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Full Symmetry Algebra for ODEs and Control Systems ⋆

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Abstract. A description of the full symmetry algebra (i.e., including higher symmetries) for a general nonlinear system of ordinary differential equations is given in terms of its general solution and differential constants. More precisely, the full symmetry algebra of a system is a module over the ring of its differential constants; the module is generated by partial derivatives of the general solution w.r.t. independent constants. Given a general solution, this description is both effective and explicit. Special solutions, such as an envelope of a family of solutions, are described naturally in this context. These results are extended to control systems; in this case, differential constants become operators on controls. Examples are provided.

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1. Introduction

The study of symmetries of ordinary differential equation (ODE) was initiated by Sophus Lie [5] and has a long history described briefly in [8]. The latest results were obtained in [7] and [3].

To find symmetries for a particular equation still remains a hard task. The present paper deals, however, with another problem. We give a full description of the symmetry algebra of a system of ODE in a nondegenerate situation using the general solution whose (local) existence is guaranteed by classical theorems. For a linear system of ODEs, this result was obtained in [8] and it was recently generalized to the normal form scalar ODEs in [3].⋆⋆

Given a general solution, our description of the symmetry algebra is both effective and explicit: the full symmetry algebra of a system is a module over the ring of its differential constants; the module is generated by partial derivatives of the general solution with respect to independent constants. Special solutions, such as the envelope of a family of solutions, are described naturally in this context. (We note that [1] contains an implicit description of ODE symmetries, see Remark 3 below.)

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⋆⋆ See also Yumaguzhin's paper in this special issue (editor's note).

Of course, these results are of little practical importance since there is no need in symmetries when a general solution is known. Symmetries are used to obtain new solutions, not the other way round. Yet the interconnection between differential invariants, symmetries and a general solution is quite transparent in the case of ODEs and sometimes may be used as a model applicable in other situations.

In this paper, we give two such applications. First, we describe the symmetries of a boundary/initial value problem for a one-dimensional wave equation. The main second application deals with symmetries of control systems. In both cases, differential invariants become nonlocal ones.

The paper is organized as follows. Section 2 describes the full symmetry algebra for a general nonlinear system of ordinary differential equations. It also contains a description of special solutions as invariants of basic symmetries for a given general solution (Subsection 2.3) and examples (Subsection 2.4). Section 3 is an application of this approach to control systems; examples are also provided.

2. Full Symmetry Algebra for a General Nonlinear Ordinary Differential Equation and a System of Equations

2.1. GENERAL SOLUTION AND DIFFERENTIAL CONSTANTS

We begin with trivialities to introduce notation. Let $\mathcal{E}$ denote a general scalar ordinary differential equation of nth order:

$$y^{(n)} - F(x, y, y', \ldots, y^{(n-1)}) = 0. \tag{1}$$

Its *general solution* (or a *general integral*) is of the form

$$\Phi(x, y, c_1, c_2, \ldots, c_n) = 0. \tag{2}$$

When (2) is solved with respect to y, we get

$$y = f(x, c_1, \ldots, c_n); \tag{3}$$

almost any solution of (1) is obtained from (3) by a proper choice of constants c_i. (The solution that is not produced by the general one is called a *special solution*. Such solutions are discussed below.)

Remark 1. Existence of a general solution of (1) is by no means guaranteed. Yet if F is continuously differentiable, the classical theorem on a differentiable dependence of a solution of ODE on initial data guaranties existence of a local form of (2) in a neighborhood of a chosen solution. In this local form the initial datum $y^{(k)}(x_0)$ is taken as a differential constant c_k, $k = 0, \ldots, n-1$. Below we proceed with a global general solution, but it is always possible to make the corresponding local statement.

Differentiating (3) by x, we obtain the following system of n independent equations

$$\begin{aligned} & y = f(x, c_1, \dots, c_n), \\ & y' = f'(x, c_1, \dots, c_n), \\ & \dots\dots\dots\dots\dots\dots\dots \\ & y^{(n-1)} = f^{(n-1)}(x, c_1, \dots, c_n) \end{aligned} \tag{4}$$

(further differentiation produces dependent equations, since $y^{(k)}$, $k \geqslant n$, are expressed in $y^{(i)}$, $i < n$, via (1)).

