar{X}}$, E, $S = \partial_s$ span the tangent space to P at that point and we have

$$[E, \partial_s] = 0 \quad [E, \bar{\bar{X}}] = 0 \quad [\partial_s, \bar{\bar{X}}] = 0 \quad [\bar{\bar{X}}, \bar{\bar{Y}}] = \overline{\overline{[X, Y]}} - p^*\omega(X, Y)E.$$

The formulas for ∇^P are:

$$\nabla^P_{\bar{\bar{X}}}\bar{\bar{Y}} = \overline{\overline{\nabla_X Y}} - \tfrac{1}{2}p^*(\omega(X, Y))E - p^*(\hat{s}(X, Y))\partial_s,$$

$$\nabla^P_E\bar{\bar{X}} = \nabla^P_{\bar{\bar{X}}}E = 2\overline{\overline{\sigma X}} + p^*(\omega(X, U))\partial_s,$$

$$\nabla^P_{\partial_s}\bar{\bar{X}} = \nabla^P_{\bar{\bar{X}}}\partial_s = \overline{\overline{X}},$$

$$\nabla^P_E E = p^* f\,\partial_s - 2\overline{\overline{U}},$$

$$\nabla^P_E\partial_s = \nabla^P_{\partial_s}E = E,$$

$$\nabla^P_{\partial_s}\partial_s = \partial_s,$$

where f is a function on M, U is a vector field on M, $\hat{s}$ is a symmetric 2-tensor on M, and σ is the endomorphism of TM associated to s, hence $\hat{s}(X, Y) = \omega(X, \sigma Y)$.

Notice first that these formulas have the correct linearity properties and yield a torsion free linear connection on P. One checks readily that $\nabla^P\mu = 0$ so that ∇^P is a symplectic connection on (P, μ).

We now compute the curvature R^{∇^P} of this connection ∇^P. We get

$$\begin{aligned}
R^{\nabla^P}(\bar{\bar{X}}, \bar{\bar{Y}})\bar{\bar{Z}} = {} & \overline{\overline{R^{\nabla}(X, Y)Z}}, \\
& +\overline{\overline{2\omega(X, Y)\sigma Z - \omega(Y, Z)\sigma X + \omega(X, Z)\sigma Y - \hat{s}(Y, Z)X + \hat{s}(X, Z)Y}}, \\
& +p^*[\omega(X, D(\sigma, U)(Y, Z)) - \omega(Y, D(\sigma, U)(X, Z)]\partial_s,
\end{aligned}$$

$$\begin{aligned}
R^{\nabla^P}(\bar{\bar{X}}, \bar{\bar{Y}})E = {} & \overline{\overline{2D(\sigma, U)(X, Y) - 2D(\sigma, U)(Y, X)}} \\
& +p^*[\omega(X, \tfrac{1}{2}fY - \nabla_Y U - 2\sigma^2 Y) - \omega(Y, \tfrac{1}{2}fX - \nabla_X U - 2\sigma^2 X)]\partial_s,
\end{aligned}$$

$$R^{\nabla^P}(\bar{\bar{X}}, E)\bar{\bar{Y}} = \overline{\overline{2D(\sigma, U)(X, Y)}} - p^*[\omega(Y, \tfrac{1}{2}fX - \nabla_X U - 2\sigma^2 X)]\partial_s,$$

$$R^{\nabla^P}(\bar{\bar{X}}, E)E = 2\overline{\overline{\tfrac{1}{2}fX - \nabla_X U - 2\sigma^2 X}} + p^*[Xf + 4s(X, u)]\partial_s,$$

$$\begin{array}{lll}
R^{\nabla^P}(\bar{\bar{X}}, \bar{\bar{Y}})\partial_s = 0 & R^{\nabla^P}(\bar{\bar{X}}, E)\partial_s = 0 & \\
R^{\nabla^P}(\bar{\bar{X}}, \partial_s)\bar{\bar{Y}} = 0 & R^{\nabla^P}(\bar{\bar{X}}, \partial_s)E = 0 & R^{\nabla^P}(\bar{\bar{X}}, \partial_s)\partial_s = 0 \\
R^{\nabla^P}(E, \partial_s)\bar{\bar{X}} = 0 & R^{\nabla^P}(E, \partial_s)E = 0 & R^{\nabla^P}(E, \partial_s)\partial_s = 0,
\end{array}$$

where

$$D(\sigma, U)(Y, Y') := (\nabla_Y\sigma)Y' + \tfrac{1}{2}\omega(Y', U)Y - \tfrac{1}{2}\omega(Y, Y')U.$$

The Ricci tensor r^{∇^P} of the connection ∇^P is given by

$$\begin{aligned}
r^{\nabla^P}(\bar{\bar{X}}, \bar{\bar{Y}}) &= r^{\nabla}(X, Y) + 2(n+1)\hat{s}(X, Y),\\
r^{\nabla^P}(\bar{\bar{X}}, E) &= -(2n+1)\omega(X, u) - 2\,\mathrm{Tr}[Y \to (\nabla_Y \sigma)(X)],\\
r^{\nabla^P}(\bar{\bar{X}}, \partial_s) &= 0,\\
r^{\nabla^P}(E, E) &= 4\,\mathrm{Tr}(\sigma^2) - 2nf + 2\,\mathrm{Tr}[X \to \nabla_X U],\\
r^{\nabla^P}(E, \partial_s) &= 0,\\
r^{\nabla^P}(\partial_s, \partial_s) &= 0.
\end{aligned}$$

Theorem 4.1 *In the framework described above, ∇^P is a symplectic connection on (P, μ) for any choice of $\hat{s}$, U and f. The vector field E on P is affine ($L_{\tilde{E}}\nabla^P = 0$) and symplectic ($L_{\tilde{E}}\mu = 0$); the vector field ∂_s on P is affine and conformal ($L_{\partial_s}\mu = 2\mu$).*

Furthermore, choosing

$$\begin{aligned}
\hat{s} &= \frac{-1}{2(n+1)} r^{\nabla},\\
\underline{U} := \omega(U, \cdot) &= \frac{2}{2n+1}\,\mathrm{Tr}[Y \to \nabla_Y \sigma],\\
f &= \frac{1}{2n(n+1)^2}\,\mathrm{Tr}(\rho^{\nabla})^2 + \frac{1}{n}\,\mathrm{Tr}[X \to \nabla_X U],
\end{aligned}$$

we have:

- *the connection ∇^P on (P, μ) is Ricci-flat (i.e., has zero Ricci tensor);*
- *if the symplectic connection ∇ on (M, ω) is of Ricci-type, then the connection ∇^P on (P, μ) is flat;*
- *if the connection ∇^P is locally symmetric, the connection ∇ is of Ricci-type, hence ∇^P is flat.*

Proof. The first point is an immediate consequences of the formulas above for r^{∇^P}. The second point is a consequence of the differential identities satisfied by the Ricci-type symplectic connections (which appear in M. Cahen, S. Gutt, J. Horowitz and J. Rawnsley, Homogeneous symplectic manifolds with Ricci-type curvature, *J. Geom. Phys.* **38** (2001) 140–151).

The third point comes from the fact that $(\nabla^P_{\bar{\bar{Z}}} R^{\nabla^P})(\bar{\bar{X}}, \bar{\bar{Y}})\bar{\bar{T}}$ contains only one term in E whose coefficient is $\frac{1}{2} W^{\nabla^P}(X, Y, T, Z)$. □

5 A reduction construction

We present here a procedure to construct symplectic connections on some reduced symplectic manifolds; this is a generalisation of the construction given by P. Baguis and M. Cahen [*Lett. Math. Phys.* **57** (2001), pp. 149–160]. Let (P, μ) be a symplectic manifold of dimension $(2n + 2)$. Assume P admits a complete conformal vector field S:

$$L_S\mu = 2\mu; \qquad \text{define } \alpha := \tfrac{1}{2} i(S)\mu \qquad \text{so that } d\alpha = \mu.$$

Assume also that P admits a symplectic vector field E commuting with S,

$$L_E\mu = 0, \qquad [S, E] = 0, \qquad (\Rightarrow L_E\alpha = 0).$$

Then $S\mu(S, E) = (L_S\mu)(S, E) = 2\mu(S, E)$, so if x is a point of P where $\mu_x(S, E) \neq 0$ and if s is a parameter along the integral line γ of S passing through x and taking value 0 at x, we have $\mu_{\gamma(s)}(S, E) = e^{2s}\mu_x(S, E)$.

Assume $P' := \{x \in P | \mu_x(S, E) > 0\} \neq \emptyset$ and let:

$$\Sigma = \{x \in P \mid \mu_x(S, E) = 1\} = \{x \in P \mid f_E(x) = \tfrac{1}{2}\}$$

where $f_E = -i(E)\alpha = -\frac{1}{2}\mu(S, E)$ so that $df_E = -L_E\alpha + i(E)d\alpha = i(E)\mu$. Thus $\Sigma \neq \emptyset$ and it is a closed hypersurface (called the constraint hypersurface). Remark that $P' \cong \Sigma \times \mathbb{R}$. The tangent space to the hypersurface Σ is given by

$$T_x\Sigma = \ker(df_E)_x = \ker(i(E)\mu)_x = E^{\perp_\mu}.$$

The restriction of μ_x to $T_x\Sigma$ has rank $2n - 2$ and a radical spanned by E_x. Remark thus that the restriction of α to Σ is a contact 1-form on Σ.

Let $\sim$ be the equivalence relation defined on Σ by the flow of E. Assume that the quotient $\Sigma/\sim$ has a $2n$-dimensional manifold M structure so that $\pi : \Sigma \to \Sigma/\sim = M$ is a smooth submersion. Define on Σ a "horizontal" distribution of dimension $2n$, $\mathcal{H}$, by

$$\mathcal{H} = > E, S <^{\perp_\mu},$$

and remark that $\pi_{*|\mathcal{H}_y} : \mathcal{H}_y \to T_{x=\pi(y)}M$ is an isomorphism. Define as usual the reduced 2-form ω on M by

$$\omega_{x=\pi(y)}(Y_1, Y_2) = \mu_y(\bar{Y}_1, \bar{Y}_2)$$

where $\bar{Y}_i$ $(i = 1, 2)$ is defined by (i) $\pi_*\bar{Y}_i = Y_i$ (ii) $\bar{Y}_i \in \mathcal{H}_y$. Notice that $\pi_*[E, \bar{Y}] = 0$, and $\mu(S, [E, \bar{Y}]) = -L_E\mu(S, \bar{Y}) + E\mu(S, \bar{Y}) = 0$ hence

$$[E, \bar{Y}] = 0.$$

The definition of ω_x does not depend on the choice of y. Indeed

$$E\mu(\bar{Y}_1, \bar{Y}_2) = L_E\mu(\bar{Y}_1, \bar{Y}_2) + \mu([E, \bar{Y}_1], \bar{Y}_2) + \mu(\bar{Y}_1, [E, \bar{Y}_2]) = 0.$$

Clearly ω is of maximal rank $2n$ as $\mathcal{H}$ is a symplectic subspace. Finally

$$\begin{aligned} \pi^*(d\omega(Y_1, Y_2, Y_3)) &= \circlearrowleft\!\!\!\!\!\bigoplus_{123} (Y_1\omega(Y_2, Y_3) - \omega([Y_1, Y_2], Y_3)) \\ &= \circlearrowleft\!\!\!\!\!\bigoplus_{123} (\bar{Y}_1\mu(\bar{Y}_2, \bar{Y}_3) - \mu(\overline{[Y_1, Y_2]}, \bar{Y}_3)) \end{aligned}$$

and

$$[\bar{Y}_1, \bar{Y}_2] = \overline{[Y_1, Y_2]} + \mu(S, [\bar{Y}_1, \bar{Y}_2])E.$$

Hence ω is closed and thus symplectic. Clearly $\pi^*\omega = \mu_{|\Sigma} = d(\alpha_{|\Sigma})$.

Remark 5.1 The symplectic manifold (M, ω) is the first element of a contact quadruple $(M, \Sigma, \frac{1}{2}\alpha_{|\Sigma}, \pi)$ and the associated symplectic $(2n+2)$-dimensional manifold is $(P', \mu_{|P'})$.

We shall now consider the reduction of a connection. Let (P, μ), E, S, Σ, M, ω be as above. Let ∇^P be a symplectic connection on P and assume that the vector field E is affine ($L_E \nabla^P = 0$).

Then define a connection ∇^Σ on Σ by

$$\nabla^\Sigma_A B := \nabla^P_A B - \mu(\nabla^P_A B, E)S = \nabla^P_A B + \mu(B, \nabla^P_A E)S.$$

Then:

$$\begin{aligned}\nabla^\Sigma_A B - \nabla^\Sigma_B A - [A, B] &= (\mu(B, \nabla^P_A E) - \mu(A, \nabla^P_B E))S \\ &= (\mu(B, \nabla^P_E A + [A, E]) - \mu(A, \nabla^P_E B + [B, E]))S \\ &= (E\mu(B, A) - \mu(B, [E, A]) - \mu([E, B], A))S \\ &= (L_E \mu(B, A))S = 0.\end{aligned}$$

Also

$$\begin{aligned}(L_E \nabla^\Sigma)_A B &= [E, \nabla^P_A B + \mu(B, \nabla^P_A E)S] \\ &\quad - \nabla^P_{[E,A]} B - \mu(B, \nabla^P_{[E,A]} E)S - \nabla^P_A [E, B] - \mu([E, B], \nabla^P_A E)S \\ &= (L_E \nabla^P)_A B + (E\mu(B, \nabla^P_A E) - \mu(B, \nabla^P_{[E,A]} E) - \mu([E, B], \nabla^P_A E))S \\ &= (L_E \mu)(B, \nabla^P_A E) + \mu(B, [E, \nabla^P_A E] - \nabla^P_{[E,A]} E)S = 0,\end{aligned}$$

i.e., ∇^Σ is a torsion free connection and E is an affine vector field for ∇^Σ.

Define a connection ∇^M on M by:

$$\overline{\nabla^M_{Y_1} Y_2}(y) = \nabla^\Sigma_{\bar{Y}_1} \bar{Y}_2(y) - \mu(\bar{Y}_2, \nabla^P_{\bar{Y}_1} S)E.$$

If $x \in M$, this definition does not depend on the choice of $y \in \pi^{-1}(x)$. Also

$$\begin{aligned}\overline{\nabla^M_{Y_1} Y_2} - \overline{\nabla^M_{Y_2} Y_1} - \overline{[Y_1, Y_2]} &= \nabla^\Sigma_{\bar{Y}_1} \bar{Y}_2 - \nabla^\Sigma_{\bar{Y}_2} \bar{Y}_1 - \overline{[Y_1, Y_2]} \\ &\quad + (-\mu(\bar{Y}_2, \nabla^P_{\bar{Y}_1} S) + \mu(\bar{Y}_1, \nabla^P_{\bar{Y}_2} S))E \\ &= \mu(S, [\bar{Y}_1, \bar{Y}_2])E + (\mu(\nabla^P_{\bar{Y}_1} \bar{Y}_2, S) - \mu(\nabla^P_{\bar{Y}_2} \bar{Y}_1, S))E = 0.\end{aligned}$$

Finally

$$\begin{aligned}\pi^*((\nabla^M_{Y_1} \omega)(Y_2, Y_3)) &= \pi^*(Y_1 \omega(Y_2, Y_3) - \omega(\nabla^M_{Y_1} Y_2, Y_3) - \omega(Y_2, \nabla^M_{Y_1} Y_3)) \\ &= \bar{Y}_1 \mu(\bar{Y}_2, \bar{Y}_3) - \mu(\nabla^P_{\bar{Y}_1} \bar{Y}_2 + \mu(\bar{Y}_2, \nabla^P_{\bar{Y}_1} E)S - \mu(\bar{Y}_2, \nabla^P_{\bar{Y}_1} S)E, \bar{Y}_3) \\ &\quad - \mu(\bar{Y}_2, \nabla^P_{\bar{Y}_1} \bar{Y}_3 + \mu(\bar{Y}_3, \nabla^P_{\bar{Y}_1} E)S - \mu(\bar{Y}_3, \nabla^P_{\bar{Y}_1} S)E) \\ &= 0,\end{aligned}$$

i.e., the connection ∇^M is symplectic.

Lemma 5.2 *Let* (P, μ) *be a symplectic manifold admitting a symplectic connection* ∇^P*, a conformal vector field* S *which is complete, a symplectic vector field* E *which is affine and commutes with* S*. If the constraint manifold* $\Sigma = \{x \in P | \mu_x(S, E) = 1\}$ *is not empty, and if the reduction of* Σ *is a manifold* M*, this manifold admits a symplectic structure* ω *and a natural reduced symplectic connection* ∇^M*.*

In particular:

Theorem 5.3 *Let* (P, μ) *be a symplectic manifold admitting a conformal vector field* S $(L_S\mu = 2\mu)$ *which is complete, a symplectic vector field* E *which commutes with* S *and assume that, for any* $x \in P$, $\mu_x(S, E) > 0$*. If the reduction of* $\Sigma = \{x \in P \mid \mu_x(S, E) = 1\}$ *by the flow of* E *has a manifold structure* M *with* $\pi : \Sigma \to M$ *a surjective submersion, then* M *admits a reduced symplectic structure* ω *and* (P, μ) *is obtained by induction from* (M, ω) *using the contact quadruple* $(M, \Sigma, \frac{1}{2}i(S)\mu_{|\Sigma}, \pi)$*.*

In particular (P, μ) *admits a Ricci-flat connection.*

Reducing (P, μ) as above and inducing back we see that Theorem 4.1 immediately proves this.

References

1. Michel Cahen, Simone Gutt, Lorenz Schwachhöfer: Construction of Ricci-type connections by reduction and induction, preprint math.DG/0310375, in *The Breadth of Symplectic and Poisson Geometry*, Marsden, J.E. and Ratiu, T.S. (eds), Progress in Math 232, Birkhäuser, 2004, 41–57.
2. M. Cahen and L. Schachhöfer, Special symplectic connections, preprint DG0402221.
3. V. Guillemin, S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press 1984 (pp. 319–324).
4. B. Kostant, Minimal coadjoint orbits and symplectic induction, in *The Breadth of Symplectic and Poisson Geometry*, Marsden, J.E. and Ratiu, T.S. (eds), Progress in Math 232, Birkhauser, 2004.
5. I. Vaisman, Symplectic Curvature Tensors *Monats. Math.* **100** (1985) 299–327.

Local Lie Algebra Determines Base Manifold*

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Dedicated to Hideki Omori

Summary. It is proven that a local Lie algebra in the sense of A. A. Kirillov determines the base manifold up to a diffeomorphism provided the anchor map is nowhere-vanishing. In particular, the Lie algebras of nowhere-vanishing Poisson or Jacobi brackets determine manifolds. This result has been proven for different types of differentiability: smooth, real-analytic, and holomorphic.

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1 Introduction

The classical result of Shanks and Pursell [PS] states that the Lie algebra $\mathcal{X}_c(M)$ of all compactly supported smooth vector fields on a smooth manifold M determines the manifold M, i.e., the Lie algebras $\mathcal{X}_c(M_1)$ and $\mathcal{X}_c(M_2)$ are isomorphic if and only if M_1 and M_2 are diffeomorphic. A similar theorem holds for other complete and transitive Lie algebras of vector fields [KMO1, KMO2] and for the Lie algebras of all differential and pseudodifferential operators [DS, GP].

There is a huge list of papers in which special geometric situations (hamiltonian, contact, group invariant, foliation preserving, etc., vector fields) are concerned. Let us mention the results of Omori [O1] (Ch. X) and [O2] ([Ch. XII), or [Ab, AG, FT, HM, Ry, G5], for which specific tools were developed in each case. There is however a case when the answer is more or less complete in the whole generality. These are the Lie algebras of vector fields which are modules over the corresponding rings of functions (we shall call them *modular*). The standard model of a modular Lie algebra of vector

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fields is the Lie algebra $\mathcal{X}(\mathcal{F})$ of all vector fields tangent to a given (generalized) foliation $\mathcal{F}$. If Pursell–Shanks-type results are concerned in this context, let us recall the work of Amemiya [Am] and our paper [G1], where the developed algebraic approach made it possible to consider analytic cases as well. The method of Shanks and Pursell consists of the description of maximal ideals in the Lie algebra $\mathcal{X}_c(M)$ in terms of the points of M: maximal ideals are of the form $\widetilde{p}$ for $p \in M$, where $\widetilde{p}$ consists of vector fields which are flat at p. This method, however, fails in analytic cases, since analytic vector fields flat at p are zero on the corresponding component of M. Therefore in [Am, G1] maximal finite-codimensional subalgebras are used instead of ideals. A similar approach is used in [GG] for proving that the Lie algebras associated with Lie algebroids determine base manifolds.

The whole story for modular Lie algebras of vector fields has been in a sense finished by the brilliant purely algebraic result of Skryabin [S1], where one associates the associative algebra of functions with the Lie algebra of vector fields without any description of the points of the manifold as ideals. This final result implies in particular that, in the case when modular Lie algebras of vector fields contain finite families of vector fields with no common zeros (we say that they are *strongly non–singular*), isomorphisms between them are generated by isomorphisms of corresponding algebras of functions, i.e., by diffeomorphisms of underlying manifolds.

On the other hand, there are many geometrically interesting Lie algebras of vector fields which are not modular, e.g., the Lie algebras of hamiltonian vector fields on a Poisson manifold etc. For such algebras the situation is much more complicated and no analog of Skryabin's method is known in these cases. In [G6] a Pursell–Shanks-type result for the Lie algebras associated with Jacobi structures on a manifold has been announced. The result suggests that the concept of a Jacobi structure should be developed for sections of an arbitrary line bundle rather than for the algebra of functions, i.e., sections of the trivial line bundle. This is exactly the concept of *local Lie algebra* in the sense of A. A. Kirillov [Ki] which we will call also *Jacobi–Kirillov bundle*.

In the present note we complete the Lie algebroid result of [GG] by proving that the local Lie algebra determines the base manifold up to a diffeomorphism if and only if the anchor map is nowhere-vanishing (Theorem 7). The methods, however, are more complicated (due to the fact that the Lie algebra of Jacobi–hamiltonian vector fields is not modular) and different from those in [GG]. A part of these methods is a modification of what has been sketched in [G6]. However, the full generalization of [G6] for local Lie algebras on arbitrary line bundles, i.e., the description of isomorphisms of local Lie algebras, is much more delicate and we postpone it to a separate paper. Note also that in our approach we admit different categories of differentiability: smooth, real-analytic, and holomorphic (Stein manifolds).

2 Jacobi modules

What we will call *Jacobi module* is an algebraic counterpart of geometric structures which include *Lie algebroids* and *Jacobi structures* (or, more generally, local Lie algebras in the sense of Kirillov [Ki]). For a short survey one can see [G7], where these geometric structures appeared under the name of *Lie QD-algebroids*.

The concept of a *Lie algebroid* (or its pure algebraic counterpart—a *Lie pseudoalgebra*) is one of the most natural concepts in geometry.

Definition 1 Let R be a commutative and unitary ring, and let $\mathcal{A}$ be a commutative R-algebra. A *Lie pseudoalgebra* over R and $\mathcal{A}$ is an $\mathcal{A}$-module $\mathcal{E}$ together with a bracket $[\cdot,\cdot]:\mathcal{E}\times\mathcal{E}\to\mathcal{E}$ on the module $\mathcal{E}$, and an $\mathcal{A}$-module morphism $\alpha:\mathcal{E}\to\mathrm{Der}(\mathcal{A})$ from $\mathcal{E}$ to the $\mathcal{A}$-module $\mathrm{Der}(\mathcal{A})$ of derivations of $\mathcal{A}$, called the *anchor* of $\mathcal{E}$, such that

(i) the bracket on $\mathcal{E}$ is R-bilinear, alternating, and satisfies the Jacobi identity:

$$[[X,Y],Z]=[X,[Y,Z]]-[Y,[X,Z]].$$

(ii) For all $X,Y\in\mathcal{E}$ and all $f\in\mathcal{A}$ we have

$$[X,fY]=f[X,Y]+\alpha(X)(f)Y; \tag{1}$$

(iii) $\alpha([X,Y])=[\alpha(X),\alpha(Y)]_c$ for all $X,Y\in\mathcal{E}$, where $[\cdot,\cdot]_c$ is the commutator bracket on $\mathrm{Der}(\mathcal{A})$.

A *Lie algebroid* on a vector bundle E over a base manifold M is a Lie pseudoalgebra on the $(\mathbb{R},C^\infty(M))$-module $\mathcal{E}=\mathrm{Sec}(E)$ of smooth sections of E. Here the anchor map is described by a vector bundle morphism $\alpha:E\to TM$ which induces the bracket homomorphism from $(\mathcal{E},[\cdot,\cdot])$ into the Lie algebra $(\mathcal{X}(M),[\cdot,\cdot]_{vf})$ of vector fields on M. In this case, as in the case of any faithful $\mathcal{A}$-module $\mathcal{E}$, i.e., when $fX=0$ for all $X\in\mathcal{E}$ implies $f=0$, the axiom (iii) is a consequence of (i) and (ii). Of course, we can consider Lie algebroids in the real-analytic or holomorphic (on complex holomorphic bundles over Stein manifolds) category as well.

Lie pseudoalgebras appeared first in a paper by Herz [He] but one can find similar concepts under more than a dozen names in the literature (e.g., Lie modules, (R,A)-Lie algebras, Lie–Cartan pairs, Lie–Rinehart algebras, differential algebras, etc.). Lie algebroids were introduced by Pradines [Pr] as infinitesimal parts of differentiable groupoids. In the same year a book by Nelson [Ne] was published, where a general theory of Lie modules together with a big part of the corresponding differential calculus can be found. We also refer to a survey article by Mackenzie [Ma2].

Note that Lie algebroids on a singleton base space are Lie algebras. Another canonical example is the tangent bundle TM with the canonical bracket $[\cdot,\cdot]_{vf}$ on the space $\mathcal{X}(M)=\mathrm{Sec}(TM)$ of vector fields.

The property (1) of the bracket in the $\mathcal{A}$-module $\mathcal{E}$ can be expressed as the fact that $ad_X=[X,\cdot]$ is a *quasi-derivation* in $\mathcal{E}$, i.e., an R-linear operator D in $\mathcal{E}$ such that $D(fY)=fD(Y)+\widehat{D}(f)Y$ for any $f\in\mathcal{A}$ and certain derivation $\widehat{D}$ of $\mathcal{A}$ called the *anchor* of D. The concept of quasi-derivation can be traced back to N. Jacobson [J1, J2] as a special case of his *pseudo-linear endomorphism*. It has appeared also in [Ne] under the name of a *module derivation* and used to define linear connections in the algebraic setting. In the geometric setting, for Lie algebroids, it has been studied in [Ma1], Ch. III, under the name *covariant differential operator*.

Starting with the notion of Lie pseudoalgebra we obtain the notion of *Jacobi module* when we drop the assumption that the anchor map is $\mathcal{A}$-linear.

Definition 2 Let R be a commutative and unitary ring, and let $\mathcal{A}$ be a commutative R-algebra. A *Jacobi module* over $(R, \mathcal{A})$ is an $\mathcal{A}$-module $\mathcal{E}$ together with a bracket $[\cdot,\cdot] : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ on the module $\mathcal{E}$, and an R-module morphism $\alpha : \mathcal{E} \to \mathrm{Der}(\mathcal{A})$ from $\mathcal{E}$ to the $\mathcal{A}$-module $\mathrm{Der}(\mathcal{A})$ of derivations of $\mathcal{A}$, called the *anchor* of $\mathcal{E}$, such that (i)–(iii) of Definition 1 are satisfied. Again, for faithful $\mathcal{E}$, the axiom (iii) follows from (i) and (ii). This concept is in a sense already present in [He], although in [He] it has been assumed that $\mathcal{A}$ is a field. It has been observed in [He] that every Jacobi module (over a field) of dimension > 1 is just a Lie pseudoalgebra.

Definition 3 (cf. [G7]) A *Lie QD-algebroid* is a Jacobi module structure on the $(\mathbb{R}, C^\infty(M)$-module $\mathcal{E} = \mathrm{Sec}(E)$ of sections of a vector bundle E over a manifold M.

The case $\mathrm{rank}(E) = 1$ is special for many reasons and it was originally studied by A. A. Kirillov [Ki]. For a trivial bundle, well-known examples are those given by Poisson or, more generally, Jacobi brackets (cf. [Li]). In [Ki] such structures on line bundles are called *local Lie algebras* and in [Mr] *Jacobi bundles*. We will refer to them also as *local Lie algebras* or *Jacobi–Kirillov bundles* and to the corresponding brackets as to *Jacobi–Kirillov brackets*.

Definition 4 A *Jacobi–Kirillov bundle* (*local Lie algebra* in the sense of Kirillov) is a Lie QD-algebroid on a vector bundle of rank 1. In other words, a *Jacobi–Kirillov bundle* is a Jacobi module structure on the $(\mathbb{R}, C^\infty(M))$-module $\mathcal{E}$ of sections of a line bundle E over a smooth manifold M. The corresponding bracket on $\mathcal{E}$ we call a *Jacobi–Kirillov bracket* and the values of the anchor map $\alpha : \mathcal{E} \to \mathcal{X}(M)$ we call *Jacobi–hamiltonian vector fields*.

It is easy to see (cf. [G7]) that any Lie QD-algebroid on a vector bundle of rank > 1 must be a Lie algebroid. Of course, we can consider Lie QD-algebroids and Jacobi–Kirillov bundles in real-analytic or in holomorphic category as well.

Since quasi-derivations are particular first-order differential operators in the algebraic sense, it is easy to see that, for a Jacobi module $\mathcal{E}$ over $(R, \mathcal{A})$, the anchor map $\alpha : \mathcal{E} \to \mathrm{Der}(A)$ is also a first-order differential operator, i.e.,

$$\alpha(fgX) = f\alpha(gX) + g\alpha(fX) - fg\alpha(X) \tag{2}$$

for any $f, g \in \mathcal{A}$ and $X \in \mathcal{E}$. Denoting the Jacobi–hamiltonian vector field $\alpha(X)$ shortly by $\widehat{X}$, we can write for any $f, g \in \mathcal{A}$ and $X, Y \in \mathcal{E}$,

$$\begin{aligned}[gX, fY] &= \widehat{gX}(f)Y - f \cdot \widehat{Y}(g)X + fg[X, Y] \\ &= g \cdot \widehat{X}(f)Y - \widehat{fY}(g)X + fg[X, Y],\end{aligned} \tag{3}$$

so that for the map $\Lambda_X : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined by $\Lambda_X(g, f) := \widehat{gX}(f) - g\widehat{X}(f)$ we have

$$\Lambda_X(g, f)Y = -\Lambda_Y(f, g)X. \tag{4}$$

The above identity implies clearly that, roughly speaking, $\mathrm{rank}_{\mathcal{A}}\mathcal{E} = 1$ 'at points where Λ is non-vanishing' (cf. [G7]) and that

$$\Lambda_X(g, f)X = -\Lambda_X(f, g)X. \tag{5}$$

The identity (5) does not contain much information about Λ_X if there is 'much torsion' in the module $\mathcal{E}$. But if, for example, there is a torsion-free element in $\mathcal{E}$, say X_0, (this is the case for the module of sections of a vector bundle), then the situation is simpler. In view of (5), Λ_{X_0} is skew-symmetric and, in turn, by (4) every Λ_X is skew-symmetric. Every Λ_X is by definition a derivation with respect to the second argument, so, being skew-symmetric, it is a derivation also with respect to the first argument. Since in view of (3),

$$[gX, fX] = \left(g\widehat{X}(f) - f \cdot \widehat{X}(g) + \Lambda_X(g, f)\right) X,$$

and since Λ_X and $\widehat{X}$ respect the annihilator $\mathrm{Ann}(X) = \{f \in \mathcal{A} : fX = 0\}$, we get easily the following.

Proposition 1 *If $\mathcal{E}$ is a Jacobi module over $(R, \mathcal{A})$, then, for every $X \in \mathcal{E}$, the map $\Lambda_X : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ induces a skew-symmetric bi-derivation of $\mathcal{A}/\mathrm{Ann}(X)$, the derivation $\widehat{X}$ of $\mathcal{A}$ induces a derivation of $\mathcal{A}/\mathrm{Ann}(X)$ and the bracket*

$$\{\overline{f}, \overline{g}\}_X = \Lambda_X(\overline{f}, \overline{g}) + \overline{f} \cdot \widehat{X}(\overline{g}) - \overline{g} \cdot \widehat{X}(\overline{f}), \tag{6}$$

where $\overline{f}$ denotes the class of $f \in \mathcal{A}$ in $\mathcal{A}/\mathrm{Ann}(X)$, is a Jacobi bracket on $\mathcal{A}/\mathrm{Ann}(X)$ associated with the Jacobi structure $(\Lambda_X, \widehat{X})$. Moreover, $\mathcal{A}/\mathrm{Ann}(X) \ni \overline{f} \mapsto fX \in \mathcal{E}$ is a Lie algebra homomorphism of the bracket $\{\cdot, \cdot\}_X$ into $[\cdot, \cdot]$.

For pure algebraic approaches to Jacobi brackets we refer to [S2, S3, G4].

Corollary 1 *If the $\mathcal{A}$-module $\mathcal{E}$ is generated by torsion-free elements, then for every $X \in \mathcal{E}$, the map $\Lambda_X : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a skew-symmetric bi-derivation and the bracket*

$$\{f, g\}_X = \Lambda_X(f, g) + f \cdot \widehat{X}(g) - g \cdot \widehat{X}(f), \tag{7}$$

is a Jacobi bracket on $\mathcal{A}$ associated with the Jacobi structure $(\Lambda_X, \widehat{X})$. Moreover, $\mathcal{A} \ni f \mapsto fX \in \mathcal{E}$ is a Lie algebra homomorphism of the bracket $\{\cdot, \cdot\}_X$ into $[\cdot, \cdot]$.

For any torsion-free generated Jacobi module, e.g., a module of sections of a vector bundle, we have additional identities as shows the following.

Proposition 2 *If the $\mathcal{A}$-module $\mathcal{E}$ is generated by torsion-free elements, then for all $f_1, \ldots, f_m \in \mathcal{A}$, $m \geq 2$, and all $X, Y \in \mathcal{E}$,*

(a) $$(m-1)[FX, Y] = \sum_{i=1}^{m}[F_iX, f_iY] - [X, FY]$$

and

(b) $$(m-2)[FX,Y]=\sum_{i=1}^{m-1}[F_iX,f_iY]+[F_mY,f_mX],$$

where $F=\prod_{i=1}^m f_i$, $F_k=\prod_{i\neq k} f_i$.

Proof. (a) We have (cf. (3))

$$\begin{aligned}\sum_{i=1}^m[F_iX,f_iY]&=\sum_{i=1}^m\big(F_i\widehat{X}(f_i)Y-f_i\widehat{Y}(F_i)X+F[X,Y]+\Lambda_X(F_i,f_i)Y\big)\\&=\widehat{X}(F)Y-(m-1)\widehat{Y}(F)X+mF[X,Y]+\sum_{i\neq j}F_{ij}\Lambda_X(f_j,f_i)Y\\&=[X,FY]+(m-1)[FX,Y]+\sum_{i\neq j}F_{ij}\Lambda_X(f_j,f_i)Y,\end{aligned}$$

where $F_{ij}=\prod_{k\neq i,j} f_k$. The calculations are based on the Leibniz rule for derivations:

$$\widehat{X}\left(\prod_{i=1}^m f_i\right)=\sum_{i=1}^m F_i\widehat{X}(f_i),$$

etc. Since, due to Corollary 1, Λ_X is skew-symmetric and $F_{ij}=F_{ji}$, we have

$$\sum_{i\neq j}F_{ij}\Lambda_X(f_j,f_i)Y=0$$

and (a) follows.

(b) In view of (a), we have

$$\begin{aligned}(m-2)[FX,Y]&=\sum_{i=1}^m[F_iX,f_iY]-[X,FY]-[FX,Y]\\&=\sum_{i=1}^{m-1}[F_iX,f_iY]+[F_mX,f_mY]-[X,F_mf_mY]-[F_mf_mX,Y].\end{aligned}$$

But

$$[F_mX,f_mY]-[X,F_mf_mY]-[F_mf_mX,Y]=[F_mY,f_mX]$$

is a particular case of (a). □

3 Useful facts about associative algebras

In what follows, $\mathcal{A}$ will be an associative commutative unital algebra over a field $\mathbb{K}$ of characteristic 0. Our standard model will be the algebra $\mathcal{C}(N)$ of class $\mathcal{C}$ functions on a manifold N of class $\mathcal{C}$, $\mathcal{C}=C^\infty, C^\omega, \mathcal{H}$. Here C^∞ refers to the smooth category with $\mathbb{K}=\mathbb{R}$, C^ω – to the $\mathbb{R}$-analytic category with $\mathbb{K}=\mathbb{R}$, and $\mathcal{H}$ – to the holomorphic category of Stein manifolds with $\mathbb{K}=\mathbb{C}$ (cf. [G1, AG]). All manifolds are assumed to be paracompact and second countable. It is obvious what is meant by a Lie QD-algebroid or a Jacobi–Kirillov bundle of class $\mathcal{C}$. The rings of germs of class $\mathcal{C}$ functions at a given point are noetherian in analytic cases, which is no longer true in the C^∞ case. However, all the algebras $\mathcal{C}(N)$ are in a sense noetherian in finite codimension. To explain this, let us start with the following well-known observation.

Theorem 1 *Every maximal finite-codimensional ideal of* $\mathcal{C}(N)$ *is of the form* $\overline{p} = \{f \in \mathcal{C}(N) : f(p) = 0\}$ *for a unique* $p \in N$ *and* $\overline{p}$ *is finitely generated.*

Proof. The form of such ideals is proven e.g., in [G1], Proposition 3.5. In view of embedding theorems for all types of manifolds we consider, there is an embedding $f = (f_1, \dots, f_n) : N \to \mathbb{K}^n$, $f_i \in \mathcal{C}(N)$. Then, the ideal $\overline{p}$ is generated by $\{f_i - f_i(p) \cdot 1 : i = 1, \dots, n\}$. In the smooth case it is obvious, in the analytic cases it can be proven by means of some coherent analytic sheaves and methods parallel to those in [G2], Note 2.3. □

Remark Note that in the case of a non-compact N there are maximal ideals of $\mathcal{C}(N)$ which are not of the form $\overline{p}$. They are of course of infinite codimension. It is not known if the above theorem holds also for manifolds which are not second countable (cf. [G8]).

For a subset $B \subset \mathcal{A}$, by $\mathrm{Sp}(\mathcal{A}, B)$ we denote the set of those maximal finite-codimensional ideals of $\mathcal{A}$ which contain B. For example, $\mathrm{Sp}(\mathcal{A}, \{0\})$ is just the set of all maximal finite-codimensional ideals which we denote shortly by $\mathrm{Sp}(\mathcal{A})$. Put $\overline{B} = \cap_{I \in \mathrm{Sp}(A,B)} I$. For an ideal $I \subset \mathcal{A}$, by $\sqrt{I}$ we denote the radical of I, i.e.,

$$\sqrt{I} = \{f \in \mathcal{A} : f^n \in I, \quad \text{for some } n = 1, 2, \dots\}.$$

The following easy observations will be used in the sequel.

Theorem 2 (a) *If I is an ideal of codimension k in $\mathcal{A}$, then $\sqrt{I} = \overline{I}$ and $(\overline{I})^k \subset I$.*
(b) *Every finite-codimensional prime ideal in $\mathcal{A}$ is maximal.*
(c) *If a derivation $D \in \mathrm{Der}(\mathcal{A})$ preserves a finite-codimensional ideal I in $\mathcal{A}$, then $X(\mathcal{A}) \subset \overline{I}$.*
(d) *If $I_1, \dots, I_n$ are finite-codimensional and finitely generated ideals of $\mathcal{A}$, then the ideal $I_1 \cdots I_n$ is finite-codimensional and finitely generated.*

Proof. (a) The descending series of ideals

$$I + \overline{I} \supset I + (\overline{I})^2 \supset \cdots$$

stabilizes at kth step at most, so $I + (\overline{I})^k = I + (\overline{I})^{k+1}$. Applying Nakayama's Lemma to the finite-dimensional module $(I + (\overline{I})^k)/I$ over the algebra A/I, we get $(I + (\overline{I})^k)/I = \{0\}$, i.e., $(\overline{I})^k \subset I$, thus $\overline{I} \subset \sqrt{I}$. Since for all $J \in \mathrm{Sp}(A, I)$ we have $\sqrt{I} \subset \sqrt{J} = J$, also $\overline{I} \supset \sqrt{I}$.

(b) If I is prime and finite-codimensional, $\sqrt{I} = I$ and $\sqrt{I} = \overline{I}$ by (a). But a finite intersection of maximal ideals is prime only if they coincide, so $\overline{I} = J$ for a single $J \in \mathrm{Sp}(\mathcal{A})$.

(c) By Lemma 4.2 of [G1], $D(I) \subset I$ for a finite-codimensional ideal I implies $D(\mathcal{A}) \subset J$ for each $J \in \mathrm{Sp}(\mathcal{A}, I)$.

(d) It suffices to prove (d) for $n = 2$ and to use induction. Suppose that I_1, I_2 are finite-codimensional and finitely generated by $\{u_i\}$ and $\{v_j\}$, respectively. It is easy to see that $I_1 \cdot I_2$ is generated by $\{u_i \cdot v_j\}$ and that $I_1 \cdot I_2$ is finite-codimensional in I_1. Indeed, if $c_1, \ldots, c_k \in \mathcal{A}$ represent a basis of A/I_2, then $\{c_l u_i\}$ represent a basis of $I_1/(I_1 \cdot I_2)$. □

Theorem 3 *For an associative commutative unital algebra $\mathcal{A}$ the following are equivalent:*

(a) *Every finite-codimensional ideal of A is finitely generated.*
(b) *Every maximal finite-codimensional ideal of A is finitely generated.*
(c) *Every prime finite-codimensional ideal of A is finitely generated.*

Proof. (a) $\Rightarrow$ (b) is trivial, (b) $\Rightarrow$ (c) follows from Theorem 1 (b), and (c) $\Rightarrow$ (a) is a version of Cohen's theorem for finite-dimensional ideals. □

Definition 5 We call an associative commutative unital algebra $\mathcal{A}$ *noetherian in finite codimension* if one of the above (a), (b), (c), thus all, is satisfied.

An immediate consequence of Theorem 1 is the following.

Theorem 4 *The algebra $\mathcal{A} = \mathcal{C}(N)$ is noetherian in finite codimension.*

4 Spectra of Jacobi modules

Let us fix a Jacobi module $(\mathcal{E}, [\cdot,\cdot])$ over $(\mathbb{K}, \mathcal{A})$. Throughout this section we will assume that $\mathcal{E}$ is finitely generated by torsion-free elements and that $\mathcal{A}$ is a noetherian algebra in finite codimension over a field $\mathbb{K}$ of characteristic 0. The $(\mathbb{K}, \mathcal{C}(N))$-modules of sections of class $\mathcal{C}$ vector bundles over N can serve as standard examples.

For $L \subset \mathcal{E}$, by $\widehat{L}$ denote the image of L under the anchor map: $\widehat{L} = \{\alpha(X) : X \in L\} \subset \mathrm{Der}(\mathcal{A})$. The set $\widehat{\mathcal{E}}$ is a Lie subalgebra in $\mathrm{Der}(\mathcal{A})$ with the commutator bracket $[\cdot,\cdot]_c$ and we will refer to $\widehat{\mathcal{E}}$ as to the Lie algebra of 'Jacobi–hamiltonian vector fields'. The main difference with the 'modular' case (in particular, with that of Pursell and Shanks [PS]) is that $\widehat{\mathcal{E}}$ is no longer, in general, an $\mathcal{A}$-module, so we cannot multiply by 'functions' inside $\widehat{\mathcal{E}}$. However, we still can try to translate some properties of the Lie algebra $(\mathcal{E}, [\cdot,\cdot])$ into the properties of the Lie algebra $\widehat{\mathcal{E}}$ of Jacobi–hamiltonian vector fields by means of the anchor map and to describe some 'Lie objects' in $\mathcal{E}$ or $\widehat{\mathcal{E}}$ by means of 'associative objects' in $\mathcal{A}$.

The *spectrum of the Jacobi module* $\mathcal{E}$, denoted by $\mathrm{Sp}(\mathcal{E})$, will be the set of such maximal finite-codimensional Lie subalgebras in $\mathcal{E}$ that do not contain finite-codimensional Lie ideals of $\mathcal{E}$. In nice geometric situations, $\mathrm{Sp}(\mathcal{E})$ will be interpreted as a set of points of the base manifold at which the anchor map does not vanish. Note that the method developed in [GG] for Lie pseudoalgebras fails, since Lemma 1 therein is no longer true for Jacobi modules. In fact, as easily shown by the example of a symplectic Poisson bracket on a compact manifold, $[\mathcal{E}, \mathcal{E}]$ may include no non-trivial $\mathcal{A}$-submodules of $\mathcal{E}$. Therefore we will modify the method from [G3] where Poisson brackets have been considered.

Let us fix some notation. For a linear subspace L in $\mathcal{E}$ and for $J \subset \mathcal{A}$, denote

- $\mathcal{N}_L = \{X \in \mathcal{E} : [X, L] \subset L\}$—the Lie normalizer of L;
- $U_L = \{X \in \mathcal{E} : [X, \mathcal{E}] \subset L\}$;
- $I(L) = \{f \in \mathcal{A} : \forall X \in \mathcal{E}\ [fX \in L]\}$—the largest associative ideal I in $\mathcal{A}$ such that $I\mathcal{E} \subset L$;
- $\mathcal{E}_J = \{X \in \mathcal{E} : \widehat{X}(A) \subset J\}$.

It is an easy excercise to prove the following proposition (cf. [G3], Theorem 1.6).

Proposition 3 *If L is a Lie subalgebra in $\mathcal{E}$, then $\mathcal{N}_L$ is a Lie subalgebra containing L, the set U_L is a Lie ideal in $\mathcal{N}_L$, and $\widehat{\mathcal{N}_L}(I(U_L)) \subset I(U_L)$.*

Choose now generators $X_1, \ldots, X_n$ of $\mathcal{E}$ over $\mathcal{A}$. For a fixed finite-codimensional Lie subalgebra L in $\mathcal{E}$ put $U_i = \{f \in \mathcal{A} : fX_i \in U_L\}$ and $U = \bigcap_{i=1}^n U_i$. Since U_L is clearly finite-codimensional in $\mathcal{E}$, all U_i are finite-codimensional in $\mathcal{A}$, so is U.

Lemma 1

(a) $[U^m X_j, X_k] \subset L$ *for all* $j, k = 1, \ldots, n$ *and* $m \geq 3$.
(b) $[U^m X_j, U^l X_k] \subset L$ *for all* $j, k = 1, \ldots, n$ *and* $m, l \geq 1$.

Proof. (a) Take $f_1, \ldots, f_m \in U$. Since $f_i X_k \in U_L$, Proposition 2 (b) implies $[f_1 \cdots f_m X_j, X_k] \in L$.

(b) The inclusion is trivial for $l = 1$, so suppose $l \geq 2$. Take $f_1, \ldots, f_m \in U$, $f_{m+1} \in U^l$ and put $F = f_1 \ldots f_{m+1}$, $F_i = \prod_{r \neq i} f_r$. By Proposition 2 (b)

$$[f_1 \cdots f_m X_k, f_{m+1} X_j] = (m-1)[F X_j, X_k] - \sum_{i=1}^{m} [F_i X_j, f_i X_k].$$

Since $F \in U^{m+l}$, according to (a), $[F X_j, X_k] \in L$ and $[F_i X_j, f_i X_k] \subset [\mathcal{E}, U_L] \subset L$, so the lemma follows. □

Theorem 5 *The ideal $I(U_L)$ is finite-codimensional in $\mathcal{A}$ provided L is a finite-codimensional Lie subalgebra in $\mathcal{E}$.*

Proof. Let $\mathcal{U}$ be the associative subalgebra in $\mathcal{A}$ generated by U. It is finite-codimensional and, in view of Lemma 1 (b), $[\mathcal{U}X_j, \mathcal{U}X_k] \subset L$. Being finite-codimensional in $\mathcal{A}$, the associative subalgebra $\mathcal{U}$ contains a finite-codimensional ideal J of $\mathcal{A}$ (cf. [G3], Proposition 2.1 b)). Hence $[JX_j, JX_k] \subset L$ and, since X_i are generators of $\mathcal{E}$, $[J\mathcal{E}, J\mathcal{E}] \subset L$. Note that we do not exclude the extremal case $\mathcal{U} = \mathcal{A} = J$. Applying the identity

$$[f_1 f_2 X, Y] = [f_2 X, f_1 Y] + [f_1 X, f_2 Y] - [f_1 f_2 Y, X]$$

for $f_1, f_2 \in J$, $X \in U_L$, we see that $J^2 U_L \subset U_L$. In particular, $J^2 U X_i \subset U_L$ for all $i = 1, \ldots, n$, so $J^2 U \subset U$ and hence $J^2 \mathcal{U} \subset U$ and $J^3 \mathcal{E} \subset U_L$. Consequently $J^3 \subset I(U_L)$. Since J is finite-codimensional and finitely generated, J^3 is finite-codimensional (Theorem 1 (d)), so $I(U_L)$ is finite-codimensional. □

Denote by $\mathrm{Sp}_{\mathcal{E}}(\mathcal{A})$ the set of these maximal finite-codimensional ideals $I \subset \mathcal{A}$ which do not contain $\widehat{\mathcal{E}}(\mathcal{A})$, i.e., $\mathcal{E}_I \neq \mathcal{E}$. Geometrically, $\mathrm{Sp}_{\mathcal{E}}(\mathcal{A})$ can be interpreted as the support of the anchor map. Recall that $\mathrm{Sp}(\mathcal{E})$ is the set of these maximal finite-codimensional Lie subalgebras in $\mathcal{E}$ which do not contain finite-codimensional Lie ideals.

Theorem 6 *The map $J \mapsto \mathcal{E}_J$ constitutes a bijection of $\mathrm{Sp}_{\mathcal{E}}(\mathcal{A})$ with $\mathrm{Sp}(\mathcal{E})$. The inverse map is $L \mapsto \sqrt{I(L)}$.*

Proof. Let us take $J \in \mathrm{Sp}_{\mathcal{E}}(\mathcal{A})$. In view of (2), $J^2\mathcal{E} \subset \mathcal{E}_J$ which implies that $\mathcal{E}_J$ is finite-codimensional, as J^2 is finite-codimensional and $\mathcal{E}$ is finitely generated.

We will show that $\mathcal{E}_J$ is maximal. Of course, $\mathcal{E}_J \neq \mathcal{E}$ and $\mathcal{E}_J$ is of finite codimension, so there is a maximal Lie subalgebra L containing $\mathcal{E}_J$. We have

$$J^2\mathcal{E} \subset \mathcal{E}_J \subset L \Rightarrow J^2 \subset I(L) \Rightarrow J \subset \sqrt{I(L)} \Rightarrow J = \sqrt{I(L)}.$$

Moreover, $I(L)$ is finite-codimensional, and since, due to (1), $\widehat{L}(I(L)) \subset I(L)$, then, by Theorem 2 (c), $\widehat{L}(A) \subset J$, i.e., $L \subset \mathcal{E}_J$ and finally $L = \mathcal{E}_J$.

Finally, suppose P is a finite-codimensional Lie ideal of $\mathcal{E}$ contained in $\mathcal{E}_J$. Then U_P is a Lie ideal in $\mathcal{E}$ of finite codimension and, according to Theorem 5, $I(U_P)$ is a finite-codimensional ideal in $\mathcal{A}$. Since $\widehat{\mathcal{E}}(I(U_P)) \subset I(U_P)$, and since $I(U_P) \subset I(U_L) \subset J$, we have $\widehat{\mathcal{E}}(A) \subset J$, i.e., $\mathcal{E} = \mathcal{E}_J$; a contradiction.

Suppose now that $L \in \mathrm{Sp}(\mathcal{E})$. Observe first that $\mathcal{N}_L = L$, since otherwise L would be a Lie ideal, that would, in turn, imply $U_L \subset L$ and $I(U_L) \subset I(L)$. Since U_L is finite-codimensional, Theorem 5 shows that $I(L)$ is finite-codimensional. Exactly as above we show that $\widehat{L}(\mathcal{A}) \subset \sqrt{I(L)}$, i.e., $L \subset \mathcal{E}_J$, where $J = \sqrt{I(L)}$. By Theorem 2 (a), $J^k \subset I(L)$ for some k, so if we had $\mathcal{E}_J = \mathcal{E}$, then $J^k \cdot \mathcal{E}$ would be a finite-codimensional Lie ideal contained in L. Thus $\mathcal{E}_J \neq \mathcal{E}$ and there is $I \in \mathrm{Sp}(\mathcal{A}, J)$ with $\mathcal{E}_I \neq \mathcal{E}$. We know already that in this case $\mathcal{E}_I$ is maximal. Since $L \subset \mathcal{E}_J \subset \mathcal{E}_I$ and L is maximal, we have $L = \mathcal{E}_I$ and $I = J = \sqrt{I(L)}$. □

Corollary 2 *Let $(\mathcal{E}, [\cdot,\cdot])$ be a Lie QD-algebroid of class $\mathcal{C}$ (i.e., a Jacobi module over $(\mathbb{K}, \mathcal{C}(N))$ of class $\mathcal{C}$ sections of a class $\mathcal{C}$ vector bundle) over a class $\mathcal{C}$ manifold N. Let $S \subset N$ be the open support of the anchor map, i.e., $S = \{p \in N : \widehat{X}(p) \neq 0$ for some $X \in \mathcal{E}\}$. Then the map $p \mapsto p^* = \{X \in \mathcal{E} : \widehat{X}(p) = 0\}$ constitutes a bijection of S with $\mathrm{Sp}(\mathcal{E})$.*

Let $\widehat{\mathcal{E}}$ be the image of the anchor map $\alpha : \mathcal{E} \to \mathrm{Der}(\mathcal{A})$. By definition of a Jacobi module, $\widehat{\mathcal{E}}$ is a Lie subalgebra in $(\mathrm{Der}(\mathcal{A}), [\cdot,\cdot]_c)$. Since $\alpha : \mathcal{E} \to \widehat{\mathcal{E}}$ is a surjective Lie algebra homomorphism, it induces a bijection of $\mathrm{Sp}(\mathcal{E})$ onto $\mathrm{Sp}(\widehat{\mathcal{E}})$, $L \mapsto \widehat{L} = \alpha(L)$. Thus we get the following.

Corollary 3 *Let* $(\mathcal{E}, [\cdot\,,\cdot])$ *be a Lie QD-algebroid of class* $\mathcal{C}$ *over a class* $\mathcal{C}$ *manifold* N. *Let* $S \subset N$ *be the open support of the anchor map, i.e.,* $S = \{p \in N : \widehat{X}(p) \neq 0$ *for some* $X \in \mathcal{E}\}$. *Then the map* $p \mapsto \widehat{p} = \{\xi \in \widehat{\mathcal{E}} : \xi(p) = 0\}$ *constitutes a bijection of* S *with* $\mathrm{Sp}(\widehat{\mathcal{E}})$.

5 Isomorphisms

It is clear that any isomorphism $\Psi : \mathcal{E}_1 \to \mathcal{E}_2$ of the Lie algebras associated with Jacobi modules $\mathcal{E}_i$ over $(R_i, \mathcal{A}_i)$, $i = 1, 2$, induces a bijection $\psi : \mathrm{Sp}(\mathcal{E}_2) \to \mathrm{Sp}(\mathcal{E}_1)$. Since the kernels K_i of the anchor maps $\alpha_i : \mathcal{E}_i \to \widehat{\mathcal{E}_i}$ are the intersections

$$K_i = \bigcap_{L \in \mathrm{Sp}(\mathcal{E}_i)} L, \quad i = 1, 2,$$

$\Psi(K_1) = K_2$, so Ψ induces a well-defined isomorphism

$$\widehat{\Psi} : \widehat{\mathcal{E}_1} \to \widehat{\mathcal{E}_2}, \quad \widehat{\Psi}(\widehat{X}) = \widehat{\Psi(X)}$$

with the property

$$\widehat{L} \in \mathrm{Sp}(\widehat{\mathcal{E}_1}) \Leftrightarrow \widehat{\Psi}(\widehat{L}) \in \mathrm{Sp}(\widehat{\mathcal{E}_2}). \tag{8}$$

Proposition 4 *If the Lie algebras* $(\mathcal{E}_i, [\cdot\,,\cdot]_i)$, *associated with Jacobi modules* $\mathcal{E}_i$, $i = 1, 2$, *are isomorphic, then the Lie algebras of Jacobi–hamiltonian vector fields* $\widehat{\mathcal{E}_i}$, $i = 1, 2$, *are isomorphic.*

The following theorem describes isomorphisms of the Lie algebras of Jacobi–hamiltonian vector fields.

Theorem 7 *Let* $(\mathcal{E}_i, [\cdot\,,\cdot]_i)$ *be a Lie QD-algebroid of class* $\mathcal{C}$, *over a class* $\mathcal{C}$ *manifold* N_i, *and let* $S_i \subset N_i$ *be the (open) support of the anchor map* $\alpha_i : \mathcal{E}_i \to \widehat{\mathcal{E}_i}$, $i = 1, 2$. *Then every isomorphism of the Lie algebras of Jacobi–hamiltonian vector fields* $\Phi : \widehat{\mathcal{E}_1} \to \widehat{\mathcal{E}_2}$ *is of the form* $\Phi(\xi) = \phi_*(\xi)$ *for a class* $\mathcal{C}$ *diffeomorphism* $\phi : S_1 \to S_2$.

Corollary 4 *If the Lie algebras associated with Lie QD-algebroids* E_i *of class* $\mathcal{C}$, *over class* $\mathcal{C}$ *manifolds* N_i, $i = 1, 2$, *are isomorphic, then the (open) supports* $S_i \subset N_i$ *of the anchor maps* $\alpha_i : \mathcal{E}_i \to \widehat{\mathcal{E}_i}$, $i = 1, 2$, *are* $\mathcal{C}$*-diffeomorphic. In particular,* N_1 *and* N_2 *are* $\mathcal{C}$*-diffeomorphic provided the anchors are nowhere-vanishing.*

Proof of Theorem 7. According to Corollary 3, the isomorphism Φ induces a bijection $\phi : S_1 \to S_2$ such that, for every $\xi \in \widehat{\mathcal{E}_1}$ and every $p \in S_1$,

$$\xi(p) = 0 \Leftrightarrow \Phi(\xi)(\phi(p)) = 0. \tag{9}$$

First, we will show that ϕ is a diffeomorphism of class $\mathcal{C}$. For, let $f \in \mathcal{C}(N_1)$. Since the anchor map is a first-order differential operator, for every $X \in \mathcal{E}_1$ we have $\widehat{f^2 X} = 2f \cdot \widehat{fX} - f^2 \cdot \widehat{X}$. In particular, for any $p \in N_1$,

$$\widehat{f^2X}(p) - 2f(p)\widehat{fX}(p) + f^2(p)\widehat{X}(p) = 0,$$

so that, due to (9),

$$\Phi(\widehat{f^2X})(\phi(p)) = 2f(p)\Phi(\widehat{fX})(\phi(p)) - f^2(p)\Phi(\widehat{X})(\phi(p)). \tag{10}$$

We can rewrite (9) in the form

$$\Phi(\widehat{f^2X}) = 2(f \circ \psi) \cdot \Phi(\widehat{fX}) - (f \circ \psi)^2 \cdot \Phi(\widehat{X}), \tag{11}$$

where $\psi = \phi^{-1}$ and both sides of (11) are viewed as vector fields on S_2. In a similar way one can get

$$\Phi(\widehat{f^3X}) = 3(f \circ \psi)^2 \cdot \Phi(\widehat{fX}) - 2(f \circ \psi)^3 \cdot \Phi(\widehat{X}). \tag{12}$$

To show that $f \circ \psi$ is of class $\mathcal{C}$, choose $q \in S_2$ and $X \in \mathcal{E}_1$ such that $\Phi(\widehat{X})(q) \neq 0$. Then we can choose local coordinates $(x_1, \ldots, x_n)$ around q such that $\Phi(\widehat{X}) = \partial_1 = \partial/\partial x_1$. If a is the first coefficient of the vector field $\Phi(\widehat{fX})$ in these coordinates, we get out of (11) and (12) that $(f \circ \psi)^2 - 2a(f \circ \psi)$ and $2(f \circ \psi)^3 - 3a(f \circ \psi)^2$ are of class $\mathcal{C}$ in a neighbourhood of q. But

$$(f \circ \psi)^2 - 2a(f \circ \psi) = (f \circ \psi - a)^2 - a^2 \tag{13}$$

and

$$2(f \circ \psi)^3 - 3a(f \circ \psi)^2 = 2(f \circ \psi - a)^3 + 3a(f \circ \psi - a)^2 - a^2, \tag{14}$$

so $(f \circ \psi - a)^2$ and $(f \circ \psi - a)^3$ are functions of class $\mathcal{C}$ in a neighbourhood of q, as the function a is of class $\mathcal{C}$. Now we will use the following lemma which proves that $f \circ \psi - a$, thus $f \circ \psi$, is of class $\mathcal{C}$.

Lemma 2 *If g is a $\mathbb{K}$-valued function in a neighbourhood of $0 \in \mathbb{K}^n$ such that g^2 and g^3 are of class $\mathcal{C}$, then g is of class $\mathcal{C}$.*

Proof. In the analytic cases the lemma is almost obvious, since $g = g^3/g^2$ is a meromorphic and continuous function. In the smooth case the Lemma is non-trivial and proven in [Jo]. □

To finish the proof of the theorem, we observe that $f \circ \psi$ is of class $\mathcal{C}$ for all $f \in \mathcal{C}(N_2)$ implies that ψ, thus $\phi = \psi^{-1}$, is of class $\mathcal{C}$ and we show that $\Phi = \phi_*$ or, in other words, that $\widehat{Y}(f) \circ \psi = \Phi(\widehat{Y})(f \circ \psi)$ for all $f \in \mathcal{C}(N_1)$ and all $Y \in \mathcal{E}_1$. Indeed, for arbitrary $f \in \mathcal{C}(N_1)$ and $X, Y \in \mathcal{E}_1$, the bracket of vector fields $[\widehat{Y}, \widehat{f^2X}]$ reads

$$\begin{aligned}[\widehat{Y}, \widehat{f^2X}] &= [\widehat{Y}, 2f \cdot \widehat{fX} - f^2 \cdot \widehat{X}] \\ &= 2\widehat{Y}(f) \cdot \widehat{fX} - 2f \cdot \widehat{Y}(f) \cdot \widehat{X} + 2f[\widehat{Y}, \widehat{fX}] - f^2[\widehat{Y}, \widehat{X}].\end{aligned}$$

Hence, similarly as in (11),

$$\Phi([\widehat{Y}, \widehat{f^2X}]) = 2(\widehat{Y}(f) \circ \psi) \cdot \Phi(\widehat{fX}) - 2(f \circ \psi) \cdot (\widehat{Y}(f) \circ \psi) \cdot \Phi(\widehat{X}) \\ + 2(f \circ \psi) \cdot \Phi([\widehat{Y}, \widehat{fX}]) - (f \circ \psi)^2 \cdot \Phi([\widehat{Y}, \widehat{X}]).$$

Comparing the above with

$$[\Phi(\widehat{Y}), \Phi(\widehat{f^2X})] = [\Phi(\widehat{Y}), 2(f \circ \psi) \cdot \Phi(\widehat{fX}) - (f \circ \psi)^2 \cdot \Phi(\widehat{X})],$$

we get easily

$$\left(\widehat{Y}(f) \circ \psi - \Phi(\widehat{Y})(f \circ \psi)\right)\left(\Phi(\widehat{fX}) - (f \circ \psi) \cdot \Phi(\widehat{X})\right) = 0. \tag{15}$$

After polarizing with $f := f + h$ and multiplying both sides by $\widehat{Y}(f) \circ \psi - \Phi(\widehat{Y})(f \circ \psi)$, we get the identity

$$\left(\widehat{Y}(f) \circ \psi - \Phi(\widehat{Y})(f \circ \psi)\right)^2\left(\Phi(\widehat{hX}) - (h \circ \psi) \cdot \Phi(\widehat{X})\right) = 0, \tag{16}$$

valid for all $f, h \in \mathcal{C}(N_1)$ and all $X, Y \in \mathcal{E}_1$. From (16) we get

$$(\widehat{Y}(f) \circ \psi)(q) = (\Phi(\widehat{Y})(f \circ \psi))(q)$$

for such $q = \phi(p) \in S_2$ for which in no neighbourhood of them the anchor map is a differential operator of order 0, i.e., for q which do not belong to

$$S_2^0 = \{\phi(p) \in S_2 : \widehat{hX}(p') = h(p')\widehat{X}(p') \text{ for all } h \in \mathcal{C}(N_1),\, X \in \mathcal{E}_1 \text{ and } p' \text{ close to } p\}.$$

If, on the other hand, $q \in S_2^0$, then $\Phi(\widehat{hX})(q') = (h \circ \psi)(q') \cdot \Phi(\widehat{X})(q')$ for q' from a neighbourhood of q, so that comparing in this neighbourhood

$$\Phi([\widehat{Y}, \widehat{fX}]) = (\widehat{Y}(f) \circ \psi) \cdot \Phi(\widehat{X}) + (f \circ \psi) \cdot \Phi([\widehat{Y}, \widehat{X}])$$

with

$$[\Phi(\widehat{Y}), \Phi(\widehat{fX})] = \Phi(\widehat{Y})(f \circ \psi) \cdot \Phi(\widehat{X}) + (f \circ \psi) \cdot [\Phi(\widehat{Y}), \Phi(\widehat{X})]$$

we get

$$(\widehat{Y}(f) \circ \psi)(q) \cdot \Phi(\widehat{X})(q) = \Phi(\widehat{Y})(f \circ \psi)(q) \cdot \Phi(\widehat{X})(q),$$

thus

$$(\widehat{Y}(f) \circ \psi)(q) = \Phi(\widehat{Y})(f \circ \psi)(q)$$

also for $q \in S_2^0$. □

Remark (a) For Jacobi–Kirillov bundles with all leaves of the characteristic foliation (i.e., orbits of $\widehat{\mathcal{E}}$) of dimension > 1 there is a much simpler argument showing that ψ is smooth than the one using Lemma 2. The difficulty in the general case comes from singularities of the 'bivector field' part of the anchor map and forced us to use Lemma 2.

(b) Theorem 7 has been proven for Lie algebroids in [GG], so the new (and difficult) case here is the case of Jacobi–Kirillov bundles with non-trivial 'bivector part' of the bracket. A similar result for the Lie algebras of smooth vector fields preserving a symplectic or a contact form up to a multiplicative factor has been proven by H. Omori [O1]. These Lie algebras are the Lie algebras of locally hamiltonian vector fields for the Jacobi–Kirillov brackets associated with the symplectic and the contact form, respectively.

Corollary 5

(a) *If the Lie algebras* $(\mathcal{C}(N_i), \{\cdot, \cdot\}_{\beta_i})$ *of the Jacobi contact brackets, associated with contact manifolds* (N_i, β_i), $i = 1, 2$, *of class* $\mathcal{C}$, *are isomorphic, then the manifolds* N_1 *and* N_2 *are* $\mathcal{C}$*-diffeomorphic.*
(b) *If the Lie algebras associated with nowhere-vanishing Poisson structures of class* $\mathcal{C}$ *on class* $\mathcal{C}$ *manifolds* N_i, $i = 1, 2$, *are isomorphic, then the manifolds* N_1 *and* N_2 *are* $\mathcal{C}$*-diffeomorphic.*

References

[Ab] Abe K, *Pursell–Shanks type theorem for orbit spaces and G-manifolds*, Publ. Res. Inst. Math. Sci., **18** (1982), pp. 265–282

[Am] Amemiya I, *Lie algebra of vector fields and complex structure*, J. Math. Soc. Japan, **27** (1975), pp. 545–549

[AG] Atkin C J, Grabowski J, *Homomorphisms of the Lie algebras associated with a symplectic manifold*, Compos. Math., **76** (1990), pp. 315–348

[DS] Duistermaat J J and Singer I M, *Order-preserving isomorphisms between algebras of pseudo-differential operators*, Comm. Pure Appl. Math. **29** (1976), 39–47

[FT] Fukui K and Tomita M, *Lie algebra of foliation preserving vector fields*, J. Math. Kyoto Univ. **22** (1982), pp. 685–699

[GG] Grabowska K and Grabowski J, *The Lie algebra of a Lie algebroid*, Lie algebroids and related topics in differential geometry (Warsaw, 2000), pp. 43–50, Banach Center Publ., **54**, Polish Acad. Sci., Warsaw, 2001

[G1] Grabowski J, *Isomorphisms and ideals of the Lie algebras of vector fields*, Invent. math., **50** (1978), pp. 13–33

[G2] Grabowski J, *Derivations of the Lie algebras of analytic vector fields. Compositio Math.* **43** (1981), 239–252

[G3] Grabowski J, *The Lie structure of* C^* *and Poisson algebras*, Studia Math. **81** (1985), 259–270

[G4] Grabowski J, *Abstract Jacobi and Poisson structures. Quantization and star-products*, J. Geom. Phys. **9** (1992), 45–73

[G5] Grabowski J, *Lie algebras of vector fields and generalized foliations*, Publ. Matem., **37** (1993), pp 359–367

[G6] Grabowski J, *Isomorphisms of Poisson and Jacobi brackets*, Poisson geometry (Warsaw, 1998), pp. 79–85, Banach Center Publ., 51, Polish Acad. Sci., Warsaw, 2000

[G7] Grabowski J, *Quasi-derivations and QD-algebroids*, Rep. Math. Phys. **52**, no. 3 (2003), pp. 445–451

[G8] Grabowski J, *Isomorphisms of algebras of smooth functions revisited*, Arch. Math. **85** (2005), 190–196

[GP] Grabowski J and Poncin N, *Automorphisms of quantum and classical Poisson algebras*, Compos. Math. **140**, no. 2 (2004), pp. 511–527

[HM] Hauser H and Müller G, *Affine varieties and Lie algebras of vector fields*, Manusc. Math., **80** (1993), pp. 309–337

[He] Herz J C, *Pseudo-algèbres de Lie*, C. R. Acad. Sci. Paris **236** (1953), I, pp. 1935–1937, II, pp. 2289–2291

[J1] Jacobson N, *On pseudo-linear transformations*, Proc. Nat. Acad. Sci. **21** (1935), 667–670

[J2] Jacobson N, *Pseudo-linear transformations*, Ann. Math. **38** (1937), 485–506

[Jo] Joris H, *Une $\mathcal{C}^\infty$-application non-immersive qui possède la propriété universelle des immersions*, Arch. Math. (Basel) **39** (1982), 269–277

[Ki] Kirillov A A, *Local Lie algebras* (Russian), Uspekhi Mat. Nauk **31** (1976), 57–76

[KMO1] Koriyama A, Maeda Y and Omori H, *On Lie algebras of vector fields on expansive sets*, Japan. J. Math. (N.S.) **3** (1977), 57–80

[KMO2] Koriyama A, Maeda Y and Omori H, *On Lie algebras of vector fields*, Trans. Amer. Math. Soc. **226** (1977), 89–117

[Li] Lichnerowicz A, *Les variétés de Jacobi et leurs algébres de Lie associées*, J. Math. Pures Appl. **57** (1978), 453–488

[Ma1] Mackenzie K, *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge University Press, 1987.

[Ma2] Mackenzie K, *Lie algebroids and Lie pseudoalgebras*, Bull. London Math. Soc. **27** (1995), 97–147

[Mr] Marle C M, *On Jacobi manifolds and Jacobi bundles*, Symplectic Geometry, Groupoids, and Integrable Systems (Berkeley 1998), MSRI Publ., 20, Springer, 1991, 227–246

[Ne] Nelson E, *Tensor Analysis*, Princeton University Press and The University of Tokyo Press, Princeton 1967

[O1] Omori H, *Infinite dimensional Lie transformation groups*, Lect. Notes in Math., **427** (1976), Springer Verlag

[O2] Omori H, *Infinite-dimensional Lie groups*, Translated from the 1979 Japanese original and revised by the author. Translations of Mathematical Monographs, 158. American Mathematical Society, Providence, RI, 1997

[Pr] Pradines J, *Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux*, C. R. Acad. Sci. Paris, Sér. A, **264** (1967), 245–248

[PS] Pursell L E and Shanks M E, *The Lie algebra of a smooth manifold*, Proc. Amer. Math. Soc., **5** (1954), pp. 468–472

[Ry] Rybicki T, *Lie algebras of vector fields and codimension one foliation*, Publ. Mat. UAB **34** (1990), pp. 311–321

[S1] Skryabin S M, *The regular Lie rings of derivations of commutative rings (Russian)*, preprint WINITI 4403-W87 (1987)

[S2] Skryabin S M, *Lie algebras of derivations of commutative rings: generalizations of Lie algebras of Cartan type (Russian)*, preprint WINITI 4405-W87 (1987)

[S3] Skryabin S M, *An algebraic approach to the Lie algebras of Cartan type*, Comm. Algebra **21** (1993), no. 4, pp. 1229–1336

Lie Algebroids Associated with Deformed Schouten Bracket of 2-Vector Fields

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Summary. Given a 2-vector field and a closed 1-form on a manifold, we consider the set of cotangent vectors which annihilate the deformed Schouten bracket of the 2-vector field by the closed 1-form. We show that if the space of cotangent vectors forms a vector bundle, it carries a structure of a Lie algebroid. We treat this theorem in the category of Lie algebroids. As a special case, this result contains the well-known fact that the 1-jet bundle of functions of a contact manifold has a Lie algebroid structure.

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Key words: Lie algebroid, deformed Schouten bracket.

1 Introduction

The Poisson bi-vector field on a Poisson manifold (M, π) defines a bundle morphism $\tilde{\pi} : \mathrm{T}^*(M) \to \mathrm{T}(M)$ which is given by $\alpha \mapsto \pi(\alpha, \cdot)$. The image of $\tilde{\pi}$ is called the characteristic distribution of the Poisson structure π. It is integrable and gives a generalized foliation of M consisting of leaves with symplectic structure. Moreover, $\mathrm{T}^*(M)$ has a structure of a Lie algebroid which leads to the Poisson cohomology. One can naturally ask the condition for a general 2-vector field π (not necessarily a Poisson), under which the image of $\tilde{\pi}$ is integrable and ask how special a Poisson bi-vector is. The condition for the integrability can well be seen from the formula (see Section 3)

$$[\pi(\alpha), \pi(\beta)] = \pi(\{\alpha, \beta\}) + (1/2)[\pi, \pi](\alpha, \beta) \quad \text{for 1-forms } \alpha \text{ and } \beta$$

where $\{\alpha, \beta\}$ is the bracket on $\Gamma(\mathrm{T}^*(M))$, and $\pi(\alpha)$ means precisely $\tilde{\pi}(\alpha)$, but we often use both notations interchangeably. The formula above says, if the Schouten bracket $[\pi, \pi]$ is in the image of $\tilde{\pi}$, the Frobenius conditions are satisfied and the distribution is integrable (while in the case of a Poisson structure *a fortiori* $[\pi, \pi] = 0$ holds). In [5], the authors considered the condition that $[\pi, \pi]$ is an image of a closed 3-form under the induced map of $\tilde{\pi}$ and proved $\mathrm{T}^*(M)$ has a Lie algebroid structure which they call a twisted Poisson structure. Clearly, this condition implies the integrability of the image of $\tilde{\pi}$ by the above formula.

In our previous paper ([4]), we considered the space of cotangent vectors $\mathcal{A} = \{\alpha \mid [\pi,\pi](\alpha,\cdot,\cdot) = 0\}$ and proved $\mathcal{A}$ has a natural Lie algebroid structure (provided $\mathcal{A}$ is a vector bundle of constant rank). In this paper, we generalize the discussion to the case of deformed Schouten bracket $[\pi,\pi]^{\phi}$ and show that the same result is obtained in this case too (Theorem 3.4). Also, we introduce the definition of a Jacobi–Lie algebroid. It is nothing but a Lie algebroid equipped with a specified 1-cocycle. However, this definition is sometimes preferable when we treat such objects formally. For example, one can define a homomorphism between two Jacobi–Lie algebroids. In the next section, we recall some basics on the Lie algebroids and the Schouten–Jacobi bracket. In section 3, we prove our main theorem and give a computational example of the theorem.

2 Lie algebroids and Jacobi–Lie algebroids

In this section, we review some basic ingredients of Lie algebroids for later use and introduce the notion of a Jacobi–Lie algebroid. All manifolds and functions are assumed to be smooth (C^{∞}) throughout the paper.

Definition 2.1 A vector bundle $\mathcal{L}$ over a manifold M is a *Lie algebroid* if

(a) the space of sections $\Gamma(\mathcal{L})$ is endowed with a Lie algebra bracket $[\cdot,\cdot]$ over $\mathbb{R}$,
(b) there is given a bundle map $a : \mathcal{L} \to \mathrm{T}(M)$ (called an *anchor*) which induces a Lie algebra homomorphism $a : \Gamma(\mathcal{L}) \to \Gamma(\mathrm{T}(M))$, satisfying the condition

$$[X, fY] = \langle a(X), df\rangle Y + f[X,Y], \qquad X, Y \in \Gamma(\mathcal{L}),\ f \in C^{\infty}(M).$$

Thus a Lie algebroid is a triple $(\mathcal{L}, [\cdot,\cdot], a)$, however we often call $\mathcal{L}$ a Lie algebroid when the bracket and the anchor are understood. The most popular and important example of a Lie algebroid is the tangent bundle with usual Lie bracket of vector fields. The cotangent bundle of a Poisson manifold is another example of a Lie algebroid. There are many other examples of Lie algebroids which are useful in geometry (see [1]).

Let $\mathcal{L}^*$ be the dual vector bundle of $\mathcal{L}$. We note that the anchor of $\mathcal{L}$ induces a dual morphism $a^* : \mathrm{T}^*(M) \longrightarrow \mathcal{L}^*$.

The Lie algebra bracket on $\Gamma(\mathcal{L})$ and the action of $a(X)$ on $C^{\infty}(M)$ induces an 'exterior differential' $d_{\mathcal{L}}$ on $\Gamma(\Lambda^{\bullet}\mathcal{L}^*)$ defined by a well-known formula;

$$\begin{aligned}(d_{\mathcal{L}}\omega)(X_0, X_1, \dots, X_r) := {} & \sum_{i=0}^{r} (-1)^i \langle d(\omega(\dots, \hat{X}_i, \dots)), a(X_i)\rangle \\ & + \sum_{i<j} (-1)^{i+j}\omega([X_i, X_j], \dots, \hat{X}_i, \dots, \hat{X}_j, \dots), \\ & \qquad \omega \in \Gamma(\Lambda^r \mathcal{L}^*),\ X_0, \dots, X_r \in \Gamma(\mathcal{L}),\end{aligned}$$

For example,

$$\langle d_{\mathcal{L}} f, X\rangle = \langle df, a(X)\rangle = \langle a^*(df), X\rangle, \quad f \in \Gamma(\Lambda^0\mathcal{L}^*) = C^\infty(M),\ X \in \Gamma(\mathcal{L}),$$
$$(d_{\mathcal{L}}\beta)(X, Y) = \langle d(\beta(Y)), a(X)\rangle - \langle d(\beta(X)), a(Y)\rangle - \langle \beta, [X, Y]\rangle$$
$$= L_{a(X)}(\beta(Y)) - L_{a(Y)}(\beta(X)) - \langle \beta, [X, Y]\rangle,$$
$$\beta \in \Gamma(\Lambda^1\mathcal{L}^*),\ X, Y \in \Gamma(\mathcal{L}).$$

With this differential $d_{\mathcal{L}}$, $\Gamma(\Lambda^\bullet\mathcal{L}^*)$ becomes a differential graded algebra and a^* induces a homomorphism of differential graded algebras $\Gamma(\Lambda^\bullet \mathrm{T}^*(M)) \to \Gamma(\Lambda^\bullet\mathcal{L}^*)$.

Conversely, the exterior differential $d_{\mathcal{L}}$ on $\Gamma(\Lambda^\bullet\mathcal{L}^*)$ recovers the anchor and the Lie algebra bracket on $\mathcal{L}$, hence recovers the Lie algebroid structure of $\mathcal{L}$ by the formulas

(a′) $\langle a(X), df\rangle := \langle X, d_{\mathcal{L}} f\rangle$,
(b′) $\langle [X, Y], \beta\rangle := \langle X, d_{\mathcal{L}}(\beta(Y))\rangle - \langle Y, d_{\mathcal{L}}(\beta(X))\rangle - (d_{\mathcal{L}}\beta)(X, Y), (\beta \in \Gamma(\mathcal{L}^*))$.

In [3], the authors introduced the *deformed exterior differential* and the *Schouten–Jacobi bracket* on $\Gamma(\Lambda^\bullet\mathcal{L})$ deformed by a 1-cocycle ϕ.

Definition 2.2 Let ϕ be a 1-cocycle in $\Gamma(\Lambda^\bullet\mathcal{L}^*)$ with respect to $d_{\mathcal{L}}$, i.e., $\phi \in \Gamma(\mathcal{L}^*)$ and ϕ satisfies

$$\phi([X, Y]) = L_{a(X)}(\phi(Y)) - L_{a(Y)}(\phi(X))$$

for $X, Y \in \Gamma(\mathcal{L})$. The deformed exterior differential is defined by

$$d^\phi_{\mathcal{L}}\alpha = d_{\mathcal{L}}\alpha + \phi \wedge \alpha, \quad \alpha \in \Gamma(\Lambda^\bullet\mathcal{L}^*). \tag{2.1}$$

The operator $d^\phi_{\mathcal{L}}$ satisfies

$$d^\phi_{\mathcal{L}} \circ d^\phi_{\mathcal{L}} = 0, \qquad d^\phi_{\mathcal{L}}(\alpha \wedge \beta) = d^\phi_{\mathcal{L}}\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d^\phi_{\mathcal{L}}\beta - \phi \wedge \alpha \wedge \beta,$$

where $|\alpha|$ means the degree of α, namely, $\alpha \in \Gamma(\Lambda^{|\alpha|}\mathcal{L}^*)$. On the other hand, ($\phi$-deformed) Schouten–Jacobi bracket $[\cdot, \cdot]^\phi$ is defined by

$$[P, Q]^\phi = [P, Q] + (-1)^p P(\phi) \wedge (q-1)Q + (p-1)P \wedge Q(\phi), \tag{2.2}$$
$$P \in \Gamma(\Lambda^p\mathcal{L}), Q \in \Gamma(\Lambda^q\mathcal{L}).$$

Here and hereafter, $P(\phi)$ denotes the interior product $\iota_\phi P$ or $\phi \rfloor P$ of ϕ and P. We use these notations interchangeably.

This bracket on $\Gamma(\Lambda^\bullet\mathcal{L})$ shares similar properties with the usual Schouten–Nijenhuis bracket. In our sign convention, formulas of calculation for $[\cdot, \cdot]^\phi$ are the following:

(1) $[X, Y]^\phi = [X, Y]$ (Lie algebra bracket), for $X, Y \in \Gamma(\mathcal{L})$,
(2) $[P, Q]^\phi = -(-1)^{(p-1)(q-1)}[Q, P]^\phi$,
(3) $[P, [Q, R]^\phi]^\phi = [[P, Q]^\phi, R]^\phi + (-1)^{(p-1)(q-1)}[Q, [P, R]^\phi]^\phi$ (super Jacobi identity),

(4) $[P, Q \wedge R]^\phi = [P, Q]^\phi \wedge R + (-1)^{(p-1)q} Q \wedge [P, R]^\phi + (-1)^p P(\phi) \wedge Q \wedge R$,
(5) $[f, P]^\phi = -P(d^\phi_{\mathcal{L}} f),\ f \in C^\infty(M)$,

where $P \in \Gamma(\Lambda^p \mathcal{L}),\ Q \in \Gamma(\Lambda^q \mathcal{L}),\ R \in \Gamma(\Lambda^r \mathcal{L})$.

For $\phi = 0$, these are just the formulas for the Nijenhuis–Schouten bracket. The only difference is that the deformed one does not satisfy the Leibniz property for the wedge product (see (4) above). Since $d^\phi_{\mathcal{L}} f$ in (5) above is defined by $d_{\mathcal{L}} f + f\phi = a^*(df) + f\phi$ in $\Gamma(\mathcal{L}^*)$, the action of $X \in \Gamma(\mathcal{L})$ on $C^\infty(M)$ through $[\cdot,\cdot]^\phi$ is given by $X \cdot f := [X, f]^\phi = \langle a(X), df\rangle + f\langle X, \phi\rangle$ where $\langle a(X), df\rangle$ is the usual action of Lie algebroid through the anchor map. Putting $f = 1$, we see that the 1-cocycle ϕ is recovered from the bracket since $\phi(X) = [X, 1]^\phi = X \cdot 1$ holds.

The difference of the action of X on $C^\infty(M)$ from the usual derivation leads to the different 'exterior differential' and 'Lie derivation'. The ϕ-Lie derivative operator L^ϕ for 'forms' and 'vectors' are defined by

$$L^\phi_X \alpha = (d^\phi_{\mathcal{L}} i_X + i_X d^\phi_{\mathcal{L}})\alpha = L_X \alpha + \phi(X)\alpha, \tag{2.3}$$

$$L^\phi_X P = [X, P]^\phi = [X, P] - (p-1)\phi(X)P \tag{2.4}$$

respectively. Then we have the following list of formulas.

$$L^\phi_X(\alpha \wedge \beta) = L^\phi_X \alpha \wedge \beta + \alpha \wedge L^\phi_X \beta - \phi(X)\alpha \wedge \beta \tag{2.5}$$

$$L^\phi_X(P \wedge Q) = L^\phi_X P \wedge Q + P \wedge L^\phi_X Q - \phi(X) P \wedge Q \tag{2.6}$$

$$L^\phi_X(P(\alpha)) = (L^\phi_X P)(\alpha) + P(L^\phi_X \alpha) + (|\alpha| - 1)\phi(X)P(\alpha) \tag{2.7}$$

$$L^\phi_X(\alpha(P)) = \alpha(L^\phi_X P) + (L^\phi_X \alpha)(P) + (p-1)\phi(X)\alpha(P) \tag{2.8}$$

$$L^\phi_X[P, Q]^\phi = [L^\phi_X P, Q]^\phi + [P, L^\phi_X Q]^\phi \tag{2.9}$$

$$L^\phi_{fX} P = f L^\phi_X P - X \wedge P(d_{\mathcal{L}} f) \tag{2.10}$$

Note that (2.7) or (2.8) tells us that L^ϕ_X does not commute with the contraction in general, although L_X does.

Remark 2.1 Let ϕ be a usual closed 1-form on M. We can see a cue of defining the ϕ-deformed Schouten–Jacobi bracket $[\cdot,\cdot]^\phi$ in the following observation when $\mathcal{L} = \mathrm{T}(M)$. Let $\phi = df$ locally where f is a function on M. For a p-vector field P, we put $\hat{P} = e^{-(p-1)f} P$. Note that this assignment $P \mapsto \hat{P}$ is injective and it is the identity transformation on the space of vector fields. If we compute $[\hat{P}, \hat{Q}]$, we have $e^{-(p+q-2)f}[P, Q]^\phi$.

As we will see below, one of the advantages of introducing $[\cdot,\cdot]^\phi$ is that we can treat a Jacobi structure on M as if it were a Poisson structure on M with respect to $[\cdot,\cdot]^\phi$. It seems natural here to generalize the Lie algebroid slightly and to introduce the notion of *Jacobi–Lie algebroid*.

Let $\mathcal{T}^*M$ denote the bundle of 1-jets of functions on M. $\mathcal{T}^*M$ has a natural projection onto the bundle of 0-jets which is a trivial line bundle $\varepsilon \cong M \times \mathbb{R}$. The kernel of the projection is the cotangent bundle $\mathrm{T}^*(M)$ and $\mathcal{T}^*M \cong \mathrm{T}^*(M) \oplus \varepsilon$ by $j_x^1 f \mapsto (df_x, f(x))$. Let $\mathcal{T}M$ denote the dual bundle and call it the *extended tangent bundle* of M. The sections of $\mathcal{T}M$ form the set of differential operators on $C^\infty(M)$ of order ≤ 1. Geometrically, $\mathcal{T}M$ can be identified with the tangent bundle $\mathrm{T}(M \times \mathbb{R})$ restricted to $M \times \{0\}$ (or to any level $M \times \{t\}$). Then a section $\mathbf{X}$ of $\mathcal{T}M$ is expressed as

$$\mathbf{X} = X + \lambda \frac{\partial}{\partial \tau}$$

where X is a vector field on M lifted to $M \times \mathbb{R}$ and $\frac{\partial}{\partial \tau} = \left(\frac{\partial}{\partial t}\right)_0$ is the *tangent vector* of $\mathbb{R}$ at 0. From this viewpoint, we may write 1-jet $j^1 f$ as $df + f d\tau$, where $d\tau$ is the dual of $\frac{\partial}{\partial \tau}$.

$\mathbf{X}$ acts on $C^\infty(M)$ as a first-order differential operator by

$$\mathbf{X} \cdot f = \langle \mathbf{X}, j^1 f \rangle = L_X f + \lambda f.$$

The commutator bracket of $\mathbf{X} = X + \lambda \frac{\partial}{\partial \tau}$ and $\mathbf{Y} = Y + \mu \frac{\partial}{\partial \tau}$ in $\Gamma(\mathcal{T}M)$ as operators is

$$[\mathbf{X}, \mathbf{Y}] = \left[\!\!\left[X + \lambda \frac{\partial}{\partial \tau}, Y + \mu \frac{\partial}{\partial \tau} \right]\!\!\right] = [X, Y] + (\langle X, d\mu \rangle - \langle Y, d\lambda \rangle) \frac{\partial}{\partial \tau}.$$

With this bracket on $\Gamma(\mathcal{T}M)$ and the natural projection $pr_1 : \mathcal{T}M \to \mathrm{T}(M)$ as the anchor, $(\mathcal{T}M, [\cdot, \cdot], pr_1)$ is a Lie algebroid, and the action of $\mathbf{X}$ on $f \in C^\infty(M)$ here, is through the vector field X. The difference between the two actions of $\mathbf{X}$ is the multiplication by λ. The map $\phi_0 : \mathbf{X} \mapsto \lambda = \mathbf{X} \cdot 1$ can be considered as a 1-*cocycle* of the Lie algebroid $\mathcal{T}M$. Indeed

$$(d\phi_0)(\mathbf{X}, \mathbf{Y}) = L_X \mu - L_Y \lambda - \phi_0([\mathbf{X}, \mathbf{Y}]) = 0.$$

We call this cocycle ϕ_0 of $\mathcal{T}M$ the *canonical 1-cocycle*.

Let $(\mathcal{L}, [\cdot, \cdot], a)$ be a Lie algebroid and ϕ any Lie algebroid-1-cocycle of $\mathcal{L}$. Then we have a bundle map $\bar{a} : \mathcal{L} \to \mathcal{T}M$ defined by $\bar{a}(X) = a(X) + \phi(X) \frac{\partial}{\partial \tau} \in \mathrm{T}(M) \oplus \varepsilon = \mathcal{T}M$. Using this map, we formulate a Lie algebroid with specified 1-cocycle as follows.

