УДК 517.95:517.928:512.816

У книгу вміщено 29 вибраних наукових статей видатного українського вченого, засновника київської школи симетрійного аналізу диференціальних рівнянь, члена-кореспондента НАН України професора Вільгельма Ілліча Фущича (1936–1997). Включено також спогади дружини В.І. Фущича та повний перелік його наукових праць.

Для наукових працівників — математиків і фізиків, а також аспірантів відповідних спеціальностей, які цікавляться застосуваннями теоретико-групових методів до дослідження рівнянь математичної фізики.

The book includes 29 selected scientific papers of an outstanding Ukrainian scientist, founder of the Kyiv school of symmetry analysis of differential equations, Corresponding Member of NAS of Ukraine, Professor Wilhelm Fushchych (1936–1997). It also contains memoirs of wife of W.I. Fushchych and the complete list of Fushchych's scientific works.

The book will be of interest for researchers — mathematicians and physicists, and post-graduate students involved in application of group-theoretical methods to investigation of equations of mathematical physics.

Рекомендовано до друку вченою радою Інституту математики НАН України

Видання здійснене за державним контрактом на випуск наукової друкованої продукції

Редакція фізико-математичної та технічної літератури

Редактор *Т.С. Мельник*

Ф $\frac{\mathbf{1602070100\text{-}015}}{\mathbf{2005}}$

ISBN 966-00-0426-5

Передмова

Вільгельм Ілліч Фущич був і залишається дуже помітною зіркою на світовому математичному небосхилі. Невтомний і пристрасний дослідник, він залишив наукову спадщину, яка, безумовно, перевищує можливості навіть дуже обдарованої людини. Він був також Учителем (з великої літери), що зумів запалити любов до математики у численних учнів і послідовників.

За той, на жаль, не дуже довгий час, що йому подарувала доля, Вільгельм Ілліч встиг зробити стільки, що могло б вистачити на цілий науковий інститут. Вражає навіть кількісний перелік його творчої спадщини: 9 монографій та понад 300 наукових статей, значна частина з яких опублікована в провідних міжнародних журналах (повний перелік наукових робіт наведено в даній книжці). Але ще більше враження справляє кількість дослідників вищої кваліфікації, яких він підготував для нашої держави і, так сталося, що і для багатьох інших держав. Під його керівництвом написано і успішно захищено 47 кандидатських дисертацій, 13 з його учнів стали докторами наук.

Вільгельм Ілліч створив українську школу групового аналізу диференціальних рівнянь, яка сьогодні займає одне з провідних місць у світовій математичній спільноті. Вагомим індикатором визнання цієї школи може вважатися надзвичайна популярність серії міжнародних конференцій “Symmetry in Nonlinear Mathematical Physics”, започаткованої Вільгельмом Іллічем у 1995 р. Остання, П’ята, конференція, що відбулася в червні 2003 р., зібрала 256 учасників з 38 країн світу. Ще більше представницькою обіцяє бути чергова, Шоста, конференція, що планується цього року.

Вільгельм Ілліч був всебічно обдарованою людиною: талановитим дослідником, чудовим педагогом і вмілим організатором. Створене ним продовжує жити після його смерті, і це є найкращий підсумок життя для творчої людини.

У короткій передмові ми спробуємо охарактеризувати Вільгельма Ілліча Фущича, насамперед, як науковця і Учителя в науці. З його великої і, можна навіть сказати, колосальної наукової спадщини ми відібрали лише 29 статей до цієї збірки вибраних праць. Всі ці праці дістали гідну оцінку світової наукової спільноти і цитуються в провідних журналах з математичної фізики, але вони лише частково відображають ті різноманітні та різнопланові результати Вільгельма Ілліча, що залишили вагомий слід у математичній фізиці. Ми хотіли б ще раз подякувати міжнародним видавництвам за люб’язний дозвіл на передрукування праць Вільгельма Ілліча. Ми дуже вдячні Ользі Іванівні Фущич, дружині Вільгельма Ілліча, за спогади, включені до даної книги.

Вільгельм Ілліч володів рідкісною якістю, яку доля дарує тільки справжнім вченим, — умінням вибирати дійсно важливі задачі з того нескінченного переліку проблем, що стають на шляху кожного активно працюючого дослідника. Саме цей дар дозволив йому отримати низку наукових результатів, які виявилися передвісниками нових напрямів у сучасній математичній фізиці. Деякі з цих результатів згадуються далі.

До наукових інтересів Вільгельма Ілліча належали квантова теорія поля, зображення груп і алгебр Лі, підгрупова структура груп Лі, теоретико-груповий аналіз диференціальних рівнянь тощо. Але основна тема його праць — це симетрія в математичній фізиці.

Наприкінці 60-х років минулого сторіччя Вільгельмом Іллічем Фущичем прокласифіковано і конструктивно описано незвідні зображення узагальненої групи Пуанкаре $P(1,4)$ — групи рухів $(1+4)$-вимірного простору Мінковського, а також побудовано рівняння руху, інваріантні відносно цієї групи. Трохи пізніше ним разом з В.М. Федорчуком повністю описано підгрупову структуру цієї групи і прокласифіковано всі неспряжені підалгебри відповідної алгебри Лі.

Ці результати (технічно дуже складні) спочатку здавались далекими від можливих застосувань. Але історія подальшого розвитку математичної і теоретичної фізики підтвердила виключну важливість цих досліджень. Так, майже всі сучасні моделі квантової (супер)гравітації базуються на суттєвому розширенні $(1+3)$-вимірного простору Мінковського. При цьому зображення груп $P(1,4)$ та $P(2,3)$ і відповідні інваріантні рівняння відіграють ключову роль і виникають як в результаті редукції модельних рівнянь у просторі вищих розмірностей, так і при побудові моделей безпосередньо у деситерівському чи антидеситерівському просторах.

У 70-х роках XX сторіччя Вільгельмом Іллічем та його учнями побудовано релятивістські хвильові рівняння для частинок з довільним спіном, які дозволяли формулювати конструктивні моделі для таких частинок із різними типами взаємодії. Задача побудови таких рівнянь здавалася абстрактно математичною, оскільки стабільних частинок зі спіном, не рівним нулеві чи одиниці, на той час фізики ще не відкрили. Але досить швидко, у зв'язку зі стрімким розвитком суперсиметричних теорій, попит на рівняння для частинок з іншими значеннями спіну став однією з основних тенденцій сучасної математичної фізики.

Велику увагу приділяв Вільгельм Ілліч дослідженням рівнянь, інваріантних відносно групи Галілея. Здавалося б інтерес до таких рівнянь є дуже природним, оскільки абсолютна більшість фізичних (а також хімічних, біологічних та інших, наприклад, соціальних) явищ, що досліджуються з використанням математичних моделей, відбуваються за швидкостей, набагато менших за швидкість світла. Це означає, що згадані явища повинні задовольняти принцип відносності Галілея, а відповідні моделі — бути інваріантними відносно перетворень Галілея.

Як це не дивно, у сучасній теоретичній фізиці поширене певне нехтування вимогою галілеївської інваріантності. При побудові математичних моделей така інваріантність вимагається дуже рідко — зазвичай, вона виникає як граничний випадок релятивістських теорій.

Вільгельм Ілліч добре розумів, що насправді умова галілеївської інваріантності є важливою і конструктивною, бо дозволяє різко обмежити вибір можливих моделей із використанням фізично обгрунтованої вимоги. У серії праць ним було побудовано рівняння руху частинок із довільним значенням спіну, інваріантні відносно перетворень Галілея. Виявилося, що знайдені рівняння добре описують всі основні властивості таких частинок, — не гірше від рівняння Дірака (з розумними обмеженнями на швидкості частинок). Відносно недавно ці результати було підхоплено канадськими дослідниками з Албертського університету (Канада), що має добрі традиції в дослідженні інваріантних хвильових рівнянь.

Цікаві результати, які активно цитуються, отримано Вільгельмом Іллічем при дослідженні нелінійних рівнянь другого порядку, що допускають групу Галілея. В цілому можна прогнозувати подальше зростання інтересу до результатів досліджень, пов'язаних із галілеївською симетрією.

Одна з найулюбленіших (і найбільш плідних) ідей Вільгельма Ілліча полягала в тому, що диференціальні рівняння можуть допускати набагато ширші класи симетрій, ніж ті, що можуть бути знайдені в класичному підході Лі. Ця ідея надихала його на розробку нових методів дослідження симетрійних властивостей диференціальних рівнянь, які дістали загальну назву неліївських.