One can obtain an expression (not necessary explicit) for c_i solving (4). Thus

$$c_i = c_i(x, y, y', \dots, y^{(n-1)}), \quad i = 1, \dots, n. \tag{5}$$

In this way, all c_i are differential constants of order $< n$. In other words, they are differential operators of order $n-1$, or functions on the jet space $J^{n-1}(\mathbb{R})$.

In the case of a system of m differential equations,

$$\mathbf{y}^{(n)} - \mathbf{F}(x, \mathbf{y}, \mathbf{y}', \dots, \mathbf{y}^{(n-1)}) = 0, \tag{6}$$

where $\mathbf{y} = (y_1, \dots, y_m)$, $\mathbf{F} = (F_1, \dots, F_m)$, the general solution is of the form

$$\Phi_k(x, \mathbf{y}, c_1, c_2, \dots, c_{mn}) = 0, \quad k = 1, \dots, m, \tag{7}$$

or

$$\mathbf{y} = \mathbf{f}(x, c_1, \dots, c_{mn}). \tag{8}$$

Almost any solution of (6) is obtained from (8) by a proper choice of constants c_i.

2.2. FULL SYMMETRY ALGEBRA

By definition of a solution, if the right-hand side of (3) is substituted for y in (1), we obtain the identity

$$f^{(n)} - F(x, f, f', \dots, f^{(n-1)}) \equiv 0. \tag{9}$$

Hence

$$\frac{\partial}{\partial c_i}\big(f^{(n)} - F(x, f, f', \dots, f^{(n-1)})\big) = 0 \tag{10}$$

for all i, or

$$\left(D^n - \sum_{j=1}^{n} \frac{\partial F(x, y, y', \dots, y^{(n-1)})}{\partial y_j} D^j\right)\Bigg|_{y=f(x,y,c_1,\dots,c_n)} f_{c_i} = 0, \tag{11}$$

where $D = \mathrm{d}/\mathrm{d}x$ is the total derivative with respect to x and f_{c_i} denotes the partial derivative over c_i.

Recall that

$$\mathcal{L}_{y^{(n)}-F} \stackrel{\text{def}}{=} D^n - \sum_{j=1}^{n} \frac{\partial F(x, y, y', \dots, y^{(n-1)})}{\partial y_j} D^j \tag{12}$$

is called the *universal linearization* of the operator $y^{(n)} - F$ and that a solution ϕ of the equation

$$(\mathcal{L}_{y^{(n)}-F})\varphi|_{\mathcal{E}} = 0 \tag{13}$$

is a *symmetry* of $\mathcal{E}$.

THEOREM 1. *The partial derivatives f_{c_i}, $i = 1, \dots, n$, form a full functionally independent basis of symmetries for Equation* (1).

Proof. The difference between (11) and (13) is that the same operator is restricted to formally different objects. However, note that the set

$$\{y = f(x, c_1, \dots, c_n), y' = f'(x, y, c_1, \dots, c_n), \dots \mid \forall c_i \in \mathbb{R}\} \subset J^n(\mathbb{R})$$

coincides with $\mathcal{E}$. Indeed, codim $J^n(\mathbb{R}) = n + 2$, dim $\mathcal{E} \subset J^n(\mathbb{R}) = 1$, so dim $\mathcal{E} = n + 1$. It follows from the existence theorem for ordinary differential equations that there is a solution containing any initial value point $x_0, y_0, y'_0, \dots, y_0^{n-1} \in \mathcal{E}$. Now, since (3) produces almost all solutions and

$$\dim\{y = f(x, y, \mathbf{c}), y' = f'(x, y, \mathbf{c}), \dots \mid \forall \mathbf{c} \in \mathbb{R}^n\} = n + 1,$$

we conclude that (11) coincides with the symmetry equation (13) almost everywhere on $\mathcal{E}$. Therefore, f_{c_i}, $i = 1, \dots, n$, are symmetries of Equation (1). Moreover, they form a basis of the symmetry algebra.