Definition 2.3 A Jacobi–Lie algebroid over a manifold M is a triplet $(\mathcal{L}, [\cdot, \cdot], \bar{a})$ of a vector bundle $\mathcal{L}$, a Lie algebra structure $[\cdot, \cdot]$ on $\Gamma(\mathcal{L})$, and a bundle map $\bar{a}$ of $\mathcal{L}$ into $\mathcal{T}M$ (called also an anchor), satisfying

(1) $(\mathcal{L}, [\cdot, \cdot], \mathrm{pr}_1 \circ \bar{a})$ is a Lie algebroid over M,

and

(2) $\bar{a}$ induces a Lie algebra homomorphism from $\Gamma(\mathcal{L})$ into $\Gamma(\mathcal{T}M)$.

Note that $\phi = \bar{a}^*(\phi_0)$ is a 1-cocycle of $\mathcal{L}$. Conversely, if a Lie algebroid $(\mathcal{L}, [\cdot, \cdot], a)$ has a 1-cocycle ϕ, then the map $\bar{a} : X \mapsto \bar{a}(X) = a(X) + \phi(X) \frac{\partial}{\partial \tau}$ is verified to be an anchor of a Jacobi–Lie algebroid. Indeed, for $X, Y \in \Gamma(\mathcal{L})$, we have

$$\bar{a}([X, Y]) = a([X, Y]) + \phi([X, Y]) \frac{\partial}{\partial \tau}$$

$$= [a(X), a(Y)] + (\langle a(X), d(\phi(Y))\rangle - \langle a(Y), d(\phi(X))\rangle)\frac{\partial}{\partial\tau}$$
$$= \left[\!\!\left[a(X) + \phi(X)\frac{\partial}{\partial\tau}, a(Y) + \phi(Y)\frac{\partial}{\partial\tau} \right]\!\!\right] = [\bar{a}(X), \bar{a}(Y)],$$

and

$$[X, fY] = f[X, Y] + \langle a(X), df\rangle Y.$$

Since $\mathcal{T}M \cong \mathrm{T}(M) \oplus \varepsilon$, we have an isomorphism $\Lambda^p\mathcal{T}M \cong \Lambda^p\mathrm{T}(M) \oplus \Lambda^{p-1}\mathrm{T}(M)$. Thus an element $\mathbf{P} \in \Gamma(\Lambda^p(\mathcal{T}M)$ is expressed also as a pair (P, P') of a p-vector field and a $(p-1)$-vector field. The correspondence is given by $\mathbf{P} = P + \frac{\partial}{\partial\tau}\wedge P' \leftrightarrow (P, P')$. Similarly, an element $\boldsymbol{\alpha} = \alpha + d\tau \wedge \alpha' \in \Gamma(\Lambda^p\mathcal{T}^*M)$ is given as a pair (α, α') consisting of a p-form and a $(p-1)$-form. Especially, the canonical 1-cocycle ϕ_0 is a pair (0, 1) where 0 denotes the zero 1-form and 1 is a constant function. We sometimes adopt this notation.

Example 2.1 (Jacobi structure on M) Let $\boldsymbol{\pi} = (\pi, \xi)$ be an element in $\Gamma(\Lambda^2\mathcal{T}M)$. With the above notation, we have

$$[\boldsymbol{\pi}, \boldsymbol{\pi}]^{\phi_0} = [(\pi, \xi), (\pi, \xi)]^{\phi_0} = [(\pi, \xi), (\pi, \xi)] + 2(i_{\phi_0}(\pi, \xi)) \wedge (\pi, \xi)$$
$$= ([\pi, \pi], 2[\xi, \pi]) + (2\xi \wedge \pi, 0) = ([\pi, \pi] + 2\xi \wedge \pi, 2[\xi, \pi]).$$

Thus $[\boldsymbol{\pi}, \boldsymbol{\pi}]^{\phi_0} = 0$ is equivalent to (π, ξ) being a Jacobi structure. The differential $d^{\phi_0} f$ is (df, f) and 'Hamiltonian vector field' $\boldsymbol{\pi}(d^{\phi_0} f)$ of f is a pair $\big(\pi(df) + f\xi, -\langle\xi, (df)\rangle\big)$. The bracket of functions f and g is given by

$$\{f, g\} = \boldsymbol{\pi}(d^{\phi_0} f, d^{\phi_0} g) = L^{\phi_0}_{\boldsymbol{\pi}(d^{\phi_0} f)}\, g = L_{(\pi(df)+f\xi, -\langle\xi, df\rangle)}\, g + \phi_0(\boldsymbol{\pi}(d^{\phi_0} f))g$$
$$= \pi(df, dg) + f\langle\xi, dg\rangle - g\langle\xi, df\rangle\ .$$

In the case of a contact structure, $\pi^n \wedge \xi$ is nowhere zero and the map $f \mapsto \pi(df) + f\xi$ is injective from $C^\infty(M)$ into $\Gamma(\mathrm{T}(M))$ and this vector field is called a contact Hamiltonian vector field.

3 Deformed bracket on 1-forms

Let $\mathcal{L}$ be a Lie algebroid over a manifold M whose anchor is $a : \mathcal{L} \to \mathrm{T}(M)$. We fix a 1-cocycle ϕ and consider ϕ-deformed exterior differential $d^{\phi}_{\mathcal{L}}$ and ϕ-deformed Schouten bracket $[\cdot, \cdot]^\phi$. By an abuse of language, we call $P \in \Gamma(\Lambda^p\mathcal{L})$ a *p-vector field* and $\alpha \in \Gamma(\Lambda^p\mathcal{L}^*)$ a *p-form*. In this section, we prove our main theorem. Namely, we show that $\big([\pi, \pi]^\phi\big)^0$ has a Lie algebroid structure (Theorem 3.4). (P^0 denotes the space of annihilating elements of P in $\mathcal{L}^*$.)

First we prove

Lemma 3.1 *Let $P \in \Gamma(\Lambda^p \mathcal{L})$, $Q \in \Gamma(\Lambda^q \mathcal{L})$ be a p-vector field and a q-vector field, respectively. For a 1-form α, the following equality holds;*

$$[P, Q]^\phi(\alpha) = [P(\alpha), Q]^\phi + (-1)^{p-1}[P, Q(\alpha)]^\phi + (-1)^p (P \wedge Q)(d^\phi_{\mathcal{L}}\alpha) + (-1)^{p-1} P(d^\phi_{\mathcal{L}}\alpha) \wedge Q + (-1)^{p-1} P \wedge Q(d^\phi_{\mathcal{L}}\alpha) \tag{3.1}$$

where for $p \le 1$, we understand $P(d^\phi_{\mathcal{L}}\alpha) = 0$ and similarly for $q \le 1$, $Q(d^\phi_{\mathcal{L}}\alpha) = 0$.

This immediately shows the following.

Corollary 3.2 *For a 2-vector field π and a 1-form α, we have*

$$[\pi(\alpha), \pi]^\phi = -\frac{1}{2}(\pi \wedge \pi)(d^\phi_{\mathcal{L}}\alpha) + \frac{1}{2}[\pi, \pi]^\phi(\alpha) + \pi(d^\phi_{\mathcal{L}}\alpha)\pi.$$

Proof of Lemma 3.1. In the case $\phi = 0$, the proof is seen in [4]. For general ϕ, we recall the defining equation (2.1) $d^\phi_{\mathcal{L}}\alpha = d_{\mathcal{L}}\alpha + \phi \wedge \alpha$ of $d^\phi_{\mathcal{L}}$ and the equation (2.2) of $[\cdot, \cdot]^\phi$. Using these formulas, we can check that the terms containing ϕ are equal on both sides in (3.1). Consequently, the equality is valid for a general Schouten–Jacobi bracket. □

Given a 2-vector field $\pi \in \Gamma(\Lambda^2 \mathcal{L})$ and a 1-cocycle ϕ, we define a bracket on 1-forms as follows.

$$\{\alpha, \beta\}^\phi_\pi := L^\phi_{\pi(\alpha)}\beta - L^\phi_{\pi(\beta)}\alpha - d^\phi_{\mathcal{L}}(\pi(\alpha, \beta)), \qquad \alpha, \beta \in \Gamma(\mathcal{L}^*). \tag{3.2}$$

Since $d^\phi_{\mathcal{L}}(\pi(\alpha, \beta)) = L^\phi_{\pi(\alpha)}\beta - i_{\pi(\alpha)} d^\phi_{\mathcal{L}}\beta$, we have another expression

$$\{\alpha, \beta\}^\phi_\pi = i_{\pi(\alpha)} d^\phi_{\mathcal{L}}\beta - L^\phi_{\pi(\beta)}\alpha. \tag{3.3}$$

This bracket is not a Lie algebra bracket in general. The following formula is useful in our computations.

Lemma 3.3 *For a 2-vector field π, the following equality holds:*

$$[\pi(\alpha), \pi(\beta)]^\phi = \pi(\{\alpha, \beta\}^\phi_\pi) + \frac{1}{2}[\pi, \pi]^\phi(\alpha, \beta). \tag{3.4}$$

Proof. When $\mathcal{L} = \mathrm{T}(M)$ and $\phi = 0$, the above equation is already known in [4]. Since $\{\alpha, \beta\}^\phi_\pi = i_{\pi(\alpha)} d^\phi_{\mathcal{L}}\beta - L^\phi_{\pi(\beta)}\alpha$, we have

$$\begin{aligned}\pi(\{\alpha, \beta\}^\phi_\pi) &= \pi(i_{\pi(\alpha)} d^\phi_{\mathcal{L}}\beta) - \pi(L^\phi_{\pi(\beta)}\alpha) \\ &= \pi(i_{\pi(\alpha)} d^\phi_{\mathcal{L}}\beta) + [\pi(\alpha), \pi(\beta)]^\phi + [\pi(\beta), \pi]^\phi(\alpha).\end{aligned} \tag{3.5}$$

Here, we used a general formula

$$L^{\phi}_{X}(P(\alpha)) = (L^{\phi}_{X}P)(\alpha) + P(L^{\phi}_{X}\,\alpha) + (|\alpha| - 1)\phi(X)P(\alpha)$$

for $X = \pi(\beta)$ and $P = \pi$. By Corollary 3.2, (3.5) is followed by

$$\pi(i_{\pi(\alpha)}d^{\phi}_{\mathcal{L}}\beta) + [\pi(\alpha), \pi(\beta)]^{\phi} - \frac{1}{2}[\pi, \pi]^{\phi}(\alpha, \beta) - (\pi(\alpha) \wedge \pi)(d^{\phi}_{\mathcal{L}}\beta) + \pi(d^{\phi}_{\mathcal{L}}\beta)\pi(\alpha)$$
$$= [\pi(\alpha), \pi(\beta)]^{\phi} - \frac{1}{2}[\pi, \pi]^{\phi}(\alpha, \beta).$$

Here we used the identity

$$\pi(i_{\pi(\alpha)}d^{\phi}_{\mathcal{L}}\beta) - (\pi(\alpha) \wedge \pi)(d^{\phi}_{\mathcal{L}}\beta) + \pi(d^{\phi}_{\mathcal{L}}\beta)\pi(\alpha) = 0$$

which can be verified by putting $d^{\phi}_{\mathcal{L}}\beta = \theta_1 \wedge \theta_2$ if necessary, where $\theta_1, \theta_2 \in \Gamma(\mathcal{L})$. □

Remark 3.1 Since $[X, Y]^{\phi} = [X, Y]$ for each 1-vector field, the lemma above means, for a 2-vector field π, the following equality holds:

$$[\pi(\alpha), \pi(\beta)] = \pi(\{\alpha, \beta\}^{\phi}_{\pi}) + \frac{1}{2}[\pi, \pi]^{\phi}(\alpha, \beta). \tag{3.6}$$

Theorem 3.4 Let $(\mathcal{L}, [\cdot, \cdot], a)$ be a Lie algebroid over a manifold M and ϕ be a 1-cocycle. That is, $\mathcal{L}$ has a Jacobi–Lie algebroid structure with anchor $\bar{a} : \mathcal{L} \to \mathcal{T}M, X \mapsto a(X) + \phi(X)\frac{\partial}{\partial\tau}$. Let π be an arbitrary 2-field of $\mathcal{L}$, that is $\pi \in \Gamma(\Lambda^2\mathcal{L})$. Suppose that the rank of $[\pi, \pi]^{\phi}$ is constant. Then the sub-bundle $([\pi, \pi]^{\phi})^0$ is a Jacobi–Lie algebroid with respect to the bracket

$$\{\alpha, \beta\}^{\phi}_{\pi} = L^{\phi}_{\tilde{\pi}(\alpha)}\beta - L^{\phi}_{\tilde{\pi}(\beta)}\alpha - d^{\phi}_{\mathcal{L}}(\mathcal{L}(\alpha, \beta))$$

and the anchor is given by the composition of $\bar{a}$ and $\tilde{\pi}$ restricted to $([\pi, \pi]^{\phi})^0$.

Corollary 3.5 $\mathcal{H} = a \circ \tilde{\pi}(([\pi, \pi]^{\phi})^0)$ is an integrable distribution.

Proof of Theorem 3.4. First we show the space of sections of $([\pi, \pi]^{\phi})^0$ is closed under the bracket $\{\ ,\ \}^{\phi}_{\pi}$. Let 1-forms α and β be sections of $([\pi, \pi]^{\phi})^0$ so that $\alpha \rfloor [\pi, \pi]^{\phi} = \beta \rfloor [\pi, \pi]^{\phi} = 0$. In order to prove $\{\alpha, \beta\}^{\phi}_{\pi} \rfloor [\pi, \pi]^{\phi} = 0$, we use Corollary 3.2 again. It says

$$\frac{1}{2}\{\alpha, \beta\}^{\phi}_{\pi} \rfloor [\pi, \pi]^{\phi} = [\tilde{\pi}(\{\alpha, \beta\}^{\phi}_{\pi}), \mathcal{L}]^{\phi} + \frac{1}{2}(d^{\phi}_{\mathcal{L}}\{\alpha, \beta\}^{\phi}_{\pi}) \rfloor (\mathcal{L} \wedge \mathcal{L}) - \mathcal{L}(d^{\phi}_{\mathcal{L}}\{\alpha, \beta\}^{\phi}_{\pi})\mathcal{L}$$

in general. By the same formula, α and β satisfy

$$[\tilde{\pi}(\alpha), \mathcal{L}]^{\phi} + \frac{1}{2}(d^{\phi}_{\mathcal{L}}\alpha) \rfloor (\mathcal{L} \wedge \mathcal{L}) - \mathcal{L}(d^{\phi}_{\mathcal{L}}\alpha)\mathcal{L} = 0$$

and

$$[\tilde{\pi}(\beta), \mathcal{L}]^{\phi} + \frac{1}{2}(d^{\phi}_{\mathcal{L}}\beta) \rfloor (\mathcal{L} \wedge \mathcal{L}) - \mathcal{L}(d^{\phi}_{\mathcal{L}}\beta)\mathcal{L} = 0\ .$$

Since $\tilde{\pi}(\{\alpha, \beta\}^{\phi}_{\pi}) = [\tilde{\pi}(\alpha), \tilde{\pi}(\beta)]^{\phi}$ and $\{\alpha, \beta\}^{\phi}_{\pi} = L^{\phi}_{\tilde{\pi}(\alpha)}\beta - L^{\phi}_{\tilde{\pi}(\beta)}\alpha - d^{\phi}_{\mathcal{L}}\pi(\alpha, \beta)$, we have

$$
\begin{aligned}
&\frac{1}{2}\{\alpha, \beta\}^{\phi}_{\pi} \rfloor [\pi, \pi]^{\phi} \\
&\quad = [[\tilde{\pi}(\alpha), \tilde{\pi}(\beta)]^{\phi}, \pi]^{\phi} + \frac{1}{2}\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta - L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right) \rfloor \pi^2 \\
&\qquad - \pi\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta - L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right)\pi \\
&\quad = [\tilde{\pi}(\alpha), [\tilde{\pi}(\beta), \pi]^{\phi}]^{\phi} + [[\tilde{\pi}(\alpha), \pi]^{\phi}, \tilde{\pi}(\beta)]^{\phi} \\
&\qquad + \frac{1}{2}\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta - L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right) \rfloor \pi^2 - \pi\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta - L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right)\pi \\
&\quad = L^{\phi}_{\tilde{\pi}(\alpha)}\left(-\frac{1}{2}(d^{\phi}_{\mathcal{L}}\beta) \rfloor \pi^2 + \pi(d^{\phi}_{\mathcal{L}}\beta)\pi\right) - L^{\phi}_{\tilde{\pi}(\beta)}\left(-\frac{1}{2}(d^{\phi}_{\mathcal{L}}\alpha) \rfloor \pi^2 + \pi(d^{\phi}_{\mathcal{L}}\alpha)\pi\right) \\
&\qquad + \frac{1}{2}\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta - L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right) \rfloor \pi^2 - \pi\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta - L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right)\pi \\
&\quad = -\frac{1}{2}\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta\right) \rfloor \pi^2 - \frac{1}{2} d^{\phi}_{\mathcal{L}}\beta \rfloor L^{\phi}_{\tilde{\pi}(\alpha)}\pi^2 - \frac{1}{2}\phi(\tilde{\pi}(\alpha)) d^{\phi}_{\mathcal{L}}\beta \rfloor \pi^2 \\
&\qquad + (L^{\phi}_{\tilde{\pi}(\alpha)}\pi)(d^{\phi}_{\mathcal{L}}\beta)\pi + \pi(L^{\phi}_{\tilde{\pi}(\alpha)}(d^{\phi}_{\mathcal{L}}\beta))\pi + \pi(d^{\phi}_{\mathcal{L}}\beta) L^{\phi}_{\tilde{\pi}(\alpha)}\pi \\
&\qquad + \frac{1}{2}\left(L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right) \rfloor \pi^2 + \frac{1}{2} d^{\phi}_{\mathcal{L}}\alpha \rfloor L^{\phi}_{\tilde{\pi}(\beta)}\pi^2 + \frac{1}{2}\phi(\tilde{\pi}(\beta)) d^{\phi}_{\mathcal{L}}\alpha \rfloor \pi^2 \\
&\qquad - (L^{\phi}_{\tilde{\pi}(\beta)}\pi)(d^{\phi}_{\mathcal{L}}\alpha)\pi - \pi(L^{\phi}_{\tilde{\pi}(\beta)}(d^{\phi}_{\mathcal{L}}\alpha))\pi - \pi(d^{\phi}_{\mathcal{L}}\alpha) L^{\phi}_{\tilde{\pi}(\beta)}\pi \\
&\qquad + \frac{1}{2}\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta - L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right) \rfloor \pi^2 - \pi\left(L^{\phi}_{\tilde{\pi}(\alpha)} d^{\phi}_{\mathcal{L}}\beta - L^{\phi}_{\tilde{\pi}(\beta)} d^{\phi}_{\mathcal{L}}\alpha\right)\pi \\
&\quad = -\frac{1}{2} d^{\phi}_{\mathcal{L}}\beta \rfloor L^{\phi}_{\tilde{\pi}(\alpha)}\pi^2 \qquad + (L^{\phi}_{\tilde{\pi}(\alpha)}\pi)(d^{\phi}_{\mathcal{L}}\beta)\pi + \pi(d^{\phi}_{\mathcal{L}}\beta) L^{\phi}_{\tilde{\pi}(\alpha)}\pi \\
&\qquad + \frac{1}{2} d^{\phi}_{\mathcal{L}}\alpha \rfloor L^{\phi}_{\tilde{\pi}(\beta)}\pi^2 - (L^{\phi}_{\tilde{\pi}(\beta)}\pi)(d^{\phi}_{\mathcal{L}}\alpha)\pi - \pi(d^{\phi}_{\mathcal{L}}\alpha) L^{\phi}_{\tilde{\pi}(\beta)}\pi \\
&\qquad - \frac{1}{2}\phi(\tilde{\pi}(\alpha)) d^{\phi}_{\mathcal{L}}\beta \rfloor \pi^2 + \frac{1}{2}\phi(\tilde{\pi}(\beta)) d^{\phi}_{\mathcal{L}}\alpha \rfloor \pi^2 .
\end{aligned}
$$

The sum of the 2nd and 5th terms of the right-hand sides of the equations above is zero as we see from the assumption

$$
\begin{aligned}
&(L^{\phi}_{\tilde{\pi}(\alpha)}\pi)(d^{\phi}_{\mathcal{L}}\beta)\pi - (L^{\phi}_{\tilde{\pi}(\beta)}\pi)(d^{\phi}_{\mathcal{L}}\alpha)\pi \\
&\quad = \left(-\frac{1}{2} d^{\phi}_{\mathcal{L}}\beta \rfloor d^{\phi}_{\mathcal{L}}\alpha \rfloor \pi^2 + \pi(d^{\phi}_{\mathcal{L}}\alpha)\pi(d^{\phi}_{\mathcal{L}}\beta)\right)\pi \\
&\qquad - \left(-\frac{1}{2} d^{\phi}_{\mathcal{L}}\alpha \rfloor d^{\phi}_{\mathcal{L}}\beta \rfloor \pi^2 + \pi(d^{\phi}_{\mathcal{L}}\beta)\pi(d^{\phi}_{\mathcal{L}}\alpha)\right)\pi \\
&\quad = 0
\end{aligned}
$$

and also from the assumption the sum of the 3rd and 6th terms becomes

$$
-\frac{1}{2}\pi(d^{\phi}_{\mathcal{L}}\beta) d^{\phi}_{\mathcal{L}}\alpha \rfloor \pi^2 + \frac{1}{2}\pi(d^{\phi}_{\mathcal{L}}\alpha) d^{\phi}_{\mathcal{L}}\beta \rfloor \pi^2.
$$

Thus,

$$
\begin{aligned}
&\{\alpha,\beta\}^{\phi}_{\pi}\rfloor[\pi,\pi]^{\phi}\\
&\quad = -d^{\phi}_{\mathcal{L}}\beta\rfloor(2[\tilde{\pi}(\alpha),\pi]^{\phi}\wedge\pi-\phi(\tilde{\pi}(\alpha))\pi^2)\\
&\qquad + d^{\phi}_{\mathcal{L}}\alpha\rfloor(2[\tilde{\pi}(\beta),\pi]^{\phi}\wedge\pi-\phi(\tilde{\pi}(\beta))\pi^2)\\
&\qquad -\pi(d^{\phi}_{\mathcal{L}}\beta)d^{\phi}_{\mathcal{L}}\alpha\rfloor\pi^2+\pi(d^{\phi}_{\mathcal{L}}\alpha)d^{\phi}_{\mathcal{L}}\beta\rfloor\pi^2\\
&\qquad -\phi(\tilde{\pi}(\alpha))d^{\phi}_{\mathcal{L}}\beta\rfloor\pi^2+\phi(\tilde{\pi}(\beta))d^{\phi}_{\mathcal{L}}\alpha\rfloor\pi^2\\
&\quad = d^{\phi}_{\mathcal{L}}\beta\rfloor\left((d^{\phi}_{\mathcal{L}}\alpha\rfloor\pi^2)\wedge\pi-2\pi(d^{\phi}_{\mathcal{L}}\alpha)\pi^2\right)\\
&\qquad - d^{\phi}_{\mathcal{L}}\alpha\rfloor\left((d^{\phi}_{\mathcal{L}}\beta\rfloor\pi^2)\wedge\pi-2\pi(d^{\phi}_{\mathcal{L}}\beta)\pi^2\right)\\
&\qquad -\pi(d^{\phi}_{\mathcal{L}}\beta)d^{\phi}_{\mathcal{L}}\alpha\rfloor\pi^2+\pi(d^{\phi}_{\mathcal{L}}\alpha)d^{\phi}_{\mathcal{L}}\beta\rfloor\pi^2\\
&\quad = d^{\phi}_{\mathcal{L}}\beta\rfloor\left((d^{\phi}_{\mathcal{L}}\alpha\rfloor\pi^2)\wedge\pi\right)-\pi(d^{\phi}_{\mathcal{L}}\alpha)(d^{\phi}_{\mathcal{L}}\beta\rfloor\pi^2)\\
&\qquad - d^{\phi}_{\mathcal{L}}\alpha\rfloor\left((d^{\phi}_{\mathcal{L}}\beta\rfloor\pi^2)\wedge\pi\right)\\
&\qquad +\pi(d^{\phi}_{\mathcal{L}}\beta)(d^{\phi}_{\mathcal{L}}\alpha\rfloor\pi^2).
\end{aligned}
$$

We claim that the above is identically zero. To prove this, it suffices to verify the claim in the case when $d^{\phi}_{\mathcal{L}}\alpha=\theta_1\wedge\theta_2$ and $d^{\phi}_{\mathcal{L}}\beta=\eta_1\wedge\eta_2$. By a direct and lengthy computation, we can verify that the above actually vanishes.

Proof of the Jacobi identity. Let $\alpha,\beta,\gamma\in([\pi,\pi]^{\phi})^0$. Using the definition of the bracket, we see that

$$
\begin{aligned}
\{\alpha,\{\beta,\gamma\}^{\phi}_{\pi}\}^{\phi}_{\pi} &= \tilde{\pi}(\alpha)\rfloor d^{\phi}_{\mathcal{L}}\{\beta,\gamma\}^{\phi}_{\pi}-L^{\phi}_{\tilde{\pi}(\{\beta,\gamma\}^{\phi}_{\pi})}\alpha\\
&= L^{\phi}_{\tilde{\pi}(\alpha)}\{\beta,\gamma\}^{\phi}_{\pi}-d^{\phi}_{\mathcal{L}}(\tilde{\pi}(\alpha)\rfloor\{\beta,\gamma\}^{\phi}_{\pi})-L^{\phi}_{\tilde{\pi}(\{\beta,\gamma\}^{\phi}_{\pi})}\alpha
\end{aligned}
$$

using Lemma 3.3

$$
\begin{aligned}
&= L^{\phi}_{\tilde{\pi}(\alpha)}\left(\tilde{\pi}(\beta)\rfloor d^{\phi}_{\mathcal{L}}\gamma-L^{\phi}_{\tilde{\pi}(\gamma)}\beta\right)\\
&\quad - d^{\phi}_{\mathcal{L}}\left(\tilde{\pi}(\alpha)\rfloor(\tilde{\pi}(\beta)\rfloor d^{\phi}_{\mathcal{L}}\gamma-L^{\phi}_{\tilde{\pi}(\beta)}\gamma)\right)-L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha\\
&= L^{\phi}_{\tilde{\pi}(\alpha)}\left(L^{\phi}_{\tilde{\pi}(\beta)}\gamma-d^{\phi}_{\mathcal{L}}(\tilde{\pi}(\beta)\rfloor\gamma)\right)-L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\gamma)}\beta\\
&\quad - d^{\phi}_{\mathcal{L}}\left(\tilde{\pi}(\alpha)\rfloor\tilde{\pi}(\beta)\rfloor d^{\phi}_{\mathcal{L}}\gamma-\tilde{\pi}(\alpha)\rfloor L^{\phi}_{\tilde{\pi}(\beta)}\gamma\right)-L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha\\
&= L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\beta)}\gamma-L^{\phi}_{\tilde{\pi}(\alpha)}d^{\phi}_{\mathcal{L}}(\pi(\beta,\gamma))-L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\gamma)}\beta\\
&\quad - d^{\phi}_{\mathcal{L}}\left(\tilde{\pi}(\alpha)\rfloor\tilde{\pi}(\beta)\rfloor d^{\phi}_{\mathcal{L}}\gamma-\tilde{\pi}(\alpha)\rfloor L^{\phi}_{\tilde{\pi}(\beta)}\gamma\right)-L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha\\
&= L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\beta)}\gamma-L^{\phi}_{\tilde{\pi}(\alpha)}L^{\phi}_{\tilde{\pi}(\gamma)}\beta-L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}}\alpha\\
&\quad - d^{\phi}_{\mathcal{L}}\left(L^{\phi}_{\tilde{\pi}(\alpha)}(\pi(\beta,\gamma))+\tilde{\pi}(\alpha)\rfloor\tilde{\pi}(\beta)\rfloor d^{\phi}_{\mathcal{L}}\gamma-\tilde{\pi}(\alpha)\rfloor L^{\phi}_{\tilde{\pi}(\beta)}\gamma\right)
\end{aligned}
$$

$$
\begin{aligned}
&= L^{\phi}_{\tilde{\pi}(\alpha)} L^{\phi}_{\tilde{\pi}(\beta)} \gamma - L^{\phi}_{\tilde{\pi}(\alpha)} L^{\phi}_{\tilde{\pi}(\gamma)} \beta - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}} \alpha \\
&\quad - d^{\phi}_{\mathcal{L}} \Big(\tilde{\pi}(\alpha) \rfloor d^{\phi}_{\mathcal{L}} (\pi(\beta,\gamma)) + \tilde{\pi}(\alpha) \rfloor \tilde{\pi}(\beta) \rfloor d^{\phi}_{\mathcal{L}} \gamma \\
&\qquad - \tilde{\pi}(\alpha) \rfloor \tilde{\pi}(\beta) \rfloor d^{\phi}_{\mathcal{L}} \gamma - \tilde{\pi}(\alpha) \rfloor d^{\phi}_{\mathcal{L}} (\tilde{\pi}(\beta) \rfloor \gamma) \Big) \\
&= L^{\phi}_{\tilde{\pi}(\alpha)} L^{\phi}_{\tilde{\pi}(\beta)} \gamma - L^{\phi}_{\tilde{\pi}(\alpha)} L^{\phi}_{\tilde{\pi}(\gamma)} \beta - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}} \alpha
\end{aligned}
$$

and so we have

$$
\underset{\alpha,\beta,\gamma}{\mathfrak{S}} \{\alpha, \{\beta,\gamma\}^{\phi}_{\pi}\}^{\phi}_{\pi} = \underset{\alpha,\beta,\gamma}{\mathfrak{S}} \Big((L^{\phi}_{\tilde{\pi}(\alpha)} L^{\phi}_{\tilde{\pi}(\beta)} \gamma - L^{\phi}_{\tilde{\pi}(\beta)} L^{\phi}_{\tilde{\pi}(\alpha)} \gamma) - L^{\phi}_{[\tilde{\pi}(\beta),\tilde{\pi}(\gamma)]^{\phi}} \alpha \Big) = 0
$$

using $L^{\phi}_X \circ L^{\phi}_Y - L^{\phi}_Y \circ L^{\phi}_X = L^{\phi}_{[X,Y]^{\phi}}$ on $\Gamma(\Lambda^{\bullet}\mathcal{L}^*)$ for each pair of vector fields X and Y, which is true by virtue of the closedness of ϕ.

The anchor for Lie algebroid Since

$$
\begin{aligned}
L^{\phi}_X(f\beta) &= (L^{\phi}_X f)\beta + f L^{\phi}_X \beta - \langle \phi, X \rangle f\beta, \\
L^{\phi}_{fX}\alpha &= f L^{\phi}_X \alpha + (X \rfloor \alpha) d_{\mathcal{L}} f,
\end{aligned}
$$

we have

$$
\begin{aligned}
\{\alpha, f\beta\}^{\phi}_{\pi} &= (L^{\phi}_{\tilde{\pi}(\alpha)} f)\beta + f L^{\phi}_{\tilde{\pi}(\alpha)} \beta - \langle \phi, \tilde{\pi}(\alpha) \rangle f\beta - (f L^{\phi}_{\tilde{\pi}(\beta)} \alpha + (\tilde{\pi}(\beta)) \rfloor \alpha) d_{\mathcal{L}} f \\
&\quad - (f d_{\mathcal{L}}(\pi(\alpha,\beta)) + \pi(\alpha,\beta) d_{\mathcal{L}} f - f\pi(\alpha,\beta)\phi) \\
&= f\{\alpha,\beta\}^{\phi}_{\pi} + \text{Rest},
\end{aligned}
$$

where

$$
\begin{aligned}
\text{Rest} &= (L^{\phi}_{\tilde{\pi}(\alpha)} f)\beta - \langle \phi, \tilde{\pi}(\alpha) \rangle f\beta - (\tilde{\pi}(\beta) \rfloor \alpha) d_{\mathcal{L}} f - (\pi(\alpha,\beta) d_{\mathcal{L}} f - f\pi(\alpha,\beta)\phi) \\
&= \langle \tilde{\pi}(\alpha), d_{\mathcal{L}} f \rangle \beta = \langle a(\tilde{\pi}(\alpha)), df \rangle \beta\ .
\end{aligned}
$$

Thus, we have

$$
\{\alpha, f\beta\}^{\phi}_{\pi} = f\{\alpha,\beta\}^{\phi}_{\pi} + \langle a(\tilde{\pi}(\alpha)), df \rangle \beta.
$$

This shows $a \circ \tilde{\pi}$ is the anchor for Lie algebroid $(([\pi,\pi]^{\phi})^0, \{\cdot,\cdot\}^{\phi}_{\pi})$.

Corresponding 1-cocycle We will verify $\phi \circ \tilde{\pi}$ is a 1-cocycle on $(([\pi,\pi]^{\phi})^0, \{\cdot,\cdot\}^{\phi}_{\pi}, a \circ \tilde{\pi})$. Put here $\phi \circ \tilde{\pi}$ by φ. We have to show

$$
\varphi(\{\alpha,\beta\}^{\phi}_{\pi}) = L_{b(\alpha)}(\varphi(\beta)) - L_{b(\beta)}(\varphi(\alpha)) \qquad \text{for each } \alpha, \beta \in ([\pi,\pi]^{\phi})^0\ .
$$

The right-hand side is reduced as

$$
\text{RHS} = L_{a(\tilde{\pi}(\alpha))}(\phi\tilde{\pi}(\beta)) - L_{a(\tilde{\pi}(\beta))}(\phi\tilde{\pi}(\alpha)) = \phi([\tilde{\pi}(\alpha), \tilde{\pi}(\beta)])
$$

because of ϕ being closed. Concerning the left-hand side, we have

$$\begin{aligned}\text{LHS} &= (\phi \circ \tilde{\pi})\{\alpha, \beta\}^{\phi}_{\pi} \\ &= \phi([\tilde{\pi}(\alpha), \tilde{\pi}(\beta)] - \frac{1}{2}[\pi, \pi]^{\phi}(\alpha, \beta)) \qquad \text{using (3.6)} \\ &= \phi([\tilde{\pi}(\alpha), \tilde{\pi}(\beta)]\end{aligned}$$

because of $\alpha, \beta \in ([\pi, \pi]^{\phi})^0$. Thus we have checked the equality of both sides, and $\bar{a} \circ \tilde{\pi}$ is the anchor for the Jacobi–Lie algebroid. □

Remark 3.2 In the proof above, we see that if ϕ is exact, then the corresponding 1-cocycle is also exact. In fact, assume $\phi = d_{\mathcal{L}} f$ for some f, i.e., $\langle \phi, X \rangle = \langle d_{\mathcal{L}} f, X \rangle = \langle df, a(X) \rightarrow$ for each $X \in \Gamma(\mathcal{L})$. Then, we have $\langle \varphi, \alpha \rangle = \langle \phi\tilde{\pi}, \alpha \rangle = \langle \phi, \tilde{\pi}(\alpha) \rangle = \langle df, a(\tilde{\pi}(\alpha)) \rangle$.

3.1 An example

We show an example on the 5-dimensional Euclidean space $\mathbb{R}^5$ with the Cartesian coordinates $(x^1, \ldots, x^5)$, which exhibits some difference between the ordinary bracket and the deformed one. Since the space is simply-connected, every closed 1-form is exact, and every closed 1-form ϕ is of form $\phi = df = \sum_{j=1}^{5} \frac{\partial f}{\partial x^j} dx^j = \sum_{j=1}^{5} f_j dx^j$ for some function f, where $f_j = \frac{\partial f}{\partial x^j}$.

Take the frame field $\{Z_1, \ldots, Z_5\}$ defined by

$$Z_1 = \frac{\partial}{\partial x^1} - \frac{x^2}{2}\frac{\partial}{\partial x^5}, \quad Z_2 = \frac{\partial}{\partial x^2} + \frac{x^1}{2}\frac{\partial}{\partial x^5},$$
$$Z_3 = \frac{\partial}{\partial x^3} - \frac{x^4}{2}\frac{\partial}{\partial x^5}, \quad Z_4 = \frac{\partial}{\partial x^4} + \frac{x^3}{2}\frac{\partial}{\partial x^5}, \quad Z_5 = \frac{\partial}{\partial x^5}.$$

Then, Z_5 is a central element and the bracket relations are given by

$$[Z_1, Z_2] = -[Z_2, Z_1] = [Z_3, Z_4] = -[Z_4, Z_3] = Z_5$$

and all the other brackets vanish.

Let us consider a 2-vector field π:

$$\begin{aligned}\pi = &a^{12} Z_1 \wedge Z_2 + a^{13} Z_1 \wedge Z_3 + a^{15} Z_1 \wedge Z_5 + a^{23} Z_2 \wedge Z_3 \\ &+ a^{25} Z_2 \wedge Z_5 + a^{35} Z_3 \wedge Z_5\end{aligned}$$

where $\{a^{ij}\}$ are constant. The rank of π is 4 if and only if $\Delta := a^{12}a^{35} - a^{13}a^{25} + a^{15}a^{23} \neq 0$. Hereafter, we assume that π is of rank 4. We have the following calculation:

$$\begin{aligned}[\pi, \pi] &= 2a^{12}\left(a^{12} Z_1 \wedge Z_2 + a^{13} Z_1 \wedge Z_3 + a^{23} Z_2 \wedge Z_3\right) \wedge Z_5 \\ &= 2a^{12}\left(a^{12}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + a^{13}\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^3} + a^{23}\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3}\right) \wedge \frac{\partial}{\partial x^5}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}[\pi,\pi]^{\phi} &= \frac{1}{2}[\pi,\pi] + \tilde{\pi}(\phi)\wedge\pi \\ &= -f_5\Delta\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3} + \left(a^{12}a^{12} + f_3\Delta\right)\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^5} \\ &\quad + \left(a^{12}a^{13} - f_2\Delta\right)\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^3}\wedge\frac{\partial}{\partial x^5} \\ &\quad + \left(a^{12}a^{23} + f_1\Delta\right)\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}\wedge\frac{\partial}{\partial x^5}.\end{aligned}$$

These equations above imply that $[\pi,\pi] = 0$ if and only if $a^{12} = 0$, and $[\pi,\pi]^{\phi} = [\pi,\pi]$ if and only if $\phi = df$ with $f_1 = f_2 = f_3 = f_5 = 0$ for some function f, and $[\pi,\pi]^{\phi} = 0$ if and only if

$$f_1 = -\frac{1}{\Delta}a^{12}a^{23}\,, \quad f_2 = \frac{1}{\Delta}a^{12}a^{13}\,, \quad f_3 = -\frac{1}{\Delta}a^{12}a^{12}\,, \quad f_5 = 0\,. \tag{3.7}$$

Now, we consider the following special cases.

Case 1 If $a^{12} = 0$ and $\Delta \neq 0$, then $[\pi,\pi] = 0$, and so $[\pi,\pi]^0$ is the whole cotangent bundle of $\mathbb{R}^5$ and $\dim\tilde{\pi}([\pi,\pi]^0) = 4$, $\tilde{\pi}([\pi,\pi]^0) = \mathrm{Im}\tilde{\pi}$. Choose $\phi = df$ with $f_1 \neq 0$ and $f_2 = f_3 = 0$. Then $([\pi,\pi]^{\phi})^0$ is spanned by ϕ and dx^4. $\tilde{\pi}(([\pi,\pi]^{\phi})^0)$ is spanned by

$$\begin{aligned}\tilde{\pi}(df) =& f_5\left(\frac{x^4}{2}a^{13} - a^{15}\right)\frac{\partial}{\partial x^1} + f_5\left(\frac{x^4}{2}a^{23} - a^{25}\right)\frac{\partial}{\partial x^2} \\ &+ \left(f_1 a^{13} + f_5\left(\frac{x^1}{2}a^{23} - \frac{x^2}{2}a^{13} - a^{35}\right)\right)\frac{\partial}{\partial x^3} - f_1\left(\frac{x^4}{2}a^{13} - a^{15}\right)\frac{\partial}{\partial x^5}\end{aligned}$$

and we see that this does never vanish from the assumption $\Delta \neq 0$. Thus, $\tilde{\pi}(([\pi,\pi]^{\phi})^0)$ is of dimension 1.

Case 2 Assume $a^{12} \neq 0$ and $\Delta \neq 0$. For example, choose $a^{12} = a^{35} = 1, a^{13} = a^{23} = 0$. Then $[\pi,\pi] = 2\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^5}$ and so $[\pi,\pi]^0$ is spanned by dx^3 and dx^4. Thus, $\tilde{\pi}([\pi,\pi]^0)$ is spanned by $\frac{\partial}{\partial x^5}$ and $\dim\tilde{\pi}([\pi,\pi]^0) = 1$. According to the condition (3.7), if we choose $f_1 = f_2 = 0$, $f_3 = 1$ and $f_5 = 0$, then $[\pi,\pi]^{\phi} = 0$ and so $([\pi,\pi]^{\phi})^0$ is the whole cotangent bundle and $\tilde{\pi}(([\pi,\pi]^{\phi})^0) = \mathrm{Im}\tilde{\pi}$ is of dimension 4, which is spanned by Z_1, Z_2, Z_3, Z_5.

If we choose $f_1 \neq -a^{12}a^{23}/\Delta$, (i.e., $f \neq 0$ right now), $f_2 = f_5 = 0$, and $f_3 = 1$, then $[\pi,\pi]^{\phi} = 2f_1\frac{\partial}{\partial x^2}\wedge\frac{\partial}{\partial x^3}\wedge\frac{\partial}{\partial x^5} \neq 0$. $([\pi,\pi]^{\phi})^0$ is spanned by dx^1 and dx^4. Since $\tilde{\pi}(dx^1) = \frac{\partial}{\partial x^2} + (a^{15} + \frac{x^1}{2})\frac{\partial}{\partial x^5}$ and $\tilde{\pi}(dx^4) = 0$, we see that $\tilde{\pi}(([\pi,\pi]^{\phi})^0)$ is 1-dimensional.

References

1. Cannas da Silva A. and Weinstein A. *Geometric models for noncommutative algebras*. University of California, Berkeley—AMS, Providence, 1999. Berkeley Mathematics Lecture Notes, 10.
2. Grabowski J. and Marmo G. Jacobi structures revisited. *J. Phys. A: Math. Gen.*, 34:10975–l0990, 2001.
3. Iglesias D. and Marrero J. C. Generalised Lie bialgebroids and Jacobi structure. *J. Geom. Phys.*, 40:176–199, 2001.
4. Mikami K. and Mizutani T. Integrability of plane fields defined by 2-vector fields. *International J. Math.*, 16(2):197–212, 2005.
5. Severa P. and Weinstein A. Poisson geometry with a 3-form background. *Progr. Theoret. Phys. Suppl.*, 144:145–154, 2001.

Parabolic Geometries Associated with Differential Equations of Finite Type

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Summary. We present here classes of parabolic geometries arising naturally from Se-ashi's principle to form good classes of linear differential equations of finite type, which generalize the cases of second and third order ODE for scalar functions. We will explicitly describe the symbols of these differential equations. The model equations of these classes admit nonlinear contact transformations and their symmetry algebras become finite dimensional and simple.

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1 Introduction

The geometry of ordinary differential equations for scalar functions is strongly linked to the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{sl}(\hat{V})$, where $\hat{V}$ is a vector space of dimension 2. Associated to the geometry of kth order ordinary differential equation

$$\frac{d^k y}{dx^k} = F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{k-1}y}{dx^{k-1}}\right),$$

we have the irreducible representation of $\hat{\mathfrak{l}} = \mathfrak{sl}(\hat{V})$ on $S = S^{k-1}(\hat{V}^*)$, where $S^{k-1}(\hat{V}^*)$ is the space of homogeneous polynomials of degree $k-1$ in two variables and is the solution space of the model equation $\frac{d^k y}{dx^k} = 0$ on the model space $\mathbb{P}^1(\mathbb{R}) = \mathbb{P}(\hat{V})$. It is known that the Lie algebra $\mathfrak{l} = \mathfrak{gl}(\hat{V})$ is the infinitesimal group of linear automorphisms of the model equation (cf. Proposition 4.4.1 [Sea88]). Moreover the Lie algebra $\mathfrak{g}^k = \mathfrak{g}^k(1, 1)$ of infinitesimal contact transformations of $\frac{d^k y}{dx^k} = 0$ is given as follows; (1) $\mathfrak{g}^2$ is isomorphic to $\mathfrak{sl}(3, \mathbb{R})$. (2) $\mathfrak{g}^3$ is isomorphic to $\mathfrak{sp}(2, \mathbb{R})$. (3) Otherwise, for $k \geqq 4$, $\mathfrak{g}^k = S \oplus \mathfrak{l}$ is a subalgebra of the affine Lie algebra

$\mathfrak{A}(S) = S \oplus \mathfrak{gl}(S)$ (see Section 2.1). The Lie algebra $\mathfrak{g}^k$ plays a fundamental role in the contact geometry of kth order ordinary differential equations.

Thus, when $k = 2$ and 3, special phenomena prevail and result in rich automorphism groups so that these two cases offer examples of parabolic geometries associated with differential equations. Here the *Parabolic Geometry* is a geometry modeled after the homogeneous space G/P, where G is a (semi-)simple Lie group and P is a parabolic subgroup of G (cf. [Bai93]). Precisely, in this paper, we mean, by a parabolic geometry, the geometry associated with the simple graded Lie algebra in the sense of N. Tanaka ([Tan79]). The main purpose of this paper is to seek other such special phenomena and to present other classes of parabolic geometries associated with differential equations of finite type, which naturally arise from Se-ashi's principle and generalize the above cases of $\mathfrak{g}^2$ and $\mathfrak{g}^3$.

For the geometry of differential equations of finite type, our study is based on the geometry of differential systems in the following way (cf. [YY02]): We regard a kth order differential equation as a submanifold R of the k-jet space $J^k(n, m)$ for n independent and m dependent variables. Defined on R, we have the differential system $\hat{D}$ obtained by restricting to R the canonical system C^k on $J^k(n, m)$ (see Section 2.1). Especially, when R is a kth order involutive differential equation of finite type, $p = \pi^k_{k-1} |_R \colon R \to J^{k-1}$ is an immersion so that we have a pseudo-product structure $D = E \oplus F$ on R, where D is the pullback $(p_*)^{-1}(C^{k-1})$ of C^{k-1} through p, $E = \hat{D}$ is the restriction of C^k to R and $F = \mathrm{Ker}\,(\pi^k_{k-2} |_R)_*$ is the fibre direction of $\pi^k_{k-2} |_R$.

Now, let us recall Se-ashi's procedure to form good classes of linear differential equations of finite type, following [Sea88] and [YY02]. Se-ashi's procedure starts from a reductive graded Lie algebra (GLA) $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ and a faithful irreducible $\mathfrak{l}$-module S. Then we form the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a finite dimensional *reductive GLA* of the first kind such that

(1) The ideal $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ of $\mathfrak{l}$ is a simple Lie algebra.
(2) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is contained in $\mathfrak{l}_0$.

Let S be a finite dimensional *faithful irreducible* $\mathfrak{l}$*-module*. We put

$$S_{-1} = \{s \in S \mid \mathfrak{l}_1 \cdot s = 0\}$$

and

$$S_p = \mathrm{ad}(\mathfrak{l}_{-1})^{-p-1} S_{-1} \quad \text{for } p < 0.$$

We form the semi-direct product $\mathfrak{g}$ of $\mathfrak{l}$ by S, and put

$$\mathfrak{g} = S \oplus \mathfrak{l}, \qquad [S, S] = 0,$$

$$\mathfrak{g}_k = \mathfrak{l}_k \ (k \geqq 0), \quad \mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1},$$

$$\mathfrak{g}_p = S_p \ (p < -1).$$

Then $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ enjoys the following properties (Lemma 2.1);

(1) $S = \bigoplus_{p=-1}^{-\mu} S_p$*, where* $S_{-\mu} = \{s \in S \mid [\mathfrak{l}_{-1}, s] = 0\}$.

(2) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is generated by $\mathfrak{g}_{-1}$.
(3) *S_p is naturally embedded as a subspace of $W \otimes S^{\mu+p}(\mathfrak{l}_{-1}{}^*)$ through the bracket operation in $\mathfrak{m}$, where $W = S_{-\mu}$.*

Thus $S = S_{-\mu} \oplus S_{-\mu+1} \oplus \cdots \oplus S_{-1} \subset W \oplus W \otimes V^* \oplus \cdots \oplus W \otimes S^{\mu-1}(V^*)$ defines a symbol of μth order differential equations of finite type by putting $S_0 = \{0\} \subset W \otimes S^{\mu}(V^*)$. We can construct the model linear equation R_0 of finite type, whose symbol at each point is isomorphic to S (see Section 4 [Sea88]). R_0 is a μth or der involutive differential equation of finite type. Then, we see that the symbol algebra of (R_0, D_0) is isomorphic to $\mathfrak{m}$, where D_0 is the pullback of the canonical system $C^{\mu-1}$ on the $(\mu-1)$-jet space $J^{\mu-1}$. $\mathfrak{m}$ has the splitting $\mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1}$, corresponding to the pseudo-product structure on R_0, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. In this way, $\mathfrak{m}$ is a symbol algebra of μth order differential equation of finite type, which is called the typical symbol of type $(\mathfrak{l}, S)$.

This class of higher order (linear) differential equations of finite type first appeared in the work of Y. Se-ashi [Sea88], who discussed the linear equivalence of this class of equations and gave the complete system of differential invariants of these equations, generalizing the classical theory of Laguerre–Forsyth for linear ordinary differential equations.

We ask the following question for the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$:

When is $\mathfrak{g}$ the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$?

Namely we ask whether $\mathfrak{g}$ exhausts all the infinitesimal automorphisms of the differential system (R_0, D_0) or its pseudo-product structure.

The answer to this question is given in Theorem 5.2 of [YY02] (Theorem 2.3 below), where we can find the classes of parabolic geometries, which generalize the cases of second and third order ordinary differential equations. More precisely, this theorem states : *For a pseudo-product GLA $\mathfrak{g} = \bigoplus_{p=-\mu}^{1} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{m}, \mathfrak{g})_{0,0} = 0$, $\mathfrak{g}$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for three cases.* Let $\check{\mathfrak{b}}$ be the prolongation of $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$, where $\mathfrak{b}_{-1} = S$ and $\mathfrak{b}_0 = \mathfrak{l}$. Then the three exceptional cases correspond to cases : (a) $\dim \check{\mathfrak{b}} < \infty$ and $\check{\mathfrak{b}}_1 \neq 0$, (b) $\dim \check{\mathfrak{b}} = \infty$, (c) $\mathfrak{g}$ is a pseudo-projective GLA (for the detail, see Section 2). In case (a), $\check{\mathfrak{b}} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \check{\mathfrak{b}}_1$ becomes a simple graded Lie algebra containing $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$ as a parabolic subalgebra. Thus, basically, the case (a) corresponds to the parabolic geometries that we seek. In fact, in the case of kth order ordinary differential equations for a scalar function, $\mathfrak{g}^2$ and $\mathfrak{g}^3$ belong to case (a) and $\mathfrak{g}^k$ belongs to case (c) for $k \geqq 4$.

In Section 2, we will recall the above results from [YY02]. The symbol algebras of these parabolic geometries will be given in Theorem 2.3 in terms of root space decompositions of the corresponding simple Lie algebras. We will describe these symbol algebras and the model differential equations of finite type explicitly by utilizing the explicit matrices description of the simple graded Lie algebra $\check{\mathfrak{b}}$ for the classical cases in Section 3 and by describing the structure of $\mathfrak{m}$ explicitly by use of the Chevalley basis of the exceptional simple Lie algebras in Section 4. Finally, in Section 5, we will discuss the equivalence of each parabolic geometry associated with the differential equations of finite type explicitly described in previous sections.

2 Pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$

In this section, we will summarize the results in [YY02] and explain the prolongation theorem (Theorem 2.1). We will first discuss the prolongation of symbol algebras of the pseudo-product structures associated with higher order differential equations of finite type. Moreover we will generalize this algebra to the notion of the pseudo-product GLA (graded Lie algebras) of irreducible type and introduce the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ and ask when $\mathfrak{g}$ is the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$, where $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$. In the answer to this question, we will find the classes of finite type differential equations mentioned in the introduction.

2.1 Pseudo-projective GLA of order k of bidegree (n, m)

We first consider a system of higher order differential equations of finite type of the following form :

$$\frac{\partial^k y^\alpha}{\partial x_{i_1}\cdots\partial x_{i_k}} = F^\alpha_{i_1\cdots i_k}(x_1,\ldots,x_n,y^1,\cdots,y^m,\ldots,p^\beta_i,\ldots,p^\beta_{j_1\cdots j_{k-1}})$$
$$(1 \leqq \alpha \leqq m,\ 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n),$$

where $p^\beta_{i_1\cdots i_\ell} = \frac{\partial^\ell y^\beta}{\partial x_{i_1}\cdots\partial x_{i_\ell}}$. These equations define a submanifold R in k-jets space J^k such that the restriction

p to R of the bundle projection $\pi^k_{k-1} : J^k \to J^{k-1}$ gives a diffeomorphism ;

$$p : R \to J^{k-1};\ \text{diffeomorphism.} \tag{2.1}$$

On J^k, we have the contact (differential) system C^k defined by

$$C^k = \{\varpi^\alpha = \varpi^\alpha_i = \cdots = \varpi^\alpha_{i_1\cdots i_{k-1}} = 0\},$$

where

$$\begin{cases} \varpi^\alpha = d\,y^\alpha - \sum_{i=1}^n p^\alpha_i d\,x_i, & (1 \leqq \alpha \leqq m) \\ \varpi^\alpha_i = d\,p^\alpha_i - \sum_{j=1}^n p^\alpha_{ij} d\,x_j, & (1 \leqq \alpha \leqq m,\ 1 \leqq i \leqq n) \\ \cdots\cdots\cdots\cdots, & \\ \varpi^\alpha_{i_1\cdots i_{k-1}} = d\,p^\alpha_{i_1\cdots i_{k-1}} - \displaystyle\sum_{j=1}^n p^\alpha_{i_1\cdots i_{k-1}j} d\,x_j & \\ \qquad (1 \leqq \alpha \leqq m,\ 1 \leqq i_1 \leqq \cdots \leqq i_{k-1} \leqq n). & \end{cases} \tag{2.2}$$

Then C^k gives a foliation on R when R is integrable. Namely the restriction E of C^k to R is completely integrable.

Thus, through the diffeomorphism (2.1), R defines a completely integrable differential system $E' = p_*(E)$ on J^{k-1} such that

$$C^{k-1} = E' \oplus F', \quad F' = \mathrm{Ker}\,(\pi^{k-1}_{k-2})_*$$

where $\pi^{k-1}_{k-2} : J^{k-1} \to J^{k-2}$ is the bundle projection. The triplet $(J^{k-1}; E', F')$ is called the *pseudo-product structure* associated with R.

Corresponding to the splitting $D = E \oplus F = (p^{-1})_*(C^{k-1})$, we have the splitting in the symbol algebra of the regular differential system $(R, D) \cong (J^{k-1}, C^{k-1})$ of type $\mathfrak{C}^{k-1}(n, m)$;

$$\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f},$$

where $\mathfrak{e} = V$, $\mathfrak{f} = W \otimes S^{k-1}(V^*)$. At each point $x \in R$, $\mathfrak{e}$ corresponds to $E(x)$ (the point in $R^{(1)}$ over x) and $\mathfrak{f}$ corresponds to $\mathrm{Ker}\,(\pi^{k-1}_{k-2})_*(p(x))$. Here we recall (see Section 1.3[YY02] for detail) that the fundamental graded Lie algebra (FGLA) $\mathfrak{C}^{k-1}(n, m)$ is defined by

$$\mathfrak{C}^{k-1}(n, m) = \mathfrak{C}_{-k} \oplus \cdots \oplus \mathfrak{C}_{-2} \oplus \mathfrak{C}_{-1},$$

where $\mathfrak{C}_{-k} = W$, $\mathfrak{C}_p = W \otimes S^{k+p}(V^*)$, $\mathfrak{C}_{-1} = V \oplus W \otimes S^{k-1}(V^*)$. Here V and W are vector spaces of dimension n and m respectively and the bracket product of $\mathfrak{C}^{k-1}(n, m) = \mathfrak{C}^{k-1}(V, W)$ is defined accordingly through the pairing between V and V^* such that V and $W \otimes S^{k-1}(V^*)$ are both abelian subspaces of $\mathfrak{C}_{-1}$. Here $S^r(V^*)$ denotes the rth symmetric product of V^*.

Now we put

$$\check{\mathfrak{g}}_0 = \{X \in \mathfrak{g}_0(\mathfrak{C}^{k-1}(n, m)) \mid [X, \mathfrak{e}] \subset \mathfrak{e}, [X, \mathfrak{f}] \subset \mathfrak{f}\}$$

and consider the (algebraic) prolongation $\mathfrak{g}^k(n, m)$ of $(\mathfrak{C}^{k-1}(n, m), \check{\mathfrak{g}}_0)$, which is called the *pseudo-projective GLA of order k of bidegree* (n, m) ([Tan89]). Here $\mathfrak{g}_0(\mathfrak{C}^{k-1}(n, m))$ denotes the Lie algebra of gradation preserving derivations of $\mathfrak{C}^{k-1}(n, m)$.

Let $\check{G}_0 \subset GL(\mathfrak{C}^{k-1}(n, m))$ be the (gradation preserving) automorphism group of $\mathfrak{C}^{k-1}(n, m)$ which also preserve the splitting $\mathfrak{C}_{-1} = \mathfrak{e} \oplus \mathfrak{f}$. Then $\check{G}_0$ is the Lie subgroup of $GL(\mathfrak{C}^{k-1}(n, m))$ with Lie algebra $\check{\mathfrak{g}}_0$. The pseudo-product structure on a kth order differential equation R of finite type given above, which is called the *pseudo-projective system of order k of bidegree* (n, m) in [Tan89], can be formulated as the $\check{G}^{\sharp}_0$-structure over a regular differential system of type $\mathfrak{C}^{k-1}(n, m)$ ([Tan70], [Tan89], [DKM99]). Thus the prolongation $\mathfrak{g}^k(n, m)$ of $(\mathfrak{C}^{k-1}(n, m), \check{\mathfrak{g}}_0)$ represents the Lie algebra of infinitesimal automorphisms of the (local) model kth order differential equation R_0 of finite type, where

$$R_0 = \left\{ \frac{\partial^k y^\alpha}{\partial x_{i_1} \cdots \partial x_{i_k}} = 0 \quad (1 \leqq \alpha \leqq m, 1 \leqq i_1 \leqq \cdots \leqq i_k \leqq n) \right\}.$$

The isomorphism ϕ of the pseudo-product structure on R preserves the differential system $D = E \oplus F$, which is equivalent to the canonical system C^{k-1} on J^{k-1}.

Hence, by Bäcklund's theorem (cf. [Yam83]), ϕ is the lift of a point transformation on J^0 when $m \geqq 2$ and $k \geqq 2$ and is the lift of a contact transformation on J^1 when $m = 1$ and $k \geqq 3$. When $(m, k) = (1, 2)$, ϕ is the lift of the point transformation on J^0, since ϕ preserves both D and $F = \operatorname{Ker}(\pi_0^1)_*$. Thus the equivalence of the pseudo-product structure on R is the equivalence of the kth order equation under point or contact transformations. To settle the equivalence problem for the pseudo-projective systems of order k of bidegree (n, m), N. Tanaka constructed the *normal Cartan connections of type* $\mathfrak{g}^k(n, m)$ ([Tan79], [Tan82], [Tan89]).

It is well known that $\mathfrak{g}^k(n, m)$ ($k \geqq 2$) has the following structure ([Tan89], [Yam93], [DKM99], [YY02]);

(1) $k = 2$. $\mathfrak{g}^2(n, m)$ is isomorphic to $\mathfrak{sl}(m + n + 1, \mathbb{R})$ and has the following gradation:

$$\mathfrak{sl}(m + n + 1, \mathbb{R}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

where the gradation is given by subdividing matrices as follows;

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{pmatrix} \middle| \quad \xi \in W \cong \mathbb{R}^m \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & A & 0 \end{pmatrix} \middle| \; x \in V \cong \mathbb{R}^n,\; A \in M(m, n) = W \otimes V^* \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \middle| \; \begin{array}{l} a \in \mathbb{R},\; B \in \mathfrak{gl}(V),\; C \in \mathfrak{gl}(W), \\ a + \mathrm{tr}B + \mathrm{tr}C = 0 \end{array} \right\},$$

$$\mathfrak{g}_1 = \{ {}^t X \mid X \in \mathfrak{g}_{-1} \}, \qquad \mathfrak{g}_2 = \{ {}^t X \mid X \in \mathfrak{g}_{-2} \},$$

where $V = M(n, 1)$, $W = M(m, 1)$ and $M(a, b)$ denotes the set of $a \times b$ matrices.

(2) $k = 3$ and $m = 1$. $\mathfrak{g}^3(n, 1)$ is isomorphic to $\mathfrak{sp}(n+1, \mathbb{R})$ and has the following gradation:

$$\mathfrak{sp}(n + 1, \mathbb{R}) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3.$$

First we describe

$$\mathfrak{sp}(n + 1, \mathbb{R}) = \{X \in \mathfrak{gl}(2n + 2, \mathbb{R}) \mid {}^t XJ + JX = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2n + 2, \mathbb{R}), \quad I_n = (\delta_{ij}) \in \mathfrak{gl}(n, \mathbb{R}).$$

Here $I_n \in \mathfrak{gl}(n, \mathbb{R})$ is the unit matrix and the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-3} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2a & 0 & 0 & 0 \end{pmatrix} \middle| \quad a \in \mathbb{R} \right\},$$

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & {}^t\xi & 0 & 0 \end{pmatrix} \middle| \quad \xi \in \mathbb{R}^n \cong V^* \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & -{}^t x & 0 \end{pmatrix} \middle| \quad x \in \mathbb{R}^n = V, \ A \in \mathrm{Sym}(n) \cong S^2(V^*) \right\},$$

$$\mathfrak{g}_{0} = \left\{ \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & -{}^t B & 0 \\ 0 & 0 & 0 & -b \end{pmatrix} \middle| \quad b \in \mathbb{R}, \ B \in \mathfrak{gl}(V) \right\},$$

$$\mathfrak{g}_k = \{ {}^t X \mid X \in \mathfrak{g}_{-k} \}, \ (k = 1, 2, 3),$$

where $\mathrm{Sym}(n) = \{ A \in \mathfrak{gl}(n, \mathbb{R}) \mid {}^t A = A \}$ is the space of symmetric matrices.

(3) otherwise. For vector spaces V and W of dimension n and m respectively, $\mathfrak{g}^k(n, m) = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ has the following description:

$$\mathfrak{g}_k = \{0\} \quad (k \geqq 2), \qquad \mathfrak{g}_1 = V^*, \qquad \mathfrak{g}_0 = \mathfrak{gl}(V) \oplus \mathfrak{gl}(W),$$

$$\mathfrak{g}_{-1} = V \oplus W \otimes S^{k-1}(V^*), \quad \mathfrak{g}_p = W \otimes S^{k+p}(V^*) \quad (p < -1).$$

Here the bracket product in $\mathfrak{g}^k(n, m)$ is given through the natural tensor operations.

For the structure of $\mathfrak{g}^k(n, m)$ in case (3), we observe the following points. We put

$$\begin{aligned} \mathfrak{l} = V \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 &= (V \oplus \mathfrak{gl}(V) \oplus V^*) \oplus \mathfrak{gl}(W) \\ &\cong \mathfrak{sl}(\hat{V}) \oplus \mathfrak{gl}(W), \\ S = W \otimes S^{k-1}(\hat{V}^*), &\qquad \hat{V} = \mathbb{R} \oplus V. \end{aligned} \tag{2.3}$$

where the gradation of the first kind; $\mathfrak{sl}(\hat{V}) = V \oplus \mathfrak{gl}(V) \oplus V^*$ is given by subdividing matrices corresponding to the decomposition $\hat{V} = \mathbb{R} \oplus V$. Then

$$S^{k-1}(\hat{V}^*) \cong \bigoplus_{\ell=0}^{k-1} S^\ell(V^*),$$

and S is a faithful irreducible $\mathfrak{l}$-module such that $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ is a reductive graded Lie algebra, where $\mathfrak{l}_{-1} = V$, $\mathfrak{l}_0 = \mathfrak{g}_0$, $\mathfrak{l}_1 = \mathfrak{g}_1$. Moreover $\mathfrak{g}^k(n, m) \cong S \oplus \mathfrak{l}$ is the semi-direct product of $\mathfrak{l}$ by S.

In the following sections, we will seek to find other parabolic geometries associated with differential equations of finite type, which are the generalizations of the above cases (1) and (2).

2.2 Pseudo-product GLA of type ($\mathfrak{l}$, S)

We will now give the notion of the pseudo-product GLA of type $(\mathfrak{l}, S)$, generalizing the pseudo-projective GLA of order k of bidegree (n, m).

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a (transitive) graded Lie algebra (GLA) over the field $\mathbb{K}$ such that the negative part $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ is a FGLA, i.e., $[\mathfrak{g}_p, \mathfrak{g}_{-1}] = \mathfrak{g}_{p-1}$ for $p < 0$, where $\mathbb{K}$ is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $\mathfrak{e}$ and $\mathfrak{f}$ be subspaces of $\mathfrak{g}_{-1}$. Then the system $\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{e}, \mathfrak{f})$ is called a pseudo-product GLA (PPGLA) of irreducible type if the following conditions hold:

(1) $\mathfrak{g}$ is transitive, i.e., for each $k \geqq 0$, if $X \in \mathfrak{g}_k$ and $[X, \mathfrak{g}_{-1}] = 0$, then $X = 0$.
(2) $\mathfrak{g}_{-1} = \mathfrak{e} \oplus \mathfrak{f}, \quad [\mathfrak{e}, \mathfrak{e}] = [\mathfrak{f}, \mathfrak{f}] = 0$.
(3) $[\mathfrak{g}_0, \mathfrak{e}] \subset \mathfrak{e}$ and $[\mathfrak{g}_0, \mathfrak{f}] \subset \mathfrak{f}$.
(4) $\mathfrak{g}_{-2} \neq 0$ and the $\mathfrak{g}_0$-modules $\mathfrak{e}$ and $\mathfrak{f}$ are irreducible.

It is known that $\mathfrak{g}$ becomes finite dimensional under these conditions (see [Tan85], [Yat88]).

As a typical example, starting from a reductive GLA $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ and a faithful irreducible $\mathfrak{l}$-module S, we define the pseudo-product GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ of type $(\mathfrak{l}, S)$ as follows: Let $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ be a finite dimensional *reductive GLA* of the first kind such that

(1) The ideal $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ of $\mathfrak{l}$ is a simple Lie algebra.
(2) The center $\mathfrak{z}(\mathfrak{l})$ of $\mathfrak{l}$ is contained in $\mathfrak{l}_0$.

Let S be a finite dimensional *faithful irreducible* $\mathfrak{l}$*-module*. We put

$$S_{-1} = \{s \in S \mid \mathfrak{l}_1 \cdot s = 0\}$$

and

$$S_p = \mathrm{ad}(\mathfrak{l}_{-1})^{-p-1} S_{-1} \quad \text{for } p < 0.$$

We form the semi-direct product $\mathfrak{g}$ of $\mathfrak{l}$ by S, and put

$$\mathfrak{g} = S \oplus \mathfrak{l}, \qquad [S, S] = 0,$$

$$\mathfrak{g}_k = \mathfrak{l}_k \ (k \geqq 0), \quad \mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1},$$
$$\mathfrak{g}_p = S_p \ (p < -1).$$

Namely $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{A}(S) = S \oplus \mathfrak{gl}(S)$ of infinitesimal affine transformations of S.

Then we have (Lemma 2.1 [YY02])

Lemma 2.1 *Notation being as above,*

(1) $S = \bigoplus_{p=-1}^{-\mu} S_p$, *where* $S_{-\mu} = \{s \in S \mid [\mathfrak{l}_{-1}, s] = 0\}$.
(2) $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ *is generated by* $\mathfrak{g}_{-1}$.
(3) $[S_p, \mathfrak{l}_1] = S_{p+1}$ *for* $p < -1$.
(4) S_p *is naturally embedded as a subspace of* $W \otimes S^{\mu+p}(\mathfrak{l}_{-1}{}^*)$ *through the bracket operation in* $\mathfrak{m}$, *where* $W = S_{-\mu}$.
(5) $S_{-1}, S_{-\mu}$ *are irreducible* $\mathfrak{l}_0$ *-modules.*

Thus $\mathfrak{m}$ is a graded subalgebra of $\mathfrak{C}^{\mu-1}(V, W)$, which has the splitting $\mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1}$, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. Hence $\mathfrak{m}$ is a symbol algebra of μth order differential equations of finite type, which is called the *typical symbol of type* $(\mathfrak{l}, S)$. Moreover the system $\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{l}_{-1}, S_{-1})$ becomes a PPGLA of irreducible type, which is called the *pseudo-product GLA of type* $(\mathfrak{l}, S)$.

This class of higher order (linear) differential equations of finite type first appeared in the work of Y. Se-ashi [Sea88].

2.3 Prolongation Theorem

Let $\mathfrak{G} = (\mathfrak{g}, (\mathfrak{g}_p)_{p\in\mathbb{Z}}, \mathfrak{l}_{-1}, S_{-1})$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$, i.e., $\mathfrak{g} = S \oplus \mathfrak{l}$ is endowed with the gradation $(\mathfrak{g}_p)_{p\in\mathbb{Z}}$, $\mathfrak{g} = \bigoplus_{p=-\mu}^{1} \mathfrak{g}_p$ given in Section 2.2. $\mathfrak{g}$ has also another gradation $(\mathfrak{b}_p)_{p\in\mathbb{Z}}$, $\mathfrak{g} = \bigoplus_{p=-1}^{0} \mathfrak{b}_p$, given by $\mathfrak{b}_{-1} = S$ and $\mathfrak{b}_0 = \mathfrak{l}$. Thus $\mathfrak{g}$ has a bigradation $(\mathfrak{g}_{p,q})_{p,q\in\mathbb{Z}}$, where $\mathfrak{g}_{p,q} = \mathfrak{g}_p \cap \mathfrak{b}_q$. We have the cohomology group $H^*(\mathfrak{G}) = H^*(\mathfrak{m}, \mathfrak{g})$ associated with the adjoint representation of $\mathfrak{m} = \mathfrak{g}_-$ on $\mathfrak{g}$, that is, the cohomology space of the cochain complex $C^*(\mathfrak{G}) = \bigoplus C^p(\mathfrak{G})$ with the coboundary operator $\partial : C^p(\mathfrak{G}) \longrightarrow C^{p+1}(\mathfrak{G})$, where $C^p(\mathfrak{G}) = \mathrm{Hom}(\bigwedge^p \mathfrak{g}_-, \mathfrak{g})$. We put

$$C^p(\mathfrak{G})_{r,s} = \{\omega \in C^p(\mathfrak{G}) \mid \omega(\mathfrak{g}_{i_1,j_1} \wedge \cdots \wedge \mathfrak{g}_{i_p,j_p}) \subset \mathfrak{g}_{i_1+\cdots+i_p+r,\, j_1+\cdots+j_p+s} \quad \text{for all} \quad i_1, \ldots, i_p, j_1, \ldots, j_p\}.$$

As is easily seen, $C^*(\mathfrak{G})_{r,s} = \bigoplus_p C^p(\mathfrak{G})_{r,s}$ is a subcomplex of $C^*(\mathfrak{G})$. Denoting its cohomology space by $H(\mathfrak{G})_{r,s} = \bigoplus H^p(\mathfrak{G})_{r,s}$, we obtain the direct sum decomposition

$$H^*(\mathfrak{G}) = \bigoplus_{p,r,s} H^p(\mathfrak{G})_{r,s}.$$

The cohomology space, endowed with this tri-gradation, is called the generalized Spencer cohomology space of the PPGLA $\mathfrak{G}$ of type $(\mathfrak{l}, S)$. Note that $H^1(\mathfrak{G})_{0,0} = 0$ if and only if $\mathfrak{g}_0$ coincides with the Lie algebra of derivations of $\mathfrak{m}$ such that $D(\mathfrak{g}_p) \subset \mathfrak{g}_p$ $(p < 0)$, $D(\mathfrak{l}_{-1}) \subset \mathfrak{l}_{-1}$ and $D(S_{-1}) \subset S_{-1}$.

From now on, we assume for the sake of simplicity that the ground field is the field $\mathbb{C}$ of complex numbers. For the discussion over $\mathbb{R}$, the corresponding results will be obtained easily through the argument of complexification as in Section 3.2 in [Yam93]. We set $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus [\mathfrak{l}_{-1}, \mathfrak{l}_1] \oplus \mathfrak{l}_1$ and $\mathfrak{u} = \mathcal{D}(\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}}))$; then $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{u} \oplus \mathfrak{z}(\mathfrak{l})$, $\mathcal{D}(\mathfrak{l}) = \hat{\mathfrak{l}} \oplus \mathfrak{u}$ and $\hat{\mathfrak{l}} = \mathfrak{l}_{-1} \oplus \hat{\mathfrak{l}}_0 \oplus \mathfrak{l}_1$, where $\hat{\mathfrak{l}}_0 = [\mathfrak{l}_{-1}, \mathfrak{l}_1]$, is a simple GLA. Let us take a Cartan subalgebra

$\mathfrak{h}$ of $\mathfrak{l}$ such that $\mathfrak{h} \subset \mathfrak{l}_0$. Then $\mathfrak{h} \cap \hat{\mathfrak{l}}$ (resp. $\mathfrak{h} \cap \mathfrak{u}$) is a Cartan subalgebra of $\hat{\mathfrak{l}}$ (resp. $\mathfrak{u}$). Let $\Delta = \{\alpha_1, \ldots, \alpha_\ell\}$ (resp. $\Delta' = \{\beta_1, \ldots, \beta_m\}$) be a simple root system of $(\hat{\mathfrak{l}}, \mathfrak{h} \cap \hat{\mathfrak{l}})$ (resp. $(\mathfrak{u}, \mathfrak{h} \cap \mathfrak{u})$) such that $\alpha(Z) \geqq 0$ for all $\alpha \in \Delta$, where Z is the characteristic element of the GLA $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$. We assume that $\hat{\mathfrak{l}}$ is a simple Lie algebra of type X_ℓ. We set $\Delta_1 = \{\alpha \in \Delta \mid \alpha(Z) = 1\}$. It is well known that the pair (X_ℓ, Δ_1) is one of the following type (up to a diagram automorphism) (cf. Section 3 in [Yam93]):

$$(A_\ell, \{\alpha_i\})\ (1 \leqq i \leqq [(\ell+1)/2]),\ (B_\ell, \{\alpha_1\})\ (\ell \geqq 3),\ (C_\ell, \{\alpha_\ell\})\ (\ell \geqq 2),$$
$$(D_\ell, \{\alpha_1\})\ (\ell \geqq 4),\ (D_\ell, \{\alpha_{\ell-1}\})\ (\ell \geqq 5),\ (E_6, \{\alpha_1\}),\ (E_7, \{\alpha_7\}).$$

We denote by $\{\varpi_1, \ldots, \varpi_\ell\}$ (resp. $\{\pi_1, \ldots, \pi_n\}$) the set of fundamental weights relative to Δ (resp. Δ'). Since S is a faithful $\mathfrak{l}$-module, we have $\dim \mathfrak{z}(\mathfrak{l}) \leqq 1$. Assume that $\mathfrak{z}(\mathfrak{l}) \neq \{0\}$. Let σ be the element of $\mathfrak{z}(\mathfrak{l})^*$ such that $\sigma(J) = 1$, where J is the characteristic element of the GLA $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$. Namely $J = -id_S \in \mathfrak{z}(\mathfrak{l}) \subset \mathfrak{b}_0 = \mathfrak{l}$ as the element of $\mathfrak{gl}(S)$. There is an irreducible $\hat{\mathfrak{l}}$ -module T (resp. $\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$ -module U) with highest weight χ (resp. $\eta - \sigma$) such that $S = \mathfrak{b}_{-1}$ is isomorphic to $U \otimes T$ as an $\mathfrak{l}$-module, where η is a weight of $\mathfrak{u}$. Then we have (Lemma 4.5 [YY02]).

Lemma 2.2 *$H^1(\mathfrak{G})_{0,0} = 0$ if and only if $\mathfrak{z}_{\mathfrak{l}}(\hat{\mathfrak{l}})$ is isomorphic to $\mathfrak{gl}(U)$ and $\eta = \pi_1$. Especially, when $\mathcal{D}(\mathfrak{l}) = \hat{\mathfrak{l}}$, $H^1(\mathfrak{G})_{0,0} = 0$ if and only if $\mathfrak{l} = \hat{\mathfrak{l}} \oplus \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{z}(\mathfrak{l}) = \langle J \rangle$.*

Thus, when $H^1(\mathfrak{G})_{0,0} = 0$, the semisimple GLA $\mathcal{D}(\mathfrak{l})$ is of type $(X_\ell \times A_n, \{\alpha_i\})$ and S is an irreducible $\mathcal{D}(\mathfrak{l})$-module with highest weight $\Xi = \chi + \pi_1$ when $\dim U > 1$ and $\mathcal{D}(\mathfrak{l})$ is of type $(X_\ell, \{\alpha_i\})$ and S is an irreducible $\hat{\mathfrak{l}}$-module with highest weight χ, when $\mathcal{D}(\mathfrak{l}) = \hat{\mathfrak{l}}$ (i.e., when $\dim U = 1$).

The following theorem was obtained in Theorem 5.2 [YY02] as the answer to the following question:

When is $\mathfrak{g}$ the prolongation of $\mathfrak{m}$ or $(\mathfrak{m}, \mathfrak{g}_0)$?

In the following theorem (a), the simple graded Lie algebra $\mathfrak{b} = \check{\mathfrak{g}} = \bigoplus_{p \in \mathbb{Z}} \check{\mathfrak{g}}_p$ is described by $(Y_{\ell+n+1}, \Sigma_1)$ such that $\mathfrak{g} = \bigoplus_{p=-\mu}^{1} \mathfrak{g}_p$ is a graded subalgebra of $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ satisfying $\mathfrak{g}_p = \check{\mathfrak{g}}_p$ for $p \leqq 0$.

Theorem 2.3 *Let $\mathfrak{G}$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$ satisfying the condition $H^1(\mathfrak{G})_{0,0} = 0$. Let $\mathfrak{b} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{b}_p$ be the prolongation of $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0$, where $\mathfrak{b}_{-1} = S$ and $\mathfrak{b}_0 = \mathfrak{l}$. Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is the prolongation of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ except for the following three cases.*

(a) $\dim \mathfrak{b} < \infty$ *and* $\mathfrak{b}_1 \neq 0$ ($\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$*: simple) (see Table 1). In this case* $(Y_{\ell+n+1}, \Sigma_1)$ *is the prolongation of* $\mathfrak{m}$ *except for* $(A_{\ell+n+1}, \{\gamma_1, \gamma_{\ell+1}\})$ *and* $(C_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$*. Moreover the latter two are the prolongations of* $(\mathfrak{m}, \mathfrak{g}_0)$.
(b) $\dim \mathfrak{b} = \infty$ *(see Table 2). In the* $(C_\ell, \{\alpha_\ell\})$*-case,* $\mu = 2$

$$S_{-2} = V^*, \quad S_{-1} = V, \quad \mathfrak{l}_{-1} = S^2(V^*),$$
$$\mathfrak{l}_0 = V \otimes V^* \oplus \mathcal{C}, \qquad \mathfrak{l}_1 = S^2(V).$$

Table 1.

$\mathcal{D}(\mathfrak{l}) = [\mathfrak{l}, \mathfrak{l}]$	Δ_1	$\mathfrak{b}_{-1} = S$	$\check{\mathfrak{g}} = Y_{\ell+n+1}$	Σ_1
$A_\ell \times A_n$	$\{\alpha_i\}$	$\varpi_\ell + \pi_1$	$A_{\ell+n+1}$	$\{\gamma_i, \gamma_{\ell+1}\}$
A_ℓ	$\{\alpha_i\}$	$2\varpi_l$	$C_{\ell+1}$	$\{\gamma_i, \gamma_{\ell+1}\}$
A_ℓ $(\ell \geqq 3)$	$\{\alpha_i\}$	$\varpi_{\ell-1}$	$D_{\ell+1}$	$\{\gamma_i, \gamma_{\ell+1}\}$
B_ℓ $(\ell \geqq 2)$	$\{\alpha_1\}$	ϖ_1	$B_{\ell+1}$	$\{\gamma_2, \gamma_1\}$
D_ℓ $(\ell \geqq 4)$	$\{\alpha_1\}$	ϖ_1	$D_{\ell+1}$	$\{\gamma_2, \gamma_1\}$
D_ℓ $(\ell \geqq 4)$	$\{\alpha_\ell\}$	ϖ_1	$D_{\ell+1}$	$\{\gamma_{\ell+1}, \gamma_1\}$
D_5	$\{\alpha_1\}$	ϖ_5	E_6	$\{\gamma_1, \gamma_6\}$
D_5	$\{\alpha_5\}$	ϖ_5	E_6	$\{\gamma_3, \gamma_1\}$
D_5	$\{\alpha_4\}$	ϖ_5	E_6	$\{\gamma_2, \gamma_1\}$
E_6	$\{\alpha_6\}$	ϖ_6	E_7	$\{\gamma_6, \gamma_7\}$
E_6	$\{\alpha_1\}$	ϖ_6	E_7	$\{\gamma_1, \gamma_7\}$

Table 2.

$\mathcal{D}(\mathfrak{l})$	Δ_1	$\mathfrak{b}_{-1}$	$\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$
A_ℓ	$\{\alpha_i\}$	ϖ_ℓ	$(A_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$
C_ℓ	$\{\alpha_\ell\}$	ϖ_1	$\mathfrak{g}$

(c) $\mathfrak{g}$ *is a pseudo-projective GLA, i.e.,* $\mathcal{D}(\mathfrak{l}) = (A_\ell \times A_n, \{\alpha_1\})$, $\Xi = k\varpi_\ell + \pi_1$, $(k \geqq 2, n \geqq 1)$, *or* $\mathcal{D}(\mathfrak{l}) = (A_\ell, \{\alpha_1\})$, $\chi = k\varpi_\ell$, $(k \geqq 3, n = 0)$

$$S_{-\mu} = W, \quad S_p = W \otimes S^{\mu+p}(V^*)\ (-\mu < p < 0),$$

$$\mathfrak{l}_{-1} = V, \quad \mathfrak{l}_0 = \mathfrak{gl}(V) \oplus \mathfrak{gl}(W), \quad \mathfrak{l}_1 = V^*,$$

where $\mu = k+1$, $\dim V = \ell$ *and* $\dim W = n+1$. *In this case* $\mathfrak{g}$ *is the prolongation of* $(\mathfrak{m}, \mathfrak{g}_0)$.

By Proposition 4.4.1 in [Sea88], the Lie algebra of infinitesimal linear automorphisms of the model equation of type $(\mathfrak{l}, S)$ coincides with $\mathfrak{l}$. Hence the cases (a) and (b) of the above theorem exhaust classes of the equations of type $(\mathfrak{l}, S)$, for which the model equations admit non-trivial nonlinear automorphisms. These cases correspond to the parabolic geometries associated with differential equations of finite type, which generalize the case of second and third order ordinary differential equations, mentioned in the introduction. More precisely, in the cases of $(A_{\ell+1}, \{\gamma_1, \gamma_i\})$ and $(C_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$, $\mathfrak{m}$ coincides with the symbol algebra of the canonical system of the first or second order jet spaces (cf. Section 4.5 [Yam93]) and $\mathfrak{g}_0$ determines the splitting of $\mathfrak{g}_{-1}$, hence the parabolic geometries associated with these graded Lie algebras are geometries of the pseudo-product structures on the first or second order jet spaces. In fact the parabolic geometry associated with $(A_{m+n}, \{\gamma_1, \gamma_{n+1}\})$ is the geometry of the pseudo-projective system of order 2 of bidegree (n, m) and the parabolic geometry associated with $(C_{n+1}, \{\gamma_1, \gamma_{n+1}\})$ is the geometry of the pseudo-projective system of order 3 of bidegree $(n, 1)$ (see the following section).

In the other cases of the above theorem (a), $(Y_{\ell+n+1}, \Sigma_1)$ is the prolongation of $\mathfrak{m}$. This fact implies that the parabolic geometries associated with these graded Lie algebras are geometries of regular differential systems of type $\mathfrak{m}$, which have the (almost) pseudo-product structure corresponding to $\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1}$. Moreover every isomorphism of these regular differential systems preserves this pseudo-product structure. Thus the parabolic geometries associated with $(Y_{\ell+n+1}, \Sigma_1)$ have the canonical (almost) pseudo-product structures in the regular differential system of type $\mathfrak{m}$ corresponding to the splitting $\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1}$.

In the following sections, we will calculate explicit forms of typical symbols of type $(\mathfrak{l}, S)$ of the above cases and describe the above (almost) pseudo-product structures as differential equations of finite type.

3 Symbol of the classical cases

In this section we will describe the symbol algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ explicitly as the subalgebra of $\mathfrak{C}^{\mu-1}(V, W)$, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$, by utilizing the explicit matrices description of the graded Lie algebra $\check{\mathfrak{g}}$ of type (Y_L, Σ_1). For an explicit matrices description of the graded Lie algebra (Y_L, Σ_1), we refer the reader to Section 4.4 in [Yam93]. By this calculation, we can explicitly write down the class of differential equations of finite type corresponding to the pseudo-product structure associated with the simple graded Lie algebra (Y_L, Σ_1). In this section, we shall work in the complex analytic or the real C^∞ category depending on whether $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$.

Case (1) $[(A_\ell \times A_n, \{\alpha_i\}), \varpi_\ell + \pi_1, (A_{\ell+n+1}, \{\gamma_i, \gamma_{\ell+1}\})]$ $(1 < i \leqq \ell, n \geqq 0, \ell \geqq 2)$.

This includes the first case of (b) in the above theorem as the case $n = 0$. $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(A_{\ell+n+1}, \{\gamma_{\ell+1}\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(A_{\ell+n+1}, \{\gamma_i, \gamma_{\ell+1}\})$. Hence $\mu = 2$ and we obtain the following matrix representation of $\check{\mathfrak{g}} = \mathfrak{b} = \mathfrak{sl}(\ell + n + 2, \mathbb{K})$:

$$\mathfrak{sl}(\ell + n + 2, \mathbb{K}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \check{\mathfrak{g}}_1 \oplus \check{\mathfrak{g}}_2 = S \oplus \mathfrak{l} \oplus S^*,$$

where the gradation is given by subdividing matrices as follows:

$$\mathfrak{g}_{-2} = S_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A & 0 & 0 \end{pmatrix} \middle| \; A \in M(n+1, i) \cong U \otimes T_0^* \right\},$$

$$\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

$$S_{-1} = U \otimes T_{-1}^* = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & B & 0 \end{pmatrix} \middle| \; B \in M(n+1, j) \right\},$$

$$\mathfrak{l}_{-1} = T_{-1} \otimes T_0^* = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ C & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \; C \in M(j, i) \right\},$$

$$\mathfrak{g}_0 = \check{\mathfrak{l}}_0 \oplus \mathfrak{u} = \left\{ \begin{pmatrix} F & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & H \end{pmatrix} \middle| \; \begin{array}{l} F \in \mathfrak{gl}(T_0),\ G \in \mathfrak{gl}(T_{-1}),\ H \in \mathfrak{gl}(U), \\ \mathrm{tr} F + \mathrm{tr} G + \mathrm{tr} H = 0 \end{array} \right\},$$

$$\check{\mathfrak{g}}_1 = \{ {}^t X \mid X \in \mathfrak{g}_{-1} \}, \qquad \check{\mathfrak{g}}_2 = \{ {}^t X \mid X \in \mathfrak{g}_{-2} \},$$

where $i + j = \ell + 1$, $U = \mathbb{K}^{n+1}$, $T = T_0 \oplus T_{-1} = \mathbb{K}^{\ell+1}$, $T_0 = \mathbb{K}^i$, $T_{-1} = \mathbb{K}^j$ and $M(a, b)$ denotes the set of $a \times b$ matrices. Thus we have

$$S = U \otimes T^*, \quad \mathfrak{l} = \mathfrak{sl}(T) \oplus \mathfrak{gl}(U), \quad \text{and} \quad \check{\mathfrak{g}} = \mathfrak{sl}(T \oplus U).$$

We will divide the argument into the following two cases. We first consider the typical case:

(i) $i = \ell \geqq 2, n = 0$.

We have $j = 1$ and $n = 0$ in the above matrix description. Hence $\dim \mathfrak{l}_{-1} = \dim S_{-2} = \ell$ and $\dim S_{-1} = 1$. We put $\mathfrak{l}_{-1} = S_{-2} = V$. Then

$$\mathfrak{m} = S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ y & a & 0 \end{pmatrix} = \check{y} + \hat{a} + \hat{x} \middle| \; x, y \in V = M(1, \ell), a \in \mathbb{K} \right\}.$$

By a direct calculation, we have $[\hat{a}, \hat{x}] = (\check{ax}) \in S_{-2} = V$, i.e., $y = ax$. Thus S_{-1} is embedded as the 1-dimensional subspace of scalar multiplications of $V \otimes V^* = S_{-2} \otimes (\mathfrak{l}_{-1})^*$ through the bracket operation in $\mathfrak{m}$. This implies that the model equation of our typical symbol $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^1(V, V)$ is given by

$$\frac{\partial y_p}{\partial x_q} = \delta_{pq} \frac{\partial y_1}{\partial x_1} \qquad \text{for} \quad 1 \leqq p, q \leqq \ell. \tag{3.1}$$

where $y_1, \ldots, y_\ell$ are dependent variables and $x_1, \ldots, x_\ell$ are independent variables. By a direct calculation, we see that the prolongation of the first order system (3.1) is given by

$$\frac{\partial^2 y_p}{\partial x_q \partial x_r} = 0 \qquad \text{for} \quad 1 \leqq p, q, r \leqq \ell. \tag{3.2}$$

(ii) otherwise.

We have $S_{-2} = U \otimes T_0^* \cong M(n+1, i)$, $S_{-1} = U \otimes T_{-1}^* \cong M(n+1, j)$ and $\mathfrak{l}_{-1} = T_{-1} \otimes T_0^* \cong M(j, i)$. Then

$$\mathfrak{m} = S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1})$$
$$= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ Y & A & 0 \end{pmatrix} = \check{Y} + \hat{A} + \hat{X} \;\middle|\; Y \in M(n+1, i), A \in M(n+1, j), X \in M(j, i) \right\}.$$

By a direct calculation, we have $[\hat{A}, \hat{X}] = (AX)\check{} \in S_{-2}$, i.e., $y^\alpha = \sum_{\tau=1}^{j} a_\tau^\alpha x^\tau$, where y^α is the αth row of Y, x^τ is the τth row of X and $A = (a_\tau^\alpha)$. From (i), we see that the model equation of our typical symbol $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^1(\mathfrak{l}_{-1}, S_{-2})$ is given by

$$\frac{\partial y_p^\alpha}{\partial x_q^\tau} = \delta_{pq} \frac{\partial y_1^\alpha}{\partial x_1^\tau} \qquad \text{for} \quad \alpha = 1, \dots, n+1, \quad \tau = 1, \dots, j, \quad 1 \leqq p, q \leqq i, \tag{3.3}$$

where $y_1^1, \dots, y_i^1, \dots, y_1^{n+1}, \dots, y_i^{n+1}$ are dependent variables and $x_1^1, \dots, x_i^1, \dots, x_1^j, \dots, x_i^j$ are independent variables. By a direct calculation, we see that the prolongation of the first order system (3.3) is given by

$$\frac{\partial^2 y_p^\alpha}{\partial x_q^\tau \partial x_r^\upsilon} = 0 \qquad \text{for} \quad \alpha = 1, \dots, n+1, \quad 1 \leqq \tau, \upsilon \leqq j, \quad 1 \leqq p, q, r \leqq i. \tag{3.4}$$

Case (2) $[(A_\ell \times A_n, \{\alpha_1\}), \varpi_\ell + \pi_1, (A_{\ell+n+1}, \{\gamma_1, \gamma_{\ell+1}\})]$ $(n \geqq 0, \ell \geqq 1)$.