Неліївські симетрії в загальному випадку не утворюють алгебр чи груп Лі. Але існує важливий підклас таких симетрій, що утворюють згадані структури. Вільгельмом Іллічем доведено, що подібні симетрії притаманні майже всім рівнянням релятивістської квантової теорії. Зокрема, встановлено, що, окрім релятивістської та конформної інваріантності, рівняння Максвелла допускають восьмипараметричну групу Лі, генератори якої є інтегро-диференціальними операторами.

Вагоме місце в дослідженнях Вільгельма Ілліча займали симетрії, що реалізуються диференціальними операторами вищих порядків. Такі симетрії відіграють важливу роль при описі систем координат, в яких існують розв'язки з розділеними змінними. В останні роки дослідження таких симетрій набули особливої популярності у зв'язку з поглибленим вивченням проблеми суперінтегровності. Важливим їх застосуванням є також побудова законів збереження, не пов'язаних з симетрією відносно груп Лі.

Серед багатьох вагомих результатів, отриманих Вільгельмом Іллічем при описі симетрій відносно диференціальних операторів вищих порядків, відзначимо повний опис симетрій третього порядку для рівняння Шрьодінгера з потенціалом і законів збереження другого порядку для рівнянь Максвелла. Ці результати дістали подальший розвиток у працях українських та закордонних авторів і активно використовуються в наш час.

Низку фундаментальних результатів Вільгельм Ілліч отримав у загальній теорії розділення змінних. У співавторстві з Р.З. Ждановим і І.В. Ревенком ним розроблено оригінальний і конструктивний підхід до опису рівнянь, що допускають розв'язки з розділеними змінними. Цей підхід, зокрема, дозволив суттєво розширити класи потенціалів, для яких рівняння Шрьодінгера допускає розділення змінних.

Ще одним і, можливо, найважливішим полем діяльності Вільгельма Ілліча була побудова точних розв'язків складних рівнянь математичної фізики. Під його керівництвом і при його безпосередній участі знайдено точні розв'язки рівнянь руху частинки довільного спіну, що взаємодіє з полем Кулона, полем плоскої хвилі, схрещеними постійними електричним та магнітним полями. Розв'язати відповідні складні системи рівнянь вдалося завдяки широкому застосуванню як класичних ліївських, так і вищих симетрій. Вільгельмом Іллічем отримано низку фундаментальних результатів щодо точних розв'язків нелінійних багатовимірних диференціальних рівнянь та систем таких рівнянь, зокрема, рівнянь Дірака, Нав'є–Стокса, Бусінеска, Гамільтона–Якобі, Шрьодінгера, Даламбера, рівнянь нелінійної акустики, рівнянь реакції–дифузії тощо. Він також зробив вагомий внесок у побудову нових методів знаходження таких розв'язків, які суттєво розширили можливості і межі класичного групового аналізу.

Сам Вільгельм Ілліч вважав найвагомішим серед своїх результатів створення методу умовної симетрії. Основна ідея цього методу полягає в тому, що симетрія рівняння може бути розширена, якщо на множину його розв'язків накласти деяку додаткову умову, сумісну з вихідним рівнянням. Ця додаткова симетрія (що дістала назву умовної симетрії) може дозволити побудувати точні розв'язки, які в принципі не можуть бути знайдені з використанням класичного підходу Лі.

Очевидно, що перетворення умовної симетрії не переводять будь-який розв'язок вихідного рівняння в розв'язок. Але існує така підмножина розв'язків, визначена додатковою умовою, для якої перетворений розв'язок є знову розв'язком.

Як це часто буває, шлях від загальної ідеї до її конструктивної реалізації виявився дуже нетривіальним. По-перше, треба було створити алгоритм побудови таких додаткових умов, який би гарантував їх сумісність із вихідними рівняннями і водночас приводив до розширення симетрії. По-друге, розв'язки, отримані з використанням умовної симетрії, часто співпадають з класичними груповими розв'язками, і бажано мати апріорні оцінки, коли це може статись. Але Вільгельм Ілліч та очолювана ним команда (Р.З. Жданов, М.І. Сєров, І.М. Цифра, В.І. Чопик та ін.) успішно здолали головні перешкоди, створивши новий потужний метод знаходження точних розв'язків складних нелінійних систем.

Вільгельм Ілліч зробив вагомий внесок також у розвиток класичного групового аналізу. Творчо використовуючи надбання Софуса Лі та його послідовників, він зумів суттєво розширити межі застосування групового аналізу і описати симетрії чи побудувати точні розв'язки для багатьох важливих рівнянь сучасної математичної фізики. При переліку відповідних результатів майже всюди треба вживати слово "вперше", навіть у такій загальновідомій царині як класичний груповий аналіз Фущичу та очолюваним ним учням вдалося залишити дуже і дуже помітний слід. Це стосується багатьох розділів групового аналізу, зокрема, теорії диференціальних інваріантів, реалізацій алгебр Лі векторними полями, підгрупового аналізу фундаментальних груп математичної фізики, опису підмоделей рівнянь механіки і фізики.

Повне електронне зібрання робіт Вільгельма Ілліча Фущича, фотографії з сімейного архіву сім'ї Фущичів та іншу інформацію можна знайти на інтернет-сторінці `www.imath.kiev.ua/~fushchych/`. Сподіваємося, що ця сторінка розвиватиметься та доповнюватиметься і надалі. Також віримо, що наш приклад надихне інші наукові колективи на створення подібних інтернет-ресурсів для пропаганди досягнень видатних українських учених та українських наукових шкіл.

Головне, що створена Вільгельмом Іллічем Фущичем школа — потужний творчий колектив, дружнє коло однодумців і друзів, осередок якого знаходиться в Києві в Інституті математики НАН України, продовжує жити спільними ідеями, розвиватися і поповнюватися молоддю. Сподіваємося, що закладене і розпочате Вільгельмом Іллічем примножиться новими досягненнями київської школи симетрійного аналізу.

Редакційна колегія

Спогади про Вільгельма Фущича

Ольга Іванівна ФУЩИЧ

У своїх коротких спогадах я намагаюсь донести і показати, що наука, яка лежить в основі всього того, що створено і відкрито людством, — це цікаво і талановито, а вчені варті того, щоб про них знали, як про людей, які становлять інтелект нації, людей щонайширшої ерудиції, з державним мисленням, здатністю до передбачення майбутніх змін не тільки наукових, але й суспільних процесів. Учений у суспільстві повинен зайняти своє місце як письменник, як художник, як політик чи артист.

Вільгельм належить саме до таких УЧЕНИХ, учених з великої літери, з величезною силою волі, титанічної працездатності, який мав дар охоплювати і координувати думкою цілу сукупність проблем, невтомно генерувати нові ідеї, вміло організувати науку, поєднувати дар популяризатора науки з хистом справжнього дослідника.

Я намагатимуся коротко розказати про життєвий і творчий шлях Вільгельма як професіонала високого гатунку, талановитої і яскравої особистості з власною відповідальною позицією, якою він керувався у своєму житті, який активно і плідно вів науково-дослідницьку та науково-організаційну, громадську та педагогічну діяльність. Він сам своєю працею здобув авторитет, повагу серед учнів, колег, учених і просто громадян.

Практично все трудове життя Вільгельма було пов'язане з Києвом, з Інститутом математики НАН України, де він працював протягом 37 років, пройшовши шлях від аспіранта до професора, члена-кореспондента НАН України, завідувача відділу, і немало зробив для перетворення відділу в один із кращих, в якому успішно проводились наукові дослідження з нелінійної математичної фізики.

Вільгельм Фущич — це краса і гідність української науки, будівничий однієї з шкіл українських математиків — школи симетрійного аналізу диференціальних рівнянь математичної фізики, який своїми науковими досягненнями і новаторством збагатив математичну науку XX століття.

Він був ученим, освітянином, громадським діячем, в активі якого 9 монографій, більше 300 наукових статей, педагогічна робота понад 15 років у Київському національному університеті ім. Тараса Шевченка і Національному педагогічному університеті ім. Михайла Драгоманова, створення наукової школи в галузі математичної фізики, організація нового журналу "Journal of Nonlinear Mathematical Physics", започаткування серії міжнародних конференцій "Symmetry in Nonlinear Mathematical Physics" в 1995 р. та участь у роботі Міжнародного комітету з питань науки і культури при НАН України від дня його заснування.