Indeed, let φ be a symmetry. Then it defines a flow on the set of solutions by the formula

$$\frac{\partial y}{\partial \tau} = \varphi|_y, \tag{14}$$

where $y = f(x, y, c_1, \dots, c_n)$. It can be solved (see [2]), and a solution of this equation is a one-parameter family of solutions of (1). By (3), it has the form

$$y = f(x, c_1(\tau), \dots, c_n(\tau)). \tag{15}$$

On the other hand, differentiating (15) by τ, we obtain (via (14)) that

$$\varphi|_y = \left.\left(\sum_{i=1}^{n} \frac{\partial c_i}{\partial \tau} f_{c_i}\right)\right|_y \tag{16}$$

for any solution y of Equation (1). Therefore,

$$\varphi = \sum_{i=1}^{n} \frac{\partial c_i}{\partial \tau} f_{c_i} \tag{17}$$

holds everywhere on $\mathcal{E}$.

Note that the derivatives $\partial c_i/\partial\tau|_y$ depend on y, that is, on $c_1, \dots, c_1$, which are functions on $J^{n-1}(\mathbb{R})$ by virtue of (5). Since any choice of arbitrary functions $c_i(\tau)$ define some symmetry by (15), the functions $\partial c_i/\partial\tau|_y$ are also arbitrary.

Thus, we got the general form of a symmetry for Equation (1):

$$\varphi = \sum_{i=1}^{n} A_i(c_1, \dots, c_n) \frac{\partial}{\partial c_i} f(x, y, c_1, \dots, c_n); \tag{18}$$

here f is a general solution, A_i are arbitrary functions and c_i are functions on $J^{n-1}(\mathbb{R})$ given by system (4).

Formula (18) completes the proof of the theorem.

Remark 2. A full symmetry algebra is a module over the ring of the equation differential constants. The module is generated by partial derivatives of a general solution over independent constants.

Remark 3. Formula (18) gives a representation of the algebra of vector fields on $\mathbb{R}^n$ in the full symmetry algebra of (6) by the isomorphism

$$\sum_{i=1}^{n} A_i(c_1, \dots, c_n) \frac{\partial}{\partial c_i} \longleftrightarrow \sum_{i=1}^{n} A_i(c_1, \dots, c_n) \frac{\partial}{\partial c_i} f(x, c_1, \dots, c_n) \tag{19}$$

(on the left-hand side, c_i are coordinates in $\mathbb{R}^n$; on the right-hand side they denote differential invariants (5) of (1) or special functions on $J^{n-1}(\mathbb{R})$).

It was first stated in [1, Proposition 4.6] that the symmetry algebra for an ordinary differential equation coincides with the algebra of vector fields on the solution space. Theorem 1 gives an explicit representation of this correspondence, provided the general solution is known. Yet its existence is guaranteed only locally; hence, formula (19) is also generally local.

Remark 4. Theorem 1 generalizes easily to the case of system of differential equations (6). Locally, its full symmetry algebra is isomorphic to the algebra of vector fields on $\mathbb{R}^{mn}$: the representation is given by

$$\sum_{i=1}^{mn} A_i(c_1, \dots, c_{mn}) \frac{\partial}{\partial c_i} \longleftrightarrow \partial\mathbf{f} \times \mathbf{A},$$

where $\partial\mathbf{f}$, $\mathbf{A}$ are respectively $m \times mn$ and $mn \times 1$ matrices with matrix elements given by the formulas

$$(\partial\mathbf{f})_{j,i} = \frac{\partial f_j}{\partial c_i}, \qquad (\mathbf{A})_i = A_i.$$

A version of Theorem 1 is also valid in the case of an even more general system of ordinary differential equations,

$$y_j^{(n_j)} - F_j(x, y_1, y_1', \ldots, y_1^{(n_1-1)}, \ldots, y_m, y_m', \ldots, y_1^{(n_m-1)}) = 0.$$

It is not hard to write down the corresponding isomorphism between vector fields on the solution space and symmetries in this case too. Yet the formula is awkward to read and therefore it is omitted here. See [7] for relevant technicalities.

Let us call f_{c_i}, $i = 1, \ldots, n$, *basic symmetries*. They correspond to the flows $y(\tau) = f(x, c_1, \ldots, c_i + \tau, \ldots, c_n)$. Thus, in the case of explicit general solution (3) basic symmetries are $f_{c_i} = y_{c_i}$.