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(A_{\ell+n+1}, \{\gamma_{\ell+1}\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(A_{\ell+n+1}, \{\gamma_1, \gamma_{\ell+1}\})$. Hence $\mu = 2$ and we obtain $\check{\mathfrak{g}} = \mathfrak{g}^2(\ell, n+1)$. The matrix representation is given as (1) in Section 2.1.

We have $S_{-2} = W \cong M(n+1, 1)$, $\mathfrak{l}_{-1} = V \cong M(\ell, 1)$, $S_{-1} = W \otimes V^* \cong M(n+1, \ell)$ and $\mathfrak{g}_0$ determines the splitting of $\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1}$. Thus the model equation of our typical symbol $\mathfrak{m} = \mathfrak{C}^1(V, W)$ is given by

$$\frac{\partial^2 y^\alpha}{\partial x_p \partial x_q} = 0 \qquad \text{for} \quad \alpha = 1, \dots, n+1, \quad 1 \leqq p, q \leqq \ell, \tag{3.5}$$

where $y^1, \dots, y^{n+1}$ are dependent variables and $x_1, \dots, x_\ell$ are independent variables.

Case (3) $[(A_\ell, \{\alpha_i\}), 2\varpi_l, (C_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})]$ $(1 < i \leqq \ell, \ell \geqq 2)$.

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(C_{\ell+1}, \{\gamma_{\ell+1}\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(C_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$. Hence $\mu = 3$ and $\check{\mathfrak{g}} = \mathfrak{b}$ is isomorphic to $\mathfrak{sp}(\ell+1, \mathbb{K})$. First we describe

$$\mathfrak{sp}(\ell+1, \mathbb{K}) = \{X \in \mathfrak{gl}(2\ell+2, \mathbb{K}) \mid {}^tXJ + JX = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & I_i \\ 0 & 0 & I_j & 0 \\ 0 & -I_j & 0 & 0 \\ -I_i & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2\ell+2, \mathbb{K}), \quad I_k = (\delta_{pq}) \in \mathfrak{gl}(k, \mathbb{K}).$$

Here $I_k \in \mathfrak{gl}(k, \mathbb{K})$ is the unit matrix and the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-3} = S_{-3} = S^2(T_0^*) = \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ A&0&0&0 \end{pmatrix} \middle| \; A \in \mathrm{Sym}(i) \right\},$$

$$\mathfrak{g}_{-2} = S_{-2} = T_{-1} \otimes T_0^* = \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ B&0&0&0 \\ 0&{}^tB&0&0 \end{pmatrix} \middle| \; B \in M(j, i) \right\},$$

$$\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

$$S_{-1} = S^2(T_{-1}^*) = \left\{ \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&D&0&0 \\ 0&0&0&0 \end{pmatrix} \middle| \; D \in \mathrm{Sym}(j) \right\},$$

$$\mathfrak{l}_{-1} = T_{-1} \otimes T_0^* = \left\{ \begin{pmatrix} 0&0&0&0 \\ C&0&0&0 \\ 0&0&0&0 \\ 0&0&-{}^tC&0 \end{pmatrix} \middle| \; C \in M(j, i) \right\},$$

$$\mathfrak{g}_0 = \check{\mathfrak{l}}_0 = \left\{ \begin{pmatrix} F&0&0&0 \\ 0&G&0&0 \\ 0&0&-{}^tG&0 \\ 0&0&0&-{}^tF \end{pmatrix} \middle| \; F \in \mathfrak{gl}(i, \mathbb{K}),\; G \in \mathfrak{gl}(j, \mathbb{K}) \right\},$$

$$\check{\mathfrak{g}}_k = \{ {}^tX \mid X \in \mathfrak{g}_{-k} \}, \; (k = 1, 2, 3),$$

where $i + j = \ell + 1$, $T = T_0 \oplus T_{-1} = \mathbb{K}^{\ell+1}$, $T_0 = \mathbb{K}^i$, $T_{-1} = \mathbb{K}^j$ and $\mathrm{Sym}(k) = \{ A \in \mathfrak{gl}(k, \mathbb{K}) \mid {}^tA = A \}$ is the space of symmetric matrices. Thus we have

$$S = S^2(T^*), \qquad \mathfrak{l} = \mathfrak{sl}(T), \text{ and } \qquad \check{\mathfrak{g}} = \mathfrak{sp}(T \oplus T).$$

We will divide the argument into the following two cases. We first consider the typical case:

(i) $i = \ell \geqq 2$.

We have $j = 1$ in the above matrix description. Hence $\dim \mathfrak{l}_{-1} = S_{-2} = \ell$, $\dim S_{-1} = 1$ and $\dim S_{-3} = \frac{1}{2}\ell(\ell+1)$. Then

$$\mathfrak{m} = S_{-3} \oplus S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1})$$
$$= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ \xi & a & 0 & 0 \\ Y & {}^t\xi & -{}^tx & 0 \end{pmatrix} = \hat{Y} + \check{\xi} + \hat{a} + \hat{x} \;\middle|\; \begin{array}{l} a \in \mathbb{K}, x, \xi \in \mathbb{K}^\ell = M(1, \ell), \\ Y \in \mathrm{Sym}(\ell) \end{array} \right\},$$

By calculating $[\hat{\xi}, \hat{x}]$ and $[[\hat{a}, \hat{x}], \hat{x}]$, we have

$$y_{pq} (= y_{qp}) = \xi_p x_q + \xi_q x_p = 2a x_p x_q,$$

where $Y = (y_{pq})$, $\xi = (\xi_1, \ldots, \xi_\ell)$ and $x = (x_1, \ldots, x_\ell)$. From the first equality, we can embed S_{-2} as a subspace of $S_{-3} \otimes (\mathfrak{l}_{-1})^*$ and obtain the following first order system as the model equation whose symbol coincides with this subspace:

$$\frac{\partial y_{pq}}{\partial x_r} = 0 \quad \text{for} \quad r \neq p, q, \qquad \frac{\partial y_{pq}}{\partial x_q} = \frac{1}{2} \frac{\partial y_{pp}}{\partial x_p} \qquad \text{for} \quad p \neq q, \tag{3.6}$$

where $y_{pq} = y_{qp}$ $(1 \leqq p \leqq q \leqq \ell)$ are dependent variables and $x_1, \ldots, x_\ell$ are independent variables. Moreover, by a direct calculation, we see that the prolongation of the first order system (3.6) is given by

$$\frac{\partial^2 y_{pq}}{\partial x_r \partial x_s} = 0 \quad \text{for } \{r, s\} \neq \{p, q\}, \qquad \frac{\partial^2 y_{pq}}{\partial x_p \partial x_q} = \frac{1}{2} \frac{\partial^2 y_{pp}}{\partial^2 x_p} = \frac{1}{2} \frac{\partial^2 y_{qq}}{\partial^2 x_q} \quad \text{for } p \neq q. \tag{3.7}$$

From the second equality, we observe that the above second order system is the model equation of the 1-dimensional embedded subspace S_{-1} in $S_{-3} \otimes S^2((\mathfrak{l}_{-1})^*)$. Furthermore, by a direct calculation, we see that the prolongation of this second order system (3.7) is given by

$$\frac{\partial^3 y_{pq}}{\partial x_r \partial x_s \partial x_t} = 0 \quad \text{for } 1 \leqq p, q, r, s, t \leqq \ell. \tag{3.8}$$

(ii) $1 < i < \ell$.

We have $S_{-3} = S^2(T_0^*) \cong \mathrm{Sym}(i)$, $S_{-2} = T_{-1} \otimes T_0^* \cong M(j, i)$, $S_{-1} = S^2(T_{-1}^*) \cong \mathrm{Sym}(j)$ and $\mathfrak{l}_{-1} = T_{-1} \otimes T_0^* \cong M(j, i)$. Then

$$\mathfrak{m} = S_{-3} \oplus S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1})$$
$$= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 \\ \Xi & A & 0 & 0 \\ Y & {}^t\Xi & -{}^tX & 0 \end{pmatrix} = \hat{Y} + \check{\Xi} + \hat{A} + \hat{X} \;\middle|\; \begin{array}{l} A \in \mathrm{Sym}(j), X, \Xi \in M(j, i), \\ Y \in \mathrm{Sym}(i) \end{array} \right\}.$$

By calculating $[\hat{\Xi}, \hat{X}]$ and $[[\hat{A}, \hat{X}], \hat{X}]$, we have

$$y_{pq}(=y_{qp}) = \sum_{\alpha=1}^{j}(\xi_p^\alpha x_q^\alpha + \xi_q^\alpha x_p^\alpha) = 2\sum_{\alpha,\beta=1}^{j} a_{\alpha\beta}x_p^\alpha x_q^\beta,$$

where $Y = (y_{pq})$, $\Xi = (\xi_p^\alpha)$ $A = (a_{\alpha\beta})$ and $X = (x_p^\alpha)$. From the first equality, we can embed S_{-2} as a subspace of $S_{-3} \otimes (\mathfrak{l}_{-1})^*$ and obtain the following first order system as the model equation whose symbol coincides with this subspace:

$$\frac{\partial y_{pq}}{\partial x_r^\alpha} = 0 \quad \text{for} \quad r \neq p, q, \qquad \frac{\partial y_{pq}}{\partial x_q^\alpha} = \frac{1}{2}\frac{\partial y_{pp}}{\partial x_p^\alpha} \quad \text{for} \quad p \neq q \tag{3.9}$$

where $y_{pq} = y_{qp}$ $(1 \leqq p \leqq q \leqq i)$ are dependent variables and x_p^α $(1 \leqq p \leqq i, 1 \leqq \alpha \leqq j)$ are independent variables. Moreover, by a direct calculation, we see that the prolongation of the first order system (3.9) is given by

$$\frac{\partial^2 y_{pq}}{\partial x_r^\alpha \partial x_s^\beta} = 0 \quad \text{for}\, \{r,s\} \neq \{p,q\}, \quad \frac{\partial^2 y_{pq}}{\partial x_p^\alpha \partial x_q^\beta} = \frac{1}{2}\frac{\partial^2 y_{pp}}{\partial x_p^\alpha \partial x_p^\beta} = \frac{1}{2}\frac{\partial^2 y_{qq}}{\partial x_q^\alpha \partial x_q^\beta} \quad \text{for}\, p \neq q. \tag{3.10}$$

From the second equality, we observe that the above second order system is the model equation of the embedded subspace S_{-1} in $S_{-3} \otimes S^2((\mathfrak{l}_{-1})^*)$. Furthermore, by a direct calculation, we see that the prolongation of this second order system (3.10) is given by

$$\frac{\partial^3 y_{pq}}{\partial x_r^\alpha \partial x_s^\beta \partial x_t^\gamma} = 0 \qquad \text{for}\, 1 \leqq p,q,r,s,t \leqq i,\ 1 \leqq \alpha,\beta,\gamma \leqq j. \tag{3.11}$$

Case (4) $[(A_\ell, \{\alpha_1\}), 2\varpi_l, (C_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})]$ $(\ell \geqq 1)$.

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(C_{\ell+1}, \{\gamma_{\ell+1}\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(C_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$. Hence $\mu = 3$ and we obtain $\check{\mathfrak{g}} = \mathfrak{g}^3(\ell, 1)$. The matrix representation is given as (2) in Section 2.1.

We have $S_{-3} = \mathbb{K}$, $S_{-2} = V^*$, $\mathfrak{l}_{-1} = V$, $S_{-1} = S^2(V^*)$ and $\mathfrak{g}_0$ determines the splitting of $\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1}$. Thus the model equation of our typical symbol $\mathfrak{m} = \mathfrak{C}^2(V, \mathbb{K})$ is given by

$$\frac{\partial^3 y}{\partial x_p \partial x_q \partial x_r} = 0 \qquad \text{for} \quad 1 \leqq p,q,r \leqq \ell, \tag{3.12}$$

where y is a dependent variable and $x_1, \ldots, x_\ell$ are independent variables.

Case (5) $[(B_\ell, \{\alpha_1\}), \varpi_1, (B_{\ell+1}, \{\gamma_2, \gamma_1\})]$ $(\ell \geqq 2)$ $[(D_\ell, \{\alpha_1\}), \varpi_1, (D_{\ell+1}, \{\gamma_2, \gamma_1\})]$ $(\ell \geqq 4)$.

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(BD_{\ell+1}, \{\gamma_1\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(BD_{\ell+1}, \{\gamma_2, \gamma_1\})$. Hence $\mu = 3$ and $\check{\mathfrak{g}} = \mathfrak{b}$ is isomorphic to $\mathfrak{o}(n+4)$. First we describe

$$\mathfrak{o}(n+4) = \{X \in \mathfrak{gl}(n+4, \mathbb{K}) \mid {}^tXJ + JX = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(n+4, \mathbb{K}), \quad I_n = (\delta_{ij}) \in \mathfrak{gl}(n, \mathbb{K}).$$

Here $I_n \in \mathfrak{gl}(n, \mathbb{K})$ is the unit matrix and the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-3} = S_{-3} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 & 0 \\ 0 & -y & 0 & 0 & 0 \end{pmatrix} \middle| \; y \in \mathbb{K} \right\},$$

$$\mathfrak{g}_{-2} = S_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -{}^t\xi & 0 & 0 \end{pmatrix} \middle| \; \xi \in \mathbb{K}^n = M(n, 1) \right\},$$

$$\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

$$S_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 \end{pmatrix} \middle| \; b \in \mathbb{K} \right\},$$

$$\mathfrak{l}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & x & 0 & 0 & 0 \\ 0 & 0 & -{}^t x & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; x \in \mathbb{K}^n = M(n, 1) \right\},$$

$$\mathfrak{g}_0 = \check{\mathfrak{l}}_0 = \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & -a_1 \end{pmatrix} \middle| \; a_1, a_2 \in \mathbb{K}, \; A \in \mathfrak{o}(n) \right\},$$

$$\check{\mathfrak{g}}_k = \{ {}^t X \mid X \in \mathfrak{g}_{-k} \}, (k = 1, 2, 3).$$

We have $\dim S_{-3} = \dim S_{-1} = 1$ and $\dim S_{-2} = \dim \mathfrak{l}_{-1} = n$. Then

$$\mathfrak{m} = S_{-3} \oplus S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1})$$

$$= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ \xi & x & 0 & 0 & 0 \\ y & 0 & -{}^t x & 0 & 0 \\ 0 & -y & -{}^t\xi & -a & 0 \end{pmatrix} = \hat{y} + \check{\xi} + \hat{a} + \hat{x} \;\middle|\; y, a \in \mathbb{K},\ x, \xi \in \mathbb{K}^n = M(n,1) \right\}.$$

From $[\check{\xi}, \hat{x}] = (\sum_{i=1}^n \xi_i x_i)\hat{}$ and $[[\hat{a}, \hat{x}], \hat{x}] = (-a \sum_{i=1}^n x_i^2)\hat{}$, we have $S_{-2} = V^*$, putting $S_{-3} = \mathbb{K}$ and $\mathfrak{l}_{-1} = V$. Moreover S_{-1} is embedded as the 1-dimensional subspace spanned by the unit matrix in $\mathrm{Sym}(n) \cong S^2(V^*)$ through the bracket operation in $\mathfrak{m}$. This implies that the model equation of our typical symbol $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^2(V, \mathbb{K})$ is given by

$$\frac{\partial^2 y}{\partial x_p \partial x_q} = \delta_{pq} \frac{\partial^2 y}{\partial^2 x_1} \quad \text{for } 1 \leqq p, q \leqq n, \tag{3.13}$$

where y is a dependent variable and $x_1, \ldots, x_n$ are independent variables. By a direct calculation, we see that the prolongation of the second order system (3.13) is given by

$$\frac{\partial^3 y}{\partial x_p \partial x_q \partial x_r} = 0 \quad \text{for } 1 \leqq p, q, r \leqq n. \tag{3.14}$$

Case (6) $[(D_\ell, \{\alpha_\ell\}), \varpi_1, (D_{\ell+1}, \{\gamma_{\ell+1}\}, \gamma_1)]$ ($\ell \geqq 4$).

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(D_{\ell+1}, \{\gamma_1\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(D_{\ell+1}, \{\gamma_{\ell+1}\}, \gamma_1\})$. Hence $\mu = 2$ and $\check{\mathfrak{g}} = \mathfrak{b}$ is isomorphic to $\mathfrak{o}(2\ell + 2)$. First we describe

$$\mathfrak{o}(2\ell + 2) = \{X \in \mathfrak{gl}(2\ell + 2, \mathbb{K}) \mid {}^t X J + J X = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & I_\ell & 0 \\ 0 & I_\ell & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2\ell + 2, \mathbb{K}), \quad I_\ell = (\delta_{ij}) \in \mathfrak{gl}(\ell, \mathbb{K}).$$

Here the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-2} = S_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & -{}^t y & 0 & 0 \end{pmatrix} \;\middle|\; y \in \mathbb{K}^\ell = M(\ell, 1) \right\},$$

$$\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

$$S_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -{}^t\xi & 0 \end{pmatrix} \middle|\ \xi \in \mathbb{K}^\ell = M(\ell, 1) \right\},$$

$$\mathfrak{l}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle|\ X \in \mathfrak{o}(\ell) \right\},$$

$$\mathfrak{g}_0 = \check{\mathfrak{l}}_0 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & -{}^tA & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \middle|\ a \in \mathbb{K},\ A \in \mathfrak{gl}(\ell, \mathbb{K}) \right\},$$

$$\check{\mathfrak{g}}_k = \{ {}^tX \mid X \in \mathfrak{g}_{-k} \}, \quad (k = 1, 2, 3).$$

We have $\dim S_{-2} = \dim S_{-1} = \ell$, $\dim \mathfrak{l}_{-1} = \frac{1}{2}\ell(\ell - 1)$, $\mathfrak{l} = \mathfrak{o}(2\ell)$ and $S = \mathbb{K}^{2\ell}$. Then

$$\begin{aligned} \mathfrak{m} &= S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1}) \\ &= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \\ y & X & 0 & 0 \\ 0 & -{}^ty & -{}^t\xi & 0 \end{pmatrix} = \hat{y} + \check{\xi} + \hat{X} \middle|\ y, \xi \in \mathbb{K}^\ell = M(\ell, 1),\ X \in \mathfrak{o}(\ell) \right\}. \end{aligned}$$

By calculating $[\check{\xi}, \hat{X}]$, we have

$$y_p = \sum_{q=1}^{\ell} x_{pq} \xi_q, \qquad (x_{pq} + x_{qp} = 0),$$

where $y = {}^t(y_1, \ldots, y_\ell)$, $\xi = {}^t(\xi_1, \ldots, \xi_\ell)$ and $X = (x_{pq})$. Then the model equation of our typical symbol $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^1(\mathfrak{l}_{-1}, S_{-2})$ is given by

$$\frac{\partial y_p}{\partial x_{qr}} = 0 \quad \text{for distinct} \quad p, q, r \qquad \frac{\partial y_p}{\partial x_{pq}} = \frac{\partial y_r}{\partial x_{rq}} \quad \text{for} \quad p, r \neq q, \tag{3.15}$$

where $y_1, \ldots, y_\ell$ are dependent variables and x_{pq} $(1 \leqq p < q \leqq \ell)$ are independent variables. By a direct calculation, we see that the prolongation of the first order system (3.15) is given by

$$\frac{\partial^2 y_p}{\partial x_{q_1 r_1} \partial x_{q_2 r_2}} = 0 \qquad \text{for} \quad 1 \leqq p, q_1, r_1, q_2, r_2 \leqq \ell. \tag{3.16}$$

Case (7) $[(A_\ell, \{\alpha_1\}, \varpi_{\ell-1}, (D_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})]$ $(\ell \geqq 3)$.

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(D_{\ell+1}, \{\gamma_{\ell+1}\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(D_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$. Hence $\mu = 2$ and $\check{\mathfrak{g}} = \mathfrak{b}$ is isomorphic to $\mathfrak{o}(2\ell + 2)$. First we describe

$$\mathfrak{o}(2\ell+2) = \{X \in \mathfrak{gl}(2\ell+2, \mathbb{K}) \mid {}^tXJ + JX = 0\},$$

as in Case (6) and the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-2} = S_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & -{}^ty & 0 & 0 \end{pmatrix} \middle| \; y \in \mathbb{K}^\ell = M(\ell, 1) \right\},$$

$$\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

$$S_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \Xi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; \Xi \in \mathfrak{o}(\ell) \right\},$$

$$\mathfrak{l}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -{}^tx & 0 \end{pmatrix} \middle| \; x \in \mathbb{K}^\ell = M(\ell, 1) \right\},$$

$$\mathfrak{g}_0 = \check{\mathfrak{l}}_0 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & -{}^tA & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \middle| \; a \in \mathbb{K}, \; A \in \mathfrak{gl}(\ell, \mathbb{K}) \right\},$$

$$\check{\mathfrak{g}}_k = \{{}^tX \mid X \in \mathfrak{g}_{-k}\}, \quad (k = 1, 2, 3).$$

We have $\dim S_{-2} = \dim \mathfrak{l}_{-1} = \ell$, $\dim S_{-1} = \frac{1}{2}\ell(\ell-1)$, $\mathfrak{l} = \mathfrak{sl}(T)$ and $S = \bigwedge^2 T^*$. Then

$$\begin{aligned} \mathfrak{m} &= S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1}) \\ &= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & \Xi & 0 & 0 \\ 0 & -{}^ty & -{}^tx & 0 \end{pmatrix} = \check{y} + \hat{\Xi} + \hat{x} \; \middle| \; x, y \in \mathbb{K}^\ell = M(\ell, 1), \; \Xi \in \mathfrak{o}(\ell) \right\}. \end{aligned}$$

By calculating $[\hat{\Xi}, \hat{x}]$, we have

$$y_p = \sum_{q=1}^{\ell} \xi_{pq} x_q, \qquad (\xi_{pq} + \xi_{qp} = 0),$$

where $y = {}^t(y_1, \ldots, y_\ell)$, $x = {}^t(x_1, \ldots, x_\ell)$ and $\Xi = (\xi_{pq})$.

Then the model equation of our typical symbol $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^1(\mathfrak{l}_{-1}, S_{-2})$ is given by

$$\frac{\partial y_p}{\partial x_q} + \frac{\partial y_q}{\partial x_p} = 0 \qquad \text{for} \quad 1 \leqq p < q \leqq \ell, \tag{3.17}$$

where $y_1, \ldots, y_\ell$ are dependent variables and $x_1, \ldots, x_\ell$ are independent variables. By a direct calculation, we see that the prolongation of the first order system (3.17) is given by

$$\frac{\partial^2 y_p}{\partial x_q \partial x_r} = 0 \qquad \text{for} \quad 1 \leqq p, q, r \leqq \ell. \tag{3.18}$$

Case (8) $[(A_\ell, \{\alpha_\ell\}), \varpi_{\ell-1}, (D_{\ell+1}, \{\gamma_\ell, \gamma_{\ell+1}\})](\ell \geqq 3)$.

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(D_{\ell+1}, \{\gamma_{\ell+1}\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(D_{\ell+1}, \{\gamma_\ell, \gamma_{\ell+1}\})$. Hence $\mu = 2$ and $\check{\mathfrak{g}} = \mathfrak{b}$ is isomorphic to $\mathfrak{o}(2\ell + 2)$. First we describe

$$\mathfrak{o}(2\ell + 2) = \{X \in \mathfrak{gl}(2\ell + 2, \mathbb{K}) \mid {}^tXJ + JX = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & I_\ell \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ I_\ell & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2\ell + 2, \mathbb{K}), \quad I_\ell = (\delta_{ij}) \in \mathfrak{gl}(\ell, \mathbb{K}).$$

Here the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-2} = S_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 \end{pmatrix} \middle| \; Y \in \mathfrak{o}(\ell) \right\},$$

$$\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

$$S_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & -{}^t\xi & 0 & 0 \end{pmatrix} \middle| \; \xi \in M(1, \ell) \right\},$$

$$\mathfrak{l}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -{}^tx & 0 \end{pmatrix} \middle| \; x \in \mathbb{K}^\ell = M(1, \ell) \right\},$$

$$\mathfrak{g}_0 = \check{\mathfrak{l}}_0 = \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -{}^tA \end{pmatrix} \middle| \; a \in \mathbb{K}, \; A \in \mathfrak{gl}(\ell, \mathbb{K}) \right\},$$

$$\check{\mathfrak{g}}_k = \{{}^tX \mid X \in \mathfrak{g}_{-k}\}, \quad (k = 1, 2, 3).$$

We have $\dim S_{-2} = \frac{1}{2}\ell(\ell - 1)$, $\dim S_{-1} = \dim \mathfrak{l}_{-1} = \ell$, $\mathfrak{l} = \mathfrak{sl}(T)$ and $S = \bigwedge^2 T^*$. Then

$$\mathfrak{m} = S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1})$$
$$= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ \xi & 0 & 0 & 0 \\ Y & -{}^t\xi & -{}^t x & 0 \end{pmatrix} = \hat{Y} + \check{\xi} + \hat{x} \;\middle|\; x, \xi \in \mathbb{K}^\ell = M(1, \ell),\ Y \in \mathfrak{o}(\ell) \right\}.$$

By calculating $[\check{\xi}, \hat{x}]$, we have

$$y_{pq} = \xi_q x_p - \xi_p x_q, \qquad (y_{pq} + y_{qp} = 0),$$

where $x = (x_1, \ldots, x_\ell), \xi = (\xi_1, \ldots, \xi_\ell)$ and $Y = (y_{pq})$. Then the model equation of our typical symbol $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^1(\mathfrak{l}_{-1}, S_{-2})$ is given by

$$\frac{\partial y_{pq}}{\partial x_r} = 0 \quad \text{for distinct} \quad p, q, r \qquad \frac{\partial y_{pq}}{\partial x_p} + \frac{\partial y_{qr}}{\partial x_r} = 0 \quad \text{for} \quad q \neq p, r, \tag{3.19}$$

where y_{pq} $(1 \leqq p < q \leqq \ell)$ are dependent variables and $x_1, \ldots, x_\ell$ are independent variables. By a direct calculation, we see that the prolongation of the first order system (3.19) is given by

$$\frac{\partial^2 y_{pq}}{\partial x_r \partial x_s} = 0 \qquad \text{for} \quad 1 \leqq p, q, r, s \leqq \ell. \tag{3.20}$$

Case (9) $[(A_\ell, \{\alpha_i\}), \varpi_{\ell-1}, (D_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})]$ $(2 < i < \ell, \ell \geqq 4)$.

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(D_{\ell+1}, \{\gamma_{\ell+1}\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(D_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$. Hence $\mu = 3$ and $\check{\mathfrak{g}} = \mathfrak{b}$ is isomorphic to $\mathfrak{o}(2\ell + 2)$. First we describe

$$\mathfrak{o}(2\ell + 2) = \{X \in \mathfrak{gl}(2\ell + 2, \mathbb{K}) \mid {}^t X J + J X = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & I_i \\ 0 & 0 & I_j & 0 \\ 0 & I_j & 0 & 0 \\ I_i & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2\ell + 2, \mathbb{K}), \quad I_k = (\delta_{pq}) \in \mathfrak{gl}(k, \mathbb{K}).$$

Here the gradation is given again by subdividing matrices as follows;

$$\mathfrak{g}_{-3} = S_{-3} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Y & 0 & 0 & 0 \end{pmatrix} \;\middle|\; Y \in \mathfrak{o}(i) \right\},$$

$$\mathfrak{g}_{-2} = S_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Xi & 0 & 0 & 0 \\ 0 & -{}^t\Xi & 0 & 0 \end{pmatrix} \;\middle|\; \Xi \in M(j, i) \right\},$$

$$\mathfrak{g}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

$$S_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; A \in \mathfrak{o}(j) \right\},$$

$$\mathfrak{l}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -{}^tX & 0 \end{pmatrix} \;\middle|\; X \in M(j, i) \right\},$$

$$\mathfrak{g}_0 = \check{\mathfrak{l}}_0 = \left\{ \begin{pmatrix} F & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & -{}^tG & 0 \\ 0 & 0 & 0 & -{}^tF \end{pmatrix} \;\middle|\; F \in \mathfrak{gl}(i, \mathbb{K}),\ G \in \mathfrak{gl}(j, \mathbb{K}) \right\},$$

$$\check{\mathfrak{g}}_k = \{ {}^tX \mid X \in \mathfrak{g}_{-k} \}, \quad (k = 1, 2, 3).$$

We have $i + j = \ell + 1$, $\dim S_{-3} = \frac{1}{2} i(i-1)$, $\dim S_{-2} = \dim \mathfrak{l}_{-1} = ij$, $\dim S_{-1} = \frac{1}{2} j(j-1)$, $\mathfrak{l} = \mathfrak{sl}(T)$ and $S = \bigwedge^2 T^*$. Then

$$\begin{aligned} \mathfrak{m} &= S_{-3} \oplus S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1}) \\ &= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 \\ \Xi & A & 0 & 0 \\ Y & -{}^t\Xi & -{}^tX & 0 \end{pmatrix} = \hat{Y} + \check{\Xi} + \hat{A} + \hat{X} \;\middle|\; \begin{array}{l} X, \Xi \in M(j, i),\ Y \in \mathfrak{o}(i), \\ A \in \mathfrak{o}(j) \end{array} \right\}. \end{aligned}$$

By calculating $[\check{\Xi}, \hat{X}]$ and $[[\hat{A}, \hat{X}], \hat{X}]$, we have

$$y_{pq} (= -y_{qp}) = \sum_{\alpha=1}^{j} (\xi_q^\alpha x_p^\alpha - \xi_p^\alpha x_q^\alpha) = 2 \sum_{\alpha,\beta=1}^{j} a_{\alpha\beta} x_p^\alpha x_q^\beta,$$

where $Y = (y_{pq})$, $\Xi = (\xi_p^\alpha)$ and $X = (x_p^\alpha)$. From the first equality, we can embed S_{-2} as a subspace of $S_{-3} \otimes (\mathfrak{l}_{-1})^*$ and obtain the following first order system as the model equation whose symbol coincides with this subspace:

$$\frac{\partial y_{pq}}{\partial x_r^\alpha} = 0 \quad \text{for distinct } p, q, r, \qquad \frac{\partial y_{pq}}{\partial x_p^\alpha} + \frac{\partial y_{qr}}{\partial x_r^\alpha} = 0 \quad \text{for } q \neq p, r, \tag{3.21}$$

where y_{pq} $(1 \leqq p < q \leqq i)$ are dependent variables and x_p^α $1 \leqq p \leqq i, 1 \leqq \alpha \leqq j)$ are independent variables. Moreover, by a direct calculation, we see that the prolongation of the first order system (3.21) is given by

$$\frac{\partial^2 y_{pq}}{\partial x_r^\alpha \partial x_s^\beta} = \frac{\partial^2 y_{pq}}{\partial x_p^\alpha \partial x_p^\beta} = 0 \quad \text{for } \{r, s\} \neq \{p, q\},$$

$$\frac{\partial^2 y_{pq}}{\partial x_p^\alpha \partial x_q^\beta} = \frac{\partial^2 y_{rs}}{\partial x_r^\alpha \partial x_s^\beta} \quad \text{for } (p,q) \neq (r,s). \tag{3.22}$$

From the second equality, we observe that the above second order system is the model equation of the embedded subspace S_{-1} in $S_{-3} \otimes S^2((\mathfrak{l}_{-1})^*)$. Furthermore, by a direct calculation, we see that the prolongation of this second order system (3.22) is given by

$$\frac{\partial^3 y_{pq}}{\partial x_r^\alpha \partial x_s^\beta \partial x_t^\gamma} = 0 \quad \text{for } 1 \leqq p,q,r,s,t \leqq i,\ 1 \leqq \alpha,\beta,\gamma \leqq j. \tag{3.23}$$

Case (10) $[(A_\ell, \{\alpha_2\}), \varpi_{\ell-1}, (D_{\ell+1}, \{\gamma_2, \gamma_{\ell+1}\})]$ $(\ell \geqq 3)$.

$\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is described by $(D_{\ell+1}, \{\gamma_{\ell+1}\})$ and $\check{\mathfrak{g}} = \bigoplus_{p=-\mu}^{\mu} \check{\mathfrak{g}}_p$ is described by $(D_{\ell+1}, \{\gamma_2, \gamma_{\ell+1}\})$. Hence $\mu = 3$ and $\check{\mathfrak{g}} = \mathfrak{b}$ is isomorphic to $\mathfrak{o}(2\ell+2)$. First we describe

$$\mathfrak{o}(2\ell+2) = \{X \in \mathfrak{gl}(2\ell+2, \mathbb{K}) \mid {}^tXJ + JX = 0\},$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & I_2 \\ 0 & 0 & I_{\ell-1} & 0 \\ 0 & I_{\ell-1} & 0 & 0 \\ I_2 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2\ell+2, \mathbb{K}), \qquad I_{\ell-1} = (\delta_{ij}) \in \mathfrak{gl}(\ell-1, \mathbb{K}).$$

Hence the gradation is given as in the case (9) with $i = 2$ and $j = \ell - 1$. We have $\dim S_{-3} = 1, \dim S_{-2} = \dim \mathfrak{l}_{-1} = 2(\ell-1), \dim S_{-1} = \frac{1}{2}(\ell-1)(\ell-2), \mathfrak{l} = \mathfrak{sl}(T)$ and $S = \bigwedge^2 T^*$.

Then

$$\begin{aligned} \mathfrak{m} &= S_{-3} \oplus S_{-2} \oplus (S_{-1} \oplus \mathfrak{l}_{-1}) \\ &= \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ X & 0 & 0 & 0 \\ \Xi & A & 0 & 0 \\ Y & -{}^t\Xi & -{}^tX & 0 \end{pmatrix} = \hat{Y} + \check{\Xi} + \hat{A} + \hat{X} \ \middle| \ \begin{array}{l} X, \Xi \in M(\ell-1, 2),\ Y \in \mathfrak{o}(2), \\ A \in \mathfrak{o}(\ell-1) \end{array} \right\}. \end{aligned}$$

By calculating $[\check{\Xi}, \hat{X}]$ and $[[\hat{A}, \hat{X}], \hat{X}]$, we have

$$y = \sum_{\alpha=1}^{\ell-1} (\xi_1^\alpha x_2^\alpha + \xi_2^\alpha x_1^\alpha) = 2 \sum_{\alpha,\beta=1}^{\ell-1} a_{\alpha\beta} x_1^\alpha x_2^\beta,$$

where $Y = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix}$, $A = (a_{\alpha\beta})$ $(a_{\alpha\beta} + a_{\beta\alpha} = 0)$, $\Xi = (\xi_p^\alpha)$ and $X = (x_p^\alpha)$. From the first equality, we have $S_{-2} = V^*$, putting $S_{-3} = \mathbb{K}$ and $\mathfrak{l}_{-1} = V$. Moreover, from the second equality and $a_{\alpha\beta} + a_{\beta\alpha} = 0$, we see that the model equation of our typical symbol $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^2(V, \mathbb{K})$ is given by

$$\frac{\partial^2 y}{\partial x_i^\alpha \partial x_j^\beta} + \frac{\partial^2 y}{\partial x_i^\beta \partial x_j^\alpha} = 0 \qquad \text{for } 1 \leqq i,j \leqq 2,\ 1 \leqq \alpha < \beta \leqq \ell - 1, \tag{3.24}$$

where y is a dependent variable and $x_1^1, \dots, x_1^{\ell-1}, x_2^1, \dots, x_2^{\ell-1}$ are independent variables. By a direct calculation, we see that the prolongation of the second order system (3.24) is given by

$$\frac{\partial^3 y}{\partial x_i^\alpha \partial x_j^\beta \partial x_k^\gamma} = 0 \qquad \text{for} \quad 1 \leq i, j, k \leqq 2 \quad 1 \leqq \alpha, \beta, \gamma \leqq \ell - 1. \tag{3.25}$$

4 Symbol of the exceptional cases

In this section we will describe the symbol algebra $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ explicitly as the subalgebra of $\mathfrak{C}^{\mu-1}(V, W)$, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$, by first describing the structure of $\mathfrak{m} = \bigoplus_{p<0} \mathfrak{g}_p$ explicitly by use of the Chevalley basis of Y_L.

We first recall that the graded Lie algebra $\check{\mathfrak{g}}$ of type (Y_L, Σ_1) is described in terms of the root space decomposition as follows (cf. Section 3 in [Yam93]): Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\check{\mathfrak{g}}$ and choose a simple root system $\Sigma = \{\gamma_1, \dots, \gamma_L\}$ of the root system Φ of $\check{\mathfrak{g}}$ relative to $\mathfrak{h}$. For the subset Σ_1 of Σ, we put

$$\Phi_k^+ = \left\{ \alpha = \sum_{i=1}^{L} n_i(\alpha)\gamma_i \in \Phi^+ \mid \sum_{\gamma_i \in \Sigma_1} n_i(\alpha) = k \right\} \quad \text{for } k \geqq 0,$$

where Φ^+ denotes the set of positive roots. Then the gradation $\check{\mathfrak{g}}$ is given by

$$\check{\mathfrak{g}}_{-k} = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_{-\alpha}, \qquad \check{\mathfrak{g}}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_0^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}), \qquad \check{\mathfrak{g}}_k = \bigoplus_{\alpha \in \Phi_k^+} \mathfrak{g}_\alpha \quad (k > 0),$$

where $\mathfrak{g}_\alpha$ is the root space for $\alpha \in \Phi$.

In the following, let us take a Chevalley basis $\{x_\alpha (\alpha \in \Phi); h_i (1 \leqq i \leqq L)\}$ of $\check{\mathfrak{g}}$ and put $y_\beta = x_{-\beta}$ for $\beta \in \Phi^+$ (cf. Chapter VII [Hum72]). We will describe the structure of the negative part $\mathfrak{m} = \sum_{p<0} \check{\mathfrak{g}}_p$ of (Y_L, Σ_1) in terms of $\{y_\beta\}_{\beta \in \Phi^+}$. For the property of the Chevalley basis, we recall that, for $\alpha, \beta \in \Phi^+$, if $\alpha + \beta \in \Phi$ and $\alpha - \beta \notin \Phi$, then $[y_\alpha, y_\beta] = \pm y_{\alpha+\beta}$ (see Section 25.2 in [Hum72]).

In this section, we shall treat both complex simple graded Lie algebras (Y_L, Σ_1) and their normal real forms at the same time and we shall work in the complex analytic or the real C^∞ category depending on whether we treat complex simple graded Lie algebras (Y_L, Σ_1) or their normal real forms.

Case (1) $[(D_5, \{\alpha_1\}, \varpi_5, (E_6, \{\gamma_1, \gamma_6\})]$.

For the gradation of type $(E_6, \{\gamma_1, \gamma_6\})$, we have

$$\Phi_2^+ = \Big\{ \alpha_{-7} = \begin{smallmatrix} 1&1&1&1&1\\ &&0&& \end{smallmatrix}, \quad \alpha_{-5} = \begin{smallmatrix} 1&1&1&1&1\\ &&1&& \end{smallmatrix}, \quad \alpha_{-3} = \begin{smallmatrix} 1&1&2&1&1\\ &&1&& \end{smallmatrix}, \quad \alpha_{-1} = \begin{smallmatrix} 1&1&2&2&1\\ &&1&& \end{smallmatrix},$$
$$\alpha_1 = \begin{smallmatrix} 1&2&2&1&1\\ &&1&& \end{smallmatrix}, \quad \alpha_3 = \begin{smallmatrix} 1&2&2&2&1\\ &&1&& \end{smallmatrix}, \quad \alpha_5 = \begin{smallmatrix} 1&2&3&2&1\\ &&1&& \end{smallmatrix}, \quad \alpha_7 = \begin{smallmatrix} 1&2&3&2&1\\ &&2&& \end{smallmatrix} \Big\},$$

$$\Phi_1^+ = \Psi^1 \cup \Psi^6$$

$$\Psi^1 = \Big\{ \xi_{-7} = \begin{smallmatrix}1&0&0&0&0\\&&0&&\end{smallmatrix}, \quad \xi_{-5} = \begin{smallmatrix}1&1&0&0&0\\&&0&&\end{smallmatrix}, \quad \xi_{-3} = \begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix}, \quad \xi_{-1} = \begin{smallmatrix}1&1&1&1&0\\&&0&&\end{smallmatrix},$$
$$\xi_{1} = \begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix}, \quad \xi_{3} = \begin{smallmatrix}1&1&1&1&0\\&&1&&\end{smallmatrix}, \quad \xi_{5} = \begin{smallmatrix}1&1&2&1&0\\&&1&&\end{smallmatrix}, \quad \xi_{7} = \begin{smallmatrix}1&2&2&1&0\\&&1&&\end{smallmatrix} \Big\},$$

$$\Psi^6 = \Big\{ \eta_{-7} = \begin{smallmatrix}0&0&0&0&1\\&&0&&\end{smallmatrix}, \quad \eta_{-5} = \begin{smallmatrix}0&0&0&1&1\\&&0&&\end{smallmatrix}, \quad \eta_{-3} = \begin{smallmatrix}0&0&1&1&1\\&&0&&\end{smallmatrix}, \quad \eta_{-1} = \begin{smallmatrix}0&1&1&1&1\\&&0&&\end{smallmatrix},$$
$$\eta_{1} = \begin{smallmatrix}0&0&1&1&1\\&&1&&\end{smallmatrix}, \quad \eta_{3} = \begin{smallmatrix}0&1&1&1&1\\&&1&&\end{smallmatrix}, \quad \eta_{5} = \begin{smallmatrix}0&1&2&1&1\\&&1&&\end{smallmatrix}, \quad \eta_{7} = \begin{smallmatrix}0&1&2&2&1\\&&1&&\end{smallmatrix} \Big\},$$

where $\begin{smallmatrix}a_1&a_3&a_4&a_5&a_6\\&&a_2&&\end{smallmatrix}$ stands for the root $\alpha = \sum_{i=1}^{6} a_i \gamma_i \in \Phi^+$ (see Planche V in [Bou68]).

Thus we have $\mu = 2$,

$$\mathfrak{m} = \check{\mathfrak{g}}_{-2} \oplus \check{\mathfrak{g}}_{-1} \qquad \text{and} \qquad \check{\mathfrak{g}}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

where $\check{\mathfrak{g}}_{-2} = S_{-2}$, S_{-1} and $\mathfrak{l}_{-1}$ are spanned by the root spaces $\mathfrak{g}_{-\beta}$ for $\beta \in \Phi_2^+, \Psi^6$ and Ψ^1 respectively. Hence $\dim S_{-2} = \dim S_{-1} = \dim \mathfrak{l}_{-1} = 8$.

For Φ_2^+, Ψ^1 and Ψ^6, we observe that $\alpha + \beta \notin \Phi$ for $\alpha, \beta \in \Phi_2^+ \cup \Psi^1$ or for $\alpha, \beta \in \Phi_2^+ \cup \Psi^6$ and that $\eta - \xi \notin \Phi$ for $\eta \in \Psi^6, \xi \in \Psi^1$. This implies that $[y_\alpha, y_\beta] = 0$ for $\alpha, \beta \in \Phi_2^+ \cup \Psi^1$ or for $\alpha, \beta \in \Phi_2^+ \cup \Psi^6$ and that $[y_\eta, y_\xi] = \pm y_{\eta+\xi}$ for $\eta \in \Psi^6, \xi \in \Psi^1$, if $\eta + \xi \in \Phi$, by the above mentioned property of the Chevalley basis. Hence, from Planche V in [Bou68], we readily obtain the non-trivial bracket relation among $\check{\mathfrak{g}}_{-1}$ as in (4.1) below up to signs.

We solve the problem of signs as follows. First we choose the orientation of y_β for $\beta \in \Psi^1, \Psi^6$ and Φ_2^+ as in the following: We choose the orientation of y_{γ_i} for simple roots by fixing the root vectors $y_i = y_{\gamma_i} \in \mathfrak{g}_{-\gamma_i}$. For $\xi \in \Psi^1$, we fix the orientation by the following order;

$$\begin{aligned} &y_{\xi_{-7}} = y_1, && y_{\xi_{-5}} = [y_3, y_{\xi_{-7}}], && y_{\xi_{-3}} = [y_4, y_{\xi_{-5}}], && y_{\xi_{-1}} = [y_5, y_{\xi_{-3}}], \\ &y_{\xi_1} = [y_2, y_{\xi_{-3}}], && y_{\xi_3} = [y_5, y_{\xi_1}], && y_{\xi_5} = [y_4, y_{\xi_3}], && y_{\xi_7} = [y_3, y_{\xi_5}]. \end{aligned}$$

For $\eta \in \Psi^6$, we fix the orientation by the following order;

$$\begin{aligned} &y_{\eta_{-7}} = y_6, && y_{\eta_{-5}} = [y_5, y_{\eta_{-7}}], && y_{\eta_{-3}} = [y_4, y_{\eta_{-5}}], && y_{\eta_{-1}} = [y_3, y_{\eta_{-3}}], \\ &y_{\eta_1} = [y_2, y_{\eta_{-3}}], && y_{\eta_3} = [y_3, y_{\eta_1}], && y_{\eta_5} = [y_4, y_{\eta_3}], && y_{\eta_7} = [y_5, y_{\eta_5}]. \end{aligned}$$

Finally, for $\alpha \in \Phi_2^+$, we fix the orientation by the following;

$$\begin{aligned} &y_{\alpha_{-7}} = [y_{\eta_{-1}}, y_{\xi_{-7}}], && y_{\alpha_{-5}} = [y_{\eta_3}, y_{\xi_{-7}}], && y_{\alpha_{-3}} = [y_{\eta_5}, y_{\xi_{-7}}], && y_{\alpha_{-1}} = [y_{\eta_7}, y_{\xi_{-7}}], \\ &y_{\alpha_1} = [y_{\eta_5}, y_{\xi_{-5}}], && y_{\alpha_3} = [y_{\eta_7}, y_{\xi_{-5}}], && y_{\alpha_5} = [y_{\eta_7}, y_{\xi_{-3}}], && y_{\alpha_7} = [y_{\eta_7}, y_{\xi_1}]. \end{aligned}$$

Then, for example, we calculate

$$[y_{\eta_{-1}}, y_{\xi_{-7}}] = [[y_3, y_{\eta_{-3}}], y_{\xi_{-7}}] = [[y_3, y_{\xi_{-7}}], y_{\eta_{-3}}] + [y_3, [y_{\eta_{-3}}, y_{\xi_{-7}}]] = [y_{\xi_{-5}}, y_{\eta_{-3}}].$$

In the same way, by the repeated application of Jacobi identities, we obtain

$$
\begin{aligned}
y_{\alpha_{-7}} &= [y_{\eta_{-1}}, y_{\xi_{-7}}] = -[y_{\eta_{-3}}, y_{\xi_{-5}}] = [y_{\eta_{-5}}, y_{\xi_{-3}}] = -[y_{\eta_{-7}}, y_{\xi_{-1}}],\\
y_{\alpha_{-5}} &= [y_{\eta_3}, y_{\xi_{-7}}] = -[y_{\eta_1}, y_{\xi_{-5}}] = [y_{\eta_{-5}}, y_{\xi_1}] = -[y_{\eta_{-7}}, y_{\xi_3}],\\
y_{\alpha_{-3}} &= [y_{\eta_5}, y_{\xi_{-7}}] = -[y_{\eta_1}, y_{\xi_{-3}}] = [y_{\eta_{-3}}, y_{\xi_1}] = -[y_{\eta_{-7}}, y_{\xi_5}],\\
y_{\alpha_{-1}} &= [y_{\eta_7}, y_{\xi_{-7}}] = -[y_{\eta_1}, y_{\xi_{-1}}] = [y_{\eta_{-3}}, y_{\xi_3}] = -[y_{\eta_{-5}}, y_{\xi_5}],\\
y_{\alpha_1} &= [y_{\eta_5}, y_{\xi_{-5}}] = -[y_{\eta_3}, y_{\xi_{-3}}] = [y_{\eta_{-1}}, y_{\xi_1}] = -[y_{\eta_{-7}}, y_{\xi_7}],\\
y_{\alpha_3} &= [y_{\eta_7}, y_{\xi_{-5}}] = -[y_{\eta_3}, y_{\xi_{-1}}] = [y_{\eta_{-1}}, y_{\xi_3}] = -[y_{\eta_{-5}}, y_{\xi_7}],\\
y_{\alpha_5} &= [y_{\eta_7}, y_{\xi_{-3}}] = -[y_{\eta_5}, y_{\xi_{-1}}] = [y_{\eta_{-1}}, y_{\xi_5}] = -[y_{\eta_{-3}}, y_{\xi_7}],\\
y_{\alpha_7} &= [y_{\eta_7}, y_{\xi_1}] = -[y_{\eta_5}, y_{\xi_3}] = [y_{\eta_3}, y_{\xi_5}] = -[y_{\eta_1}, y_{\xi_7}].
\end{aligned}
\tag{4.1}
$$

Thus, by fixing the basis $\{y_{\alpha_i}\}$ of S_{-2} and $\{y_{\xi_j}\}$ of $\mathfrak{l}_{-1}$, an element $A = a_1\,\mathrm{ad}(y_{\eta_{-7}}) + a_2\,\mathrm{ad}(y_{\eta_{-5}}) + a_3\,\mathrm{ad}(y_{\eta_{-3}}) + a_4\,\mathrm{ad}(y_{\eta_{-1}}) + a_5\,\mathrm{ad}(y_{\eta_1}) + a_6\,\mathrm{ad}(y_{\eta_3}) + a_7\,\mathrm{ad}(y_{\eta_5}) + a_8\,\mathrm{ad}(y_{\eta_7}) \in S_{-1} \subset S_{-2} \otimes (\mathfrak{l}_{-1})^* \cong M(8,8)$ is represented as the matrix of the following form;

$$
\begin{pmatrix}
a_4 & -a_3 & a_2 & -a_1 & 0 & 0 & 0 & 0\\
a_6 & -a_5 & 0 & 0 & a_2 & -a_1 & 0 & 0\\
a_7 & 0 & -a_5 & 0 & a_3 & 0 & -a_1 & 0\\
a_8 & 0 & 0 & -a_5 & 0 & a_3 & -a_2 & 0\\
0 & a_7 & -a_6 & 0 & a_4 & 0 & 0 & -a_1\\
0 & a_8 & 0 & -a_6 & 0 & a_4 & 0 & -a_2\\
0 & 0 & a_8 & -a_7 & 0 & 0 & a_4 & -a_3\\
0 & 0 & 0 & 0 & a_8 & -a_7 & a_6 & -a_5
\end{pmatrix}
$$

Hence the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case is given by

$$D_{\mathfrak{m}} = \{\varpi_1 = \varpi_2 = \cdots = \varpi_8 = 0\},$$

where

$$
\begin{aligned}
\varpi_1 &= dy_1 + p_4dx_1 - p_3dx_2 + p_2dx_3 - p_1dx_4,\\
\varpi_2 &= dy_2 + p_6dx_1 - p_5dx_2 + p_2dx_5 - p_1dx_6,\\
\varpi_3 &= dy_3 + p_7dx_1 - p_5dx_3 + p_3dx_5 - p_1dx_7,\\
\varpi_4 &= dy_4 + p_8dx_1 - p_5dx_4 + p_3dx_6 - p_2dx_7,\\
\varpi_5 &= dy_5 + p_7dx_2 - p_6dx_3 + p_4dx_5 - p_1dx_8,\\
\varpi_6 &= dy_6 + p_8dx_2 - p_6dx_4 + p_4dx_6 - p_2dx_8,\\
\varpi_7 &= dy_7 + p_8dx_3 - p_7dx_4 + p_4dx_7 - p_3dx_8,\\
\varpi_8 &= dy_8 + p_8dx_5 - p_7dx_6 + p_6dx_7 - p_5dx_8.
\end{aligned}
$$

Here $(y_1, \ldots, y_8, x_1, \ldots, x_8, p_1, \ldots, p_8)$ is a coordinate system of $M(\mathfrak{m}) \cong \mathbb{K}^{24}$. Thus the model equation of our typical symbol $\mathfrak{m} = \check{\mathfrak{g}}_{-2} \oplus \check{\mathfrak{g}}_{-1} \subset \mathfrak{C}^1(\mathfrak{l}_{-1}, S_{-2})$ is given by

$$\frac{\partial y_1}{\partial x_4} = \frac{\partial y_2}{\partial x_6} = \frac{\partial y_3}{\partial x_7} = \frac{\partial y_5}{\partial x_8}, \qquad \frac{\partial y_1}{\partial x_3} = \frac{\partial y_2}{\partial x_5} = -\frac{\partial y_4}{\partial x_7} = -\frac{\partial y_6}{\partial x_8},$$

$$
\begin{aligned}
&\frac{\partial y_1}{\partial x_2} = -\frac{\partial y_3}{\partial x_5} = -\frac{\partial y_4}{\partial x_6} = \frac{\partial y_7}{\partial x_8}, &\qquad& \frac{\partial y_1}{\partial x_1} = \frac{\partial y_5}{\partial x_5} = \frac{\partial y_6}{\partial x_6} = \frac{\partial y_7}{\partial x_7},\\
&\frac{\partial y_2}{\partial x_2} = \frac{\partial y_3}{\partial x_3} = \frac{\partial y_4}{\partial x_4} = \frac{\partial y_8}{\partial x_8}, && \frac{\partial y_2}{\partial x_1} = -\frac{\partial y_5}{\partial x_3} = -\frac{\partial y_6}{\partial x_4} = \frac{\partial y_8}{\partial x_7},\\
&\frac{\partial y_3}{\partial x_1} = \frac{\partial y_5}{\partial x_2} = -\frac{\partial y_7}{\partial x_4} = -\frac{\partial y_8}{\partial x_6}, && \frac{\partial y_4}{\partial x_1} = \frac{\partial y_6}{\partial x_2} = \frac{\partial y_7}{\partial x_3} = \frac{\partial y_8}{\partial x_5},\\
&\frac{\partial y_i}{\partial x_j} = 0 \qquad \text{otherwise,}
\end{aligned}
\tag{4.2}
$$

where $y_1, \dots, y_8$ are dependent variables and $x_1, \dots, x_8$ are independent variables. By a direct calculation, we see that the prolongation of the first order system (4.2) is given by

$$
\frac{\partial^2 y_i}{\partial x_j \partial x_k} = 0 \qquad \text{for} \quad 1 \leqq i, j, k \leqq 8. \tag{4.3}
$$

Case (2) $[(D_5, \{\alpha_5\}), \varpi_5, (E_6, \{\gamma_3, \gamma_1\})]$.

For the gradation of type $(E_6, \{\gamma_3, \gamma_1\})$, we have

$$
\begin{aligned}
\Phi_3^+ = \Big\{ &\alpha_1 = \begin{smallmatrix}1&2&2&1&0\\&&1&&\end{smallmatrix}, \ \alpha_2 = \begin{smallmatrix}1&2&2&1&1\\&&1&&\end{smallmatrix}, \ \alpha_3 = \begin{smallmatrix}1&2&2&2&1\\&&1&&\end{smallmatrix}, \ \alpha_4 = \begin{smallmatrix}1&2&3&2&1\\&&1&&\end{smallmatrix},\\
&\alpha_5 = \begin{smallmatrix}1&2&3&2&1\\&&2&&\end{smallmatrix} \Big\},
\end{aligned}
$$

$$
\begin{aligned}
\Phi_2^+ = \Big\{ &\eta_1 = \begin{smallmatrix}1&1&0&0&0\\&&0&&\end{smallmatrix}, \ \eta_2 = \begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix}, \ \eta_3 = \begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix}, \ \eta_4 = \begin{smallmatrix}1&1&1&1&0\\&&0&&\end{smallmatrix},\\
&\eta_5 = \begin{smallmatrix}1&1&1&1&0\\&&1&&\end{smallmatrix}, \ \eta_6 = \begin{smallmatrix}1&1&1&1&1\\&&0&&\end{smallmatrix}, \ \eta_7 = \begin{smallmatrix}1&1&1&1&1\\&&1&&\end{smallmatrix}, \ \eta_8 = \begin{smallmatrix}1&1&2&1&0\\&&1&&\end{smallmatrix},\\
&\eta_9 = \begin{smallmatrix}1&1&2&1&1\\&&1&&\end{smallmatrix}, \ \eta_{10} = \begin{smallmatrix}1&1&2&2&1\\&&1&&\end{smallmatrix} \Big\},
\end{aligned}
$$

$$
\Phi_1^+ = \Psi^1 \cup \Psi^3,
$$

$$
\begin{aligned}
\Psi^3 = \Big\{ &\xi_1 = \begin{smallmatrix}0&1&0&0&0\\&&0&&\end{smallmatrix}, \ \xi_2 = \begin{smallmatrix}0&1&1&0&0\\&&0&&\end{smallmatrix}, \ \xi_3 = \begin{smallmatrix}0&1&1&0&0\\&&1&&\end{smallmatrix}, \ \xi_4 = \begin{smallmatrix}0&1&1&1&0\\&&0&&\end{smallmatrix},\\
&\xi_5 = \begin{smallmatrix}0&1&1&1&0\\&&1&&\end{smallmatrix}, \ \xi_6 = \begin{smallmatrix}0&1&1&1&1\\&&0&&\end{smallmatrix}, \ \xi_7 = \begin{smallmatrix}0&1&1&1&1\\&&1&&\end{smallmatrix}, \ \xi_8 = \begin{smallmatrix}0&1&2&1&0\\&&1&&\end{smallmatrix},\\
&\xi_9 = \begin{smallmatrix}0&1&2&1&1\\&&1&&\end{smallmatrix}, \ \xi_{10} = \begin{smallmatrix}0&1&2&2&1\\&&1&&\end{smallmatrix} \Big\},
\end{aligned}
$$

$$
\Psi^1 = \Big\{ \gamma_1 = \begin{smallmatrix}1&0&0&0&0\\&&0&&\end{smallmatrix} \Big\}
$$

where $\begin{smallmatrix}a_1&a_3&a_4&a_5&a_6\\&&a_2&&\end{smallmatrix}$ stands for the root $\alpha = \sum_{i=1}^{6} a_i \gamma_i \in \Phi^+$ (see Planche V in [Bou68]).

Thus we have $\mu = 3$,

$$
\mathfrak{m} = \check{\mathfrak{g}}_{-3} \oplus \check{\mathfrak{g}}_{-2} \oplus \check{\mathfrak{g}}_{-1} \qquad \text{and} \qquad \check{\mathfrak{g}}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},
$$

where $\check{\mathfrak{g}}_{-3} = S_{-3}$, $\check{\mathfrak{g}}_{-2} = S_{-2}$, S_{-1} and $\mathfrak{l}_{-1}$ are spanned by the root spaces $\mathfrak{g}_{-\beta}$ for $\beta \in \Phi_3^+, \Phi_2^+, \Psi^1$ and Ψ^3 respectively. Hence $\dim S_{-3} = 5$, $\dim S_{-2} = \dim \mathfrak{l}_{-1} = 10$ and $\dim S_{-1} = 1$.

For $\Phi_3^+, \Phi_2^+, \Psi^1$ and Ψ^3, we observe that $\alpha + \beta \notin \Phi$ for $\alpha, \beta \in \Phi_3^+ \cup \Phi_2^+ \cup \Psi^1$, $\xi - \gamma \notin \Phi$ for $\xi \in \Psi^3, \gamma \in \Psi^1$ and that, if $\eta + \xi \in \Phi_3^+$ for $\eta \in \Phi_2^+, \xi \in \Psi^3$, then $\eta - \xi \notin \Phi$. This implies that $[y_\alpha, y_\beta] = 0$ for $\alpha, \beta \in \Phi_3^+ \cup \Phi_2^+ \cup \Psi^1$, $[y_\gamma, y_\xi] = \pm y_{\gamma+\xi}$ for $\xi \in \Psi^3, \gamma \in \Psi^1$, if $\gamma + \xi \in \Phi$ and $[y_\eta, y_\xi] = \pm y_{\eta+\xi}$ for $\eta \in \Phi_2^+, \xi \in \Psi^3$, if $\eta + \xi \in \Phi$, by the property of the Chevalley basis. Hence, from Planche V in [Bou68], we readily obtain the non-trivial bracket relation among $\check{\mathfrak{g}}_{-1}$ and $[\check{\mathfrak{g}}_{-2}, \mathfrak{l}_{-1}]$ as in (4.4) and (4.5) below up to signs.

We fix the signs of y_β for $\beta \in \Phi_3^+, \Phi_2^+, \Psi^3$ and Ψ^1 as follows: First we choose the orientation of y_{γ_i} for simple roots by fixing the root vectors $y_i = y_{\gamma_i} \in \mathfrak{g}_{-\gamma_i}$. For $\xi \in \Psi^3$, we fix the orientation by the following order;

$$\begin{aligned} &y_{\xi_1} = y_3, && y_{\xi_2} = [y_4, y_{\xi_1}], && y_{\xi_3} = [y_2, y_{\xi_2}], && y_{\xi_4} = [y_5, y_{\xi_2}],\\ &y_{\xi_5} = [y_5, y_{\xi_3}], && y_{\xi_6} = [y_6, y_{\xi_4}], && y_{\xi_7} = [y_2, y_{\xi_6}], && y_{\xi_8} = [y_4, y_{\xi_5}],\\ &y_{\xi_9} = [y_6, y_{\xi_8}], && y_{\xi_{10}} = [y_5, y_{\xi_9}]. \end{aligned}$$

For $\eta \in \Phi_2^+$, we fix the orientation by the following ;

$$y_{\eta_i} = [y_1, y_{\xi_i}] \qquad \text{for } i = 1, 2, \ldots, 10. \tag{4.4}$$

Finally, for $\alpha \in \Phi_3^+$, we fix the orientation by the following;

$$\begin{aligned} &y_{\alpha_1} = [y_{\eta_8}, y_{\xi_1}], && y_{\alpha_2} = [y_{\eta_9}, y_{\xi_1}], && y_{\alpha_3} = [y_{\eta_{10}}, y_{\xi_1}],\\ &y_{\alpha_4} = [y_{\eta_{10}}, y_{\xi_2}], && y_{\alpha_5} = [y_{\eta_{10}}, y_{\xi_3}]. \end{aligned}$$

Then, for example, we calculate

$$[y_{\eta_p}, y_{\xi_q}] = [[y_1, y_{\xi_p}], y_{\xi_q}] = [[y_1, y_{\xi_q}], y_{\xi_p}] = [y_{\eta_q}, y_{\xi_p}] \qquad \text{for } 1 \leqq p, q \leqq 10,$$

and

$$\begin{aligned} [y_{\eta_5}, y_{\xi_2}] &= [y_{\eta_5}, [y_4, y_{\xi_1}]] = [[y_{\eta_5}, y_4], y_{\xi_1}] = [[[y_1, y_{\xi_5}], y_4], y_{\xi_1}]\\ &= [[y_1, [y_{\xi_5}, y_4]], y_{\xi_1}] = -[[y_1, y_{\xi_8}], y_{\xi_1}] = -[y_{\eta_8}, y_{\xi_1}]. \end{aligned}$$

In the same way, by the repeated application of Jacobi identities, we obtain

$$\begin{aligned} y_{\alpha_1} &= [y_{\eta_8}, y_{\xi_1}] = -[y_{\eta_5}, y_{\xi_2}] = [y_{\eta_4}, y_{\xi_3}] = [y_{\eta_3}, y_{\xi_4}] = -[y_{\eta_2}, y_{\xi_5}] = [y_{\eta_1}, y_{\xi_8}],\\ y_{\alpha_2} &= [y_{\eta_9}, y_{\xi_1}] = -[y_{\eta_7}, y_{\xi_2}] = [y_{\eta_6}, y_{\xi_3}] = [y_{\eta_3}, y_{\xi_6}] = -[y_{\eta_2}, y_{\xi_7}] = [y_{\eta_1}, y_{\xi_9}],\\ y_{\alpha_3} &= [y_{\eta_{10}}, y_{\xi_1}] = -[y_{\eta_7}, y_{\xi_4}] = [y_{\eta_6}, y_{\xi_5}] = [y_{\eta_5}, y_{\xi_6}] = -[y_{\eta_4}, y_{\xi_7}] = [y_{\eta_1}, y_{\xi_{10}}],\\ y_{\alpha_4} &= [y_{\eta_{10}}, y_{\xi_2}] = -[y_{\eta_9}, y_{\xi_4}] = [y_{\eta_8}, y_{\xi_6}] = [y_{\eta_6}, y_{\xi_8}] = -[y_{\eta_4}, y_{\xi_9}] = [y_{\eta_2}, y_{\xi_{10}}],\\ y_{\alpha_5} &= [y_{\eta_{10}}, y_{\xi_3}] = -[y_{\eta_9}, y_{\xi_5}] = [y_{\eta_8}, y_{\xi_7}] = [y_{\eta_7}, y_{\xi_8}] = -[y_{\eta_5}, y_{\xi_9}] = [y_{\eta_3}, y_{\xi_{10}}]. \end{aligned} \tag{4.5}$$

Then, by fixing the basis $\{y_{\alpha_i}\}_{i=1}^{5}$ of S_{-3} and $\{y_{\xi_j}\}_{j=1}^{10}$ of $\mathfrak{l}_{-1}$, an element $A = \sum_{j=1}^{10} a_j\,\mathrm{ad}(y_{\eta_j}) \in S_{-2} \subset S_{-3} \otimes (\mathfrak{l}_{-1})^* \cong M(5, 10)$ is represented as the matrix of the following form;

$$\begin{pmatrix} a_8 & -a_5 & a_4 & a_3 & -a_2 & 0 & 0 & a_1 & 0 & 0 \\ a_9 & -a_7 & a_6 & 0 & 0 & a_3 & -a_2 & 0 & a_1 & 0 \\ a_{10} & 0 & 0 & -a_7 & a_6 & a_5 & -a_4 & 0 & 0 & a_1 \\ 0 & a_{10} & 0 & -a_9 & 0 & a_8 & 0 & a_6 & -a_4 & a_2 \\ 0 & 0 & a_{10} & 0 & -a_9 & 0 & a_8 & a_7 & -a_5 & a_3 \end{pmatrix}$$

Moreover, for $y_1 \in S_{-1}$, we have

$$\begin{aligned} y_{\alpha_1} &= [[y_1, y_{\xi_1}], y_{\xi_8}] = -[[y_1.y_{\xi_2}], y_{\xi_5}] = [[y_1, y_{\xi_3}], y_{\xi_4}], \\ y_{\alpha_2} &= [[y_1, y_{\xi_1}], y_{\xi_9}] = -[[y_1, y_{\xi_2}], y_{\xi_7}] = [[y_1, y_{\xi_3}], y_{\xi_6}], \\ y_{\alpha_3} &= [[y_1, y_{\xi_1}], y_{\xi_{10}}] = -[[y_1, y_{\xi_4}], y_{\xi_7}] = [[y_1, y_{\xi_5}], y_{\xi_6}], \\ y_{\alpha_4} &= [[y_1, y_{\xi_2}], y_{\xi_{10}}] = -[[y_1, y_{\xi_4}], y_{\xi_9}] = [[y_1, y_{\xi_6}], y_{\xi_8}], \\ y_{\alpha_5} &= [[y_1, y_{\xi_3}], y_{\xi_{10}}] = -[[y_1, y_{\xi_5}], y_{\xi_9}] = [[y_1, y_{\xi_7}], y_{\xi_8}]. \end{aligned}$$

Thus S_{-1} is embedded as the 1-dimensional subspace of $S_{-3} \otimes S^2((\mathfrak{l}_{-1})^*)$ spanned by the quadratic form f,

$$\begin{aligned} f(X, X) = {} & (x_1x_8 - x_2x_5 + x_3x_4)y_{\alpha_1} + (x_1x_9 - x_2x_7 + x_3x_6)y_{\alpha_2} \\ & + (x_1x_{10} - x_4x_7 + x_5x_6)y_{\alpha_3} + (x_2x_{10} - x_4x_9 + x_6x_8)y_{\alpha_4} \\ & + (x_3x_{10} - x_5x_9 + x_7x_8)y_{\alpha_5} \end{aligned}$$

for $X = \sum_{j=1}^{10} x_j y_{\xi_j} \in \mathfrak{l}_{-1}$.

By the above matrix representation, we can embed S_{-2} as a subspace of $S_{-3} \otimes (\mathfrak{l}_{-1})^* \cong M(5, 10)$ and obtain the following first order system as the model equation whose symbol coincides with this subspace:

$$\begin{aligned} &\frac{\partial y_1}{\partial x_8} = \frac{\partial y_2}{\partial x_9} = \frac{\partial y_3}{\partial x_{10}}, && -\frac{\partial y_1}{\partial x_5} = -\frac{\partial y_2}{\partial x_7} = \frac{\partial y_4}{\partial x_{10}}, && \frac{\partial y_1}{\partial x_4} = \frac{\partial y_2}{\partial x_6} = \frac{\partial y_5}{\partial x_{10}}, \\ &\frac{\partial y_1}{\partial x_3} = -\frac{\partial y_3}{\partial x_7} = -\frac{\partial y_4}{\partial x_9}, && -\frac{\partial y_1}{\partial x_2} = \frac{\partial y_3}{\partial x_6} = -\frac{\partial y_5}{\partial x_9}, && \frac{\partial y_2}{\partial x_3} = \frac{\partial y_3}{\partial x_5} = \frac{\partial y_4}{\partial x_8}, \\ &-\frac{\partial y_2}{\partial x_2} = -\frac{\partial y_3}{\partial x_4} = \frac{\partial y_5}{\partial x_8}, && \frac{\partial y_1}{\partial x_1} = \frac{\partial y_4}{\partial x_6} = \frac{\partial y_5}{\partial x_7}, && \frac{\partial y_2}{\partial x_1} = -\frac{\partial y_4}{\partial x_4} = -\frac{\partial y_5}{\partial x_5}, \\ &\frac{\partial y_3}{\partial x_1} = \frac{\partial y_4}{\partial x_2} = \frac{\partial y_5}{\partial x_3}, && \frac{\partial y_i}{\partial x_j} = 0 \quad \text{otherwise}, && \end{aligned} \tag{4.6}$$

where $y_1, \ldots, y_5$ are dependent variables and $x_1, \ldots, x_{10}$ are independent variables. Moreover, by a direct calculation, we see that the prolongation of the first order system (4.6) is given by

$$\frac{\partial^2 y_1}{\partial x_1 \partial x_8} = \frac{\partial^2 y_2}{\partial x_1 \partial x_9} = \frac{\partial^2 y_3}{\partial x_1 \partial x_{10}} = \frac{\partial^2 y_4}{\partial x_{10} \partial x_2} = \frac{\partial^2 y_5}{\partial x_{10} \partial x_3}$$

$$= \frac{\partial^2 y_2}{\partial x_3\partial x_6} = \frac{\partial^2 y_1}{\partial x_3 \partial x_4} = -\frac{\partial^2 y_3}{\partial x_4\partial x_7} = -\frac{\partial^2 y_4}{\partial x_4\partial x_9} = -\frac{\partial^2 y_5}{\partial x_9 \partial x_5}$$

$$= \frac{\partial^2 y_3}{\partial x_5\partial x_6} = -\frac{\partial^2 y_1}{\partial x_5\partial x_2} = -\frac{\partial^2 y_2}{\partial x_2\partial x_7} = \frac{\partial^2 y_5}{\partial x_7\partial x_8} = \frac{\partial^2 y_4}{\partial x_8\partial x_6},$$

$$\frac{\partial^2 y_i}{\partial x_p \partial x_q} = 0 \qquad \text{otherwise.} \tag{4.7}$$

From the above expression of f, we observe that the above second order system is the model equation of the 1-dimensional embedded subspace S_{-1} in $S_{-3} \otimes S^2((\mathfrak{l}_{-1})^*)$. Furthermore, by a direct calculation, we see that the prolo ngation of this second order system (4.7) is given by

$$\frac{\partial^3 y_i}{\partial x_p \partial x_q \partial x_r} = 0 \qquad \text{for} \quad 1 \leqq i \leqq 5, \quad 1 \leqq p, q, r \leqq 10. \tag{4.8}$$

Case (3) $[(D_5, \{\alpha_4\}), \varpi_5, (E_6, \{\gamma_2, \gamma_1\})]$. For the gradation of type $(E_6, \{\gamma_2, \gamma_1\})$, we have

$$\Phi_3^+ = \left\{\theta = \begin{smallmatrix}1&2&3&2&1\\&&2&&\end{smallmatrix}\right\},$$

$$\Phi_2^+ = \Big\{\eta_1 = \begin{smallmatrix}1&2&3&2&1\\&&1&&\end{smallmatrix},\ \eta_2 = \begin{smallmatrix}1&2&2&2&1\\&&1&&\end{smallmatrix},\ \eta_3 = \begin{smallmatrix}1&1&2&2&1\\&&1&&\end{smallmatrix},\ \eta_4 = \begin{smallmatrix}1&2&2&1&1\\&&1&&\end{smallmatrix},$$
$$\eta_5 = \begin{smallmatrix}1&1&2&1&1\\&&1&&\end{smallmatrix},\ \eta_6 = \begin{smallmatrix}1&2&2&1&0\\&&1&&\end{smallmatrix},\ \eta_7 = \begin{smallmatrix}1&1&2&1&0\\&&1&&\end{smallmatrix},\ \eta_8 = \begin{smallmatrix}1&1&1&1&1\\&&1&&\end{smallmatrix},$$
$$\eta_9 = \begin{smallmatrix}1&1&1&1&0\\&&1&&\end{smallmatrix},\ \eta_{10} = \begin{smallmatrix}1&1&1&0&0\\&&1&&\end{smallmatrix}\Big\},$$

$$\Phi_1^+ = \Psi^2 \cup \Psi^1,$$

$$\Psi^2 = \Big\{\xi_1 = \begin{smallmatrix}0&0&0&0&0\\&&1&&\end{smallmatrix},\ \xi_2 = \begin{smallmatrix}0&0&1&0&0\\&&1&&\end{smallmatrix},\ \xi_3 = \begin{smallmatrix}0&1&1&0&0\\&&1&&\end{smallmatrix},\ \xi_4 = \begin{smallmatrix}0&0&1&1&0\\&&1&&\end{smallmatrix},$$
$$\xi_5 = \begin{smallmatrix}0&1&1&1&0\\&&1&&\end{smallmatrix},\ \xi_6 = \begin{smallmatrix}0&0&1&1&1\\&&1&&\end{smallmatrix},\ \xi_7 = \begin{smallmatrix}0&1&1&1&1\\&&1&&\end{smallmatrix},\ \xi_8 = \begin{smallmatrix}0&1&2&1&0\\&&1&&\end{smallmatrix},$$
$$\xi_9 = \begin{smallmatrix}0&1&2&1&1\\&&1&&\end{smallmatrix},\ \xi_{10} = \begin{smallmatrix}0&1&2&2&1\\&&1&&\end{smallmatrix}\Big\},$$

$$\Psi^1 = \Big\{\zeta_1 = \begin{smallmatrix}1&0&0&0&0\\&&0&&\end{smallmatrix},\ \zeta_2 = \begin{smallmatrix}1&1&0&0&0\\&&0&&\end{smallmatrix},\ \zeta_3 = \begin{smallmatrix}1&1&1&0&0\\&&0&&\end{smallmatrix},\ \zeta_4 = \begin{smallmatrix}1&1&1&1&0\\&&0&&\end{smallmatrix},$$
$$\zeta_5 = \begin{smallmatrix}1&1&1&1&1\\&&0&&\end{smallmatrix}\Big\}.$$

where $\begin{smallmatrix}a_1&a_3&a_4&a_5&a_6\\&&a_2&&\end{smallmatrix}$ stands for the root $\alpha = \sum_{i=1}^{6} a_i\gamma_i \in \Phi^+$ (see Planche V in [Bou68]).

Thus we have $\mu = 3$,

$$\mathfrak{m} = \check{\mathfrak{g}}_{-3} \oplus \check{\mathfrak{g}}_{-2} \oplus \check{\mathfrak{g}}_{-1} \qquad \text{and} \qquad \check{\mathfrak{g}}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

where $\check{\mathfrak{g}}_{-3} = S_{-3}$, $\check{\mathfrak{g}}_{-2} = S_{-2}$, S_{-1} and $\mathfrak{l}_{-1}$ are spanned by the root spaces $\mathfrak{g}_{-\beta}$ for $\beta \in \Phi_3^+, \Phi_2^+, \Psi^1$ and Ψ^2 respectively. Hence $\dim S_{-3} = 1$, $\dim S_{-2} = \dim \mathfrak{l}_{-1} = 10$ and $\dim S_{-1} = 5$.

For $\Phi_3^+, \Phi_2^+, \Psi^1$ and Ψ^2, we observe that $\Phi_3^+ = \{\theta\}$, where θ is the highest root, $\alpha+\beta \notin \Phi$ for $\alpha, \beta \in \Phi_3^+ \cup \Phi_2^+ \cup \Psi^1$, $\zeta - \xi \notin \Phi$ for $\xi \in \Psi^2$, $\zeta \in \Psi^1$ and that $\eta_i + \xi_i = \theta$, $\eta_i - \xi_i \notin \Phi$, $\xi_i + \xi_j \notin \Phi$ and $\eta_i + \xi_j \notin \Phi$ if $i \neq j$ for $\eta_i \in \Phi_2^+$ and $\xi_i, \xi_j \in \Psi^2$ $(i, j = 1, \dots, 10)$. This implies that $[y_\alpha, y_\beta] = 0$ for $\alpha, \beta \in \Phi_3^+ \cup \Phi_2^+ \cup \Psi^1$, $[y_\zeta, y_\xi] = \pm y_{\zeta+\xi}$ for $\xi \in \Psi^2$, $\zeta \in \Psi^1$, if $\zeta + \xi \in \Phi$ and that $[y_{\xi_i}, y_{\xi_j}] = 0$, $[y_{\eta_i}, y_{\xi_j}] = \pm\delta_{ij} y_\theta$ for $\eta_i \in \Phi_2^+$, $\xi_i, \xi_j \in \Psi^2$ $(i, j = 1, \dots, 10)$, by the property of the Chevalley basis. Hence, from Planche V in [Bou68], we readily obtain the non-trivial bracket relation among $\check{\mathfrak{g}}_{-1}$ and $[\check{\mathfrak{g}}_{-2}, \mathfrak{l}_{-1}]$ as in (4.9) and (4.10) below up to signs.

We fix the signs of y_β for $\beta \in \Phi_3^+, \Phi_2^+, \Psi^2$ and Ψ^1 as follows: First we choose the orientation of y_{γ_i} for simple roots by fixing the root vectors $y_i = y_{\gamma_i} \in \mathfrak{g}_{-\gamma_i}$. For $\zeta \in \Psi^1$, we fix the orientation by the following order;

$$y_{\zeta_1} = y_1, \quad y_{\zeta_2} = [y_3, y_{\zeta_1}], \quad y_{\zeta_3} = [y_4, y_{\zeta_2}], \quad y_{\zeta_4} = [y_5, y_{\zeta_3}], \quad y_{\zeta_5} = [y_6, y_{\zeta_4}].$$

For $\xi \in \Psi^2$, we fix the orientation by the following order;

$$\begin{array}{llll} y_{\xi_1} = y_2, & y_{\xi_2} = [y_{\xi_1}, y_4], & y_{\xi_3} = [y_{\xi_2}, y_3], & y_{\xi_4} = [y_{\xi_2}, y_5], \\ y_{\xi_5} = [y_{\xi_3}, y_5], & y_{\xi_6} = [y_{\xi_4}, y_6], & y_{\xi_7} = [y_{\xi_5}, y_6], & y_{\xi_8} = [y_{\xi_5}, y_4], \\ y_{\xi_9} = [y_{\xi_8}, y_6], & y_{\xi_{10}} = [y_{\xi_9}, y_5]. & & \end{array}$$

For $\eta \in \Phi_2^+$, we fix the orientation by the following order;

$$\begin{array}{llll} y_{\eta_1} = [y_{\zeta_3}, y_{\xi_{10}}], & y_{\eta_2} = [y_{\zeta_2}, y_{\xi_{10}}], & y_{\eta_3} = [y_{\zeta_1}, y_{\xi_{10}}], & y_{\eta_4} = -[y_{\zeta_2}, y_{\xi_9}], \\ y_{\eta_5} = -[y_{\zeta_1}, y_{\xi_9}], & y_{\eta_6} = [y_{\zeta_2}, y_{\xi_8}], & y_{\eta_7} = [y_{\zeta_1}, y_{\xi_8}], & y_{\eta_8} = [y_{\zeta_1}, y_{\xi_7}], \\ y_{\eta_9} = -[y_{\zeta_1}, y_{\xi_5}], & y_{\eta_{10}} = [y_{\zeta_1}, y_{\xi_3}]. & & \end{array}$$

Finally, for $\theta \in \Phi_3^+$, we fix the orientation by the following;

$$y_\theta = [y_{\eta_{10}}, y_{\xi_{10}}].$$

Then, for example, we calculate

$$\begin{aligned} [y_{\eta_{10}}, y_{\xi_{10}}] &= [y_{\eta_{10}}, [y_{\xi_9}, y_5]] = [y_{\xi_9}, [y_{\eta_{10}}, y_5]] = [y_{\xi_9}, [[y_{\zeta_1}, y_{\xi_3}], y_5]] \\ &= [y_{\xi_9}, [y_{\zeta_1}, [y_{\xi_3}, y_5]]] = [y_{\xi_9}, [y_{\zeta_1}, y_{\xi_5}]] = [y_{\xi_9}, -y_{\eta_9}] = [y_{\eta_9}, y_{\xi_9}], \end{aligned}$$

and obtain

$$[y_{\eta_i}, y_{\xi_j}] = \delta_{ij} y_\theta, \qquad [y_{\eta_i} . y_{\eta_j}] = [y_{\xi_i}, y_{\xi_j}] = 0 \qquad \text{for} \quad 1 \leqq i, j \leqq 10. \tag{4.9}$$

Moreover we calculate as in

$$[y_{\zeta_3}, y_{\xi_7}] = [[y_4, y_{\zeta_2}], y_{\xi_7}] = [[y_4, y_{\xi_7}], y_{\zeta_2}] = [[y_4, [y_{\xi_5}, y_6]], y_{\zeta_2}]$$

$$= [[[y_4, y_{\xi_5}], y_6], y_{\zeta_2}] = [[-y_{\xi_8}, y_6], y_{\zeta_2}] = -[y_{\xi_9}, y_{\zeta_2}] = -y_{\eta_4},$$

and obtain

$$\begin{aligned}
y_\theta &= [[y_{\zeta_1}, y_{\xi_3}], y_{\xi_{10}}] = -[[y_{\zeta_1}, y_{\xi_5}], y_{\xi_9}] = [[y_{\zeta_1}, y_{\xi_7}], y_{\xi_8}],\\
y_\theta &= [[y_{\zeta_2}, y_{\xi_2}], y_{\xi_{10}}] = -[[y_{\zeta_2}, y_{\xi_4}], y_{\xi_9}] = [[y_{\zeta_2}, y_{\xi_6}], y_{\xi_8}],\\
y_\theta &= [[y_{\zeta_3}, y_{\xi_1}], y_{\xi_{10}}] = -[[y_{\zeta_3}, y_{\xi_4}], y_{\xi_7}] = [[y_{\zeta_3}, y_{\xi_5}], y_{\xi_6}],\\
y_\theta &= [[y_{\zeta_4}, y_{\xi_1}], y_{\xi_9}] = -[[y_{\zeta_4}, y_{\xi_2}], y_{\xi_7}] = [[y_{\zeta_4}, y_{\xi_3}], y_{\xi_6}],\\
y_\theta &= [[y_{\zeta_5}, y_{\xi_1}], y_{\xi_8}] = -[[y_{\zeta_2}, y_{\xi_2}], y_{\xi_5}] = [[y_{\zeta_5}, y_{\xi_3}], y_{\xi_4}].
\end{aligned} \tag{4.10}$$

From (4.9), we have $S_{-2} = V^*$, by fixing the base of $S_{-3} \cong \mathbb{K}$ and putting $\mathfrak{l}_{-1} = V$. Moreover, from (4.10), S_{-1} is embedded as the 5-dimensional subspace of $S^2(V^*)$ spanned by the following quadratic forms $f_1, \dots, f_5$;

$$\begin{aligned}
f_1(X, X) &= x_3x_{10} - x_5x_9 + x_7x_8, & f_2(X, X) &= x_2x_{10} - x_4x_9 + x_6x_8,\\
f_3(X, X) &= x_1x_{10} - x_4x_7 + x_5x_6, & f_4(X, X) &= x_1x_9 - x_2x_7 + x_3x_6,\\
f_5(X, X) &= x_1x_8 - x_2x_5 + x_3x_4,
\end{aligned}$$

for $X = \sum_{i=1}^{10} x_i y_{\xi_i} \in \mathfrak{l}_{-1}$. Thus, by fixing the basis $\{y_\theta\}$ of S_{-3} and $\{y_{\xi_1}, \dots, y_{\xi_{10}}\}$ of $\mathfrak{l}_{-1}$, an element $A = \sum_{i=1}^{5} a_i \operatorname{ad}(y_{\zeta_i}) \in S_{-1} \subset S^2(V^*) \cong \mathrm{Sym}(10)$ is represented as the symmetric matix of the following form;

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & a_4 & a_3\\
0 & 0 & 0 & 0 & -a_5 & 0 & -a_4 & 0 & 0 & a_2\\
0 & 0 & 0 & a_5 & 0 & a_4 & 0 & 0 & 0 & a_1\\
0 & 0 & a_5 & 0 & 0 & 0 & -a_3 & 0 & -a_2 & 0\\
0 & -a_5 & 0 & 0 & 0 & a_3 & 0 & 0 & -a_1 & 0\\
0 & 0 & a_4 & 0 & a_3 & 0 & 0 & a_2 & 0 & 0\\
0 & -a_4 & 0 & -a_3 & 0 & 0 & 0 & a_1 & 0 & 0\\
a_5 & 0 & 0 & 0 & 0 & a_2 & a_1 & 0 & 0 & 0\\
a_4 & 0 & 0 & -a_2 & -a_1 & 0 & 0 & 0 & 0 & 0\\
a_3 & a_2 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Hence the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case is given by

$$D_{\mathfrak{m}} = \{\varpi = \varpi_1 = \varpi_2 = \cdots = \varpi_{10} = 0\},$$

where

$$\begin{aligned}
\varpi &= dy - p_1dx_1 - \cdots - p_{10}dx_{10},\\
\varpi_1 &= dp_1 + q_5dx_8 + q_4dx_9 + q_3dx_{10}, & \varpi_2 &= dp_2 - q_5dx_5 - q_4dx_7 + q_2dx_{10},\\
\varpi_3 &= dp_3 + q_5dx_4 + q_4dx_6 + q_1dx_{10}, & \varpi_4 &= dp_4 + q_5dx_3 - q_3dx_7 - q_2dx_9,\\
\varpi_5 &= dp_5 - q_5dx_2 + q_3dx_6 - q_1dx_9, & \varpi_6 &= dp_6 + q_4dx_3 + q_3dx_5 + q_2dx_8,
\end{aligned}$$

$$\varpi_7 = dp_7 - q_4 dx_2 - q_3 dx_4 + q_1 dx_8, \qquad \varpi_8 = dp_8 + q_5 dx_1 + q_2 dx_6 + q_1 dx_7,$$
$$\varpi_9 = dp_9 + q_4 dx_1 - q_2 dx_4 - q_1 dx_5, \qquad \varpi_{10} = dp_{10} + q_3 dx_1 + q_2 dx_2 + q_1 dx_3.$$

Here $(x_1, \dots, x_{10}, y, p_1, \dots, p_{10}, q_1, \dots, q_5)$ is a coordinate system of $M(\mathfrak{m}) \cong \mathbb{K}^{26}$. Thus the model equation of our typical symbol $\mathfrak{m} = \check{\mathfrak{g}}_{-3} \oplus \check{\mathfrak{g}}_{-2} \oplus \check{\mathfrak{g}}_{-1} \subset \mathfrak{C}^2(\mathfrak{l}_{-1}, \mathbb{K})$ is given by

$$\begin{aligned}
&\frac{\partial^2 y}{\partial x_3 \partial x_{10}} = -\frac{\partial^2 y}{\partial x_5 \partial x_9} = \frac{\partial^2 y}{\partial x_7 \partial x_8}, \qquad && \frac{\partial^2 y}{\partial x_2 \partial x_{10}} = -\frac{\partial^2 y}{\partial x_4 \partial x_9} = \frac{\partial^2 y}{\partial x_6 \partial x_8},\\
&\frac{\partial^2 y}{\partial x_1 \partial x_{10}} = -\frac{\partial^2 y}{\partial x_4 \partial x_7} = \frac{\partial^2 y}{\partial x_5 \partial x_6}, && \frac{\partial^2 y}{\partial x_1 \partial x_9} = -\frac{\partial^2 y}{\partial x_2 \partial x_7} = \frac{\partial^2 y}{\partial x_3 \partial x_6},\\
&\frac{\partial^2 y}{\partial x_1 \partial x_8} = -\frac{\partial^2 y}{\partial x_2 \partial x_5} = \frac{\partial^2 y}{\partial x_3 \partial x_4}, && \frac{\partial^2 y}{\partial x_i \partial x_j} = 0 \qquad \text{otherwise},
\end{aligned} \tag{4.11}$$

where y is a dependent variable and $x_1, \dots, x_{10}$ are independent variables. By a direct calculation, we see that the prolongation of the second order system (4.11) is given by

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0 \qquad \text{for} \quad 1 \leqq i, j, k \leqq 10. \tag{4.12}$$

Case (4) $[(E_6, \{\alpha_6\}), \varpi_6, (E_7, \{\gamma_6, \gamma_7\})]$.