Вільгельм народився на Закарпатті. Він палко любив його камені, ліси, сніги, його голос і душу народу — пісні. Вільгельм був сином своєї держави і своєї епохи, яка змінювалася в громадському, політичному і психологічному розумінні. Він жив у той час, коли українці наполегливо добивались незалежності, і такі люди, як Вільгельм, були серед них найактивнішими.

Вільгельм присвятив усе своє життя науці, вихованню молодого освіченого покоління, молодих учених на благо України, а відтак — і національному відродженню України.

Деякі штрихи з біографії. Вільгельм народився 18 грудня 1936 р. в селі Сільце Іршавського району Закарпатської області, яка в той час входила до складу Чехословаччини. Батько, Ілько Михайлович, 1905 року народження, був селянином. У молоді роки впродовж кількох років працював у вугільних копальнях Франції, де заробив гроші, за які після повернення в Сільце купив землю, виноградники, магазин і одружився з Марією Іванівною Івегеш, 1908 р. народження. Марія народила у 1928 р. доньку Ганну, у 1932 р. — сина Михайла, а в 1936 р. — наймолодшого Вільгельма (Вілія). Батько і мати були дуже працьовитими. Робота на землі, у власному магазині і на виноградниках давала можливість жити сім'ї заможно і в достатку.

Батько був вимогливим, строгим, розумним, щирим, доброзичливим і чесним — уособленням послідовності і чесності перед собою і людьми. Він мріяв дати своїм дітям освіту, щоб вони у майбутньому змогли продовжити справу, для започаткування якої йому, бідняку, селянському хлопцеві без гроша, без кола прийшлося тяжко працювати у Франції і втратити здоров'я: у копальнях батько захворів на туберкульоз, який став причиною його смерті в 1946 р. Лікування у Чехії та у купальнях Синяка (Закарпатська обл.) не допомогли відновити втрачене здоров'я. Незважаючи на важку хворобу, батько багато працював, активно і вміло керував своїм господарством. Мати Вільгельма — унікальна жінка, яка жила по совісті, багато і чесно працювала. В ній була цінна, необхідна для формування характеру дітей частка гордості, міри, благородства, людяності, інтелігентності та енергійності. Батько і мати — це дві розумні, працьовиті, миролюбні, тверді в досягненні своєї мети ПОСТАТІ. Вони і були визначальними в розумінні Вільгельмом СВІТУ.

Після смерті батька, коли на Закарпатті, як у свій час на Україні в 30-х роках, розпочалось розкуркулювання заможних селян, у матері Вільгельма, на руках якої залишилося троє дітей, відібрали все — землю, магазин, виноградники, а сім'ю зарахували до куркулів і запланували виселити із Закарпаття. Тільки несамовита відвага Марії Іванівни, її вміння налагоджувати добрі стосунки і відповідні зв'язки з необхідними для цього людьми врятували сім'ю від розкуркулювання і виселення.

Про яке багатство могла йти мова? Вілі розповідав, що іноді не було, що їсти, як і у багатьох сім'ях у ті повоєнні роки. Але сім'я дружно трималася. Діти вчилися і допомагали матері садити картоплю, кукурудзу, квасолю, копали, збирали урожай на тій дільниці землі, яку їм залишили. Марія Іванівна робила все можливе, щоб дітям дати освіту. Вона вступила до колгоспу і тяжко працювала, отримуючи за свою працю на трудодні мізерну платню. Вона жила своїми дітьми. Діти радували її своєю слухняністю, працьовитістю, успішністю в школі. Це надавало їй сили боротися зі скрутою.

Не так багато жінок, які залишились після 1947 р. практично без засобів до існування, можуть похвалитися таким героїзмом — виховати дітей, дати їм освіту, як про це мріяв її чоловік Ілько, і вивести їх у люди.

Після закінчення середньої школи у м. Іршаві Ганна поступила в педагогічне училище у м. Хусті, а Михайло — на фінансові річні курси у м. Чернівці. Вілі продовжував навчатися у середній школі, щодня долаючи пішки 5 км. Ганна після закінчення педучилища аж до виходу на пенсію працювала учителькою початкових класів у рідному селі Сільці. Михайло після служби в армії закінчив Ленінградське морехідне училище і на протязі багатьох років працював на морському флоті.

Після закінчення середньої школи у м. Іршаві в 1953 р. Вільгельм поступив до Ужгородського державного університету на фізико-математичний факультет. Щоб Вілі міг поїхати в Ужгород на вступні іспити, мати позичила гроші, купила чемодан і дала 100 крб, після чого сказала: "Вілі, а що буде, якщо ти не поступиш? Як ми повернемо гроші?" Взявши з собою, крім чемодану і 100 крб, ще і деякі харчі, Вілі відправився в Ужгород. Іспити він склав успішно і був зарахований на перший курс фізико-математичного факультету за спеціальністю фізика.

Під час вступних іспитів з Вілі сталося декілька кумедних пригод, одну з яких я не можу не пригадати. Вілію дуже захотілося їсти. А тут один з абітурієнтів із заможної сім'ї попросив Вілія скласти замість нього за шматок ковбаси іспити з фізики і математики. Вілі згодився. Спочатку склав фізику на п'ять за себе, а наступний день — теж на п'ять за того хлопця, причому тому самому викладачу (Буш). До речі, згодом Буш викладав у Вілія загальну фізику на першому і другому курсах. Наступним іспитом була математика. Приймав його викладач Хічі. Перші випускники фізико-математичного факультету добре пам'ятають цього добросовісного, пунктуального, сумлінного, вимогливого викладача математичного аналізу. Ті, кому він читав лекції, можуть констатувати: номер, який пройшов у Вілія з Бушом, з Хічі не міг пройти. Вілі склав іспит з математики на п'ять, а за того хлопця не пішов, відмовився, повернувся вже від дверей, побачивши, що в аудиторії сидить той самий викладач, що приймав у нього іспит у попередній день, хоча з письмовою роботою допоміг, сидячи біля нього. Серце підказало Вілію не робити цього, бо може статися біда — впізнає його викладач! І що тоді буде? Прощавай університет! Бажання учитися в університеті виявилося сильнішим від бажання з'їсти шматок ковбаси.

Університет, відкритий в 1945 р., в 1953 р. був малочисельним. Кожен курс більшості факультетів нараховував близько 25 студентів. Найбільш багаточисельним факультетом на той час був медичний — 200 осіб на курсі. Під час навчання в університеті Вілі жив у гуртожитку на Московській набережній. Перші три курси — у кімнаті, в якій, крім нього, проживало ще 29 студентів з різних факультетів — медичного, біологічного, філологічного, фізико-математичного. Проживання в такій "комунальній" кімнаті та товариський, компанійський характер Вілія зробили його знаним серед студентства університету.

Враховуючи, що в гуртожитку вчитися практично не було можливості, Вілі працював у бібліотеці. Харчувався в студентській їдальні на талони, які купував із стипендії зразу після її отримання. Повертався в гуртожиток пізно, біля 23-ї години. І так щодня, крім вихідних, коли всі хлопці грали у футбол, волейбол, а вечорами ходили на танці в клуб університету. Вілі грав у футбол і волейбол за збірну фізико-математичного факультету, а під час роботи в університеті (1959–1960 рр.) — за волейбольну команду викладачів університету, яку тренував і очолював відомий волейболіст, у минулому член збірної України з волейболу, викладач фізкультури Ужгородського університету Скрябін. Щоб якось допомогти матері фінансово, Вілі під час кожних канікул працював.

Студентське життя хоча через матеріальні труднощі було не легким, але дуже цікавим, веселим, щасливим, свідомим, наповненим мріями, новими знаннями не лише з фізики і математики, але й з філософії і літератури, насиченим культурними і духовними цінностями та новими знайомствами. Вілі вчився на відмінно. Ще в студентські роки при зустрічі друзі віталися з ним: “Привіт, професоре!”