Remark 5. If a general solution of (1) is given in implicit form (2), then

$$\frac{\mathrm{d}\Phi}{\mathrm{d}c_i} = \frac{\partial \Phi}{\partial c_i} + \frac{\partial \Phi}{\partial y}\frac{\partial y}{\partial c_i} = 0.$$

It follows immediately that basic symmetries are given by

$$y_{c_i} = -\left(\frac{\partial \Phi}{\partial c_i}\right) \Big/ \left(\frac{\partial \Phi}{\partial y}\right). \tag{20}$$

This formula generalizes in a straightforward way for the case of a system of equations.

2.3. SPECIAL AND INVARIANT SOLUTIONS

Invariant or *self-similar* solution y of (1) is a solution that satisfies the condition $\varphi(y) = 0$ for some symmetry φ of the form (18). Hence, an invariant solution satisfy the system of equations

$$\begin{aligned} &\mathcal{E}(f) = y^{(n)} - F(x, y, y', \ldots, y^{(n-1)}) = 0, \\ &\phi(y) = \sum_{i=1}^{n} A_i(c_1(y), \ldots, c_n(y)) \frac{\partial}{\partial c_i} f(x, y, c_1(y), \ldots, c_n(y)) = 0. \end{aligned} \tag{21}$$

Since c_i are constants on solutions of (1), so are $A_i(c_1(y), \ldots, c_n(y))$. Thus (21) is simply

$$\begin{aligned} &\mathcal{E}(f) = y^{(n)} - F(x, y, y', \ldots, y^{(n-1)}) = 0, \\ &\phi(y) = \sum_{i=1}^{n} A_i f_{c_i}(x, y, c_1, \ldots, c_n) = 0 \end{aligned} \tag{22}$$

with constant A_i and c_i. The second condition in (22) means that basic symmetries are linearly dependent of an invariant solution. If rank $\{f_{c_1}, \ldots, f_{c_n}\}|_y = n - k$, it is natural to introduce the notion of a k-invariant solution.

Remark 6. Recall that f_{c_i} represent independent vector fields on $\mathbb{R}^n$. In this way the structure of invariant solutions of ordinary differential equation is connected with the structure of degenerate points of a system of n independent vector fields on $\mathbb{R}^n$.

Consider a simple case of (22),

$$\begin{aligned} & y^{(n)} - F(x, y, y', \dots, y^{(n-1)}) = 0, \\ & f_{c_i} = 0. \end{aligned} \tag{23}$$

Its solution is a fixed point of the flow $c_i \to c_i + \tau$. Geometrically, such a solution is the envelope for the family of solutions generated by this flow, see Subsection 2.4.

2.4. EXAMPLES

EXAMPLE 1. Consider the equation $y'' + \frac{9}{8}(y')^4 = 0$. It is invariant with respect to the translations in both x and y. Its general solution is

$$\Phi(x, y, c_1, c_2) = (y + c_1)^3 - (x + c_2)^2 = 0,$$

or

$$y = f(x, c_1, c_2) = (x + c_2)^{2/3} - c_1.$$

Therefore, its basic symmetries are $f_{c_1} = -1$, $f_{c_2} = \frac{2}{3}(x + c_2)^{-1/3}$. They depend on the differential constants c_1, c_2 that may be found from system (4),

$$(y + c_1)^3 = (x + c_2)^2, \qquad 3y'(y + c_1)^2 = 2(x + c_2).$$

It follows that

$$c_1 = \left(\frac{2}{3y'}\right)^2 - y, \qquad c_2 = \left(\frac{2}{3y'}\right)^3 - x.$$

Now, basic symmetries come to $f_{c_1} = -1$, $f_{c_2} = y'$, which are (not surprisingly) translations in y and x respectively.

So the general symmetry for this equation is of the form (18)

$$\begin{aligned} \varphi &= A_1(c_1, c_2) f_{c_1} + A_2(c_1, c_2) f_{c_2} \\ &= -A_1\left(\left(\frac{2}{3y'}\right)^2 - y, \left(\frac{2}{3y'}\right)^3 - x\right) + \\ &\quad + A_2\left(\left(\frac{2}{3y'}\right)^2 - y, \left(\frac{2}{3y'}\right)^3 - x\right) y', \end{aligned}$$

where A_1, A_2 are arbitrary functions in two variables.