For the gradation of type $(E_7, \{\gamma_6, \gamma_7\})$, we have

$$\begin{aligned}
\Phi_3^+ = \Big\{ & \alpha_1 = \begin{smallmatrix}0&1&2&2&2&1\\&&1\end{smallmatrix},\ \alpha_2 = \begin{smallmatrix}1&1&2&2&2&1\\&&1\end{smallmatrix},\ \alpha_3 = \begin{smallmatrix}1&2&2&2&2&1\\&&1\end{smallmatrix},\ \alpha_4 = \begin{smallmatrix}1&2&3&2&2&1\\&&1\end{smallmatrix},\\
& \alpha_5 = \begin{smallmatrix}1&2&3&3&2&1\\&&1\end{smallmatrix},\ \alpha_6 = \begin{smallmatrix}1&2&3&2&2&1\\&&2\end{smallmatrix},\ \alpha_7 = \begin{smallmatrix}1&2&3&3&2&1\\&&2\end{smallmatrix},\ \alpha_8 = \begin{smallmatrix}1&2&4&3&2&1\\&&2\end{smallmatrix},\\
& \alpha_9 = \begin{smallmatrix}1&3&4&3&2&1\\&&2\end{smallmatrix}, \alpha_{10} = \begin{smallmatrix}2&3&4&3&2&1\\&&2\end{smallmatrix}\Big\},
\end{aligned}$$

$$\begin{aligned}
\Phi_2^+ = \Big\{ & \eta_1 = \begin{smallmatrix}0&0&0&0&1&1\\&&0\end{smallmatrix},\ \eta_2 = \begin{smallmatrix}0&0&0&1&1&1\\&&0\end{smallmatrix},\ \eta_3 = \begin{smallmatrix}0&0&1&1&1&1\\&&0\end{smallmatrix},\ \eta_4 = \begin{smallmatrix}0&0&1&1&1&1\\&&1\end{smallmatrix},\\
& \eta_5 = \begin{smallmatrix}0&1&1&1&1&1\\&&0\end{smallmatrix},\ \eta_6 = \begin{smallmatrix}0&1&1&1&1&1\\&&1\end{smallmatrix},\ \eta_7 = \begin{smallmatrix}1&1&1&1&1&1\\&&0\end{smallmatrix},\ \eta_8 = \begin{smallmatrix}1&1&1&1&1&1\\&&1\end{smallmatrix},\\
& \eta_9 = \begin{smallmatrix}0&1&2&1&1&1\\&&1\end{smallmatrix}, \eta_{10} = \begin{smallmatrix}1&1&2&1&1&1\\&&1\end{smallmatrix},\ \eta_{11} = \begin{smallmatrix}0&1&2&2&1&1\\&&1\end{smallmatrix}, \eta_{12} = \begin{smallmatrix}1&2&2&1&1&1\\&&1\end{smallmatrix},\\
& \eta_{13} = \begin{smallmatrix}1&1&2&2&1&1\\&&1\end{smallmatrix}, \eta_{14} = \begin{smallmatrix}1&2&2&2&1&1\\&&1\end{smallmatrix},\ \eta_{15} = \begin{smallmatrix}1&2&3&2&1&1\\&&1\end{smallmatrix}, \eta_{16} = \begin{smallmatrix}1&2&3&2&1&1\\&&2\end{smallmatrix}\Big\},
\end{aligned}$$

$$\Phi_1^+ = \Psi^6 \cup \Psi^7,$$

$$\begin{aligned}
\Psi^6 = \Big\{ & \xi_1 = \begin{smallmatrix}0&0&0&0&1&0\\&&0\end{smallmatrix},\ \xi_2 = \begin{smallmatrix}0&0&0&1&1&0\\&&0\end{smallmatrix},\ \xi_3 = \begin{smallmatrix}0&0&1&1&1&0\\&&0\end{smallmatrix},\ \xi_4 = \begin{smallmatrix}0&0&1&1&1&0\\&&1\end{smallmatrix},\\
& \xi_5 = \begin{smallmatrix}0&1&1&1&1&0\\&&0\end{smallmatrix},\ \xi_6 = \begin{smallmatrix}0&1&1&1&1&0\\&&1\end{smallmatrix},\ \xi_7 = \begin{smallmatrix}1&1&1&1&1&0\\&&0\end{smallmatrix},\ \xi_8 = \begin{smallmatrix}1&1&1&1&1&0\\&&1\end{smallmatrix},\\
& \xi_9 = \begin{smallmatrix}0&1&2&1&1&0\\&&1\end{smallmatrix}, \xi_{10} = \begin{smallmatrix}1&1&2&1&1&0\\&&1\end{smallmatrix},\ \xi_{11} = \begin{smallmatrix}0&1&2&2&1&0\\&&1\end{smallmatrix}, \xi_{12} = \begin{smallmatrix}1&2&2&1&1&0\\&&1\end{smallmatrix},\\
& \xi_{13} = \begin{smallmatrix}1&1&2&2&1&0\\&&1\end{smallmatrix}, \xi_{14} = \begin{smallmatrix}1&2&2&2&1&0\\&&1\end{smallmatrix},\ \xi_{15} = \begin{smallmatrix}1&2&3&2&1&0\\&&1\end{smallmatrix}, \xi_{16} = \begin{smallmatrix}1&2&3&2&1&0\\&&2\end{smallmatrix}\Big\},
\end{aligned}$$

$$\Psi^7 = \Big\{ \gamma_7 = \begin{smallmatrix}0&0&0&0&0&1\\&&0\end{smallmatrix} \Big\},$$

where $\begin{smallmatrix} a_1 & a_3 & a_4 & a_5 & a_6 & a_7 \\ & & a_2 & & & \end{smallmatrix}$ stands for the root $\alpha = \sum_{i=1}^{7} a_i \gamma_i \in \Phi^+$ (see Planche VI in [Bou68]).

Thus we have $\mu = 3$,

$$\mathfrak{m} = \check{\mathfrak{g}}_{-3} \oplus \check{\mathfrak{g}}_{-2} \oplus \check{\mathfrak{g}}_{-1} \qquad \text{and} \qquad \check{\mathfrak{g}}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

where $\check{\mathfrak{g}}_{-3} = S_{-3}$, $\check{\mathfrak{g}}_{-2} = S_{-2}$, S_{-1} and $\mathfrak{l}_{-1}$ are spanned by the root spaces $\mathfrak{g}_{-\beta}$ for $\beta \in \Phi_3^+, \Phi_2^+, \Psi^7$ and Ψ^6 respectively. Hence $\dim S_{-3} = 10$, $\dim S_{-2} = \dim \mathfrak{l}_{-1} = 16$ and $\dim S_{-1} = 1$.

For $\Phi_3^+, \Phi_2^+, \Psi^7$ and Ψ^6, we observe that $\alpha + \beta \notin \Phi$ for $\alpha, \beta \in \Phi_3^+ \cup \Phi_2^+ \cup \Psi^7$, $\xi - \gamma \notin \Phi$ for $\xi \in \Psi^6, \gamma \in \Psi^7$ and that, if $\eta + \xi \in \Phi_3^+$ for $\eta \in \Phi_2^+, \xi \in \Psi^6$, then $\eta - \xi \notin \Phi$. This implies that $[y_\alpha, y_\beta] = 0$ for $\alpha, \beta \in \Phi_3^+ \cup \Phi_2^+ \cup \Psi^7$, $[y_\gamma, y_\xi] = \pm y_{\gamma+\xi}$ for $\xi \in \Psi^6, \gamma \in \Psi^7$, if $\gamma + \xi \in \Phi$ and $[y_\eta, y_\xi] = \pm y_{\eta+\xi}$ for $\eta \in \Phi_2^+, \xi \in \Psi^6$, if $\eta + \xi \in \Phi$, by the property of the Chevalley basis. Hence, from Planche VI in [Bou68], we readily obtain the non-trivial bracket relation among $\check{\mathfrak{g}}_{-1}$ and $[\check{\mathfrak{g}}_{-2}, \mathfrak{l}_{-1}]$ as in (4.13) and (4.14) below up to signs.

We fix the signs of y_β for $\beta \in \Phi_3^+, \Phi_2^+, \Psi^7$ and Ψ^6 as follows: First we choose the orientation of y_{γ_i} for simple roots by fixing the root vectors $y_i = y_{\gamma_i} \in \mathfrak{g}_{-\gamma_i}$. For $\xi \in \Psi^6$, we fix the orientation by the following order;

$$\begin{array}{llll} y_{\xi_1} = y_6, & y_{\xi_2} = [y_{\xi_1}, y_5], & y_{\xi_3} = [y_{\xi_2}, y_4], & y_{\xi_4} = [y_{\xi_3}, y_2], \\ y_{\xi_5} = [y_{\xi_3}, y_3], & y_{\xi_6} = [y_{\xi_5}, y_2], & y_{\xi_7} = [y_{\xi_5}, y_1], & y_{\xi_8} = [y_{\xi_7}, y_2], \\ y_{\xi_9} = [y_{\xi_6}, y_4], & y_{\xi_{10}} = [y_{\xi_9}, y_1], & y_{\xi_{11}} = [y_{\xi_9}, y_5], & y_{\xi_{12}} = [y_{\xi_{10}}, y_3], \\ y_{\xi_{13}} = [y_{\xi_{11}}, y_1], & y_{\xi_{14}} = [y_{\xi_{13}}, y_3], & y_{\xi_{15}} = [y_{\xi_{14}}, y_4], & y_{\xi_{16}} = [y_{\xi_{15}}, y_2]. \end{array}$$

For $\eta \in \Phi_2^+$, we fix the orientation by the following ;

$$y_{\eta_p} = [y_7, y_{\xi_p}] \quad \text{for } p = 1, 2, \ldots, 16. \tag{4.13}$$

Finally, for $\alpha \in \Phi_3^+$, we fix the orientation by the following;

$$\begin{array}{llll} y_{\alpha_1} = [y_{\eta_{11}}, y_{\xi_1}], & y_{\alpha_2} = [y_{\eta_{13}}, y_{\xi_1}], & y_{\alpha_3} = [y_{\eta_{14}}, y_{\xi_1}], & y_{\alpha_4} = [y_{\eta_{15}}, y_{\xi_1}], \\ y_{\alpha_5} = [y_{\eta_{15}}, y_{\xi_2}], & y_{\alpha_6} = [y_{\eta_{16}}, y_{\xi_1}], & y_{\alpha_7} = [y_{\eta_{16}}, y_{\xi_2}], & y_{\alpha_8} = [y_{\eta_{16}}, y_{\xi_3}], \\ y_{\alpha_9} = [y_{\eta_{16}}, y_{\xi_5}], & y_{\alpha_{10}} = [y_{\eta_{16}}, y_{\xi_7}]. & & \end{array}$$

Then, for example, we calculate

$$[y_{\eta_p}, y_{\xi_q}] = [[y_7, y_{\xi_p}], y_{\xi_q}] = [[y_7, y_{\xi_q}], y_{\xi_p}] = [y_{\eta_q}, y_{\xi_p}] \quad \text{for } 1 \leqq p, q \leqq 16,$$

and

$$\begin{aligned} [y_{\eta_{11}}, y_{\xi_1}] &= [[y_7, y_{\xi_{11}}], y_{\xi_1}] = [[y_7, [y_{\xi_9}, y_5]], y_{\xi_1}] = [[[y_7, y_{\xi_9}], y_5], y_{\xi_1}] \\ &= [[y_{\eta_9}, y_5], y_{\xi_1}] = [y_{\eta_9}, [y_5, y_{\xi_1}]] = -[y_{\eta_9}, y_{\xi_2}]. \end{aligned}$$

In the same way, by the repeated application of Jacobi identities, we obtain

$$
\begin{aligned}
y_{\alpha_1} &= [y_{\eta_{11}}, y_{\xi_1}] = -[y_{\eta_9}, y_{\xi_2}] = [y_{\eta_6}, y_{\xi_3}] = -[y_{\eta_5}, y_{\xi_4}],\\
y_{\alpha_2} &= [y_{\eta_{13}}, y_{\xi_1}] = -[y_{\eta_{10}}, y_{\xi_2}] = [y_{\eta_8}, y_{\xi_3}] = -[y_{\eta_7}, y_{\xi_4}],\\
y_{\alpha_3} &= [y_{\eta_{14}}, y_{\xi_1}] = -[y_{\eta_{12}}, y_{\xi_2}] = [y_{\eta_8}, y_{\xi_5}] = -[y_{\eta_7}, y_{\xi_6}],\\
y_{\alpha_4} &= [y_{\eta_{15}}, y_{\xi_1}] = -[y_{\eta_{12}}, y_{\xi_3}] = [y_{\eta_{10}}, y_{\xi_5}] = -[y_{\eta_9}, y_{\xi_7}],\\
y_{\alpha_5} &= [y_{\eta_{15}}, y_{\xi_2}] = -[y_{\eta_{14}}, y_{\xi_3}] = [y_{\eta_{13}}, y_{\xi_5}] = -[y_{\eta_{11}}, y_{\xi_7}],\\
y_{\alpha_6} &= [y_{\eta_{16}}, y_{\xi_1}] = -[y_{\eta_{12}}, y_{\xi_4}] = [y_{\eta_{10}}, y_{\xi_6}] = -[y_{\eta_9}, y_{\xi_8}],\\
y_{\alpha_7} &= [y_{\eta_{16}}, y_{\xi_2}] = -[y_{\eta_{14}}, y_{\xi_4}] = [y_{\eta_{13}}, y_{\xi_6}] = -[y_{\eta_{11}}, y_{\xi_8}],\\
y_{\alpha_8} &= [y_{\eta_{16}}, y_{\xi_3}] = -[y_{\eta_{15}}, y_{\xi_4}] = [y_{\eta_{13}}, y_{\xi_9}] = -[y_{\eta_{11}}, y_{\xi_{10}}],\\
y_{\alpha_9} &= [y_{\eta_{16}}, y_{\xi_5}] = -[y_{\eta_{15}}, y_{\xi_6}] = [y_{\eta_{14}}, y_{\xi_9}] = -[y_{\eta_{12}}, y_{\xi_{11}}],\\
y_{\alpha_{10}} &= [y_{\eta_{16}}, y_{\xi_7}] = -[y_{\eta_{15}}, y_{\xi_8}] = [y_{\eta_{14}}, y_{\xi_{10}}] = -[y_{\eta_{13}}, y_{\xi_{12}}].
\end{aligned}
\tag{4.14}
$$

Then, by fixing the basis $\{y_{\alpha_i}\}_{i=1}^{10}$ of S_{-3} and $\{y_{\xi_j}\}_{j=1}^{16}$ of $\mathfrak{l}_{-1}$, an element $A = \sum_{j=1}^{16} a_j \,\mathrm{ad}(y_{\eta_j}) \in S_{-2} \subset S_{-3} \otimes (\mathfrak{l}_{-1})^* \cong M(10, 16)$ is represented as the matrix of the following form;

$$
\begin{pmatrix}
a_{11} & a_9{}^* & a_6 & a_5{}^* & a_4{}^* & a_3 & 0 & 0 & a_2{}^* & 0 & a_1 & 0 & 0 & 0 & 0 & 0\\
a_{13} & a_{10}{}^* & a_8 & a_7{}^* & 0 & 0 & a_4{}^* & a_3 & 0 & a_2{}^* & 0 & 0 & a_1 & 0 & 0 & 0\\
a_{14} & a_{12}{}^* & 0 & 0 & a_8 & a_7{}^* & a_6{}^* & a_5 & 0 & 0 & 0 & a_2{}^* & 0 & a_1 & 0 & 0\\
a_{15} & 0 & a_{12}{}^* & 0 & a_{10} & 0 & a_9{}^* & 0 & a_7{}^* & a_5 & 0 & a_3{}^* & 0 & 0 & a_1 & 0\\
0 & a_{15} & a_{14}{}^* & 0 & a_{13} & 0 & a_{11}{}^* & 0 & 0 & 0 & a_7{}^* & 0 & a_5 & a_3{}^* & a_2 & 0\\
a_{16} & 0 & 0 & a_{12}{}^* & 0 & a_{10} & 0 & a_9{}^* & a_8{}^* & a_6 & 0 & a_4{}^* & 0 & 0 & 0 & a_1\\
0 & a_{16} & 0 & a_{14}{}^* & 0 & a_{13} & 0 & a_{11}{}^* & 0 & 0 & a_8{}^* & 0 & a_6 & a_4{}^* & 0 & a_2\\
0 & 0 & a_{16} & a_{15}{}^* & 0 & 0 & 0 & 0 & a_{13} & a_{11}{}^* & a_{10}{}^* & 0 & a_9 & 0 & a_4{}^* & a_3\\
0 & 0 & 0 & 0 & a_{16} & a_{15}{}^* & 0 & 0 & a_{14} & 0 & a_{12}{}^* & a_{11}{}^* & 0 & a_9 & a_6{}^* & a_5\\
0 & 0 & 0 & 0 & 0 & 0 & a_{16} & a_{15}{}^* & 0 & a_{14} & 0 & a_{13}{}^* & a_{12}{}^* & a_{10} & a_8{}^* & a_7
\end{pmatrix}
$$

where $a_i{}^* = -a_i$. Moreover, for $y_7 \in S_{-1}$, we have

$$
\begin{aligned}
y_{\alpha_1} &= [[y_7, y_{\xi_1}], y_{\xi_{11}}] = -[[y_7, y_{\xi_2}], y_{\xi_9}] = [[y_7, y_{\xi_3}], y_{\xi_6}] = -[[y_7.y_{\xi_4}], y_{\xi_5}],\\
y_{\alpha_2} &= [[y_7, y_{\xi_1}], y_{\xi_{13}}] = -[[y_7, y_{\xi_2}], y_{\xi_{10}}] = [[y_7, y_{\xi_3}], y_{\xi_8}] = -[[y_7, y_{\xi_4}], y_{\xi_7}],\\
y_{\alpha_3} &= [[y_7, y_{\xi_1}], y_{\xi_{14}}] = -[[y_7, y_{\xi_2}], y_{\xi_{12}}] = [[y_7, y_{\xi_5}], y_{\xi_8}] = -[[y_7, y_{\xi_6}], y_{\xi_7}],\\
y_{\alpha_4} &= [[y_7, y_{\xi_1}], y_{\xi_{15}}] = -[[y_7, y_{\xi_3}], y_{\xi_{12}}] = [[y_7, y_{\xi_5}], y_{\xi_{10}}] = -[[y_7, y_{\xi_7}], y_{\xi_9}],\\
y_{\alpha_5} &= [[y_7, y_{\xi_2}], y_{\xi_{15}}] = -[[y_7, y_{\xi_3}], y_{\xi_{14}}] = [[y_7, y_{\xi_5}], y_{\xi_{13}}] = -[[y_7, y_{\xi_7}], y_{\xi_{11}}],\\
y_{\alpha_6} &= [[y_7, y_{\xi_1}], y_{\xi_{16}}] = -[[y_7.y_{\xi_4}], y_{\xi_{12}}] = [[y_7, y_{\xi_6}], y_{\xi_{10}}] = -[[y_7.y_{\xi_8}], y_{\xi_9}],\\
y_{\alpha_7} &= [[y_7, y_{\xi_2}], y_{\xi_{16}}] = -[[y_7, y_{\xi_4}], y_{\xi_{14}}] = [[y_7, y_{\xi_6}], y_{\xi_{13}}] = -[[y_7, y_{\xi_8}], y_{\xi_{11}}],\\
y_{\alpha_8} &= [[y_7, y_{\xi_3}], y_{\xi_{16}}] = -[[y_7, y_{\xi_4}], y_{\xi_{15}}] = [[y_7, y_{\xi_9}], y_{\xi_{13}}] = -[[y_7, y_{\xi_{10}}], y_{\xi_{11}}],\\
y_{\alpha_9} &= [[y_7, y_{\xi_5}], y_{\xi_{16}}] = -[[y_7, y_{\xi_6}], y_{\xi_{15}}] = [[y_7, y_{\xi_9}], y_{\xi_{14}}] = -[[y_7, y_{\xi_{11}}], y_{\xi_{12}}],\\
y_{\alpha_{10}} &= [[y_7, y_{\xi_7}], y_{\xi_{16}}] = -[[y_7, y_{\xi_8}], y_{\xi_{15}}] = [[y_7, y_{\xi_{10}}], y_{\xi_{14}}] = -[[y_7, y_{\xi_{12}}], y_{\xi_{13}}].
\end{aligned}
$$

Thus S_{-1} is embedded as the 1-dimensional subspace of $S_{-3} \otimes S^2((\mathfrak{l}_{-1})^*)$ spanned by the following quadratic form f:

$$\begin{aligned} f(X,X) = {} & (x_1x_{11} - x_2x_9 + x_3x_6 - x_4x_5)y_{\alpha_1} \\ & + (x_1x_{13} - x_2x_{10} + x_3x_8 - x_4x_7)y_{\alpha_2} \\ & + (x_1x_{14} - x_2x_{12} + x_5x_8 - x_6x_7)y_{\alpha_3} \\ & + (x_1x_{15} - x_3x_{12} + x_5x_{10} - x_7x_9)y_{\alpha_4} \\ & + (x_2x_{15} - x_3x_{14} + x_5x_{13} - x_7x_{11})y_{\alpha_5} \\ & + (x_1x_{16} - x_4x_{12} + x_6x_{10} - x_8x_9)y_{\alpha_6} \\ & + (x_2x_{16} - x_4x_{14} + x_6x_{13} - x_8x_{11})y_{\alpha_7} \\ & + (x_3x_{16} - x_4x_{15} + x_9x_{13} - x_{10}x_{11})y_{\alpha_8} \\ & + (x_5x_{16} - x_6x_{15} + x_9x_{14} - x_{11}x_{12})y_{\alpha_9} \\ & + (x_7x_{16} - x_8x_{15} + x_{10}x_{14} - x_{12}x_{13})y_{\alpha_{10}} \end{aligned}$$

for $X = \sum_{j=1}^{16} x_j y_{\xi_j} \in \mathfrak{l}_{-1}$.

By the above matrix representation, we can embed S_{-2} as a subspace of $S_{-3} \otimes (\mathfrak{l}_{-1})^* \cong M(10, 16)$ and obtain the following first order system as the model equation whose symbol coincides with this subspace:

$$\begin{aligned}
&\frac{\partial y_1}{\partial x_{11}} = \frac{\partial y_2}{\partial x_{13}} = \frac{\partial y_3}{\partial x_{14}} = \frac{\partial y_4}{\partial x_{15}} = \frac{\partial y_6}{\partial x_{16}},\\
&-\frac{\partial y_1}{\partial x_9} = -\frac{\partial y_2}{\partial x_{10}} = -\frac{\partial y_3}{\partial x_{12}} = \frac{\partial y_5}{\partial x_{15}} = \frac{\partial y_7}{\partial x_{16}},\\
&\frac{\partial y_1}{\partial x_6} = \frac{\partial y_2}{\partial x_8} = -\frac{\partial y_4}{\partial x_{12}} = -\frac{\partial y_5}{\partial x_{14}} = \frac{\partial y_8}{\partial x_{16}},\\
&\frac{\partial y_1}{\partial x_5} = \frac{\partial y_2}{\partial x_7} = \frac{\partial y_6}{\partial x_{12}} = \frac{\partial y_7}{\partial x_{14}} = \frac{\partial y_8}{\partial x_{15}},\\
&-\frac{\partial y_1}{\partial x_4} = \frac{\partial y_3}{\partial x_8} = \frac{\partial y_4}{\partial x_{10}} = \frac{\partial y_5}{\partial x_{13}} = \frac{\partial y_9}{\partial x_{16}},\\
&\frac{\partial y_1}{\partial x_3} = -\frac{\partial y_3}{\partial x_7} = \frac{\partial y_6}{\partial x_{10}} = \frac{\partial y_7}{\partial x_{13}} = -\frac{\partial y_9}{\partial x_{15}},\\
&-\frac{\partial y_2}{\partial x_4} = -\frac{\partial y_3}{\partial x_6} = -\frac{\partial y_4}{\partial x_9} = -\frac{\partial y_5}{\partial x_{11}} = \frac{\partial y_{10}}{\partial x_{16}},\\
&\frac{\partial y_2}{\partial x_3} = \frac{\partial y_3}{\partial x_5} = -\frac{\partial y_6}{\partial x_9} = -\frac{\partial y_7}{\partial x_{11}} = -\frac{\partial y_{10}}{\partial x_{15}},\\
&-\frac{\partial y_1}{\partial x_2} = -\frac{\partial y_4}{\partial x_7} = -\frac{\partial y_6}{\partial x_8} = \frac{\partial y_8}{\partial x_{13}} = \frac{\partial y_9}{\partial x_{14}},\\
&-\frac{\partial y_2}{\partial x_2} = \frac{\partial y_4}{\partial x_5} = \frac{\partial y_6}{\partial x_6} = -\frac{\partial y_8}{\partial x_{11}} = \frac{\partial y_{10}}{\partial x_{14}},\\
&\frac{\partial y_1}{\partial x_1} = -\frac{\partial y_5}{\partial x_7} = -\frac{\partial y_7}{\partial x_8} = -\frac{\partial y_8}{\partial x_{10}} = -\frac{\partial y_9}{\partial x_{12}},\\
&\frac{\partial y_3}{\partial x_2} = \frac{\partial y_4}{\partial x_3} = \frac{\partial y_6}{\partial x_4} = \frac{\partial y_9}{\partial x_{11}} = \frac{\partial y_{10}}{\partial x_{13}},
\end{aligned}$$

$$\frac{\partial y_2}{\partial x_1} = \frac{\partial y_5}{\partial x_5} = \frac{\partial y_7}{\partial x_6} = \frac{\partial y_8}{\partial x_9} = -\frac{\partial y_{10}}{\partial x_{12}},$$
$$\frac{\partial y_3}{\partial x_1} = -\frac{\partial y_5}{\partial x_3} = -\frac{\partial y_7}{\partial x_4} = \frac{\partial y_9}{\partial x_9} = \frac{\partial y_{10}}{\partial x_{10}},$$
$$\frac{\partial y_4}{\partial x_1} = \frac{\partial y_5}{\partial x_2} = -\frac{\partial y_8}{\partial x_4} = -\frac{\partial y_9}{\partial x_6} = -\frac{\partial y_{10}}{\partial x_8},$$
$$\frac{\partial y_6}{\partial x_1} = \frac{\partial y_7}{\partial x_2} = \frac{\partial y_8}{\partial x_3} = \frac{\partial y_9}{\partial x_5} = \frac{\partial y_{10}}{\partial x_7},$$
$$\frac{\partial y_i}{\partial x_j} = 0 \qquad \text{otherwise,} \tag{4.15}$$

where $y_1, \ldots, y_{10}$ are dependent variables and $x_1, \ldots, x_{16}$ are independent variables. Moreover, by a direct calculation, we see that the prolongation of the first order system (4.15) is given by

$$\begin{aligned}
\frac{\partial^2 y_1}{\partial x_1 \partial x_{11}} &= \frac{\partial^2 y_2}{\partial x_1 \partial x_{13}} = \frac{\partial^2 y_3}{\partial x_1 \partial x_{14}} = \frac{\partial^2 y_4}{\partial x_1 \partial x_{15}} = \frac{\partial^2 y_6}{\partial x_1 \partial x_{16}} = \frac{\partial^2 y_7}{\partial x_{16} \partial x_2} \\
&= \frac{\partial^2 y_8}{\partial x_{16} \partial x_3} = \frac{\partial^2 y_9}{\partial x_{16} \partial x_5} = \frac{\partial^2 y_{10}}{\partial x_{16} \partial x_7} = -\frac{\partial^2 y_5}{\partial x_7 \partial x_{11}} = -\frac{\partial^2 y_4}{\partial x_7 \partial x_9} \\
&= -\frac{\partial^2 y_3}{\partial x_7 \partial x_6} = -\frac{\partial^2 y_2}{\partial x_7 \partial x_4} = -\frac{\partial^2 y_6}{\partial x_4 \partial x_{12}} = -\frac{\partial^2 y_7}{\partial x_4 \partial x_{14}} = -\frac{\partial^2 y_8}{\partial x_4 \partial x_{15}} \\
&= -\frac{\partial^2 y_1}{\partial x_4 \partial x_5} = \frac{\partial^2 y_3}{\partial x_5 \partial x_8} = \frac{\partial^2 y_4}{\partial x_5 \partial x_{10}} = \frac{\partial^2 y_5}{\partial x_5 \partial x_{13}} = \frac{\partial^2 y_7}{\partial x_{13} \partial x_6} = \frac{\partial^2 y_8}{\partial x_{13} \partial x_9} \\
&= -\frac{\partial^2 y_{10}}{\partial x_{13} \partial x_{12}} = -\frac{\partial^2 y_9}{\partial x_{12} \partial x_{11}} = -\frac{\partial^2 y_4}{\partial x_{12} \partial x_3} = -\frac{\partial^2 y_3}{\partial x_{12} \partial x_2} = \frac{\partial^2 y_5}{\partial x_2 \partial x_{15}} \\
&= -\frac{\partial^2 y_2}{\partial x_2 \partial x_{10}} = -\frac{\partial^2 y_1}{\partial x_2 \partial x_9} = -\frac{\partial^2 y_6}{\partial x_9 \partial x_8} = \frac{\partial^2 y_9}{\partial x_9 \partial x_{14}} = -\frac{\partial^2 y_5}{\partial x_{14} \partial x_3} \\
&= \frac{\partial^2 y_{10}}{\partial x_{14} \partial x_{10}} = -\frac{\partial^2 y_8}{\partial x_{10} \partial x_{11}} = \frac{\partial^2 y_6}{\partial x_{10} \partial x_6} = -\frac{\partial^2 y_9}{\partial x_6 \partial x_{15}} = \frac{\partial^2 y_1}{\partial x_6 \partial x_3} \\
&= \frac{\partial^2 y_2}{\partial x_3 \partial x_8} = -\frac{\partial^2 y_7}{\partial x_8 \partial x_{11}} = -\frac{\partial^2 y_{10}}{\partial x_8 \partial x_{15}},
\end{aligned}$$
$$\frac{\partial^2 y_i}{\partial x_p \partial x_q} = 0 \qquad \text{otherwise.} \tag{4.16}$$

From the above expression of f, we observe that the above second order system is the model equation of the 1-dimensional embedded subspace S_{-1} in $S_{-3} \otimes S^2((\mathfrak{l}_{-1})^*)$. Furthermore, by a direct calculation, we see that the prolo ngation of this second order system (4.16) is given by

$$\frac{\partial^3 y_i}{\partial x_p \partial x_q \partial x_r} = 0 \qquad \text{for} \quad 1 \leqq i \leqq 10, \quad 1 \leqq p, q, r \leqq 16. \tag{4.17}$$

Case (5) $[(E_6, \{\alpha_1\}), \varpi_6, (E_7, \{\gamma_1, \gamma_7\})]$.

For the gradation of type $(E_7, \{\gamma_1, \gamma_7\})$, we have

$$\Phi_3^+ = \left\{\theta = \begin{smallmatrix}2&3&4&3&2&1\\&&2&&&\end{smallmatrix}\right\},$$

$$\begin{aligned}\Phi_2^+ = \Big\{&\eta_1 = \begin{smallmatrix}1&3&4&3&2&1\\&&2&&&\end{smallmatrix}, \quad \eta_2 = \begin{smallmatrix}1&2&4&3&2&1\\&&2&&&\end{smallmatrix}, \quad \eta_3 = \begin{smallmatrix}1&2&3&3&2&1\\&&2&&&\end{smallmatrix}, \quad \eta_4 = \begin{smallmatrix}1&2&3&3&2&1\\&&1&&&\end{smallmatrix},\\ &\eta_5 = \begin{smallmatrix}1&2&3&2&2&1\\&&2&&&\end{smallmatrix}, \quad \eta_6 = \begin{smallmatrix}1&2&3&2&2&1\\&&1&&&\end{smallmatrix}, \quad \eta_7 = \begin{smallmatrix}1&2&3&2&1&1\\&&2&&&\end{smallmatrix}, \quad \eta_8 = \begin{smallmatrix}1&2&3&2&1&1\\&&1&&&\end{smallmatrix},\\ &\eta_9 = \begin{smallmatrix}1&2&2&2&2&1\\&&1&&&\end{smallmatrix}, \quad \eta_{10} = \begin{smallmatrix}1&2&2&2&1&1\\&&1&&&\end{smallmatrix}, \quad \eta_{11} = \begin{smallmatrix}1&1&2&2&2&1\\&&1&&&\end{smallmatrix}, \quad \eta_{12} = \begin{smallmatrix}1&1&2&2&1&1\\&&1&&&\end{smallmatrix},\\ &\eta_{13} = \begin{smallmatrix}1&2&2&1&1&1\\&&1&&&\end{smallmatrix}, \quad \eta_{14} = \begin{smallmatrix}1&1&2&1&1&1\\&&1&&&\end{smallmatrix}, \quad \eta_{15} = \begin{smallmatrix}1&1&1&1&1&1\\&&1&&&\end{smallmatrix}, \quad \eta_{16} = \begin{smallmatrix}1&1&1&1&1&1\\&&0&&&\end{smallmatrix}\Big\},\end{aligned}$$

$$\Phi_1^+ = \Psi^1 \cup \Psi^7,$$

$$\begin{aligned}\Psi^1 = \Big\{&\xi_1 = \begin{smallmatrix}1&0&0&0&0&0\\&&0&&&\end{smallmatrix}, \quad \xi_2 = \begin{smallmatrix}1&1&0&0&0&0\\&&0&&&\end{smallmatrix}, \quad \xi_3 = \begin{smallmatrix}1&1&1&0&0&0\\&&0&&&\end{smallmatrix}, \quad \xi_4 = \begin{smallmatrix}1&1&1&0&0&0\\&&1&&&\end{smallmatrix},\\ &\xi_5 = \begin{smallmatrix}1&1&1&1&0&0\\&&0&&&\end{smallmatrix}, \quad \xi_6 = \begin{smallmatrix}1&1&1&1&0&0\\&&1&&&\end{smallmatrix}, \quad \xi_7 = \begin{smallmatrix}1&1&1&1&1&0\\&&0&&&\end{smallmatrix}, \quad \xi_8 = \begin{smallmatrix}1&1&1&1&1&0\\&&1&&&\end{smallmatrix},\\ &\xi_9 = \begin{smallmatrix}1&1&2&1&0&0\\&&1&&&\end{smallmatrix}, \quad \xi_{10} = \begin{smallmatrix}1&1&2&1&1&0\\&&1&&&\end{smallmatrix}, \quad \xi_{11} = \begin{smallmatrix}1&2&2&1&0&0\\&&1&&&\end{smallmatrix}, \quad \xi_{12} = \begin{smallmatrix}1&2&2&1&1&0\\&&1&&&\end{smallmatrix},\\ &\xi_{13} = \begin{smallmatrix}1&1&2&2&1&0\\&&1&&&\end{smallmatrix}, \quad \xi_{14} = \begin{smallmatrix}1&2&2&2&1&0\\&&1&&&\end{smallmatrix}, \quad \xi_{15} = \begin{smallmatrix}1&2&3&2&1&0\\&&1&&&\end{smallmatrix}, \quad \xi_{16} = \begin{smallmatrix}1&2&3&2&1&0\\&&2&&&\end{smallmatrix}\Big\},\end{aligned}$$

$$\begin{aligned}\Psi^7 = \Big\{&\zeta_1 = \begin{smallmatrix}0&0&0&0&0&1\\&&0&&&\end{smallmatrix}, \quad \zeta_2 = \begin{smallmatrix}0&0&0&0&1&1\\&&0&&&\end{smallmatrix}, \quad \zeta_3 = \begin{smallmatrix}0&0&0&1&1&1\\&&0&&&\end{smallmatrix}, \quad \zeta_4 = \begin{smallmatrix}0&0&1&1&1&1\\&&0&&&\end{smallmatrix},\\ &\zeta_5 = \begin{smallmatrix}0&1&1&1&1&1\\&&0&&&\end{smallmatrix}, \quad \zeta_6 = \begin{smallmatrix}0&0&1&1&1&1\\&&1&&&\end{smallmatrix}, \quad \zeta_7 = \begin{smallmatrix}0&1&1&1&1&1\\&&1&&&\end{smallmatrix}, \quad \zeta_8 = \begin{smallmatrix}0&1&2&1&1&1\\&&1&&&\end{smallmatrix},\\ &\zeta_9 = \begin{smallmatrix}0&1&2&2&1&1\\&&1&&&\end{smallmatrix}, \quad \zeta_{10} = \begin{smallmatrix}0&1&2&2&2&1\\&&1&&&\end{smallmatrix}\Big\},\end{aligned}$$

where $\begin{smallmatrix}a_1&a_3&a_4&a_5&a_6&a_7\\&&a_2&&&\end{smallmatrix}$ stands for the root $\alpha = \sum_{i=1}^{7} a_i\gamma_i \in \Phi^+$ (see Planche VI in [Bou68]).

Thus we have $\mu = 3$,

$$\mathfrak{m} = \check{\mathfrak{g}}_{-3} \oplus \check{\mathfrak{g}}_{-2} \oplus \check{\mathfrak{g}}_{-1} \qquad \text{and} \qquad \check{\mathfrak{g}}_{-1} = S_{-1} \oplus \mathfrak{l}_{-1},$$

where $\check{\mathfrak{g}}_{-3} = S_{-3}$, $\check{\mathfrak{g}}_{-2} = S_{-2}$, S_{-1} and $\mathfrak{l}_{-1}$ are spanned by the root spaces $\mathfrak{g}_{-\beta}$ for $\beta \in \Phi_3^+, \Phi_2^+, \Psi^7$ and Ψ^1 respectively. Hence $\dim S_{-3} = 1$, $\dim S_{-2} = \dim \mathfrak{l}_{-1} = 16$ and $\dim S_{-1} = 10$.

For $\Phi_3^+, \Phi_2^+, \Psi^7$ and Ψ^1, we observe that $\Phi_3^+ = \{\theta\}$, where θ is the highest root, $\alpha+\beta \notin \Phi$ for $\alpha, \beta \in \Phi_3^+ \cup \Phi_2^+ \cup \Psi^7$, $\zeta - \xi \notin \Phi$ for $\xi \in \Psi^1$, $\zeta \in \Psi^7$ and that $\eta_i + \xi_i = \theta$, $\eta_i - \xi_i \notin \Phi$, $\xi_i + \xi_j \notin \Phi$ and $\eta_i + \xi_j \notin \Phi$ if $i \neq j$ for $\eta_i \in \Phi_2^+$ and $\xi_i, \xi_j \in \Psi^2$ $(i, j = 1, \ldots, 16)$. This implies that $[y_\alpha, y_\beta] = 0$ for $\alpha, \beta \in \Phi_3^+ \cup \Phi_2^+ \cup \Psi^7$, $[y_\zeta, y_\xi] = \pm y_{\zeta+\xi}$ for $\xi \in \Psi^2$, $\zeta \in \Psi^7$, if $\zeta + \xi \in \Phi$ and that $[y_{\xi_i}, y_{\xi_j}] = 0$, $[y_{\eta_i}, y_{\xi_j}] = \pm\delta_{ij} y_\theta$ for $\eta_i \in \Phi_2^+$, $\xi_i, \xi_j \in \Psi^2$ $(i, j = 1, \ldots, 16)$, by the property of the Chevalley basis. Hence, from Planche VI in [Bou68], we readily obtain the non-trivial bracket relation among $\check{\mathfrak{g}}_{-1}$ and $[\check{\mathfrak{g}}_{-2}, \mathfrak{l}_{-1}]$ as in (4.18) and (4.19) below up to signs.

We fix the signs of y_β for $\beta \in \Phi_3^+, \Phi_2^+, \Psi^7$ and Ψ^1 as follows: First we choose the orientation of y_{γ_i} for simple roots by fixing the root vectors $y_i = y_{\gamma_i} \in \mathfrak{g}_{-\gamma_i}$. For $\zeta \in \Psi^7$, we fix the orientation by the following order;

$$\begin{aligned} &y_{\zeta_1} = y_7, && y_{\zeta_2} = [y_6, y_{\zeta_1}], && y_{\zeta_3} = [y_5, y_{\zeta_2}], && y_{\zeta_4} = [y_4, y_{\zeta_3}],\\ &y_{\zeta_5} = [y_3, y_{\zeta_4}], && y_{\zeta_6} = [y_2, y_{\zeta_4}], && y_{\zeta_7} = [y_2, y_{\zeta_5}], && y_{\zeta_8} = [y_4, y_{\zeta_7}],\\ &y_{\zeta_9} = [y_5, y_{\zeta_8}], && y_{\zeta_{10}} = [y_6, y_{\zeta_9}]. \end{aligned}$$

For $\xi \in \Psi^1$, we fix the orientation by the following order;

$$\begin{aligned} &y_{\xi_1} = y_1, && y_{\xi_2} = [y_{\xi_1}, y_3], && y_{\xi_3} = [y_{\xi_2}, y_4], && y_{\xi_4} = [y_{\xi_3}, y_2],\\ &y_{\xi_5} = [y_{\xi_3}, y_5], && y_{\xi_6} = [y_{\xi_5}, y_2], && y_{\xi_7} = [y_{\xi_5}, y_6], && y_{\xi_8} = [y_{\xi_7}, y_2],\\ &y_{\xi_9} = [y_{\xi_6}, y_4], && y_{\xi_{10}} = [y_{\xi_9}, y_6], && y_{\xi_{11}} = [y_{\xi_9}, y_3], && y_{\xi_{12}} = [y_{\xi_{11}}, y_6],\\ &y_{\xi_{13}} = [y_{\xi_{10}}, y_5], && y_{\xi_{14}} = [y_{\xi_{13}}, y_3], && y_{\xi_{15}} = [y_{\xi_{14}}, y_4], && y_{\xi_{16}} = [y_{\xi_{15}}, y_2]. \end{aligned}$$

For $\eta \in \Phi_2^+$, we fix the orientation by the following order;

$$\begin{aligned} &y_{\eta_1} = -[y_{\zeta_5}, y_{\xi_{16}}], && y_{\eta_2} = -[y_{\zeta_4}, y_{\xi_{16}}], && y_{\eta_3} = -[y_{\zeta_3}, y_{\xi_{16}}], && y_{\eta_4} = [y_{\zeta_3}, y_{\xi_{15}}],\\ &y_{\eta_5} = -[y_{\zeta_2}, y_{\xi_{16}}], && y_{\eta_6} = [y_{\zeta_2}, y_{\xi_{15}}], && y_{\eta_7} = -[y_{\zeta_1}, y_{\xi_{16}}], && y_{\eta_8} = [y_{\zeta_1}, y_{\xi_{15}}],\\ &y_{\eta_9} = -[y_{\zeta_2}, y_{\xi_{14}}], && y_{\eta_{10}} = -[y_{\zeta_1}, y_{\xi_{14}}], && y_{\eta_{11}} = [y_{\zeta_2}, y_{\xi_{13}}], && y_{\eta_{12}} = [y_{\zeta_1}, y_{\xi_{13}}],\\ &y_{\eta_{13}} = [y_{\zeta_1}, y_{\xi_{12}}], && y_{\eta_{14}} = -[y_{\zeta_1}, y_{\xi_{10}}], && y_{\eta_{15}} = [y_{\zeta_1}, y_{\xi_8}], && y_{\eta_{16}} = -[y_{\zeta_1}, y_{\xi_7}]. \end{aligned}$$

Finally, for $\theta \in \Phi_3^+$, we fix the orientation by the following;

$$y_\theta = [y_{\eta_{16}}, y_{\xi_{16}}].$$

Then, for example, we calculate

$$[y_{\eta_{16}}, y_{\xi_{16}}] = [-[y_{\zeta_1}, y_{\xi_7}], y_{\xi_{16}}] = [-[y_{\zeta_1}, y_{\xi_{16}}], y_{\xi_7}] = [y_{\eta_7}, y_{\xi_7}]$$

and obtain

$$[y_{\eta_p}, y_{\xi_q}] = \delta_{pq} y_\theta, \quad [y_{\eta_p} . y_{\eta_q}] = [y_{\xi_p}, y_{\xi_q}] = 0 \qquad \text{for } 1 \leqq p, q \leqq 16. \tag{4.18}$$

Moreover we calculate as in

$$\begin{aligned} [y_{\zeta_3}, y_{\xi_{12}}] &= [[y_5, y_{\zeta_2}], y_{\xi_{12}}] = [[y_5, y_{\xi_{12}}], y_{\zeta_2}] = [y_{\zeta_2}, [y_{\xi_{12}}, y_5]] = [y_{\zeta_2}, y_{\xi_{14}}]\\ &= -y_{\eta_9}, \end{aligned}$$

and obtain

$$\begin{aligned} y_\theta &= -[[y_{\zeta_1}, y_{\xi_7}], y_{\xi_{16}}] = [[y_{\zeta_1}, y_{\xi_8}], y_{\xi_{15}}] = -[[y_{\zeta_1}, y_{\xi_{10}}], y_{\xi_{14}}] = [[y_{\zeta_1}, y_{\xi_{12}}], y_{\xi_{13}}],\\ y_\theta &= -[[y_{\zeta_2}, y_{\xi_5}], y_{\xi_{16}}] = [[y_{\zeta_2}, y_{\xi_6}], y_{\xi_{15}}] = -[[y_{\zeta_2}, y_{\xi_9}], y_{\xi_{14}}] = [[y_{\zeta_2}, y_{\xi_{11}}], y_{\xi_{13}}],\\ y_\theta &= -[[y_{\zeta_3}, y_{\xi_3}], y_{\xi_{16}}] = [[y_{\zeta_3}, y_{\xi_4}], y_{\xi_{15}}] = -[[y_{\zeta_3}, y_{\xi_9}], y_{\xi_{12}}] = [[y_{\zeta_3}, y_{\xi_{10}}], y_{\xi_{11}}],\\ y_\theta &= -[[y_{\zeta_4}, y_{\xi_2}], y_{\xi_{16}}] = [[y_{\zeta_4}, y_{\xi_4}], y_{\xi_{14}}] = -[[y_{\zeta_4}, y_{\xi_6}], y_{\xi_{12}}] = [[y_{\zeta_4}, y_{\xi_8}], y_{\xi_{11}}], \end{aligned}$$

$$
\begin{aligned}
y_\theta &= -[[y_{\zeta_5}, y_{\xi_1}], y_{\xi_{16}}] = [[y_{\zeta_5}, y_{\xi_4}], y_{\xi_{13}}] = -[[y_{\zeta_5}, y_{\xi_6}], y_{\xi_{10}}] = [[y_{\zeta_5}, y_{\xi_8}], y_{\xi_9}],\\
y_\theta &= -[[y_{\zeta_6}, y_{\xi_2}], y_{\xi_{15}}] = [[y_{\zeta_6}, y_{\xi_3}], y_{\xi_{14}}] = -[[y_{\zeta_6}, y_{\xi_5}], y_{\xi_{12}}] = [[y_{\zeta_6}, y_{\xi_7}], y_{\xi_{11}}],\\
y_\theta &= -[[y_{\zeta_7}, y_{\xi_1}], y_{\xi_{15}}] = [[y_{\zeta_7}, y_{\xi_3}], y_{\xi_{13}}] = -[[y_{\zeta_7}, y_{\xi_5}], y_{\xi_{10}}] = [[y_{\zeta_7}, y_{\xi_7}], y_{\xi_9}],\\
y_\theta &= -[[y_{\zeta_8}, y_{\xi_1}], y_{\xi_{14}}] = [[y_{\zeta_8}, y_{\xi_2}], y_{\xi_{13}}] = -[[y_{\zeta_8}, y_{\xi_5}], y_{\xi_8}] = [[y_{\zeta_8}, y_{\xi_6}], y_{\xi_7}],\\
y_\theta &= -[[y_{\zeta_9}, y_{\xi_1}], y_{\xi_{12}}] = [[y_{\zeta_9}, y_{\xi_2}], y_{\xi_{10}}] = -[[y_{\zeta_9}, y_{\xi_3}], y_{\xi_8}] = [[y_{\zeta_9}, y_{\xi_4}], y_{\xi_7}],\\
y_\theta &= -[[y_{\zeta_{10}}, y_{\xi_1}], y_{\xi_{11}}] = [[y_{\zeta_{10}}, y_{\xi_2}], y_{\xi_9}] = -[[y_{\zeta_{10}}, y_{\xi_3}], y_{\xi_6}] = [[y_{\zeta_{10}}, y_{\xi_4}], y_{\xi_5}].
\end{aligned}
\tag{4.19}
$$

From (4.18), we have $S_{-2} = V^*$, by fixing the base of $S_{-3} \cong \mathbb{K}$ and putting $\mathfrak{l}_{-1} = V$. Moreover, from (4.19), S_{-1} is embedded as the 10-dimensional subspace of $S^2(V^*)$ spanned by the following quadratic forms $f_1, \dots, f_{10}$;

$$
\begin{aligned}
f_1(X) &= -x_7x_{16} + x_8x_{15} - x_{10}x_{14} + x_{12}x_{13},\\
f_2(X) &= -x_5x_{16} + x_6x_{15} - x_9x_{14} + x_{11}x_{13},\\
f_3(X) &= -x_3x_{16} + x_4x_{15} - x_9x_{12} + x_{10}x_{11},\\
f_4(X) &= -x_2x_{16} + x_4x_{14} - x_6x_{12} + x_8x_{11},\\
f_5(X) &= -x_1x_{16} + x_4x_{13} - x_6x_{10} + x_8x_9,\\
f_6(X) &= -x_2x_{15} + x_3x_{14} - x_5x_{12} + x_7x_{11},\\
f_7(X) &= -x_1x_{15} + x_3x_{13} - x_5x_{10} + x_7x_9,\\
f_8(X) &= -x_1x_{14} + x_2x_{13} - x_5x_8 + x_6x_7,\\
f_9(X) &= -x_1x_{12} + x_2x_{10} - x_3x_8 + x_4x_7,\\
f_{10}(X) &= -x_1x_{11} + x_2x_9 - x_3x_6 + x_4x_5
\end{aligned}
$$

for $X = \sum_{i=1}^{16} x_i y_{\xi_i} \in \mathfrak{l}_{-1}$. Thus, by fixing the basis $\{y_\theta\}$ of S_{-3} and $\{y_{\xi_1}, \dots, y_{\xi_{16}}\}$ of $\mathfrak{l}_{-1}$, an element $A = \sum_{i=1}^{10} a_i \operatorname{ad}(y_{\zeta_i}) \in S_{-1} \subset S^2(V^*) \cong \mathrm{Sym}(16)$ is represented as the symmetric matix of the following form;

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & a_9 & 0 & a_8 & a_7 & a_5\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & {a_{10}}^* & {a_9}^* & 0 & 0 & {a_8}^* & 0 & a_6 & a_4\\
0 & 0 & 0 & 0 & 0 & a_{10} & 0 & a_9 & 0 & 0 & 0 & 0 & {a_7}^* & {a_6}^* & 0 & a_3\\
0 & 0 & 0 & 0 & {a_{10}}^* & 0 & {a_9}^* & 0 & 0 & 0 & 0 & 0 & {a_5}^* & {a_4}^* & {a_3}^* & 0\\
0 & 0 & 0 & {a_{10}}^* & 0 & 0 & 0 & a_8 & 0 & a_7 & 0 & a_6 & 0 & 0 & 0 & a_2\\
0 & 0 & a_{10} & 0 & 0 & 0 & {a_8}^* & 0 & 0 & a_5 & 0 & a_4 & 0 & 0 & {a_2}^* & 0\\
0 & 0 & 0 & {a_9}^* & 0 & {a_8}^* & 0 & 0 & {a_7}^* & 0 & {a_6}^* & 0 & 0 & 0 & 0 & a_1\\
0 & 0 & a_9 & 0 & a_8 & 0 & 0 & 0 & {a_5}^* & 0 & {a_4}^* & 0 & 0 & 0 & {a_1}^* & 0\\
0 & {a_{10}}^* & 0 & 0 & 0 & 0 & {a_7}^* & {a_5}^* & 0 & 0 & 0 & a_3 & 0 & a_2 & 0 & 0\\
0 & {a_9}^* & 0 & 0 & a_7 & a_5 & 0 & 0 & 0 & 0 & {a_3}^* & 0 & 0 & a_1 & 0 & 0\\
a_{10} & 0 & 0 & 0 & 0 & 0 & {a_6}^* & {a_4}^* & 0 & {a_3}^* & 0 & 0 & {a_2}^* & 0 & 0 & 0\\
a_9 & 0 & 0 & 0 & a_6 & a_4 & 0 & 0 & a_3 & 0 & 0 & 0 & {a_1}^* & 0 & 0 & 0\\
0 & {a_8}^* & {a_7}^* & {a_5}^* & 0 & 0 & 0 & 0 & 0 & 0 & {a_2}^* & {a_1}^* & 0 & 0 & 0 & 0\\
a_8 & 0 & {a_6}^* & {a_4}^* & 0 & 0 & 0 & 0 & a_2 & a_1 & 0 & 0 & 0 & 0 & 0 & 0\\
a_7 & a_6 & 0 & {a_3}^* & 0 & {a_2}^* & 0 & {a_1}^* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
a_5 & a_4 & a_3 & 0 & a_2 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

where ${a_i}^* = -a_i$. Hence the standard differential system $(M(\mathfrak{m}), D_{\mathfrak{m}})$ of type $\mathfrak{m}$ in this case is given by

$$D_{\mathfrak{m}} = \{\varpi = \varpi_1 = \varpi_2 = \cdots = \varpi_{16} = 0\,\},$$

where

$$\begin{aligned}
\varpi &= dy - p_1 dx_1 - \cdots - p_{16} dx_{16},\\
\varpi_1 &= dp_1 + q_{10} dx_{11} + q_9 dx_{12} + q_8 dx_{14} + q_7 dx_{15} + q_5 dx_{16},\\
\varpi_2 &= dp_2 - q_{10} dx_9 - q_9 dx_{10} - q_8 dx_{13} + q_6 dx_{15} + q_4 dx_{16},\\
\varpi_3 &= dp_3 + q_{10} dx_6 + q_9 dx_8 - q_7 dx_{13} - q_6 dx_{14} + q_3 dx_{16},\\
\varpi_4 &= dp_4 - q_{10} dx_5 - q_9 dx_7 - q_5 dx_{13} - q_4 dx_{14} - q_3 dx_{15},\\
\varpi_5 &= dp_5 - q_{10} dx_4 + q_8 dx_8 + q_7 dx_{10} + q_6 dx_{12} + q_2 dx_{16},\\
\varpi_6 &= dp_6 + q_{10} dx_3 - q_8 dx_7 + q_5 dx_{10} + q_4 dx_{12} - q_2 dx_{15},\\
\varpi_7 &= dp_7 - q_9 dx_4 - q_8 dx_6 - q_7 dx_9 - q_6 dx_{11} + q_1 dx_{16},\\
\varpi_8 &= dp_8 + q_9 dx_3 + q_8 dx_5 - q_5 dx_9 - q_4 dx_{11} - q_1 dx_{15},\\
\varpi_9 &= dp_9 - q_{10} dx_2 - q_7 dx_7 - q_5 dx_8 + q_3 dx_{12} + q_2 dx_{14},\\
\varpi_{10} &= dp_{10} - q_9 dx_2 + q_7 dx_5 + q_5 dx_6 - q_3 dx_{11} + q_1 dx_{14},\\
\varpi_{11} &= dp_{11} + q_{10} dx_1 - q_6 dx_7 - q_4 dx_8 - q_3 dx_{10} - q_2 dx_{13},\\
\varpi_{12} &= dp_{12} + q_9 dx_1 + q_6 dx_5 + q_4 dx_6 + q_3 dx_9 - q_1 dx_{13},\\
\varpi_{13} &= dp_{13} - q_8 dx_2 - q_7 dx_3 - q_5 dx_4 - q_2 dx_{11} - q_1 dx_{12},\\
\varpi_{14} &= dp_{14} + q_8 dx_1 - q_6 dx_3 - q_4 dx_4 + q_2 dx_9 + q_1 dx_{10},\\
\varpi_{15} &= dp_{15} + q_7 dx_1 + q_6 dx_2 - q_3 dx_4 - q_2 dx_6 - q_1 dx_8,\\
\varpi_{16} &= dp_{16} + q_5 dx_1 + q_4 dx_2 + q_3 dx_3 + q_2 dx_5 + q_1 dx_7.
\end{aligned}$$

Here $(x_1, \ldots, x_{16}, y, p_1, \ldots, p_{16}, q_1, \ldots, q_{10})$ is a coordinate system of $M(\mathfrak{m}) \cong \mathbb{K}^{43}$. Thus the model equation of our typical symbol $\mathfrak{m} = \check{\mathfrak{g}}_{-3} \oplus \check{\mathfrak{g}}_{-2} \oplus \check{\mathfrak{g}}_{-1} \subset \mathfrak{C}^2(\mathfrak{l}_{-1}, \mathbb{K})$ is given by

$$\begin{aligned}
\frac{\partial^2 y}{\partial x_7 \partial x_{16}} &= -\frac{\partial^2 y}{\partial x_8 \partial x_{15}} = \frac{\partial^2 y}{\partial x_{10} \partial x_{14}} = -\frac{\partial^2 y}{\partial x_{12} \partial x_{13}},\\
\frac{\partial^2 y}{\partial x_5 \partial x_{16}} &= -\frac{\partial^2 y}{\partial x_6 \partial x_{15}} = \frac{\partial^2 y}{\partial x_9 \partial x_{14}} = -\frac{\partial^2 y}{\partial x_{11} \partial x_{13}},\\
\frac{\partial^2 y}{\partial x_3 \partial x_{16}} &= -\frac{\partial^2 y}{\partial x_4 \partial x_{15}} = \frac{\partial^2 y}{\partial x_9 \partial x_{12}} = -\frac{\partial^2 y}{\partial x_{10} \partial x_{11}},\\
\frac{\partial^2 y}{\partial x_2 x_{16}} &= -\frac{\partial^2 y}{\partial x_4 \partial x_{14}} = \frac{\partial^2 y}{\partial x_6 \partial x_{12}} = -\frac{\partial^2 y}{\partial x_8 \partial x_{11}},\\
\frac{\partial^2 y}{\partial x_1 \partial x_{16}} &= -\frac{\partial^2 y}{\partial x_4 \partial x_{13}} = \frac{\partial^2 y}{\partial x_6 \partial x_{10}} = -\frac{\partial^2 y}{\partial x_8 \partial x_9},\\
\frac{\partial^2 y}{\partial x_2 \partial x_{15}} &= -\frac{\partial^2 y}{\partial x_3 \partial x_{14}} = \frac{\partial^2 y}{\partial x_5 \partial x_{12}} = -\frac{\partial^2 y}{\partial x_7 \partial x_{11}},
\end{aligned}$$

$$\frac{\partial^2 y}{\partial x_1 \partial x_{15}} = -\frac{\partial^2 y}{\partial x_3 \partial x_{13}} = \frac{\partial^2 y}{\partial x_5 \partial x_{10}} = -\frac{\partial^2 y}{\partial x_7 \partial x_9},$$
$$\frac{\partial^2 y}{\partial x_1 \partial x_{14}} = -\frac{\partial^2 y}{\partial x_2 \partial x_{13}} = \frac{\partial^2 y}{\partial x_5 \partial x_8} = -\frac{\partial^2 y}{\partial x_6 \partial x_7},$$
$$\frac{\partial^2 y}{\partial x_1 \partial x_{12}} = -\frac{\partial^2 y}{\partial x_2 \partial x_{10}} = \frac{\partial^2 y}{\partial x_3 \partial x_8} = -\frac{\partial^2 y}{\partial x_4 \partial x_7},$$
$$\frac{\partial^2 y}{\partial x_1 \partial x_{11}} = -\frac{\partial^2 y}{\partial x_2 \partial x_9} = \frac{\partial^2 y}{\partial x_3 \partial x_6} = -\frac{\partial^2 y}{\partial x_4 \partial x_5},$$
$$\frac{\partial^2 y}{\partial x_i \partial x_j} = 0 \qquad \text{otherwise,} \tag{4.20}$$

where y is a dependent variable and $x_1, \ldots, x_{16}$ are independent variables. By a direct calculation, we see that the prolongation of the second order system (4.20) is given by

$$\frac{\partial^3 y}{\partial x_i \partial x_j \partial x_k} = 0 \qquad \text{for } 1 \leqq i, j, k \leqq 16. \tag{4.21}$$

5 Equivalence of Parabolic Geometries

In this section, we will discuss the equivalence of each parabolic geometry associated with the differential equations of finite type explicitly described in Section 3 and Section 4.

In the following, we will first show a common property of the typical symbol $\mathfrak{m}$ of type $(\mathfrak{l}, S)$. Here $\mathfrak{m} = \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p$ is a graded subalgebra of $\mathfrak{C}^{\mu-1}(V, W)$, which has the splitting $\mathfrak{g}_{-1} = \mathfrak{l}_{-1} \oplus S_{-1}$, where $V = \mathfrak{l}_{-1}$ and $W = S_{-\mu}$. In particular $S_p \subset W \otimes S^{\mu+p}(V^*)$. Thus we have the notion of the algebraic prolongation $\rho(S_p)$ of S_p, which is defined by

$$\rho(S_p) = S_p \otimes V^* \cap W \otimes S^{\mu+p+1}(V^*) \qquad \text{for} \quad -\mu + 1 \leqq p \leqq -1.$$

We will show the following Proposition 5.1 concerning the property of the prolongations of S_p for the typical symbol $\mathfrak{m}$ of type $(\mathfrak{l}, S)$.

Let $\mathfrak{g} = \bigoplus_{p=-\mu}^{1} \mathfrak{g}_p$ be a pseudo-product GLA of type $(\mathfrak{l}, S)$.

Lemma 5.1 *Let p be an integer with $-\mu + 1 \leqq p \leqq -1$. If $H^1(\mathfrak{m}, \mathfrak{g})_{p+1,-1} = 0$, then the algebraic prolongation $\rho(S_p)$ of S_p is equal to S_{p+1}, where we put $S_0 = 0$.*

Proof. Since the fact $S_{p+1} \subset \rho(S_p)$ is clear, it is sufficient to prove that $\rho(S_p) \subset S_{p+1}$. Let φ be an element of $\rho(S_p)$. The space $\rho(S_p)$ can be considered as a subspace of $\mathrm{Hom}(\mathfrak{l}_{-1}, S_p)$. We define an element $\tilde{\varphi}$ of $C^1(\mathfrak{m}, \mathfrak{g})_{p+1,-1}$ as follows:

$$\tilde{\varphi}(X) = \varphi(X) \ (X \in \mathfrak{l}_{-1}), \quad \tilde{\varphi}(S) = 0.$$

Then we have

$$\partial\tilde{\varphi}(X_1, X_2) = [X_1, \varphi(X_2)] - [X_2, \varphi(X_1)] \quad \text{for } X_1, X_2 \in \mathfrak{l}_{-1}.$$

Since $\varphi \in \rho(S_p)$, we get $\partial\tilde{\varphi} = 0$. Also since $H^1(\mathfrak{m}, \mathfrak{g})_{p+1,-1} = 0$, there exits an element $s \in S_{p+1}$ such that $\partial s = \tilde{\varphi}$. Hence $\rho(S_p) \subset S_{p+1}$. □

For a pseudo-product GLA $\mathfrak{g}$ of type $(\mathfrak{l}, S)$, we furthermore assume that the prolongation $\check{\mathfrak{g}} = \bigoplus_{p\in\mathbb{Z}} \check{\mathfrak{g}}_p$ of $(\mathfrak{m}, \mathfrak{g}_0)$ is a simple graded Lie algebra (SGLA), where $\mathfrak{m} = \mathfrak{g}_-$.

Now we investigate the space $H^1(\mathfrak{m}, \mathfrak{g})_{r,-1}$. Note that, from $\check{\mathfrak{g}}_p = \mathfrak{g}_p$ for $p \leqq 0$, $H^1(\mathfrak{m}, \mathfrak{g})_{r,-1} = H^1(\mathfrak{m}, \check{\mathfrak{g}})_{r,-1}$ for $r \leqq 1$. Also we know that $H^1(\mathfrak{m}, \mathfrak{g})_{r,-1}$ is isomorphic to $H^1(\mathfrak{l}_{-1}, S)_r$ as a $\mathfrak{g}_0$-module(see Section 5 in [YY02]). Let $\Sigma = \{\gamma_1, \dots, \gamma_L\}$ be a simple root system of $\check{\mathfrak{g}}$ and let θ be the highest root of $\check{\mathfrak{g}}$. Assume that

(i) The SGLA $\check{\mathfrak{g}}$ is of type $(Y_L, \{\gamma_a, \gamma_b\})$;
(ii) $\mathfrak{l}_{-1}$ is a $\mathfrak{g}_0$-module with highest weight $-\gamma_a$;
(iii) S_{-1} is a $\mathfrak{g}_0$-module with highest weight $-\gamma_b$.

By Kostant's theorem, $H^1(\mathfrak{m}, \mathfrak{g})_{r,-1}$ is an irreducible $\mathfrak{g}_0$-module with lowest weight $\sigma_a(-\theta - \delta) + \delta$, where we use the notation in [Yam93]. Let E be the characteristic element of the GLA $\check{\mathfrak{g}}$; then

$$(\sigma_a(-\theta - \delta) + \delta)(E) = -\mu + \langle\theta, \gamma_a\rangle + 1.$$

Hence $H^1(\mathfrak{m}, \check{\mathfrak{g}})_{r,-1} \neq 0$ if and only if $r = -\mu + \langle\theta, \gamma_a\rangle + 1$. From the table in Theorem 2.1 (a) and [Bou68], we obtain the following lemma.

Lemma 5.2 *Under the above assumptions, we have*

(1) *Assume that* $(Y_L, \{\gamma_a, \gamma_b\})$ *is one of the following types:* $(A_{\ell+n+1}, \{\gamma_1, \gamma_{\ell+1}\})$ $(n \geqq 0, \ell \geqq 1)$, $(C_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$ $(\ell \geqq 1)$. *Then* $H^1(\mathfrak{m}, \mathfrak{g})_{r,-1} \neq 0$ *if and only if* $r = 0$.
(2) *Assume that* $(Y_L, \{\gamma_a, \gamma_b\})$ *is one of the following types:* $(A_{\ell+n+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(1 < i \leqq \ell, \ell \geqq 2, n \geqq 0)$, $(B_{\ell+1}, \{\gamma_2, \gamma_1\})$ $(\ell \geqq 2)$, $(D_{\ell+1}, \{\gamma_2, \gamma_1\})$ $(\ell \geqq 4)$, $(D_{\ell+1}, \{\gamma_{\ell+1}, \gamma_1\})$ $(\ell \geqq 4)$, $(D_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$, $(D_{\ell+1}, \{\gamma_\ell, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$, $(D_{\ell+1}, \{\gamma_2, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$, $(E_6, \{\gamma_1, \gamma_6\})$, $(E_6, \{\gamma_2, \gamma_1\})$, $(E_7, \{\gamma_1, \gamma_7\})$. *Then* $H^1(\mathfrak{m}, \mathfrak{g})_{r,-1} \neq 0$ *if and only if* $r = -1$.
(3) *Assume that* $(Y_L, \{\gamma_a, \gamma_b\})$ *is one of the following types:* $(C_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(1 < i \leqq \ell, \ell \geqq 2)$, $(D_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(2 < i < \ell, \ell \geqq 4)$, $(E_6, \{\gamma_3, \gamma_1\})$, $(E_7, \{\gamma_6, \gamma_7\})$. *Then* $H^1(\mathfrak{m}, \mathfrak{g})_{r,-1} \neq 0$ *if and only if* $r = -2$.

By Lemmas 5.1 and 5.2, we get the following proposition.

Proposition 5.1 *Under the above assumptions, we have:*

(1) *Unless* $(Y_L, \{\gamma_a, \gamma_b\})$ *is* $(A_{\ell+n+1}, \{\gamma_1, \gamma_{\ell+1}\})$ $(n \geqq 0, \ell \geqq 1)$ *or* $(C_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$ $(\ell \geqq 1)$, *the algebraic prolongation* $\rho(S_{-1})$ *of* S_{-1} *is* $\{0\}$.
(2) *Assume that* $(Y_L, \{\gamma_a, \gamma_b\})$ *is one of the following types:* $(C_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(1 < i \leqq \ell, \ell \geqq 2)$, $(D_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(2 < i < \ell, \ell \geqq 4)$, $(E_6, \{\gamma_3, \gamma_1\})$, $(E_7, \{\gamma_6, \gamma_7\})$. *Then* $\rho(S_{-2}) = S_{-1}$ *and* $\rho(S_{-1}) = 0$.

Actually we can check these properties by direct calculations in each cases in the previous sections. By these properties of the typical symbols, we can classify our parabolic geometries into the following four groups.

(**A**) The parabolic geometry associated with $(A_{\ell+n+1}, \{\gamma_1, \gamma_{\ell+1}\})$ $(n \geqq 0, \ell \geqq 1)$ is the geometry of the pseudo-projective systems of second order of bidegree $(\ell, n+1)$, i.e., the geometry of the second order equations of ℓ independent and $n+1$ dependent variables by point transformations. The parabolic geometry associated with $(C_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$ $(\ell \geqq 1)$ is the geometry of the pseudo-projective systems of third order of bidegree $(\ell, 1)$, i.e., the geometry of the third order equations of ℓ independent and one dependent variables by contact transformations.

(**B**) The parabolic geometries associated with $(A_{\ell+n+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(2 \leqq i \leqq \ell, n \geqq 0)$, $(D_{\ell+1}, \{\gamma_{\ell+1}, \gamma_1\})$ $(\ell \geqq 4)$, $(D_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$, $(D_{\ell+1}, \{\gamma_\ell, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$ and $(E_6, \{\gamma_1, \gamma_6\})$ are the contact geometries of finite type equations of the first order in the following sense.

In this case $\mu = 2$ and the typical symbol $\mathfrak{m}$ has the following description: $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^1(V, W)$, where $W = S_{-2}$ and $V = \mathfrak{l}_{-1}$. Moreover $\mathfrak{g}_{-1} = V \oplus S_{-1}$ and $S_{-1} \subset W \otimes V^*$. Let $J^k(n, m)$ be the space of k-jets of n independent and m dependent variables, where $n = \dim V$ and $m = \dim W$. We consider a submanifold R of $J^1(n, m)$ such that $\pi_0^1 \mid_R \colon R \to J^0(n, m)$ is a submersion. Let D be the restriction to R of the canonical system C^1 on $J^1(n, m)$ and $R^{(1)} \subset J^2(n, m)$ be the first prolongation of R (cf. Section 4.2 [Yam82]). We assume that $p^{(1)} : R^{(1)} \to R$ is onto. This assumption is equivalent to saying that (R, D) has an (n-dimensional) integral element (transversal to the fibre $\mathrm{Ker}\,(\pi_0^1 \mid_R)_*$) at each point of R. Under this integrability condition, (R, D) is a regular differential system of type $\mathfrak{m}$ if and only if the symbols of this equation R are isomorphic to $S_{-1} \subset W \otimes V^*$ at each point of R (see Section 2.1 in [SYY97] for the precise meaning of the isomorphism of the symbol). In this case, by (1) of Proposition 5.1, integral elements of (R, D) are unique at each point of R so that $p^{(1)} : R^{(1)} \to R$ is a diffeomorphism. Thus (R, D) has the (almost) pseudo-product structure corresponding to the splitting $\mathfrak{g}_{-1} = V \oplus S_{-1}$. In fact S_{-1} corresponds to the fibre direction $\mathrm{Ker}\,(\pi_0^1 \mid_R)_*$ and V corresponds to the restriction to $R^{(1)}$ of the canonical system C^2 on $J^2(n, m)$. Since $\check{\mathfrak{g}}$ is the prolongation of $\mathfrak{m}$, an isomorphism of (R, D) preserves the pseudo-product structure. In particular an isomorphism of (R, D) preserves the projection $\pi_0^1 \mid_R \colon R \to J^0(n, m)$. Hence a local isomorphism of (R, D) is the lift of a local point transformation of $J^0(n, m)$.

In this class (**B**), we can discuss the duality of our pseudo-product structures as in [Tan89]. For example, $(A_{\ell+n+1}, \{\gamma_{n+1}, \gamma_{n+j+1}\})$ corresponds to the dual of $(A_{\ell+n+1}, \{\gamma_i, \gamma_{\ell+1}\})$.

By Theorem 2.7 and 2.9 [Tan79] and Proposition 5.5 [Yam93], we observe that parabolic geometries associated with $(A_{\ell+n+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(3 \leqq i \leqq \ell - 1, n \geqq 2)$, $(D_{\ell+1}, \{\gamma_\ell, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$ and $(E_6, \{\gamma_1, \gamma_6\})$ have no local invariants. Hence in these cases, (R, D), satisfying the integrability condition, is always locally isomorphic to the model equation given in Case (1), (8) of Section 3 or Case (1) of Section 4 respectively.

(**C**) The parabolic geometries associated with $(B_{\ell+1}, \{\gamma_2, \gamma_1\})$ $(\ell \geqq 2)$, $(D_{\ell+1}, \{\gamma_2, \gamma_1\})$ $(\ell \geqq 4)$, $(D_{\ell+1}, \{\gamma_2, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$, $(E_6, \{\gamma_2, \gamma_1\})$ and $(E_7, \{\gamma_1, \gamma_7\})$ are the contact geometries of finite type equations of the second order in the following sense.

In this case $\mu = 3$ and the typical symbol $\mathfrak{m}$ has the following description: $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^2(V, W)$, where $W = \mathbb{K}$, $V = \mathfrak{l}_{-1}$ and $\dim V = n$. Moreover we have $\mathfrak{g}_{-2} = V^*$, $\mathfrak{g}_{-1} = V \oplus S_{-1}$ and $S_{-1} \subset S^2(V^*)$. In this case, we note that the standard differential system $(M_{\mathfrak{g}}, D_{\mathfrak{g}})$ of type $(Y_L, \{\gamma_a\})$ is the standard contact manifold of type Y_L (see Section 4 in [Yam93]).

We consider a submanifold R of $J^2(n, 1)$ such that $\pi_1^2 \mid_R \colon R \to J^1(n, 1)$ is a submersion. Let D be the restriction to R of the canonical systetem C^2 on $J^2(n, 1)$ and $R^{(1)} \subset J^3(n, 1)$ be the first prolongation of R. We assume that $p^{(1)} : R^{(1)} \to R$ is onto. Under this integrability condition, (R, D) is a regular differential system of type $\mathfrak{m}$ if and only if the symbols of this equation R are isomorphic to $S_{-1} \subset S^2(V^*)$ at each point of R. In this case, by (1) of Proposition 5.1, integral elements of (R, D) are unique at each point of R so that $p^{(1)} : R^{(1)} \to R$ is a diffeomorphism. Thus (R, D) has the (almost) pseudo-product structure corresponding to the splitting $\mathfrak{g}_{-1} = V \oplus S_{-1}$. In fact S_{-1} corresponds to the fibre direction $\operatorname{Ker}(\pi_1^2 \mid_R)_*$ and V corresponds to the restriction to $R^{(1)}$ of the canonical system C^3 on $J^3(n, 1)$. Since $\check{\mathfrak{g}}$ is the prolongation of $\mathfrak{m}$, an isomorphism of (R, D) preserves the pseudo-product structure. In particular an isomorphism of (R, D) preserves the projection $\pi_1^2 \mid_R \colon R \to J^1(n, 1)$ and $\partial D = (\pi_1^2)_*^{-1}(C^1)$. Hence a local isomorphism of (R, D) is the lift of a local contact transformation of $J^1(n, 1)$.

By Theorem 2.7 and 2.9 [Tan79] and Proposition 5.5 [Yam93], we observe that parabolic geometries associated with $(D_{\ell+1}, \{\gamma_2, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$, $(E_6, \{\gamma_2, \gamma_1\})$ and $(E_7, \{\gamma_1, \gamma_7\})$ have no local invariants. Hence in these cases, (R, D), satisfying the integrability condition, is always locally isomorphic to the model equation given in Case (10) of Section 3 or Case (3), (5) of Section 4 respectively. The rigidity of the parabolic geometry associated with $(D_{\ell+1}, \{\gamma_2, \gamma_{\ell+1}\})$ $(\ell \geqq 3)$ is already discussed in [YY02] in connection with the Plücker embedding equations.

(**D**) The parabolic geometries associated with $(C_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(1 < i \leqq \ell, \ell \geqq 2)$, $(D_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(2 < i < \ell, \ell \geqq 4)$, $(E_6, \{\gamma_3, \gamma_1\})$ and $(E_7, \{\gamma_6, \gamma_7\})$ are the geometries of finite type equations of the first order in the following sense.

In this case $\mu = 3$ and the typical symbol $\mathfrak{m}$ has the following description: $\mathfrak{m} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \subset \mathfrak{C}^2(V, W)$, where $W = S_{-3}$ and $V = \mathfrak{l}_{-1}$. Moreover $\mathfrak{g}_{-2} = S_{-2}$, $\mathfrak{g}_{-1} = V \oplus S_{-1}$, $S_{-2} \subset W \otimes V^*$, $S_{-1} \subset W \otimes S^2(V^*)$ and $\dim S_{-2} = \dim V$. In this case we first consider a submanifold R of $J^1(n, m)$ such that $\pi_0^1 \mid_R \colon R \to J^0(n, m)$ is a submersion, where $n = \dim V$ and $m = \dim W$. Let D be the restriction to R of the canonical systetem C^1 on $J^1(n, m)$ and $R^{(1)} \subset J^2(n, m)$ be the first prolongation of R. We assume that the symbols of this equation R are isomorphic to $S_{-2} \subset W \otimes V^*$ at each point of R and also assume that $p^{(1)} : R^{(1)} \to R$ is onto. Then (R, D) is a regular differential system of type $\hat{\mathfrak{m}} = \hat{\mathfrak{g}}_{-2} \oplus \hat{\mathfrak{g}}_{-1}$, where $\hat{\mathfrak{g}}_{-2} = W$ and $\hat{\mathfrak{g}}_{-1} = V \oplus S_{-2}$. Here the symbol algebra $\hat{\mathfrak{m}}$ is the negative part of the simple graded Lie algebra of type $(Y_L, \{\gamma_a\})$, i.e., of type $(C_{\ell+1}, \{\gamma_i\})$ $(2 \leqq i \leqq \ell)$, $(D_{\ell+1}, \{\gamma_i\})$ $(2 < i < \ell)$, $(E_6, \{\gamma_3\})$ and $(E_7, \{\gamma_6\})$ respectively. Furthermore, by (2) of Proposition 5.1, the symbols of this equation $R^{(1)}$ are isomorphic to $\rho(S_{-2}) = S_{-1} \subset W \otimes S^2(V^*)$. Let $D^{(1)}$ be the restriction to $R^{(1)}$ of the canonical system C^2 on $J^2(n, m)$ and $R^{(2)} \subset J^3(n, m)$ be the prolongation of $R^{(1)}$. We further assume that $p^{(2)} : R^{(2)} \to R^{(1)}$ is onto. Under these integrability conditions, $(R^{(1)}, D^{(1)})$ becomes a regular differential system of type $\mathfrak{m}$. Actually the set of n-dimensional integral elements of (R, D) forms a bundle over R, which contains $R^{(1)}$ as an open dense subset such that $D^{(1)}$ coincides with the canonical system induced by this Grassmanian construction (cf. Section 2 in [Yam82], Section 1 in [Yam99]). Moreover, by (1) of Proposition 5.1, integral elements of $(R^{(1)}, D^{(1)})$ are unique at each point of $R^{(1)}$ so that $p^{(2)} : R^{(2)} \to R^{(1)}$ is a diffeomorphism. Thus $(R^{(1)}, D^{(1)})$ has the (almost) pseudo-product structure corresponding to the splitting $\mathfrak{g}_{-1} = V \oplus S_{-1}$. In fact S_{-1} corresponds to the fibre direction $\mathrm{Ker}\,(p^{(1)})_*$ and V corresponds to the restriction to $R^{(2)}$ of the canonical system C^3 on $J^3(n, m)$. Since $\check{\mathfrak{g}}$ is the prolongation of $\mathfrak{m}$, an isomorphism of $(R^{(1)}, D^{(1)})$ preserves the pseudo-product structure. In particular an isomorphism of $(R^{(1)}, D^{(1)})$ preserves the projection $p^{(1)} : R^{(1)} \to R$ and $\partial D^{(1)} = (p^{(1)})_*^{-1}(D)$. Thus a local isomorphism of $(R^{(1)}, D^{(1)})$ induces that of (R, D) and coincides with the local lift of this isomorphism of (R, D). Hence the local equivalence of $(R^{(1)}, D^{(1)})$ is reducible to that of (R, D).

By Theorem 2.7 and 2.9 [Tan79] and Proposition 5.5 [Yam93], we observe that parabolic geometries associated with $(C_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(2 < i < \ell)$, $(D_{\ell+1}, \{\gamma_i, \gamma_{\ell+1}\})$ $(2 < i < \ell)$, $(E_6, \{\gamma_3, \gamma_1\})$ and $(E_7, \{\gamma_6, \gamma_7\})$ have no local invariants.

Hence in these cases, (R, D), satisfying the integrability conditions, is always locally isomorphic to the model equation given in Case (3), (9) of Section 3 or Case (2), (4) of Section 4 respectively.

Remark 5.1 Among the cases in (**A**) and (**C**), notable omissions are the parabolic geometries asociated with $(A_{\ell+1}, \{\gamma_1, \gamma_{i+1}, \gamma_{\ell+1}\})$ $(0 < i < \ell)$, which do not show up in the exceptional lists in Theorem 2.1, but are associated with differential equations of finite type as follows. In fact the standard contact manifold of type $A_{\ell+1}$ is given by $(A_{\ell+1}, \{\gamma_1, \gamma_{\ell+1}\})$. Hence the typical symbol $\mathfrak{m}$ in this case has the same description as in (**C**). The model equation in this case, as the second order system, is given by

$$\frac{\partial^2 y}{\partial x_{i_1} \partial x_{i_2}} = \frac{\partial^2 y}{\partial x_{j_1} \partial x_{j_2}} = 0 \qquad \text{for} \quad 1 \leqq i_1, i_2 \leqq i, \quad \text{and} \quad i < j_1, j_2 < \ell + 1,$$

where y is a dependent variable and $x_1, \ldots, x_\ell$ are independent variables.

References

[Bai93] T. N. Baily, Parabolic Invariant Theory and Geometry in "The Penrose Transform and Analytic Cohomology in Representation Theory" Contemp. Math. **154**, Amer. Math. Soc., 1993.

[Bou68] N. Bourbaki, *Groupes et algebres de Lie, Chapitres 4, 5 et 6*, Hermann Paris (1968).

[Bou75] N. Bourbaki, *Groupes et algebres de Lie, Chapitres 7 et 8*, Hermann Paris (1975).

[BCG91] R. Bryant, S. Chern, R. Gardner, H. Goldschmidt and P. Griffiths, *Exterior Differential Systems*, MSRI Publ. vol. **18**, Springer Verlag, Berlin 1991.

[Car10] E. Cartan : Les systèmes de Pfaff à cinq variables et les équations aux dérivées partielles du second ordre, Ann. École Normale, **27** (1910), 109–192.

[DKM99] B. Doubrov, B. Komrakov and T. Morimoto, *Equivalence of holonomic differential equations*, Lobachevskii J. of Math. 3 (1999), 39–71.

[Hum72] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag 1972.

[Kos61] B. Kostant, *Lie algebra cohomology and generalized Borel–Weil theorem*, Ann. of Math. 74 (1961), 329–397.

[Sea88] Y. Se-ashi, *On differential invariants of integrable finite type linear differential equations*, Hokkaido Math. J. 17 (1988), 151–195.

[SYY97] T. Sasaki, K. Yamaguchi and M. Yoshida, *On the Rigidity of Differential Systems modeled on Hermitian Symmetric Spaces and Disproofs of a Conjecture concerning Modular Interpretations of Configuration Spaces*, Advanced Studies in Pure Math. 25 (1997), 318–354.

[SY98] H. Sato and A. Y. Yoshikawa, *Third order ordinary differential equations and Legendre connection*, J. Math. Soc. Japan, 50 (1998), 993–1013.

[Tan70] N. Tanaka, *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ. 10 (1970), 1–82.

[Tan79] N. Tanaka, *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J. 8 (1979), 23–84.

[Tan82] N. Tanaka, *On geometry and integration of systems of second order ordinary differential equations*, Proc. Symposium on Differential Geometry, 1982, pp. 194–205 (in Japanese).

[Tan85] N. Tanaka, *On affine symmetric spaces and the automorphism groups of product manifolds*, Hokkaido Math. J. 14 (1985), 277–351.

[Tan89] N. Tanaka, *Geometric theory of ordinary differential equations*, Report of Grant-in-Aid for Scientific Research MESC Japan (1989).

[Yam82] K. Yamaguchi, Contact geometry of higher order, Japan. J. Math **8**(1982), 109–176.

[Yam83] K. Yamaguchi, Geometrization of jet bundles, Hokkaido Math. J. **12** (1983), 27–40.

[Yam93] K. Yamaguchi, Differential Systems associated with Simple Graded Lie Algebras, Advanced Studies in Pure Mathematics **22** (1993), 413–494.

[Yam99] K. Yamaguchi, G_2-geometry of overdetermined systems of second order, Trends in Mathematics (Analysis and Geometry in Several Complex Variables) (1999), Birkhäuser, Boston, 289–314.

[YY02] K. Yamaguchi and T. Yatsui, Geometry of Higher Order Differential Equations of Finite Type associated with Symmetric Spaces, Advanced Studies in Pure Mathematics **37** (2002), 397-458.

[Yat88] T. Yatsui, *On pseudo-product graded Lie algebras*, Hokkaido Math. J. 17 (1988), 333–343.

[Yat92] T. Yatsui, *On completely reducible transitive graded Lie algebras of finite depth*, Japan. J. Math. 18 (1992), 291–330.

Part IV

Quantizations and Noncommutative Geometry

Toward Geometric Quantum Theory

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Summary. The notion of μ-regulated algebras is given in a slightly revised version. Notions of their localizations, limit localizations and intertwiners among them are defined. A family of limit localizations parameterized by Σ is viewed as the space of deformations of expressions of the algebra obtained by a limit localization. The space has a flat connection defined through infinitesimal intertwiners. Several examples of localized algebra with finite generator system are given. In concrete calculation, there appear several strange phenomena such as an element having two different inverses, or an expression of some elements having sign ambiguity. However, such phenomena are related with Jacobi's theta functions, where Jacobi's theta function is redefined as the bilateral power series in a $*$-algebra.

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A little ant digs a tiny hole in a big established bank. No one cares about this tiny hole. But in some rainy season, this may grow to a big hole to cause the bank itself to collapse.

In this note, the author wants to propose a rough *scenario* for the geometrical quantum theory, which may be acceptable by differential geometers, although not by topologists, because the underlying object is *not* a topological space.

Section 1 and the first part of Section 2 up to Section 2.1 are devoted to giving a short summary of notions and several results which have already appeared in our papers. At first, a notion of μ-regulated algebra is given abstractly in a slightly revised version.

In Section 2.2 and in what follows, notions of their localizations, limit and extremal localizations, intertwiners, and infinitesimal intertwiners are defined so as to provide a new philosophical key to open the quantum world.

The original algebra is understood as a patchwork of a collection of localizations. However, the collection does not satisfy the cocycle condition. The discordance of patchwork is defined.

In Section 3, the parameter space of all limit localizations are viewed as the parameter space of possible deformations of expressions of a limit localized algebra. This is viewed as if it were a moduli-space of a limit localized algebra.

In Section 4, Section 5, we give an example of deformation of expressions of algebra generated only by one variable. It is a little surprising that Jacobi's theta functions are expressed simply as bilateral power series of an exponential function.

In Section 6, we treat several examples of localizations which might be interesting objects for study in the future. However in this paper, we have reached only the front door of the geometrical quantum world. Detailed description of the inside will be given in forthcoming papers.

1 μ-regulated algebras

In this section we introduce a notion of *μ-regulated algebras*, revised a little from that given in [17], as an abstraction of the algebra of all pseudo-differential operators of order 0 investigated in [16]. μ-regulated algebras give abstract models of algebras obtained by non-formal deformation quantizations.

1.1 Primitive axioms

In an associative algebra $(\mathcal{O}, *)$, we denote by $[a, b]$ the commutator $a * b - b * a$. A topological associative algebra $(\mathcal{O}, *)$ over $\mathbb{R}$ or $\mathbb{C}$ (denote this by $\mathbb{K}$) with the multiplicative unit 1 is called a *Fréchet algebra*, when $\mathcal{O}$ is a Fréchet space as a topological linear space. Here a Fréchet space means a complete topological vector space defined by a countable system of seminorms. A Fréchet algebra $(\mathcal{O}, *)$ is called a *μ-regulated algebra*, if there is an element μ ($\neq$0) in $\mathcal{O}$, called a *regulator*, and $(\mathcal{O}, \mu, *)$ satisfies the following axioms:

(A:1) $[\mu, \mathcal{O}] \subset \mu * \mathcal{O} * \mu$.
(A:2) $[\mathcal{O}, \mathcal{O}] \subset \mu * \mathcal{O}$.
(A:3) $\mu * \mathcal{O}$ is a closed subspace and there is a closed linear subspace B of $\mathcal{O}$ such that $\mathcal{O} = B \oplus \mu * \mathcal{O}$ (topological direct sum).
(A:4) Mappings $\mu* : \mathcal{O} \to \mu * \mathcal{O}$, $*\mu : \mathcal{O} \to \mathcal{O} * \mu$ defined by $a \to \mu * a$, $a \to a * \mu$ respectively are linear isomorphisms over $\mathbb{K}$.

If $(\mathcal{O}, *)$ is defined over $\mathbb{K} = \mathbb{C}$, we often take an additional axiom

(A:5) There is an involutive anti-automorphism $a \to \bar{a}$ such that $\bar{\mu} = \mu$ or $\bar{\mu} = -\mu$.

The most typical example of μ-regulated algebra is the symbol calculus of all pseudo-differential operators (ΨDO) of order 0 on the cotangent bundle T^*_N of a closed Riemannian manifold N. In this calculus, the regulator μ is the symbol of the operator such as $\sqrt{1 + \Delta}$, which plays the role of determining the notion of "order" of operators.

The simplest example of μ-regulated algebra is the space $C^\infty(\mathbb{R})$ of all C^∞ functions of the variable μ with the C^∞-topology, where $C^\infty(\mathbb{R}) = \mathbb{C} \oplus \mu * C^\infty(\mathbb{R})$.

Let Hol($\mathbb{C}$) be the space of all entire functions on $\mathbb{C}$ with the compact open topology, and let μ be a complex coordinate function on $\mathbb{C}$. Viewing Hol($\mathbb{C}$) as an algebra over $\mathbb{R}$, we define an associative but non-commutative product $*$ by requesting $\mu^n * i = (-1)^n i * \mu$. Then, (Hol($\mathbb{C}$), $*$) is a μ-regulated algebra such that $\mathrm{Hol}(\mathbb{C}) = \mathbb{C} \oplus \mu * \mathrm{Hol}(\mathbb{C})$.

Note that μ^2 is a central element and $(\mu^2 + 1)$ is the maximal two - sided ideal such that the quotient algebra $(\mathrm{Hol}(\mathbb{C}), *)/(\mu^2 + 1)$ is the standard quaternion field, $\mathbb{H}$, such that $\mu = j, i * \mu = k$.

We next state several facts induced easily by the axioms. By (A.1), we see $a * \mu = \mu * (a - b * \mu), b \in \mathcal{O}$, hence $\mathcal{O} * \mu \subset \mu * \mathcal{O}$. Similarly, we have $\mathcal{O} * \mu \supset \mu * \mathcal{O}$. Thus $\mathcal{O} * \mu = \mu * \mathcal{O}$, and this is a closed two-sided ideal of $\mathcal{O}$.

By (A.2), (A.3), $*$-product on $\mathcal{O}$ defines naturally a topological commutative algebra structure on the quotient space $\mathcal{O}/\mu * \mathcal{O}$ which is identified with B. This commutative algebra is denoted by $(B, \cdot)$.