Вілі мріяв зайнятися наукою. Все почалося з вересня 1955 р., коли в Ужгород приїхав випускник аспірантури Фізичного інституту ім. Лебедєва АН СРСР, кандидат фіз.-мат. наук (а згодом доктор фізико-математичних наук, професор) Ю.М. Ламсадзе і очолив кафедру теоретичної фізики. Вілія захопили його лекції, семінари, під впливом яких він і вирішив пов’язати своє життя з наукою, обравши собі за спеціальність теоретичну фізику. Він переконався в правильності вибору після поїздки в Москву на виробничу практику в Московський університет, в якому викладали відомі вчені фізики-теоретики. Вілі неодноразово розповідав своїй сім’ї про цю практику на кафедрі теоретичної фізики, яка проходила протягом одного семестру після четвертого курсу. Він і ще семеро його однокурсників відвідували лекції академіка Л.Д. Ландау з квантової механіки та академіка І.Е. Тамма з теорії електромагнітних явищ. З певними труднощами вони пробивались через існуючу тоді пропускну систему на наукові семінари в Інститут фізичних проблем АН СРСР та Фізичний інститут ім. Лебедєва, де часто доповідали всесвітньо відомі учені в галузі квантової теорії поля і теорії елементарних частинок, як радянські, так і закордонні.

У 1958 р. Вілі закінчив кафедру теоретичної фізики Ужгородського університету. Ю.М. Ламсадзе, помітивши у Вілія такі риси, як прагнення до знань, цілеспрямованість, хист до наукової роботи та відмінні знання, дав йому рекомендацію в аспірантуру. Однак через певні обставини (на цей час ми були одружені, а я ще вчилася) Вілі вирішив попрацювати в університеті. Тому після закінчення університету два роки (до 1960 р.) працював на викладацькій роботі на кафедрі теоретичної фізики. Розуміючи, що імовірність надбання знань у столиці через наявність відомих учених, наукових шкіл, чудових бібліотек набагато вища, ніж в Ужгороді, Вілі у листопаді 1960 р. поступив до аспірантури Інституту математики Академії наук України. Вільгельма я зустріла ще в студентські роки на фізико-математичному факультеті, студенткою якого я також була. Мріючи з Вілі про науку в столиці, я не заперечувала щодо його від’їзду до Києва. Навпаки, з’явилась надія на те, що з часом, як тільки підросте дочка, по моєму переїзді до Києва я також зможу влаштуватися в один з інститутів Академії наук. Наша мрія збулася. Я знайшла в Києві улюблену роботу в Інституті проблем матеріалознавства НАН України, в якому працюю і сьогодні, захистила кандидатську дисертацію, виконану під керівництвом відомого ученого академіка НАН України І.М. Федорченка. У Києві прожила з Вілі і дітьми, Маріанною і Богданом, велику і кращу частину свого життя.

Після закінчення аспірантури з листопада 1963 р. Вільгельм працював молодшим, а з січня 1965 р. — старшим науковим співробітником Інституту математики АН України.

У лютому 1964 р. Вільгельм захистив кандидатську дисертацію на тему “Аналітичні властивості амплітуд народження як функції переданого імпульсу”. Кандидатську дисертацію Вільгельм виконав під керівництвом видатного ученого, члена-кореспондента (згодом академіка) АН України, доктора фізико-математичних наук професора О.С. Парасюка. Остап Степанович — зірка в науці. Він намагався вкласти у своїх учнів те, що з нього самого било джерелом: високу думку, знання і велику фантазію при виконанні досліджень. Остап Степанович прекрасно знав всю наукову літературу (і не тільки) та намагався прищепити своїм учням любов до книг, до диспутів, що виникали на наукових семінарах, які він проводив щотижня і на яких народжувались нові ідеї і шляхи їх реалізації. Пропускати наукові семінари без поважної причини не дозволялось, бо семінари для Остапа Степановича були святою справою. Вільгельм перейняв від Остапа Степановича цю любов до книг, до нових знань, методи роботи з учнями. До кінця життя він був відданий своєму УЧИТЕЛЮ, любив і шанував його, постійно спілкувався і радився з ним.

У квітні 1971 р. Вільгельм на вченій раді Інституту математики захистив докторську дисертацію на тему “Теоретико-групові основи узагальненої релятивістської квантової механіки і P-, T-, C-перетворення”, консультантом якої був академік О.С. Парасюк. Офіційними опонентами дисертації були: член-кореспондент, доктор фізико-математичних наук, професор В.П. Шелест (м. Київ, ІТФ АН України), доктор фізико-математичних наук, професор О.О. Боргардт (м. Донецьк, ФТІ) і доктор фізико-математичних наук, професор В.Г. Кадишевський (м. Дубна, ОІЯД). У докторській дисертації підсумовано результати, пов’язані з класифікацією незвідних зображень узагальнених груп Пуанкаре в багатовимірних просторах, зокрема, у просторах де Сіттера. На основі цих досліджень побудовано математичні основи квантової механіки для частинок із змінною масою.

З 1978 р. до кінця свого життя Вільгельм був завідувачем відділу прикладних досліджень Інституту математики НАН України. Звання професора йому присуджено у 1980 р.

Якщо на початку наукової діяльності коло наукових інтересів Вільгельма становили проблеми квантової теорії поля, то починаючи з 70-х років основною темою його праць стає теоретико-груповий аналіз рівнянь математичної фізики та квантової механіки. Саме в 70-х роках Вільгельм дав розв’язок фундаментальної проблеми математичної фізики, над якою раніше працювали такі видатні вчені світу, як Вігнер, Баргман, Швінгер та ін., яка полягає в пошуку та описі систем диференціальних та інтегро-диференціальних рівнянь, інваріантних відносно груп Галілея і Пуанкаре. Для розв’язання цієї проблеми він запропонував нелагранжевий підхід до побудови та дослідження рівнянь руху в квантовій теорії.

Без перебільшення можна стверджувати, що Вільгельм зробив значний внесок у новий розділ сучасної математичної фізики, який спеціалісти називають симетрійним аналізом диференціальних рівнянь. Такі загальноприйняті тепер терміни, як неліївська симетрія, нелокальна симетрія, неліївський підхід, нелінійна математична фізика вперше введено в працях Фущича. Коли з’явились перші праці Вільгельма з симетрійного аналізу в наукових журналах, то багатьом здавалось, що вони не будуть мати широкого продовження, бо всі спеціалісти знаходилися під впливом досліджень з теорії симетрій, виконаних раніше такими всесвітньо відомими вченими, як Альберт Енштейн, Анрі Пуанкаре, Поль Дірак, Софус Лі та ін., і що їх результатами ці проблеми уже вичерпані.

Подальші праці Вільгельма та учнів його школи в цій галузі показали, що Фущич запропонував нові підходи до дослідження симетрії рівнянь математичної і теоретичної фізики, які знайшли розвиток в працях вчених багатьох країн світу. Наукові результати Вільгельма і його учнів у галузі теорій симетрій, на мою думку, є значним досягненням української математичної науки.

Слід підкреслити, що для того щоб прожити в гармонії і злагоді з чоловіком-ученим, який повністю віддається роботі, треба його розуміти і разом з ним відчувати все те, що відчуває він у кращі чи гірші моменти свого життя, бо у справжнього ученого вихідних, свят і 8-годинного робочого дня, а часто й відпуски, просто немає. Є улюблена робота, робота і ще раз робота, є учні — студенти, аспіранти, докторанти, наукові співробітники, яких він учив проводити фундаментальні дослідження і за яких він переживав як за своїх рідних дітей, переймався їхніми проблемами як своїми.

Крім "дітей-учнів" (всіх учнів Вільгельм називав дітьми), у нас з Вілі двоє дітей — Маріанна (тепер Маріанна Ойлер), 1961 року, і Богдан, 1972 року народження. Вілі любив своїх дітей, пишався ними і підтримував їх. Він був завжди їм другом і порадником, розумів цінність батьківського виховання, бо саме воно формує дитину як особистість. Він ніколи не забував про необхідність емоційної, моральної підтримки дітей і про їх потребу в самовираженні. Служив дітям прикладом у шляхетності, був для них великим авторитетом.

Діти були свідками нашої поваги один до одного, поваги до них як особистостей, гармонії в сім'ї. Гармонія між дітьми і батьками зрівнює світ так само, як плюс і мінус, добро і зло, біле і чорне. Це допомогло виховати їх чистими, щирими, добрими, розумними, збудити і прищепити в них з дитинства, замолоду духовні і людські якості. Вілі намагався не тиснути на дітей, не наполягав, а тільки злегка направляв і керував ними. Він мав достатньо твердих принципів, щоб сформувати дітей свідомими українцями, справжніми патріотами, виховати у них пошану до роду, розвинути в них духовність.