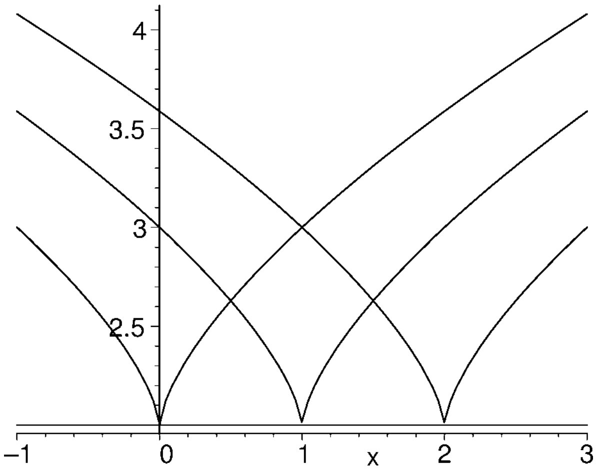

Figure 1. Enveloping invariant solution $y = 2$.

Invariant solutions must satisfy system (22)

$$A + y'B = 0, \qquad y'' + \frac{9}{8}(y')^4 = 0$$

for some constants A, B. It follows that $y' = 0$, so $y = \text{const}$. This is a special solution (i.e., it is not obtained from the general integral). Each special solution is the envelope for the family $(y - \text{const})^3 - (x + c_2)^2 = 0$ for all c_2, see Figure 1.

EXAMPLE 2. Consider the equation

$$yy'' + 2(y'^2 + 1) = 0.$$

The general integral in this case is as follows:

$$\Phi = \int \frac{y^2\,\mathrm{d}y}{\sqrt{c_1 - y^4}} \pm x + c_2.$$

Basic symmetries are obtained here by formula (20):

$$\begin{aligned} \varphi_1 &= -\frac{\Phi_{c_1}}{\Phi_y} = \frac{1}{2}\frac{\sqrt{c_1 - y^4}}{y^2}\int \frac{y^2\,\mathrm{d}y}{(\sqrt{c_1 - y^4})^3}, \\ \varphi_2 &= -\frac{\Phi_{c_2}}{\Phi_y} = -\frac{\sqrt{c_1 - y^4}}{y^2}. \end{aligned}$$

To obtain the final form for these symmetries it remains to express differential constants as functions on $J^1(\mathbb{R})$ using (4):

$$\int \frac{y^a\,\mathrm{d}y}{\sqrt{c_1 - y^{2a}}} \pm x + c_2 = 0, \qquad y'\frac{\sqrt{c_1 - y^4}}{y^2} \pm 1 = 0.$$

It follows immediately that

$$c_1 = y^4(y'^2 + 1), \qquad c_2 = \pm \int \mathrm{d}x \mp x = c_2.$$

Substituting these expressions into basic symmetries, we obtain

$$\varphi_1 = \frac{y'}{2} \int \frac{\mathrm{d}y}{y'^3 y^4}, \qquad \varphi_2 = y'.$$

Note that φ_1 is a nonlocal symmetry.

EXAMPLE 3. Linear equations (cf. [7]) $y^{(n)} + \sum_{i=0}^{n-1} a_i(x) y^{(i)} = 0$. Here the general integral is if the form $y = \sum_{i=1}^{n} c_i f_i(x)$, where $f_i(x)$ are independent solutions, i.e., their Wronskian is nonzero:

$$W = W(f_1, \dots, f_i, \dots, f_n) = \begin{vmatrix} f_1 & \dots & f_i & \dots & f_n \\ f_1' & \dots & f_i' & \dots & f_n' \\ \dots & & & & \\ f_1^{(n-1)} & \dots & f_i^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix} \neq 0.$$

Independent solutions f_i coincide with basic symmetries in this case: $f_i = f_{c_i}$.

Differential constant c_i is given by the formula

$$c_i(y, y', \dots, y^{(n-1)}) = \frac{W_i}{W},$$

where W_i is obtained from W by changing the entries of the ith column of W for $y, y', \dots, y^{(n-1)}$ in respective order.