The symmetric product $a \circ b = (1/2)(a * b + b * a)$ makes $(\mathcal{O}, \circ)$ a special Jordan algebra. Since

$$a\circ(b\circ c) - (a\circ b)\circ c = \frac{1}{4}[b, [c, a]] \in \mu^2 * \mathcal{O}, \tag{1}$$

the symmetric product induced on $\mathcal{O}/\mu^2 * \mathcal{O}$ is associative and commutative.

The axiom (A:4) is crucial for making it possible to consider the inverse μ^{-1} of μ by setting $[\mu^{-1}, a] = -\mu^{-1}*[\mu, a]*\mu^{-1}$. Let $\mathcal{O}[\mu^{-1}]$ be the space of all polynomials

$$\mu^{-k} * a_{-k} + \cdots + \mu^{-1} * a_{-1} + a_0, \quad a_j \in \mathcal{O}, \quad \text{where } \mu^{-k} = (\mu^{-1})^k. \tag{2}$$

The product $*$ on $\mathcal{O}$ extends $\mathcal{O}[\mu^{-1}]$ to form an associative algebra. It is easy to see that

$$[\mu^{-1} * \mathcal{O}, \mathcal{O}] \subset \mathcal{O}, \quad [\mu^{-1} * \mathcal{O}, \mu^{-1} * \mathcal{O}] \subset \mu^{-1} * \mathcal{O}.$$

By these relations we have the following:

Proposition 1 $(\mathcal{O}[\mu^{-1}], *)$ *is an associative algebra and* $ad(\mu^{-1} * \mathcal{O})$ *is a Lie algebra of derivations of* $(\mathcal{O}, *)$.

By the property (A.3), $\mathcal{O}$ is decomposed for every positive integer N into

$$\mathcal{O} = B \oplus \mu * B \oplus \cdots \oplus \mu^{N-1} * B \oplus \mu^N * \mathcal{O}. \tag{3}$$

Set $\mathrm{ad}(\mu^{-1})(a) = [\mu^{-1}, a]$, and $\mathrm{Ad}(\mu^{-1})(a) = \mu^{-1} * a * \mu$. According to the decomposition (3), we may write uniquely as follows for any $a, b \in B$:

$$\begin{aligned} a * b &= \pi_0(a, b) + \mu * \pi_1(a, b) + \cdots + \mu^k * \pi_k(a, b) + \cdots, \\ \mathrm{ad}(\mu^{-1})(a) &= \xi_0(a) + \mu * \xi_1(a) + \cdots + \mu^k * \xi_k(a) + \cdots, \\ \mathrm{Ad}(\mu^{-1})(a) &= a + \xi_0(a) * \mu + \mu * \xi_1(a) * \mu + \cdots + \mu^k * \xi_k(a) * \mu + \cdots. \end{aligned} \tag{4}$$

Definition 1 A continuous surjective homomorphism $p : (\mathcal{O}, *) \to \mathbb{K}$ such that $p(\mu) = 0$ is called a *classical point*. Ker p is denoted by $\mathcal{I}_p$. A continuous homomorphism $\tilde{p} : (\mathcal{O}, *) \to \mathbb{K}[[\mu]]$ with dense image is called a *semiclassical* point, where $\mathbb{K}[[\mu]]$ is the formal power series ring $\prod_k \mathbb{K}\mu^k$ with the direct product topology.

The totality M of classical points may be viewed as the phase space. The closed two-sided ideal $\mathcal{R} = \cap_{p\in M}\mathcal{I}_p$ is called the *radical* of $\mathcal{O}$. Every $\tilde{f} \in \mathcal{O}/\mathcal{R}$ is viewed as a function on M. The topology on M is given in such a way that every $\tilde{f}$ is viewed as a continuous function.

Definition 2 A topological automorphism ψ of $(\mathcal{O}, *)$ is called a *μ-automorphism*, if $\psi(\mu * \mathcal{O}) = \mu * \mathcal{O}$. By $\mathrm{Aut}_\mu(\mathcal{O})$, we denote the group of all μ-automorphisms.

It is obvious that $\mathrm{Aut}_\mu(\mathcal{O})\mu * \mathcal{O} = \mu * \mathcal{O}$, $\mathrm{Aut}_\mu(\mathcal{O})\mathcal{R} = \mathcal{R}$, and every $\psi \in \mathrm{Aut}_\mu(\mathcal{O})$ induces a homeomorphism of M. Let $\mathrm{Aut}_\mu(\mathcal{O})_M$ be the normal subgroup consisting of all ψ which leaves each classical point fixed. Since $\mu * \mathcal{O} \subset \mathcal{R}$, $\mathrm{Aut}_\mu(\mathcal{O})_M$ acts on $\mathcal{R}/\mu * \mathcal{O}$.

To consider a smooth structure on M, we have to assume several nice properties for $\mathrm{Aut}_\mu(\mathcal{O})$ and its action on $\mathcal{O}$, which will be given only as an optional assumption:

Option 1 $\mathrm{Aut}_\mu(\mathcal{O})$ is a generalized Lie group with the Lie algebra $\mathfrak{g}$ contained in $\mathrm{ad}(\mu^{-1} * \mathcal{O})$ as a real Lie subalgebra, and $\mathrm{Aut}_\mu(\mathcal{O})$ acts smoothly on $\mathcal{O}$.

A generalized Lie group defined in [16], [18] is not a genuine notion of Lie groups, but a pair consisting of a topological group and a Lie algebra, which has several amenable properties as Lie groups and the following conceptual advantage:

- Every closed subgroup of a generalized Lie group is a generalized Lie group.
- The quotient group of a generalized Lie group by a closed normal subgroup is a generalized Lie group.
- Finite dimensional Lie groups are generalized Lie groups, and the converse is true if the exponential mapping gives a local surjective mapping.

We do not repeat the detail, but recall the following fundamental fact:

Lemma 1 *If the equation* $\frac{d}{dt} f_t = \mathrm{ad}(\mu^{-1}{*}a)(f_t)$, $f_0 = f$, *has a real analytic solution for every* $t \in \mathbb{R}$, *and the continuity holds for initial conditions, then the fundamental solution denoted by* $e^{t\,\mathrm{ad}(\mu^{-1}*a)} : (\mathcal{O}, *) \to (\mathcal{O}, *)$ *is a one-parameter subgroup of* $\mathrm{Aut}_\mu(\mathcal{O})$.

1.2 Classical notions defined on B, or $B \oplus \mu B$

Recall $(B, \cdot)$ is the commutative algebra over $\mathbb{K}$ defined by $\mathcal{O}/\mu * \mathcal{O}$, and $\pi_0(a, b) = a \cdot b$. Note that the commutator bracket $[a, b]$ is a biderivation of $\mathcal{O} \times \mathcal{O}$ into $\mathcal{O}$. Hence the skew part π_1^- of π_1 gives a skew biderivation of $B \times B$ into B, which is denoted by $\{a, b\}$.

1.2.1 Characteristic vector fields

The derivation $\mathrm{ad}(\mu^{-1})$ of $\mathcal{O}$ is decomposed for $a \in B$ into

$$\mathrm{ad}(\mu^{-1})(a) = \xi_0(a) + \cdots + \mu^k * \xi_k(a) + \cdots .$$

ξ_0 is also a derivation of $(B, \cdot)$, which is called the *characteristic vector field*. We easily have

$$\xi_0(\{a, b\}) = \{\xi_0(a), b\} + \{a, \xi_0(b)\} \tag{5}$$

and since $\mu^{-1} * \mathcal{O}$ forms a Lie algebra, we see that Liouville bracket

$$\{a, b\}_L = a\xi_0(b) - \xi_0(a)b + \{a, b\} \tag{6}$$

defines a Lie algebra structure on B, i.e., skew-symmetric with Jacobi identity.

Definition 3 A μ-regulated algebra $(\mathcal{O}, *)$ is called a *q-Jacobi algebra*, if $\xi_0 \neq 0$. $(\mathcal{O}, *)$ is called a *q-Poisson algebra*, if $[\mu, \mathcal{O}] = \{0\}$.

1.2.2 From q-Jacobi to q-Poisson

Set $\mathcal{O}_\mu = \{a; [\mu^{-1}, a] = 0\}$. Then $\mathcal{O}_\mu$ may be viewed as a q-Poisson algebra. There is also another way to obtain a q-Poisson algebra from a q-Jacobi algebra.

Let r be the coordinate function of an open interval $(1 - \epsilon, 1 + \epsilon)$. Let $\mathcal{O}\{r, \epsilon\}$, $B\{r, \epsilon\}$ be Fréchet spaces of all C^∞ mappings from $(1 - \epsilon, 1 + \epsilon)$ into a μ-regulated algebra $\mathcal{O}$, and into its subspace B respectively. Give them the C^∞-topology. Set $\mu = \nu r^{-1}$, and treat ν as a central element, i.e., $[\nu, \mathcal{O}\{r, \epsilon\}] = 0$. It is easy to see that $\mathcal{O}\{r, \epsilon\} = B\{r, \epsilon\} \oplus \nu\mathcal{O}\{r, \epsilon\}$. $\mathcal{O}\{r, \epsilon\}$ is ν-regulated algebra under the naturally extended commutation relation $[r, f] = \nu\, \mathrm{ad}(\mu^{-1})(f)$.

By this observation, the case where μ is in the center is fundamental. When μ is in the center, one may set $\mu = i\hbar$ for any complex number $\hbar$. This is the procedure for making the quotient Fréchet algebra $\mathcal{O}/\bar{I}_\hbar$ by the closure $\bar{I}_\hbar$ of the two-sided ideal $I_\hbar = \mathcal{O} * (\mu - i\hbar)$ of $\mathcal{O}$.

We call $\mathcal{O}/\bar{I}_\hbar$ *the restricted algebra* at $\mu = i\hbar$. However if $(1 - \frac{1}{i\hbar}\mu)^{-1} \in \mathcal{O}$, such as in the case when μ is a formal parameter, the quotient algebra collapses $\mathcal{O}/\bar{I}_\hbar = \{0\}$ (cf. [11]).

1.2.3 $\mathbb{Z}_2$-graded structure

Suppose a μ-regulated algebra $(\mathcal{O}; \mu)$ is decomposed into $\mathcal{O} = \mathcal{O}^0 \oplus \mathcal{O}^1$ $(= \mathcal{O}^{ev} \oplus \mathcal{O}^{od})$ such that

$$\mathcal{O}^0 * \mathcal{O}^0 \subset \mathcal{O}^0, \quad \mathcal{O}^1 * \mathcal{O}^1 \subset \mu * \mathcal{O}^0, \quad \mathcal{O}^0 * \mathcal{O}^1,\ \mathcal{O}^1 * \mathcal{O}^0 \subset \mathcal{O}^1, \quad \mathcal{O}^1 \circ \mathcal{O}^1 \subset \mu^2 * \mathcal{O}^0,$$

where $\circ$ means the symmetric product (cf. (1)). In this situation, we often use the *graded commutator* defined by $[a, b]_\pm = a * b - (-1)^{|a||b|} b * a$, where $|a|$ means the parity of a. For $a \in \mathcal{O}^i$ $(i = 0, 1)$, we define $D(a)(f) = a * f - (-1)^{|a||f|} f * a$. $D(a)$ is a *graded derivation* similar to the exterior derivative, i.e.,

$$D(a)(f * g) = D(a)(f) * g + (-1)^{|f|} f * D(a)(g), \tag{7}$$

and in particular if $a \in \mathcal{O}^{od}$, then $D(a)\big(D(a)(f)\big) = D(a^2)(f) \in \mu^3 * \mathcal{O}$.

2 Deformation quantizations and localizations

Like many other theories, our ultimate aim is the theory of gravity. But here, we give only few examples by assuming that $(B, \cdot) \cong (C^\infty(M), \cdot)$; the space of all C^∞ functions on a manifold M with the ordinary commutative product: Suppose there are a derivation ξ_0 and a skew biderivation $\{f, g\}$ on $C^\infty(M)$. $(C^\infty(M), \xi_0, \{\ ,\ \})$ is called a *Jacobi algebra* (cf. [7]), if it satisfies (5) and $\{f, g\}_L$ defined by (6) gives a Lie algebra structure on $C^\infty(M)$. A Jacobi algebra with $\xi_0 = 0$ is called a *Poisson algebra*. A Poisson algebra is called a *symplectic algebra*, if the rank of $\{\ ,\ \}$ is dim$M$ at every point. $(C^\infty(M), \xi_0, \{\ ,\ \})$ is a *contact algebra*, if ξ_0 vanishes nowhere and rank$\{\ ,\ \} = \dim M - 1$ everywhere; M is odd dimensional in particular. For a symplectic algebra $(C^\infty(M), \{\ ,\ \})$, the theorem of Darboux shows that there is a canonical local coordinate system $x_1, \dots, x_m, y_1, \dots, y_m$ on a neighborhood of p such that

$$\{x_i, x_j\} = \{y_i, y_j\} = 0, \quad \{x_i, y_j\} = \delta_{ij}.$$

Exterior algebra Let $(\mathcal{F}^*(M), \wedge)$ be the exterior algebra of all smooth differential forms on a finite dimensional manifold. Let $(\mathcal{F}^*(M)[[\mu]], \wedge)$ be the algebra defined naturally on the space $\mathcal{F}^*(M)[[\mu]] = \prod_k \mathcal{F}^*(M)\mu^k$ with the direct product topology. Define a new product $*$ on $\mathcal{O} = \mathcal{F}^*(M)[[\mu]]$ by

$$\omega * \omega' = \begin{cases} \omega\wedge\omega', & \omega \text{ or } \omega' \in \mathcal{F}^{ev}(M), \\ \mu\omega\wedge\omega', & \omega \text{ and } \omega' \in \mathcal{F}^{od}(M). \end{cases}$$

Proposition 2 $(\mathcal{F}^*(M)[[\mu]], *)$ *is a μ-regulated algebra such that* $B = \mathcal{F}^{ev}(M)$.

If we want to take a Riemannian structure into account, we have to set $dx^i * dx^j = \mu^2 g^{ij}$. If M is the cotangent bundle T^*_N of a Riemannian manifold N, it is possible to include such a structure in a μ-regulated algebra. (See [18].)

We have seen various structures in classical differential geometry that are involved in the first few terms of the expansions (4). Thus, it is natural to ask the following (*quantization problem*):

Do these structures defined by classical terms come from μ-regulated algebras?

If this is the sole question, one has only to make a μ-regulated algebra where μ is a formal parameter, and then the quantization problems would have almost been settled.

Namely, Kontsevich [10] showed *every Poisson algebra is quantizable*. (See also [21], where we proved that all symplectic algebra is deformation quantizable.)

Thus, as a corollary, we see also that every Jacobi algebra is deformation quantizable [11].

However, if μ is not a formal but a central element of an algebra, then there appear several anomalous phenomena in the restricted algebra at $\mu = i\hbar$, $(\hbar{\neq}0)$, such as the spontaneous break-down of the associativity (cf. Section 5.4). Some of them have been discussed in [9], [12], [13] in the case of the simplest Weyl algebra of 2 generators.

2.1 Notion of vacuums and left-regular representations

By putting the traditional quantum theory in mind, a μ-regulated algebra has to be represented by a left-representation κ on a fixed Fréchet space Ψ over $\mathbb{C}$ with a Hermitian inner product structure (pre-Hilbert space). This is indeed a left-regular representation by considering a closed left ideal defined by $\{a{\in}\mathcal{O};\, a|0\rangle = 0\}$. In general, $|0\rangle$ is not an element of $\mathcal{O}$, but an element of an $\mathcal{O}$-bimodule (cf. Section 2.2.1). $|0\rangle$ is sometimes called the *vacuum*.

Writing the quotient space by $\Psi = \mathcal{O}|0\rangle$, and assuming there is a closed subspace $\Psi \subset \mathcal{O}$ representing Ψ, the left-regular representation κ is given as an operator

$$\kappa(a): \quad \psi|0\rangle \to a * \psi|0\rangle.$$

If the complex conjugation is defined and one can set $\overline{\psi|0\rangle} = \langle 0|\psi^*$, then $\kappa^*(a)$: $\langle 0|\psi^* \to \langle 0|\psi^* * a$ gives a right-regular representation as an anti-homomorphism. In other words, letting $L(\Psi)$ be the space of all continuous linear operators of Ψ into itself (with a compact open topology), we get that κ (resp. κ^*) become (continuous) homomorphisms (resp. anti-homomorphisms) of $(\mathcal{O}, *)$ into $L(\Psi)$. An element $\psi \in \Psi$ is called a *state function*.

For the sole purpose of obtaining left-, right-regular representations, one may choose $|0\rangle = 1 \in \mathcal{O}$. However, in the quantum theory it is required that $\langle 0|a|0\rangle \in \mathbb{C}$. Although this is not enough to make an inner product structure, this condition means in particular that

$$\mathrm{codim}\big(\{a;\, a|0\rangle = 0\} + \{a;\, \langle 0|a = 0\}\big) = 1.$$

Remarking that $\mathcal{O} = \Psi \oplus \{a;\, a|0\rangle = 0\} = \Psi^* \oplus \{a;\, \langle 0|a = 0\}$ and $\Psi \cap \Psi^* = \mathbb{C}$, we see that the representation space Ψ has approximately *half* the dimension of $\mathcal{O}$.

For simplicity, we denote $\kappa(a)$ by a_κ. By such a representation, the μ-regulated algebra $(\mathcal{O}, *)$ is represented as a subalgebra $\mathcal{O}_\kappa = \{a_\kappa;\, a \in \mathcal{O}\}$ of $L(\Psi)$.

For an element $H_\kappa{\in}\mathcal{O}_\kappa$, the Schrödinger equation is written as $\frac{d}{dt}\psi_t = iH_\kappa\psi_t$. However, since the fundamental solution $\exp tH_\kappa$ may not be included in $\mathcal{O}_\kappa$, we have to consider a certain extended system of $\mathcal{O}_\kappa$ by constructing an exponential calculus so that the fundamental solution is included.

2.2 Notion of localizations

It is easy to consider a *localization* of $\mathcal{O}$, if $B \cong C^\infty(M)$, $\bigcap_k \mu^k * \mathcal{O} = \{0\}$ and every π_k and ξ_k appearing in the expansion (4) is respectively a bilinear or a linear differential operator.

Namely, on every open subset U of M one can make $C^\infty(U)[[\mu]]$ a μ-regulated algebra by using the *same* π_k, ξ_k, $k \in \mathbb{N}$. The naturally defined restriction $\mathcal{O} \to C^\infty(U)[[\mu]]$ is an algebra homomorphism. $C^\infty(U)[[\mu]]$ may be called a localization of $\mathcal{O}$.

Furthermore, if $\mathcal{O} = C^\infty(M)[[\mu]]$, then the localization theorem given in [16] shows that one can replace B in the splitting $\mathcal{O} = B \oplus \mu * \mathcal{O}$ of $(A.3)$ in such a way that $\pi_k, \xi_k, k \in \mathbb{N}$ in (4) can be replaced by bilinear and linear differential operators.

However, if some of π_k or ξ_k are not local operators, or if $\mathcal{O}^\infty = \bigcap_k \mu^k * \mathcal{O}$ is a nontrivial two-sided ideal, then there is no such effective notion of localization. In spite of such difficulties, we want to know the "local generator system" of $\mathcal{O}$ in order to analyze the detailed structure of $\mathcal{O}$.

Recall that in algebraic geometry, the localization of algebra is considered by joining the inverse a^{-1} of the element a called the *divisor*. The inclusion homomorphism $\mathcal{O} \to \mathcal{O}[a^{-1}]$ is naturally defined. The notion of localizations defined in the next subsection is a mixture of these notions of localizations mentioned above. From my personal view point, this is similar to the notion of so-called "second quantization" in quantum physics.

2.2.1 $\mathcal{O}$-bimodules

Here we give the notion of localization of an associative algebra $(\mathcal{O}, *)$ with 1. We consider first another Fréchet space F, and a system $\{(\kappa_l, \kappa_r); \kappa \in K\}$ of left- and right-representations of the algebra $(\mathcal{O}, *)$ onto F.

(F, κ_l, κ_r) is called an $\mathcal{O}$-*bimodule*, if the following conditions are satisfied:

(R:1) There is $1 \in F$ such that $\kappa_l(a)(1) = \kappa_r(a)(1)$ for every $a \in \mathcal{O}$.
(R:2) $\kappa_l(1)(f) = f = \kappa_r(1)$ for every $f \in F$.
(R:3) $\kappa_r(a)(\kappa_l(b)(f)) = \kappa_l(b)(\kappa_r(a)(f))$.
(R:4) If $\kappa_l(a)(1) = 0$, then $\kappa_l(a)(f) = \kappa_r(a)(f) = 0$ for every $f \in F$.

Obviously $I_\kappa = \{a \in \mathcal{O}; \kappa_l(a)(1) = 0\}$ is a closed two - sided ideal. Denote $\mathcal{O}/I_\kappa$ by $\mathcal{O}_\kappa$ and the induced product by $*_\kappa$. $(\mathcal{O}_\kappa, *_\kappa)$ is an associative algebra. We denote by π^κ the natural projection $(\mathcal{O}, *) \to (\mathcal{O}_\kappa, *_\kappa)$.

The left- and right-representations (κ_l, κ_r) of the algebra $(\mathcal{O}, *)$ onto F induces naturally the left- and right-representations $(\tilde{\kappa}_l, \tilde{\kappa}_r)$ of the algebra $(\mathcal{O}_\kappa, *_\kappa)$ onto F. Hence, an $\mathcal{O}$-bimodule $(F, \tilde{\kappa}_l, \tilde{\kappa}_r)$ is an $\mathcal{O}_\kappa$-bimodule. $(\mathcal{O}_\kappa, *_\kappa)$ is called the *effective quotient* algebra of $(\mathcal{O}, *)$.

By denoting left-, right-multiplications by ι_l, ι_r, $(\mathcal{O}_\kappa, \iota_l, \iota_r)$ becomes also an $\mathcal{O}_\kappa$-bimodule. We denote $\kappa_l(a)(1) = \pi^\kappa(a)$ by $a_\kappa \in \mathcal{O}_\kappa$, and $\kappa_l(a)(f), \kappa_r(a)(f)$, respectively, by $a_\kappa *_\kappa f$, $f *_\kappa a_\kappa$.

It is often convenient to regard an $\mathcal{O}$-bimodule (F, κ_l, κ_r) as an extension of the effective quotient algebra algebra $(\mathcal{O}_\kappa, *_\kappa)$. In such a case, we denote this by $(F, \mathcal{O}_\kappa, *_\kappa)$.

Proposition 3 $(F, \mathcal{O}_\kappa, *_\kappa)$ *has the following properties:*

- $f *_\kappa g$ *is defined if* f *or* g *is in* $\mathcal{O}_\kappa$.
- $*_\kappa$ *are continuous bi-linear mapping of* $\mathcal{O}_\kappa \times F \to F$ *and* $F \times \mathcal{O}_\kappa \to F$.
- *The associativity* $f *_\kappa (g *_\kappa h) = (f *_\kappa g) *_\kappa h$ *holds if two of* f, g, h *are in* $\mathcal{O}_\kappa$.

A concrete example of such an extension will be given in Theorem 1 below. The following shows that there are many examples of $\mathcal{O}$-bimodules:

Proposition 4 *If* $(F, \mathcal{O}_\kappa, *_\kappa)$ *is an* $\mathcal{O}$*-bimodule, then the tensor product* $F \otimes M(n)$ *with the matrix algebra* $M(n)$*, and the dual space* F^* *are* $\mathcal{O}_\kappa$*-bimodules, providing* F^* *is a Fréchet space.*

As for notations for $\mathcal{O}$-bimodules, we often use $(F_\kappa, \mathcal{O}_\kappa, *_\kappa)$ instead of $(F_\kappa, \kappa_l, \kappa_r)$, where the suffix κ is the label of the representation when we consider a system of $\mathcal{O}$-bimodules, and $(\mathcal{O}_\kappa, *_\kappa)$ is the effective quotient algebra of $(\mathcal{O}, *)$.

Definition 4 An $\mathcal{O}$-bimodule $(F_\kappa, \mathcal{O}_\kappa, *_\kappa)$ becomes a *localization* of a μ-regulated algebra $(\mathcal{O}, *, \mu)$, if the following holds:

(L.1) By setting $\mu_\kappa = \pi^\kappa(\mu)$, $B_\kappa = \pi^\kappa(B)$, $(\mathcal{O}_\kappa, *_\kappa, \mu_\kappa)$ is a μ_κ-regulated algebra such that $\mathcal{O}_\kappa = B_\kappa \oplus \mu_\kappa *_\kappa \mathcal{O}_\kappa$.
(L.2) $\mathcal{O}_\kappa$ is dense in F_κ, and $\mu_\kappa *_\kappa F_\kappa$, $F_\kappa *_\kappa \mu_\kappa$ are closed subspaces of F_κ.
(L.3) $\mu_\kappa *_\kappa : F_\kappa \to \mu *_\kappa F_\kappa$, and $*_\kappa \mu : F_\kappa \to F_\kappa *_\kappa \mu$ are continuous linear isomorphisms.
(L.4) The closure $\overline{\pi^\kappa(B)}$ in F_κ is a direct summand of $\mu_\kappa *_\kappa F_\kappa$.

By continuity, we see $[\mu_\kappa, F_\kappa] \subset \mu_\kappa *_\kappa F_\kappa *_\kappa \mu_\kappa$, $[\mathcal{O}_\kappa, F_\kappa] \subset \mu_\kappa *_\kappa F_\kappa$, and by using (L.3) for every $a \in \mathcal{O}_\kappa$, $\mathrm{ad}(\mu_\kappa^{-1} *_\kappa a)$ is defined as a continuous linear mapping of F_κ into itself.

Let $\mathcal{K}_\mathcal{O}$ be the set of all localizations of a μ-regulated algebra $(\mathcal{O}, *, \mu)$. An element of $\mathcal{K}_\mathcal{O}$ is denoted by $(F_\kappa, \mathcal{O}_\kappa, *_\kappa, \mu_\kappa)$. $(\mathcal{O}, \mathcal{O}, *, \mu)$ is a member of $\mathcal{K}_\mathcal{O}$ as the trivial localization. Sometimes we denote a member of $\mathcal{K}_\mathcal{O}$ simply by κ, and $1 = (\mathcal{O}, \mathcal{O}, *, \mu)$.

For two localizations $\kappa = (F_\kappa, \mathcal{O}_\kappa, *_\kappa, \mu_\kappa)$, $\kappa' = (F_{\kappa'}, \mathcal{O}_{\kappa'}, *_{\kappa'}, \mu_{\kappa'})$, we consider the following:

Definition 5 A continuous bimodule homomorphism $\psi : (F_\kappa, \mathcal{O}_\kappa, *_\kappa) \to (F_{\kappa'}, \mathcal{O}_{\kappa'}, *_{\kappa'})$ is a *morphism*, if the following conditions are fulfilled:

- ψ induces a continuous homomorphism $(\mathcal{O}_\kappa, *_\kappa)$ onto $(\mathcal{O}_{\kappa'}, *_{\kappa'})$ such that $\psi(\mu_\kappa) = \mu_{\kappa'}$.
- $\psi(B_\kappa)$ is a direct summand of $\mu_{\kappa'} *_{\kappa'} \mathcal{O}_{\kappa'}$.
- The closure $\overline{\psi(B_\kappa)}$ in $F_{\kappa'}$ is a direct summand of $\mu_{\kappa'} *_\kappa F_{\kappa'}$.

For every $\kappa \in \mathcal{K}_{\mathcal{O}}$ we have a morphism $\pi^{\kappa} : (\mathcal{O}, \mathcal{O}, *, \mu) \to (F_{\kappa}, \mathcal{O}_{\kappa}, *_{\kappa}, \mu_{\kappa})$, called the *projection*. For localizations $\kappa, \kappa' \in \mathcal{K}_{\mathcal{O}}$, we say that κ' is a *localization* of κ, and denote $\kappa \succ \kappa'$, if there is a morphism $\pi_{\kappa}^{\kappa'} : (F_{\kappa}, \mathcal{O}_{\kappa}, *_{\kappa}, \mu_{\kappa}) \to (F_{\kappa'}, \mathcal{O}_{\kappa'}, *_{\kappa'}, \mu_{\kappa'})$ such that $\pi_{\kappa}^{\kappa'}\pi^{\kappa} = \pi^{\kappa'}$.

For later use we denote

$$I_{\kappa} = \operatorname{Ker}\pi^{\kappa}, \quad I_{\kappa}^{B} = B \cap I_{\kappa} \quad I_{\kappa}^{\mu} = (\mu * \mathcal{O}) \cap I_{\kappa}. \tag{8}$$

$\mathcal{K}_{\mathcal{O}}$ is partially ordered by $\succ$. For $\kappa, \kappa' \in \mathcal{K}_{\mathcal{O}}$, let $\kappa = (F_{\kappa}, \mathcal{O}_{\kappa}, *_{\kappa}, \mu_{\kappa})$, $\kappa' = (F_{\kappa'}, \mathcal{O}_{\kappa'}, *_{\kappa'}, \mu_{\kappa'})$. By definition, we see $F_{\kappa} = \overline{\mathcal{O} * 1}$ etc. Set $\mathcal{O}_{\kappa\cup\kappa'} = \mathcal{O}/\operatorname{Ker}\pi^{\kappa} \cap \operatorname{Ker}\pi^{\kappa'}$, and consider the $\mathcal{O}$-bimodule $(F_{\kappa} \oplus F_{\kappa'}, \mathcal{O}_{\kappa\cup\kappa'}, *_{\kappa\cup\kappa'})$, where $*_{\kappa\cup\kappa'}$ is the product which is naturally induced on $\mathcal{O}/\operatorname{Ker}\pi^{\kappa} \cap \operatorname{Ker}\pi^{\kappa'}$. Take the closure $\overline{\mathcal{O} * (1, 1)}$ in the space $F_{\kappa} \oplus F_{\kappa'}$ and denote this by $F_{\kappa\cup\kappa'}$. Then, it is not difficult to check that

$$\kappa\cup\kappa' = (F_{\kappa\cup\kappa'}, \mathcal{O}_{\kappa\cup\kappa'}, *_{\kappa\cup\kappa'}, \mu_{\kappa\cup\kappa'}), \quad \mu_{\kappa\cup\kappa'} = \text{the image of } \mu$$

is a localization such that $\kappa\cup\kappa' \succ \kappa, \kappa'$. Morphisms $\kappa\cup\kappa' \to \kappa, \kappa'$ are induced by projections $F_{\kappa} \oplus F_{\kappa'} \to F_{\kappa}, F_{\kappa'}$, but one should be careful, for $\pi_{\kappa\cup\kappa'}^{\kappa} : F_{\kappa\cup\kappa'} \to F_{\kappa}$ may not be surjective.

κ and κ' are called *mutually independent*, if there is no κ'' such that $\kappa, \kappa' \succ \kappa''$.

Recall that a localization $\kappa = (F_{\kappa}, \mathcal{O}_{\kappa}, \mu_{\kappa})$ is defined by a pair of left-, right-representations (κ_l, κ_r) which makes F_{κ} an $\mathcal{O}/I_{\kappa}$-bimodule. For an automorphism $\varphi \in \operatorname{Aut}_{\mu}(\mathcal{O})$, the pull-back $\varphi^*\kappa = (\varphi^*\kappa_l, \varphi^*\kappa_r)$ is a pair of left-, right-representations which makes F_{κ} an $\mathcal{O}/\varphi^{-1}(I_{\kappa})$-bimodule by defining

$$a *_{\varphi^*(\kappa)} f = \varphi(a) *_{\kappa} f, \quad f *_{\varphi^*(\kappa)} a = f *_{\kappa} \varphi(a), \quad \text{for } a \in \mathcal{O},\ f \in F_{\kappa}. \tag{9}$$

We define $\varphi^*(\kappa) = (F_{\varphi^*(\kappa)}, \mathcal{O}_{\varphi^*(\kappa)}, *_{\varphi^*(\kappa)}, \mu_{\varphi^*(\kappa)}) \in \mathcal{K}_{\mathcal{O}}$ by $(F_{\kappa}, \mathcal{O}/\varphi^{-1}I_{\kappa}, *_{\varphi^*(\kappa)}, \pi^{\kappa}(\varphi^{-1}(\mu)))$, where $*_{\varphi^*(\kappa)}$ is the product induced naturally on $\mathcal{O}/\varphi^{-1}I_{\kappa}$.

In this way, $\operatorname{Aut}_{\mu}(\mathcal{O})$ acts naturally on $\mathcal{K}_{\mathcal{O}}$ as order preserving isomorphisms. We denote the *isotropy subgroup* at κ by $\operatorname{Aut}_{[\kappa]}(\mathcal{O}) = \{\phi \in \operatorname{Aut}_{\mu}(\mathcal{O}); \phi^*(\kappa) = \kappa\}$. Note that $\phi^*(\kappa) = \kappa$ if and only if $\phi : \mathcal{O} \to \mathcal{O}$ induces an isomorphism $\phi_{\kappa} : \mathcal{O}_{\kappa} \to \mathcal{O}_{\kappa}$ which extends to an isomorphism $F_{\kappa} \to F_{\kappa}$.

2.2.2 Typical examples

For making an example, we have to start with a little more general setting. Let $(\mathcal{O}; \hat{*})$ be a μ-regulated algebra, and let $\mathcal{O}[\boldsymbol{u}]$ be the totality of polynomials of $u_1, \dots, u_n$ with coefficients in $\mathcal{O}$. $\mathcal{O}[\boldsymbol{u}]$ is an associative algebra under the naturally extended $\hat{*}$-product. Here each u_i is treated as a central element.

For an arbitrarily fixed $(n \times n)$-complex matrix Λ, we define the product

$$f *_\Lambda g = f e^{\frac{\mu}{2}(\sum \overleftarrow{\partial_{u_i}} \hat{*} \Lambda^{ij} \overrightarrow{\partial_{u_j}})} g = \sum_k \frac{\mu^k}{k!2^k} \Lambda^{i_1 j_1} \cdots \Lambda^{i_k j_k} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \hat{*} \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g. \tag{10}$$

This defines an associative algebra $(\mathcal{O}[\boldsymbol{u}], *_\Lambda)$.

In what follows we denote $\partial_{u_i}\partial_{u_j}\cdots\partial_{u_k}$ by simplified notation $\partial_{u_i u_j \cdots u_k}$. It is easy to see ∂_{u_i} is a derivation of $(\mathcal{O}[\boldsymbol{u}], *_\Lambda)$.

Equality (10) is also written by a little strange notation

$$\sum_k \frac{(\mu)^k}{k!2^k} f \overleftarrow{\partial_{u_{i_1}\cdots u_{i_k}}} \Lambda^{i_1 j_1} \hat{*} \cdots \hat{*} \Lambda^{i_k j_k} \overrightarrow{\partial_{u_{j_1}\cdots u_{j_k}}} g$$

The over left/right arrows indicate to which side the differentiation operator acts.

Let $\mathrm{Hol}(\mathbb{C}^n, \mathcal{O})$ be the space of all holomorphic mappings from $\mathbb{C}^n$ into $\mathcal{O}$. For every $p>0$, we set a subspace as follows:

$$\mathcal{E}_p(\mathbb{C}^n, \mathcal{O}) = \{f(\boldsymbol{u}) \in \mathrm{Hol}(\mathbb{C}^n, \mathcal{O});\ \|f\|_{p,s;\lambda} = \sup \|f\|_\lambda e^{-s|\boldsymbol{u}|^p} < \infty,\ \forall s > 0\} \tag{11}$$

where $|\boldsymbol{u}|^2 = u_1^2 + \cdots + u_n^2$ and $\|\cdot\|_\lambda$ is the family of seminorms defining the Fréchet topology of $\mathcal{O}$. The family of seminorms $\{\|f\|_{p,s;\lambda}\}_{\lambda,s>0}$ induces a topology of $\mathcal{E}_p(\mathbb{C}^n, \mathcal{O})$.

We denote $\mathcal{E}_{p+}(\mathbb{C}^n, \mathcal{O}) = \bigcap_{q>p} \mathcal{E}_q(\mathbb{C}^n, \mathcal{O})$ with the intersection (projective limit) topology.

The following result can be obtained by almost the same proof as in [11], [12], in which the case $\mathcal{O} = \mathbb{C}$ has been treated:

Theorem 1 *For every pair* (p, p') *such that* $\frac{1}{p} + \frac{1}{p'} \geq 1$ *the product* (10) *extends to define a continuous bilinear mapping* $\mathcal{E}_p(\mathbb{C}^n, \mathcal{O}) \times \mathcal{E}_{p'}(\mathbb{C}^n, \mathcal{O}) \to \mathcal{E}_{p \vee p'}(\mathbb{C}^n, \mathcal{O})$.

If $\mathcal{O}$ is a μ-regulated algebra over $\mathbb{C}$, then $\mathcal{E}_p(\mathbb{C}^n, \mathcal{O}) = \mathcal{E}_p(\mathbb{C}^n, B) \oplus \mu *_\Lambda \mathcal{E}_p(\mathbb{C}^n, \mathcal{O})$. Thus, if $0 < p \leq 2$, then Theorem 1 shows that $(\mathcal{E}_p(\mathbb{C}^n, \mathcal{O}), *_\Lambda)$ is a μ-regulated algebra. However, if $p>2$, then $\mathcal{E}_p(\mathbb{C}^n, \mathcal{O})$ is only an $\mathcal{E}_{p'}(\mathbb{C}^n, \mathcal{O})$-bimodule for $p'>0$ such that $\frac{1}{p} + \frac{1}{p'} \geq 1$. Thus, $\mathcal{E}_p(\mathbb{C}^n, \mathcal{O})$ is a localization of the μ-regulated algebra $\mathcal{E}_{p'}(\mathbb{C}^n, \mathcal{O})$.

For another constant matrix $\tilde{\Lambda}$, we define another product on $(\mathcal{O}[\boldsymbol{u}], *_\Lambda)$ by

$$f \tilde{*} g = f e^{\frac{\mu}{2}(\sum \overleftarrow{\partial_{u_i}} \tilde{\Lambda}^{ij} *_\Lambda \overrightarrow{\partial_{u_j}})} g = \sum_k \frac{(\mu)^k}{k!2^k} \tilde{\Lambda}^{i_1 j_1} \cdots \tilde{\Lambda}^{i_k j_k} \partial_{u_{i_1}\cdots u_{i_k}} f *_\Lambda \partial_{u_{j_1}\cdots u_{j_k}} g.$$

This defines also an associative algebra $(\mathcal{O}[\boldsymbol{u}], \tilde{*})$.

Since $\Lambda, \hat{\Lambda}$ are constant matrices, and compositions of matrices are not used in the calculus, we may exchange the order of differentiations $\partial_{u_{i_1}\cdots u_{i_k}}$; the new product is written as

$$f\tilde{*}g = \sum_k \frac{(\mu)^k}{k!2^k}(\tilde{\Lambda}+\Lambda)^{i_1j_1}\cdots(\tilde{\Lambda}+\Lambda)^{i_kj_k}\partial_{u_{i_1}\cdots u_{i_k}} f\hat{*}\partial_{u_{j_1}\cdots u_{j_k}} g.$$

This may be rewritten as follows:

$$fe^{\frac{\mu}{2}(\sum \overleftarrow{\partial_{u_i}}(\tilde{\Lambda}+\Lambda)^{ij}\hat{*}\overrightarrow{\partial_{u_j}})}g = fe^{\frac{\mu}{2}(\sum \overleftarrow{\partial_{u_i}}\tilde{\Lambda}^{ij}e^{\frac{\mu}{2}(\sum\overleftarrow{\partial_{u_k}}\Lambda^{kl}\hat{*}\overrightarrow{\partial_{u_k}})}\overrightarrow{\partial_{u_j}})}g. \tag{12}$$

On the other hand, using a symmetric matrix K we consider $\frac{1}{k!}(\frac{\mu}{4}\sum K^{ij}\partial_{u_iu_j})^k(f*_\Lambda g)$. Splitting this into

$$\sum_{p+q+r=k}\frac{(\mu)^r}{r!2^r}K^{i_1j_1}\cdots K^{i_rj_r}\left\{\partial_{u_{i_1}\cdots u_{i_r}}\frac{1}{p!}\left(\frac{\mu}{4}\sum K^{ij}\partial_{u_iu_j}\right)^p f\right\}$$
$$*_\Lambda\left\{\partial_{u_{j_1}\cdots u_{j_r}}\frac{1}{q!}\left(\frac{\mu}{4}\sum K^{ij}\partial_{u_iu_j}\right)^q g\right\},$$

we have the following useful formula:

$$e^{\frac{\mu}{4}\sum K^{ij}\partial_{u_iu_j}}\left(\left(e^{-\frac{\mu}{4}\sum K^{ij}\partial_{u_i}\partial_{u_j}}f\right)*_\Lambda\left(e^{-\frac{\mu}{4}\sum K^{ij}\partial_{u_iu_j}}g\right)\right)$$
$$= fe^{\frac{\mu}{2}(\sum\overleftarrow{\partial_{u_i}}*_\Lambda K^{ij}*_\Lambda\overrightarrow{\partial_{u_j}})}g = f*_{\Lambda+K}g. \tag{13}$$

Namely, the new product obtained by using a symmetric matrix does not change the algebraic structure. Thus, the equality (13) gives:

Proposition 5 $e^{\frac{\mu}{4}\sum K^{ij}\partial_{u_iu_j}}: (\mathcal{O}[\boldsymbol{u}]; *_\Lambda) \to (\mathcal{O}[\boldsymbol{u}]; *_{\Lambda+K})$ *is an isomorphism. Namely, the algebraic structure depends only on the skew symmetric part J of Λ.*

Let $\kappa, \kappa' = \kappa + K$ be symmetric parts of $\Lambda, \Lambda+K$ respectively, and let J be the skew symmetric part of Λ. We call $e^{(\mu/4)\sum(\kappa'-\kappa)^{ij}\partial_{u_iu_j}}$ the *intertwiner*, and denote it by $I_\kappa^{\kappa'}$.

The following result can be obtained by the same proof as in [11], [12]:

Theorem 2 *If $p \le 2$, the intertwiner $I_\kappa^{\kappa'}$ extends a topological algebra isomorphism from $(\mathcal{E}_p(\mathbb{C}^n, \mathcal{O}), *_{\kappa+J})$ onto $(\mathcal{E}_p(\mathbb{C}^n, \mathcal{O}), *_{\kappa'+J})$.*

2.3 Abstract notion of intertwiners and sogo

These are also the most fundamental notions besides the notion of localization. To give an abstract notion of *intertwiners*, we have to take a little smaller subset $\widetilde{\mathcal{K}}_\mathcal{O}$ of $\mathcal{K}_\mathcal{O}$ having the following property:

- If $\kappa, \kappa' \succ \kappa_0$, then there is $\tilde{\kappa}$ such that $\tilde{\kappa} \succ \kappa, \kappa'$ and $\pi_\kappa^{\kappa_0}F_\kappa \cap \pi_{\kappa'}^{\kappa_0}F_{\kappa'} = \pi_{\tilde{\kappa}}^{\kappa_0}F_{\tilde{\kappa}}$ holds.

$\mathrm{Aut}_\mu(\mathcal{O})$ acts naturally on the partially ordered subset $\widetilde{\mathcal{K}}_\mathcal{O}$ preserving the partial order $\succ$.

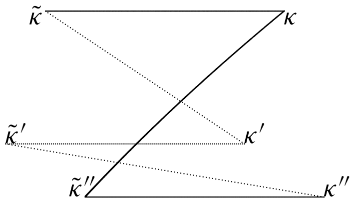
Definition 6 For $\kappa, \kappa' \in \widetilde{\mathcal{K}}_{\mathcal{O}}$, a densely defined linear one-to-one mapping $I_\kappa^{\kappa'}$ from F_κ into $F_{\kappa'}$ is called an *intertwiner*, if:

- $I_\kappa^{\kappa'} : (\mathcal{O}_\kappa, *_\kappa, \mu_\kappa) \to (\mathcal{O}_{\kappa'}, *_{\kappa'}, \mu_{\kappa'})$ is an isomorphism.
- If $\tilde{\kappa} \succ \kappa, \kappa'$, then the equality $I_\kappa^{\kappa'} \pi_{\tilde{\kappa}}^{\kappa}(a) = \pi_{\tilde{\kappa}}^{\kappa'}(a)$ holds for all $a \in F_{\tilde{\kappa}}$.
- For $\kappa, \kappa' \in \widetilde{\mathcal{K}}_{\mathcal{O}}$, suppose $\kappa_i \succ \kappa, \kappa'$ for $i = 1, 2, \ldots, m$. Then the defining domain of $I_\kappa^{\kappa'}$ must contain the linear hull of $\cap_{i=1}^m \pi_{\kappa_i}^{\kappa} F_{\kappa_i}$.

It is clear that $(I_\kappa^{\kappa'})^{-1} = I_{\kappa'}^{\kappa}$, and $I_\kappa^\kappa = 1$. We can find such intertwiners in several concrete examples (cf. Theorem 2). However, the existence of intertwiners is not so obvious in general. In general, intertwiners do not satisfy the cocycle condition, i.e., $I_{\kappa''}^{\kappa} I_{\kappa'}^{\kappa''} I_\kappa^{\kappa'}(f)$ is not necessarily f. Therefore it should deserve a definition of sogo.

Definition 7 $D_{\kappa\kappa'\kappa''} = I_{\kappa''}^{\kappa} I_{\kappa'}^{\kappa''} I_\kappa^{\kappa'}$ is called the *discordance*, or *sogo* [1].

An example of sogo will be given in Section 4. But, remark that known examples of sogo appear only as multiplicative constants, or additive constants (see [25]). Here we give only the reason why sogo appears. For $\kappa, \kappa', \kappa'' \in \widetilde{\mathcal{K}}_{\mathcal{O}}$, suppose there are $\tilde{\kappa}, \tilde{\kappa}', \tilde{\kappa}''$ such that $\tilde{\kappa} \succ \kappa, \kappa'$, $\tilde{\kappa}' \succ \kappa', \kappa''$, $\tilde{\kappa}'' \succ \kappa'', \kappa$. Even if the following identities hold $\pi_{\tilde{\kappa}}^{\kappa'}(f) = \pi_{\tilde{\kappa}'}^{\kappa'}(f')$, $\pi_{\tilde{\kappa}'}^{\kappa''}(f') = \pi_{\tilde{\kappa}''}^{\kappa''}(f'')$, $\pi_{\tilde{\kappa}''}^{\kappa}(f'') = \pi_{\tilde{\kappa}}^{\kappa}(\hat{f})$, we can not conclude $\pi_{\tilde{\kappa}}^{\kappa}(\hat{f}) = \pi_{\tilde{\kappa}}^{\kappa}(f)$.



Proposition 6 *Intertwiners have sogo in general, even though these give isomorphisms between $\mathcal{O}_\kappa$'s. Suppose $\kappa_0 \succ \kappa, \kappa', \kappa''$. Then $D_{\kappa\kappa'\kappa''}$ is a densely defined monomorphism which is the identity on $\pi_{\kappa_0}^{\kappa} F_{\kappa_0}$.*

When a sogo appears in localizations, we have to treat $f \in F_\kappa$ such that $D_{\kappa\kappa'\kappa''} f \neq f$ as a *multi-valued*, or a *set-valued* element. Hence the disjoint union $\coprod_\kappa F_\kappa$ can never have a vector bundle structure. The notion of bundle gerbe (cf. [3]) may be useful for that purpose. Although the cohomological aspect is stressed in the notion of gerbes, we think the idea is based on much more primitive phenomena.

In general, representation spaces of localized algebras may not be glued together to form a global representation space (cf. [14]).

In spite of this, we have to treat the space of "sections" of $\coprod_\kappa F_\kappa$ in order to construct the field theory. For that purpose we have to restrict the "domain" of localizations so that the sogo disappears. We shall give later an example of elements whose domain of localizations are restricted to the complex "right half" space (cf. (49)).

[1] Sogo means in Japanese the discordance of upper and lower arrangement of teeth.

2.3.1 Exponential calculus on a localization

In a suitably localized system, it is natural to expect that we have a *generator system*.

Definition 8 A finite dimensional linear subspace $\mathfrak{L}_\kappa$ of $\mathcal{O}_\kappa$ is a *local generator system* of $(\mathcal{O}_\kappa, *_\kappa)$, if the subalgebra generated by $\mathfrak{L}_\kappa$ is dense in $\mathcal{O}_\kappa$.

However, since $(\mathcal{O}_\kappa, *_\kappa)$ is non-commutative in general, it is difficult to express an element in a univalent way. In the beginning of quantum theory, this was called the *ordering problem*. In fact, there are few concrete examples of algebras given by an explicit formula of $*_\kappa$-product by using a system of generators. Thus, a non-commutative algebra with which we are concerned needs to be restricted to the one that is linearly isomorphic to a certain well-known algebra having univalent expression rule such as the usual polynomial algebra or Grassmann algebra, on which non-commutative product structure is given by a concrete product formula. The ordering problem in the Weyl algebra has been discussed in [11], and in [13] as K-orderings, but since this is the same context as localizations, it is better to call them K-expressions. Since K moves continuously, K may be viewed as a *deformation parameter* or the parameter of *variations*. However, since both "deformation" and "variation" have certain specific meanings in [12] and [6], we use "deformation of expressions" .

See also Section 4 for its simplest version.

Suppose a localization $(F_\kappa, \mathcal{O}_\kappa, *_\kappa)$ is given. For every $a \in \mathcal{O}_\kappa$, the left-/right-multiplication $f \to a *_\kappa f$,/ $f \to f *_\kappa a$ are regarded as vector fields on the space F_κ. Equations of integral curves through $1 \in F_\kappa$ are respectively

$$\frac{d}{dt} f_t(\kappa) = a *_\kappa f_t(\kappa), \;\; f_0(\kappa) = 1, \qquad \frac{d}{dt} g_t(\kappa) = g_t(\kappa) *_\kappa a, \;\; g_0(\kappa) = 1. \quad (14)$$

The existence of solutions is not ensured in general, even if a is a member of $\mathfrak{L}_\kappa$, but the uniqueness holds for real analytic solutions.

If $f_t(\kappa)$ is a real analytic solution of the left equation of (14), then we see $f_t(\kappa) * a = a * f_t(\kappa)$ by the uniqueness and the associativity $(a * f_t(\kappa)) * a = a * (f_t(\kappa) * a)$. Using this, we see $f_t(\kappa)$ is also the solution of the right equation. We denote the real analytic solution by $e_{*_\kappa}^{ta}$ if it exists in F_κ. By the uniqueness, the exponential law holds which will be denoted by

$$e_{*_\kappa}^{(s+t)a} = e_{*_\kappa}^{sa} *_\kappa e_{*_\kappa}^{ta}, \quad e_{*_\kappa}^{z+ta} = e^z e_{*_\kappa}^{ta}, \quad s, t, z \in \mathbb{C}. \quad (15)$$

On the other hand, $[a, f] = a *_\kappa f - f *_\kappa a$ is defined for every $a \in \mathcal{O}_\kappa$ as a continuous linear mapping of F_κ into itself. Denote this by $\mathrm{ad}(a)$ and consider the equation

$$\frac{d}{dt} g_t(\kappa) = \mathrm{ad}(a) g_t(\kappa), \quad g_0(\kappa) = g. \quad (16)$$

If the solution exists, it will be denoted by $e^{t\,\mathrm{ad}(a)} g$. If the solution $e_{*_\kappa}^{ta}$ of (14) exists, then we have the identity $e^{t\,\mathrm{ad}(a)} g = e_{*_\kappa}^{ta} *_\kappa g *_\kappa e_{*_\kappa}^{-ta}$, if the associativity $(e_{*_\kappa}^{ta} *_\kappa g) *_\kappa$

$e_{*_\kappa}^{-ta} = e_{*_\kappa}^{ta} *_\kappa (g *_\kappa e_{*_\kappa}^{-ta})$ holds. We denote the right-hand side by $\mathrm{Ad}(e_{*_\kappa}^{ta})(g)$ if this is the case.

Recall that we do not have a good criterion for the associativity. The way of checking the associativity is mainly based on the fact that the associativity holds when the regulator μ is treated as a formal parameter. Therefore, if the subsystem with which we are concerned is embedded in another subsystem in which the regulator μ is formal, then we obtain the desired associativity. See [18] for detail. However, in a general setting, one can not say anything further without knowledge of a concrete expression such as (50) obtained in the sequel by using a generator system.

In the last part of this section, we indicate a strange fact which often appears in the exponential calculus. (Cf. [12].) Loosely speaking, this is the fact that for "almost all" elements $a \in \mathcal{O}_\kappa$, the integral $\int_{-\infty}^{\infty} e_*^{ta} dt$, and hence $\sum_{n\in\mathbb{Z}} e_*^{na}$ converge in a certain F_κ (see Section 4 for concrete examples). Since $\lim_{t\to\pm\infty} e_*^{ta} = 0$ in such a case, these imply that $a \in \mathcal{O}_\kappa$, and $e_*^a - 1$ have two different two-sided inverses respectively: Namely, we have inverses as follows:

$$-\int_0^\infty e_*^{ta} dt, \quad \int_{-\infty}^0 e_*^{ta} dt, \quad -\sum_{n=0}^{\infty} e_*^{na}, \quad \sum_{n=-\infty}^{0} e_*^{na}, \quad \text{(cf. Section 5.4)}.$$

For instance, the continuity of $a*$ gives $a * \int_{-\infty}^0 e_*^{ta} dt = \int_{-\infty}^0 a * e_*^{ta} dt = \int_{-\infty}^0 \frac{d}{dt} e_*^{ta} dt = 1$, and similarly we see $\int_{-\infty}^0 e_*^{ta} dt * a = 1$.

These violate the associativity. Recall that $a^{-1} = a^{-1} * (a * (a')^{-1}) = (a^{-1} * a) * (a')^{-1} = (a')^{-1}$ if the associativity holds. In particular, write the last two quantities as

$$\sum_{n=0}^{\infty} e_*^{na} = \frac{1}{1 - e_*^a}, \qquad -\sum_{-\infty}^{1} e_*^{na} = \frac{e_*^{-a}}{e_*^{-a} - 1}$$

to see how the associativity is broken.

However, such a phenomenon is necessary to define a closed left ideal $\{a \in \mathcal{O};\, a * |0\rangle = 0\}$. Suppose $|0\rangle = \int_{-\infty}^{\infty} e_*^{ta} dt$ is an element of F_κ. Then, we see that

$$a * |0\rangle = \int_{-\infty}^{\infty} \frac{d}{dt} e_*^{ta} dt = \lim_{t\to\infty} e_*^{ta} - \lim_{t\to-\infty} e_*^{ta} = 0$$

must hold, but the breakup of associativity makes it possible to avoid collapsing such that $|0\rangle = (a^{-1} * a) * |0\rangle = 0$.

The main difference of $-\int_0^\infty e_*^{ta} dt$ and $\sum_{n=0}^{\infty} e_*^{na}$ appears in the following calculations:

$$-\sum_{k=1}^{\infty} \left(\frac{\partial}{\partial\beta} \log\right) \int_0^\infty e_*^{t\beta ka} dt, \qquad \sum_{k=1}^{\infty} \left(\frac{\partial}{\partial\beta} \log\right) \sum_{n=0}^{\infty} e_*^{n\beta ka}, \tag{17}$$

where $(\frac{\partial}{\partial\beta} \log) f(\beta) = f'(\beta) * f(\beta)^{-1}$. For the first one, the integration by parts gives

$$\frac{\partial}{\partial\beta}\int_0^\infty e_*^{t\beta ka}dt = \beta^{-1}\int_0^\infty t(k\beta a)*e_*^{tk\beta a}dt = -\beta^{-1}\int_0^\infty e_*^{t\beta ka}dt.$$

Thus, the first quantity of (17) is $\sum_{k=1}^{\infty}\beta^{-1}$, and there is no way to avoid divergence. On the other hand for the second one, the identity

$$\frac{\partial}{\partial\beta}\sum_{n=0}^{\infty}e_*^{n\beta ka} = ka*\sum_{n=0}^{\infty}ne_*^{n\beta ka} = ka*\left(\sum_{n=0}^{\infty}e_*^{n\beta ka}\right)*\left(\sum_{n=1}^{\infty}e_*^{n\beta ka}\right),$$

gives $\sum_{k=1}^{\infty}(\sum_{n=1}^{\infty}e_*^{n\beta ka})*ka$. This converges under certain K-expressions (cf. τ-expression of Section 5.1).

3 Limit, extremal localizations, infinitesimal intertwiners

There are infinitely long series of localizations $\kappa_0 \succ \kappa_1 \succ \kappa_2 \succ \cdots$. We call such a series a *countable chain* of localizations, and denote by $C(\widetilde{\mathcal{K}}_{\mathcal{O}})$ the set of all countable chains. Countable chains $\sigma = \{\kappa_i\}_i, \sigma' = \{\kappa_i'\}_i$ are said to be *co-final*, if there is a countable chain including both σ, σ' in an order preserving manner.

Given a countable chain $\sigma = \{\kappa_i\}_i \in C(\widetilde{\mathcal{K}}_{\mathcal{O}})$, we consider the following series:

$$\mathcal{O}_{\kappa_0}\to\mathcal{O}_{\kappa_1}\to\mathcal{O}_{\kappa_2}\to\cdots,\quad F_{\kappa_0}\to F_{\kappa_1}\to F_{\kappa_2}\to\cdots.$$

Recall that $\mathcal{O}_{\kappa_i} = \mathcal{O}/I_{\kappa_i}$, $B_{\kappa_i} = B/I^B_{\kappa_i}$, $\mu_\kappa *_\kappa \mathcal{O}_{\kappa_i} = \mu*\mathcal{O}/I^\mu_{\kappa_i}$ where $I_{\kappa_i} = \mathrm{Ker}\pi^{\kappa_i}$, and $I_{\kappa_i}\subset I_{\kappa_{i+1}}$. We set $\mathcal{O}_\sigma = \mathcal{O}/\overline{\cap_i I_{\kappa_i}}$, $B_\sigma = B/\overline{\cap_i I^B_{\kappa_i}}$, $\mu *_\sigma \mathcal{O}_\sigma = \mu*\mathcal{O}/\overline{\cap_i I^\mu_{\kappa_i}}$, where $\overline{A}$ is the closure of A in $\mathcal{O}$. Let $\pi^\sigma_{\kappa_i}:\mathcal{O}_{\kappa_i}\to\mathcal{O}_\sigma$ be the natural projection, and set $\mu_\sigma = \pi^\sigma_{\kappa_i}(\mu_{\kappa_i})$. Denote by $*_\sigma$ the product induced naturally on $\mathcal{O}_\sigma$. Then, we see that $(\mathcal{O}_\sigma, *_\sigma, \mu_\sigma)$ is a μ_σ-regulated algebra.

Recall also $\mathrm{Ker}\pi^{\kappa_j}_{\kappa_i}\subset\mathrm{Ker}\pi^{\kappa_{j+1}}_{\kappa_i}$. Let $\tilde F_{\kappa_i} = F_{\kappa_i}/\overline{\cap_{j\ge i}\mathrm{Ker}\pi^{\kappa_j}_{\kappa_i}}$. Then we have an increasing sequence of Fréchet spaces $\{\tilde F_{\kappa_i}\}_i$. Let $\tilde F_\sigma = \varinjlim \tilde F_{\kappa_i} = \cap\tilde F_{\kappa_i}$ the injective limit.

We call the countable chain $\{\kappa_i\}_i$ *finite type*, if $\tilde F_\sigma = \cap\tilde F_{\kappa_i} = \tilde F_{\kappa_j}$ for some κ_j so that $\tilde F_\sigma$ is a Fréchet space. Note that this does not necessarily imply that $\overline{\cap_{k\ge i}\mathrm{Ker}\pi^{\kappa_k}_{\kappa_i}} = \cap_{j\ge k\ge i}\mathrm{Ker}\pi^{\kappa_k}_{\kappa_i}$.

Let $\mathcal{F}_{\mathcal{O}}$ be the subset of $C(\tilde{\mathcal{K}}_{\mathcal{O}})$ consisting of all countable chains of finite type.

Lemma 2 *For a countable chain of finite type, we see that $(\tilde F_\sigma, \mathcal{O}_\sigma, *_\sigma, \pi^\sigma(\mu))$ is a localization of $(\mathcal{O}, *, \mu)$. This will be called the limit localization.*

Lemma 3 $\mathrm{Aut}_\mu(\mathcal{O})$ *acts naturally on $\mathcal{F}_{\mathcal{O}}$.*

Proof. Let $\varphi\in\mathrm{Aut}_\mu(\mathcal{O})$. Recall that the action of φ on an $\mathcal{O}$-bimodule F_{κ_i} etc. is given by regarding F_{κ_i} as a $\varphi(\mathcal{O})$-bimodule. Hence, everything is traced as $\varphi(\mathcal{O})$-bimodules. □

3.1 Topology and a smooth structure on Σ

Apparently, we are concerned only with co-final classes of $\mathcal{F}_{\mathcal{O}}$. We denote the set of all co-final classes by $\Sigma = \mathcal{F}_{\mathcal{O}}/\!\sim$. An element $\sigma \in \Sigma$ is called the *co-final point* of the series $\{\kappa_i\}_i$.

If we imagine the limit "point" σ of such an infinitely long sequence, then totality of $\sigma = \lim_n \kappa_n$ may be viewed as a *continuum* Σ. For a fixed κ we have many countable chains of finite type starting with κ. Taking all co-final points we have a subset Σ_κ of Σ. Using the cascade structure on $\mathcal{F}_{\mathcal{O}}$, we make Σ a topological space. Namely for a co-final point σ of $\{\kappa_i\}_i$, $\{\Sigma_{\kappa_i}, i = 0, 1, 2, \dots\}$ gives the basis of neighborhoods of σ.

Proposition 7 $\mathrm{Aut}_\mu(\mathcal{O})$ *acts on the set* Σ *as homeomorphisms.*

For a co-final point σ, we denote the isotropy group at σ by $\mathrm{Aut}_{[\sigma]}(\mathcal{O}) = \{\phi\in\mathrm{Aut}_\mu(\mathcal{O}); \phi(\sigma) = \sigma\}$.

If we take Option 1, we can define the smoothness by saying that $\mathrm{Aut}_\mu(\mathcal{O})$ acts smoothly on the set Σ as "diffeomorphisms":

Definition 9 A function $f : \Sigma \to \mathbb{R}$ is said to be *smooth*, if for every one-parameter subgroup φ_t of $\mathrm{Aut}_\mu(\mathcal{O})$, generated by a member of $\mathfrak{g}$, $f(\varphi_t(\sigma))$ is C^∞ with respect to t and every derivative is continuous with respect to (t, σ).

Intuitively speaking, Σ is the underlying space where we have to develop differential geometry. However, as a matter of fact, this is far from the notion of manifolds. Here, recall the Option 1 of general Lie groups. Every $\mathrm{Aut}_\mu(\mathcal{O})$-orbit may be viewed as a *manifold*.

It is important to remark that Σ is *not* viewed as the phase space, but as a *parameter space of limit localizations*.

Consider now a neighborhood V_σ of $\sigma\in\Sigma$. Suppose for a moment that $V_\sigma = V_\sigma^{(o)}\times V_\sigma^{(m)}$ where $V_\sigma^{(o)} = V_\sigma \cap \mathrm{Aut}_\mu(\mathcal{O})(\sigma)$. Then, $V_\sigma^{(m)}$ is viewed as the parameter space of deformations of algebraic system $(\tilde{F}_\sigma, *_\sigma, \mu_\sigma)$ like the *moduli-space*. However, since algebraic systems are mutually isomorphic along $V_\sigma^{(o)}$, we think what is deformed along $V_\sigma^{(o)}$ is the expression of the algebraic system $(\tilde{F}_\sigma, *_\sigma, \mu_\sigma)$ given by a limit localization of $(\mathcal{O}, *, \mu)$, although the precise definition of deformation of expressions will be given later.

3.1.1 Infinitesimal intertwiners

Let $\sigma = \varinjlim \kappa_i\in\Sigma$. For every $f\in\tilde{F}_\sigma$, there is κ_i (depending on f) such that $f \in \pi^\sigma_{\kappa_i} F_{\kappa_i}$.

Suppose the intertwiner $\varinjlim_j I_{\kappa_i}^{\kappa'_j}(f)$ is defined for another $\sigma' = \varinjlim \kappa'_j\in\Sigma$ such that $\sigma' \in \Sigma_{\kappa_i}$. We denote this by

$$I_\sigma^{\sigma'}(f) = \varinjlim_j I_{\kappa_i}^{\kappa'_j}(f). \tag{18}$$

If $\sigma(t)$ is a continuous curve with $\sigma(0) = 0$ in Σ. The above observation shows that for every $f \in \tilde{F}_\sigma$, there is $\epsilon > 0$ such that $I_\sigma^{\sigma(t)}(f)$ is defined for every t, $|t| < \epsilon$.

Let ϕ_t be a one-parameter subgroup of $\mathrm{Aut}_\mu(\mathcal{O})$ whose infinitesimal generator is $X : \mathcal{O} \to \mathcal{O}$. We denote $\phi_t = \exp tX$. By definition, $\exp tX(\sigma)$ is viewed as a "smooth" curve in Σ starting at σ. Since the action is continuous by definition, there is $\epsilon > 0$ for any neighborhood V_σ of σ such that $\exp tX(\sigma) \in V_\sigma$ for every t, $|t| < \epsilon$.

Using (18), we define the *infinitesimal intertwiner* as follows:

$$dI_\sigma(X)(f) = \frac{d}{dt}\Big|_{t=0} I_\sigma^{\phi_t(\sigma)}(f), \quad \text{for all } f \in \tilde{F}_\sigma. \tag{19}$$

By defining the *tangent space* $T_\sigma \Sigma$ at σ as the symbolic set $\{\frac{d}{dt}|_{t=0} \exp tX(\sigma); \mathfrak{g}\}$, $dI_\sigma(X)$ is defined for all $X \in T_\sigma \Sigma$. The tangent space $T_\sigma \Sigma$ is more realistic, if we take Option 1 stated before. However, $dI_\sigma(X)$ is *not* a derivation in general as it will be seen in Section 3.2.1 and Section 5.3.

On the other hand, let $\tilde{F}^*_{\kappa_i}$ be the dual space of $\tilde{F}_{\kappa_i}$. Since $\tilde{F}_{\kappa_i}$ is dense in $\tilde{F}_{\kappa_{i+1}}$, we see $\tilde{F}^*_{\kappa_i} \supset \tilde{F}^*_{\kappa_{i+1}}$. Denote the projective limit space $\varprojlim \tilde{F}^*_{\kappa_i} = \cap \tilde{F}^*_{\kappa_i}$ by $\tilde{F}^*_\sigma$. This is the dual space of $\tilde{F}_\sigma$, and a Fréchet space if $\tilde{F}^*_{\kappa_i}$ is a Fréchet space. By this duality, we see

Lemma 4 *The infinitesimal intertwiner $dI_\sigma(X)$ is defined also on $\tilde{F}^*_\sigma$.*

Remark For a descending sequence $U_0 \supset U_1 \supset U_2 \supset \cdots$ of open subsets shrinking to a point $\sigma \in \mathbb{R}^n$, we have the sequence $C^\infty(\mathbb{R}^n) \to C^\infty(U_0) \to C^\infty(U_1) \to \cdots$, hence the inductive limit $\varinjlim C^\infty(U_k)$ is the space of germs of C^∞-functions at σ. However, since we take the closure of $\cap_i I_{\kappa_i}$ in our case, $\mathcal{O}_\sigma$ is the space of all formal power series. The dual spaces make a projective system $\mathcal{D}_0(M) \leftarrow \mathcal{D}_0(U_0) \leftarrow \mathcal{D}_0(U_1) \leftarrow \cdots$ where $\mathcal{D}_0(U_i)$ is the space of the Schwartz distributions with compact support on U_i, and $F^*_\sigma = \varprojlim \mathcal{D}_0(U_i)[\mu]$ is also considered. However, our $\tilde{F}^*_\sigma$ is the space of all polynomials of μ with coefficients of distributions supported on a single point σ.

Elements of $(\mathcal{O}, *, \mu)$ are represented naturally as a "smooth section" of bimodule bundles

$$\coprod_{\sigma \in \Sigma} (\mathcal{O}_\sigma, *_\sigma, \mu_\sigma), \quad \coprod_{\sigma \in \Sigma} (\tilde{F}_\sigma, *_\sigma, \mu_\sigma), \quad \coprod_{\sigma \in \Sigma} (\tilde{F}^*_\sigma, *_\sigma, \mu_\sigma).$$

Although local triviality does not hold in general, we denote these respectively by $\mathcal{O}_\Sigma$, $\tilde{F}_\Sigma$, $\tilde{F}^*_\Sigma$ for simplicity.

Since $\mathcal{O}$ contains 1 and $C^\infty(\Sigma)$ is viewed as a subspace of $\boldsymbol{\Gamma}(\mathcal{O}_\Sigma)$, we can define a probability measure on Σ (cf. [27]).

Let $\boldsymbol{\Gamma}(\mathcal{O}_\Sigma)$, $\boldsymbol{\Gamma}(\tilde{F}_\Sigma)$, $\boldsymbol{\Gamma}_0(\tilde{F}^*_\Sigma)$ be respectively the space of "smooth" sections of $\mathcal{O}_\Sigma$, $\tilde{F}_\Sigma$, and of $\tilde{F}^*_\Sigma$ with compact support. Every $f^*_\sigma \in \tilde{F}^*_\sigma$ may be viewed as a distribution supported only at the point σ. Such an element f^*_σ is called a *particle function* in [22].

Any field theoretic quantities are expressed by *sections* of these bundles. The second quantization is formulated by setting a family of vacuums on the bimodule bundle $\coprod_{\sigma \in \Sigma} (\mathcal{O}_\sigma, *_\sigma, \mu_\sigma)$.

For the computations using distributions, we define the pairing as follows:

$$\langle \tilde{F}_\sigma, \tilde{F}^*_{\sigma'} \rangle = \{0\}, \qquad \text{if} \quad \sigma \neq \sigma'. \tag{20}$$

Note that for every "smooth" function $\phi(\sigma)$, the pairing

$$\langle h, \int_\Sigma d\sigma \phi(\sigma) f^*_\sigma \rangle = \int_\Sigma d\sigma \langle h(\sigma), \phi(\sigma) f^*_\sigma \rangle, \quad h \in \boldsymbol{\Gamma}(\tilde{F}_\Sigma)$$

makes sense. Hence $\int_\Sigma d\sigma \phi(\sigma) f^*_\sigma$ is defined as an element of $\boldsymbol{\Gamma}_0(\tilde{F}^*_\Sigma)$.

3.1.2 Extremal localizations

Definition 10 $(\mathcal{O}_\sigma, \tilde{F}_\sigma, *_\sigma, \mu_\sigma)$ is called an *extremal localization*, if $\tilde{F}_\sigma$ is a Fréchet space, on which infinitesimal intertwiner $dI_\sigma(X)$ is defined for every continuous one-parameter μ-automorphism group $\exp tX$, and there is no further localization of it.

An example of extremal localization. Let $C^\infty(\mathbb{R}^n)$ be the algebra of all C^∞ functions with the C^∞-topology. Let $\mathcal{O} = C^\infty(\mathbb{R}^n)[[\mu]]$ be the space of all formal power series of μ with coefficients in $C^\infty(\mathbb{R}^n)$. By defining $[\mu, x_i] = \mu * x_i * \mu$, $[x_i, x_j] = 0$, we can make $(\mathcal{O}, *, \mu)$ a μ-regulated algebra. We see that $\mathcal{O}_\sigma = \tilde{F}_\sigma$ is the space of all formal power series $\Psi(n)[[\mu]]$. Since the direct product topology is the weakest topology to make projections to each component continuous, we see there is no further localization.

The classification of extremal localizations are not fixed yet. But, note that even in an extremal localization σ, we may have a nontrivial isotropy subgroup. Suppose the isotropy subgroup $\mathrm{Aut}_{[\kappa]}(\mathcal{O})$ of a certain localization κ contains $\varphi = cI$ such as $\varphi(f) = cf, c \neq 1$. Then, the automorphism $\varphi : \mathcal{O}_\kappa \to \mathcal{O}_\kappa$ must extend to a continuous linear isomorphism of F_κ. We see there is no localization of κ' of κ excluding cI from the isotropy subgroup, that is, $\varphi(\kappa') \neq \kappa'$, because if such κ' exists, then φ must *not* extend to $F_{\kappa'}$, but this is impossible.

3.2 Infinitesimal intertwiners as a flat connection

As was mentioned before, intertwiners are not defined on the whole space, but infinitesimal intertwiners are defined on the whole space. In this subsection, we show that infinitesimal intertwiners may be regarded as infinitesimal parallel translations of some flat connections on an algebra bundle over Σ.

Recall that $\mathrm{Aut}_\mu(\mathcal{O})$ acts on the bundle $\tilde{F}_\Sigma = \coprod_{\sigma \in \Sigma} \tilde{F}_\sigma$, where Σ is the space of all limit localizations. We regard an *infinitesimal intertwiner as a horizontal distribution* $H_{\sigma,f}$ defined by

$$H_{\sigma,f} = \{(X, dI_\sigma(X)(f));\ X \in T_\sigma\Sigma\}, \quad H_\sigma : T_\sigma\Sigma \times \tilde{F}_\sigma \to \tilde{F}_\sigma. \tag{21}$$

Along a smooth curve $\sigma(t)$ in Σ, the equation of parallel translation is given by

$$\partial_t f_t = dI_{\sigma(t)}(\dot\sigma(t))(f_t), \quad f_0 = f \in \tilde F_\sigma. \tag{22}$$

If f_t satisfies (22), then f_t is said to be *parallel* along the curve $\sigma(t)$. However, if f is general, (22) may not have a solution. The following is easy to see.

Proposition 8 *The linear connection* (21) *defined on $\tilde F_\Sigma$ is a flat connection, and for every $a \in \mathcal{O}$, $a_\sigma = \pi^\sigma(a)$, $\sigma \in \Sigma$ is a parallel section of $\tilde F_\Sigma$.*

If σ is fixed, then $a \to \pi^\sigma(a)$ is only a homomorphism in general. However, as we take *all* possible limit localizations, it is natural to think that the mapping $a \to \{\pi^\sigma(a); \sigma \in \Sigma\}$ is *faithful*, i.e., an into-isomorphism.

For $a \in \mathcal{O}_\kappa$, $\{a *_\sigma 1; \sigma \in \Sigma\}$ gives only a *locally defined* parallel section defined on Σ_κ $(\subset \Sigma)$. We may regard *q-number functions* mentioned in [1] as maximally defined parallel sections of $\tilde F_\Sigma$ (cf. Section 5.1), although the explicit definition is not given in [1].

3.2.1 The flow of ad(μ^{-1}) on Σ

Now, let Σ be the set of all limit localization of $(\mathcal{O}, *, \mu)$, and suppose that elements $f \in \mathcal{O}$ are represented faithfully as parallel sections (q-number functions) of $\tilde F_\Sigma$.

For a while, we assume that $e^{t\mathrm{ad}(\mu^{-1})}$ exists as a one-parameter automorphism group of $(\mathcal{O}, *)$. This induces a "smooth" one-parameter flow ϕ_t on the base space Σ. $e^{t\mathrm{ad}(\mu^{-1})}$ gives an isomorphism of $(\mathcal{O}_\sigma, *_\sigma)$ onto $(\mathcal{O}_{\sigma(t)}, *_{\sigma(t)})$ where $\sigma(t) = \phi_t(\sigma)$. We denote this by $e^{t\mathrm{ad}(\mu^{-1})}(\sigma) : (\mathcal{O}_\sigma, *_\sigma) \to (\mathcal{O}_{\phi_t(\sigma)}, *_{\phi_t(\sigma)})$. In a limit localization, it is natural to think that $\phi_t(\sigma) \neq \sigma$ for every $t \neq 0$. Thus, $\frac{d}{dt}\big|_{t=0}\phi_t$ may be viewed as a non-vanishing "vector field" Ξ_0, just like the characteristic vector field. We denote $\Xi_0(\sigma) = \frac{d}{dt}\big|_{t=0}\phi_t(\sigma)$.

Now we have to consider how the family of isomorphisms

$$e^{t\mathrm{ad}(\mu^{-1})}(\sigma) : (\tilde F_\sigma, \mathcal{O}_\sigma, *_\sigma) \to (\tilde F_{\phi_t(\sigma)}, \mathcal{O}_{\phi_t(\sigma)}, *_{\phi_t(\sigma)})$$

relates to the family of intertwiners $I_\sigma^{\phi_t(\sigma)}$. At this stage, it is natural to consider its infinitesimal version by setting $(\mathrm{ad}(\mu^{-1})(f))(\sigma) = \frac{d}{dt}\big|_{t=0}(e^{t\mathrm{ad}(\mu^{-1})}f)(\sigma)$, since $I_\sigma^{\phi_t(\sigma)}$ is not defined for all $\tilde F_\sigma$.

However, there are many ways to consider derivatives. It depends how $\tilde F_{\sigma(t)}$ is identified with $\tilde F_\sigma$. As in the classical differential geometry, we define

$$\nabla_{\Xi_0(\sigma)} f_* = (\nabla_{\Xi_0} f_*)(\sigma) = \frac{d}{dt}\bigg|_{t=0} \big(e^{t\mathrm{ad}(\mu^{-1})} I^{\cdot}_{\phi_t(\cdot)}(f_*)\big)(\sigma), \quad \text{(covariant derivative)},$$

where $I^{\cdot}_{\phi_t(\cdot)}(f_*)$ is a parallel section defined by $I^{\sigma'}_{\phi(t)(\sigma')}(f_{\sigma'})$. We define also

$$(\mathfrak{L}_{\Xi_0} f_*)(\sigma) = \frac{d}{dt}\bigg|_{t=0} e^{t\mathrm{ad}(\mu_\kappa^{-1})}(f_*(e^{-t\mathrm{ad}(\mu_\kappa^{-1})}(\sigma))), \quad \text{(Lie derivative)}.$$

Both of them are derivations. However, let us note here the following computation:

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0} & e^{t\,\mathrm{ad}(\mu_\kappa^{-1})}(\sigma)(f *_\sigma g) \\ &= \frac{d}{dt}\Big|_{t=0} \left(e^{t\,\mathrm{ad}(\mu_\kappa^{-1})}(\sigma)(f) *_{\sigma(t)} e^{t\,\mathrm{ad}(\mu_\kappa^{-1})}(\sigma)(g)\right) \\ &= \left(\frac{d}{dt}\Big|_{t=0} e^{t\,\mathrm{ad}(\mu_\kappa^{-1})}(\sigma)(f)\right) *_\sigma g + f *_\sigma \left(\frac{d}{dt}\Big|_{t=0} e^{t\,\mathrm{ad}(\mu_\kappa^{-1})}(\sigma)(g)\right) \\ &\quad + \frac{d}{dt}\Big|_{t=0} f *_{\sigma(t)} g.\end{aligned}$$

This makes sense, if both $\mathcal{O}_{\sigma(t)}$ and $\mathcal{O}_\sigma$ are identified respectively with subspaces of a common Fréchet space. Here we think that the notion of *deformation of expressions* gives this identification.

Here "deformation" means neither the traditional deformation theory of Gerstenhaber (cf. [6] and references therein) nor its noncommutative generalization (cf. [12]).

The usual theory of deformation of algebras mainly concerns deformations of algebraic structures, written in terms of Hochschild cohomology groups. In our theory of deformation, the underlying algebraic structure is fixed, but this structure plays only a supplemental role in order to give the univalent expression for elements, and to give several operations which will be used for the construction of algebras. Algebraic systems to be considered are given separately by the product formula written by using the underlying algebraic structure.

We are assuming that $(\tilde{F}_{\sigma(t)}, \mathcal{O}_{\sigma(t)}, *_\sigma(t))$ is given by a *deformation* of expressions of $(\mathcal{O}_\sigma, *_\sigma)$.

Thus, what is deformed is product formulas written on the space $(\tilde{F}_\sigma, \mathcal{O}_\sigma)$. Such a notion has been extensively discussed in [12],[13].

We denote $(\mathfrak{D}_{\Xi_0} f) = \frac{d}{dt}\big|_{t=0} e^{t\,\mathrm{ad}(\mu_\kappa^{-1})} f$ and call this a *deformation derivative*. In general $\mathfrak{D}_{\Xi_0}$ is not a derivation. The typical example is D^2 considered in (26) below.

In the case where intertwiners are given on a common underlying algebra, $\mathfrak{D}_{\Xi_0} = d\boldsymbol{I_\bullet}(\Xi_0)$ gives a typical example. But in general, $d\boldsymbol{I_\bullet}(\Xi_0)$ is a linear combination of $\mathfrak{D}_{\Xi_0}$ and $\nabla_{\Xi_0(\sigma)}, \mathfrak{L}_{\Xi_0}$.

We see that $\mathrm{ad}(\mu_\kappa^{-1})(\sigma)$ must satisfy

$$(\mathfrak{D}_{\Xi_0})(\sigma)(f *_\sigma g)) = (\mathfrak{D}_{\Xi_0} f)(\sigma)(f) *_\sigma g + f *_\sigma (\mathfrak{D}_{\Xi_0} f)(\sigma)(g) + \langle f, g\rangle_\sigma \tag{23}$$

where $\langle f, g\rangle_\sigma$ is the bilinear mapping defined by $\langle f, g\rangle_\sigma = \frac{d}{dt}\big|_{t=0} f *_{\sigma(t)} g$.

The polynomial algebra of one variable is very rigid, but even in such a case, there are many ways of expressions. We discuss such deformations in the next section together with a more precise definition of (23).

4 Deformation by one variable

So far, we treated the notion of localizations, and limit localizations. In this section, we give a basic idea of *deformation* which may appear in such a localization. Especially, we want to give a precise meaning to (23). For that purpose we have to start with a little more generality. The precise meaning of (23) will be given in the last paragraph of Section 4.1.

4.1 Basic formula

Let $(\mathcal{A}, \hat{*})$ be a Fréchet algebra and let $(F, \mathcal{A}, \hat{*})$ be an $(\mathcal{A}, \hat{*})$-bimodule such that $\mathcal{A}$ itself is an effective quotient algebra and densely included in F.

Definition 11 A continuous linear mapping $D : F \to F$ is a *derivation*, if D gives a continuous derivation of $(\mathcal{A}, \hat{*})$ and $D(f\hat{*}g) = D(f)\hat{*}g + f\hat{*}D(g)$ holds if f or g is in $\mathcal{A}$.

Given such a derivation D, we define a new product $*_\tau$ by the following formula:

$$f *_\tau g = \sum_{k\geq 0} \frac{\tau^k}{2^k k!} D^k f \hat{*} D^k g = f e^{\frac{\tau}{2}\overleftarrow{D}\hat{*}\overrightarrow{D}} g \quad \text{(see also (10), (50))} \tag{24}$$

where $\tau \in \mathbb{C}$ is viewed as the *deformation parameter*. If $\tau = 0$, then $*_0 = \hat{*}$. Here we stress that $\hat{*}$-product is used in order to define $*_\tau$. The usual deformation theory does *not* take this formulation.

Under a suitable condition for D ensuring the convergence, $(\mathcal{A}, *_\tau)$ is an associative Fréchet algebra and $(F, \mathcal{A}, *_\tau)$ is an $(\mathcal{A}, *_\tau)$-bimodule. A sufficient condition for convergence is that $\{D^k\}_{k=0,1,2\ldots}$ is equi-continuous, i.e.:

Let $\{\|\cdot\|_\lambda\}_\lambda$ be a family of seminorms which give the topology of $\mathcal{A}$. $\{D^k\}_k$ is called *equi-continuous*, if for every $\|\cdot\|_\lambda$ there are $\|\cdot\|_{\lambda'}$ and a constant $C_{\lambda,\lambda'}$ such that $\|D^k a\|_\lambda \leq C^k_{\lambda,\lambda'}\|a\|_{\lambda'}$ for every k.

Under the condition that $\{D^k\}_{k=0,1,2\ldots}$ is equi-continuous, the mapping $e^{(\tau/4)D^2} : \mathcal{A} \to \mathcal{A}$ is a linear isomorphism. In general, $e^{(\tau/4)D^2}$ does not extend to F. Since $\mathcal{A}$ is dense in F, $e^{(\tau/4)D^2} : F \to F$ is only densely defined. However, $D^2 : F \to F$ is a continuous linear mapping.

Moreover, by splitting $\frac{1}{k!}(\frac{\tau}{4}D^2)^k(f\hat{*}g)$ into

$$\sum_{p+q+r=k} \frac{\tau^r}{r!2^r} D^r \left(\frac{1}{p!}\left(\frac{\tau}{4}D^2\right)^p f\right) \hat{*} D^r \left(\frac{1}{q!}\left(\frac{\tau}{4}D^2\right)^q g\right), \tag{25}$$

we see that

$$e^{\frac{\tau}{4}D^2} : (\mathcal{A}, \hat{*}) \to (\mathcal{A}, *_\tau) \tag{26}$$

is an algebra isomorphism. Hence the new algebra $(\mathcal{A}, *_\tau)$ is isomorphic to the original algebra.

Remark Let $\mathcal{A}' = \{a \in \mathcal{A};\ D^k a = 0, \text{ for some } k = k(a)\}$. Then $\mathcal{A}'$ is a subalgebra of $\mathcal{A}$ and F is viewed as an $\mathcal{A}'$-bimodule. The details about the convergence will appear in [19]. See also the next section for the simplest example.

We see that D acts as a derivation of $(F, \mathcal{A}, *_\tau)$, and for any other constant $\hat{\tau}$, another product formula

$$f e^{\frac{\hat{\tau}}{2}\sum \overleftarrow{D} *_\tau \overrightarrow{D}} g = \sum_k \frac{1}{k!2^k} \hat{\tau}^k D^k f *_\tau D^k g$$

defines also an associative algebra $(\mathcal{A}, *_{\hat{\tau}})$ and $(F, \mathcal{A}, *_{\hat{\tau}})$ is an $(\mathcal{A}, *_\tau)$-bimodule. Since $\tau, \hat{\tau}$ are constants, this formula can be written as

$$f e^{\frac{\hat{\tau}}{2}(\overleftarrow{D} e^{\frac{1}{2}(\overleftarrow{D} *_\tau \overrightarrow{D})} \overrightarrow{D})} g = \sum_k \frac{1}{k!2^k} (\hat{\tau} + \tau)^k D^k f \hat{*} D^k g = f *_{\tau+\hat{\tau}} g. \tag{27}$$

Moreover, the same argument as in (25) gives the following formula:

$$e^{\frac{\hat{\tau}}{4}D^2}\Big(\big(e^{-\frac{\tau}{4}D^2} f\big) *_\tau \big(e^{-\frac{\tau}{4}D^2} g\big)\Big) = f e^{\frac{\hat{\tau}}{2}(\overleftarrow{D} *_\tau \overrightarrow{D})} g = f *_{\tau+\hat{\tau}} g. \tag{28}$$

This means $*_{\tau+\hat{\tau}}$-product is isomorphic to $*_\tau$-product by the isomorphism $e^{(1/4)(\hat{\tau}-\tau)D^2}$. Hence intertwiners are given as follows (cf. Theorem 2):

Proposition 9 *For every τ, τ', the intertwiner is defined by*

$$I_\tau^{\tau'}(f) = \exp\left(\frac{1}{4}(\tau' - \tau) D^2\right) f \ \ (= I_0^{\tau'}(I_0^\tau)^{-1}(f)), \tag{29}$$

and by (28) *it gives an algebra isomorphism* $I_\tau^{\tau'} : (\mathcal{A}; *_\tau) \rightarrow (\mathcal{A}, *_{\tau'})$.

4.1.1 Combination with automorphisms

Note that (24) is only a typical example of deformation. One can combine this with arbitrary automorphisms $\psi : (\mathcal{A}; \hat{*}) \rightarrow (\mathcal{A}; \hat{*})$, $\psi' : (\mathcal{A}; *_\tau) \rightarrow (\mathcal{A}; *_\tau)$. Namely, we define

$$f *_{\psi,\tau,\psi'} g = \psi'(\psi(f) *_\tau \psi(g)). \tag{30}$$

The associativity is checked by noting

$$f *_{\psi,\tau,\psi'} (g *_{\psi,\tau,\psi'} h) = \psi'(\psi(f) \hat{*} \psi'^{-1}(g *_{\psi,\tau,\psi'} h)) = \psi'(\psi(f) \hat{*} (\psi(g) \hat{*} \psi(h)).$$

The intertwiner $\hat{I}_0^\tau$ of the combined deformation is

$$\hat{I}_0^\tau(f) = \psi' \circ I_0^\tau \circ \psi(f). \tag{31}$$

4.1.2 Relation with limit localizations

Now, go back to the family of localizations $(\tilde{F}_{\sigma(t)}, \mathcal{O}_{\sigma(t)}, *_{\sigma(t)})$ given in the last section. This was defined by the one-parameter automorphisms $\psi_t = e^{\mathrm{ad}(\mu^{-1})} \in \mathrm{Aut}_\mu(\mathcal{O})$ which were temporarily assumed to exist. In the previous section, we viewed ψ_t as inducing deformations of $(\tilde{F}_\sigma, \mathcal{O}_\sigma, *_\sigma)$. Namely we regard t as a deformation parameter and set $e^{t\mathrm{ad}(\mu^{-1})}(\sigma) = \hat{I}_\sigma^{\sigma(t)}$.

Taking the derivative at $t = 0$, we think that $\mathrm{ad}(\mu^{-1})(\sigma)$ is given by

$$\mathrm{ad}(\mu^{-1})(\sigma) = D^2 + X \quad \text{(cf. (39))} \tag{32}$$

where D, X are derivations of $\mathcal{O}_\sigma$, one of which is possibly zero.

5 The case where *D* is the ordinary differential

Let $\mathrm{Hol}(\mathbb{C}, \mathcal{A}_0)$ be the Fréchet space of all holomorphic mappings of $\mathbb{C}$ into a Fréchet space $\mathcal{A}_0$. Let $\|\cdot\|_\lambda$ be the family of seminorms defining the topology of $\mathcal{A}_0$. For every $p > 0$, we define the Fréchet space $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0)$ by (11) replaced $\mathcal{O}$ by $\mathcal{A}_0$.

It is easily seen that for $0 < p < p'$, there is a continuous embedding $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0) \subset \mathcal{E}_{p'}(\mathbb{C}, \mathcal{A}_0)$ as Fréchet spaces (cf. [4]). Every element of $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0)$ may be written as an $\mathcal{A}_0$-valued function $f(\zeta)$, where ζ is the complex coordinate function of $\mathbb{C}$.

Suppose now that $\mathcal{A}_0$ is a Fréchet algebra with 1 over $\mathbb{C}$. Denote by $\hat{*}$ the product defined on $\mathcal{A}_0$. Here, ζ is viewed naturally as a member of $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0)$ as $\zeta \to \zeta \cdot 1$. It is easy to see $(\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0), \hat{*})$ is an associative Fréchet algebra.

The theory of deformation can be constructed more concretely by setting $D = \partial_\zeta$. The following are the special cases of Theorems 1 and 2:

Theorem 3 *For every pair* (p, p') *such that* $\frac{1}{p} + \frac{1}{p'} \geq 1$, *the product formula* (24) *gives a continuous bi-linear mapping of* $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0) \times \mathcal{E}_{p'}(\mathbb{C}, \mathcal{A}_0) \to \mathcal{E}_{p \vee p'}(\mathbb{C}, \mathcal{A}_0)$.