Коли у Вілія траплялись важкі ситуації, я і діти були завжди поряд і закликали його до спокою, до посмішки, згуртовувались і вірили, що разом обов'язково подолаємо труднощі.

Вілі зумів прищепити дітям любов до математики, до науки, до праці. Маріанна закінчила фізико-математичну школу № 145 при Київському університеті в 1978 р., а в 1983 р. — механіко-математичний факультет цього ж університету. Аспірантуру закінчила при Інституті математики НАН України. У 1988 р. захистила дисертацію на тему "Застосування асимптотичних методів до розв'язання деяких нелінійних хвильових рівнянь" за спеціальністю "диференціальні рівняння і математична фізика" і отримала наукову ступінь кандидата фізико-математичних наук. З 1997 р. працює у Швеції в Технологічному університеті м. Лулеа викладачем математики. Богдан у 1988 р. закінчив фізико-математичну школу № 145 при Київському університеті, а в 1993 р. — механіко-математичний факультет того ж університету. Рік навчався у Мюленберзькому коледжі (США) і рік — в аспірантурі Гарвардського університету (США). У 2000 р. закінчив юридичний факультет Київського університету. Працює в страховому бізнесі.

Спогади Маріанни і Богдана. "У нас склалася така думка, що будь-які наукові проблеми з участю Батька розв'язувались на найвищому рівні. Наша пам'ять про нього — це вічна дискусія з ним та вічне очікування підтримки та поради від нього, згадки про різні миті його життя, випадки і слова.

Переконані, що Батько був зразком лицарського служіння Істині, Добру, Науці, Україні і своїм життям завоював почесне звання — Людина. Віримо, що все те, що залишив Батько в науці, ім'я якої нелінійна математична фізика, залишиться для майбутніх поколінь. Сподіваємось, що учні Батька будуть надалі сприяти поширенню його наукових, суспільних, морально-етичних ідей та принципів.”

Наукова робота. Вільгельм любив Інститут математики, був прив’язаний до нього. Він шанобливо і з повагою ставився до всіх співробітників інституту незалежно від посади, яку вони обіймали, в тому числі і до своїх учнів. В інституті Вільгельм знаходив життєдайний ковток свіжого повітря, який живив його душу і серце.

Вільгельм вважав, що наукова робота — це особлива професія. Йому подобалось учити молодих спеціалістів і своїх учнів, творити разом з ними нові формули, рівняння, які випереджали час. Він часто повторював, що учений-педагог — це прекрасна професія, якій притаманні риси творця, який крок за кроком “теше”, немов скульптор, нових учених.

Працюючи з учнями, формуючи їх як учених, Вільгельм отримував велике, ні з чим незрівнянне моральне задоволення. Наукова творчість стала для Вільгельма його життям, про механізм якої він сказав так: “Спочатку починаю хворіти новою ідеєю, новою задачею, потім заглиблююсь у неї, спираючись на попередні результати, а потім розв’язую її з учнями, після чого видужую. І часто буває так, що народжується несподівано для всіх нас першокласний результат і він видається всім нам приголомшливим і неочікуваним.”

Вільгельм ніколи жорстко не розмовляв з своїми учнями з приводу їх наукової роботи, але вимагав відповідальності при її виконанні. Двері його кабінетів удома і в інституті завжди були відкритими. Він робив з учнями одну справу, тому його спілкування з ними неможливе було без довірливого тону; між ними завжди були добрі, чисті, чесні, людські стосунки.

Монографії і наукові статті — це творчий підсумок напрацювань Вільгельма з учнями, який об’єднав учителя і учнів, керівника і співробітників. Я пам’ятаю, як вони, працюючи разом, народжували нові, більш досконалі результати. І ці митті були у них прекрасні і неповторні. Заради них Вільгельм жертвував у житті багато чим.

Однак, наука, як і будь-яка творчість, будується на гонористості. Тому Вільгельм не завжди був задоволений результатом. Але не засмучувався, бо розумів, що над одержанням вагомого результату потрібно багато працювати, знав, що деякі результати неможливо отримати швидко та легко, як здається на перший погляд. “Ідея довго зріє, накопичується всередині і в один прекрасний день проривається,” — так говорив він.

Вільгельм завжди намагався зберегти і примножити високий творчий потенціал і наукову атмосферу колективу, яким він керував і трудовими буднями якого жив, створюючи умови творчої взаємодії між співробітниками. Широта наукових інтересів Вільгельма та постійне піклування про молодь вплинуло на склад та якість відділу прикладних досліджень Інституту математики, який він очолював майже два десятка років.

Віддаючись науці, поринаючи в роботу цілком і повністю без залишку, на мою думку, Вільгельм був для учнів прикладом самокритичності і працездатності, доброти і щирості, справедливості і людяності.

Науковий доробок Вільгельма і його учнів — це, безумовно, *цілий світ, появу якого спричинила особлива свобода — свобода творити нові закони симетрійного аналізу, нові рівняння, нові теореми нелінійної математичної фізики, розробля-*

ти нові методи класифікації рівнянь. Жодна книга, жодна робота не заперечує іншої і в кожній відразу упізнається наукова школа, створена Вільгельмом.

Про значимість отриманих Вільгельмом і його школою результатів найкраще свідчать друковані праці, перелік яких наведено в цій книзі і з повним текстом яких можна ознайомитися на інтернет-сторінці `www.imath.kiev.ua/~fushchych/`, а також цитування цих результатів провідними вченими світу.

З науковими працями та робочими матеріалами до них, біографічними документами та документами про вченого, документами про діяльність, листуванням, фотодокументами та дарчими надписами можна познайомитися в Інституті архівознавства НАН України (фонд № 315, опис № 1 за 1956–2001 рр.).

Педагогічна робота. Природна риса Вільгельма — це потяг до спілкування з науковою молоддю, яка завжди його оточувала. Тому свою наукову роботу Вільгельм поєднував з педагогічною. Протягом багатьох років він читав курси лекцій з математичної фізики на механіко-математичному факультеті Київського національного університету ім. Тараса Шевченка та фізико-математичному факультеті Національного педагогічного університету ім. Михайла Драгоманова.

Вільгельм завжди намагався налагодити з першого разу те енергійне, емоційне спілкування професор–студент, від якого залежав успіх їхньої роботи — процес навчання. Вибір матеріалу лекцій належною мірою визначався актуальними питаннями математичної фізики, зокрема теорією зображень груп і алгебр Лі. Розуміючи значення їх застосування до задач фізики, поряд з іншими темами він викладав основи теорії груп і алгебр та їх класифікацію.

На своїх лекціях і семінарах Вільгельм намагався дати максимально необхідну кількість знань, ознайомити студентів з останніми досягненнями в математиці та фізиці, які не можна було прочитати в навчальних посібниках. Аналіз і вивчення новітніх робіт допомагали студентам поглибити своє розуміння проблеми, опанувати сучасний математичний апарат, який надалі вони могли успішно використовувати в науковій роботі. Вільгельм давав можливість студентам брати активну участь у семінарах в Інституті математики НАН України, які він проводив у своєму відділі прикладних досліджень із своїми учнями — науковими співробітниками, аспірантами, докторантами, кандидатами і докторами наук. На цих семінарах студенти мали можливість опанувати нові знання, які вони застосовували до розв'язування задач, поставлених Вільгельмом для дипломних чи дисертаційних робіт. Хочу відзначити, що з більшістю аспірантів Вільгельм працював ще з студентської лави.

Вільгельм вчив своїх студентів допитливості, цілеспрямованості, умінню проводити фундаментальні теоретичні дослідження, творити нові математичні знання і ефективно їх застосовувати, бо був переконаний, що математика — це той ключ, з допомогою якого відкриваються двері до багатьох, часом зовсім нових галузей природничих знань.

Міжнародна наукова діяльність. Справжній Фущич заявив про себе в працях 70-х років. Вільгельм остаточно утвердився як новатор у вибраній галузі математичної фізики у 80-і роки, коли прийшов до науки з оригінальними роботами про умовну симетрію, які інтенсивно цитуються. Він стає знаним у науковому світі, його запрошують не тільки провідні наукові центри Росії (Об'єднаний інститут ядерних досліджень, Математичний інститут ім. Стєклова РАН), з якими Вільгельм до кінця життя підтримував тісні наукові контакти, але й відомі університети світу.