The general form of a symmetry is

$$\varphi = \sum_{i=1}^{n} A_i\left(\frac{W_1}{W}, \dots, \frac{W_i}{W}, \dots, \frac{W_n}{W}\right) f_i(x).$$

EXAMPLE 4. Linear boundary problem

$$u_{tt} - u_{xx} = 0, \qquad u|_{x=0} = u|_{x=\pi} = 0.$$

This example is a rather wide generalization of the previous one. Fourier general solution on $[0, \pi]$ for this string is

$$u = \sum_{n=0}^{\infty} \sin nx(a_n \cos nt + b_n \sin nt),$$

where a_n, b_n are constants, but neither differential nor local: the Fourier coefficient formula states that

$$a_n = \frac{2}{\pi} \int_0^{\pi} u|_{t=0} \sin nx \, \mathrm{d}x, \qquad b_n = \frac{2}{\pi n} \int_0^{\pi} u_t|_{t=0} \sin nx \, \mathrm{d}x. \tag{24}$$

A general form of symmetries is given by

$$\varphi = \sum_{n=0}^{\infty} \sin nx (A_n(a_1, b_1, \dots, a_i, b_i, \dots) \cos nt + \\ + B_n(a_1, b_1, \dots, a_i, b_i, \dots) \sin nt).$$

Here A_n, B_n are arbitrary functions depending on any finite number of a_i, b_j given by (24).

3. Full Symmetry Algebra for a General Control System

3.1. GENERAL SOLUTION AND DIFFERENTIAL CONSTANTS

Consider a first-order control system

$$\mathbf{y}' = \mathbf{F}(x, \mathbf{y}, \mathbf{v}(x)), \tag{25}$$

where $\mathbf{y} \in \mathbb{R}^m$ is an m-vector of unknown functions and $\mathbf{v}(\mathbf{x}) \in \mathbb{R}^k$ is a k-vector of control functions.

With any fixed choice of controls, (25) comes to (6), where $n = 1$. Thus, the general solution of (25) is of the form

$$\mathbf{y} = \mathbf{f}(x, c_1, \dots, c_m, \mathbf{v}(x)), \tag{26}$$

where c_i are constants. From (26) it follows that there exists (at least an implicit) dependence

$$c_i = c_i(x, \mathbf{y}(x), \mathbf{y}'(x), \mathbf{v}(x)), \quad i = 1, \dots, m, \tag{27}$$

of constants c_i on x, $\mathbf{y}(x)$, $\mathbf{y}'(x)$, $\mathbf{v}(x)$. Both $\mathbf{f}$ and c_i are operators on $\mathbf{v}$. Examples below show that these operators may be nonlocal.

3.2. FULL SYMMETRY ALGEBRA

Technically, Equation (25) is an equation with two types of dependent variables, that is, $\mathbf{y}$ and $\mathbf{v}$. Let us put this equation in the form

$$\mathcal{H}(\mathbf{y}, \mathbf{v}) = \mathbf{y}' - \mathbf{F}(x, \mathbf{y}, \mathbf{v}(x)) = 0.$$

The symmetry equation in this case is as follows:

$$(D - \mathbf{F_y})\mathbf{A} - \mathbf{F_v}\mathbf{B}|_{\mathcal{H}=0} = 0, \tag{28}$$

where $(\mathbf{A}, \mathbf{B})$ is a symmetry (if it defines a flow, then $\mathbf{y}_\tau = \mathbf{A}$, $\mathbf{v}_\tau = \mathbf{B}$). Besides, $\mathbf{F_y}$ is an $m \times m$ matrix with the entries $(F_i)_{y_j}$ and $\mathbf{F_v}$ is an $m \times k$ matrix with the entries $(F_i)_{v_j}$.

It is convenient to put (28) in the vector form,

$$(D - \mathbf{F_y}, -\mathbf{F_v}) \cdot \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \Bigg|_{\mathcal{H}=0} = 0. \tag{29}$$

The left factor in this formula is the linearization of $\mathcal{H}$ denoted by $\mathcal{L}_{\mathcal{H}}$.

THEOREM 2. *Partial derivative vectors*

$$\begin{pmatrix} \mathbf{f_c} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} \mathbf{f_v} \\ \mathrm{I} \end{pmatrix} \tag{30}$$

form a full functionally independent basis of symmetries for Equation (25).