If $0 < p \leq 2$, the product formula (24) extends to make the space $(\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0), *_\tau)$ a complete non-commutative topological associative algebra over $\mathbb{C}$ (cf. [11]), but for $p > 2$, $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0)$ is only a $\mathcal{E}_{p'}(\mathbb{C}, \mathcal{A}_0)$-bimodule for every $p' > 0$ such that $\frac{1}{p} + \frac{1}{p'} \geq 1$. By taking the limit of the associativity of $\mathcal{E}_{p'}(\mathbb{C}, \mathcal{A}_0)$, the associativity

$$f *_\tau (g *_\tau h) = (f *_\tau g) *_\tau h \tag{33}$$

holds if any two of f, g, h are in $\mathcal{E}_{p'}(\mathbb{C}, \mathcal{A}_0)$.

Theorem 4 $I_\tau^{\tau'}$ *gives an algebra isomorphism of* $(\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0), *_\tau)$ *onto* $(\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0), *_{\tau'})$ *for every* $p \leq 2$ (cf. [12]).

It is easily seen that the following identities hold on $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0)$, $p \leq 2$,

$$I_{\tau'}^{\tau} I_\tau^{\tau'} = 1, \quad I_{\tau'}^{\tau''} I_\tau^{\tau'} = I_\tau^{\tau''}. \tag{34}$$

Suppose that our algebra $(\mathcal{O}_\sigma, *_\sigma)$ is linearly isomorphic to $\mathcal{E}_{1+}(\mathbb{C}, \mathcal{A}_0)$, and F_σ is linearly isomorphic to $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0)$ for some $p > 2$. This implies that if $f, g \in \mathcal{E}_{1+}(\mathbb{C}, \mathcal{A}_0)$, then the product $f *_\tau g$ defined by (24) converges, and if one of $f(\zeta)$, $g(\zeta)$ is in $\mathcal{E}_{1+}(\mathbb{C}, \mathcal{A}_0)$ and another is in $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0)$, then $f *_\tau g$ converges in $\mathcal{E}_p(\mathbb{C}, \mathcal{A}_0)$.

However, the product $*_\tau$ extends for a fairly wide class of functions. For instance, Taylor expansion gives the following for every holomorphic mapping f of $\mathbb{C}$ into $\mathcal{A}_0$:

$$e^{2s\zeta} *_\tau f(\zeta) = e^{2s\zeta} \hat{*} f(\zeta + s\tau) = e^{2s\zeta} \hat{*} e^{s\tau\partial_\zeta} f(\zeta). \tag{35}$$

5.1 *q*-number functions

By a direct calculation of the intertwiner, we see that $I_\tau^{\tau'}(e^{s\zeta}) = e^{(1/4)(\tau'-\tau)s^2} e^{s\zeta}$. Hence, we have $I_\tau^{\tau'}(e^{(1/4)s^2\tau} e^{s\zeta}) = e^{(1/4)s^2\tau'} e^{s\zeta}$.

We shall denote the set $\{e^{(1/4)s^2\tau} e^{s\zeta};\ \tau \in \mathbb{C}\}$ symbolically by $e_*^{s\zeta}$, and we regard $e_*^{s\zeta}$ as the exponential function in the world of $*$-product. In this context, $e_*^{s\zeta}$ is called a *q-number exponential function* or a *$*$-exponential function*. We use also the notation $:\cdot:_\tau$ for the τ-expression. $: e_*^{s\zeta} :_\tau$ is viewed as the τ-expression of $e_*^{s\zeta}$,

$$: e_*^{s\zeta} :_\tau = e^{\frac{1}{4}s^2\tau} e^{s\zeta} = e^{\frac{1}{4}s^2\tau + s\zeta}. \tag{36}$$

Using the product formula (24), we easily see the exponential law

$$: e_*^{s\zeta} :_\tau *_\tau : e_*^{t\zeta} :_\tau =: e_*^{(s+t)\zeta} :_\tau, \quad \forall\, \tau \in \mathbb{C}. \tag{37}$$

This may be written as $e_*^{s\zeta} * e_*^{t\zeta} = e_*^{(s+t)\zeta}$. Moreover for every τ, $: e_*^{s\zeta} :_\tau$ is the solution of $\frac{d}{dt} g(t) = \zeta *_\tau g(t)$ with initial data $g(0) = 1$. $e_*^{s\zeta}$ forms a one-parameter group of elements which are mutually intertwined. By the uniqueness of the real analytic solution, we obtain a little more general exponential law (15) than (37). In general, we have the formula

$$: e_*^{2s\zeta} * h_*(\zeta) :_\tau = e^{2s\zeta + s^2\tau} h(\zeta + s\tau), \quad h \in \mathcal{E}_\infty(\mathbb{C}). \tag{38}$$

In this context, we may write as follows:

$$: a\zeta + b :_\tau = a\zeta + b, \quad : 2{\zeta_*}^2 :_\tau = 2\zeta^2 + \tau, \quad : 2{\zeta_*}^3 :_\tau = 2\zeta^3 + 3\tau\zeta, \quad \text{etc.}$$

Putting Theorem 4 in mind, we set $f_* = \{I_0^\tau(f);\ \tau \in \mathbb{C}\}$ for every $f \in \mathcal{E}_p(\mathbb{C})$, $p \le 2$, and call f_* a *q-number function* or *$*$-function*. $\sin_* \zeta$, $\cos_* \zeta$ are defined in this way.

Moreover, any $f \in \mathcal{E}_\infty(\mathbb{C})$ may be seen as the τ-expression of a q-number function f_*. That is, one may write formally $: f_* :_\tau = f$, although $: f_* :_{\tau'}$ may not be defined for $\tau' \neq \tau$.

We confirm (32) by the following:

$$\begin{aligned}
&\partial_\tau|_{\tau=0}\, e_*^{2a\zeta + a^2\tau} *_\tau e_*^{2b\zeta + b^2\tau} \\
&\quad = a^2 e_*^{2a\zeta} *_0 e_*^{2b\zeta} + b^2 e_*^{2a\zeta} *_0 e_*^{2b\zeta} + \partial_\tau|_{\tau=0}\, e_*^{2a\zeta} *_\tau e_*^{2b\zeta}
\end{aligned}$$

$$= a^2 e_*^{2a\zeta} *_0 e_*^{2b\zeta} + b^2 e_*^{2a\zeta} *_0 e_*^{2b\zeta} + 2ab e_*^{2a\zeta} *_0 e_*^{2b\zeta}$$
$$= \frac{1}{4}\partial_\zeta^2 \big(e_*^{2a\zeta} *_0 e_*^{2b\zeta}\big). \qquad (39)$$

We are thinking that if we forget about μ_σ, an extremal localization $(F_\sigma, \mathcal{O}_\sigma, *_\sigma)$ is a system something like

$$(\mathcal{E}_\infty(\mathbb{C}), \mathcal{E}_{1+}(\mathbb{C}), *_\tau),$$

and the original $\mathcal{O}$ is given by the space of parallel sections of $\coprod_\tau (\mathcal{E}_{1+}(\mathbb{C}), *_\tau)$.

5.2 Star-exponential functions of quadratics and intertwiners

Consider now the exponential function $e_*^{t\zeta_*^2}$ where $\zeta_*^2 = \zeta * \zeta$. Remark that $: \zeta_*^2 :_\tau = \zeta^2 + \frac{\tau}{2}$. Such exponential functions are extensively investigated in [9], [12], [13]. Consider the differential equation

$$\frac{d}{dt} f_t = \zeta_*^2 * f_t, \quad f_0 = 1.$$

Remembering the uniqueness of the real analytic solution, we set $: f_t :_\tau = g(t)e^{h(t)\zeta^2}$. By using $: \zeta_*^2 :_\tau = \zeta^2 + \frac{\tau}{2}$ and (24), the equation turns out to be a system of ordinary differential equations for every $\tau \in \mathbb{C}$:

$$\begin{cases} \dfrac{d}{dt} h(t) = (1 + \tau h(t))^2, & h(0) = 0, \\ \dfrac{d}{dt} g(t) = \dfrac{1}{2}(\tau^2 h(t) + \tau) g(t), & g(0) = 1. \end{cases}$$

Solving this, we have

$$: e_*^{t\zeta_*^2} :_\tau = \frac{1}{\sqrt{1-\tau t}} e^{\frac{t}{1-\tau t}\zeta^2}, \ \text{for every } \tau, \ t\tau \neq 1. \qquad (40)$$

Because of the $\sqrt{\ }$ in this formula, we have to treat $e_*^{t\zeta_*^2}$ as a two-valued element with $\pm$ ambiguity. The reason is as follows: Let $\tau(\theta) = e^{i\theta}(\tau - \frac{1}{t}) + \frac{1}{t}$. Then, we see that

$$: e_*^{t\zeta_*^2} :_{\tau(\theta)} = \frac{1}{\sqrt{e^{i\theta}(1-\tau t)}} \, e^{\frac{t}{e^{i\theta}(1-\tau t)}\zeta^2}.$$

Tracing θ from 0 to 2π, we see that $: e_*^{t\zeta_*^2} :_\tau = \pm : e_*^{t\zeta_*^2} :_\tau$. This gives indeed the *sogo* in $\mathbb{Z}_2$.

In the above tracing, we fix the element $e_*^{t\zeta_*^2}$ and move the τ-expressions, but reversing the situation we can fix the expression and move the $*$-functions. In a similar way, we can make a one-parameter element $e_*^{a(\theta)\zeta_*^2}$ such that $a(0) = a(2\pi)$, $a(\theta) \neq -1$, and $: e_*^{a(\theta)\zeta_*^2} :_{-1}$ is defined but

$$:e_*^{a(0)\zeta_*^2}:_{-1} = -:e_*^{a(2\pi)\zeta_*^2}:_{-1}.$$

These are *not* contradictions, but imply that $e_*^{t\zeta_*^2}$ is a two-valued function. (See also [13].) Since the solution is real analytic where they are defined, the exponential law $e_*^{s\zeta_*^2} * e_*^{t\zeta_*^2} = e_*^{(s+t)\zeta_*^2}$ holds by the uniqueness. Rewriting this we have

$$\frac{1}{\sqrt{1-\tau s}}e^{\frac{s}{1-\tau s}\zeta^2} *_\tau \frac{1}{\sqrt{1-\tau t}}e^{\frac{t}{1-\tau t}\zeta^2} = \frac{1}{\sqrt{1-\tau(s+t)}}e^{\frac{s+t}{1-\tau(s+t)}\zeta^2}.$$

Setting $\frac{s}{1-\tau s}=a$, $\frac{t}{1-\tau t}=b$, we have

$$e^{a\zeta^2} *_\tau e^{b\zeta^2} = \frac{1}{\sqrt{1-ab\tau^2}}e^{\frac{a+b+ab\tau}{1-ab\tau^2}\zeta^2}. \tag{41}$$

Note that the $\pm$ ambiguity of the $\sqrt{\ }$ can not be eliminated. Note also the following equalities which can very easily lead to mistakes:

$$:e_*^{\frac{1}{\tau}\zeta_*^2}:_\tau = \infty, \quad :e_*^{\frac{1}{2\tau}\zeta_*^2}:_\tau = \sqrt{2}e^{\frac{1}{\tau}\zeta^2}.$$

Since ζ_*^2 and ζ commutes, we see that $e_*^{t\zeta_*^2+2s\zeta} = e_*^{t\zeta_*^2} * e_*^{2s\zeta}$. Hence by (40)

$$:e_*^{t\zeta_*^2+2s\zeta}:_\tau = \frac{1}{\sqrt{1-t\tau}}e^{s^2\tau}e^{\frac{t}{1-t\tau}\zeta^2} *_\tau e^{2s\zeta} = \frac{1}{\sqrt{1-t\tau}}e^{\frac{t}{1-t\tau}(\zeta+s\tau)^2}e^{s^2\tau}e^{2s\zeta}.$$

Multiplying $e^{(1/t)s^2}$ to both sides and setting $\alpha = s/t$, we have that

$$:e_*^{t(\zeta+\alpha)_*^2}:_\tau = \frac{1}{\sqrt{1-t\tau}}\ e^{\frac{t}{1-t\tau}(\zeta+\alpha)^2}. \tag{42}$$

This confirms that the formula (40) is invariant by parallel translation.

Note that functions such as $e^{a\zeta^2}$ are not contained in $\mathcal{E}_2(\mathbb{C})$, but in $\mathcal{E}_{2+}(\mathbb{C})$. Furthermore, for $p>2$, intertwiners are not defined on the whole space $\mathcal{E}_p(\mathbb{C})$. However we can define the intertwiner on the space $\mathbb{C}e^{\mathbb{C}\zeta^2}$ of exponential functions of quadratic forms (cf. [12]). The formulas for intertwiners are obtained by solving the evolution equation by setting

$$e^{t\partial_\zeta^2}(ge^{a\zeta^2}) = g(t)e^{q(t)\zeta^2}, \quad g(0)=g\in\mathbb{C},\ q(0)=a\in\mathbb{C}. \tag{43}$$

A direct calculation gives $\partial_\zeta^2\big(g(t)e^{q(t)\zeta^2}\big) = g(t)\Big(2q(t)+4q(t)^2\zeta^2\Big)e^{q(t)\zeta^2}$. Thus, we have

$$\begin{cases} \dfrac{d}{dt}q(t) = 4q(t)^2 \\ \dfrac{d}{dt}g(t) = 2g(t)q(t) \end{cases} \qquad q(0)=a, \quad g(0)=g.$$

Hence we have $q(t) = \frac{a}{1-4at}$, $g(t) = \frac{g}{\sqrt{1-4at}}$. Setting $t=\frac{1}{4}$, we have the intertwiner I_0^τ:

$$I_0^\tau(ge^{a\zeta^2}) = \frac{g}{\sqrt{1-a\tau}} e^{\frac{a}{1-a\tau}\zeta^2} =: ge_*^{a\zeta_*^2} :_\tau \,. \tag{44}$$

This confirms that $e_*^{a\zeta_*^2}$ is a $*$-exponential function, but its τ-expression is given only for $\tau \neq a^{-1}$ and it is two-valued.

5.3 Infinitesimal intertwiners viewed as linear connections

Intertwiners may not be defined for $f \in \mathcal{E}_p(\mathbb{C})$, $p > 2$. However, infinitesimal intertwiners are defined.

Define the infinitesimal intertwiner at $\tau \in \mathbb{C}$ to the direction τ' as follows:

$$dI_\tau(\tau')(f) = \frac{d}{dt}\Big|_{t=0} I_\tau^{\tau+t\tau'}(f) = \frac{1}{4}\tau'\partial_\zeta^2 f, \quad \left((\partial_\tau I)f = \frac{1}{4}\partial_\zeta^2 f\right) \quad \text{(cf. (19))}. \tag{45}$$

Infinitesimal intertwiners are defined for all $f \in \mathcal{E}_\infty(\mathbb{C})$. Define the *horizontal distribution* $H_\tau(f)$ at the point $(\tau; f)$ by $H_\tau(f) = \{(\tau', dI_\tau(\tau')(f));\ \tau \in \mathbb{C}\}$. This is viewed as the linear connection defined on the trivial algebra bundle $\coprod_{\tau\in\mathbb{C}} \mathcal{E}_\infty(\mathbb{C})$. The curvature of this connection vanishes obviously. Along a curve $\tau(t)$ in $\mathbb{C}$, the equation of parallel translation is given by $\frac{\partial}{\partial t} f = \frac{1}{4}\frac{d}{dt}\tau(t)\partial_\zeta^2 f$. If t is a complex variable, this is written by setting $t = \tau$ as

$$\frac{\partial}{\partial \tau} f = \frac{1}{4}\partial_\zeta^2 f. \qquad \text{(Cf.(22).)} \tag{46}$$

If $f(t,\zeta)$ satisfies (46), then $f(t,\zeta)$ is said to be *parallel* along the curve $\tau(t)$. Typical parallel sections are

$$e^{2a\zeta + a^2\tau}, \quad a(4\tau + \zeta^2) \quad \text{for an arbitrary } a \in \mathbb{C}. \tag{47}$$

Hence, for every $f \in \mathcal{E}_p(\mathbb{C})$, $p \le 2$, the set $f_*(\zeta) = \{I_0^\tau(f);\ \tau \in \mathbb{C}\}$ is a globally defined parallel section. However, since one can not solve (46) for all initial elements, one can not give local trivializations by using locally defined parallel sections.

For parallel sections $: f_* :_\tau, : g_* :_\tau$, the product $: f_* :_\tau *_\tau : g_* :_\tau$ is also a parallel section, whenever they are defined. This may be written as $: f_* * g_* :_\tau$, on which the family of infinitesimal intertwiners $dI_\tau = \frac{1}{4}\partial_\zeta^2$ acts as a derivation, for $\partial_\tau f_* = \frac{1}{4}\partial_\zeta^2 f_*$ holds for every parallel section. In other words, on the space of parallel sections, the operator $\frac{1}{4}\partial_\zeta^2$ is replaced by ∂_τ.

Computing the inverse $I_\tau^0 = (I_0^\tau)^{-1}$, and the composition $I_0^{\tau'} I_\tau^0$, we easily see

$$I_\tau^{\tau'}(ge^{a\zeta^2}) = g\frac{1}{\sqrt{1-a(\tau'-\tau)}} e^{\frac{a}{1-a(\tau'-\tau)}\zeta^2}.$$

Proposition 10 *If $a \neq 0$, then $\frac{g}{\sqrt{(1-a\tau)}} e^{\frac{a}{1-a\tau}\zeta^2}$ is a two-valued parallel section defined on $\mathcal{D}_a = \{\tau \in \mathbb{C};\ 1 - a\tau \neq 0\}$.*

Such a monodromic phenomenon gives an example of discordance $I_{\tau''}^{\tau} I_{\tau'}^{\tau''} I_\tau^{\tau'} \neq 1$ taking place on $\mathcal{E}_{2+}(\mathbb{C})$. The sign ambiguity of this is called $\mathbb{Z}_2$-sogo.

5.4 Strange phenomena

Since $:e_*^{s(z+\zeta)}:_\tau = e^{(\tau/4)t^2+t(z+\zeta)}$, we have easily the following strange fact mentioned in the last paragraph of Section 2.3.1.

Proposition 11 *If* $\mathrm{Re}\tau<0$ *and* $f(s)$ *is a smooth function of exponential growth, then for every* $z\in\mathbb{C}$*, the integral* $\int_{-\infty}^{\infty} f(s)e_*^{s(z+\zeta)}ds$ *converges absolutely for every fixed* ζ*. Similarly, for every* $z\in\mathbb{C}$*, the bilateral power series* $\sum_{n=-\infty}^{\infty} f(n)e_*^{n(z+\zeta)}ds$ *converges absolutely.*

By the continuity we see

$$\int_{-\infty}^{\infty} e^{\frac{\tau}{4}t^2+t(z+\zeta)}dt = \int_{-\infty}^{\infty} :e_*^{t(z+\zeta)}:_\tau\, dt =: \int_{-\infty}^{\infty} e_*^{t(z+\zeta)}dt :_\tau .$$

A little more careful estimate shows that $\int_{-\infty}^{\infty} f(s)e_*^{s(z+\zeta)}ds\in\mathcal{E}_{2+}(\mathbb{C})$ for any τ with $\mathrm{Re}\tau<0$. It is not hard to see that

$$\int_0^{\infty} e_*^{s(z+\zeta)}ds, \quad \int_{-\infty}^{0} e_*^{s(z+\zeta)}ds \quad \in\ \mathcal{E}_{2+}(\mathbb{C}).$$

Hence by Theorem 3, we see the following:

Corollary 1 *Both* $-\int_0^{\infty} e_*^{s(z+\zeta)}ds$ *and* $\int_{-\infty}^{0} e_*^{s(z+\zeta)}ds$ *are inverses of* $z+\zeta$*. Similarly both* $\sum_0^{\infty} e_*^{n(z+\zeta)}$ *and* $-\sum_1^{\infty} e_*^{-n(z+\zeta)}$ *are inverses of* $1-e_*^{z+\zeta}$*.*

However, such a strange phenomenon seems to be necessary for supporting the multi-valuedness of elements.

Proposition 12 $e^{i\theta}\int_{-\infty}^{\infty} e_*^{e^{i\theta}s(z+\zeta)}ds$ *does not depend on* θ *whenever* $Re\, e^{i\theta}\tau<0$*. The* τ*-expression of this integral is*

$$:e^{i\theta}\int_{-\infty}^{\infty} e_*^{e^{i\theta}s(z+\zeta)}ds :_\tau = e^{i\theta}\int_{-\infty}^{\infty} e^{e^{\frac{\tau}{4}e^{2i\theta}s^2}} e^{e^{i\theta}s(z+\zeta)}ds.$$

Proof. Since the integral converges absolutely, the differentiation by θ gives

$$ie^{i\theta}\int_{-\infty}^{\infty} e_*^{e^{i\theta}s(z+\zeta)}ds + e^{i\theta}\int_{-\infty}^{\infty} ie^{i\theta}s(z+\zeta)*e_*^{e^{i\theta}s(z+\zeta)}ds.$$

The integration by parts gives 0. □

Thus, replacing τ by $e^{-2i\theta}\tau$, we see that $\mathrm{Re}e^{-2i\theta}\tau e^{2i\theta}=\mathrm{Re}\,\tau<0$, and therefore

$$:e^{i\theta}\int_{-\infty}^{\infty} e_*^{e^{i\theta}s(z+\zeta)}ds :_{e^{-2i\theta}\tau} = \int_{-\infty}^{\infty} e^{-\frac{1}{4}|\tau|s^2} e^{e^{i\theta}s(z+\zeta)}d(e^{i\theta}s)$$

exists for any $\tau\neq 0$. Tracing $\theta=0$ to π, we see that the parallel transform of $:\int_{-\infty}^{\infty} e_*^{s(z+\zeta)}ds :_\tau$ along the closed curve $e^{i\theta}\tau$, $0\le\theta\le 2\pi$ of expressions gives $-:\int_{-\infty}^{\infty} e_*^{s(z+\zeta)}ds :_\tau$ at $\theta=2\pi$.

This implies that $\int_{-\infty}^{\infty} e_*^{s(z+\zeta)}ds$ should be treated as a two-valued element. Another concrete example is seen in [12], [13].

By a similar proof, we can show also the following:

Proposition 13 *There is a closed curve $\tau(t)$, $0 \le t \le 2\pi$ such that the parallel translation of* $:-\int_0^{\infty} e_*^{s(z+\zeta)}ds :_{\tau(0)}$ *given in Corollary 1 along $\tau(t)$ gives* $:\int_{-\infty}^{0} e_*^{s(z+\zeta)}ds :_{\tau(0)}$ *at $t = 2\pi$.*

Note also that Propositions 12, 13 hold for $\int_{-\infty}^{\infty} f(s)e_*^{s(z+\zeta)}ds$ when $f(s)$ is an entire function of exponential order, (precisely $f \in \mathcal{E}_{1+}(\mathbb{C})$).

However, it is seen in Section 5.5 that two inverses $\sum_0^{\infty} e_*^{n(z+\zeta)}$ and $-\sum_1^{\infty} e_*^{-n(z+\zeta)}$ of $1 - e_*^{z+\zeta}$ can not be connected by a parallel translation.

5.5 Theta functions

We show in this subsection that such a strange phenomena relates to elliptic theta functions of Jacobi. Consider the following bilateral power series:

$$\begin{aligned} \theta_1(\zeta,*) &= \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n e_*^{(2n+1)i\zeta}, & \theta_2(\zeta,*) &= \sum_{n=-\infty}^{\infty} e_*^{(2n+1)i\zeta}, \\ \theta_3(\zeta,*) &= \sum_{n=-\infty}^{\infty} e_*^{2ni\zeta}, & \theta_4(\zeta,*) &= \sum_{n=-\infty}^{\infty}(-1)^n e_*^{2ni\zeta}. \end{aligned} \tag{48}$$

Suppose $\mathrm{Re}\,\tau>0$. Then, we see that τ-expressions $\theta_i(\zeta,\tau) =: \theta_1(\zeta,*) :_\tau$ converge absolutely for every fixed ζ, and these are nothing but Jacobi's elliptic theta functions (cf. [2]).

For instance, the τ-expression $\theta_3(\zeta,\tau)$ of $\theta_3(\zeta,*)$ is given as follows:

$$\theta_3(\zeta,\tau) =: \theta_3(\zeta,*) :_\tau = \sum_{n\in\mathbb{Z}} e^{-n^2\tau+2ni\zeta} = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2ni\zeta}, \quad q = e^{-\tau}. \tag{49}$$

The τ-expression of the trivial identity $e_*^{2ni\zeta} * \theta_3(\zeta,*) = \theta_3(\zeta,*)$ gives the formula of quasi-periodicity of $\theta_3(\zeta,\tau)$. (Use (35) in order to compute $e^{-n^2\tau+2ni\zeta} *_\tau \theta_3(\zeta,\tau)$.)

Although $\theta_3(\zeta,0)$ diverges, $\sum_n e^{2ni\zeta}$ may be regarded as the delta function $\delta_0(\zeta)$ on S^1. Hence, $\theta_3(\zeta,*)$ may be regarded as a $*$-delta function on S^1. The parameter τ in the theory of Jacobi's theta functions means quasi-period other than 2π, but here τ is only a deformation parameter of expressions of $*$-delta function on S^1. This might imply that $\theta_i(\zeta,*)$ is the *genuine* physical existence of 2π-periodic q-number functions.

The important feature of theta functions $\theta_i(\zeta,*)$ is the fact that the domain of τ-expression is restricted to the right half-plane $\mathrm{Re}\,\tau>0$. The famous Hadamard's gap theorem shows that $|q| = 1$ in the expression (49) is the natural boundary with respect to the holomorphic function of q. Consequently, $\theta_i(\zeta,*)$ are *single valued q-number functions*. There is no monodromic phenomenon such as Proposition 13.

Corollary 2 *There is no closed curve of τ-expression along which two inverses $\sum_{n=0}^{\infty} e_*^{2ni\zeta}$ and $\sum_{-\infty}^{1} e_*^{2ni\zeta}$ of $1-e_*^{2i\zeta}$ are connected by a parallel translation.*

Proof. Note that

$$:\sum_{n=0}^{\infty} e_*^{2ni\zeta} :_\tau = \sum_{n=0}^{\infty} e^{-n^2\tau} e^{2ni\zeta},$$

and

$$:\sum_{n=-\infty}^{1} e_*^{2ni\zeta} :_\tau = \sum_{n=1}^{\infty} e^{-n^2\tau} e^{2ni(-\zeta)}.$$

Consider a curve such as $\sum_{n=0}^{\infty} e^{e^{2i\theta} n^2\tau} e^{2nie^{i\theta}\zeta}$. To ensure the convergence, we have to keep the inequality $\mathrm{Re}\, e^{2i\theta}\tau > 0$. Hence one can not form a closed curve to obtain $e^{2i\theta}=1, e^{i\theta}=-1$. □

In general, we define as follows:

Definition 12 For a pair (a,b) of complex numbers, a $*$-*theta function of type* (a,b) is an element $f_* \in \mathcal{E}_\infty(\mathbb{C})$ satisfying

$$e_*^{a\zeta} * f_*(\zeta) = e^{\alpha} f_*(\zeta), \quad e_*^{b\partial_\zeta} f_*(\zeta) = e^{\beta} f_*(\zeta), \quad \text{for some } \alpha, \beta \in \mathbb{C}.$$

By $\Theta(a,b)$ we denote the totality of $*$-theta functions of type (a,b).

The second equality gives the quasi-periodicity $f_*(\zeta+b) = e^{\beta} f_*(\zeta)$ and the first equality is rewritten as $e_*^{a(\zeta-(\alpha/a))} * f_*(\zeta) = f_*(\zeta)$. This gives the quasi-periodicity in the τ-expression as follows: $e^{a\zeta+(a^2/4)\tau-\alpha} f(\zeta+\frac{a}{2}\tau) = f(\zeta)$. Setting $g_*(\zeta) = e_*^{-(\beta/b)\zeta} f_*(\zeta)$, we have $g_*(\zeta+b) = g(\zeta)$ and $e_*^{a\zeta} * g_*(\zeta) = e^{\alpha} g_*(\zeta)$, and this is $e^{a\zeta+(1/4)a^2\tau} g(\zeta+\frac{1}{2}a\tau) = e^{\alpha} g(\zeta)$.

Lemma 5 $\Theta(a,b) \neq \{0\}$, *if and only if* $ab \in 2\pi i\mathbb{Z}$.

Proof. Since the product formula is translation invariant, we see $e_*^{a(\zeta+b)} * g_*(\zeta+b) = e^{\alpha} g_*(\zeta+b)$. Hence $e^{ab} e_*^{a\zeta} * g(\zeta) = e^{\alpha} g(\zeta)$. It follows that $e^{ab}=1$. □

6 Special localizations

In this section we treat the case where a localization $\kappa = (F_\kappa, \mathcal{O}_\kappa, *_\kappa, \mu_\kappa)$ has certain nice properties. Roughly speaking, localizations are classified by the property of μ_κ:

- There exists $\zeta \in \mathcal{O}_\kappa$ such that $[\mu_\kappa^{-1}, \zeta] = i$.
- There exists $X \in \mathcal{O}_\kappa$ such that $[\mu_\kappa^{-1}, X] = X$.

6.1 Localizations where the canonical conjugate of μ_κ^{-1} exists

First, we take the case where there is a special element $\zeta\in\mathcal{O}_\kappa$ such that $[\zeta,\mu_\kappa]_*=i\mu_\kappa^2$. In view of Proposition 1, this may be written as $[\mu_\kappa^{-1},\zeta]_*=i$.

Let $\mathcal{O}_{\mu,\zeta}=\{f\in\mathcal{O}_\kappa;[\mu_\kappa,f]_*=[\zeta,f]_*=0\}$, and $F_{\mu,\zeta}=\{f\in F_\kappa;[\mu_\kappa,f]_*=[\zeta,f]_*=0\}$. We denote the restricted product $*_\kappa$ simply by $\hat{*}$. Here we assume the following:

Option 2 Set $u=\mu_\kappa^{-1}$, $v=\zeta$ for simplicity. $(\mathcal{O}_\kappa,*_\kappa)$ is given by a certain algebra of $\mathcal{O}_{\mu,\zeta}$-valued ordinary (but very restricted in the variable u) C^∞-functions $f(u,v)$ defined on $\mathbb{R}^2$ with the following product formula, called the *Moyal product formula*:

$$f(u,v)*_\kappa g(u,v)=f(u,v)\exp\left\{\frac{\kappa}{2}(\overleftarrow{\partial_u}\,\hat{*}\,\overrightarrow{\partial_v}-\overleftarrow{\partial_v}\,\hat{*}\,\overrightarrow{\partial_u})\right\}g(u,v), \tag{50}$$

where $\hat{*}$ is the product restricted on $\mathcal{O}_{\mu,\zeta}$ and the arrow indicates to which side the derivation operators act. Similarly, F_κ is given by a certain space of $F_{\mu,\zeta}$-valued ordinary functions $f(u,v)$. Recall again that u may not be a member of $\mathcal{O}_\kappa$.

Remark The product formula (50) is not unique for obtaining the algebra $(\mathcal{O}_\kappa,*_\kappa)$. Just as in the previous section, we have seen in [12] that these can be *deformed* in such a way that the algebra structure does not change, but the expression changes.

The product $f(u,v)*_\kappa g(u,v)$ is well-defined at least if one of f,g is an $\mathcal{O}_{\mu,\zeta}$-valued polynomial, and the associativity $(f*_\kappa g)*_\kappa h=f*_\kappa(g*_\kappa h)$ holds if any two of f,g,h are $\mathcal{O}_{\mu,\zeta}$-valued polynomials.

One can define the $*$-exponential function e_*^{au+ibv} by solving the differential equation $\frac{d}{dt}f_t=(au+ibv)*_\kappa f_t$. However, since the product formula (50) gives

$$(au+ibv)*_\kappa g(au+ibv)=(au+ibv)g(au+ibv),$$

the uniqueness of a real analytic solution, and (50) give

$$e_*^{au+ibv}=e^{au+ibv},\quad e_*^{au+ibv}*_\kappa e_*^{a'u+ib'v}=e^{\frac{i}{2}(ab'-ba')}e_*^{(a+a')u+i(b+b')v}. \tag{51}$$

The following formula can be directly proved from (50)

$$\begin{aligned}e^{-2isu-2tv}*_\kappa f(u,v)&=e^{-2isu-2tv}f(u+t,v+s)\\&=f(u,v)*_\kappa e^{2isu+2tv},\quad s,t\in\mathbb{R}\end{aligned} \tag{52}$$

whenever $f(u,v)$ is real analytic. In general, we extend the $*_\kappa$-product by (52).

Let $\hat{f}(u,t)=\int_{\mathbb{R}}ds\,f(u,s)e^{-its}$ be the partial Fourier transform. $f(u,v)$ is recovered by

$$f(u,v)=\int_{\mathbb{R}}dt\,\hat{f}(u,t)e^{itv}=\int_{\mathbb{R}}dt\,\hat{f}(u+\frac{t}{2},t)*_\kappa e^{itv}. \tag{53}$$

Now we suppose an element $\varpi_{00}=2e^{2iuv}$, called a *vacuum*, is contained in F_κ. Consider a closed left-ideal $\{f\in\mathcal{O}_\kappa;f*\varpi_{00}=0\}$ of $\mathcal{O}_\kappa$. By the product formula

(50) we easily see that $v *_\kappa \varpi_{00} = 0$. Moreover, using the uniqueness of real analytic solutions, we see $e^{itv} *_\kappa \varpi_{00} = \varpi_{00}$, and

$$e^{itv} *_\kappa \phi(u) *_\kappa \varpi_{00} = \phi(u + \frac{t}{2}) *_\kappa \varpi_{00},$$

and

$$f(u, v) *_\kappa \varpi_{00} = \int_{\mathbb{R}} \bar{d}t \hat{f}(u - \frac{t}{2}, t) *_\kappa \varpi_{00} = f(u, 0) *_\kappa \varpi_{00}. \tag{54}$$

Using (54) to $f(u, v) *_\kappa \phi(u) *_\kappa \varpi_{00}$, we have

$$\begin{aligned} f(u, v) *_\kappa \phi(u) *_\kappa \varpi_{00} &= \int_{\mathbb{R}} \bar{d}t \hat{f}(u + \frac{t}{2}, t) *_\kappa e^{itv} *_\kappa \phi(u) *_\kappa \varpi_{00} \\ &= \int_{\mathbb{R}} \bar{d}t \hat{f}(u + \frac{t}{2}, t) \phi(u + \frac{t}{2}) *_\kappa \varpi_{00}. \end{aligned}$$

Hence we have the formula for the regular representation $\phi(u) *_\kappa \varpi_{00} \to f(u, v) *_\kappa \phi(u) *_\kappa \varpi_{00}$ in the form of pseudo differential operators (ΨDO):

$$f(u, v) *_\kappa \phi(u) *_\kappa \varpi_{00} = \int_{\mathbb{R}} \bar{d}t \int_{\mathbb{R}} \bar{d}s f(u + \frac{t}{2}, s) e^{-its} \phi(u + \frac{t}{2}) *_\kappa \varpi_{00}.$$

6.2 Localization to the periodic phenomena

One can consider a localization where only periodical phenomena are concerned. In the physical phenomena, the periodicity always means the time periodicity of state functions. As in the previous subsection, we assume the following:

Option 3 $\mu_\kappa^{-1} \in F_\kappa$, and there are elements $e_*^{in\zeta} \in \mathcal{O}_\kappa$ such that $[\mu_\kappa^{-1}, e_*^{in\zeta}] = -ne_*^{in\zeta}$ for every $n \in \mathbb{Z}$, that is, $\mu_\kappa^{-1} *_\kappa e_*^{in\zeta} = e_*^{in\zeta} *_\kappa \mu_\kappa^{-1} - ne_*^{in\zeta}$.

Set $w = \mu_\kappa^{-1} *_\kappa e_*^{i\zeta}$, and $\bar{w} = e_*^{-i\zeta}$ so that $[w, \bar{w}] = 1$, $\mu_\kappa^{-1} = w *_\kappa \bar{w}$. Note that w is only a member of F_κ, while $w^{-1}, \bar{w} \in \mathcal{O}_\kappa$.

We assume now there is an element $\varpi_{00} \in F_\kappa$ such that $\mu_\kappa *_\kappa \varpi_{00} = \varpi_{00}$, and for every $a \in \mathcal{O}_\kappa$, the identity $(a *_\kappa \mu_\kappa^{-1}) *_\kappa \varpi_{00} = a *_\kappa \varpi_{00}$ holds. (Note that $\varpi = e^{2i(u-1)v}$ in the notation of (50) satisfies $u *_\kappa \varpi = \varpi$.) If the associativity holds, then the identity $(a *_\kappa \mu_\kappa^{-1}) *_\kappa \varpi_{00} = a *_\kappa \varpi_{00}$ is trivial. But let us first note the following:

Lemma 6 *ϖ_{00} can not be in $\mathcal{O}_\kappa$.*

Proof. Suppose an element $\Omega \in \mathcal{O}_\kappa$ satisfies $\mu_\kappa^{-1} *_\kappa \Omega = \Omega$. Then we have $w *_\kappa \Omega = 0$ by using $\mu_\kappa^{-1} *_\kappa e^{i\zeta} *_\kappa \Omega = [\mu_\kappa^{-1}, e^{i\zeta}] *_\kappa \Omega + e^{i\zeta} *_\kappa \Omega$. Since $w^{-1}, \Omega \in \mathcal{O}_\kappa$, the associativity gives

$$\Omega = (w^{-1} *_\kappa w) *_\kappa \Omega = w^{-1} *_\kappa (w *_\kappa \Omega) = 0.$$

□

It is easy to see the following:

$$w *_\kappa \varpi_{00} = 0, \quad (w *_\kappa w) *_\kappa \varpi_{00} = \varpi_{00}, \quad w *_\kappa \varpi_{00} = w^{-1} *_\kappa \varpi_{00}.$$

Moreover, we have $w^{-1} *_\kappa (w *_\kappa \varpi_{00}) = 0$, but this does not imply $\varpi_{00} = 0$, because of the break-down of the associativity.

Define a closed left ideal of $\mathcal{O}_\kappa$ by $\{a \in \mathcal{O}; a *_\kappa \varpi_{00} = 0\}$. Then, $w, \mu_\kappa, \mu_\kappa^{-1}$ disappear in the factor space $\Psi = \mathcal{O}_\kappa *_\kappa \varpi_{00}$. Since by the assumption, we have

$$\begin{aligned}(\mu_\kappa^{-1} *_\kappa e_*^{-in\zeta}) *_\kappa \varpi_{00} &= [\mu_\kappa^{-1}, e_*^{-in\zeta}] *_\kappa \varpi_{00} + (e_*^{-in\zeta} * \mu_\kappa^{-1}) *_\kappa \varpi_{00} \\ &= (n+1) e_*^{-in\zeta} *_\kappa \varpi_{00},\end{aligned}$$

$(w)^n *_\kappa \varpi_{00}$ are eigenstate functions of μ_κ^{-1}.

By these observations, it is natural to think that Ψ consists of $\mathcal{O}_{\mu,\zeta}$-valued holomorphic functions of w on $\mathbb{C} - \{0\}$. Since $w = e_*^{-i\zeta}$, this implies every state function is not only 2π-periodic as a function of ζ but also a holomorphic function of w.

Note that $w *_\kappa \phi(w) *_\kappa \varpi_{00} = \phi'(w) *_\kappa \varpi_{00}$. Hence, w, w are called respectively the *creation*, and the *annihilation* operators.

Now, in addition to the assumption that $\phi(w)$ is holomorphic on $\mathbb{C} - \{0\}$, we consider the following scale-transformation property for state functions: For some a, $\mathrm{Re}\, a > 0$,

$$\phi(aw) *_\kappa \varpi_{00} = (\alpha w + \beta)\phi(w) *_\kappa \varpi_{00}. \tag{55}$$

If $\alpha = 0$, then $\beta = a^\ell \in \mathbb{Z}$ and $\phi(w) = c(w)^\ell$.

To treat the case $\alpha \neq 0$, we first take note of the formula

$$e_*^{s\mu_\kappa^{-1}} *_\kappa \phi(w) *_\kappa \varpi_{00} = \mathrm{Ad}(e^{s\mu_\kappa^{-1}})(\phi(w)) *_\kappa e^s *_\kappa \varpi_{00}.$$

Since $\mathrm{Ad}(e^{sn\mu_\kappa^{-1}})(\phi(e^{i\zeta})) = \phi(e^{i(\zeta+si)})$, the equality we have to consider is

$$\phi(aw)a^{-1} *_\kappa \varpi_{00} = (\alpha w + \beta)\phi(w) *_\kappa \varpi_{00}, \quad a = e^{-s}. \tag{56}$$

Rewriting this as a function of ζ by setting $\phi(e^{i\zeta}) = \psi(\zeta)$, we have

$$\psi(\zeta + si) *_\kappa \varpi_{00} = e^s(\alpha e^{i\zeta} + \beta)\psi(\zeta) *_\kappa \varpi_{00}.$$

Thus, such a scale-translation property of state functions gives theta functions.

6.3 Localization where no canonical conjugate of μ_κ^{-1} exists

In this subsection we fix a localization $\kappa = (F_\kappa, \mathcal{O}_\kappa, \mu_\kappa)$ of $(\mathcal{O}, \mu)$, and denote this by $\kappa = (F, \mathcal{O}, \mu)$ omitting the subscript κ. Let Σ_κ be the set of all extremal localizations of κ.

Recall the definition of characteristic vector field (5). If its orbit behaves chaotically, it seems impossible to find an element ζ such that $[\mu^{-1}, \zeta] = i$. Classical statistical mechanics is based on such a situation.

We assume here the convergence and the nontriviality of

$$\lim_{k\to\infty} \mu^k = \varpi_{00} \in \bigcap_k \mu^k * \mathcal{O}.$$

Hence we have $\mu^{-1} * \varpi_{00} = \varpi_{00}$ and by the uniqueness of the real analytic solution of $\frac{d}{dt} f_t = i\mu^{-1} * f_t$, we have $e_*^{it\mu^{-1}} * \varpi_{00} = e^{it}\varpi_{00}$. In particular $e_*^{2\pi i\mu^{-1}} * \varpi_{00} = \varpi_{00}$.

Thus, Lemma 6 shows that there is no $\zeta \in \mathcal{O}$ such that $e_*^{it\zeta} \in \mathcal{O}$ and $[\mu^{-1}, \zeta] = i$. However, we suppose $f \to \mathrm{Ad}(e_*^{it\mu^{-1}})(f)$ is defined for all $t \in \mathbb{R}$ and a continuous linear mapping of F into itself for each t, and it gives a one-parameter automorphism group of $(\mathcal{O}, *)$.

Proposition 14 *Under the assumption as above, if* $\lim_{n\to\infty} \frac{1}{n}\sum_0^n \mathrm{Ad}(e_*^{2\pi ik\mu^{-1}})(\phi)$ *converges in the weak topology to an element* $X \in F$*, then the identity* $\mathrm{Ad}(e_*^{2\pi i\mu^{-1}})(X) = X$ *holds.*

Proof. Let F^* be the dual space of F regarded as an $\mathcal{O}$-bimodule. We show for every $\psi \in F^*$ that $\langle \psi | \mathrm{Ad}(e_*^{2\pi i\mu^{-1}})(X)\rangle = \langle \psi | X\rangle$. Since the continuity of $\mathrm{Ad}(e_*^{it\mu^{-1}})$ is assumed, we compute as follows:

$$\begin{aligned}
\left\langle \psi | \mathrm{Ad}\left(e_*^{2\pi i\mu^{-1}}\right)(X)\right\rangle &= \left\langle \mathrm{Ad}^*\left(e_*^{2\pi i\mu^{-1}}\right)(\psi) \middle| \lim_{n\to\infty}\frac{1}{n}\sum_0^n \mathrm{Ad}\left(e_*^{2\pi ki\mu^{-1}}\right)(\phi)\right\rangle \\
&= \lim_{n\to\infty}\frac{1}{n}\left\langle \mathrm{Ad}^*\left(e_*^{2\pi i\mu^{-1}}\right)(\psi) \middle| \sum_0^n \mathrm{Ad}\left(e_*^{2\pi ik\mu^{-1}}\right)(\phi)\right\rangle \\
&= \lim_{n\to\infty}\frac{1}{n}\left\langle \psi \middle| \sum_1^{n+1} \mathrm{Ad}\left(e_*^{2\pi ik\mu^{-1}}\right)(\phi)\right\rangle \\
&= \lim_{n\to\infty}\left\langle \psi \middle| \frac{1}{n}\sum_0^n \mathrm{Ad}\left(e_*^{2\pi ik\mu^{-1}}\right)(\phi)\right\rangle.
\end{aligned}$$

□

Using this, we have a lot of elements $X \in F$ satisfying $e_*^{2\pi i\mu^{-1}} * (X * \varpi_{00}) = X * \varpi_{00}$.

Since $\varpi_{00} \in \mathcal{O}$ and $e_*^{2\pi i\mu^{-1}} * \varpi_{00} = \varpi_{00}$, the continuity of $*\varpi_{00}$ and the uniqueness of real analytic solutions give $X * \varpi_{00} = \lim_{n\to\infty} \frac{1}{n}\sum_0^n e_*^{2\pi ik\mu^{-1}} * \phi * \varpi_{00}$. Hence we have

$$e_*^{2\pi i\mu^{-1}} * (X * \varpi_{00}) = \lim_{n\to\infty}\frac{1}{n}\sum_1^{n+1} e_*^{2\pi ik\mu^{-1}} * \phi * \varpi_{00} = X * \varpi_{00}.$$

6.4 Relativistic situation

Suppose the derivation D is given by ∂_ζ. In this subsection, we show the case $\mathrm{ad}(\mu_\kappa^{-1}) = \partial_\zeta^2$ might have some relativistic explanation.

We assume that a suitably localized algebra $\mathcal{O}_\kappa$ is given by a generator system

$$u_1, \dots, u_m, v_1, \dots, v_m$$

together with the fundamental relation $[u_i, v_j] = i\hbar\delta_{ij}$, $[u_i, u_j] = [v_i, v_j] = 0$. Suppose now there is an element $\varpi_{00} \in F_\kappa$ such that $v_i *_\kappa \varpi_{00} = 0$.

Consider the case $\mu^{-1} = \sum_{ij} g^{ij}(\boldsymbol{u}) * v_i v_j + V(\boldsymbol{u})$. The vacuum representation of $*$-exponential function $e_*^{t(1/i\hbar)\mu^{-1}}$ is obtained by setting $\psi * \varpi_{00} = e_*^{t(1/i\hbar)\mu^{-1}} * \psi_0 * \varpi_{00}$ and solving the evolution equation

$$\partial_t \psi * \varpi_{00} = \frac{1}{i\hbar}\mu^{-1} * \psi * \varpi_{00}.$$

Since $\mu^{-1} * \varpi_{00} = V * \varpi_{00}$, this turns out to be the (nonrelativistic) Schrödinger equation:

$$\partial_t \psi(t, \boldsymbol{u}) * \varpi_{00} = \left(-i\hbar \sum_{ij} g^{ij}(\boldsymbol{u})\partial_i\partial_j + \frac{1}{i\hbar}V(\boldsymbol{u})\right)\psi(t, \boldsymbol{u}) * \varpi_{00}.$$

Set $\tau = -i\hbar t$ and suppose that the time parameter of Schrödinger's equation is the imaginary part of the deformation parameter τ.

In the spirit of q-number functions, a deformation is only a change of expression of the "same" object. Intuitively, it should be permitted to think that the time evolution is something like a deformation (24).

Suppose now $\psi(t, \boldsymbol{u})$ is always given in the form $f(\tau, 2\zeta, \boldsymbol{u})$ of parallel section with respect to τ, ζ. That is, we assume that f always satisfies

$$\partial_\tau f(\tau, 2\zeta, \boldsymbol{u}) = \partial_\zeta^2 f(\tau, 2\zeta, \boldsymbol{u}), \quad \text{(cf.(46))}.$$

Thus, replacing $\partial_\tau f$ by $\partial_\zeta^2 f$, we have a relativistic equation

$$\partial_\zeta^2 f(\tau, 2\zeta, \boldsymbol{u}) * \varpi_{00} = \left(\sum_{ij} g^{ij}(\boldsymbol{u})\partial_i\partial_j - \frac{1}{\hbar^2}V(\boldsymbol{u})\right) f(\tau, 2\zeta, \boldsymbol{u}) * \varpi_{00}, \tag{57}$$

if we regard ζ as the (universal) time.

At this stage the vacuum ϖ_{00} can be eliminated from both left- and right-hand sides. If we set : $f_*(2\zeta, \boldsymbol{u}) :_\tau = f(\tau, 2\zeta, \boldsymbol{u})$, then we have

$$\partial_\zeta^2 f_*(2\zeta, \boldsymbol{u}) = \left(\sum_{ij} g^{ij}(\boldsymbol{u})\partial_i\partial_j - \frac{1}{\hbar^2}V(\boldsymbol{u})\right) f_*(2\zeta, \boldsymbol{u}).$$

Since τ is a deformation parameter, variables involved in the equation are only ζ and the variables contained in $\mathcal{O}$. Hence, if g^{ij} or V in the nonrelativistic Schrödinger equation contain the variable t, then these must contain also the variable ζ in the beginning so that $g^{ij}(\tau, \zeta, \boldsymbol{u})$ and $V(\tau, \zeta, \boldsymbol{u})$ are parallel sections with respect to τ.

6.4.1 Another application

In the last part, we note that the notion of deformation can be used as a *modifier*, a technique that makes everything smooth. Though it is not directly relevant to the relativity, such technique will be useful especially in the relativity.

Consider the case $\mu_\kappa^{-1} = p(\langle \boldsymbol{a}, \boldsymbol{v}\rangle)$ where $p(z)$ is a polynomial of order n and $\langle \boldsymbol{a}, \boldsymbol{v}\rangle = \sum_{i=1}^m a_i v_i$. The equation we want to solve is

$$\partial_t^2 \psi_*(t, \boldsymbol{u}) * \varpi_{00} = p(\langle \boldsymbol{a}, \boldsymbol{v}\rangle) * \psi_*(t, \boldsymbol{u}) * \varpi_{00}. \tag{58}$$

Choose a vector $\boldsymbol{b}$ so that $[\langle \boldsymbol{a}, \boldsymbol{v}\rangle, \langle \boldsymbol{b}, \boldsymbol{u}\rangle]_* = i$ and set

$$\psi(t, \langle \boldsymbol{b}, \boldsymbol{u}\rangle) = \int g(t, s) e_*^{is\langle \boldsymbol{b}, \boldsymbol{u}\rangle} đs$$

by the partial Fourier transform.

Here we set $\zeta = \langle \boldsymbol{b}, \boldsymbol{u}\rangle$ and consider the τ-expression of $e_*^{is\zeta}$ as in Section 5. Namely, we set

$$: e_*^{is\langle \boldsymbol{b}, \boldsymbol{u}\rangle} :_\tau = e^{-\frac{1}{4}s^2\tau} e^{is\langle \boldsymbol{b}, \boldsymbol{u}\rangle}.$$

Since $e^{-(1/4)s^2\tau}$ is rapidly decreasing whenever $\mathrm{Re}\,\tau > 0$, this makes calculations very smooth. Thus, we have only to solve

$$\partial_t^2 g(t, s) = p(i\partial_s) g(t, s).$$

In order to solve this, we set $g(t, s) = \int \rho(t, \xi) e^{is\xi} đ\xi$ and consider the equation

$$\partial_t^2 \rho(t, \xi) = p(i\xi)\rho(t, \xi), \quad \rho(0, \xi) = 1, \tag{59}$$

for every fixed ξ. The solution is given by $\rho(t, \xi) = e^{t\sqrt{p(i\xi)}}$, after fixing a branch of the 2-valued function $\sqrt{p(i\xi)}$. Hence, in the form of τ-expression such that $\mathrm{Re}\,\tau > 0$, a solution of (58) is given by

$$e_*^{tp(\frac{1}{i}\langle \boldsymbol{b}, \boldsymbol{u}\rangle)_*^{1/2}}(P) = \iint_P e^{tp(i\xi)^{1/2}} e^{is\xi} e_*^{s\frac{1}{i}\langle \boldsymbol{b}, \boldsymbol{u}\rangle} đs đ\xi, \tag{60}$$

where the path P of integration by ξ is a curve from $-\infty$ to ∞ chosen in such a way that this does not hit the branching points of $\sqrt{p(i\xi)}$. However, because of the 2-valued character of $\sqrt{p(i\xi)}$, we have a lot of such paths of integration. Fixing any one of them, we easily see that the τ-expression is

$$: e_*^{tp(\frac{1}{i}\langle \boldsymbol{b}, \boldsymbol{u}\rangle)_*^{1/2}}(P) :_\tau = \iint_P e^{tp(i\xi)^{1/2}} e^{is\xi} e^{-\frac{1}{4}s^2\tau} e^{s\frac{1}{i}\langle \boldsymbol{b}, \boldsymbol{u}\rangle} đs đ\xi.$$

Since the τ-expression $: e_*^{is\langle \boldsymbol{b}, \boldsymbol{u}\rangle} :_\tau$ with $\mathrm{Re}\,\tau > 0$ is rapidly decreasing with respect to s, so is its Fourier transform

$$\int e^{is\xi} e_*^{is\langle \boldsymbol{b},\boldsymbol{u}\rangle} \bar{d}s. \tag{61}$$

Thus, if $e^{tp(i\xi)^{1/2}}$ is a temperate distribution, (60) is well-defined. Recall that Fourier transform for temperate distributions is well-defined.

Proposition 15 *If* $p(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ *is a polynomial such that* $a_0 > 0$, *then* $e^{tp(i\xi)^{1/2}}$ *is a temperate distribution.*

Proof. On a domain $|z| < 2D$, $e^{p(z)^{1/2}}$ is bounded together with its derivatives. Set $p(z) = z^n(a_0 + a_1 z^{-1} + \cdots + a_n z^{-n})$. $q(z) = a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}$ is bounded on a domain $|z| > D$. Hence $e^{q(z)^{1/2}}$ is bounded together with its derivatives. Taking $a_0^{1/2}$ as a real number and setting $p(i\xi)^{1/2} = i a_0^{1/2} \xi q(i\xi)^{1/2}$, we see that $e^{p(i\xi)^{1/2}}$ is a temperate distribution. □

Let $\xi = C(\eta)$ be one of such paths P. Then, $\sqrt{p(iC(\eta)}$ is C^∞ on C. The value of the integral does not change by any slight move of C in a compact region without hitting the branching point. However, if C crosses a branching point, this causes the drastic change of path of integration by switching branches.

Moreover, we have another problem that the Fourier transform (61) has the sign ambiguity by the same reasoning as in Propositions 12, 13, since $e^{is\xi}$ is an entire function of exponential order with respect to s. Thus, we see the following important result, which shows we have to treat many-valued functions:

Proposition 16 *The solution given* $e_*^{tp(i\langle \boldsymbol{b},\boldsymbol{u}\rangle)_*^{1/2}}$ *given by* (60) *is defined as a 2-valued parallel section with respect to the deformation parameter* τ.

We remark now that even for multi-valued functions, one can ask whether it is holomorphic on a domain which does not contain a branching point.

Now regard the constant term a_n of $p(z)$ in Proposition 15 as a variable and set $p(z) = p_0(z) + w$. Though $e_*^{tp(i\langle \boldsymbol{b},\boldsymbol{u}\rangle)_*^{1/2}}$ is a 2-valued parallel section, we have the following:

Proposition 17 $e_*^{t(w+p_0((1/i\hbar)\langle \boldsymbol{b},\boldsymbol{u}\rangle))_*^{1/2}}$ *can be defined so that it is holomorphic with respect to* w *on a neighborhood of* ∞.

Acknowledgments In spite of the fact that this paper is far from the final version, the editors gave me permission to publish it, considering the time that is left for me to enjoy. Thus, I have to promise that the time left for me will be devoted to open the front door and reveal the beautiful garden of the quantum world to the public geometers.

References

1. G. S. Agawal, E. Wolf, *Calculus for functions of noncommuting operators and general phase-space method of functions*, Physcal Review D, vol.2, no.10, 1970, 2161–2186.
2. G. Andrews, R. Askey, R. Roy, *Special functions*, Encyclopedia Math, Appl. 71, Cambridge, 2000.
3. J. L. Brylinski, *Loop spaces, characteristic classes and geometric quantization*. Birkhäuser, 1993.
4. A. Connes *Noncommutative geometry*. Academic Press, 1994.
5. I. M. Gel'fand, G. E. Shilov, *Generalized Functions, 2* Academic Press, 1968.
6. M. Gerstenhaber, A. Giaquinto, Deformation associated to rigid algebras, preprint.
7. J. Grabowski, Abstract Jacobi and Poisson structures: Quantization and star-product, J. Geometry and Physics 9, 1992, 45–73.
8. V. Guillemin, S. Sternberg, *Geometric Asymptotics*. A.M.S. Mathematical Surveys, 14, 1977.
9. N. Hitchin, *Lectures on special Lagrangian submanifolds*, arXiv:math.DG/9907034vl 6Jul, 1999.
10. M. Kontsevitch, Deformation quantization of Poisson manifolds, I, qalg/9709040.
11. J. M. Maillard, *Star exponential functions for any ordering of the elements of the inhomogeneous symplectic Lie algebra*, J. Math. Phys. 45 (2004), 785–794.
12. F. Nadaud, Generalized deformation and Hochschild cohomology, Lett. Math. Physics 58, 2001, 41–55.
13. H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, *Deformation quantization of Fréchet-Poisson algebras: Convergence of the Moyal product*, in Conférence Moshé Flato 1999, Quantizations, Deformations, and Symmetries, Vol II, Math. Phys. Studies 22, Kluwer Academic Press, 2000, 233–246.
14. H. Omori, *One must break symmetry in order to keep associativity,* Banach Center Publ. vol.55, 2002, 153–163.
15. H. Omori, *Beyond point set topology*, Informal preprint.
16. H. Omori, *Infinite dimensional Lie groups*, AMS Translation Monograph 158, 1997.
17. H. Omori, *Noncommutative world, and its geometrical picture*, A.M.S translation of Sugaku expositions, 2000.
18. H. Omori, *Physics in mathematics*, (in Japanese) Tokyo Univ. Publications, 2004.
19. H. Omori, *A note on deformation calculus (A point of pointless calculus)*, in preparation.
20. H. Omori, T.Kobayashi, *Singular star-exponential functions,* SUT Jour, Mathematics 37, no.2, (2001), 137–152.
21. H. Omori, Y. Maeda, and A. Yoshioka, *Weyl manifolds and deformation quantization*, Advances in Math., Vol 85, No 2, pp. 224–255, 1991.
22. H. Omori, Y. Maeda, and A. Yoshioka. *Global calculus on Weyl manifolds*, Japanese J. Math. vol 17, pp. 57–82, 1991.
23. H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Strange phenomena related to ordering problems in quantizations*, Jour. Lie Theory vol. 13, no 2, 481–510, 2003.
24. H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Star exponential functions as two-valued elements*, Progress in Math. 232, Birkhäuser, 2004, 483–492.
25. H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Geometric objects in an approach to quantum geometry*, 303–323 in this volume.
26. A. Weinstein, The Maslov gerbe, Lett. Math. Phys. 63, 5–9, 2004.
27. K. Yoshida, *Functional Analysis*, Springer 1966

Resonance Gyrons and Quantum Geometry

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Dedicated to Hideki Omori

Summary. We describe irreducible representations, coherent states and star-products for algebras of integrals of motions (symmetries) of two-dimensional resonance oscillators. We demonstrate how the quantum geometry (quantum Kähler form, metric, quantum Ricci form, quantum reproducing measure) arises in this problem. We specifically study the distinction between the isotropic resonance $1:1$ and the general $l:m$ resonance for arbitrary coprime l, m. A quantum gyron is a dynamical system in the resonance algebra. We derive its Hamiltonian in irreducible representations and calculate the semiclassical asymptotics of the gyron spectrum via the quantum geometrical objects.

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1 Introduction

For complicated dynamical systems, it is important to be able to abstract from studying concrete motions or states and to observe surrounding structures, like spaces, algebras, etc., which carry essential properties of the variety of motions in the whole.

For quantum (wave) systems, the standard accompanying mathematical structures are algebras of "observables," i.e., functions on phase spaces, and representations of these algebras in Hilbert vector spaces of "states." This is the starting viewpoint for the mathematical quantization theory [1]–[10]. The more complicated systems are studied the more complicated algebras and phase spaces (symplectic manifolds) have to be used. Note that for general symplectic and even Kählerian manifolds the quantization problem is still unsolved.

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It was demonstrated in [11, 12] that for general symplectic manifolds it is possible to approximate the symplectic potential by its quadratic part (the oscillator!), then to use this quadratic part in order to define the standard Groenewold–Moyal [13, 14] product on the tangent spaces, and to construct a formal $*$-product on the original manifold by a perturbation theory. Such oscillator-generated quantum manifolds were called the "Weyl manifolds" in [11].

In quantum and wave mechanics, one often meets a situation similar in a certain sense: the dynamics of a system is, in general, chaotic, but there are some exclusive invariant submanifolds (for instance, equilibrium points) in the phase space around which the dynamics is regular and can be approximated by the oscillator motion in directions transversal to the submanifold. Thus the given system contains inside a built-in harmonic oscillator plus a certain anharmonic part near the equilibrium:

$$\frac{1}{2}\sum(p_j^2 + \omega_j^2 q_j^2) + \text{cubic} + \text{quartic} + \cdots . \tag{1.1}$$

If the frequencies ω_j of the harmonic part are incommensurable (not in a resonance), then in a small neighborhood of the submanifold the anharmonic part just slightly perturbs these frequencies, and the whole motion is performed along the perturbed Liouville tori. This is the well-investigated situation both on the classical and quantum levels [15]–[21].

If the frequencies ω_j are in a *resonance* then all standard approaches do not work and the picture occurs to be much more interesting from the viewpoint of quantum geometry. Here we will follow the works [22]–[25].

First of all, in the resonance case the Liouville tori are collapsed (to a smaller dimension), and the anharmonic part generates a nontrivial "averaged" motion in the new phase spaces: in the symplectic leaf Ω of the commutant $\mathcal{F}_\omega$ of the harmonic part. The new phase spaces represent certain hidden dynamics committed to the resonance. This dynamics describes a *precession* of the parameters of the resonance harmonic motion under the action of the anharmonic part. We call this dynamical system a *gyron* (from the Greek word "gyro," i.e., "rotating").

In the simplest case of the isotropic 1:1 resonance for two degrees of freedom the gyron system is just the Euler top system from the theory of rigid body rotations, which is related to the linear Poisson brackets. For the general $l : m$ resonance, *the gyron is described by a nonlinear Poisson brackets polynomial of degree* $l + m - 1$, see in [24, 25].

Of course, in the quantum case the resonance function algebra $\mathcal{F}_\omega$ has to be replaced by a resonance operator algebra $\mathfrak{F}_\omega$ which consists of operators commuting with the quantum oscillator $\frac{1}{2}\sum_j(\hat{p}_j^2 + \omega_j^2 q_j^2)$, where $\hat{p}_j = -i\hbar\partial/\partial q_j$. This algebra is described by nonlinear commutation relations of polynomial type, see in [24, 25]. It is the dynamic algebra for *quantum gyrons*.

Note that there is a variety of important physical models containing inside the resonance Hamiltonians like (1.1). The quantum gyrons in these models can be considered as an analog of known quasiparticles similar to polarons, rotons, excitons, etc.[1] As the

[1] Attention to this was paid by V. Maslov.

simplest example, we mention the models of nano-physics (quantum dots, artificial atoms, quantum wires, see examples in [24]). Another example is fiber waveguides in optics; they are described by the Hamiltonian

$$p^2 - n^2(q), \qquad q, p \in \mathbb{R}^3, \tag{1.2}$$

where $n(q)$ is the refraction index having maximum value along the waveguide axis, that is, along an arbitrary smooth curve in $\mathbb{R}^3$. The quadratic part of $n^2(q)$ in directions transversal to this curve is assumed to have commensurable frequencies in a certain resonance proportion $\omega_1 : \omega_2 = l : m$, where l, m are coprime integers. The quantum gyron in this model describes certain hidden "polarization" of the light beam along the given curve in the optical medium, see in [24]. The propagation of such *optical gyrons* and their spectrum depend on the anharmonic part of the refraction index, and so one can control the properties of the gyron waves by changing the geometry of the curve just by bending the optical fiber.

The aim of the given paper is to describe the quantum geometry of the gyron phase spaces in the case of the $l : m$ resonance.

If $l = m = 1$, then these phase spaces Ω are just homogeneous spheres $\mathbb{S}^2$, that is, the coadjoint su(2) orbits. The quantum geometry in this case coincides with the classical symplectic (Kählerian) geometry generated by linear Lie–Poisson brackets.

If at least one of the integers l or m exceeds 1, then, as we will see below, the quantum geometry occurs to be unusual. The quantum phase spaces are still diffeomorphic to $\mathbb{S}^2$, but the classical symplectic form is singular on them. The correct symplectic (Kählerian) form and the reproducing measure of the quantum phase space are chosen from the nontrivial condition that the operators of irreducible representations of the quantum resonance algebras $\mathfrak{F}_\omega = \mathfrak{F}_{l,m}$ have to be differential operators, not pseudodifferential (the maximal order of these operators is $\max(l, m)$).

Thus the geometry [26, 27] determining the Wick–Klauder–Berezin $*$-product on the gyron phase space has a purely quantum behavior and the $*$-product itself cannot be obtained by a formal deformation technique from the classical data.

Note that here we mean the phase spaces corresponding either to low energy levels of the oscillator (i.e., to the nano-zone near its equilibrium point, in the terminology of [24]) or to excited levels (i.e., to the micro-zone). Thus one can talk about *quantum nano- or micro-geometry* generated by the $l : m$ frequency resonance.

The distinction between the specific case $l = m = 1$ and the generic case $\max(l, m) > 1$ is the distinction between algebras with linear and nonlinear commutation relations. We see that the nonlinearity of relations in the algebra $\mathfrak{F}_{l,m}$ (the absence of a Lie group of symmetries) for the resonance oscillator implies the quantum character of the phase spaces in nano- and micro-zones near the ground state. The motion in these spaces is the gyron dynamics. In the nano-zone, this dynamics is purely quantum and does not have a classical analog at all. In the micro-zone, the gyron dynamics and the gyron spectrum can be described by semiclassical methods [23, 24] if one at first fixes the quantum geometry of the gyron phase space.

Applying this theory, for instance, to optical gyrons, we come to the conclusion that the *light beam propagating near the axis of a resonance fiber waveguide cannot* *be described by purely geometric optics and carry essentially quantum properties*. This opens an opportunity to apply such simple optical devices, for example, in constructing elements of quantum computers.

Also note that the $l : m$ resonance oscillators, which we discuss here, can be presented in the form

$$\hat{l} + \hat{m}, \tag{1.3}$$

where $\hat{l}$ and $\hat{m}$ are mutually commutating action operators with spectra $l \cdot \mathbb{Z}_+$ and $m \cdot \mathbb{Z}_+$ in the Hilbert space $\mathcal{L} = L^2(\mathbb{R} \times \mathbb{R})$. The operators $\hat{l}$ and $\hat{m}$ can be considered as "quantum integer numbers" and their sum as a quantum sum of integers. Then the representation theory of the algebra $\mathfrak{F}_{l,m}$ and the corresponding quantum geometry could be considered as a brick for construction of something like "quantum arithmetics."

2 Commutation relations and Poisson brackets for $l : m$ resonance

The Hamiltonian of the resonance oscillator (1.3) can be written as

$$\mathbf{E} = l\mathbf{b}_1^*\mathbf{b}_1 + m\mathbf{b}_2^*\mathbf{b}_2. \tag{2.1}$$

Here l, m are coprime integers, $\mathbf{b}_1, \mathbf{b}_2$ are annihilation operators in the Hilbert space $\mathcal{L}$, and $\mathbf{b}_1^*, \mathbf{b}_2^*$ are the conjugate creation operators. The commutation relations are

$$[\mathbf{b}_1, \mathbf{b}_1^*] = [\mathbf{b}_2, \mathbf{b}_2^*] = \hbar\mathbf{I},$$

all other commutators are zero.

In the algebra generated by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_1^*, \mathbf{b}_2^*$, let us consider the commutant of the element (2.1). This commutant is a nontrivial, noncommutative subalgebra. We call it a *resonance algebra*. It is related to quantum gyrons.

Note that the resonance algebra is generated by the following four elements:

$$\mathbf{A}_1 = \mathbf{b}_1^*\mathbf{b}_1, \qquad \mathbf{A}_2 = \mathbf{b}_2^*\mathbf{b}_2, \qquad \mathbf{A}_+ = (\mathbf{b}_2^*)^l\mathbf{b}_1^m, \qquad \mathbf{A}_- = \mathbf{A}_+^*. \tag{2.2}$$

Let us define the polynomials

$$\rho(A_1, A_2) \stackrel{\text{def}}{=} \prod_{j=1}^{m}(A_1 + j\hbar) \cdot \prod_{s=1}^{l}(A_2 - s\hbar + \hbar), \tag{2.3}$$

$$\varkappa(A_1, A_2) \stackrel{\text{def}}{=} lA_1 + mA_2.$$

Lemma 2.1 *Elements* (2.2) *obey the commutation relations*

$$\begin{gathered}[\mathbf{A}_1, \mathbf{A}_2] = 0,\\ [\mathbf{A}_1, \mathbf{A}_\pm] = \mp\hbar m\mathbf{A}_\pm, \qquad [\mathbf{A}_2, \mathbf{A}_\pm] = \pm\hbar l\mathbf{A}_\pm,\\ [\mathbf{A}_-, \mathbf{A}_+] = \rho(\mathbf{A}_1 - \hbar m, \mathbf{A}_2 + \hbar l) - \rho(\mathbf{A}_1, \mathbf{A}_2).\end{gathered} \tag{2.4}$$

Lemma 2.2 *In the abstract algebra* $\mathfrak{F}_{l,m}$ *with relations* (2.4) *there are two Casimir elements*

$$\kappa = \varkappa(\mathbf{A}_1, \mathbf{A}_2), \qquad \mathbf{C} = \mathbf{A}_+\mathbf{A}_- - \rho(\mathbf{A}_1, \mathbf{A}_2).$$

In realization (2.2) *the Casimir element* $\mathbf{C}$ *is identically zero, and the Casimir element* κ *coincides with the oscillator Hamiltonian* $\mathbf{E}$ (2.1).

Note that the operators $\mathbf{A}_1$, $\mathbf{A}_2$ (2.2) are self-adjoint, but $\mathbf{A}_+$ is not. Let us introduce the self-adjoint operators $\mathbf{A}_3$, $\mathbf{A}_4$ by means of the equalities

$$\mathbf{A}_\pm = \mathbf{A}_3 \mp i\mathbf{A}_4.$$

Then commutation relations (2.4) read

$$\begin{aligned} &[\mathbf{A}_1, \mathbf{A}_2] = 0, \qquad [\mathbf{A}_1, \mathbf{A}_3] = i\hbar m\mathbf{A}_4, \qquad [\mathbf{A}_1, \mathbf{A}_4] = -i\hbar m\mathbf{A}_3,\\ &\qquad\qquad\qquad\quad [\mathbf{A}_2, \mathbf{A}_3] = -i\hbar l\mathbf{A}_4, \qquad [\mathbf{A}_2, \mathbf{A}_4] = i\hbar l\mathbf{A}_3,\\ &[\mathbf{A}_3, \mathbf{A}_4] = \frac{i}{2}\big(\rho(\mathbf{A}_1 - \hbar m, \mathbf{A}_2 + \hbar l) - \rho(\mathbf{A}_1, \mathbf{A}_2)\big). \end{aligned} \tag{2.4a}$$

Let us denote by A_j the classical variable (a coordinate on $\mathbb{R}^4$) corresponding to the quantum operator $\mathbf{A}_j$. Then the relations (2.4a) are reduced to the following Poisson brackets on $\mathbb{R}^4$:

$$\begin{aligned} &\{A_1, A_2\} = 0,\\ &\{A_1, A_3\} = -mA_4, \qquad \{A_1, A_4\} = mA_3,\\ &\{A_2, A_3\} = lA_4, \qquad \{A_2, A_4\} = -lA_3,\\ &\{A_4, A_3\} = \frac{1}{2}(l^2A_1 - m^2A_2)A_1^{m-1}A_2^{l-1}. \end{aligned} \tag{2.5}$$

Lemma 2.3 *Relations* (2.5) *determine the Poisson brackets on* $\mathbb{R}^4$ *with the Casimir functions*

$$\varkappa = lA_1 + mA_2, \qquad C = A_3^2 + A_4^2 - A_1^m A_2^l.$$

Lemma 2.4 *In the subset in* $\mathbb{R}^4$ *determined by the inequalities* $A_1 \geq 0$ *and* $A_2 \geq 0$, *there is a family of surfaces*

$$\Omega = \{\varkappa = E, C = 0\}, \qquad E > 0, \tag{2.6}$$

which coincide with the closure of symplectic leaves Ω_0 *of the Poisson structure* (2.5). *These surfaces are homeomorphic to the sphere*: $\Omega \approx \mathbb{S}^2$.

The topology of the symplectic leaves Ω_0 *is the following*:

- *if* $l = m = 1$, *then* $\Omega_0 = \Omega$;
- *if* $l = 1$, $m > 1$ *or* $l > 1$, $m = 1$, *then* Ω_0 *is obtained from* Ω *by deleting the point* $(0, \frac{E}{m}, 0, 0)$ *or the point* $(\frac{E}{l}, 0, 0, 0)$;
- *if* $l > 1$, $m > 1$, *then* Ω_0 *is obtained from* Ω *by deleting both the points* $(0, \frac{E}{m}, 0, 0)$ *and* $(\frac{E}{l}, 0, 0, 0)$.

Lemma 2.5 *If $l > 1$ or $m > 1$, then the Kirillov symplectic form ω_0 on the leaf $\Omega_0 \subset \Omega$ has a weak (integrable) singularity at the point $A_2 = 0$ or $A_1 = 0$. The symplectic volume of Ω_0 is finite*

$$\frac{1}{2\pi}\int_{\Omega_0}\omega_0 = \frac{E}{lm}. \tag{2.7}$$

Lemma 2.6 *On the subset $A_1 > 0$ the complex coordinate*

$$z_0 = \frac{A_3 + iA_4}{A_1^m} \tag{2.8}$$

determines a partial complex structure consistent with the brackets (2.5) *in the sense of* [32]. *On each symplectic leaf Ω_0, this partial complex structure generates the Kählerian structure with the potential*

$$\Phi_0 = \int_0^{|z_0|^2}\left(\frac{E}{2lm} + \alpha_E(x)\right)\frac{dx}{x}, \qquad \omega_0 = i\bar{\partial}\partial\Phi_0. \tag{2.9}$$

Here ∂ is the differential by z_0 and $\alpha_E = \alpha_E(x)$ is the solution of the equation

$$x = \left(\frac{E}{2m} + l\alpha_E\right)^l\left(\frac{E}{2l} - m\alpha_E\right)^{-m} \tag{2.10}$$

with values on the interval $-\frac{E}{2lm} \le \alpha_E \le \frac{E}{2lm}$.

The singular points of ω_0 on Ω_0 correspond to the poles

$$A_2 = 0 \iff z_0 = 0, \qquad \omega_0 \sim \frac{1}{l^2}\left(\frac{E}{l}\right)^{m/l}\frac{dx\wedge d\varphi}{x^{1-1/l}} \quad \text{as} \quad z_0 \to 0, \tag{2.11}$$
$$A_1 = 0 \iff z_0 = \infty, \qquad \omega_0 \sim \frac{1}{m^2}\left(\frac{E}{m}\right)^{l/m}\frac{dx\wedge d\varphi}{x^{1+1/m}} \quad \text{as} \quad z_0 \to \infty,$$

where (x,φ) are polar coordinates, $z_0 = x^{1/2}\exp\{i\varphi\}$.

The restrictions of coordinate functions to the surface (2.6) *are given by*

$$A_1\Big|_{\Omega_0} = \frac{E}{2l} - m\alpha_E(|z_0|^2), \qquad A_2\Big|_{\Omega_0} = \frac{E}{2m} + l\alpha_E(|z_0|^2), \tag{2.12}$$
$$(A_3 + iA_4)\Big|_{\Omega_0} = z_0\left(\frac{E}{2l} - m\alpha_E(|z_0|^2)\right)^m.$$

Note that the properties of classical symplectic leaves of the $l : m$ resonance algebra, described in Lemmas 2.4–2.6, are a particular case of the topology and geometry of *toric varieties* (in our case the torus $\mathbb{T}^1 = \mathbb{S}^1$ is the cycle); about this see general theorems in [28]–[30]. The Poisson extension (2.5) by means of polynomial brackets was first described in [22, 23] for the case of $1 : 2$ resonance and in [24, 25] for the $l : m$ case, as well for the general multidimensional resonances. A type of Poisson extension was also considered in [31] for some specific class of resonance proportions (which does not include, for instance, the $1 : 2 : 3$ resonance).

3 Irreducible representations of $l : m$ resonance algebra

First of all, let us discuss the basic problems in constructing irreducible representations of algebras like (2.4), (2.4a). Following the standard geometric quantization program [6] one has to choose a line bundle over symplectic leaves Ω_0 of the Poisson algebra related to (2.4a), that is, the Poisson algebra (2.5). Then this bundle is endowed with the Hermitian connection whose curvature is $i\omega_0$, and a Hilbert space $\mathcal{H}_0$ of antiholomorphic sections of the bundle is introduced. In this Hilbert space, the operators of irreducible representation of the algebra (2.4a) are supposed to act and to be self-adjoint.

However, there are two principle difficulties. First, we do not know which measure on Ω_0 to take in order to determine the Hilbert norm in the space $\mathcal{H}_0$. The choice of measure should imply the *reproducing property* [32, 33]

$$\omega_0 = i\bar{\partial}\partial \ln \sum_k |\varphi_0^{(k)}|^2, \tag{3.1}$$

where $\{\varphi_0^{(k)}\}$ is an orthonormal basis in $\mathcal{H}_0$. For the inhomogeneous case, where the commutation relations (2.4a) are not linear and no Lie group acts on Ω_0, the existence of such a reproducing measure is, in general, unknown. This difficulty was discovered in [34] (more precisely, it was observed in [34] that the Liouville measure generated by the symplectic form ω_0 does not obey the property (3.1) in general).

Secondly, even if one knows the reproducing measure, there is still a problem: the operators of the irreducible representation constructed canonically by the geometric quantization scheme would be pseudodifferential, but not differential operators. There are additional nontrivial conditions on the complex structure (polarization) that make the generators of the algebra be differential operators (of order greater than 1, in general). About such highest analogs of the Blattner–Kostant–Sternberg conditions for the polarization to be “invariant” see in [35, 36].

Taking these difficulties into account, we modify the quantization scheme. From the very beginning, we look for an appropriate complex structure and the scalar product in the space of antiholomorphic functions that guarantee the existence of an Hermitian representation of the given algebra by differential operators, and then introduce a “quantum” Kählerian form ω on Ω, a “quantum” measure and the “quantum” Hilbert space $\mathcal{H}$ which automatically obeys the reproducing property like (3.1) (without “classical” label 0). This approach is explained in [32, 33, 37].

Note that the polynomial structure of the right-hand sides of relations (2.4), (2.4a) is critically important in this scheme to obtain representations by differential operators.

Denote by $\mathcal{P}_r$ the space of all polynomials $\varphi(\lambda) = \sum_{n=0}^{r} \varphi_n \lambda^n$ of degree $r \geq 0$ with complex coefficients.

Lemma 3.1 *Let f_+, f_- be two complex functions on $\mathbb{Z}_+$ such that*

$$\begin{aligned} &f_+ f_- > 0 \qquad \textit{on the subset} \quad \{1, \dots, r\} \subset \mathbb{Z}_+, \\ &f_-(0) = f_+(r+1) = 0. \end{aligned} \tag{3.2}$$

Then the differential operators

$$\mathbf{a}_+ = f_+\Big(\lambda\frac{d}{d\lambda}\Big)\cdot\lambda, \qquad \mathbf{a}_- = \frac{1}{\lambda}\cdot f_-\Big(\lambda\frac{d}{d\lambda}\Big) \tag{3.3}$$

leave the space $\mathcal{P}_r$ invariant and they are conjugate to each other with respect to the following scalar product in $\mathcal{P}_r$:

$$(g, g') \stackrel{\text{def}}{=} \sum_{n=0}^{r}\prod_{s=1}^{n}\overline{\frac{f_-(s)}{f_+(s)}}\varphi_n\overline{\varphi'_n}. \tag{3.4}$$

Any operator $f\big(\lambda\frac{d}{d\lambda}\big)$, where f is a real function on $\mathbb{Z}_+$, is self-adjoint in $\mathcal{P}_r$ with respect to this scalar product.

Now we consider a map

$$\gamma : \mathbb{R}^k \to \mathbb{R}^k$$

and a real function ρ on $\mathbb{R}^k$. Denote by $R_r \subset \mathbb{R}^k$ the subset of all points a_0 such that

$$\begin{aligned} \rho(\gamma^{r+1}(a_0)) &= \rho(a_0), \\ \rho(\gamma^n(a_0)) &> \rho(a_0) \qquad (n = 1, \dots, r). \end{aligned} \tag{3.5}$$

For any $a_0 \in R_r$ we define real functions f_j $(j = 1, \dots, k)$ on $\mathbb{Z}_+$ by the formula $f_j(n) \stackrel{\text{def}}{=} \gamma^n(a_0)_j$, and introduce mutually commuting operators in the space $\mathcal{P}_r$:

$$\mathbf{a}_j \stackrel{\text{def}}{=} f_j\Big(\lambda\frac{d}{d\lambda}\Big). \tag{3.6}$$

Lemma 3.2 *Let $a_0 \in R_r$, and let there be a factorization*

$$\rho(\gamma^n(a_0)) - \rho(a_0) = f_+(n)f_-(n), \qquad 0 \le n \le r+1, \tag{3.7}$$

where the factors $f_\pm$ obey the property (3.2). *Then the operator $\mathbf{a}_+$* (3.3) *and $\mathbf{a}_j$* (3.6) *in the space $\mathcal{P}_r$ with the scalar product* (3.4) *satisfy the relations*

$$\mathbf{a}_+^* = \mathbf{a}_-, \qquad \mathbf{a}_j^* = \mathbf{a}_j \qquad (j = 1, \dots, k),$$

and

$$\begin{aligned} &[\mathbf{a}_j, \mathbf{a}_s] = 0, \\ &\mathbf{a}_j\mathbf{a}_+ = \mathbf{a}_+\gamma_j(\mathbf{a}), \qquad \mathbf{a}_-\mathbf{a}_j = \gamma_j(\mathbf{a})\mathbf{a}_- \qquad (j = 1, \dots, k), \\ &[\mathbf{a}_-, \mathbf{a}_+] = \rho(\gamma(\mathbf{a})) - \rho(\mathbf{a}). \end{aligned} \tag{3.8}$$

Lemma 3.3 *Consider the abstract algebra $\mathfrak{F}$ with relations* (3.8). *The element $\mathbf{C} = \mathbf{a}_+\mathbf{a}_- - \rho(\mathbf{a})$ belongs to the center of $\mathfrak{F}$. If a function $\varkappa$ on $\mathbb{R}^k$ is γ-invariant, then the element $\kappa = \varkappa(\mathbf{a})$ belongs to the center of $\mathfrak{F}$.*

In the representation (3.3), (3.6), *these central elements are scalar: $\mathbf{C} = \rho(a_0)\cdot\mathbf{I}$, $\kappa = \varkappa(a_0)\cdot\mathbf{I}$. This representation of the algebra $\mathfrak{F}$ is irreducible and Hermitian.*

If the map γ has no fixed points, then all irreducible Hermitian representations of the algebra $\mathfrak{F}$ can be obtained in this way. All such representations of dimension $r+1$ are parameterized by elements of the set R_r $(r = 0, 1, 2, \dots)$.

Now let us return to commutation relations (2.4). In this case $k = 2$, the function ρ is given by (2.3), and the mapping $\gamma \equiv \Gamma^\hbar : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$\Gamma^\hbar \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} A_1 - \hbar m \\ A_2 + \hbar l \end{pmatrix}. \tag{3.9}$$

It follows from (2.2) that we have to be interested in a subset $A_1 \geq 0$, $A_2 \geq 0$ in $\mathbb{R}^2$. Also in view of Lemma 2.2, the values of the Casimir element $\mathbf{C} = \rho(a_0) \cdot \mathbf{I}$ must be zero. From (3.5) we obtain

$$\rho(a_0) = \rho(\Gamma^{\hbar(r+1)}(a_0)) = 0,$$
$$\rho(\Gamma^{\hbar n}(a_0)) > 0 \qquad (n = 1, \dots, r).$$

Using (2.3) let us factorize:

$$\rho = \rho_+\rho_-, \qquad \rho_+(A) \stackrel{\text{def}}{=} \prod_{j=1}^{m}(A_1 + \hbar j), \qquad \rho_-(A) \stackrel{\text{def}}{=} \prod_{s=1}^{l}(A_2 - \hbar s + \hbar). \tag{3.10}$$

It is possible to satisfy (3.7) by choosing

$$f_\pm(n) = \rho_\pm(\Gamma^{\hbar n}(a_0)).$$

In this case, the set $R_r \subset \mathbb{R}^2$ consists of all points $a_0 = \begin{pmatrix} \hbar(rm + p) \\ \hbar q \end{pmatrix}$ for which the pair of integers p, q obeys the inequalities

$$0 \leq q \leq l - 1, \qquad 0 \leq p \leq m - 1. \tag{3.11}$$

The γ-invariant function $\varkappa$ in our case (3.9) is just $\varkappa(A) = lA_1 + mA_2$. In view of Lemma 3.3, the value of the second Casimir element $\kappa = \varkappa(\mathbf{a})$ in the irreducible representation (3.3), (3.6) is $\varkappa(a_0) = E_{r,q,p}$, where

$$E_{r,q,p} \stackrel{\text{def}}{=} \hbar(lmr + lp + mq). \tag{3.12}$$

From Lemma 2.2 we conclude that these numbers coincide with eigenvalues of the oscillator $\mathbf{E}$ (2.1).

Also from (3.4) we see that the scalar product in the space $\mathcal{P}_r$ is given by

$$(\varphi, \varphi') = \sum_{n=0}^{r} \hbar^{(l-m)n} \frac{(q + nl)!(p + (r - n)m)!}{q!(p + rm)!} \varphi_n \overline{\varphi'_n}. \tag{3.13}$$

Thus the vector space of the irreducible representation depends on the number r only, but its Hilbert structures are parameterized by the pairs q, p from (3.11). That is why below we will use the notation $\mathcal{P}_r \equiv \mathcal{P}_{r,q,p}$.

Let us summarize the obtained results.

Theorem 3.4 *The commutant of the $l : m$ resonance oscillator* $\mathbf{E}$ (2.1) *is generated by operators* (2.2) *obeying commutation relation* (2.4). *The irreducible representation of the algebra* (2.4), *corresponding to the eigenvalue $E_{r,q,p}$* (3.12) *of the operator* $\mathbf{E}$, *is given by the following ordinary differential operators $\mathbf{a} = (\mathbf{a}_1, \mathbf{a}_2)$ and $\mathbf{a}_\pm$:*

$$\mathbf{a} = \Gamma^{\hbar\lambda \frac{d}{d\lambda}}(a_0), \qquad \mathbf{a}_+ = \rho_+(\mathbf{a}) \cdot \lambda, \qquad \mathbf{a}_- = \frac{1}{\lambda} \cdot \rho_-(\mathbf{a}). \tag{3.14}$$

Here $a_0 = \begin{pmatrix} \hbar(rm+p) \\ \hbar q \end{pmatrix}$, the flow Γ on $\mathbb{R}^2$ is defined by (3.9) *and the factors $\rho_\pm$ are defined by* (3.10). *The representation* (3.14) *acts in the space $\mathcal{P}_{r,q,p}$ of polynomials in λ of degree r, and it is Hermitian with respect to the scalar product* (3.13). *The dimension of this representation is $r+1$.*

In fact, formula (3.14) determines just the matrix representations of the algebra (2.4): elements $\mathbf{a}$ are represented by a diagonal matrix and $\mathbf{a}_\pm$ by near-diagonal matrices with respect to the orthonormal basis of monomials

$$\varphi^{(k)}(\lambda) = \hbar^{(m-l)k/2}\left(\frac{q!(p+rm)!}{(q+kl)!(p+(r-k)m)!}\right)^{1/2} \cdot \lambda^k \qquad (k = 0, \dots, r) \tag{3.15}$$

in the space $\mathcal{P}_{r,q,p}$. These matrices are real-valued and determined by the integer numbers l, m (from the resonance proportion) and r, p, q (labeling the representation):

$$\begin{aligned} (\mathbf{a}_1)_{ns} &= \hbar(p + (r-n)m)\delta_{n,s}, \qquad (\mathbf{a}_2)_{ns} = \hbar(q + nl)\delta_{n,s}, \\ (\mathbf{a}_+)_{ns} &= \hbar^{(l+m)/2}\left(\frac{(q+nl)!(p+(r-s)m)!}{(q+sl)!(p+(r-n)m)!}\right)^{1/2}\delta_{n-1,s}, \\ (\mathbf{a}_-)_{ns} &= (\mathbf{a}_+)_{sn}. \end{aligned} \tag{3.16}$$

Here the matrix indices n, s run over the set $\{0, \dots, r\}$ and $\delta_{n,s}$ are the Kronecker symbols.

In the particular case $l = m = 1$, from (3.16) one obtains the well-known Hermitian matrix irreducible representations of the "spin" Lie algebra su(2) with cyclic commutation relation between generators $\frac{1}{2}(\mathbf{A}_1 - \mathbf{A}_2)$, $\frac{1}{2}(\mathbf{A}_+ + \mathbf{A}_-)$, $\frac{i}{2}(\mathbf{A}_+ - \mathbf{A}_-)$.

4 Quantum geometry of the $l : m$ resonance

Now we give a geometric interpretation of the obtained representations of the resonance algebra.

It follows from (3.4) that the element $\rho_+(\mathbf{A})^{-1}(\mathbf{A}_3 - i\mathbf{A}_4)$, in the algebra generated by relations (2.4), is represented by the multiplication by λ in each irreducible representation (3.14). If we denote

$$\mathbf{z} = (\mathbf{A}_3 + i\mathbf{A}_4)\rho_+(\mathbf{A})^{-1}, \tag{4.1}$$

then the conjugate operator $\mathbf{z}^*$ in each irreducible representation can be taken equal to the multiplication by a complex variable $\overline{z}$:

$$\mathbf{z}^* = \overline{z}.$$

Thus, here we change our notation and use $\overline{z}$ instead of λ. From now on, $\mathcal{P}_{r,q,p}$ is the space of anti-holomorphic functions (polynomials in $\overline{z}$ of degree $\leq r$) on $\mathbb{R}^2$.

Let us assume that the scalar product (3.13) in the space $\mathcal{P}_{r,q,p}$ can be written in the integral form

$$(\varphi, \varphi') = \frac{1}{2\pi\hbar} \int_{\mathbb{R}^2} \varphi(\overline{z}(a)) \overline{\varphi'(\overline{z}(a))} L(a)\, da, \tag{4.2}$$

where $da = |d\overline{z}(a) \wedge dz(a)|$ and $a \to z(a)$ is the complex coordinate on $\mathbb{R}^2$.