Вільгельм отримує запрошення на участь у міжнародних конгресах, симпозіумах, конференціях та семінарах від провідних наукових центрів світу для виступів і обговорення найбільш актуальних проблем і задач у теоретичній і математичній фізиці. Він розумів і усвідомлював, наскільки важливою і перспективною справою є участь у таких представницьких міжнародних форумах, де проводили аналіз досягнень у математичній і теоретичній фізиці і підсумовували роботу учених за минулий період, окреслювалися найбільш перспективні напрями проведення досліджень. Участь у роботі таких форумів давала можливість познайомитись з науковими досягненнями провідних наукових шкіл і водночас пропагувати результати, отримані в Інституті математики НАН України.

Вільгельма слухали в наукових центрах Америки, Англії, Німеччини, Швеції, Японії, Південно-Африканської Республіки та ін. Теми лекцій були такі: “Про нові симетрії рівнянь”, “Теоретико-групові методи у математичній фізиці”, “Симетрія і точні розв’язки багатомірних нелінійних хвильових рівнянь Даламбера, Ліувілля, Шрьодінгера, Дірака”, “Група Лі і точні розв’язки рівнянь Даламбера, Ліувілля, Шрьодінгера, Дірака”, “Про нові та старі симетрії рівнянь Максвелла і Дірака”, “Симетрія і точні розв’язки нелінійних рівнянь Даламбера і Дірака”, “Симетрія і точні розв’язки нелінійних рівнянь математичної фізики”, “Симетрія і точні розв’язки спінорних нелінійних рівнянь”, “Умовні симетрії нелінійних рівнянь акустики”, “Симетрія і точні розв’язки багатомірних нелінійних рівнянь Дірака і Шрьодінгера”, “Симетрія і точні розв’язки нелінійних рівнянь для спінорних і векторних полів”. Це лише частина з великого переліку лекцій, прочитаних Вільгельмом під час його поїздок.

Поїздки Вільгельма в наукові центри світу сприяли встановленню тісних наукових контактів з науковими школами, які займались аналогічною тематикою. Ось що пише про це Вільгельм: “У теперішній час у сучасній математичній фізиці склалася така ситуація, коли багато наукових колективів у світі розробляють близькі теми. Основні напрями цих досліджень — нелінійні проблеми математичної і теоретичної фізики. Часто буває так, що один і той же результат отримують незалежно різні автори різними методами. Тому виникає нагальна необхідність отримати інформацію про останні досягнення найкоротшим шляхом, з перших рук, тобто від тих учених, які цей результат отримали вперше, а не чекати рік–два, коли з’явиться стаття про ці дослідження в науковій літературі. У багатьох наукових центрах ведуться інтенсивні і широкомасштабні дослідження з нелінійних проблем математичної і теоретичної фізики. Це пов’язано з тим, що реальні процеси квантової фізики є нелінійними, тому математичний опис цих явищ повинен моделюватися нелінійними диференціальними і інтегральними рівняннями. Лінійний опис таких процесів у багатьох відношеннях незадовільний. З цієї ж причини в таких державах, як США, Англія, ФРН, Канада, створено наукові центри з математичного дослідження нелінійних процесів у фізиці, хімії, біології.”

Спілкування Вільгельма з провідними ученими світу як на міжнародних форумах, так під час поїздок у різні наукові центри засвідчили, що застосування методів Лі, теорії представлень алгебр Лі та інших теоретико-алгебраїчних методів до нелінійних рівнянь математичної фізики і квантової теорії поля є конструкктивним напрямом у сучасній математичній і теоретичній фізиці. Цей напрям широко і інтенсивно розвивається також у багатьох західних наукових центрах.

Наведу два приклади наукових зв'язків Вільгельма із зарубіжними науковими школами: з Інститутом теоретичної фізики Технічного університету м. Клаусталь (ФРГ) і особисто з його директором, професором Г.Д. Дойбнером, яке розпочалось у кінці 70-х років, та Пітером Олвером, професором Інституту математики університету м. Міннесота (США). На перших порах це співробітництво було заочним і полягало в обміні інформацією за результатами досліджень.

Особисте знайомство Вільгельма з Дойбнером відбулося в листопаді 1984 р. під час його поїздки в Клаусталь на запрошення Дойбнера. Ось що написав Дойбнер Вільгельмові: "Я запрошую Вас від імені Теоретичного інституту м. Клаусталь відвідати нас. Всі витрати, пов'язані з проживанням у нас, будуть оплачені Інститутом. Ми вважаємо Ваш вклад у теорію симетрії різноманітних розв'язків нелінійних рівнянь у частинних похідних як найбільш важливий і істотний. Ці питання становлять для нас великий інтерес. Ми сподіваємось, що Ваш приїзд покладе початок плідному співробітництву із застосувань теоретико-групових методів до нелінійних проблем. У липні у нас відбудеться міжнародна конференція "Фізика на нескінченновимірних многовидах із спеціальними застосуваннями до гідродинаміки", яка, можливо, також зацікавить Вас. Будь ласка, повідомте мені Ваші плани якнайскоріше. Із сердечним вітанням до Вас. Щиро Ваш, професор Г.Д. Дойбнер."

Вільгельм під час цієї поїздки у ФРН відвідав Інститут теоретичної фізики в м. Бонні на запрошення директора професора Конрада Блойдера і познайомився з його науковою роботою.

Вільгельм був вражений організацією проведення досліджень: "Для швидкого обміну науковими досягненнями в університеті під керівництвом професора Г.Д. Дойбнера організуються щорічно невеликі (30–40 чоловік) колективи, які тісно пов'язані з педагогічним процесом університету. Інформацію про останні досягнення вони отримують, як правило, під час участі в різних конференціях і за рахунок особистих контактів. Читання лекцій і знайомство з дослідженнями, які ведуться в університетах Клаусталя і Бонна, підтверджують важливість і актуальність інтенсивного розвитку теоретико-алгебраїчних методів у математичній фізиці. Особливо це стосується нелінійних багатовимірних диференціальних рівнянь у частинних похідних. Потрібно більш інтенсивно розвивати в нашій країні методи диференціальної геометрії та їх застосування до проблем математичної фізики, оперативніше отримувати інформацію про останні досягнення за кордоном, всіляко підтримувати пропозиції про проведення спільних наукових робіт і конференцій. Розумно проводити невеликі конференції по вузьких напрямках, частіше запрошувати зарубіжних учених для роботи в наших інститутах на 1–2 місяці".

Співробітництво з Інститутом теоретичної фізики і професором Дойбнером особисто продовжувалось до кінця Вільгельмого життя. Воно проявлялось в участі Вільгельма практично у всіх симпозіумах, нарадах, конференціях, які проводив Інститут теоретичної фізики м. Клаусталя під керівництвом Г.Д. Дойбнера. Використовуючи результати Вільгельма, Г.Д. Дойбнер у співавторстві з Г.А. Голдіном (Rutgers University) запропонував рівняння, які тепер носять назву "Doebner–Goldin equations". За рекомендацією Вільгельма Дойбнер прийняв на 2 роки на роботу в Інститут теоретичної фізики ім. А. Зоммерфельда як стипендіата фонду Гумбольда молодого доктора фізико-математичних наук, учня Вільгельма Рената Жданова, що дало можливість здійснювати більш тісне співробітництво між двома школами. Професор Г.Д. Дойбнер тісно співпрацював з Вільгельмом і у видавничій роботі, він був членом редколегії міжнародного журналу "Journal of Nonlinear Mathematical Physics", започаткованого Вільгельмом. На запрошення НАН України Дойбнер відвідав декілька разів Інституту математики НАН України з метою виступити на наукових семінарах, організованих Вільгельмом, і поділитися останніми новими результатами, отриманими у м. Клаусталь.