Proof. In terms of the general solution, the general form of a flow on the set of solutions of Equation (25) is given by the formula

$$\mathbf{y} = \mathbf{f}(x, c_1(\tau), \dots, c_m(\tau), \mathbf{v}(x, \tau)), \tag{31}$$

where τ is a parameter of the flow. Since (31) is a solution for any τ, we have

$$\begin{aligned} &\mathbf{f}'(x, c_1(\tau), \dots, c_m(\tau), \mathbf{v}(x, \tau)) - \\ &\quad - \mathbf{F}(x, \mathbf{f}(x, c_1(\tau), \dots, c_m(\tau), \mathbf{v}(x, \tau)), \mathbf{v}(x, \tau)) = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tau}\big(\mathbf{f}'(x, c_1(\tau), \dots, c_m(\tau), \mathbf{v}(x, \tau)) - \\ &\quad - \mathbf{F}(x, \mathbf{f}(x, c_1(\tau), \dots, c_m(\tau), \mathbf{v}(x, \tau)), \mathbf{v}(x, \tau))\big) = 0. \end{aligned}$$

It follows that

$$\begin{aligned} &((D - \mathbf{F_y})(\mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau) - \mathbf{F_v}\mathbf{v}_\tau)|_{\mathcal{H}=0} \\ &\quad = (D - \mathbf{F_y}, -\mathbf{F_v}) \cdot \begin{pmatrix} \mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau \\ \mathbf{v}_\tau \end{pmatrix} \Bigg|_{\mathcal{H}=0} \\ &\quad = \mathcal{L}_{\mathcal{H}} \begin{pmatrix} \mathbf{f_c} \cdot \mathbf{c}_\tau + \mathbf{f_v} \cdot \mathbf{v}_\tau \\ \mathbf{v}_\tau \end{pmatrix} \Bigg|_{\mathcal{H}=0} = 0. \end{aligned} \tag{32}$$

Thus, the general solution of the symmetry equation is (cf. (17))

$$\begin{pmatrix} \mathbf{f_c} \\ 0 \end{pmatrix} \cdot \mathbf{c}_\tau + \begin{pmatrix} \mathbf{f_v} \\ \mathrm{I} \end{pmatrix} \cdot \mathbf{v}_\tau. \tag{33}$$

Here $\mathbf{f_c} = (f_i)_{c_j}$ is an $m \times m$ matrix, $\mathbf{f_v}$ is an $m \times k$ matrix and I is the $k \times k$ identity matrix.

To obtain the general form of the symmetry for Equation (25) it remains to notice that

(i) $\mathbf{v}_\tau$ is an arbitrary vector-function;

(ii) for any fixed $\mathbf{v}$, Equation (25) coincides with (6), so $c_{i\tau}$ are the components of a vector field on the solution space for this $\mathbf{v}$. Therefore, $c_{i\tau} = \mathcal{A}_i(\mathbf{c}, \mathbf{v})$ are arbitrary functions;

(iii) c_i are constants on solution of (25) given by (27).

Finally, we can write down the general form of a symmetry for (25):

$$\varphi = \begin{pmatrix} \mathbf{f_c} \\ 0 \end{pmatrix} \cdot \mathcal{A}(\mathbf{c}, \mathbf{v}(x)) + \begin{pmatrix} \mathbf{f_v} \\ \mathrm{I} \end{pmatrix} \cdot \mathbf{u}(x). \tag{34}$$

Here $\mathcal{A}(\mathbf{c}, \mathbf{v}(x))$ and $\mathbf{u}(x)$ are arbitrary proper-sized matrices.

Remark 7. Generally, solution (26) and its derivatives, as well as expressions of the type $\mathcal{A}(\mathbf{c}, \mathbf{v}(x))$ or $\mathbf{u}(x)$, are operators on $\mathbf{v}(x)$. If they are differential operators of order l, we obtain lth order higher symmetries by formula (34).