Lemma 4.1 *The explicit formula for the density L in* (4.2) *is*

$$L(a) = \frac{1}{4\hbar^{rm+p+q+1}(p+rm)!q!x} \times \int_0^\infty A_1^{rm+p} A_2^q \left(\frac{l^2}{A_2} + \frac{m^2}{A_1}\right)^{-1} \exp\left\{-\frac{A_1 + A_2}{2\hbar}\right\} dE,$$

where $A_1 = \frac{E}{2l} - m\alpha_E(x)$, $A_2 = \frac{E}{2m} + l\alpha_E(x)$, α_E is taken from (2.10), *and $x = |z(a)|^2$.*

These are first steps to assign some geometry to the quantum algebra (2.4) and its irreducible representations. The next step is to consider the multiplication operation in this algebra.

Note that linear operators in $\mathcal{P}_{r,q,p}$ can be presented by their kernels. So, the algebra of operators is naturally isomorphic to $\mathcal{S}_{r,q,p} \stackrel{\text{def}}{=} \mathcal{P}_{r,q,p} \otimes \overline{\mathcal{P}}_{r,q,p}$. The operator product is presented by the convolution of kernels which is generated by pairing between $\overline{\mathcal{P}}_{r,q,p}$ and $\mathcal{P}_{r,q,p}$ given by the scalar product (3.13).

The algebra $\mathcal{S}_{r,q,p}$ consists of functions in $\overline{z}$, z, they are polynomials on $\mathbb{R}^2$. On this function space we have a noncommutative product (convolution), but the unity element of this convolution is presented by the function

$$K = \sum_{k=0}^{r} \varphi^{(k)} \otimes \overline{\varphi^{(k)}}, \tag{4.3}$$

where $\varphi^{(k)}$ is the orthonormal basis in $\mathcal{P}_{r,q,p}$. This function is called a reproducing kernel [38, 39], it is independent of the choice of the basis $\{\varphi^{(k)}\}$. From (3.15) we see the explicit formula for the reproducing kernel

$$K = k(|z|^2), \qquad k(x) \stackrel{\text{def}}{=} \sum_{n=0}^{r} \hbar^{(m-l)n} \frac{q!(p+rm)!}{(q+nl)!(p+(r-n)m)!} x^n. \tag{4.4}$$

In order to give a Gelfand type spectral–geometric interpretation of some algebra, we, first of all, have to ensure that the unity element of this algebra is presented by the unity function. It is not so for the algebra $\mathcal{S}_{r,q,p}$. That is why we have to divide the "kernel elements" from $\mathcal{S}_{r,q,p}$ by the reproducing kernel (4.4). The correct function algebra consists of ratios of the type

$$f = \frac{\varphi \otimes \overline{\varphi'}}{K}, \tag{4.5}$$

where $\varphi, \varphi' \in \mathcal{P}_{r,q,p}$. The product of two functions of this type generated by the convolution of kernels is given by

$$(f_1 * f_2)(a) = \frac{1}{2\pi\hbar} \int_{\text{phase space}} f_1^{\#}(a|b) f_2^{\#}(b|a) p_a(b)\, dm(b). \tag{4.6}$$

Here

$$dm(b) \stackrel{\text{def}}{=} L(b) K(b)\, db, \tag{4.7}$$

$$p_a(b) \stackrel{\text{def}}{=} |K^{\#}(a|b)|^2 K(a)^{-1} K(b)^{-1}, \tag{4.8}$$

and the operation $f \to f^{\#}$ denotes the analytic continuation holomorphic with respect to the "right" argument and anti holomorphic with respect to the "left" argument in the notation $f^{\#}(\cdot|\cdot)$. The product (4.6) possesses the desirable property: $1 * f = f * 1 = f$.

Let us look at formula (4.5). Since f is going to be a function on an invariant geometric space, φ and φ' have to be sections of a Hermitian line bundle with the curvature form

$$\omega = i\hbar\overline{\partial}\partial \ln K \equiv i g d\overline{z} \wedge dz. \tag{4.9}$$

Here ∂ denotes the differential by z. Formula (4.9) means that the measure dm (4.7) is the reproducing measure with respect to the Kählerian form ω in the sense [33].

Note that formula (4.9) defines both the *quantum form* ω and the *quantum metric* $g = g(|z|^2)$, $g(x) = \hbar \frac{d}{dx}(x \frac{d}{dx}(\ln k(x)))$ via the polynomial (4.4).

After the quantum form ω appears, the "probability" factor p_a in the noncommutative product (4.6) can be written as

$$p_a(b) = \exp\left\{ \frac{i}{\hbar} \int_{\sum(a,b)} \omega \right\}. \tag{4.10}$$

Here $\sum(a, b)$ is a membrane in the complexified space whose boundary consists of four paths connecting points $a \to b|a \to b \to a|b \to a$ along leaves of the complex polarization and its conjugate [26, 40].

Note that the set of functions (4.10) makes up a resolution of unity:

$$\frac{1}{2\pi\hbar} \int_{\text{phase space}} p_a\, dm(a) = 1, \tag{4.11}$$

and each p_a is the "eigenfunction" of the operators of left or right multiplication:

$$f * p_a = f(\cdot|a)p_a, \qquad p_a * f = f(a|\cdot)p_a. \tag{4.12}$$

The details about such a way to establish a correspondence between quantum algebras and Kählerian geometry can be found in [33].

Let us discuss global aspects of this quantum geometry. The Kählerian form ω (4.9) is actually well defined on the compactified plane $\mathbb{R}^2 \cup \{\infty\}$ which includes the infinity point $z = \infty$. To see this, we just can make the change of variables $z' = 1/z$ and observe that ω is smooth near $z' = 0$.

Thus the actual phase space is diffeomorphic to $\mathbb{S}^2$ and we have

$$\frac{1}{2\pi\hbar}\int_{\mathbb{S}^2}\omega = r, \qquad \frac{1}{2\pi\hbar}\int_{\mathbb{S}^2} dm = r + 1. \tag{4.13}$$

The first formula (4.13) follows from the fact that $K \sim \text{const} \cdot |z|^{2r}$ as $z \to \infty$ (see in (4.4)). It means that the cohomology class $\frac{1}{2\pi\hbar}[\omega]$ is integer, and this is the necessary condition for the Hermitian bundle with the curvature $i\omega$ over $\mathbb{S}^2$ to have global sections [41].

The second formula (4.13) follows from the definition (4.3) which implies

$$\frac{1}{2\pi\hbar}\int dm = \sum_{k=0}^{r} \|\varphi^{(k)}\|^2,$$

where the norm of each $\varphi^{(k)}$ is taken in the sense (4.3) and is equal to 1 by definition. The number $r + 1$ in (4.13) is the dimension of the irreducible representation of the resonance algebra.

We stress that the quantum Kählerian form ω, given by (4.4), (4.9), and the quantum measure dm, given by (4.7) and Lemma 4.1, are essentially different from the classical form ω_0 (2.9) and the classical Liouville measure $dm_0 = |\omega_0|$. The main difference is that ω is smooth and dm is regular at poles while ω_0 and dm_0 are not. Some information regarding asymptotics of the quantum objects as $\hbar \to 0$ and asymptotics near the poles is summarized in the following lemma.

Lemma 4.2 (a) *In the classical limit $\hbar \to 0$, $E_{r,q,p} \to E > 0$, out of neighborhoods of the poles $z = 0$ and $z = \infty$ on the sphere, the quantum geometrical objects are approximated by the classical ones:*

$$\omega = \omega_0 + O(\hbar), \qquad dm = dm_0(1 + O(\hbar)).$$

(b) *The behavior of the quantum reproducing measure near the poles is the following:*

$$\begin{aligned} dm &\sim \text{const} \cdot \frac{dx \wedge d\varphi}{x^{1-(q+1)/l}} \qquad \textit{as} \quad x \to 0, \\ dm &\sim \text{const} \cdot \frac{dx \wedge d\varphi}{x^{1+(p+1)/m}} \qquad \textit{as} \quad x \to \infty, \end{aligned} \tag{4.14}$$

where $z = x^{1/2}\exp\{i\varphi\}$. *Thus the reproducing measure has weak singularities at poles.*

(c) *Near the poles, the quantum Kählerian form looks as*

$$\omega \sim \hbar^{m-l+1}\frac{(p+rm)!q!}{(p+rm-m)!(q+l)!}id\bar{z}\wedge dz \qquad as \quad z\to 0,$$
$$\omega \sim \hbar^{l-m+1}\frac{p!(q+rl)!}{(p+m)!(q+rl-l)!}\frac{id\bar{z}\wedge dz}{|z|^4} \qquad as \quad z\to \infty.$$

Thus, near the poles, the asymptotics of ω *as* $\hbar \to 0$ *is*

$$\omega \sim \mathrm{const}\hbar^{1-l}id\bar{z}\wedge dz \qquad (z\sim 0), \tag{4.15}$$
$$\omega \sim \mathrm{const}\hbar^{1-m}\frac{id\bar{z}\wedge dz}{|z|^4} \qquad (z\sim\infty).$$

Comparing (4.14) with (2.11) we see that, near poles, dm is not approximated by dm_0 as $\hbar \to 0$ if $q > 0$ or $p > 0$. So, the usual deformation theory (starting with classical data) *cannot be applied to compute the reproducing measure* globally on the phase space.

Formulas (4.15) demonstrate that the quantum ω is not approximated by ω_0 as $\hbar \to 0$ near the poles; the classical form ω_0 must be singular at $z = 0$ if $l > 1$ and be singular at $z = \infty$ if $m > 1$. This statement is in agreement with (2.11).

Note that the cohomology class of the classical symplectic form ω_0 on the classical leaf with the quantized energy $E = E_{r,q,p}$ (3.12) is given by (2.7):

$$\frac{1}{2\pi\hbar}\int_{\Omega_0}\omega_0 = r + \frac{q}{l} + \frac{p}{m}. \tag{4.16}$$

Here $r \sim \hbar^{-1}$ is the main quantum number which controls the dimension of the quantum Hilbert space $\mathcal{P}_{r,q,p}$. The integers q, p vary on the intervals (3.11), they control the fine structure of the scalar product (4.2) in $\mathcal{P}_{r,q,p}$.

In the case of "ground states," where $q = p = 0$, the condition (4.16) becomes standard for the geometric quantization. In the "excited" case where $q \geq 1$ or $p \geq 1$, we observe something like an index contribution to the geometric quantization picture appearing due to an additional holonomy around the conical poles in Ω_0. Because of these "excitations," the leaves Ω_0 with quantized energies are distant from each other by $\frac{1}{l}$ or $\frac{1}{m}$ fractions of the parameter $\hbar$.

To conclude this section, let us discuss what quantum leaves of the algebra (2.4) are. To each element $\mathbf{F}$ of the algebra one can assign the corresponding operator $\mathbf{f}$ in the irreducible representation. This operator acts in the Hilbert space $\mathcal{P}_{r,q,p}$ of anti-holomorphic sections over the phase space. Thus we can compose the function

$$f \overset{\mathrm{def}}{=} \frac{1}{K}\mathbf{f}(K). \tag{4.17}$$

Here K is the reproducing kernel (4.4) and the operator $\mathbf{f}$ acts by $\bar{z}$. The function f (4.17) is called the *Wick symbol* of the operator $\mathbf{f}$, for more details see in [3, 34, 42, 43].

The product of symbols in the sense of (4.6) corresponds to the product of operators. Moreover, one can reconstruct the operator by its symbol using the simple formula

$$\mathbf{f} = f(\overset{2}{\mathbf{z}}{}^*, \overset{1}{\mathbf{z}}),$$

where $\mathbf{z}^*$ is the operator of multiplication by $\overline{z}$ and $\mathbf{z}$ is the conjugate operator.

To generators of the algebra (2.4) we now can assign functions on the phase space:

$$a_j \overset{\text{def}}{=} \frac{1}{K}\mathbf{a}_j(K) \quad (j = 1, 2), \qquad a_\pm \overset{\text{def}}{=} \frac{1}{K}\mathbf{a}_\pm(K). \tag{4.18}$$

We can consider them as quantum analogs of the coordinate functions A_1, A_2, $A_\pm = A_3 \mp iA_4$ on classical symplectic leaves of the Poisson algebra (2.5).

Theorem 4.3 (a) *The quantum coordinate functions obey the Casimir identities*

$$ka_1 + ma_2 = E_{r,q,p},$$
$$a_+ * a_- = (a_1 + \hbar) * \cdots * (a_1 + m\hbar) * a_2 * (a_2 - \hbar) * \cdots * (a_2 - l\hbar + \hbar).$$

Here $*$ is the quantum product (4.6).

(b) *In the classical limit $\hbar \to 0$ (and $r \sim \hbar^{-1} \to \infty$) the quantum coordinate functions coincide with the classical coordinate functions* (2.12) *on the closure* Ω (2.6) *of the symplectic leaves* Ω_0.

Taking into account this theorem, we below identify the quantum phase space $\mathbb{S}^2$ with the closure Ω of the symplectic leaf (2.6), where $E = E_{r,q,p}$. We will call Ω endowed with this structure a *quantum leaf*.

Each element $\mathbf{F}$ of the algebra (2.4) can be represented as a polynomial in generators:

$$\mathbf{F} = F(\mathbf{A}), \qquad \mathbf{A} = (\overset{3}{\mathbf{A}}_+, \overset{2}{\mathbf{A}}_1, \overset{2}{\mathbf{A}}_2, \overset{1}{\mathbf{A}}_-). \tag{4.19}$$

Here F is a function on $\mathbb{R}^4$. The operation of multiplication of elements (4.19) determines a product operation $\odot$ in the algebra of polynomials over $\mathbb{R}^4$:

$$F(\mathbf{A})G(\mathbf{A}) = (F \odot G)(\mathbf{A})$$

(see details in [32]).

Following [32, 33], one can define the *quantum restriction* of the function F onto the leaves Ω:

$$F\Big|_{\hat{\Omega}} \overset{\text{def}}{=} \frac{1}{K}F(\mathbf{a})(K). \tag{4.20}$$

From [33] one knows the following assertion.

Theorem 4.4 (a) *The quantum restriction* (4.20) $F \to F\Big|_{\hat{\Omega}}$ *is a homomorphism of algebras*:

$$(F \odot G)\Big|_{\hat{\Omega}} = F\Big|_{\hat{\Omega}} * G\Big|_{\hat{\Omega}}.$$

The equivalent formula for the quantum restriction is

$$F\Big|_{\hat{\Omega}} = F(a*)1,$$

where $a*$ *are the operators of left multiplication by the quantum coordinate functions* $a = (a_+, a_1, a_2, a_-)$ (4.18) *in the algebra* (4.6).

(b) *The asymptotics as* $\hbar \to 0$ *of the quantum restriction can be derived from*

$$F\Big|_{\hat{\Omega}} = F\big(a - i\hbar\,\mathrm{ad}_-(a) + O(\hbar^2)\big)1 = F(a) + \hbar e_1(F) + O(\hbar^2). \tag{4.21}$$

Here $\mathrm{ad}_-(\cdot)$ *denotes the anti-holomorphic part of the Hamiltonian field:* $\mathrm{ad}_-(\cdot) = ig^{-1}\partial(\cdot)\overline{\partial}$, *where* g *is the quantum metric* (4.9). *The* $\hbar$*-correction* e_1 *in* (4.21) *is the second order operator* $e_1 = \frac{1}{2}\langle R\frac{\partial}{\partial a}, \frac{\partial}{\partial a}\rangle$ *determined by the symmetric tensor* $R_{jl} = \Re(g^{-1}\partial a_j\overline{\partial} a_l)$.

5 Coherent states and gyron spectrum

In the Hilbert space $\mathcal{P}_{r,q,p}$ of anti-holomorphic sections of the Hermitian line bundle with the curvature $i\omega$ over the phase space $\Omega \approx \mathbb{S}^2$ we have the irreducible representation of the resonance algebra (2.4) by differential operators

$$\mathbf{a}_1 = \hbar(rm + p) - \hbar m\overline{z}\overline{\partial}, \qquad \mathbf{a}_2 = \hbar q + \hbar l\overline{z}\overline{\partial}, \tag{5.1}$$
$$\mathbf{a}_+ = \hbar^m \prod_{j=1}^{m}(rm + p + j - m\overline{z}\overline{\partial})\cdot\overline{z}, \qquad \mathbf{a}_- = \frac{\hbar^l}{\overline{z}}\prod_{s=1}^{l}(q - s + 1 + l\overline{z}\overline{\partial}),$$

where $\overline{\partial} = \partial/\partial\overline{z}$.

The unity section $1 = \overline{z}^0$ is the vacuum vector for this representation in the sense that it is the eigenvector of the operators $\mathbf{a}_1, \mathbf{a}_2$ and it is annulled by the operator $\mathbf{a}_-$. Now let us take the vacuum vector $\mathfrak{P}_0$ in the original Hilbert space $\mathcal{L} = L^2(\mathbb{R}^2)$ which corresponds to the representation (2.2):

$$\mathbf{A}_1\mathfrak{P}_0 = \hbar(rm + p)\cdot\mathfrak{P}_0, \qquad \mathbf{A}_2\mathfrak{P}_0 = \hbar q\cdot\mathfrak{P}_0, \qquad \mathbf{A}_-\mathfrak{P}_0 = 0.$$

Definition 5.1 The *coherent states* of the algebra (2.4) is the holomorphic family of vectors $\mathfrak{P}_z \in \mathcal{L}$ defined by

$$\mathfrak{P}_z = \sum_{n=0}^{r}\frac{q!}{(q+ln)!}\left(\frac{z}{\hbar^l}\right)^n \mathbf{A}_+^n\mathfrak{P}_0, \qquad z \in \mathbf{C}.$$

For each $a \in \Omega$ let us denote by $\mathbf{\Pi}_a$ the projection onto the one-dimensional subspace in $\mathcal{L}$ generated by $\mathfrak{P}_{z(a)}$. We call Π_a a *coherent projection*.

Regarding these definitions, may be, it is useful to note the following: if one takes the Hilbert space $\mathcal{P}_{r,q,p}$ instead of $\mathcal{L}$ and the vacuum 1 instead of $\mathfrak{P}_0$, then instead of coherent states $\mathfrak{P}_z$ and the coherent projection $\mathbf{\Pi}_a$ one would see the reproducing kernel $K^{\#}(\cdot|z)$ and the probability function p_a.

In the following theorem we collect the basic properties of the coherent states $\mathfrak{P}_z$. In the general context of quantization theory, see more details in [32, 33, 34].

Theorem 5.1 (a) *The scalar product of two coherent states coincides with the reproducing kernel* (4.4):

$$\|\mathfrak{P}_{z(a)}\|^2 = K(a), \qquad a \in \Omega.$$

(b) *One has the resolution of unity by coherent projections*:

$$\frac{1}{2\pi\hbar}\int_\Omega \mathbf{\Pi}_a \, dm(a) = \mathbf{I}_{r,q,p}.$$

Here $\mathbf{I}_{r,q,p}$ is the projection in $\mathcal{L}$ onto the Hilbert subspace $\mathcal{L}_{r,q,p}$ spanned by all vectors $\mathbf{A}_+^n \mathfrak{P}_0$, $n = 0, \ldots, r$.

(c) *The whole Hilbert space $\mathcal{L}$ is the direct sum of the irreducible subspaces:*

$$\mathcal{L} = \bigoplus_{\substack{r \geq 0 \\ 0 \leq q \leq l-1 \\ 0 \leq p \leq m-1}} \mathcal{L}_{r,q,p}.$$

(d) *The coherent transform $\mathcal{L}_{r,q,p} \xrightarrow{\nu} \mathcal{P}_{r,q,p}$ defined by*

$$\nu(\psi)(\overline{z}) = (\psi, \mathfrak{P}_z), \tag{5.2}$$

has the inverse

$$\nu^{-1}(\varphi) = \frac{1}{2\pi\hbar}\int_\Omega \frac{\mathfrak{P} \otimes \varphi}{K} \, dm. \tag{5.3}$$

The mappings (5.2), (5.3) *intertwine the representations* (2.2) *and* (5.1) *of the algebra* (2.4).

(e) *Let* $\mathbf{F}$ *be an element of the algebra* (2.4) *realized in the Hilbert space $\mathcal{L}$ via the generators* (2.2) *as in* (4.19)*, and let $\mathbf{f} = \nu \circ \mathbf{F} \circ \nu^{-1}$ be the coherent transformation of* $\mathbf{F}$ *realized in the Hilbert space $\mathcal{P}_{r,q,p}$. Then the Wick symbol f* (4.17) *coincides with the Wick symbol of* $\mathbf{F}$ *given by*

$$f(a) = \mathrm{tr}(\mathbf{F}\mathbf{\Pi}_a), \qquad a \in \Omega.$$

The operators $\mathbf{F}, \mathbf{f}$ *are reconstructed via their symbols using the formulas*

$$\mathbf{F} = F(\mathbf{A}) = f\left(\overset{2}{\mathbf{z}}{}^*, \overset{1}{\mathbf{z}}\right), \qquad \mathbf{f} = F(\mathbf{a}) = f\left(\overset{2}{\overline{z}}, \overset{1}{\overline{z}}{}^*\right), \tag{5.4}$$

where $\mathbf{z}$ *is the operator of complex structure* (4.1)*,* $\mathbf{a}$ *are the operators of irreducible representation* (5.1)*. The Wick symbol of the coherent projection $\mathbf{\Pi}_a$ is the probability function p_a* (4.10).

Now following [36], [44]–[48] we explain how to reduce the coherent transform to closed curves (Lagrangian submanifolds) in the phase space.

Let $\Lambda \subset \Omega$ be a smooth closed curve, which obeys the quantization condition

$$\frac{1}{2\pi\hbar}\int_{\Sigma}\Big(\omega - \frac{\hbar}{2}\rho\Big) - \frac{1}{2} \in \mathbb{Z}, \tag{5.5}$$

where $\omega = igd\overline{z}\wedge dz$ is the quantum Kählerian form (4.9), $\rho = i\overline{\partial}\partial \ln g$ is the quantum Ricci form, and Σ is a membrane in Ω with the boundary $\partial\Sigma = \Lambda$.

We choose certain parameterization of the curve expressed via the complex coordinate on the leaf as follows:

$$\Lambda = \{z = z(t) \mid 0 \le t \le T\},$$

and define the following basis of smooth functions on the curve:

$$\phi^{(j)}(t) = \sqrt{\dot{z}(t)}\exp\Big\{-\frac{i}{\hbar}\int_0^t\Big(\overline{\theta} - \frac{\hbar}{2}\overline{\varkappa}\Big)\Big\}\varphi^{(j)}(\overline{z}(t)), \qquad j = 0,\dots,r. \tag{5.6}$$

Here $\theta \stackrel{\text{def}}{=} i\hbar\partial\ln K$ and $\varkappa = i\partial\ln g$ are primitives of the quantum Kählerian form $\omega = d\theta$ and the quantum Ricci form $\rho = d\varkappa$, the integral in (5.6) is taken over a segment of the curve Λ, and the monomials $\varphi^{(j)}$ are defined in (3.15).

Let us denote by $\mathcal{L}_\Lambda$ the vector subspace in $C^\infty(\Lambda)$ spanned by $\phi^{(j)}$ $(j = 0,\dots,r)$ and introduce the Hilbert structure in $\mathcal{L}_\Lambda$ by means of the following norm:

$$\|\phi\|_\Lambda \stackrel{\text{def}}{=} \frac{1}{\sqrt[4]{2\pi\hbar}}\Big(\sum_{j=0}^{r}\big|(\phi,\phi^{(j)})_{L^2}\big|^2\Big)^{1/2}, \tag{5.7}$$

where the scalar product $(\cdot,\cdot)_{L^2}$ is taken in the L^2-space over Λ.

For any smooth function $\phi \in C^\infty(\Lambda)$ we define

$$\mu_\Lambda(\phi) = \frac{1}{\sqrt[4]{2\pi\hbar}}\int_\Lambda \phi(t)\sqrt{\dot{z}(t)}\exp\Big\{\frac{i}{\hbar}\int_0^t\Big(\theta - \frac{\hbar}{2}\varkappa\Big)\Big\}\mathfrak{P}_{z(t)}\,dt, \tag{5.8}$$

where $\mathfrak{P} \in \mathcal{L}$ are coherent states of algebra (2.4) corresponding to its (r,q,p)-irreducible representation.

Theorem 5.2 (a) *The mapping* μ_Λ *defined by* (5.8) *is an isomorphism of Hilbert spaces*

$$\mu_\Lambda : \mathcal{L}_\Lambda \to \mathcal{L}_{r,q,p} \subset \mathcal{L}.$$

(b) *Under the isomorphism* (5.8) *the representation of the algebra* (2.4) *in the Hilbert space* $\mathcal{L}$ *is transformed to the irreducible representation in the Hilbert space* $\mathcal{L}_\Lambda$:

$$\mathbf{F} \to \mathbf{F}_\Lambda \stackrel{\text{def}}{=} \mu_\Lambda^{-1}\circ\mathbf{F}\circ\mu_\Lambda. \tag{5.9}$$

(c) *In the classical limit as* $\hbar \to 0$ *the Hilbert structure* (5.7) *coincides with the* L^2*-structure*:

$$\|\phi\|_\Lambda = \Big(\int_\Lambda |\phi(t)|^2\,dt\Big)^{1/2} + O(\hbar). \tag{5.10}$$

(d) *Let* f *be the Wick symbol* (5.4) *of the operator* $\mathbf{F}$*, then the asymptotics of the operator* (5.9) *as* $\hbar \to 0$ *is given by*

$$\mathbf{F}_\Lambda = \mathcal{F}\Big|_\Lambda - i\hbar\Big(v + \frac{1}{2}\mathrm{div}\,v\Big) + O(\hbar^2). \tag{5.11}$$

Here $\mathcal{F} = f - \frac{\hbar}{4}\Delta f$*, by* Δ *we denote the Laplace operator with respect to the quantum Kählerian metric* g*, and* $v = \mathrm{ad}_+(\mathcal{F})\big|_\Lambda$ *is the restriction to* Λ *of the holomorphic part of the Hamiltonian field* $\mathrm{ad}_+(\mathcal{F}) = -ig^{-1}\overline{\partial}\mathcal{F}\cdot\partial$.

The next terms of the asymptotic expansion (5.11) are also known (see in [36]).

In Theorem 5.2, the curve Λ is arbitrary except it has to obey the quantization condition (5.5).

Let us now choose Λ specifically to be a closed curve on the energy level

$$\Lambda \subset \{\mathcal{F} = \lambda\}, \tag{5.12}$$

and choose the coordinate t to be time on the trajectory Λ of the Hamiltonian field $\mathrm{ad}(\mathcal{F})$. Then $v = \mathrm{ad}(\mathcal{F})\big|_\Lambda = \frac{d}{dt}$, $\mathrm{div}\,v = 0$, and we have

$$\mathbf{F}_\Lambda = \lambda - i\hbar\frac{d}{dt} + O(\hbar^2). \tag{5.13}$$

This formula implies the asymptotics of eigenvalues of the operator $\mathbf{F}_\Lambda$:

$$\lambda + \hbar\frac{2\pi k}{T} + O(\hbar^2), \tag{5.14}$$

where $T = T(\lambda)$ is the period of the trajectory $\Lambda = \Lambda(\lambda)$ (5.12) and λ is determined by the quantization condition (5.5).

Note that the contribution $\frac{2\pi k}{T}$ added to λ in (5.14) can be transformed to adding the number k to the integer number on the right-hand side of condition (5.5). Thus one can omit the summand $\hbar\frac{2\pi k}{T}$ in (5.16) without loss of generality.

Corollary 5.3 *Let* $\mathbf{F}$ *be an operator commuting with the oscillator* $\mathbf{E}$ (2.1)*. Up to* $O(\hbar^2)$*, the asymptotics of its eigenvalues* λ *is determined by the quantization condition*:

$$\frac{1}{2\pi\hbar}\int_\Sigma\Big(\omega - \frac{\hbar}{2}\rho\Big) - \frac{1}{2} \in \mathbb{Z}. \tag{5.15}$$

Here Σ *is a membrane in* Ω *with the boundary* $\Lambda = \partial\Sigma$ (5.12); *the curve* Λ *is the energy level of the function* $\mathcal{F} = f - \frac{\hbar}{4}\Delta f$*, where* f *is the Wick symbol of* $\mathbf{F}$ *and* Δ *is the Laplace operator. The operator* Δ *and the forms* ω*,* ρ *are generated by the quantum Kählerian metric* g (4.9).

Now we can apply the obtained results in studying quantum gyrons. Assume one has a Hamiltonian of the type

$$\mathbf{E} + \varepsilon \mathbf{B}, \tag{5.16}$$

where $\mathbf{E}$ is the oscillator (2.1) and $\mathbf{B}$ is a perturbation presented as a function in operators $\mathbf{b}$, $\mathbf{b}^*$,

$$\mathbf{B} = \sum \beta_{\mu,\nu} \mathbf{b}^{*\nu} \mathbf{b}^{\mu}. \tag{5.17}$$

There is an *operator averaging* procedure [47, 48], which is a unitary transformation reducing (5.16) (up to $O(\varepsilon^N)$) to the Hamiltonian

$$\mathbf{E} + \varepsilon \mathbf{B} \sim \mathbf{E} + \varepsilon \mathbf{F}_N + O(\varepsilon^N), \qquad [\mathbf{F}_N, \mathbf{E}] = 0. \tag{5.18}$$

For instance, if $N = 1$, then

$$\mathbf{F}_1 = \sum_{l\nu_1 + m\nu_2 = l\mu_1 + m\mu_2} \beta_{\mu,\nu} \mathbf{b}^{*\nu} \mathbf{b}^{\mu} \tag{5.19}$$

(see also the Appendix in [25]). For any $N \geq 1$ in (5.18), the operator $\mathbf{F}_N$, commuting with $\mathbf{E}$, is uniquely determined and can be presented in the form (4.19):

$$\mathbf{F}_N = F_N(\mathbf{A}),$$

and after this in the form (5.4):

$$\nu \circ \mathbf{F}_N \circ \nu^{-1} = F_N(\mathbf{a}) = f_N(\overset{2}{\bar{z}}, \overset{1}{\bar{z}^*}). \tag{5.20}$$

Thus the study of the operator (5.16) up to $O(\varepsilon^N)$ is reduced to the study of the properties of the operator (5.20) in each irreducible representation of the algebra (2.4).

The symbols F_N or f_N are *gyron Hamiltonians*. In the (r, q, p)-irreducible representation, the gyron is described by the operator $F_N(\mathbf{a}) = F_N(\overset{3}{\mathbf{a}}_+, \overset{2}{\mathbf{a}}_1, \overset{2}{\mathbf{a}}_2, \overset{1}{\mathbf{a}}_-)$ acting in $\mathcal{P}_{r,q,p}$, where the generators $\mathbf{a}$ are given by (5.1).

In the semiclassical approximation $\hbar \to 0$ the gyron system can be reduced to (5.11) and even to (5.13) over the trajectory Λ of the effective Hamiltonian $\mathcal{F}_N = f_N - \frac{\hbar}{4} \Delta f_N + O(\hbar^2)$ on the leaf $\Omega \approx \mathbb{S}^2$. The asymptotics of the gyron spectrum was described in Corollary 5.3 by means of the membrane versions (5.15) of the Bohr–Sommerfeld quantization condition.

The quantum Kählerian geometry (via the measure dm and the forms ω, ρ) is essentially presented in all these results regarding the gyron spectrum.

The gyron is a model. It is very simple, since it arises from the "textbook" oscillator Hamiltonian. At the same time, it already contains many nontrivial aspects of the quantization theory and, of course, it has a variety of important physical applications. About more complicated models of this type and about further ideas on the quantum geometry we refer to [25, 32, 33], [51]–[63].

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References

1. I. E. Segal, *Quantization of nonlinear systems*, J. Math. Phys., **1** (1960), 468–488.
2. G. W. Mackey, *Mathematical Foundations of Quantum Mechanics*, Benjamin, New York, 1963.
3. J. R. Klauder, *Continuous representation theory*, J. Math. Phys., **4** (1963), 1055–1073.
4. V. P. Maslov, *Perturbation Theory and Asymptotic Methods*, Moscow State Univ., 1965 (in Russian).
5. J.-M. Souriau, *Quantification geometrique*, Comm. Math. Phys., **1** (1966), 374–398.
6. B. Kostant, *Quantization and unitary representations*, Lect. Notes Math., **170** (1970), 87–208.
7. A. Kirillov, *Constructions of unitary irreducible representations of Lie groups*, Vestnik Moskov. Univ. Ser. I Mat. Mekh., **2** (1970), 41–51 (in Russian); English transl. in Moscow Univ. Math. Bull.
8. F. A. Berezin, *Quantization*, Izv. Akad. Nauk SSSR Ser. Mat., **38** (1974) 1116–1175; English transl., Math. USSR-Izv., **8** (1974), 1109–1165.
9. F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Quantum mechanics as a deformation of classical mechanics*, Lett. Math. Phys., **1** (1975/77), 521–530.
10. M. Rieffel, *Deformation quantization for actions of* $\mathbb{R}^d$, Mem. Amer. Math. Soc., **106** (1993), 1–93.
11. H. Omori, Y. Maeda, and A. Yoshioka, *Weyl manifolds and deformation quantization*, Adv. Math., **85** (1991), 224–255.
12. B. Fedosov, *A simple geometrical construction of deformation quantization*, J. Diff. Geom., **40** (1994), 213–238.
13. H. J. Groenewold, *On the principles of elementary quantum mechanics*, Physica, **12** (1946), 405–460.
14. J. E. Moyal, *Quantum mechanics as a statistical theory*, Proc. Cambridge Phil. Soc., **45** (1949), 99–124.
15. V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*. In: *Modern Problems in Math*., Vol. 3, Moscow, VINITI, 1985, 5–303 (in Russian).
16. V. M. Babich and V. S. Buldyrev, *Asymptotic Methods in Problems of Diffraction of Short Waves*, Nauka, Moscow, 1972 (in Russian).
17. V. Guillemin, *Symplectic spinors and partial differential equations*, Colloques Intern. C.V.R.S., N237, Geom Sympl. & Phys. Math., 1975.
18. V. Guillemin and A. Weinstein, *Eigenvalues associated with closed geodesics*, Bull. Amer. Math. Soc., **82** (1976), 92–94.
19. J. V. Ralston, *On the construction of quasimodes associated with stable periodic orbits*, Comm. Math. Phys., **51** (1976), 219–242.
20. V. P. Maslov, *Complex WKB-Method*, Moscow, Nauka, 1976 (in Russian); English transl., Birkhäuser, Basel–Boston, 1994.
21. Y. Colin de Verdiere, *Quasi-modes sur les varietes Riemanniennes*, Invent. Math., **43** (1977), 15–52.
22. M. V. Karasev, *Resonances and quantum method of characteristics*, Intern. Conference “Differential Equations and Related Topics” (Moscow, 16–22 May, 2004), Petrovskii Seminar and Moscow Math. Society, Book of Abstracts, Publ. Moscow Univ., Moscow, 2004, 99–100 (in Russian).
23. M. V. Karasev, *Birkhoff resonances and quantum ray method*, Proc. Intern. Seminar “Days of Diffraction – 2004”, St. Petersburg University and Steklov Math. Institute, St. Petersburg, 2004, 114–126.

24. M. V. Karasev, *Noncommutative algebras, nano-structures, and quantum dynamics generated by resonances*, I. In: *Quantum Algebras and Poisson Geometry in Mathematical Physics* (M. Karasev, ed.), Amer. Math. Soc. Transl. Ser. 2, Vol. 216, Providence, RI, 2005, pp. 1–18. Preprint version in arXiv: math.QA/0412542.
25. M. V. Karasev, *Noncommutative algebras, nano-structures, and quantum dynamics generated by resonances*, II, Adv. Stud. Contemp. Math., **11** (2005), 33–56.
26. M. V. Karasev, *Formulas for noncommutative products of functions in terms of membranes and strings*, Russ. J. Math. Phys., **2** (1994), 445–462.
27. M. V. Karasev, *Geometric coherent states, membranes, and star products*. In: *Quantization, Coherent States, Complex Structures* J.-P. Antoine et al., eds., Plenum, New York, 1995, 185–199.
28. W. Fulton, *Introduction to Toric Varieties*, Ann. of Math. Stud., Princeton Univ., **131** (1993).
29. G. W. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology, **14** (1975), 63–68.
30. V. Poénaru, *Singularités C^∞ en présence de symmétrie*, Lect. Notes Math., **510** (1976).
31. A. S. Egilsson, *Newton polyhedra and Poisson structures from certain linear Hamiltonian circle actions*, Preprint version in arXiv: math.SG/0411398
32. M. V. Karasev, *Advances in quantization: quantum tensors, explicit star-products, and restriction to irreducible leaves*, Diff. Geom. and Its Appl., **9** (1998), 89–134.
33. M. V. Karasev, *Quantum surfaces, special functions, and the tunneling effect*, Lett. Math. Phys., **56** (2001), 229–269.
34. M. Cahen, S. Gutt, and J. Rawnsley, *Quantization of Kähler manifolds*, I, J. Geom. Phys., **7** (1990), 45–62; II, Trans. Amer. Math. Soc., **337** (1993), 73–98; III, Lett. Math. Phys., **30** (1994), 291–305; IV, Lett. Math. Phys., **180** (1996), 99–108.
35. R. Brylinski and B. Kostant, *Nilpotent orbits, normality, and Hamiltonian group actions*, J. Amer. Math. Soc., **7** (1994), 269–298.
36. M. V. Karasev, *Quantization and coherent states over Lagrangian submanifolds*, Russ. J. Math. Phys., **3** (1995), 393–400.
37. M. V. Karasev and E. M. Novikova, *Non-Lie permutation relations, coherent states, and quantum embedding*. In: *Coherent Transform, Quantization, and Poisson Geometry* (M. Karasev, ed.), Amer. Math. Soc. Transl. Ser. 2, Vol. 187, Providence, RI, 1998, pp. 1–202.
38. S. Bergmann, *The kernel functions and conformal mapping*, Math. Surveys Monographs, Vol. 5, Amer. Math. Soc., Providence, RI, 1950.
39. V. Bargmann, *On a Hilbert space of analytic functions and associated integral transform*, Comm. Pure Appl. Math., **14** (1961), 187–214.
40. M. V. Karasev, *Integrals over membranes, transitions amplitudes and quantization*, Russ. J. Math. Phys., **1** (1993), 523–526.
41. S. Chern, *Complex manifolds*, Bull Amer. Math. Soc., **62** (1956), 101–117.
42. F. A. Berezin, *Wick and anti-Wick symbols of operators*, Mat. Sb., **86** (1971), 578–610 (in Russian); English transl. in Math. USSR-Sb., **15** (1971).
43. F. A. Berezin, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR, Ser. Mat., **36** (1972), 1134–1167 (in Russian); English transl., Math. USSR Izv., **8** (1974), 1109–1165.
44. M. V. Karasev, *Connections over Lagrangian submanifolds and certain problems of semiclassical approximation*, Zapiski Nauch. Sem. Leningrad. Otdel. Mat. Inst. (LOMI), **172** (1989), 41–54 (in Russian); English transl., J. Sov. Math., **59** (1992), 1053–1062.
45. M. V. Karasev, *Simple quantization formula*. In: *Symplectic Geometry and Mathematical Physics, Actes du colloque en l'honneur de J.-M.Souriau* (P. Donato et al., eds.), Birkhäuser, Basel–Boston, 1991, 234–243.

46. M. V. Karasev and M. B. Kozlov, *Exact and semiclassical representation over Lagrangian submanifolds in* su(2)*, so(4)*, *and* su(1, 1)*, J. Math. Phys., **34** (1993), 4986–5006.
47. M. V. Karasev and M. V. Kozlov, *Representation of compact semisimple Lie algebras over Lagrangian submanifolds*, Funktsional. Anal. i Prilozhen., **28** (1994), no. 4, 16–27 (in Russian); English transl., Functional Anal. Appl., **28** (1994), 238–246.
48. M. V. Karasev, *Quantization by means of two-dimensional surfaces (membranes): Geometrical formulas for wave-functions*, Contemp. Math., **179** (1994), 83–113.
49. A. Weinstein, *Asymptotics of eigenvalue clusters for the Laplacian plus a potential*, Duke Math. J., **44** (1977), 883–892.
50. M. V. Karasev and V. P. Maslov, *Asymptotic and geometric quantization*, Uspekhi Mat. Nauk, **39** (1984), no. 6, 115–173 (in Russian); English transl. Russian Math. Surveys, **39** (1984), no. 6, 133–205.
51. B. Mielnik, *Geometry of quantum states*, Comm. Math. Phys., **9** (1968), 55–80.
52. A. Weinstein, *Noncommutative geometry and geometric quantization*. In: *Symplectic Geometry and Mathematical Physics, Actes du colloque en l'honneur de J.-M.Souriau* (P. Donato et al., eds.), Birkhäuser, Basel–Boston, 1991, 446–462.
53. A. Weinstein, *Classical theta-functions and quantum tori*, Publ. RIMS, Kyoto Univ., **30** (1994), 327–333.
54. A. Connes, *Noncommutative Geometry*, Academic Press, London, 1994.
55. M. V. Karasev and E. M. Novikova, *Representation of exact and semiclassical eigenfunctions via coherent states. The Hydrogen atom in a magnetic field*, Teoret. Mat. Fiz., **108** (1996), no. 3, 339–387 (in Russian); English transl. in Theoret. Math. Phys., **108** (1996).
56. H. Omori, Y. Maeda, N. Miyazaki, and A. Yoshioka, *Poincare–Cartan class and deformation quantization of Kähler manifolds*, Comm. Math. Phys., **194** (1998), 207–230.
57. D. Sternheimer, *Deformation quantization: Twenty years after*. In: *Particles, Fields, and Gravitation* (J. Rembielinski, ed.), AIP Press, New York, 1998, 107–145.
58. S. Gutt, *Variations on deformation quantization*. In: *Conference Moshe Flato, 1999* (G. Dito and D. Sternheimer, eds.), Vol. 1, Kluwer Acad. Publ., 2000, 217–254.
59. M. Kontsevich, *Deformation quantization of algebraic varieties*, Lett. Math. Phys., **56** (2001), no. 3, 271–294.
60. Y. Manin, *Theta functions, quantum tori, and Heisenberg groups*, Lett. Math. Phys., **56** (2001), no. 3, 295–320.
61. M. V. Karasev, *Quantization and intrinsic dynamics*. In: *Asymptotic Methods for Wave and Quantum Problems* (M. Karasev, ed.), Amer. Math. Soc. Transl. Ser. 2, Vol. 208, Providence, RI, 2003, pp. 1–32.
62. M. V. Karasev, *Intrinsic dynamics of manifolds: quantum paths, holonomy, and trajectory localization*, Russ. J. Math. Phys., **11** (2004), 157–176.
63. M. V. Karasev and E. M. Novikova, *Algebras with polynomial commutation relations for a quantum particle in electric and magnetic fields*. In: *Quantum Algebras and Poisson Geometry in Mathematical Physics* (M. Karasev, ed.), Amer. Math. Soc. Transl. Ser. 2, Vol. 216, Providence, RI, 2005, pp. 19–135.

A Secondary Invariant of Foliated Spaces and Type III$_\lambda$ von Neumann Algebras

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Dedicated to Professor Hideki Omori

Summary. There exist foliated T^2-bundles on close surfaces whose foliation W^*-algebras are isomorphic to the hyperfinite factor of type III$_\lambda$. We introduce a secondary invariant called a K-set for such foliations. The K-set can detect the value λ determined from Connes' S-set and its behavior for such foliations are quite similar to that of the S-set. This suggests that the K-set can be considered as a geometric counterpart of the S-set.

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Introduction

As is well known, the von Neumann algebras are classified into factors of type I, II and III. Factors of type III are further classified into type III$_\lambda$ $(0 \leq \lambda \leq 1)$ according to the value λ determined from the S-set, which is an invariant of von Neumann algebras introduced by A. Connes. There is a unique hyperfinite factor R_λ of type III$_\lambda$. Therefore, it is definitely an interesting problem to study such factors also from a geometric point of view. In fact the hyperfinite factor of type III$_1$ can be constructed from the Anosov foliation on the unit tangent bundle of a close surface. For $0 < \lambda < 1$ there exists a foliated space $(M_\mu, \mathcal{F}_\mu)$ whose foliation W^*-algebra $W^*(M_\mu, \mathcal{F}_\mu)$ is isomorphic to R_λ with $\lambda = \mu^2$. A description of such construction is given in Connes [3]. We shall provide other descriptions of such foliations in Sections 1 and 2, which are suitable in order to define a new secondary invariant of foliations. We then introduce a secondary invariant called the k-class in Section 3. We also define a numerical invariant of foliations called the K-set, which is a priori a subgroup of $\mathbb{R}$. The K-set for $(M_\mu, \mathcal{F}_\mu)$ is computed in Section 3 and it turns out that the isomorphism class of foliation $(M_\mu, \mathcal{F}_\mu)$ can be detected by the K-set. In fact, the K-set for $(M_\mu, \mathcal{F}_\mu)$

is $(\log \lambda)\mathbb{Z} \subset \mathbb{R}$ with $\lambda = \mu^2$. On the other hand, the S-set of R_λ is known to be $\{0,\ \lambda^n \mid n \in \mathbb{Z}\}$. Thus, it seems that the behaviors of these invariants are quite similar to each other for hyperfinite factors of type III_λ. This suggests that the K-set can be considered as a geometric counterpart of the S-set, which plays the central role in the classification of type III_λ factors.

1 Foliations that yield type III_λ factors

Let G denote the special linear group $SL_2(\mathbb{R})$ of rank 2 and $\widehat{G}$ the general linear group $GL_2^+(\mathbb{R})$ of rank 2 with positive determinants. We denote by Γ a cocompact discrete subgroup of G that is a central $\mathbb{Z}/2\mathbb{Z}$-extension of the fundamental group $\pi_1(\Sigma)$ of a closed surface Σ:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \Gamma \to \pi_1(\Sigma) \to 1.$$

Namely, Γ is the inverse image of $\pi_1(\Sigma)$ with respect to the projection map:$G \to PSL_2(\mathbb{R})$. Given $0 < \mu < 1$, we put

$$\widehat{\Gamma} = \Gamma \times \left\{ \begin{bmatrix} \mu^k & 0 \\ 0 & \mu^k \end{bmatrix} \;\middle|\; k \in \mathbb{Z} \right\} \subset \widehat{G},$$

which is a discrete subgroup of $\widehat{G}$. Let M_μ denote the right coset space $\widehat{\Gamma}\backslash\widehat{G}$ and $\widehat{H}$ the following subgroup of $\widehat{G}$:

$$\widehat{H} = \left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \;\middle|\; a > 0,\ b \in \mathbb{R} \right\}.$$

Note that translations by $\widehat{H}$ from the right induces a locally free action on M_μ.

Definition 1.1 A foliated space $(M_\mu, \mathcal{F}_\mu)$ for $0 < \mu < 1$ is defined by the locally free action of $\widehat{H}$.

Note that M_μ is diffeomorphic to $ST\Sigma \times S^1$, where $ST\Sigma$ denotes the sphere bundle of unit tangent vectors in $T\Sigma$; see Proposition 2.2. However, foliations $\mathcal{F}_\mu$ are not isomorphic to each other due to the following theorem; see Connes [3] for instance.

Theorem 1.2 *The foliation W^*-algebra $W^*(M_\mu, \mathcal{F}_\mu)$ associated to $(M_\mu, \mathcal{F}_\mu)$ is isomorphic to the hyperfinite factor of type III_λ with $\lambda = \mu^2$.*

According to the S-set, which was introduced by A. Connes [1], type III factors are further classified into the type III_λ factors where $0 < \lambda < 1$. In the present case the foliation W^*-algebra of $(M_\mu, \mathcal{F}_\mu)$ is isomorphic to the crossed product $L^\infty(M_\mu) \rtimes \widehat{H}$, which is isomorphic to the hyperfinite factor of type III_λ with $\lambda = \mu^2$. As a consequence, it follows that foliations $(M_\mu, \mathcal{F}_\mu)$ are not isomorphic to each other even in a measurable sense.

2 Lifted Anosov foliations and foliated T^2-bundles

In this section we shall consider the foliation $(M_\mu, \mathcal{F}_\mu)$ from a different point of view. We recapture $(M_\mu, \mathcal{F}_\mu)$ as a lifted foliation of the Anosov foliation on $ST\Sigma$ and also as a foliated T^2-bundle on Σ. This point of view will be exploited to define a new secondary invariant on $(M_\mu, \mathcal{F}_\mu)$ in the next section.

Let $\mathbb{R}_+$ denote the multiplicative group of positive real numbers. Then $\widehat{G}$ is isomorphic to $G \times \mathbb{R}_+$ via the isomorphism:

$$\phi : G \times \mathbb{R}_+ \to \widehat{G}, \qquad \phi(g, c) = g \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}.$$

The inverse map is given by

$$\psi : \widehat{G} \to G \times \mathbb{R}_+, \qquad \psi(g) = \left(\begin{bmatrix} 1/d & 0 \\ 0 & 1/d \end{bmatrix} g \, , \; d \right)$$

with $d = \sqrt{\det g}$. Take the following subgroup in G:

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix} \;\middle|\; a > 0, \; b \in \mathbb{R} \right\}.$$

We then define a right action of H on $G \times \mathbb{R}_+$ as

$$(g, c) \cdot h = (gh, ca) \qquad \text{for} \quad h = \begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix} \in H,$$

where $(g, c) \in G \times \mathbb{R}_+$. We identify the right coset $(\Gamma \times \mu^{\mathbb{Z}}) \backslash (G \times \mathbb{R}_+)$ with $(\Gamma \backslash G) \times (\mathbb{R}_+ / \mu^{\mathbb{Z}})$. Then the corresponding H-action is nothing but

$$(g, c) \cdot h = (gh, c\delta(h))$$

for $h \in H$ and $(g, c) \in (\Gamma \backslash G) \times (\mathbb{R}_+ / \mu^{\mathbb{Z}})$, where $\delta : H \to \mathbb{R}_+$ is a homomorphism given by $\delta(h) = a$. The orbits of the diagonal H-action above yields a foliation on $(\Gamma \backslash G) \times (\mathbb{R}_+ / \mu^{\mathbb{Z}})$. This is the description given by Connes [3, p. 58].

Proposition 2.1 *The isomorphism ϕ induces an equivariant diffeomorphism*

$$\varphi_1 : (\Gamma \backslash G) \times (\mathbb{R}_+ / \mu^{\mathbb{Z}}) \to M_\mu$$

with respect to the isomorphism $\rho : H \to \widehat{H}$ given by

$$\rho(h) = \begin{bmatrix} a^2 & ab \\ 0 & 1 \end{bmatrix} \qquad \textit{for} \quad h = \begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix} \in H.$$

Therefore, $(M_\mu, \mathcal{F}_\mu)$ is isomorphic to the foliation on $(\Gamma \backslash G) \times (\mathbb{R}_+ / \mu^{\mathbb{Z}})$ given by the diagonal H-action above.

Proof. It is obvious that ϕ induces a diffeomorphism φ_1. Since we have

$$\phi((g,c)\cdot h) = \phi(gh, ca) = ghac = g\begin{bmatrix} a & b \\ 0 & 1/a \end{bmatrix} ac = g\begin{bmatrix} a^2 & ab \\ 0 & 1 \end{bmatrix} c = \phi(g,c)\rho(h),$$

it follows that φ_1 is equivariant. Hence it yields an isomorphism between those foliations. □

Due to the proposition above we obtain another description of $(M_\mu, \mathcal{F}_\mu)$. Recall that $\Gamma\backslash G$ is diffeomorphic to $ST\Sigma$. It is known that the orbits in $\Gamma\backslash G$ of right translations by H corresponds to the Anosov foliation on $ST\Sigma$. Thus $(M_\mu, \mathcal{F}_\mu)$, which is isomorphic to the foliation given by the diagonal H-action on $(\Gamma\backslash G) \times (\mathbb{R}_+/\mu^{\mathbb{Z}})$, can be considered as a lifted Anosov foliation on $ST\Sigma \times S^1$.

Next we shall describe $(M_\mu, \mathcal{F}_\mu)$ as a foliated bundle. Put

$$J = \left\{ y\frac{\partial}{\partial x} \in TS^1 \;\middle|\; y > 0 \right\},$$

where $\partial/\partial x$ is the standard tangent vector on S^1. Let T_g denote the fractional linear transformation

$$T_g(z) = \frac{az+b}{cz+d} \qquad \text{for} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G,$$

which induces the actions on the upper half plane $\mathbb{H}$ as well as on S^1. Here S^1 is identified with the one-point compactification of $\mathbb{R}$. We then define an action of $\Gamma \times \mu^{\mathbb{Z}}$ on $\mathbb{H} \times J$ such that

$$\gamma \cdot (z, v) = \left(T_g(z), \lambda^k dT_g(v) \right)$$

for $\gamma = (g, \mu^k) \in \Gamma \times \mu^{\mathbb{Z}}$ and $(z, v) \in \mathbb{H} \times J$, where $\lambda = \mu^2$. Here dT_g denotes the differential of T_g. We also define an action of $\Gamma \times \mu^{\mathbb{Z}}$ on $G \times \mathbb{R}_+$ such that

$$\gamma \cdot (h, c) = \left(gh, \mu^k c \right)$$

for $(h, c) \in G \times \mathbb{R}_+$. Let $\varphi_2 : G \times \mathbb{R}_+ \to \mathbb{H} \times J$ be the map defined by

$$\varphi_2(g, c) = \left(T_g(i), c^2 dT_g(v) \right),$$

where $v = \partial/\partial x \in T_p S^1$ and $p \in S^1$ is the point corresponding to ∞. Then we have:

Proposition 2.2 *The map φ_2 defined above is an equivariant diffeomorphism with respect to the $(\Gamma \times \mu^{\mathbb{Z}})$-actions. Furthermore, it induces a diffeomorphism on each H-orbit in $G \times \mathbb{R}_+$ onto a slice $\mathbb{H} \times \{*\}$ in $\mathbb{H} \times J$.*

Proof. It is easy to verify that φ_2 is a diffeomorphism. We then have

$$\varphi_2(\gamma \cdot (h, c)) = \varphi_2 \left(gh, \mu^k c \right)$$

$$
\begin{aligned}
&= \left(T_{gh}(i), \lambda^k c^2 dT_{gh}(v)\right) \\
&= \gamma \cdot \left(T_h(i), c^2 dT_h(v)\right) \\
&= \gamma \cdot \varphi_2(g, c)
\end{aligned}
$$

since $T_{gh} = T_g T_h$. This proves that φ_2 is equivariant. Furthermore, we have

$$
\begin{aligned}
\varphi_2((g, c) \cdot k) &= \varphi_2(gk, ca) \\
&= \left(T_{gk}(i), c^2 a^2 dT_{gk}(v)\right) \\
&= \left(T_g T_k(i), c^2 dT_g(v)\right)
\end{aligned}
$$

for $k = \left[\begin{smallmatrix} a & b \\ 0 & 1/a \end{smallmatrix}\right] \in H$ since $dT_k(v) = a^{-2} v$, which proves the second claim. □

Let μ be the generator of $\mu^{\mathbb{Z}}$ and act on J as previously:

$$\mu \cdot (z, v) = (z, \lambda v)\,.$$

Thus the orbit space is $J/\lambda^{\mathbb{Z}}$, which is diffeomorphic to the 2-dimensional torus T^2. We then define the diagonal action of Γ on $\mathbb{H} \times J/\lambda^{\mathbb{Z}}$ by $\gamma \cdot (z, w) = \left(T_\gamma(z), dT_\gamma(w)\right)$ for $(z, w) \in \mathbb{H} \times J/\lambda^{\mathbb{Z}}$. The orbit space turns out to be a foliated bundle

$$\mathbb{H} \underset{\Gamma}{\times} J/\lambda^{\mathbb{Z}} \to \Sigma$$

with the typical fiber T^2, whose leaves are images of the slices $\mathbb{H} \times \{*\}$. Then Proposition 2.2 claims that there exists an isomorphism from the foliated bundle to the foliation on $(\Gamma\backslash G) \times (\mathbb{R}_+/\mu^{\mathbb{Z}})$ and hence to $(M_\mu, \mathcal{F}_\mu)$ as foliated spaces.

Remark 2.3 Taking J instead of $J/\lambda^{\mathbb{Z}}$ we can construct a foliated bundle

$$\mathbb{H} \underset{\Gamma}{\times} J \to \Sigma$$

with leaves which are the images of $\mathbb{H} \times \{*\}$, $(* \in J)$. As is mentioned previously, the resulting foliated bundle can be considered as a lift of the Anosov foliation $(ST\Sigma, \mathcal{F})$. The foliation W^*-algebra $W^*(\mathbb{H} \underset{\Gamma}{\times} J, \mathcal{F})$ is then isomorphic to the crossed product of $W^*(ST\Sigma, \mathcal{F}) \rtimes_\sigma \mathbb{R}$ with respect to the modular automorphisms $\{\sigma_t\}$ on $W^*(ST\Sigma, \mathcal{F})$, and the dual action $\widehat{\sigma_t}$ corresponds to the action induced from the translations by $\mathbb{R}_+$ on J; see Moriyoshi [5] for instance.

3 A secondary invariant associated to $(M_\mu, \mathcal{F}_\mu)$

Let $(M, \mathcal{F})$ be a C^∞-foliation of codimension q. Take a covering space $\widetilde{M} \to M$ with Π the deck transformation group, and denote by $\widetilde{\mathcal{F}}$ the induced foliation on $\widetilde{M}$ from $\mathcal{F}$. Suppose that there exists a differential form ω on $\widetilde{M}$ with $\deg \omega = q$ such that:

T) ω is a transverse invariant volume form for $\widetilde{\mathcal{F}}$, namely, it satisfies that

$$\iota_X \omega = 0, \qquad \mathcal{L}_X \omega = 0$$

for any vector field X along the foliation $\widetilde{\mathcal{F}}$ on $\widetilde{M}$ and that ω is nowhere vanishing on $\widetilde{M}$. Here $\mathcal{L}_X$ denotes the Lie derivative with respect to X.

P) ω is projectively invariant with respect to the Π-action, namely, there exists $c_g \in \mathbb{R}_+$ for each $g \in \Pi$ such that $g^*\omega = c_g \omega$.

Remark 3.1 In general, a differential form ω on M is called $\mathcal{F}$-basic if it satisfies that

$$\iota_X \omega = 0, \qquad \mathcal{L}_X \omega = 0$$

for any vector field X along $\mathcal{F}$. Then the condition T) is equivalent to saying that ω is a nowhere vanishing $\widetilde{\mathcal{F}}$-basic q-form on $\widetilde{M}$.

Observe that there exists a group homomorphism:

$$\rho_\omega : \Pi \to \mathbb{R}_+, \qquad \rho_\omega(g) = c_g$$

due to the condition P). It is then verified that $\log \rho_\omega : \Pi \to \mathbb{R}$ is a group 1-cocycle:

$$\delta \log \rho_\omega(g, h) = \log \rho_\omega(h) - \log \rho_\omega(gh) + \log \rho_\omega(g) = 0.$$

Put

$$C^{p,q} = C^p(\Pi, \Omega^q \widetilde{M}),$$

where $C^{p,q}$ is the set of cochains on the group Π with values in the differential form $\Omega^q \widetilde{M}$ of degree q. We then introduce the following double cochain complex $\{C^{p,q}, d, \delta\}$.

$$\begin{array}{ccccccc}
\uparrow & & & & & & \\
C^0(\Gamma, \Omega^3 \widetilde{M}) & \longrightarrow & & & & & \\
\uparrow & & \uparrow & & & & \\
C^0(\Gamma, \Omega^2 \widetilde{M}) & \longrightarrow & C^1(\Gamma, \Omega^2 \widetilde{M}) & \longrightarrow & & & \\
\uparrow & & \uparrow & & \uparrow & & \\
C^0(\Gamma, \Omega^1 \widetilde{M}) & \longrightarrow & C^1(\Gamma, \Omega^1 \widetilde{M}) & \longrightarrow & C^2(\Gamma, \Omega^1 \widetilde{M}) & & \\
\uparrow d & & \uparrow & & \uparrow & & \\
C^0(\Gamma, \Omega^0 \widetilde{M}) & \xrightarrow{\delta} & C^1\Gamma, \Omega^0(\widetilde{M}) & \longrightarrow & C^2(\Gamma, \Omega^0 \widetilde{M}) & \longrightarrow &
\end{array}$$

Here we denote by δ the coboundary map for chains of Π and by d the exterior differentiation on $\Omega^q \widetilde{M}$. Observe that the double complex is acyclic with respect to δ since the Π-action on $\widetilde{M}$ is proper. It then follows that the cohomology group of the total complex $\{C^{\bullet,\bullet}, d+\delta\}$ is isomorphic to the cohomology group of Π-invariant differential forms on $\widetilde{M}$, which is isomorphic to the de Rham cohomology group $H^*_{dR}(M)$:

$$H^*(C^{\bullet,\bullet}) \cong H^*_{dR}(M).$$

Recall that $\log \rho_\omega$ is closed with respect to δ. It is obvious that

$$d \log \rho_\omega = 0$$

since the value of $\log \rho_\omega$ is a constant. Thus it yields a 1-cocycle in the total complex $C^{\bullet,\bullet}$ and hence a cohomology class in $H^*(C^{\bullet,\bullet})$.

Definition 3.2 The k-class associated to ω is the cohomology class

$$k_\omega = [\log \rho_\omega] \in H^1_{dR}(M) \cong H^1(C^{\bullet,\bullet}).$$

Let ω and ω' be differential forms on $\widetilde{M}$ of degree q satisfying the condition T) and P) above. This implies that there exists a smooth function f on $\widetilde{M}$ with values in $\mathbb{R}_+$ such that $\omega' = f\omega$. It then follows that

$$\begin{aligned} \log \rho_{\omega'}(g) - \log \rho_\omega(g) &= \log(g^*(f\omega)/(f\omega)) - \log(g^*\omega/\omega) \\ &= \log g^* f - \log f \\ &= \delta \log f(g), \end{aligned}$$

namely,

$$\log \rho_{\omega'} - \log \rho_\omega = \delta \log f.$$

It also follows that

$$\delta d \log f = d\delta \log f = d(\log \rho_{\omega'} - \log \rho_\omega) = 0$$

and hence $d \log f$ is a closed 1-form on $\widetilde{M}$ that is Π-invariant, which can be considered as a closed 1-form on M.

$$\begin{array}{ccc} d\log f & \xrightarrow{\ \delta\ } & 0 \\ {\scriptstyle d}\uparrow & & \uparrow{\scriptstyle d} \\ \log f & \xrightarrow[\ \delta\ ]{} & \log \rho_{\omega'} - \log \rho_\omega \end{array}$$

Given a vector field X along the foliation $\widetilde{\mathcal{F}}$, we obtain

$$0 = \mathcal{L}_X \omega' = (Xf)\omega + f\mathcal{L}_X\omega = (Xf)\omega$$

since $\mathcal{L}_X\omega' = \mathcal{L}_X\omega = 0$. Hence we have $Xf = 0$ since ω is a transverse volume form. It then follows that

$$\iota_X d \log f = X(\log f) = 0$$

$$\mathcal{L}_X d\log f = d\iota_X d\log f = 0,$$

which implies that $d\log f$ is a $\mathcal{F}$-basic closed 1-form on M.

Now we have

$$k_{\omega'} - k_\omega = [\log \rho_{\omega'}] - [\log \rho_\omega] = [\delta \log f] = [-d\log f] \tag{1}$$

in the total complex $C^{\bullet,\bullet}$. Observe that the space $\Omega_b^*(M,\mathcal{F})$ of $\mathcal{F}$-basic forms on M is a subcomplex of $\Omega^*(M)$ with respect to d. Hence we can take the cohomology group, which is called a basic cohomology group of foliation $(M,\mathcal{F})$ and denote it by $H_b^*(M,\mathcal{F})$. We also denote by $\overline{H}_b^*(M,\mathcal{F})$ the image of $H_b^*(M,\mathcal{F})$ in $H_{dR}^*(M)$ via the natural inclusion from $\Omega_b^*(M,\mathcal{F})$ into $\Omega^*(M)$. The identity (1) then proves that the k-class is independent of the choice of ω modulo $\overline{H}_b^*(M,\mathcal{F})$.

It may appear that it depends on the choice of the preferred covering space $\widetilde{M}$. Given such a covering space $\widetilde{M}$, we can take a surjective covering projection to $\widetilde{M}$ from the universal covering space $\widetilde{M}_o$. We then consider the pullback of ω to $\widetilde{M}_o$ instead of ω. Applying the same argument as above, we can conclude that the resulting class is also independent of the choice of $\widetilde{M}$. We have thus proved the following:

Theorem 3.3 *Let $(M,\mathcal{F})$ be a C^∞-foliation satisfying the conditions* T) *and* P) *above. The k-class associated to ω is independent of the choice of ω and $\widetilde{M}$ when it is considered as the following element:*

$$k = [\log \rho_\omega] \in H_{dR}^1(M)/\overline{H}_b^1(M,\mathcal{F}).$$

Put

$$H = \bigcap_\alpha \ker[\alpha : H_1(M;\mathbb{Z}) \to \mathbb{R}] \subset H_1(M;\mathbb{Z})$$

where α is arbitrary $\mathcal{F}$-basic closed 1-form on M. We then obtain a numerical invariant of $(M,\mathcal{F})$.

Definition 3.4 The K-set of C^∞-foliation $(M,\mathcal{F})$ is defined by

$$K(M,\mathcal{F}) = \mathrm{im}\,[k : H \to \mathbb{R}] \subset \mathbb{R}.$$

If there does not exist ω satisfying the conditions, we set $K(M,\mathcal{F}) = \emptyset$.

Obviously the K-set is a subgroup of $\mathbb{R}$ and an invariant of C^∞-foliation $(M,\mathcal{F})$ satisfying the conditions T) and P). Recall that k is defined as a de Rham cohomology class modulo the image of the basic cohomology group of $(M,\mathcal{F})$; however, we note that k yields a dual map without ambiguity once it is restricted to H.

In general it is not easy to calculate the basic cohomology group of $(M,\mathcal{F})$. However, the situation is very simple in our case.

Example 3.5 Let $(M,\mathcal{F})$ be the foliated bundle $\mathbb{H} \times_\Gamma J/\lambda^{\mathbb{Z}} \to \Sigma$ described in Section 2. First we shall prove that the foliation $\mathcal{F}$ satisfy the conditions T) and P).

There is a natural identification between the jet bundle J and the product space $S^1 \times \mathbb{R}_+$:

$$y\partial/\partial x \in J \subset T_x S^1 \longleftrightarrow (x, y) \in S^1 \times \mathbb{R}_+.$$

Let γ denote an orientation-preserving diffeomorphism of S^1 which acts on J. The induced action on J is given by

$$\gamma \cdot (x, y) = (\gamma(x), \gamma'(x)y).$$

Here $\gamma'(x)$ denotes the derivative of γ at $x \in S^1$, which is a positive real number. The action is a multiplication. Let ω be a volume form on J given by

$$\omega = \frac{dx \wedge dy}{y^2}.$$

It yields

$$\gamma^*\omega = \frac{d\gamma(x) \wedge d(\gamma'(x)y)}{\gamma'(x)^2 y^2} = \frac{dx \wedge dy}{y^2}$$

and follows that ω is invariant with the induced action of γ on J. Let $\mathbb{H} \underset{\Gamma}{\times} J$ denote the orbit space with respect to the diagonal action of Γ. We then take $\mathbb{H} \underset{\Gamma}{\times} J$ as $\widetilde{M}$, which is a covering space of M with the deck transformation group $\mathbb{Z}$. The translation $\lambda \cdot (x, y) = (x, \lambda y)$ on J gives rise to a generator of $\mathbb{Z}$. Then ω yields a transverse invariant volume form $(\widetilde{M}, \widetilde{\mathcal{F}})$. Since

$$\lambda^*\omega = \frac{dx \wedge \lambda dy}{\lambda^2 y^2} = \lambda^{-1}\omega,$$

it follows that

$$\log \rho_\omega(n) = -n \log \lambda$$

for $n \in \mathbb{Z}$. Let $d \log y$ be the Haar measure on $\mathbb{R}_+$. Note that it yields a closed form on M. Set

$$f : \mathbb{H} \underset{\Gamma}{\times} J \to \mathbb{R}, \qquad f(z, x, y) = -\log y.$$

We then obtain

$$\delta f(n) = n^* f - f = -\log \lambda^n y + \log y = -n \log \lambda$$

and hence

$$[\log \rho_\omega] = [\log \rho_\omega - (d + \delta) f] = [d \log y].$$

Thus the k-class is represented by $d \log y$ on M.

With basic differential forms on $(M, \mathcal{F})$, we can prove that there do not exist such forms on M except for the trivial ones. Thus the quotient group $H_{dR}(M)/\overline{H}_b(M, \mathcal{F})$ is isomorphic to the de Rham cohomology group and it follows that $H = H_1(M; \mathbb{Z})$. Note that M is diffeomorphic to $ST\Sigma \times \mathbb{R}_+/\lambda^{\mathbb{Z}}$. Obviously the evaluation of the k-class with homology classes in $H_1(ST\Sigma; \mathbb{Z})$ is trivial. It then yields

$$\langle k,\ [\mathbb{R}_+/\lambda^{\mathbb{Z}}]\rangle = \int_1^\lambda d\log y = \log\lambda$$

and hence the K-set is given by

$$K(M, \mathcal{F}) = (\log\lambda)\mathbb{Z}.$$

On the other hand the foliation W^*-algebra $W^*(M_\mu, \mathcal{F}_\mu)$ is isomorphic to the hyperfinite factor R_λ of type III_λ and the S-set of R_λ is equal to $\{0,\ \lambda^n \mid n \in \mathbb{Z}\}$; see Connes [3] for instance. It seems that these invariants have similar behavior at least for R_λ. However, the author does not know whether there is a direct relationship between the S-set and the K-set at this point.

References

1. A. Connes, *Une classification des facteurs de type* III, Ann. Sci. École Norm. Sup., **6** (1973), 133–252.
2. ———, *Non-commutative differential geometry* I , II, Publ. Math. IHES, **62** (1986), 257–360.
3. ———, *Noncommutative Geometry*, Academic Press, 1994.
4. A. Connes and H. Moscovici, *The L^2-index theorem for homogeneous spaces of Lie groups*, Ann. of Math. **115** (1982), 291–330.
5. H. Moriyoshi, *Operator algebras and the index theorem on foliated manifolds*, Proceedings of Foliations: Geometry and Dynamics, World Scientific (2002), 127–155.

The Geometry of Space-Time and Its Deformations from a Physical Perspective

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Summary. We start with an epistemological introduction on the evolution of the concepts of space and time and more generally of physical concepts in the context of the relation between mathematics and physics from the point of view of deformation theory. The concepts of relativity, including anti de Sitter space-time, and of quantization, are important paradigms; we briefly present these and some consequences. The importance of symmetries and of space-time in fundamental physical theories is stressed. The last section deals with "composite elementary particles" in anti de Sitter space-time and ends with speculative ideas around possible quantized anti de Sitter structures in some parts of the universe.

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1 Epistemological introduction

Our representation of the universe evolved with time, based on experimental data and the interpretation we gave them. That is particularly true of the concepts of space and time, around which this text is centered—even if the definition of the word *time* may not be quite the same in both instances.

More generally, mathematics arose as an abstraction of our representations of the physical universe. The language it developed was in turn seminal for a better formulation of that representation, but a Babel tower effect can be perceived almost from the start. Indeed, as Sir Michael Atiyah said (after Oscar Wilde in 1887 about UK and US) in his closing lecture of the 2000 International Congress in Mathematical Physics (with examples taken from algebraic geometry), "Mathematics and Physics are two communities separated by a common language." In the best cases, physicists speak the mathematical language with a distinctive accent that mathematicians may have a hard time to understand while, as said by Goethe, *mathematicians are like Frenchmen: they translate everything into their own language, and henceforth it is something entirely different.* Being originally a French mathematician, I shall do my best to bring a counterexample to that affirmation.

Mathematics proceeds by logical deduction: *If A, and A implies B, then B.* In other words, A is a sufficient condition for B to hold. As simple as that sentence may seem, it is often distorted in ordinary life where (for external reasons) one is tempted to take for necessary a sufficient condition. Schematically that can be expressed as follows: *Given that A implies B, if I find B nice (thus want A because it will give me B), then A.* The subtle logical mistake is perpetrated by almost all in experimental sciences when building models.

The need for modelling is as old as Science: more and more data are being collected and it is natural to try and put some order there. So from experimental data E one imagines a model M that can explain them. Eventually (with deeper intuition) it may be possible to show that the model M is a consequence of more fundamental principles, a theory T. That is the implicit part, taken for granted by experimental scientists (the part A implies B above).

Now if new data $E_1 \supsetneq E$ are found that can also be derived from T, i.e., B becomes nice, the model or theory receives experimental confirmation (then A). One does not argue with success. The confusion between necessary and sufficient conditions may go as far as saying that abstract entities involved in T or M were "directly observed" with the new data: in fact, what has been observed is only a consequence of these entities in some model. The confusion is enhanced by the fact that nowadays our interpretation of the raw experimental data is often made within existing models or theories, so that what we call an experimental result may, in fact, be theory-dependent.

But it often happens that with a larger data set E', the new data will not be easily cast in the existing model. [Cf. a quote attributed [FerW] to Fermi: "There are two possible outcomes: if the result confirms the hypothesis, you've made a measurement. If the result is contrary to the hypothesis, you've made a discovery."] Then there will be a need to develop a new model M', if possible deriving from a new theory T', that can explain everything observed *so far* (one should not hope for a definitive theory of everything). Occasionally a far-sighted scientist may (triggered by some intuition or logical deduction) imagine the new theory even when there are not yet experimental data that make it necessary. That can be a dangerous attitude but it may prove prophetic. A scientist should therefore, even (especially) when everything seems for the best in the best of possible worlds and many are sure that we can now explain everything, be always prepared for surprises and have, in the back of his mind, a tune playing *it ain't necessarily so* in relation with the best accepted theories. That is even more true when trying to block some avenues with "no go" theorems, overlooking the hypotheses (sometimes hidden) on which they rely or the lack of rigor in their proofs (see an example in (2.2.2) below).

Towards the end of the XIXth century, with classical Newtonian mechanics and electromagnetism, many thought we had achieved our understanding of Nature—at least of Physics. What happened soon afterwards, in particular with quantum mechanics and relativity, shows that deformation theory, developed in an appropriate context, can lead us to "deformed" models and theories that fit better newly discovered (or yet undiscovered) data.

Physical theories have their domain of applicability defined by the relevant distances, velocities, energies, etc. involved. But the passage from one domain (of distances, etc.) to another does not happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict accepted theories. Eventually a new fundamental constant enters and the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism. The question is therefore, in which category do we seek for deformations? Usually physics is rather conservative and if we start e.g., with the category of associative or Lie algebras, we tend to deform in the same category. But there are important examples of generalization of this principle: e.g., quantum groups are deformations of (some commutative) Hopf algebras.

That is the basis for Flato's *deformation philosophy* [Fl82]. The main mathematical language for it was developed in 1964 by Gerstenhaber [Ge64] with his theory of deformation of algebras, though its origin can be traced back to Riemann's surface theory in the XIXth century, generalized in 1957 in the short paper by Froelicher and Nijenhuis [FN57] identifying the infinitesimal deformations that led (the fact is acknowledged in [KS58]) to the monumental works of Kodaira and Spencer [KS58] on deformations of complex analytic structures.

Since the 1970s we have been developing that philosophy in three interrelated directions. The first one, our main concern here, deals with deformations of the underlying space-time geometry. Then, at some point, one has to deal with quantum phenomena: In a nutshell the idea is to deform algebras (a linear structure) of physical observables from commutative to noncommutative, what we called *deformation quantization*. Incidentally the strategy of noncommutative geometry [Co94] proceeds in a similar fashion: the idea is to formulate usual (commutative) geometry in a somewhat unusual way using algebras and related concepts, so as to be able to "plug in" noncommutativity in a natural way. But one cannot do physics (which requires measurements) without interactions, and their mathematical expression calls for nonlinearities: The idea is then to deform (in a generalized sense) mathematical structures such as linear representations of symmetries into nonlinear ones. That is the third aspect of our trilogy [S05C]. It has to be tackled also (if not mainly) at the quantized level; that brings in formidable mathematical questions (even more so in hyperbolic theories, on noncompact space-times) posed by the need of renormalization—extracting finite results from quantities that are, at mathematical face value, infinite. Here again, subtle avatars of the deformation philosophy are proving seminal, see e.g., [Co06].

2 From Atlas to Galileo and Newton to Einstein and Planck

2.1 Deformations

The discovery of the non-flat nature of Earth may be the first example of the appearance of deformation theory in our representation of the physical space. Interestingly at first, in contradistinction with the commonly accepted idea until that time (ca. 550 BC), Pythagoras emitted the theory that all celestial bodies (including the earth) are spherical. He did that apparently on aesthetic grounds (nowadays we would say that this was a theoretical prediction). Two centuries later, Aristotle provided (indirect) physical evidence for a spherical Earth. Finally, around 240 BC, Eratosthenes proved experimentally that our Earth is not a plate, carried by a giant called Atlas in Greek mythology, but is spherical; he even used mathematics (geometry) to interpret the data and evaluate its circumference to be 252,000 stades, very close to our present knowledge. So we have a theoretician who comes up with a revolutionary idea, later indirectly proved by a phenomenologist and finally confirmed by direct observation. (The case of parity violation in particle physics is not very different, except for the time scale between events!)

Closer to us, the paradox coming from the Michelson and Morley experiment (1887) was resolved in 1905 by Einstein with the special theory of relativity. In modern language one can express that by saying that the Galilean geometrical symmetry group of Newtonian mechanics ($SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$) is deformed to the Poincaré group ($SO(3, 1) \cdot \mathbb{R}^4$) of special relativity, the new fundamental constant being c^{-1} where c is the velocity of light in vacuum. Time has to be treated on the same footing as space, expressed mathematically as a purely imaginary dimension. Here, experimental need triggered the theory.

All this is by now well known and a century old, so we shall not develop it any further. But, interestingly, only after the work of Gerstenhaber [Ge64] was it realized that the passage from nonrelativistic to relativistic physics can be interpreted as a deformation in that precise mathematical sense, even if a kind of inverse (a "contraction" [IW53, WW00] $c^{-1} \to 0$) has been intuitively understood for many years. The fact triggered strong interest for deformation theory in France among a number of theoretical physicists, including Flato who had just arrived from the Racah school and knew well the effectiveness of symmetry in physical problems. He was soon to realize that, however important symmetry is as a notion and a tool in a mathematical treatment of physical problems, it is not the only one and should be complemented with other (often related) concepts. The notion of deformation can be applied to a variety of categories that are used to express mathematically the physical reality.

For completeness, let us give a concise form of the definition of deformations of algebras, in the sense of Gerstenhaber [Ge64, BGGS] (more general forms exist, see e.g., [Na98]):

Definition 1 A deformation of an algebra A over a field $\mathbb{K}$ is an algebra $\tilde{A}$ (flat) over $\mathbb{K}[[\nu]]$ such that $\tilde{A}/\nu\tilde{A} \approx A$. Two deformations $\tilde{A}$ and $\tilde{A}'$ are said to be equivalent if they are isomorphic over $\mathbb{K}[[\nu]]$ and $\tilde{A}$ is said to be trivial if it is isomorphic to the original algebra A considered by base field extension as a $\mathbb{K}[[\nu]]$-algebra.

2.2 Some facts around symmetries and elementary particle physics

2.2.1 Symmetries and generations

At the same time (if I may write so), i.e., in the mid-1960s, particle physicists were interested in "internal" symmetries of elementary particles. These were introduced empirically in an attempt to put some order in the increasing number of "elementary" particles that were discovered in accelerators, a number that was getting so large already in the early 1950s that one day Enrico Fermi is said [FerW] to have told his student (and future Nobel Laureate) Leon Lederman: "Young man, if I could remember the names of all these particles, I would have been a botanist!"

Symmetries (groups and their representations) have proved seminal in a variety of physical problems, especially since the advent of quantum mechanics [We28]. In this domain the feedback from physics into mathematics, and vice-versa, has been essential (cf. e.g., [Wi39] and the monumental works of Harish Chandra, originally a physicist). In molecular and atomic spectroscopy the forces are well understood and the symmetries dictated by the physical problems studied (e.g., $SU(2)$, the spin group of 3-space, and finite subgroups associated with crystals, studied by the Racah school [Fl65]). The idea was to use similar methods in what can be called particle spectroscopy and regroup them in "supermultiplets" based on finite dimensional unitary irreducible representations (UIR) of compact Lie groups—hopefully as a first step towards a more dynamical theory.

At first, because of the isospin I, a quantum number separating proton and neutron introduced (in 1932, after the discovery of the neutron) by Heisenberg, $SU(2)$ was tried; then in 1947 a second generation of "strange" particles started to appear and in 1952 Pais suggested a new quantum number, the strangeness S. (See a nice account of the situation in the early 1960s in [Sa64]; in 1975 a third generation was discovered, associated e.g., with the τ lepton, and its neutrino ν_τ first observed in 2000.) In the context of what was known in the 1960s, a rank 2 group was the obvious thing to try and introduce in order to describe these "internal" properties; that is how in particle physics theory appeared the simplest simple group of rank 2, $SU(3)$, which subsists until now in various forms.

2.2.2 Space-time and internal symmetries

A natural question was then to study the relation (if any) of this internal world with space-time (relativity). That was, and still is a hard question. (E.g., combining the present Standard Model of elementary particles with gravitation is until now some quest for a Holy Grail.) Negating any connection, at least at the symmetry level, was a comfortable way out—especially when one has to convince politicians to fund the expensive apparatus needed in high energy physics (which turned out to have very positive if unexpected by-products, from medical physics to the World Wide Web).

For many, the proof of a trivial relation was achieved by what is often called the O'Raifeartaigh Theorem [OR65], a "no go theorem" stating that any finite-dimensional Lie algebra containing the Poincaré Lie algebra and an "internal" Lie algebra must contain these two as a direct product. The proof was based on the nilpotency of the Poincaré energy-momentum generators but implicitly assumed the existence of a common invariant domain of differentiable vectors, something which Wigner was careful to state as an assumption in his seminal paper [Wi39] and was proved later for Banach Lie group representations by (in his own words) "a Swedish gentleman" [Gå47]. Indeed one has to be careful with no go theorems. Shortly after [OR65] a couple of trouble-makers showed in a provocative Letter [FS65] that the result was not proved in the generality it was stated, then exhibited a number of counterexamples [FS66, FS69]. In the latter paper we also mention that another, more sophisticated, attempt to prove a direct product relation [CM67] contained an implicit hypothesis, hidden in the notation, that basically presupposed the result claimed to be proved.

We know at present that the situation is much more complex, especially when dynamics has to be introduced in the theory. Nevertheless one cannot and should not rule out a priori any relation between space-time and internal symmetries. We shall sketch in Section 3 some recent and ongoing research based on such a nontrivial subtle relation.

2.2.3 Infinite-dimensional groups

The above mentioned counterexamples are basically infinite-dimensional groups, either generated by the one-parameter groups of "local" (i.e., nonexponentiable to representations of the corresponding Lie group) representations of finite-dimensional Lie algebras containing the Poincaré (inhomogeneous Lorentz) Lie algebra, or explicitly infinite-dimensional Lie algebras exponentiable to Banach or Fréchet Lie groups. In spite of the fact that fields with an infinity of components are known to exhibit some problems, they appear recurrently in theoretical physics in a variety of contexts.

Interestingly that period (the second half of the 1960s) saw a strong renewal of interest, from a variety of perspectives, in infinite-dimensional Lie groups, a subject that had been more or less dormant since the fundamental works of Lie and Cartan at the beginning of last century. See for instance (in the Lie–Cartan line) [GS64, Ri66], the now classic Kac–Moody algebras [Ka68, Mo68] and their many avatars, and the very original works by Omori from the same time, a nice exposition of which can be found e.g., in [Om74].

2.3 Quantization as a deformation

The need for quantization appeared for the first time around 1900 when, faced with the impossibility to explain otherwise the black body radiation, Planck proposed the quantum hypothesis: the energy of light is not emitted continuously but in quanta proportional to its frequency. He wrote h for the proportionality constant which bears his name. This paradoxical situation got a beginning of a theoretical basis when, in 1905, Einstein came with the theory of the photoelectric effect. Around 1920 Louis de Broglie was introduced (among other things) to the photoelectric effect in the laboratory of his much older brother, Maurice de Broglie. This led him, in 1923, to his discovery of the duality of waves and particles, which he described in his celebrated Thesis published in 1925, and to what he called 'mécanique ondulatoire'. Physicists publishing in German, in particular Weyl, Heisenberg and Schrödinger, and Niels Bohr, transformed it into the quantum mechanics that we know, where the observables are operators in Hilbert spaces of wave functions—and were led to its probabilistic interpretation that neither Einstein nor de Broglie were at ease with.

Intuitively, classical mechanics is the limit of quantum mechanics when $\hbar = \frac{h}{2\pi}$ goes to zero. But how can this be realized when in classical mechanics the observables are functions over phase space (a Poisson manifold) and not operators? The deformation philosophy promoted by Flato shows the way: one has to look for deformations of algebras of classical observables, functions over Poisson manifolds, and realize there quantum mechanics in an autonomous manner. That is what we have done since the 1970s and is now called deformation quantization. Some recent reviews on the subject including its many avatars, from various perspectives with appropriate details, can be found in [DS01, S05A, S05C, S05L]. We shall not repeat them here. Among its avatars are quantum groups, noncommutative geometry [Co94] and quantized manifolds. These are a central theme in the program, much of which remains to be developed, that we shall sketch at the end of the next section.

3 Possible quantized anti de Sitter structures in the microworld

3.1 The context and an overview

At our distances, for most practical matters, the theory of special relativity is relevant. The corresponding space-time is Minkowski space, a 4-dimensional flat space $\mathbb{R}^4$ endowed with a hyperbolic metric. A natural question is therefore to ask whether that structure can be deformed. General relativity, introduced by Einstein in 1916, has done just that, somewhat like the passage from flat to spherical Earth but with different motivation (incorporating gravitation): Space-time is a curved Lorentzian manifold. Those with constant nonzero curvature are of special interest. At cosmological distances, there is at present reasonable experimental evidence that the curvature (or cosmological constant in Einstein equations) is positive. If constant, that is called a de Sitter universe and the Poincaré group of special relativity is deformed to the de Sitter group $SO(1,4)$, one of the two simple groups it can be deformed to, and "the buck stops there" in the category of Lie groups or algebras.

Elementary particles are traditionally (since Wigner [Wi39]) associated, in Minkowski space, with UIR of the Poincaré group (massive or massless). In these the energy operator (generator of time translations) is bounded below, as it should be. That does not happen with $SO(1,4)$ but it does with the ultrahyperbolic version $SO(2,3)$, symmetry of anti de Sitter universe AdS_4 with negative curvature. We (and others) have therefore suggested that, at least at "small" distances, our universe might have a tiny negative curvature. At both the kinematical and dynamical levels this brings in very interesting new features, such as the possibility of considering the photon as composite of more fundamental particles (the Dirac singletons) in a manner compatible with Quantum Electrodynamics [FF88], and maybe also e.g., the leptons [Fr00].

But we know that, in the category of Hopf algebras, algebras of functions over a simple Lie group or their topological duals, completed universal enveloping algebras (see e.g., [BGGS]) can be deformed to quantum groups. It is thus tempting to try and introduce the quantum group $SO_q(3,2)$. It turns out that again new features appear, such as the existence of finite-dimensional UIR for q an even root of unity [FHT93, Sc98]. One is thus tempted to deform also space-time once more, to a quantum analogue of AdS_4, an ultrahyperbolic version (to be developed) of the quantum spheres recently studied extensively by Connes and Dubois-Violette (see e.g., [CoDV]). It is in line with recent attempts aiming at developing field theory on quantized space-time, which could be the structure needed at very small distances, e.g., around the Planck length ($\simeq 10^{-32}$ cm). That is the program we shall now present.

3.2 Deforming Minkowski to anti de Sitter

In line with our deformation philosophy one is led to consider the possibility that our Minkowski flat space-time is deformed with a tiny curvature. In the spirit of the strategy of deformation quantization and noncommutative geometry, albeit here in the commutative context, that intuitive geometric notion may be expressed by deforming in a subtle way (because the Harrison cohomology can be trivial) as in [Fr01] the coordinate algebra $A = \mathbb{C}_0[x^0, \dots, x^3]$ of polynomial functions over Minkowski space without constant term into a subalgebra of the coordinate algebra of polynomial functions on AdS_4. Such an approach could be useful in quantizing AdS_4 (see (3.3) below).

However, dealing with elementary particles, it is natural to see first what happens with the UIR of the Poincaré group, especially those associated with free particles as described by Wigner [Wi39]. As we have explained above, there are problems with that interpretation when a positive curvature is introduced. This does not contradict the fact that a positive curvature can be present at cosmological distances, e.g., due to the presence of matter; it only means that a group like $SO(1,4)$ does not have a good particle interpretation, consistent with the flat space limit.

It turns out that with the negative curvature deformation, not only these problems do not appear, but there are significant advantages (see e.g., [AFFS, FF88, FFS99, Fr00]). The strategy is the following. $SO(2,3)$ group representation theory shows us that the UIR which, for many good reasons (see e.g., [AFFS]), should be called massless, are (in contradistinction with the flat space limit) composed of two more degenerate UIR of (the covering of) $SO(2,3)$, discovered by Dirac [Di63] who called them singletons. We denoted them Di and Rac, on the pattern of Dirac's "bra" and "ket". They are the massless representations of the Poincaré group in 1+2 dimensional space-time, where $SO(2,3)$ is the conformal group ($\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence).

3.2.1 Kinematically composite massless particles in anti de Sitter space

More precisely, we denote by $D(E_0, s)$ the minimal weight representations of the twofold covering of the connected component of the identity of $SO(2,3)$. Here E_0 is the minimal $SO(2)$ eigenvalue and the half-integer s is the spin. These irreducible representations are unitary provided $E_0 \geq s+1$ for $s \geq 1$ and $E_0 \geq s+\frac{1}{2}$ for $s = 0$ and

$s = \frac{1}{2}$. The massless representations of $SO(2,3)$ are defined (for $s \geq \frac{1}{2}$) as $D(s+1,s)$ and (for helicity zero) $D(1,0) \oplus D(2,0)$. At the limit of unitarity (when going down in the values of E_0) the Harish Chandra module $D(E_0,s)$ becomes indecomposable and the physical UIR appears as a quotient, a hallmark of gauge theories. For $s \geq 1$ we get in the limit an indecomposable representation $D(s+1,s) \to D(s+2,s-1)$, a shorthand notation [FF88] for what mathematicians would write as a short exact sequence of modules.