Особисте знайомство Вільгельма з професором Пітером Олвером відбулося в 1989 р., коли Вільгельм на запрошення директора Інституту математики професора А. Фрідмана і професора П. Олвера відвідав університет у м. Міннесота. Декілька слів з цього запрошення: "Шановний професоре, Інститут математики при університеті Міннесота здійснює програму з дослідження нелінійних хвиль протягом 1988–1989 рр. Від імені Координаційного комітету запрошуємо Вас взяти участь у цій програмі. Дата і тривалість Вашого візиту можуть бути відкоректовані так, як Вам буде зручно. Інститут бере на себе всі витрати з Вашого перебування в США, включаючи внутрішні авіаперельоти. Ми також забезпечимо Вас приміщенням для роботи і допомогою в діловодстві. Найбільш бажаним періодом Вашого візиту є осінь 1989 року, оскільки в цей час особлива увага буде приділена інтегрованим системам, збуренням і модуляціям".

На той час Вільгельм з учнями вперше побудував широкі класи точних розв'язків багатовимірних нелінійних рівнянь Даламбера, Шрьодінгера, Дірака і чітко продемонстрував ефективність методів теорії груп Лі при дослідженні нелінійних процесів. Зміст лекцій і виступів Вільгельма був пов'язаний саме з цим.

Перебування в університеті м. Міннесота ще раз переконало Вільгельма в тому, що нелінійні процеси є головним напрямком у математичній фізиці, на який у США виділяється найбільша кількість людей і засобів. На думку Вільгельма, саме цей напрямок необхідно розвивати в Інституті математики НАН України і ширше представляти на міжнародних конференціях.

Праця в бібліотеках при університетах у США показала, що радянські спеціалізовані бібліотеки знаходяться на рівні 50-х років. Тому, на думку Вільгельма, необхідно вжити термінові заходи з комплектування наших спеціалізованих бібліотек міжнародними журналами та комп'ютерною технікою. Наукові контакти Вільгельма з ученими Америки тривали впродовж його життя. Олвер був членом редколегії журналу "Journal of Nonlinear Mathematical Physics" і за запрошенням Академії наук України відвідав Інститут математики, на семінарах якого виступив з циклом лекцій щодо дослідження нелінійних рівнянь. Вільгельм неодноразово зустрічався з Пітером Олвером на міжнародних наукових форумах.

Хочу зазначити, що Вільгельм тісно співпрацював ще з одним американським ученим, професором В. Захарі (Говардський університет). Вони часто зустрічались у США і в Україні для обговорення отриманих результатів, активно листувалися, обмінюючись результатами досліджень.

Громадська та суспільна робота. Серед низки найкращих рис особистості Вільгельмові був притаманний патріотизм. Вільгельм був делегатом установчого з'їзду Народного руху України за перебудову, який відбувся 8–10 вересня 1989 р.

у конференц-залі Київського політехнічного інституту. Президією НАН України в присутності другого секретаря міському партії КПУ було прийнято рішення про недопустимість участі членів і співробітників НАН України в роботі з'їзду (за їх словами, з'їзду націоналістів). Незважаючи на це рішення, свідома частина учених, які не мали запрошення і не могли попасти до зали, підтримала з'їзд своєю присутністю перед будинком КПІ протягом всіх днів його роботи, уважно слухаючи трансляцію виступів і супроводжуючи їх бурхливими оплесками. Намагання влади розігнати це велике зібрання людей-патріотів не мало успіху. Вільгельм разом з академіками НАН України А.В. Скороходом і О.Г. Сітенком як делегати з'їду стали свідками початку нової ери в історії України — початку встановлення незалежної держави.

Пам'ятаю, як після закінчення роботи з'їзду його учасники пішки попрямували проспектом Перемоги, а далі — бульваром Шевченка до пам'ятника Тарасу Шевченку. Ці хвилюючі моменти перекликаються з подіями Помаранчевої Революції, ніби говорять самі за себе: "Нас багато і нас не подолати! Ми тебе, владо, не боїмося!". Після з'їзду деякі члени НАН України підходили до Вільгельма зі словами: "Я не думал, что Вы (или ты), Вильгельм, такой несерьезный человек. Разве можно посещать такие сборища?" Але після здобуття Україною незалежності ці вчені швидко перефарбувались у жовто-блакитний колір, щоб ще вище піднятися службовою драбиною і не нехтували високими урядовими нагородами. І мушу зазначити, що, на превеликий жаль, їм це вдалося.

При всій зайнятості наукою Вільгельма як патріота України завжди хвилювало національне питання. Він брав активну участь у міжнародних наукових конференціях і конгресах, присвячених цим питанням: у Першому конгресі міжнародної асоціації україністів (Київ, вересень 1990 року), у Міжнародній науковій конференції "Українські Карпати: етнос, історія, культура" (Ужгород, серпень 1991 року) та ін.

Під час Першого міжнародного з'їзду україністів у Києві, в Інституті математики окремо відбулася зустріч учасників з'їзду із Закарпаття, Пряшівщини (Словакія), Войоводини (Югославія) та вихідців з цих країв, що живуть за океаном (у США та Австралії). Організатором зустрічі був Вільгельм. Відбулася нарада стосовно відродження Закарпаття та споріднених спільнот в Югославії й Чехословаччині, обговорено питання щодо виниклого "русинства" в Закарпатті та спроб деяких людей просувати концепцію окремої "русинської народності". Вищезгадану конференцію в Ужгороді 1991 р. проведено за ухвалою зустрічі в Інституті математики. На Третьому з'їзді закарпатських українців, який відбувся 25–27 вересня 1992 р. в оселі ім. Ольжича в Лігайтоні, Вільгельм говорив про піднесення національної свідомості населення свого краю — Закарпаття, про необхідність сприяння вищій освіті через розбудову Ужгородського державного університету, про засудження чужих і штучно підтримуваних заходів з розколу закарпатського етносу бацилами русинізму, про культурні й наукові проблеми Закарпаття.

Вільгельм був патріотом України і за її межами. Під час перебування в наукових відрядженнях за кордоном ще за час існування СРСР Вільгельм зустрічався і активно спілкувався з українською діаспорою і вченими з української діаспори: професорами І. Фізером, В. Маркусем, Р.Р. Андрушківом, Р. Воронкою, В. Петришиним, О. Бедрієм, О. Біланюком, А. Кіпою, І. Капом, А. Гарасем, Р. Куцем, Й. Данком, Л. Майстренком, Д. Антоновичем, Ю. Даревичем та ін. Українську діаспору в далекому зарубіжжі цікавило все, що стосувалося подій в Україні. Темою спілкування чи виступів Вільгельма в ті роки була розповідь про єдність всіх українців, про появу взаємної довіри між українцями, власного сильного національного духу, віри у власні сили, необхідність послідовної жертовної праці, боротьби і виразної волі народу України до встановлення національного ідеалу — здобуття незалежності та створення політичного спрямованого руху НРУ, що здатен до витворення незалежної України. Після розвалу СРСР Вільгельм розповідав про своє бачення розбудови національної держави та відродження політичного, культурного, суспільного та економічного життя в Україні.

В незалежній Україні Вільгельм був постійним членом Міжнародного комітету з питань науки і культури при НАН України і намагався втілити в життя ідеї, задуми і мрії наукової співпраці із ученими з діаспори.

Щоб краще зрозуміти ставлення Вільгельма до всього українського, наведу уривки із спогадів осіб, які його добре знали.

Уривки з листа Діани Стасюк (тепер Діана Парке), яка проживала по сусідству з нами, була другом сім'ї і подругою нашої доньки Маріанни: "Після 35 моїх найкращих років на моїй рідній землі я переїхала в США, де мешкав мій чоловік. Невдовзі не стало Вільгельма Ілліча, якого я знала все моє свідоме життя. Але дорогий моєму серцю українець залишився для мене живим, тільки десь далеко, де до болю все рідне — земля, сонце, вітер, вишні, вишивані рушники і любий Київ з таким смачним українським хлібом.

Є люди, зустріч з якими дуже важлива. Такою людиною для мене був Вільгельм Ілліч. Трудар науки. Учений — велетень у математиці. Все життя, без перебільшення, пам'ятатиму його. Працювати, працювати, працювати — це було його традиційним вітанням при зустрічі. І він працював для своєї науки, для математики, яку він любив. Його праці — книжки, статті — самі говорять про те, краще за мене.

Учні, яких він залишив по всьому світу і найбільше на рідній землі Україні, впевнена, скажуть слова подяки цій чудовій людині. Учні, для яких він завжди мав час і душу, щоб допомогти, підперти, роз'яснити, з'ясувати, вибігати, відшукати, подзвонити і бути теплим та життєрадісним, але вимогливим і принциповим водночас.