3.3. EXAMPLES

EXAMPLE 5. A linear scalar equation

$$y' = xy + v(x). \tag{35}$$

The general solution in this case is

$$y = \mathrm{e}^{x^2/2} \int_{x_0}^{x} \mathrm{e}^{-t^2/2} v(t)\,\mathrm{d}t + c\mathrm{e}^{x^2/2}.$$

Thus,

$$c = y\mathrm{e}^{-x^2/2} - I(x), \quad \text{where } I(x) = \int_{x_0}^{x} \mathrm{e}^{-t^2/2} v(t)\,\mathrm{d}t,$$

is constant on any solution of (35).

Therefore, from (34) it follows that the general form of the symmetry in this example is

$$\varphi = \begin{pmatrix} \mathrm{e}^{x^2/2} \\ 0 \end{pmatrix} \mathcal{A}(y\mathrm{e}^{-x^2/2} - I(x), v(x)) + \begin{pmatrix} \mathrm{e}^{x^2/2} \int_{x_0}^{x} \mathrm{e}^{-t^2/2}[\bullet]\,\mathrm{d}t \\ 1 \end{pmatrix} u(x). \tag{36}$$

Here $\mathcal{A}(c, v(x))$ and $u(x)$ are arbitrary operator and function respectively;

$$f_v = \mathrm{e}^{x^2/2} \int_{x_0}^{x} \mathrm{e}^{-t^2/2}[\bullet]\,\mathrm{d}t$$

is an operator acting on $u(x)$ by the formula

$$\left(\mathrm{e}^{x^2/2} \int_{x_0}^{x} \mathrm{e}^{-t^2/2}[\bullet]\,\mathrm{d}t\right) u(x) = \mathrm{e}^{x^2/2} \int_{x_0}^{x} \mathrm{e}^{-t^2/2} u(t)\,\mathrm{d}t.$$

This example shows that, since a general solution $f = f(v)$ of a control system is an operator on controls, f_v in formula (34) is a linearization of this operator.

In Theorem 2 the flow of the control function v is arbitrary, so v is a free functional parameter. Suppose now it is subjected to some differential constraint $v_\tau = G(x, v, v', \dots, v^{(r)})$.*

If r is the maximal order of the derivative of v entering such a constraint, then y_τ can depend on $v^{(s)}$, $s \leqslant r-1$, only, cf. [6]. The next example is an illustration of this general statement.

EXAMPLE 6. $v_\tau = v$, $c_\tau = 0$. From (29) and (33) we obtain

$$(D - F_y, -F_v) \cdot \begin{pmatrix} f_v v \\ v \end{pmatrix}\Bigg|_{\mathcal{H}=0} = 0. \tag{37}$$

The highest-order derivative of v entering this equation is v'. It enters linearly and its coefficient is $f_v + v f_{vv}$, so it must vanish. Solving $f_v + v f_{vv} = 0$, we obtain $f_v = 1/v$ and $y_\tau = f_v v = 1$. In particular, it does not depend on v in a perfect accordance with the result of [6].

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References

1. Bocharov, A. V., Chetverikov, V. N., Duzhin, S. V., Khor'kova, N. G., Krasil'shchik, I. S., Samokhin, A. V., Torkhov, Yu. N., Verbovetsky, A. M. and Vinogradov, A. M.: *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, Amer. Math. Soc., Providence, RI, 1999. Edited and with a preface by Krasil'shchik and Vinogradov.
2. Chetverikov, V. N.: On the structure of integrable $\mathcal{C}$-fields, *Differential Geom. Appl.* **1** (1991), 309–325.
3. Doyle, P. W.: Symmetry and ordinary differential constraints, *Internat J. Nonlinear Mech.* **34** (1999), 1089–1102.
4. Kersten, P. H. M.: The general symmetry algebra structure of the underdetermined equation $u_x = v_{xx}$, *J. Math. Phys.* **32** (1991), 2043–2050.
5. Lie, S.: Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichung, *Arch. Math.* **6** (1881), 328–368.
6. Samokhin, A. V.: Symmetries of control systems, *Banach Center Publ.* **33** (1996), 337–342.
7. Samokhin, A. V.: Symmetries of linear and linearizable systems of differential equations, *Acta Appl. Math.* **56** (1999), 253–300.
8. Samokhin, A. V.: Symmetries of linear ordinary differential equations, *Amer. Math. Soc. Transl. (2)* **167** (1995).

* This constitutes an alternative approach, cf. [4].