For $s = 0$ and $s = \frac{1}{2}$, the above mentioned gauge theory appears not at the level of the massless representations $D(1,0) \oplus D(2,0)$ and $D(\frac{3}{2},\frac{1}{2})$ but at the limit of unitarity, the singletons $Rac = D(\frac{1}{2},0)$ and $Di = D(1,\frac{1}{2})$. These UIR remain irreducible on the Lorentz subgroup SO(1, 3) and on the (1+2)-dimensional Poincaré group, of which $SO(2,3)$ is the conformal group. The singleton representations have a fundamental property: $(Di \oplus Rac) \otimes (Di \oplus Rac) = (D(1,0) \oplus D(2,0)) \oplus 2\bigoplus_{s=1/2}^{\infty} D(s+1,s)$. Note that all the representations that appear in the decomposition are massless representations. Thus, in contradistinction with flat space, in AdS_4, massless states are "composed" of two singletons. The flat space limit of a singleton is a vacuum and, even in AdS_4, the singletons are very poor in states: their (E,J) diagram has a single trajectory (hence their name). In normal units a singleton with angular momentum J has energy $E = (J + \frac{1}{2})\rho$, where ρ is the curvature of the AdS_4 universe. This means that only a laboratory of cosmic dimensions can detect a J large enough for E to be measurable. Elementary particles would then be composed of two, three or more singletons and/or anti singletons (the latter being associated with the contragredient representations). As with quarks, several (at present three) flavors of singletons (and anti singletons) should eventually be introduced to account for all elementary particles.

3.2.2 Quantum Electrodynamics with composite photons

Dynamics require in particular the consideration of field equations, initially at the first quantized level, in particular the analogue of the Klein–Gordon equation in AdS_4 for the Rac. There, as can be expected of massless (in 1+2 space) representations, gauges appear, and the physical states of the singletons are determined by the value of their fields on the cone at infinity of AdS_4 (see below; we have here a phenomenon of holography [tH93], in this case an AdS_4/CFT_3 correspondence).

We thus have to deal with indecomposable representations, triple extensions of UIR, as in the Gupta–Bleuler (GB) theory, and their tensor products. [It is also desirable to take into account conformal covariance at these GB-triplets level, which in addition permits distinguishing between positive and negative helicities (in AdS_4, the time variable being compact, the massless representations of $SO(2,3)$ of helicity $s > 0$ contract (resp. extend in a unique way) to massless representations of helicity $\pm s$ of the Poincaré (resp. conformal) group.] The situation gets therefore much more involved, quite different from the flat space limit, which makes the theory even more interesting.

In order to test the procedure it is necessary to make sure that it is compatible with conventional Quantum Electrodynamics (QED), the best understood quantum field theory, at least at the physical level of rigor since from the point of view of strict mathematical rigor there is still work to be done. [Only recently was classical electrodynamics rigorously understood; by this we mean the proof of asymptotic completeness and global existence for the coupled Maxwell–Dirac equations, and a study of the infrared problem; that was done [FST97] with the third aspect of the trilogy mentioned at the end of Section 1, a theory of nonlinear group representations, plus a lot of hard analysis using spaces of initial data suggested by the group representations.]

One is therefore led to see whether QED is compatible with a massless photon composed of two scalar singletons. For reasons explained e.g., in [FFS99] and references quoted therein, we consider for the Rac, the dipole equation $(\square - \frac{5}{4}\rho)^2\phi = 0$ with the boundary conditions $r^{1/2}\phi < \infty$ as $r \to \infty$, which carries the indecomposable representation $D(\frac{1}{2}, 0) \to D(\frac{5}{2}, 0)$. A remarkable fact is that this theory is a *topological field theory*; that is [FF81], the physical solutions manifest themselves only by their boundary values at $r \to \infty$: $\lim r^{1/2}\phi$ defines a field on the 3-dimensional boundary at infinity. There, on the boundary, gauge invariant interactions are possible and make a 3-dimensional conformal field theory (CFT).

However, if massless fields (in four dimensions) are singleton composites, then singletons must come to life as 4-dimensional objects, and this requires the introduction of unconventional statistics (neither Bose–Einstein nor Fermi–Dirac). The requirement that the bilinears have the properties of ordinary (massless) bosons or fermions tells us that the statistics of singletons must be of another sort. The basic idea is [FF88] that we can decompose the Rac field operator as $\phi(x) = \sum_{-\infty}^{\infty} \phi^j(x)a_j$ in terms of positive energy creation operators $a^{*j} = a_{-j}$ and annihilation operators a_j (with $j > 0$) without so far making any assumptions about their commutation relations. The choice of commutation relations comes later, when requiring that photons, considered as 2-Rac fields, be Bose–Einstein quanta, i.e., their creation and annihilation operators satisfy the usual canonical commutation relations (CCR). The singletons are then subject to unconventional statistics (which is perfectly admissible since they are naturally confined), the total algebra being an interesting infinite-dimensional Lie algebra of a new type, a kind of "square root" of the CCR. An appropriate Fock space can then be built. Based on these principles, a (conformally covariant) composite QED theory was constructed [FF88], with all the good features of the usual theory—however about 40 years after QED was developed by Schwinger, Feynman, Tomonaga and Dyson.

3.2.3 Composite leptons and massive neutrinos

After QED the natural step is to introduce compositeness in electroweak theory. Along the lines described above, that would require finding a kind of "square root of an infinite-dimensional superalgebra," with both CAR (canonical anticommutation relations) and CCR included: The creation and annihilation operators for the naturally confined Di or Rac need not satisfy CAR or CCR; they can be subject to unusual statistics, provided that the two-singleton states satisfy Fermi–Dirac or Bose–Einstein statistics depending on their nature. We would then have a (possibly $\mathbb{Z}$-)graded algebra where only the two-singleton states creation and annihilation operators satisfy CCR or CAR. That has yet to be done. Some steps in that direction have been initiated but the mathematical problems are formidable, even more so since now the three generations of leptons have to be considered.

But here a more pragmatic approach can be envisaged [Fr00], triggered by recent experimental data which indicate that there are oscillations between various flavors of neutrinos. The latter would thus not be massless. This is not as surprising as it seems from the AdS point of view, because one of the attributes of masslessness is the presence of gauges. These are group theoretically associated with the limit of unitarity in the representations diagram, and the neutrino is above that limit in AdS: the Di is at the limit. Thus, all nine leptons can be treated on an equal footing. One is then tempted to arrange them in a square table and consider them as composites, writing $L_B^A = R^A D_B$. (We know, but do not necessarily tell phenomenologists in order not to scatter them away with a high brow theory, that they are *Rac*-*Di* composites.) In this empirical approach, the vector mesons of the electroweak model are *Rac*-*Rac* composites and the model predicts a new set of vector mesons that are *Di*-*Di* composites and play exactly the same role for the flavor symmetry $U_F(2)$ as the weak vector bosons do for the weak group $U_W(2)$. A set of (maybe five pairs of) Higgs fields would have Yukawa couplings to the leptons currents and massify the leptons (and the vector mesons and the new mesons.) This attempt has been developed in part in [Fr00] (Frønsdal and I are still pursuing that direction) and is qualitatively promising. In addition to the neutrino masses it could explain why the Higgs has so far escaped detection: instead of one "potato" one has a gross purée of five, far more difficult to isolate from background. Quantitatively however its predictive power is limited by the presence of too many free parameters.

Maybe the addition to the picture of a deformation induced by the strong force and of the 18 quarks, which (with the nine leptons) could be written in a cube and also considered composite (of maybe three constituents when the strong force is introduced), would make this "composite Standard Model" more predictive. But introducing the hadrons (strongly interacting particles) brings in a significant quantitative change that could require a qualitative change, e.g., some further deformation.

Moreover it is one thing to explain abstractly that matter is composed of initially massless particles that are massified by some mechanism, and another to describe where, when and how that baryogenesis occurs. In the next subsection we shall sketch a framework in which these questions could be addressed.

3.3 Quantizing locally anti de Sitter

Since around 1980 and until now, 't Hooft has been interested in combining quantum mechanics and black holes, using tools that a theoretical physicist can understand; see [tH85, tH05] among many. The first time I heard about it was at a conference in Stockholm in September 1980 when, dry jokingly, he called that "quantum meladynamics." In contrast with what was (until recently) conventional wisdom, he came early to the conclusion that one can get some information on black holes by communication at their surface, albeit with information loss. ("That is what we found about Nature's book keeping system: the data can be written onto a surface, and the pen with which the data are written has a finite size" [tH00].) This has lead him to two important notions, also in which he was a pioneer. The first is the principle of holography [tH93, tH00], which tells us that in some circumstances physics "in the bulk" is determined by what happens at the boundary; the AdS/CFT correspondence is a manifestation of it, being a very elaborate version of the fact that the anti de Sitter group $SO(2,n)$ is the conformal group of $(1+n)$-dimensional space-time (we have seen it above for $n=3$; it was given that name after its appearance in String Theory, for $n=4$, conjectured by Maldacena in 1997 and proved in part by Witten in 1998, see also [FF98] in a form closer to our context). The second is the idea (shared with many) that at very small distances, space-time should be quantized, see e.g., [tH96].

Until now we have seen a number of instruments making use of various aspects of deformation theory. For the "finale" of this paper we shall play all of them together—hopefully in a way that will reflect Kepler's *Harmonia Mundi*. At this stage the motivation (like that of Pythagoras, if I may use the comparison) is essentially aesthetic. Some may call it Science Fiction.

One Ansatz is that, at least in some regions of our universe, our Minkowski space-time is, at very small distances, both deformed to anti de Sitter and quantized, to qAdS. These regions would appear as black holes, from which matter would emerge. That matter could then be responsible, at very large distances, for a positive cosmological constant, consistent with recent data. For cosmological experimental reasons, there would be few (if any) such black holes in our extended neighborhood of the universe. But there could be many of these at the edge of our expanding universe. In line with 't Hooft's ideas, we would get an idea of what is inside through interactions at the boundary.

Another Ansatz is that "inside" these qAdS black holes, some kind of singletons would exist or be created. At their boundary (where both $q \to 1$ and curvature would vanish) massless 2-singleton states would interact with dark matter which according to what is now believed constitute about 23% of the universe, and/or with dark energy which constitutes 73% of it (the matter that we know representing only 4%). In a way similar to the Higgs mechanism, that would massify these states and create matter. It is a picture of a universe in constant creation.

There are a number of mathematical questions, interesting in themselves, to address in order to make this "double deformation theory" plausible. First, one should study the ultrahyperbolic version $q\mathrm{AdS}_4$ of the 4-spheres considered in [CoDV]. That is in progress (in particular with Pierre Bieliavsky and coworkers); first indications are that the theory could be simpler than in the case of quantized spheres. One should look more closely at the representation theory of the quantum group $SO_q(2,3)$, in particular study what becomes there of the singletons and what are those special finite dimensional UIR (when q is even root of unity). The existence of the latter suggests that these quantized spaces $q\mathrm{AdS}_4$ might be considered in some sense as "q-compact" or "q-bounded" (in general topological vector spaces the two notions are not equivalent). That also needs to be studied more carefully, as well as what happens at the double limit.

On the physical side, a possibly new field theory has to be developed in relation with the above deformed and quantized space-time. It could be that "inside" (whatever that may mean) these qAdS black holes are the extra dimensions of String Theory and other higher dimensional theories, the field theories of which would need to be adapted to the present construct. It may even be that most of our present ideas on black holes will have to be revised, as in the challenging approach which is now being developed by Frønsdal [Fr05]. The attempts made to a field theory of singletons would also need to be adapted to the q-deformed context. The hope, of course, is that deforming and quantizing space-time would reduce the ambiguities and infinities of the usual theories. That is easier said than done, but these problems are worthy of attack—and are likely to prove their worth by hitting back. Whatever the physical outcome is, some very nice mathematics can be expected. That's enough for us (*dayenu* in Hebrew, as is traditionally said at Passover).

References

[AFFS] E. Angelopoulos, M. Flato, C. Fronsdal and D. Sternheimer, *Massless particles, conformal group and De Sitter universe*, Phys. Rev. **D23** (1981), 1278–1289.

[BGGS] P. Bonneau, M. Gerstenhaber, A. Giaquinto and D. Sternheimer, *Quantum groups and deformation quantization: explicit approaches and implicit aspects*, J. Math. Phys. **45** (2004), no. 10, 3703–3741.

[CM67] S. Coleman and J. Mandula, *All possible symmetries of the S matrix*, Phys. Rev. **159** (1967), 1251–1256.

[Co94] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego 1994.

[Co06] A. Connes, *2006 Lectures at Collège de France* and related material posted on http://www.alainconnes.org/downloads.html

[CoDV] A. Connes and M. Dubois-Violette, *Moduli space and structure of noncommutative 3-spheres*, Lett. Math. Phys. **66** (2003), 99–121.

[Di63] P.A.M. Dirac, *A remarkable representation of the* $3+2$ *de Sitter group*, J. Math. Phys. **4** (1963), 901–909.

[DS01] G. Dito and D. Sternheimer, *Deformation quantization: genesis, developments and metamorphoses*, pp. 9–54 in: *Deformation quantization* (Strasbourg 2001), IRMA Lect. Math. Theor. Phys., 1, Walter de Gruyter, Berlin 2002 (math.QA/0201168).

[FerW] E. Fermi, as quoted e.g., in http://en.wikiquote.org/wiki/Enrico_Fermi and http://www.nbi.dk/~petersen/Quotes/Physics/physics.html.

[Fl65] M. Flato, *Ionic energy levels in trigonal and tetragonal fields*, J. Mol. Spec. **17** (1965), 300–324.

[Fl82] M. Flato, *Deformation view of physical theories*, Czechoslovak J. Phys. **B32** (1982), 472–475.

[FF81] M. Flato and C. Fronsdal, *Quantum field theory of singletons. The Rac*, J. Math. Phys. **22** (1981), 1100–1105.

[FF88] M. Flato and C. Fronsdal, *Composite Electrodynamics*, J. Geom. Phys. **5** (1988), 37–61.
M. Flato, C. Fronsdal and D. Sternheimer, *Singletons as a basis for composite conformal quantum electrodynamics*, pp. 65–76, in *Quantum Theories and Geometry*, Mathematical Physics Studies **10**, Kluwer Acad. Publ. Dordrecht (1988).

[FF98] M. Flato and C. Frønsdal, *Interacting Singletons*, Lett. Math. Phys. **44** (1998), 249–259.

[FFS99] M. Flato, C. Frønsdal and D. Sternheimer, *Singletons, physics in AdS universe and oscillations of composite neutrinos*, Lett. Math. Phys. **48** (1999), 109–119.

[FHT93] M. Flato, L.K. Hadjiivanov and I.T. Todorov, *Quantum deformations of singletons and of free zero-mass fields*, Found. Phys. **23** (1993), 571–586.

[FST97] M. Flato, J.C.H. Simon and E. Taflin, *Asymptotic completeness, global existence and the infrared problem for the Maxwell–Dirac equations*, Mem. Amer. Math. Soc. **127** (1997), no. 606, x+311 pp.

[FS65] M. Flato and D. Sternheimer, *Remarks on the connection between external and internal symmetries*, Phys. Rev. Letters **15** (1965), 934–936.

[FS66] M. Flato and D. Sternheimer, *Local representations and mass-spectrum*, Phys. Rev. Letters **16** (1966), 1185–1186.

[FS69] M. Flato and D. Sternheimer, *Poincaré partially integrable local representations and mass-spectrum*, Comm. Math. Phys. **12** (1969), 296–303; and *On an infinite dimensional group*, ibid. **14** (1969), 5–12.

[FN57] A. Froelicher and A. Nijenhuis, *A theorem on stability of complex structures*, Proc. Nat. Acad. Sci USA **43** (1957), 239–241.

[Fr00] C. Frønsdal, *Singletons and neutrinos*, Conférence Moshé Flato 1999 (Dijon), Lett. Math. Phys. **52** (2000) 51–59.

[Fr01] C. Frønsdal, *Harrison cohomology and abelian deformation quantization on algebraic varieties*, pp. 149–161 in *Deformation quantization* (Strasbourg 2001), IRMA Lect. Math. Theor. Phys., **1**, Walter de Gruyter, Berlin 2002 (hep-th/0109001).

[Fr05] C. Frønsdal, *Growth of a Black Hole*, gr-qc/0508048 (in press in J. Geom. Phys. **57** (2007) doi:10.1016/j.geomphys.2006.02.008)

[Ge64] M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. Math. **79** (1964), 59–103; and *IV*, ibid. **99** (1974), 257–276.

[GS64] V. Guillemin and S. Sternberg, *An algebraic model of transitive differential geometry*, Bull. Amer. Math. Soc. **70** (1964), 16–47.

[Gå47] L. Gårding, *Note on continuous representations of Lie groups*, Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 331–332.

[tH85] G. 't Hooft, *On the quantum structure of a black hole*, Nucl. Phys. B **256** (1985), 727–745.

[tH93] G. 't Hooft, *Dimensional reduction in quantum gravity*, pp. 284–296 in *Salamfestschrift: a collection of talks* (A. Ali, J. Ellis and S. Randjbar-Daemi eds.), World Scientific 1993 (gr-qc/9310026).

[tH96] G. 't Hooft, *Quantization of space and time in 3 and in 4 space-time dimensions*, pp. 151–163 in Cargese 1996, Quantum fields and quantum space time (gr-qc/9608037).

[tH00] G. 't Hooft, *The holographic principle: opening lecture*, pp. 72–86 in *Erice 1999, Basics and highlights in fundamental physics* (edited by Antonino Zichichi), World Scientific, Singapore 2001 (hep-th/0003004).

[tH05] G. 't Hooft, *The holographic mapping of the standard model onto the black hole horizon. Part I. Abelian vector field, scalar field and BEH mechanism*, Class. Quant. Grav. **22** (2005), 4179–4188 (gr-qc/0504120).

[IW53] E. Inönü and E.P. Wigner, *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. U. S. A. **39** (1953), 510–524.

[Ka68] V.G. Kac, *Simple irreducible graded Lie algebras of finite growth*, Izv. Akad. Nauk SSSR Ser. Mat. **32** (1968), 1323–1367.

[KS58] K. Kodaira and D.C. Spencer, *On deformations of complex analytic structures I, II,* Ann. of Math. **67** (1958), 328–466; *III. Stability theorems for complex structures*, ibid. **71** (1960), 43–76.

[Mo68] R.V. Moody, *A new class of Lie algebras*, J. Algebra **10** (1968), 211–230.

[Na98] F. Nadaud, *Generalized deformations, Koszul resolutions, Moyal Products*, Reviews Math. Phys. **10** (5) (1998), 685–704. *Thèse*, Dijon (janvier 2000).

[Om74] H. Omori, *Infinite dimensional Lie transformation groups*, Lecture Notes in Mathematics, Vol. 427. Springer-Verlag, Berlin-New York, 1974.

[OR65] L. O'Raifeartaigh, *Mass differences and Lie algebras of finite order*, Phys. Rev. Lett. **14** (1965), 575–577.

[Ri66] D.S. Rim, *Deformation of transitive Lie algebras*, Ann. of Math. (2) **83** (1966), 339–357.

[Sa64] J.J. Sakurai, *Invariance principles and elementary particles*, Princeton University Press 1964.

[Sc98] H. Steinacker, *Finite-dimensional unitary representations of quantum anti-de Sitter groups at roots of unity*, Comm. Math. Phys. **192** (1998), 687–706. *Quantum anti-de Sitter space and sphere at roots of unity*, Adv. Theor. Math. Phys. **4** (2000), 155–208.

[S05A] D. Sternheimer, *Quantization is deformation*, pp. 331–352 in (J. Fuchs et al. eds.) *Noncommutative Geometry and Representation Theory in Mathematical Physics*, Contemporary Mathematics **391**, Amer. Math. Soc. 2005.

[S05C] D. Sternheimer, *Deformation theory: a powerful tool in physics modelling*, in: Poisson geometry, deformation quantisation and group representations, 325–354, London Math. Soc. Lecture Note Ser., **323**, Cambridge Univ. Press, Cambridge, 2005.

[S05L] D. Sternheimer, *Quantization: Deformation and/or Functor?*, Lett. Math. Phys. **74** (2005), 293–309.

[WW00] E. Weimar-Woods, *Generalized Inönü-Wigner contractions and deformations of finite-dimensional Lie algebras*, Rev. Math. Phys. **12** (2000), 1505–1529.

[We28] H. Weyl, *The theory of groups and quantum mechanics*, Dover, New-York 1931, edited translation of *Gruppentheorie und Quantenmechanik*, Hirzel Verlag, Leipzig 1928.

[Wi39] E. Wigner, *On unitary representations of the inhomogeneous Lorentz group*, Ann. of Math. (2) **40** (1939), 149–204.

Geometric Objects in an Approach to Quantum Geometry

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Summary. Ideas from deformation quantization applied to algebra with one generator lead to the construction of non-linear flat connection, whose parallel sections have algebraic significance. The moduli space of parallel sections is studied as an example of bundle-like objects with discordant (sogo) transition functions, which suggests a method to treat families of meromorphic functions with smoothly varying branch points.

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1 Introduction

The aim of this paper is to show that deformation quantization provides us with a new geometric idea going beyond classical geometry. In fact, there have been several attempts to describe "quantum objects" in a geometric way (cf. [3], [5], [6]), although no treatment has been accepted as definitive. Motivated by these attempts, we produce a description of objects which arise from the deformation of algebras, as one approach to describing quantum mechanics mathematically is via deformation quantization, which is a deformation of Poisson algebras. Through the construction of the star exponential

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functions of the quadratic forms in the complex Weyl algebra, we found several strange phenomena which cannot be treated as classical geometric objects (cf. [9], [11], [12], [13]). Our main concern is to understand how to handle these objects geometrically, and we hope that our results are a step toward quantum geometry. However, similar questions arise even for deformations of commutative algebras, as in the case of deformation quantizations. For this reason, in this paper we deal with the simplest case of the deformation of the associative commutative algebra of polynomials of one variable.

In §2.1, we construct an algebra $\mathbb{C}_*[\zeta]$ whose elements of $\mathbb{C}[\zeta]$ are parametrized by the indeterminate κ.

Motivated by deformation quantization, we introduce associative commutative products on $\mathbb{C}[\zeta]$ parametrized by a complex number κ (cf. Definition 2.1), which gives both a deformation of the canonical product and a representation parameterized by κ of $\mathbb{C}$.

Our standpoint formulated in § 2.1 is to view elements in the abstract algebra $\mathbb{C}_*[\zeta]$ as a family of elements. The deformation parameter κ is viewed as an indeterminate.

One method of treating this family of elements as geometric objects is to introduce the notion of infinitesimal intertwiners, which play the role of a connection. In fact, elements of $\mathbb{C}_*[\zeta]$ can be viewed as parallel sections with respect to this connection. These elements are called q-number polynomials.

In § 2.2 and § 2.3, we extend this setting to a class of transcendental elements such as exponential functions. In this setting, the notion of densely defined multi-valued parallel sections appears crucially. We also call these q-number functions in analogy with [1]. However, the only geometrical setting possible is to extend the infinitesimal intertwiners to a linear connection on a trivial bundle over $\mathbb{C}$ with a certain Fréchet space of entire functions.

In § 3 we investigate the moduli space of densely defined parallel sections consisting of exponential functions of quadratic forms. We show that the moduli space is not an ordinary bundle, as it contains fuzzy transition functions. This has similarities to the theory of gerbes (cf. [2], [8]).

However, our construction has a different flavor from the differential geometric point of view, since gerbes are classified by the Dixmier–Douady classes in the third cohomology over $\mathbb{Z}$, while our example is constructed on the 2-sphere or the complex plane. We prefer to call this fuzzy object a *pile*, although $\mathbb{Z}_2$-gerbes have been proposed as similar notions (cf.[14]).

We run into a similar situation in quantizing non-integral closed 2-forms on manifolds. As for integral symplectic forms on symplectic manifolds, we can construct a prequantum bundle, which is a line bundle with connection whose curvature is given by the symplectic form. We attempt the prequantization of a non-integral closed 2-form by mimicing our examples describing the moduli space of densely defined multi-valued parallel sections. We note that Melrose [7] proposed a method handling a type of prequantization of non-integral closed 2-forms, which seems closely related to our approach.

In § 5, we give a simple example for treating solution spaces of ordinary differential equations with movable branch singularities. We introduce an associative product on the space of parallel sections of exponential functions of quadratic forms, but this product is "broken" in the sense that for every κ, there is a singular set on which the product diverges. Thanks to the movable singularities, this broken product defines an associative product by treating κ as an indeterminate. This computation provides a novel aspect of the noncommutative calculus. We also hope that our attempt will help with the study for solutions of ordinary differential equations with movable branch singularities.

In the end, our work seems to extend the notion of *points* as established elements of a fixed set to a more flexible notion of elements.

2 Deformation of a commutative product

We give an algebra $\mathbb{C}_*[\zeta]$ whose elements of $\mathbb{C}[\zeta]$ are parametrized by the indeterminate κ. For convenience, we denote by $*$ the product on the algebra $\mathbb{C}_*[\zeta]$. The algebra $\mathbb{C}_*[\zeta]$ is isomorphic to the algebra $\mathbb{C}[\zeta]$ of polynomials in ζ over $\mathbb{C}$, but we will view $\mathbb{C}_*[\zeta]$ as a family of algebras which are mutually *isomorphic*.

2.1 A deformation of commutative product on $\mathbb{C}[\zeta]$

We denote the set of polynomials of ζ viewed as a linear space by $\mathcal{P}(\mathbb{C})$. We introduce a family of product $*_\kappa$ on $\mathcal{P}(\mathbb{C})$ parametrized by $\kappa \in \mathbb{C}$ as follows.

Definition 2.1 For every $f, g \in \mathcal{P}(\mathbb{C})$, we set

$$f *_\kappa g = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\Big(\frac{\kappa}{2}\Big)^\ell \partial_\zeta^\ell f(\zeta) \cdot \partial_\zeta^\ell g(\zeta).$$

Then $(\mathcal{P}(\mathbb{C}), *_\kappa)$ is an associative commutative algebra for every $\kappa \in \mathbb{C}$. Since putting $\kappa = 0$ gives the algebra $\mathbb{C}[\zeta]$, the family of algebras $\{(\mathcal{P}(\mathbb{C}), *_\kappa)\}_{\kappa\in\mathbb{C}}$ gives a deformation of $\mathbb{C}[\zeta]$ within associative commutative algebras. We note the following.

Lemma 2.2 *For every $\kappa, \kappa' \in \mathbb{C}$, the algebras $(\mathcal{P}(\mathbb{C}), *_\kappa)$ and $(\mathcal{P}(\mathbb{C}), *_{\kappa'})$ are mutually isomorphic. Namely, the mapping $T_\kappa^{\kappa'} : \mathcal{P}(\mathbb{C}) \to \mathcal{P}(\mathbb{C})$ given by*

$$T_\kappa^{\kappa'}(f) = \Big(\exp \frac{1}{4}(\kappa' - \kappa)\partial_\zeta^2\Big) f(\zeta) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\Big(\frac{1}{4}(\kappa' - \kappa)\Big)^\ell (\partial_\zeta^{2\ell}) f(\zeta) \qquad (1)$$

satisfies $T_\kappa^{\kappa'}(f *_\kappa g) = T_\kappa^{\kappa'}(f) *_{\kappa'} T_\kappa^{\kappa'}(g)$.

Definition 2.3 The isomorphism $T_\kappa^{\kappa'}$ given by (1) is called the *intertwiner* between the algebras $(\mathcal{P}(\mathbb{C}), *_\kappa)$ and $(\mathcal{P}(\mathbb{C}), *_{\kappa'})$.

Taking the derivative in κ' for $T_\kappa^{\kappa'}$ defines an *infinitesimal intertwiner*. Namely, for $\kappa \in \mathbb{C}$ we set

$$t_\kappa(u)(f) = \frac{d}{ds}\Big|_{s=0} T_\kappa^{\kappa+su}(f) = \frac{1}{4}u\partial_\zeta^2 f. \tag{2}$$

The infinitesimal intertwiner gives a realization of $\mathbb{C}_*[\zeta]$ as follows. Let $\pi : \mathbb{C}\times\mathcal{P}(\mathbb{C}) \to \mathbb{C}$ be the trivial bundle over $\mathbb{C}$, and $\Gamma(\mathbb{C}\times P(\mathbb{C}))$ the set of sections of this bundle. Using the infinitesimal intertwiner defined by (2), we introduce a connection ∇ on $\Gamma(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$: For a smooth curve $c(s)$ in $\mathbb{C}$ and $\gamma \in \Gamma(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$, we set

$$\nabla_{\dot{c}}\gamma(s) = \frac{d}{ds}\gamma(c(s)) - t_{c(s)}(\dot{c}(s))(\gamma(c(s))), \quad \text{where } \dot{c}(s) = \frac{d}{ds}c(s). \tag{3}$$

Definition 2.4 A section $\gamma \in \Gamma(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$ is *parallel* if $\nabla\gamma = 0$. We denote by $\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$ the set of all parallel sections $\gamma \in \Gamma(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$.

Let us consider an element $f_* \in \mathbb{C}_*[\zeta]$. Corresponding to the unique expression of an element $f_* \in \mathbb{C}_*[\zeta]$ as

$$f_* = \sum a_j \underbrace{\zeta * \cdots * \zeta}_{j\text{-times}} \quad \text{(finite sum)}, a_j \in \mathbb{C},$$

we set the element $f_\kappa \in \mathcal{P}(\mathbb{C})$ for $\kappa \in \mathbb{C}$ by

$$f_\kappa = \sum a_j \underbrace{\zeta *_\kappa \cdots *_\kappa \zeta}_{j\text{-times}} \quad \text{(finite sum)}, a_j \in \mathbb{C}.$$

The section $\gamma_{f_*}(\kappa) = f_\kappa$ gives a parallel section of the bundle $\pi : \mathbb{C}\times\mathcal{P}(\mathbb{C})\to\mathbb{C}$.

Using the product formula $*_\kappa$, we define a product $*$ on $\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$ by

$$(\gamma_1 * \gamma_2)(\kappa) = \gamma_1(\kappa) *_\kappa \gamma_2(\kappa), \quad \gamma_1, \ \gamma_2 \in \mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C})). \tag{4}$$

Lemma 2.5 $(\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C})), *)$ *is an associative commutative algebra.*

This procedure gives an identification of the algebra $(\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C})), *)$ with $\mathbb{C}_*[\zeta]$. Elements of $(\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C})), *)$ will be called q-number polynomials. Although the space of parallel sections could also be defined as the space of leaves of a foliation, we attempt to give examples via deformations as alternative geometric objects.

2.2 Strange exponential functions

We now extend this procedure to exponential functions. For $f_* \in \mathbb{C}_*[\zeta]$, we want to describe the star exponential functions $\exp_* f_*$, which may be a highly transcendental element.

Let $\mathcal{E}(\mathbb{C})$ be the set of all entire functions on $\mathbb{C}$. For $p > 0$, we set

$$\mathcal{E}_p(\mathbb{C}) = \{f \in \mathcal{E}(\mathbb{C}) \mid ||f||_{p,\delta} = \sup_{\zeta\in\mathbb{C}} e^{-\delta|\zeta|^p}|f(\zeta)| < \infty, \forall\delta > 0\}, \tag{5}$$

and also set $\mathcal{E}_{p+}(\mathbb{C}) = \cap_{q>p}\mathcal{E}_q(\mathbb{C})$. Then $(\mathcal{E}_p(\mathbb{C}), *_\kappa)$ is a Fréchet commutative associative algebra for $p \leq 2$ (cf. [11]). Recalling the intertwiner $T_\kappa^{\kappa'}$ given by (1), we have the following [12]:

Lemma 2.6 *Let $p \leq 2$. The intertwiner $T_\kappa^{\kappa'}$ in* (1) *canonically extends to a map $T_\kappa^{\kappa'}$: $\mathcal{E}_p(\mathbb{C}) \to \mathcal{E}_p(\mathbb{C})$ satisfying*

$$T_\kappa^{\kappa'}(f *_\kappa g) = T_\kappa^{\kappa'}(f) *_{\kappa'} T_\kappa^{\kappa'}(g) \quad \textit{for every } f,\ g \in \mathcal{E}_p(\mathbb{C}). \tag{6}$$

We note that while the product $*_\kappa$ does not give an associative commutative product and the intertwiner $T_\kappa^{\kappa'}$ does not extend to $\mathcal{E}_p(\mathbb{C})$ for $p > 2$, the notion of the connection ∇ is still defined.

Namely, we consider the trivial bundle $\pi : \mathbb{C}\times\mathcal{E}(\mathbb{C}) \to \mathbb{C}$ over $\mathbb{C}$ with the fiber $\mathcal{E}(\mathbb{C})$, and the set of sections $\Gamma(\mathbb{C}\times\mathcal{E}(\mathbb{C}))$. For $\gamma \in \Gamma(\mathbb{C}\times\mathcal{E}(\mathbb{C}))$, we define a covariant derivative $\nabla_{\dot{c}}\gamma$ as the natural extension of (3). It is easily seen that ∇ is well defined for $\Gamma(\mathbb{C}\times\mathcal{E}_p(\mathbb{C}))$ and $\Gamma(\mathbb{C}\times\mathcal{E}_{p+}(\mathbb{C}))$ for every $p\geq 0$. As before, we denote by $\mathcal{S}(\mathbb{C}\times\mathcal{E}_p(\mathbb{C}))$, $\mathcal{S}(\mathbb{C}\times\mathcal{E}_{p+}(\mathbb{C}))$ the sets of parallel sections.

We wish to treat the star exponential function $\exp_* f_*$ for $f_* \in \mathbb{C}_*[\zeta]$. As in §2.1, we have the realization $\{f_\kappa\}_{\kappa\in\mathbb{C}}$ of $f_* \in \mathbb{C}_*[\zeta]$, where $f_\kappa \in \mathcal{P}(\mathbb{C})$. Fixing the $*_\kappa$ product gives the star exponential functions of $f_\kappa \in \mathcal{P}(\mathbb{C})$ with respect to $*_\kappa$ as follows. We consider the evolution equation

$$\begin{cases} \partial_t F_\kappa(t) = f_\kappa(\zeta) *_\kappa F_\kappa(t), \\ F_\kappa(0) = g_\kappa. \end{cases} \tag{7}$$

If (7) has a real analytic solution in t, then this solution is unique. Thus, we may set $\exp_{*_\kappa} f_\kappa = F_\kappa(1)$ when (7) has an analytic solution with $F_\kappa(0) = 1$.

By letting κ vary in $\mathbb{C}$, the totality of the star exponential functions $\{\exp_{*_\kappa} f_\kappa\}_{\kappa\in\mathbb{C}}$ may be viewed as a natural representation of the star exponential function $\exp_* f_*$.

As an example, we consider the linear function $f(\zeta) = a\zeta$, where $a \in \mathbb{C}$. Then the evolution equation (7) is expressed as

$$\begin{cases} \partial_t F_\kappa(t) = a\zeta F_\kappa + \frac{\kappa}{2}a\partial_\zeta F_\kappa, \\ F_\kappa(0) = 1. \end{cases} \tag{8}$$

By a direct computation, we have

Lemma 2.7 *The equation* (8) *has the solution $F_\kappa(t) = \exp(at\zeta + \frac{\kappa}{4}a^2t^2)$. Thus, we may set*

$$\exp_{*_\kappa} t\zeta = \exp(t\zeta + \frac{\kappa}{4}t^2) \tag{9}$$

which is contained in $\mathcal{E}_{1+}(\mathbb{C})$ for every $\kappa \in \mathbb{C}$.

Since the intertwiner $T_\kappa^{\kappa'}$ is defined on $\mathcal{E}_{1+}(\mathbb{C})$, and $T_\kappa^{\kappa'}(\exp_{*_\kappa} a\zeta) = \exp_{*_{\kappa'}} a\zeta$, we see that $\{\exp_{*_\kappa} a\zeta\}_{\kappa\in\mathbb{C}}$ is an element of $\mathcal{S}(\mathbb{C}\times\mathcal{E}_{1+}(\mathbb{C}))$. As in §2.1, it is natural to regard $\{\exp_{*_\kappa} a\zeta\}_{\kappa\in\mathbb{C}}$ as the star exponential function $\exp_* a\zeta$, which may be called a q-number exponential function.

From the star exponential functions $\exp_{*_\kappa} a\zeta$, we construct a type of *delta function* via the star Fourier transform: Namely, we call

$$\delta_{*_\kappa}(\zeta) = \int_{-\infty}^{\infty} \exp_{*_\kappa} it\zeta \, dt \tag{10}$$

the $*_\kappa$-delta function. Using (9), we have

Lemma 2.8 *The $*_\kappa$-delta function $\delta_{*_\kappa}(\zeta)$ is well defined as an element of $\mathcal{E}_{2+}(\mathbb{C})$ for every $\kappa \in \mathbb{C}$ such that* $\mathrm{Re}(\kappa) > 0$.

Using integration by parts, we easily see that

$$e^{i\theta} \int_{-\infty}^{\infty} \exp_{*_\kappa} e^{i\theta} it\zeta \, dt, \qquad \mathrm{Re}\, e^{2i\theta}\kappa > 0$$

does not depend on θ whenever $\mathrm{Re}(e^{2i\theta}\kappa) > 0$. This allows us to define $\delta_{*_\kappa}(\zeta) \in \mathcal{E}_{2+}(\mathbb{C})$ for $\kappa \in \mathbb{C} - \{0\}$.

Lemma 2.9 *The mapping $\delta_* : \mathbb{C}-\{0\} \to \mathcal{E}_{2+}(\mathbb{C})$ defined by $\kappa \to \delta_{*_\kappa}$ is double-valued.*

Proof. We set

$$\delta(\zeta; e^{i\theta}, \kappa) = e^{i\theta} \int_{-\infty}^{\infty} \exp(ie^{i\theta}t\zeta - \frac{\kappa}{4}e^{2i\theta}t^2)dt. \tag{11}$$

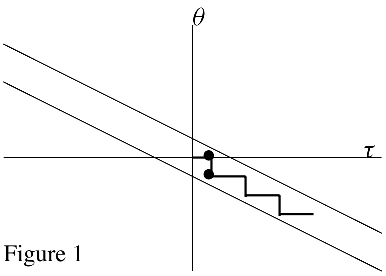

Figure 1

Setting $\kappa = e^{i\tau}$ gives that $\delta(\zeta; e^{i\theta}, e^{i\tau})$ is well defined on the strip bounded by $\theta = -\frac{\tau}{2} \pm \frac{\pi}{4}$ given in Figure 1. Note that $\delta(\zeta; e^{i\theta}, \kappa)$ depends only on τ in this strip $-\pi/2 < \tau + 2\theta < \pi/2$ and $\delta(\zeta; e^{i\theta}, \kappa)$ is a parallel section with respect to κ. By varying θ, we may move τ from 0 to 2π such that (τ, θ) is contained in the strip as indicated in the figure. Moving along such a path from $\tau = 0$ to $\tau = 2\pi$ gives

$$\delta(\zeta; 1, c) = \int_{-\infty}^{\infty} \exp(it\zeta - \frac{1}{4}ct^2)dt = -\int_{-\infty}^{\infty} \exp(-it\zeta - \frac{1}{4}ct^2)dt = -\delta(\zeta; 1, c).$$

Let us consider the trivial vector bundle over $\mathbb{C} - \{0\}$. Lemma 2.9 tells us that $\delta_*(\zeta)$ can be viewed as a double-valued holomorphic parallel section over $\mathbb{C} - \{0\}$. Note that $\delta(\zeta; 1, c) = \frac{2\sqrt{\pi}}{\sqrt{c}} e^{-\frac{1}{c}\zeta^2}$, and $\lim_{c\to 0} \delta(\zeta; 1, c)$ gives us the ordinary delta function.

As seen in the construction of the star delta functions, the notion of *densely defined multi-valued parallel sections* arises naturally, which could be handled as leaves of a foliation. However, as mentioned in §2.1, we prefer to interpret this object as an alternative geometric notion.

2.3 Star exponential functions of quadratic functions

We set

$$P^{(2)}(\mathbb{C}) = \{f(\zeta) = a\zeta^2 + b \mid a, b \in \mathbb{C}\},$$
$$\mathbb{C}_*^{(2)}[\zeta] = \{f_*(\zeta) = a\zeta{*}\zeta \in \mathbb{C}_*[\zeta] \mid a \in \mathbb{C}\}.$$

Thus, we view $a\zeta{*}\zeta$ as the section $\gamma(\kappa) = a\zeta^2 + \frac{a}{2}\kappa \in \Gamma(\mathbb{C}\times P^{(2)}(\mathbb{C}))$, where π : $\mathbb{C}\times P^{(2)}(\mathbb{C})\to\mathbb{C}$ is the trivial bundle over $\mathbb{C}$ with fiber $P^{(2)}(\mathbb{C})$. We now attempt to give a meaning to the star exponential function $\exp_* a\zeta{*}\zeta$, $a \in \mathbb{C}$ along the argument in §2.1.

We consider a quadratic element $f_* \in \mathbb{C}_*^{(2)}[\zeta]$. Then the corresponding polynomial f_κ is given by

$$f_\kappa = \zeta{*_\kappa}\zeta = \zeta^2 + \frac{\kappa}{2}. \tag{12}$$

As in §2.1, we view $\{f_\kappa\}_{\kappa\in\mathbb{C}}$ as a parallel section of $\mathbb{C}\times P^{(2)}(\mathbb{C})$. We consider the following evolution equation.

$$\partial_t F_\kappa(t) = f_\kappa(\zeta) *_\kappa F_\kappa(t), \quad F_\kappa(0) = g_\kappa, \tag{13}$$

where f_κ can be given by (12). (13) is rewritten as

$$\partial_t F_\kappa(t) = (\zeta^2 + \frac{\kappa}{2})F_\kappa + \kappa\zeta\partial_\zeta F_\kappa + \frac{\kappa^2}{4}\partial_\zeta^2 F_\kappa, \quad F_\kappa(0) = g_\kappa. \tag{14}$$

We assume that the initial condition g_κ is given by the form $g_\kappa = \rho_{\kappa,0} \exp a_{\kappa,0}\zeta^2$, where $\rho_{\kappa,0} \in \mathbb{C}_\times = \mathbb{C} - \{0\}$ and $a_{\kappa,0} \in \mathbb{C}$. Putting $g_\kappa = 1$ gives the star exponential function $\exp_{*_\kappa} f_\kappa(\zeta)$. To solve (14) explicitly, we assume that F_κ is of the following form:

$$F_\kappa(t) = \rho_\kappa(t) \exp a_\kappa(t)\zeta^2. \tag{15}$$

Plugging (15) into (14), we have

$$\begin{cases} \partial_t a_\kappa = 1 + 2a_\kappa\kappa + a_\kappa^2\kappa^2, \\ \partial_t\rho_\kappa = \frac{\kappa}{2}\big(1 + \kappa a_\kappa\big)\rho, \\ a_\kappa(0) = a_{\kappa,0}, \quad \rho_\kappa(0) = \rho_{\kappa,0}. \end{cases} \tag{16}$$

Proposition 2.10 *The solution of* (16) *is given by*

$$a_\kappa(t) = \frac{a_{\kappa,0} + t(1 + \kappa a_{\kappa,0})}{1 - \kappa t(1 + \kappa a_{\kappa,0})}, \quad \rho_\kappa(t) = \frac{\rho_{\kappa,0}}{\sqrt{1 - \kappa t(1 + \kappa a_{\kappa,0})}}, \tag{17}$$

where we note the ambiguity in choosing the sign of the square root in (17). We define a subset $\mathcal{E}^{(2)}(\mathbb{C})$ of $\mathcal{E}(\mathbb{C})$ by

$$\mathcal{E}^{(2)}(\mathbb{C}) = \{f = \rho \exp a\zeta^2 \mid \rho \in \mathbb{C}_\times,\ a \in \mathbb{C}\}.$$

Identifying $f = \rho \exp a\zeta^2 \in \mathcal{E}^{(2)}(\mathbb{C})$ with (ρ, a) gives $\mathcal{E}^{(2)}(\mathbb{C}) \cong \mathbb{C}_\times \times \mathbb{C}$. Note that $\mathcal{E}^{(2)}(\mathbb{C})$ is not contained in $\mathcal{E}_2(\mathbb{C})$ but in $\mathcal{E}_{2+}(\mathbb{C})$, on which the product $*_\kappa$ may give rise to strange phenomena (cf. [12]).

Consider the trivial bundle $\pi : \mathbb{C}\times\mathcal{E}^{(2)}(\mathbb{C})$ over $\mathbb{C}$ with fiber $\mathcal{E}^{(2)}(\mathbb{C})$. In particular, putting $a_{\kappa,0} = 0$, $\rho_{\kappa,0} = 1$ and $t = a$ in Proposition 2.10, we see that

$$\exp_{*_\kappa} a\zeta{*_\kappa}\zeta = \frac{1}{\sqrt{1 - a\kappa}} \exp \frac{a}{1 - a\kappa}\zeta^2 \tag{18}$$

where the right-hand side of (18) still has an ambiguous choice for the sign of the square root.

Keeping this ambiguity in mind, we have a kind of fuzzy one-parameter group property for the exponential function of (18). Namely, for $g_\kappa = \exp_{*_\kappa} b\zeta{*_\kappa}\zeta$, where $b \in \mathbb{C}$, the solutions of (14) yield the exponential law:

$$\exp_{*_\kappa} a\zeta{*_\kappa}\zeta *_\kappa \exp_{*_\kappa} b\zeta{*_\kappa}\zeta = \frac{1}{\sqrt{1-(a+b)\kappa}} e^{\frac{a+b}{1-(a+b)\kappa}\zeta^2} = \exp_{*_\kappa}(a+b)\zeta{*_\kappa}\zeta, \tag{19}$$

where (19) still contains an ambiguity in the sign of the square root.

Recall the connection ∇ on the trivial bundle $\pi : \mathbb{C}\times\mathcal{E}(\mathbb{C})\to\mathbb{C}$. It is easily seen that the connection ∇ gives a specific trivialization of the bundle $\pi : \mathbb{C}\times\mathcal{E}^{(2)}(\mathbb{C})\to\mathbb{C}$. According to the identification $\mathcal{E}^{(2)}(\mathbb{C}) \cong \mathbb{C}_\times\times\mathbb{C}$, we write $\gamma(\kappa) = \rho(\kappa)\exp a(\kappa)\zeta^2$ as $(\rho(\kappa), a(\kappa))$. Then the equation $\nabla_{\partial_t}\gamma = 0$ gives

$$\begin{cases} \partial_t a(t) = a(t)^2, \\ \partial_t \rho(t) = \frac{1}{2}\rho(t)a(t). \end{cases} \tag{20}$$

We easily see that (18) gives a densely defined parallel section. As seen in [12], it should also be considered as a densely defined multi-valued section of this bundle. Thus, we may view the star exponential function $\exp_* a\zeta{*}\zeta$ as a family

$$\Big\{F_\kappa(\zeta) = \frac{1}{\sqrt{1-a\kappa}} \exp\frac{a}{1-a\kappa}\zeta^2\Big\}_{\kappa\in\mathbb{C}}.$$

This realization of $\exp_* a\zeta{*}\zeta$ is a densely defined and multi-valued parallel section $\gamma(\kappa) = \rho(\kappa)\exp a(\kappa)\zeta^2$ of the bundle $\pi : \mathbb{C}\times\mathcal{E}^{(2)}(\mathbb{C})\to\mathbb{C}$. In the next section, we investigate the solution of (20) more closely.

3 Bundle gerbes as a non-cohomological notion

The bundle $\pi : \mathbb{C}\times\mathcal{E}(\mathbb{C})\to\mathbb{C}$ with the flat connection ∇ gave us the notion of parallel sections, where we extended this notion to be densely defined and multi-valued sections. This is in fact the notion of leaves of the foliation given by the flat connection

∇. We now analyze the moduli space of densely defined multi-valued parallel sections of the bundle $\pi : \mathbb{C}\times\mathcal{E}^{(2)}(\mathbb{C})\to\mathbb{C}$ with respect to the connection ∇. The moduli space has an unusual bundle structure, which we would call a *pile*. We analyse the evolution equation (20) for parallel sections as a toy model of the phenomena of movable branch singularities.

3.1 Non-linear connections

First, consider a non-linear connection on the trivial bundle $\coprod_{\kappa\in\mathbb{C}}\mathbb{C} = \mathbb{C}\times\mathbb{C}$ over $\mathbb{C}$ given by a holomorphic horizontal distribution

$$H(\kappa; y) = \{(t; y^2 t); t \in \mathbb{C}\} \quad \text{(independent of } \kappa). \tag{21}$$

The first equation of parallel translation (20) is given by $\frac{dy}{d\kappa} = y^2$. Hence, parallel sections are given in general by

$$(\kappa; y(\kappa)) = \left(\kappa; \frac{1}{c-\kappa}\right) = \left(\kappa; \frac{c^{-1}}{1-c^{-1}\kappa}\right). \tag{22}$$

There is also the singular solution $(\kappa; 0)$, corresponding to $c^{-1} = 0$. Note that $(\kappa, -\frac{1}{\kappa})$ is not a singular solution. For consistency, we think that the singular point of the section $(\kappa, 0)$ is at ∞.

Let $\mathcal{A}$ be the set of parallel sections including the singular solution $(\kappa, 0)$. Every $f \in \mathcal{A}$ has one singular point at a point $c \in S^2 = \mathbb{C}\cup\{\infty\}$. The assignment of $f \in \mathcal{A}$ to its singular point $\sigma(f) = c$ gives a bijection $\sigma : \mathcal{A}\to S^2 = \mathbb{C}\cup\{\infty\}$. Namely, $\mathcal{A}$ is parameterized by S^2 by

$$\sigma(f) = c \Leftrightarrow f = \left(\kappa, \frac{1}{c-\kappa}\right), \quad \sigma(f) = \infty \Leftrightarrow f = (\kappa, 0) \in \mathcal{A}. \tag{23}$$

In this way, we give a topology on $\mathcal{A}$.

Let $T_\kappa^{\kappa'}(y)$ be the parallel translation of $(\kappa; y)$ along a curve from κ to κ'. Since (21) is independent of the base point κ, $T_\kappa^{\kappa'}(y)$ is given by

$$T_\kappa^{\kappa'}(y) = \frac{y}{1-y(\kappa'-\kappa)}, \quad T_\kappa^{\kappa'}(\infty) = \frac{1}{\kappa-\kappa'}.$$

We easily see that $T_\kappa^{\kappa''} = T_{\kappa'}^{\kappa''}T_\kappa^{\kappa'}$, $T_\kappa^\kappa = I$. Every $f \in \mathcal{A}$ satisfies $T_\kappa^{\kappa'} f(\kappa) = f(\kappa')$ where they are defined.

3.1.1 Extension of the non-linear connection

We now extend the non-linear connection H defined by (21) to the space $\mathbb{C}\times\mathbb{C}^2$ by giving the holomorphic horizontal distributions

$$\tilde{H}(\kappa; y, z) = \{(t; y^2t, -yt); t \in \mathbb{C}\} \quad \text{(independent of } \kappa, z). \tag{24}$$

Parallel translation with respect to (24) is given by the following equations:

$$\frac{dy}{d\kappa} = y^2, \quad \frac{dz}{d\kappa} = -y. \tag{25}$$

For the equation (25), multi-valued parallel sections are given in both ways

$$\left(\kappa, \frac{a}{1-a\kappa}, z + \log(1-a\kappa)\right), \quad \left(\kappa, \frac{1}{b-\kappa}, w + \log(\kappa - b)\right), \quad (a, b \in \mathbb{C}) \tag{26}$$

although they are infinitely valued. The singular solution $(\kappa; 0, z)$ occurs in the first expression. The set-to-set correspondence

$$(a, z + 2\pi i\mathbb{Z}) \overset{\iota}{\Longleftrightarrow} (b, w + 2\pi i\mathbb{Z}) = (a^{-1}, z + \log a + \pi i + 2\pi i\mathbb{Z}) \tag{27}$$

identifies these two sets of parallel sections, which gives multi-valued parallel sections. However, because of the ambiguity of $\log a$, we can not make this correspondence a univalent correspondence (cf. Proposition 3.1).

Denote by $\tilde{\mathcal{A}}$ the set of all parallel sections written in the form (26). Denote by $\pi_3 : \tilde{\mathcal{A}} \to \mathcal{A}$ be the mapping which forgets the last component. This is surjective. For every $v \in \mathcal{A}$ such that $\sigma(v) = b = a^{-1} \in S^2$, we see

$$\begin{aligned}\pi_3^{-1}(v) &= \left\{\left(\kappa, \frac{1}{b-\kappa}, w + \log(\kappa - b)\right); w \in \mathbb{C}\right\}\\ &= \left\{\left(\kappa, \frac{a}{1-a\kappa}, z + \log(1-a\kappa)\right); z \in \mathbb{C}\right\}.\end{aligned}$$

Since there is one-dimensional freedom of moving, $\pi_3^{-1}(v)$ should be parameterized by $\mathbb{C}$. However, there is no natural parameterization and there are many technical choices.

3.1.2 Tangent spaces of $\tilde{\mathcal{A}}$

For an element $f = (\kappa, \frac{a}{1-a\kappa}, z+\log(1-a\kappa)) = (\kappa, \frac{1}{b-\kappa}, w+\log(\kappa-b))$, the *tangent space* $T_f\tilde{\mathcal{A}}$ of $\tilde{\mathcal{A}}$ at f is

$$\begin{aligned}T_f\tilde{\mathcal{A}} &= \left\{\frac{d}{ds}\Big|_{s=0}\left(\frac{a(s)}{1-a(s)\kappa}, z(s) + \log(1-a(s)\kappa)\right); (a(0), z(0)) = (a, z)\right\}\\ &= \left\{\left(\frac{\dot{a}}{(1-a\kappa)^2}, \dot{z} - \frac{\dot{a}\kappa}{1-a\kappa}\right); \dot{a}, \dot{z} \in \mathbb{C}\right\}\\ &= \left\{\left(\frac{-\dot{b}}{(b-\kappa)^2}, \dot{w} - \frac{\dot{b}}{\kappa - b}\right); \dot{b}, \dot{w} \in \mathbb{C}\right\}.\end{aligned}$$

Hence
$$\begin{bmatrix}\dot b\\ \dot w\end{bmatrix}=\begin{bmatrix}-a^{-2} & 0\\ a^{-1} & 1\end{bmatrix}\begin{bmatrix}\dot a\\ \dot z\end{bmatrix}=(d\iota)_{(a,z)}\begin{bmatrix}\dot a\\ \dot z\end{bmatrix},\quad \text{and}\quad T_f\tilde{\mathcal{A}}\cong\mathbb{C}^2.$$

Consider now a subspace H_f of $T_f\tilde{\mathcal{A}}$ obtained by setting $\dot z=0$ in the definition of $T_f\tilde{\mathcal{A}}$. Then, $\{H_f;\, f\in\tilde{\mathcal{A}}\}$ is defined without ambiguity $2\pi i\mathbb{Z}$, and obviously $H_f\cong\mathbb{C}$. We regard $\{H_f;\, f\in\tilde{\mathcal{A}}\}$ an unambiguously defined horizontal distribution on $\pi_3:\tilde{\mathcal{A}}\to\mathcal{A}$.

The invariance in the vertical direction gives that $\{H_f;\, f\in\tilde{\mathcal{A}}\}$ may be viewed as an *infinitesimal trivialization* of $\pi_3:\tilde{\mathcal{A}}\to\mathcal{A}$.

Parallel translation $I_\kappa^{\kappa'}$ for (25) is given by

$$\begin{aligned}I_\kappa^{\kappa'}(y,z)&=\left(\frac{y}{1-y(\kappa'-\kappa)},\, z+\log(1-y(\kappa'-\kappa))\right),\\ &=\left(\frac{1}{y^{-1}-\kappa'+\kappa},\, z+\log y+\log(y^{-1}-\kappa'+\kappa)\right),\end{aligned}\tag{28}$$

which is obtained by solving for (25) under the initial data (κ,y,z).

By definition we see $I_\kappa^\kappa=I$, and $I_\kappa^{\kappa''}=I_{\kappa'}^{\kappa''}I_\kappa^{\kappa'}$, as a set-to-set mapping Every $f\in\tilde{\mathcal{A}}$ satisfies $I_\kappa^{\kappa'}f(\kappa)=f(\kappa')$ where they are defined.

Proposition 3.1 *Parallel translation via the horizontal distribution* $\{H_f : f\in\tilde{\mathcal{A}}\}$ *does not give a local trivialization of* $\pi_3:\widetilde{\mathcal{A}}\to\mathcal{A}$.

Proof. For a point $g=(\kappa,\frac{a}{1-a\kappa})$ of $\mathcal{A}$, and a small neighborhood V_a of a, $\tilde V_a=\{\frac{a'}{1-a'\kappa};\, a'\in V_a\}$ is a neighborhood of f in $\mathcal{A}$. Consider the set

$$\pi_3^{-1}(\tilde V_a)=\left\{\left(\kappa,\,\frac{a'}{1-a'\kappa},\, z+\log(1-a'\kappa)\right);\, a'\in V_a,\ z\in\mathbb{C}\right\}.$$

The horizontal lift of the curve $\frac{a'(s)}{1-a'(s)\kappa}$, $a'(s)=a+s(a'-a)$ along the infinitesimal trivialization is given by solving the equation

$$\frac{d}{ds}z(s)=-\frac{(a'-a)\kappa}{1-a'(s)\kappa},\quad z(0)\in\log(1-a\kappa).$$

Hence $z(s)=\log(1-(a+s(a'-a))\kappa)$, and $z(1)=\log(1-a'\kappa)$. Thus it is impossible to eliminate the ambiguity of $\log(1-a'\kappa)$ on V_a, no matter how small the neighborhood V_a is. □

Proposition 3.1 shows that $\pi_3:\widetilde{\mathcal{A}}\to\mathcal{A}$ is not an affine bundle. In spite of this, one may say that the curvature of its connection vanishes.

3.1.3 Affine bundle gerbes

Although $\pi_3 : \widetilde{\mathcal{A}} \to \mathcal{A}$ does not have a bundle structure, we can consider *local trivializations* by restricting the domain of κ.

(a) Let $V_\infty = \{b; |b|>3\} \subset S^2$ be a neighborhood of ∞. First, we define a fiber preserving mapping $p_{\infty,D}$ from the trivial bundle $\pi : V_\infty \times \mathbb{C} \to V_\infty$ into $\pi_3 : \widetilde{\mathcal{A}} \to \mathcal{A}$ such that $\pi_3 p_{\infty,D} = \sigma^{-1}\pi$ by restricting the domain of κ in a unit disk D: Consider $(\kappa, \frac{a}{1-a\kappa}, z + \log(1-a\kappa))$ for $(\kappa, a^{-1}) \in D \times V_\infty$. Since $|a\kappa|<1/3$, $\log(1-a\kappa)$ is defined as a univalent function $\log(1-a\kappa) = \log|1-a\kappa| + i\theta$, $-\pi<\theta<\pi$ on this domain by setting $1-a\kappa = |1-a\kappa|e^{i\theta}$, which will be denoted by $\log(1-a\kappa)_{D\times V_\infty}$. We define

$$p_{\infty,D}(b,z) = (\kappa, \frac{a}{1-a\kappa}, z + \log(1-a\kappa)), \quad a^{-1} = b \in V_\infty, \quad z \in \mathbb{C} \tag{29}$$

where $\log(1-a\kappa)$ in the right-hand side is the analytic continuation of $\log(1-a\kappa) = \log(1-a\kappa)_{D\times V_\infty}$.

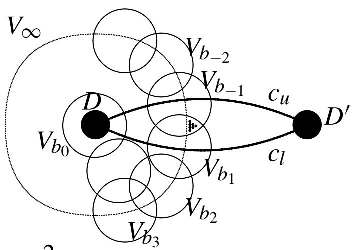

Figure 2

(b) We take a simple covering of the domain $|z| \le 3$ by unit disks $V_{b_{-k}}, \ldots, V_{b_{-1}}, V_{b_0}, V_{b_1}, \ldots, V_{b_\ell}$ as in Figure 2, and fix a unit disk D' apart from all V_{b_i}. We define a fiber preserving mapping $p_{V_{b_i},D'}$ from the trivial bundle $\pi : V_{b_i} \times \mathbb{C} \to V_{b_i}$ to the bundle $\pi_3 : \widetilde{\mathcal{A}} \to \mathcal{A}$ such that $\pi_3 p_{V_{b_i},D'} = \sigma^{-1}\pi$ by restricting the domain of κ in a unit disk D'.

We see that setting $\kappa - b = |\kappa - b|e^{i\theta}$, $\log(\kappa - b)$ is defined as a univalent function on the domain $D' \times V_{b_i}$ as $\log|\kappa - b| + i\theta$, $-\pi<\theta<\pi$, which is denoted by $\log(\kappa - b)_{D'\times V_{b_i}}$.

Consider $(\kappa, \frac{1}{b-\kappa}, w + \log(\kappa - b))$ for $(\kappa, b) \in D' \times V_{b_i}$. We define

$$p_{V_{b_i},D'}(b', w) = \left(\kappa, \frac{1}{b'-\kappa}, w + \log(\kappa - b')\right), \quad (b', w) \in V_{b_i} \times \mathbb{C} \tag{30}$$

where $\log(\kappa - b)$ on the r.h.s. is the analytic continuation of $\log(\kappa - b)_{V_{b_i}\times D'}$.

(c) Suppose $c \in V_{b_i} \cap V_{b_j}$ and $p_{V_{b_i},D'}(c, w) = p_{V_{b_j},D'}(c, w')$. Then we see that there exists a unique $n(i,j) \in \mathbb{Z}$ such that $w' = w + 2\pi i n(i,j)$. For the above covering, we see $n(i,j) = 0$ for every pair (i,j).

Let $c \in V_{b_i} \cap V_\infty$ and $p_{V_{b_i},D'}(c, w) = p_{\infty,D}(c, z)$. To fix the coordinate transformation, we have to choose the identification of two sets of values $\log(\kappa - b)_{D'\times V_{b_i}}$ and $\log(1-a\kappa)_{D\times V_\infty}$. For b_i, except b_1, we identify these through the analytic continuation along the (lower) curve c_l joining D and D', but for b_1, we identify $\log(\kappa - b)_{D'\times V_{b_1}}$ and $\log(1-a\kappa)_{D\times V_\infty}$ through the analytic continuation along the (upper) curve c_u joining D and D'.

Therefore there is a positive integer $n(i,\infty)$ such that $w' = z + 2\pi i n(i,\infty)$ by the same argument. For the above covering we see in fact that if $n(1,\infty) = m$ for $i = 1$, then $n(i,\infty) = m + 1$ for every $i \neq 1$.

These give coordinate transformations. However, the collection of these local trivializations do not glue together, for these do not satisfy the cocycle condition on the triple intersection marked with the triangle in Figure 2.

We denote by $\coprod_{b\in S^2} \mathbb{C}_b$ the collection of these local trivializations. Thus we have a commutative diagram

$$\begin{array}{ccc} \coprod_{b\in S^2} \mathbb{C}_b & \stackrel{p(\bullet)}{\longrightarrow} & \tilde{\mathcal{A}} \\ \pi \downarrow & & \pi_3 \downarrow \\ S^2 & \stackrel{\sigma}{\longleftarrow} & \mathcal{A}\,. \end{array} \tag{31}$$

One can consider various local trivializations of the bundle-like object of the left-hand side. $\coprod_{b\in S^2} \mathbb{C}_b$ is not an affine bundle, but an "affine bundle gerbe" with a holomorphic flat connection (cf. [8]). However, the geometric realization of a holomorphic parallel section is nothing but an element of $\widetilde{\mathcal{A}}$ given by (26).

3.2 Geometric notions on $\tilde{\mathcal{A}}$

Recall that the discordance (the Japanese word *sogo* is the term used in [10]) of patching of three local coordinate neighbourhood occurs only on the small dotted triangle in Figure 2.

In this section, we construct two examples which give almost the same phenomena as in the previous section for gluing local bundles.

3.2.1 Geometric quantization for a non-integral 2-form

Consider the standard volume form dV on S^2 with total volume 4π. Let Ω be a non-integral, closed smooth 2-form (current) on S^2 such that $\int_{S^2} \Omega = 4\pi\lambda$, and with the support of Ω concentrated on a small disk neighborhood of the north pole N. For $\{U_i\}_{i\in I}$ a simple cover of S^2, on each U_i, Ω is of the form $\Omega = d\omega_i$, and hence $\omega_{ij} = \omega_i - \omega_j$ on $U_{ij} = U_i \cap U_j$ is a closed 1-form (current), and is written by $\omega_{ij} = df_{ij}$ on U_{ij} for a smooth 0-form (current) f_{ij}.

Now we want to make a $U(1)$-vector bundle using $e^{\sqrt{-1}f_{ij}}$ as transition functions. However, since on $U_{ijk} = U_i \cap U_j \cap U_k$ we only have

$$e^{\sqrt{-1}f_{ij}} e^{\sqrt{-1}f_{jk}} e^{\sqrt{-1}f_{ki}} = e^{\sqrt{-1}(f_{ij}+f_{jk}+f_{ki})},$$

$e^{\sqrt{-1}f_{ij}}$ cannot be used as patching diffeomorphisms. In spite of these difficulties, we see that the horizontal distributions defined by ω_i glue together.

Thus, we can define a linear connection on such a *broken* vector bundle, which is precisely the notion of *bundle gerbes*. Since $\Omega = d\omega_i$, the curvature form of this connection is given by Ω. Note that we can make a parallel translation along any smooth curve $c(t)$ in S^2.

The support of Ω is concentrated in a small neighborhood V_N of the north pole N. Therefore any closed curve in $S^2 - V_N$ can be shrunk to a point in $S^2 - V_N$. In spite of this, the homotopy lifting of parallel translation does not succeed, because of the discordance (sogo) of the patching diffeomorphisms.

If U_i does not intersect V_N, then we have a product bundle $U_i \times \mathbb{C}$ with the trivial flat connection. Since $\omega_i = d \log e^{h_i}$, the integral submanifold of the horizontal distribution of ω_i is given by $\log e^{h_i}$. This looks like a *pile*. Thus, even if the object is restricted to $S^2 - V_N$, we have a *non-trivial* bundle gerbe which is apparently not classified by a cohomology class.

We note that this gives also a concrete example of the local line bundles over a manifold treated by [7].

3.2.2 A simple example

The simplest example of objects we propose in this paper is given by the Hopf-fibering $S^3 \xrightarrow{S^1} S^2$. Viewing $S^3 = \coprod_{q\in S^2} S^1_q$ (disjoint union), we consider the ℓ-covering $\tilde{S}^1_q$ of each fiber S^1_q, and denote by $\tilde{S}^3$ the disjoint union $\coprod_{q\in S^2} \tilde{S}^1_q$. We are able to define local trivializations of $\tilde{S}^3|_{U_i} \cong U_i \times \tilde{S}^1$ naturally through the trivializations $S^3|_{U_i}$ given on a simple open covering $\{U_i\}_{i\in\Gamma}$ of S^2. This structure permits us to treat $\tilde{S}^3$ as a local Lie group, and hence it looks like a topological space. On the other hand, we have a projection

$$\pi \ : \ \tilde{S}^3 = \coprod_{q\in S^2} \tilde{S}^1_q \to S^3 = \coprod_{q\in S^2} S^1_q$$

as the union of fiberwise projections, as if it were a non-trivial ℓ- covering. However $\tilde{S}^3$ cannot be a manifold, since S^3 is simply connected. In particular, the *points* of $\tilde{S}^3$ should be regarded as *ℓ-valued elements*.

We now consider a 1-parameter subgroup S^1 of S^3 and the inverse image $\pi^{-1}(S^1)$. Since all points of $\widetilde{S}^3$ are "ℓ-valued", this simply looks like a combined object of $S^1 \times \mathbb{Z}_\ell$ and the ℓ covering group, i.e., in some restricted region, this object can be regarded as a point set in several ways. In such a region, the ambiguity is caused simply by the reason that two pictures of point sets are mixed up.

3.2.3 Conceptual difficulties beyond ordinary mathematics

Let P_c be the parallel translation along a closed curve. Let $c_s(t)$ be a family of closed curves. Suppose $c_s(0) = c_s(1) = p$ and $c_1(t) = p$. We see that there is $(p; v)$ such that $P_{c_s}(p; v) \neq v$. Therefore there must be somewhere a singular point for the homotopy chasing, caused by the discordance. However the position of singular point can not be specified.

Even though the parallel translation is defined for every fixed curve, these parallel translations are in general set-to-set mappings when one-parameter families of closed curves are considered.

Thus, we have some conceptual difficulty that may be explained as follows: a parallel translation along a curve has a definite meaning, but when we think this in a family of curves, then we have to think *suddenly* this is a set-to-set mapping. Recall here the "Schrödinger's cat".

Such a strange phenomenon is caused in $\tilde{\mathcal{A}}$ by movable branch singularities. In §3.1, we considered a non-linear connection on the trivial bundle $S^2 \times \mathbb{C}$, and an extended connection to treat the amplitude of the star exponential functions of the quadratic form.

4 Broken associative products and extensions

In this section we give an example where such fuzzy phenomena play a crucial role in defining a concrete algebraic structure. We consider the product bundle $\coprod_{\kappa \in \mathbb{C}} \mathbb{C}$, and we define in each fiber an associative product which is *broken* in the sense that each product is not necessarily defined for all pairs (a, b).

4.1 Associative products combined with the Cayley transform

First of all, we give such a product on the fiber at $\kappa = 0$. Let S^2 be the 2-sphere identified with $\mathbb{C} \cup \{\infty\}$. Consider the Cayley transform $C_0 : S^2 \to S^2$, $C_0(X) = \frac{1-X}{1+X}$, and define the product by

$$a \bullet_0 b = \frac{a+b}{1+ab} \sim C_0^{-1}(C_0(a)C_0(b)). \tag{32}$$

Here $\sim$ means algebraic equality where defined: an algebraic procedure through the calculations such as follows:

$$\frac{1 - \frac{1-a}{1+a} \cdot \frac{1-b}{1+b}}{1 + \frac{1-a}{1+a} \cdot \frac{1-b}{1+b}} \sim \frac{(1+a)(1+b) - (1-a)(1-b)}{(1+a)(1+b) + (1-a)(1-b)} = \frac{a+b}{1+ab}.$$

The product is defined for every pair (a, b) such that $ab \neq -1$, and is commutative and associative whenever they are defined. Note also that

$$a \bullet_0 b = \frac{a+b}{1+ab} \sim \frac{a^{-1} + b^{-1}}{1 + (ab)^{-1}} = a^{-1} \bullet_0 b^{-1}. \tag{33}$$

Hence we set $\infty \bullet_0 b = b^{-1}$, $\infty \bullet_0 \infty = 0$, in particular.

One can extend this broken product to pairs $(a : g) \in \mathbb{C} \times \mathbb{C}$ as follows:

$$(a : g) \bullet_0 (b : g') = (a \bullet_0 b : gg'(1+ab)).$$

This is an associative product, which follows from (32).

$$(1+bc)\left(1 + a\frac{b+c}{1+bc}\right) = \left(1 + \frac{a+b}{1+ab}c\right)(1+ab).$$

It is worthwhile to write this identity in the logarithmic form

$$\log(1+bc)+\log\left(1+a\frac{b+c}{1+bc}\right)$$
$$=\log\left(1+\frac{a+b}{1+ab}c\right)+\log(1+ab), \quad \bmod 2\pi i\mathbb{Z} \tag{34}$$

although the logarithmic form uses infinitely-valued functions. If one sets $C(a,b)=\log(1+ab)$, then (34) is the Hochschild 2-cocycle condition:

$$C(b,c)-C(a\bullet_0 b,c)+C(a,b\bullet_0 c)-C(a,b)=0, \quad \bmod 2\pi i\mathbb{Z}.$$

We extend the product as follows:

$$(a:g)\bullet_{ln}(b:g')=(a\bullet_0 b:g+g'+\log(1+ab)). \tag{35}$$

This is associative as a set-to-set mapping. By using (27), (35) is rewritten as

$$(a^{-1}:g)\bullet_{ln}(b^{-1}:g')=(a^{-1}\bullet_0 b^{-1}:g+g'+\log(1+a^{-1}b^{-1})).$$

Next we define a family of products defined on each fiber at κ. To define such a product, we use the twisted Cayley transform defined by $C_\kappa \sim C_0T^0_\kappa$, where T^0_κ is given in the equality (1). The result is

$$C_\kappa(y)=\frac{1-y(1-\kappa)}{1+y(1+\kappa)}, \tag{36}$$

and we define

$$a\bullet_\kappa b=\frac{a+b+2ab\kappa}{1+ab(1-\kappa^2)}\sim C_\kappa^{-1}(C_\kappa(a)C_\kappa(b)). \tag{37}$$

The point is that the singular set of the product depends on κ. $a\bullet_\kappa b$ is defined for every pair (a,b) such that $ab(1-\kappa^2)\neq -1$. In other words, for an arbitrary pair $(a,b)\in\mathbb{C}^2$, the product $a\bullet_\kappa b$ is defined for some κ in an open dense domain.

For the parallel sections given in (22), we see that

$$\frac{a}{1-a\kappa}\bullet_\kappa\frac{b}{1-b\kappa}=\frac{a+b}{1-(a+b)\kappa+ab}. \tag{38}$$

In particular,

$$-\kappa^{-1}\bullet_\kappa -\kappa^{-1}=0, \quad -\kappa^{-1}\bullet_\kappa\frac{1}{b^{-1}-\kappa}=\frac{1}{b-\kappa}.$$

For simplicity, we denote by $f(\kappa)$ the section f of the bundle $\pi: S^2\times\mathbb{C}\to S^2$.

Proposition 4.1 *For parallel sections $f(\kappa)$, $g(\kappa)$ defined on open subsets, the product $f(\kappa)\bullet_\kappa g(\kappa)$ is also a parallel section where defined.*

4.1.1 Extension of the product

Using (35), one can extend the product $a \bullet_\kappa b$ by the formula

$$(a; g) \bullet_\kappa (b; g') \sim I_0^\kappa\big((I_\kappa^0(a; g)) \bullet_{ln} (I_\kappa^0(a; g'))\big).$$

Indeed, we see how the algebraic trick works:

$$\begin{aligned}(a : g) \bullet_\kappa (b : g') &= (a \bullet_\kappa b : g + g' + \log(1 + ab(1-\kappa^2)) \\ &= \left(\frac{a + b + 2ab\kappa}{1 + ab(1-\kappa^2)} : g + g' + \log(1 + ab(1-\kappa^2))\right).\end{aligned} \tag{39}$$

Proposition 4.2 *The extended product $(a : g) \bullet_\kappa (b : g')$ is defined with a $2\pi i\mathbb{Z}$ ambiguity. However, the $\bullet_\kappa$ product is associative where defined.*

The point of such a fiberwise product is the following:

Proposition 4.3 *For parallel sections $f(\kappa)$, $g(\kappa)$ defined on open subsets, the product $f(\kappa) \bullet_\kappa g(\kappa)$ is also a parallel section where defined.*

Proof. We have only to prove $I_\kappa^{\kappa'}(f \bullet_\kappa h) = I_\kappa^{\kappa'}(f) \bullet_{\kappa'} I_\kappa^{\kappa'}(h)$.

For $f = (\frac{a}{1-a\kappa}, \log(1-a\kappa))$, $h = (\frac{b}{1-b\kappa}, \log(1-b\kappa))$, we see that

$$\begin{aligned} f \bullet_\kappa h = &\Bigg(\frac{a+b}{1-(a+b)\kappa + ab}, \\ &\log((1-a\kappa)(1-b\kappa)\left(1 + \frac{a}{1-a\kappa}\frac{b}{1-b\kappa}(1-\kappa^2)\right)\Bigg) \\ = &\left(\frac{a+b}{1-(a+b)\kappa+ab}, \log(1-(a+b)\kappa + ab)\right).\end{aligned}$$

It is easily seen that $I_\kappa^{\kappa'}(f \bullet_\kappa h) = (\frac{a+b}{1-(a+b)\kappa'+ab}, \log(1-(a+b)\kappa' + ab))$. □

5 The notion of q-number functions

Using Propositions 4.1, 4.3, we define a multiplicative structure on the sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ of parallel sections. A notion of q-number functions which describe quantum observables was introduced in [1], and our notion of parallel sections is stimulated by this idea. From this point of view, we may employ the notation $: f :_\kappa$ for a section f of the bundle $\pi : \prod_{\kappa\in\mathbb{C}} \mathbb{C} \to \mathbb{C}$.

For $f \in \mathcal{A}$, we view κ as an indeterminate. For every $f, g \in \mathcal{A}$, excluding the pair $(f, g) = (\frac{1}{1-\kappa}, \frac{-1}{1+\kappa})$, we define an element $f \bullet g \in \mathcal{A}$ by

$$: f \bullet g :_\kappa = f(\kappa) \bullet_\kappa g(\kappa). \tag{40}$$

Some product formulas on $\mathcal{A}$ are given as follows:

$$0\bullet f=f,\quad \frac{-1}{\kappa}\bullet\frac{-1}{\kappa}=0,\quad \frac{1}{1-\kappa}\bullet f=\frac{1}{1-\kappa},\quad \frac{-1}{1+\kappa}\bullet f=\frac{-1}{1+\kappa},$$

where 0 stands for the singular solution $(\kappa,0)$. These formulas say that $\frac{\pm1}{1\mp\kappa}$ acts like 0 or ∞. Hence $\mathcal{A}$ is viewed naturally as the Riemann sphere with standard multiplicative structure such that $a0=0$, $a\infty=\infty$, but 0∞ is not defined. By the definition of $\bullet_\kappa$, we have $C_\kappa(f\bullet_\kappa g)=C_\kappa(f)C_\kappa(g)$.

Here the correspondence is given by the family of twisted Cayley transforms $\coprod_{\kappa\in\mathbb{C}}C_\kappa:\mathcal{A}\to\mathbb{C}\cup\{\infty\}$. We view $\mathcal{A}$ as a topological space through the identification $\coprod_{\kappa\in\mathbb{C}}C_\kappa$.

The table of correspondence is as follows:

$\mathcal{A}$	0	$\frac{-1}{\kappa}$	$\frac{1}{1-\kappa}$	$\frac{-1}{1+\kappa}$	$\frac{a}{1-a\kappa}$	$\frac{1-a}{1-\kappa+a(1+\kappa)}$	$f(\kappa)$
Image C_κ	1	-1	0	∞	$\frac{1-a}{1+a}$	a	$\frac{1-f(\kappa)(1-\kappa)}{1+f(\kappa)(1+\kappa)}$
singular point	∞	0	1	-1	$\frac{1}{a}$	$\frac{1+a}{1-a}$	—

Note that

$$C_\kappa^{-1}(a)=\frac{1-a}{1-\kappa+a(1+\kappa)}\sim\frac{\frac{1-a}{1+a}}{1-\frac{1-a}{1+a}\kappa}\sim T_0^\kappa C_0^{-1}(a)$$

is a parallel section, and $\frac{1-f(\kappa)(1-\kappa)}{1+f(\kappa)(1+\kappa)}$ is independent of κ for every parallel section f.

5.1 A product on $\tilde{\mathcal{A}}$

Let $\tilde{\mathcal{A}}$ be the space of all parallel sections given in (26), and consider the product $\bullet$ on $\tilde{\mathcal{A}}$ is given by the product formula (39). For $f,f'\in\tilde{\mathcal{A}}$, we set $f=(\kappa,y(\kappa),z(\kappa))$, $f'=(\kappa,y'(\kappa),z'(\kappa))$. $f\bullet g$ is defined as a parallel section defined on the open dense domain where $y(\kappa),y'(\kappa)\neq\infty$.

Note that

$$\begin{aligned}&\Big(\kappa,\frac{a}{1-a\kappa},\log(1-a\kappa)\Big)\bullet\Big(\kappa,\frac{-1}{1+\kappa},\log(1+\kappa)\Big)\\&\quad=\Big(\kappa,\frac{-1}{1+\kappa},\log(1-a)+\log(1+\kappa)\Big),\\&\Big(\kappa,\frac{a}{1-a\kappa},\log(1-a\kappa)\Big)\bullet\Big(\kappa,\frac{1}{1-\kappa},\log(1-\kappa)\Big)\\&\quad=\Big(\kappa,\frac{1}{1-\kappa},\log(1+a)+\log(1-\kappa)\Big).\end{aligned}$$

Although $\frac{\pm1}{1\mp\kappa}$ plays the role of 0 or ∞, the third component depends on a.

For simplicity, we denote in particular

$$\varpi_c = \left(\kappa; \frac{1}{1-\kappa}, c + \log\frac{1}{2}(1-\kappa)\right), \quad \bar{\varpi}_c = \left(\kappa; \frac{-1}{1+\kappa}, c + \log\frac{1}{2}(1+\kappa)\right). \tag{41}$$

It is easy to see that

$$\varpi_c \bullet \varpi_{c'} = \varpi_{c+c'}, \quad \bar{\varpi}_c \bullet \bar{\varpi}_{c'} = \bar{\varpi}_{c+c'},$$

but $\varpi_c \bullet \bar{\varpi}_{c'}$ diverges.

Let $\tilde{\mathcal{A}}_\times$ be the subset of $\tilde{\mathcal{A}}$ excluding the parallel sections $(\kappa; \frac{\pm 1}{1\mp\kappa}, c + \log(1\mp\kappa))$. We also set

$$\tilde{\mathcal{A}}_0 = \tilde{\mathcal{A}}_\times \cup \{\varpi_c\}, \quad \tilde{\mathcal{A}}_\infty = \tilde{\mathcal{A}}_\times \cup \{\bar{\varpi}_c\}.$$

Proposition 5.1 *$\widetilde{\mathcal{A}}$ is closed under the extended product $\bullet_\kappa$, where defined. In particular, $\widetilde{\mathcal{A}}_\times$, $\widetilde{\mathcal{A}}_0$ $\widetilde{\mathcal{A}}_\infty$ are each closed respectively under the $\bullet$-product.*

5.2 The infinitesimal left action

Note that the singular solution $\mathbf{1} = (\kappa, 0, 0) \in \tilde{\mathcal{A}}$ is the multiplicative identity. A neighborhood of $\mathbf{1}$ is given by $(\kappa, \frac{a}{1-a\kappa}, g + \log(1 - a\kappa))$ by taking (a, g) in a small neighborhood of 0. For $g = 0$, we set $f_a = (\kappa, \frac{a}{1-a\kappa}, \log(1 - a\kappa))$. For a parallel section $h = (\kappa, y(\kappa), z(\kappa)) \in \tilde{\mathcal{A}}$, the product $f_a \bullet h$ is given by

$$f_a \bullet h = \left(\frac{a + y(\kappa) + ay(\kappa)\kappa}{1 - a\kappa + ay(\kappa)(1-\kappa^2)}, z + \log(1 - a\kappa + ay(\kappa)(1-\kappa^2))\right),$$

Consider the infinitesimal action

$$\frac{d}{ds}\Big|_{s=0} f_{as} \bullet_\kappa (y, z) = (a(1 + 2y\kappa - y^2(1-\kappa^2)), a(-\kappa + y(1-\kappa^2)).$$

Define for every fixed κ the invariant distribution

$$\tilde{L}_\kappa(y, z) = \{(a((1 + y\kappa)^2 - y^2), a(-\kappa + y(1-\kappa^2))); a \in \mathbb{C}\}.$$

By Proposition 5.1, we have $dI_0^\kappa \tilde{L}_0 I_\kappa^0(y, z) = \tilde{L}_\kappa$.

5.3 The exponential mapping

The equation for the integral curves of the invariant distribution $\tilde{L}_\kappa$ through the identity $(0, 0)$ is

$$\frac{d}{dt}(y(t), z(t)) = (a((1+y(t)\kappa)^2 - y(t)^2), a(-\kappa + y(t)(1-\kappa^2)), \quad (y(0), z(0)) = (0, 0).$$

For the case $\kappa = 0, a = 1$, we have $(y(t), z(t)) = (\tanh t, \log\cosh t)$.

We define $\mathrm{Exp}_\bullet : \mathbb{C} \to \mathcal{A}_\times$ by the family of Exp_κ:

$$\mathrm{Exp}_\kappa t = T_0^\kappa(\tanh t) = \frac{\sinh t}{\cosh t - (\sinh t)\kappa},$$
$$\mathrm{Exp}_\bullet t = (\kappa; T_0^\kappa(\tanh t)) = \Big(\kappa; \frac{\sinh t}{\cosh t - (\sinh t)\kappa}\Big). \tag{42}$$

For a fixed t, $\mathrm{Exp}_\bullet t$ is a parallel section with the exponential law

$$\mathrm{Exp}_\bullet s \bullet \mathrm{Exp}_\bullet t = \mathrm{Exp}_\bullet(s+t), \quad \text{and} \quad \mathrm{Exp}_\bullet(s+2\pi i) = \mathrm{Exp}_\bullet s.$$

For the extended product, let $\widetilde{\mathrm{Exp}}_0\, t = (\tanh t; \log\cosh t)$, and let

$$\widetilde{\mathrm{Exp}}_\kappa\, t = I_0^\kappa \widetilde{\mathrm{Exp}}_0\, t = \Big(\frac{\sinh t}{\cosh t - (\sinh t)\kappa}, \log(\cosh t - (\sinh t)\kappa)\Big).$$

Although $\widetilde{\mathrm{Exp}}_\kappa$ is not defined for all $t \in \mathbb{C}$, viewing κ as an indeterminate permits us to define the exponential mapping $\widetilde{\mathrm{Exp}}_\bullet : \mathbb{C} \to \widetilde{\mathcal{A}}_\times$ by

$$\widetilde{\mathrm{Exp}}_\bullet t = \left(\kappa; \frac{\sinh t}{\cosh t - (\sinh t)\kappa}, \log(\cosh t - (\sinh t)\kappa)\right). \tag{43}$$

This is a parallel section with the exponential law

$$\widetilde{\mathrm{Exp}}_\bullet s \bullet \widetilde{\mathrm{Exp}}_\bullet\, t = \widetilde{\mathrm{Exp}}_\bullet(s+t).$$

For a closer look at $\widetilde{\mathrm{Exp}}_\bullet$, we define the logarithmic function $\log(\cosh t - (\sinh t)\kappa)$ by the integral

$$\log(\cosh t - (\sinh t)\kappa) = \int_0^t \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds \tag{44}$$

by setting the initial condition $\widetilde{\mathrm{Exp}}_\bullet 0 = (\kappa, 0, 0)$.

Using this, we see the following:

Proposition 5.2 *If the initial condition* $\widetilde{\mathrm{Exp}}_\bullet 0 = (\kappa, 0, 0)$ *is requested,* $\widetilde{\mathrm{Exp}}_\bullet : \mathbb{C} \to \widetilde{\mathcal{A}}_\times$ *is an injective homomorphism, that is,* $\widetilde{\mathrm{Exp}}_\bullet z = \widetilde{\mathrm{Exp}}_\bullet w$ *in* $\widetilde{\mathcal{A}}$ *implies* $z = w$ *and the exponential law* $\widetilde{\mathrm{Exp}}_\bullet(z+w) = \widetilde{\mathrm{Exp}}_\bullet z \bullet \widetilde{\mathrm{Exp}}_\bullet w$ *holds.*

Proof. We have only to show the injectivity. Suppose $\widetilde{\mathrm{Exp}}_\bullet z = \widetilde{\mathrm{Exp}}_\bullet w$. Then, we see that for every κ,

$$e^z = e^w, \quad \int_0^z \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds = \int_0^w \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds.$$

It follows that $w = z + 2\pi i n$ for some n. We will show that $n = 0$. So suppose $n \neq 0$, but we assume $n = 1$ for simplicity. The second identity gives

$$\int_0^{z+2\pi i} \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds - \int_0^z \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds$$
$$= \int_z^{z+2\pi i} \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds = 0.$$

Set $z = x + iy$. Then, $\int_z^{z+2\pi i} \frac{\sinh s-(\cosh s)\kappa}{\cosh s-(\sinh s)\kappa} ds$ is the contour integral

$$\int_{|w|=1} \frac{(1-\kappa)e^{2x}w-(1+\kappa)}{(1-\kappa)e^{2x}w+(1+\kappa)} \cdot \frac{dw}{w} = \begin{cases} -2\pi i, & e^{-2x}\frac{|\kappa+1|}{|\kappa-1|} > 1, \\ 2\pi i, & e^{-2x}\frac{|\kappa+1|}{|\kappa-1|} < 1. \end{cases}$$

Since this does not vanish, we must have $n = 0$. □

We see also that for every $\alpha \in \mathbb{C}$,

$$\widetilde{\mathrm{Exp}}_{\bullet}^{(\alpha)} s = \Big(\kappa; \frac{\sinh s}{\cosh t - (\sinh s)\kappa}, \log e^{\alpha s}(\cosh s - (\sinh s)\kappa)\Big)$$

satisfies the exponential law.

Using this formula, it is easily seen that for $t \in \mathbb{R}$,

$$: \varpi_0 :_\kappa = \lim_{t\to\infty} \Big(\frac{\sinh t}{\cosh t - (\sinh t)\kappa}, \log e^{-t}(\cosh t - (\sinh t)\kappa)\Big),$$

$$: \bar{\varpi}_0 :_\kappa = \lim_{t\to-\infty} \Big(\frac{\sinh t}{\cosh t - (\sinh t)\kappa}, \log e^{t}(\cosh t - (\sinh t)\kappa)\Big).$$

We end by noting that $\tilde{\mathcal{A}}_\times$ is a strange object, which one cannot treat as a usual manifold. $\tilde{\mathcal{A}}_\times$ is a group-like object and the mapping which forgets the last component for the map $\tilde{\mathcal{A}}_\times \to \mathcal{A}_\times$ is a homomorphism onto $\mathcal{A}_\times \cong \mathbb{C}_\times$.

References

1. G. S. Agarwal and E. Wolf, *Calculus for functions of noncommuting operators and general phase-space method of functions*, Physical Review D, **2** (1970), 2161-2186.
2. J.-L. Brylinski, *Loop spaces, Characteristic classes and Geometric quantization*. Progress in Mathematics **107**, Birkhäuser, 1992.
3. A. Connes, *Noncommutative geometry*. Academic Press, 1994.
4. I. M. Gel'fand, and G. E. Shilov, *Generalized Functions, II*. Academic Press, 1968.
5. G. Landi, *An introduction to noncommutative spaces and their geometries*, Lecture Note Series in Physics, New Series in: Monographs, vol. 51, Springer-Verlag, 1997.
6. Y. Manin, *Real Multiplication and noncommutative geometry*, arXiv math. AG/0202109,
7. R. Melrose, *Star products and local line bundles,* Annales de l'Institut Fourier, **54** (2004), 1581–1600
8. M. Murray, *Bundle gerbes,* J. London Math. Soc. (2), **54** (1996), 403–416.
9. H. Omori, *One must break symmetry in order to keep associativity,* Banach Center Publ. **55** (2002), 153–163.
10. H. Omori, *Toward Geometric Quantum Theory*, From Geometry to Quantum Mechanics, Progr. in Math. **252** (2006), 213–251.
11. H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, *Deformation quantization of Fréchet-Poisson algebras—Convergence of the Moyal product*, Math. Phys. Studies **22** (2000), 233–246.

12. H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, *Strange phenomena related to ordering problems in quantizations*, J. Lie Theory **13**, (2003), 481–510.
13. H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka, *Star exponential functions as two-valued elements*, The breadth of Symplectic and Poisson geometry, Progress in Mathematics **232** (2004), 483–492.
14. A. Weinstein *The Maslov gerbe*, Lett. Math. Phys., **69**, (2004), 3–9.