Батько, якого пошукати. Сильний, мудрий, розумний, але терплячий, добрий і справедливий. І неможливо було б говорити про Вільгельма Ілліча і промовчати за те, що є для мене дуже важливим, що було важливим і для нього. За те, що вважається у великій літературі за обов'язок і честь лицарів. Це була людина — ПАТРІОТ. ПАТРІОТ, де кожна літера велика. Патріот України! Я згадала велику літературу, ніби беручи її в помічниці, бо чомусь Україні відмовлено в тих великих ідеалах і мірках, в яких майже усім іншим народам, а особливо росіянам, не відмовлено. Це почуття любові і гордості за свою мову, історію, за своє коріння, за свою землю, просто тому, що ти СИН цієї країни.

Велика людина жила життям простим, але повним змісту, жила одним подихом із своєю країною, яка тільки піднімається на ноги, коли їй так тяжко і багато хто бажає, щоб вона впала. Він любив до нестями свою рідну Україну, любив в усі часи і незгоди, коли це було і є до цього часу не до вподоби багатьом. Почуттям чистим і чесним.

До речі, чесність і несхибність Вільгельма Ілліча теж заслуговують на те, щоб сказати за них особливо. Чесність у науці, чесність у стосунках на роботі, у сім'ї, чесність у великому і малому, у поглядах і стосунках. Впевнена, що якби пан Вільгельм не був таким, то за часів СРСР він досягнув би набагато вищого соціального стану. Але він був несхибним і принциповим, яскравим і сильним у науці, у любові до УКРАЇНИ.

Шана Тобі, пане Вільгельме, від твоєї землі. Вона, вистраждана земля України, знає, що ТИ її любив, як ніхто. Ти, пане Вільгельме, живеш у моєму серці. Молодий і сильний. Сповнений розуму, любові, сили і тепла. Тепла, яке мав до своїх друзів, знайомих, сусідів.

Вільгельм Ілліч — це людина, зустріч з якою робить нас кращими. Це син України, яким Україна може пишатися."

Наведу ще кілька рядків із спогадів про Вільгельма, як про людину і людину-патріота України.

Віктор Григорович Мартинюк, інженер-теплоенергетик, президент бадмінтонного клубу "Олімп": "Вільгельм Ілліч, або як ми його всі ласкаво і з великою повагою називали просто Віля, був для нас взірцем культури, шляхетності і порядності. Познайомились ми з ним на бадмінтонних кортах, які споруджено на Республіканському стадіоні за моєї ініціативи і участі. Однак особисто для мене Віля був не просто партнер по бадмінтону. Він був порадником і в спортивних справах, і в побутових, і в політичних, і в ділових. Ми всі бадмінтоністи, і я особисто, безжалісно використовували його глибокі знання не тільки з питань науки, а і з тих, часом не дуже критичних, ситуацій, які у кожного можуть виникнути. Віля нікому не відмовив. Завжди згадую один тривожний для мене час і Вілю, який мене врятував своєю прагматичною порадою. Віля — єдина людина, з якою я спілкувався українською мовою. Він був великим патріотом України. Підросло нове покоління бадмінтоністів, але ми, ветерани, з великою теплотою і повагою згадуємо нашого спортивного колегу, професора Вілю Фущича."

Васильєви: "З 1971 року ми жили по-сусідськи з сім'єю Вільгельма Фущича. З самого початку ми відчули, що це сім'я освічених, культурних, привітних учених-інтелігентів, щирих українців-закарпатців. Нас приємно вразило, що вони спілкуються українською мовою. Вільгельм намагався прищепити любов до України багатьом, з ким він спілкувався. Він говорив: "Процес людської свідомості рухається повільніше, ніж рухається прогрес науки, тому держава і нація можуть бути знаними у світі тільки завдяки їх видатним досягненням у різних галузях науки". Він любив українську націю і виховував українських учених, досягненнями яких примножено вклад у математику, а через неї і визнання держави України" .

У вирі напружених буднів, інтенсивної роботи на науковій, педагогічній і суспільно-громадській ниві Вільгельм не поривав зв'язків із рідним Закарпаттям, цікавився життям земляків, допомагав чим міг у розв'язанні проблем краю, зокрема, у розвитку науки, підготовці наукових кадрів.

Вільгельм був патріотом свого Ужгородського університету. Він ніколи не обходив його стороною. Під час перебування в Ужгороді завжди відвідував рідні йому математичний і фізичний факультети. Після поїздок у престижні університети світу і участі в міжнародних конгресах, симпозіумах, конференціях він завжди знаходив можливість поділитися новими результатами, отриманими різними науковими школами. Вільгельм обговорював з друзями-колегами потенціальні теми дисертаційних робіт, перспективні наукові напрямки, проблеми, теми, які можуть бути успішно виконані на факультетах. Підтримував порадою молодих спеціалістів та аспірантів.

Вільгельм запрошував друзів — колег з Ужгородського університету — приїжджати в Київ на семінари в Інститут математики для обговорення отриманих результатів, представляв на вченій раді Інституту математики їх роботи до захисту, виступав опонентом їх кандидатських і докторських робіт не тільки в Інституті математики, але й в інших інститутах НАН України.

Вільгельм був ініціатором утворення Товариства закарпатців, які проживають у м. Києві, і став разом з членом-кореспондентом НАН України, професором Олексою Мишаничем першим його співголовою. Перше зібрання земляків відбулося 8 грудня 1994 року в Будинку вчених (вул. Володимирська 45), на якому були присутні близько 200 чоловік. Спілкуючись, обмінюючись вітаннями, водночас обговорювали питання, важливі для Закарпаття: 120-річчя народження Августина Волошина, 55-річчя подій у Карпатській Україні в березні 1939 р., 50-річчя возз'єднання Закарпаття з Україною.

Започаткувавши видання міжнародного журналу "Journal of Nonlinear Mathematical Physics" і ставши його головним редактором і видавцем, Вільгельм зробив важливий внесок у популяризацію досягнень українських учених серед світової наукової спільноти. Вільгельм усвідомлював, що збереження і розвиток наукового потенціалу нашої України — один із стратегічних напрямків її політики, бо держава, в якій не розвивається наука, не має перспективи і залишиться на узбіччі цивілізації.

Коли Національна академія наук України, де сконцентровані потужні наукові сили, опинилася в катастрофічному стані, він став одним з організаторів проведення загальних зборів Відділення математики НАН України. Учасники зборів звернулися з відкритим листом до Президента і голови Уряду України із закликом негайного здійснення ряду кардинальних заходів, спрямованих на розвиток потенціалу нашої держави, зокрема, на збереження високого міжнародного наукового рівня НАН України, який є загальновизнаним у всьому світі.

Вільгельм сповідував українство, працював на національну ідею через виховання покоління українських учених.

Життя — це пошук точки опори, аби з неї при нагоді підстрибнути і встигнути вхопити своє щастя. Впевнено можу сказати, що такими точками опори для Вільгельма були три чудові людини, прекрасні вчені і педагоги: професор Юрій Мілітонович Ламсадзе, академік НАН України, професор Остап Степанович Парасюк і академік Російської академії наук, професор Володимир Георгійович Кадишевський.

Саме лекції Ю.М. Ламсадзе під час навчання в Ужгородському університеті з квантової механіки, квантової електродинаміки, квантової теорії поля та ряду інших спеціальних курсів, навчальні та наукові семінари Остапа Степановича з найбільш фундаментальних, ключових проблем теоретичної і математичної фізики стали основою формування Вільгельма як науковця.

А ось круті підйоми в науці і житті Вільгельмові протягом всього наукового життя допомагав долати видатний учений Росії, професор, академік Російської Академії наук, директор Об'єднаного інституту ядерних досліджень (м. Дубна

Московської області), член Президії РАН В.Г. Кадишевський. Вільгельм познайомився з тоді ще молодим професором Кадишевським (Володимиром, як Вілі його називав) у 1971 р., коли поїхав у Дубну в ОІЯД доповісти результати докторської дисертації на науковому семінарі Лабораторії теоретичної фізики, яку в той час очолював член-корреспондент АН СРСР, професор Д.І. Блохінцев. Ставши одним з опонентів докторської дисертації, Володимир надалі став Вільгельмові другом, який завжди в скрутні для Вілія моменти життя підставляв своє плече, допомагаючи порадою, словом і ділом. Досить сказати, що незважаючи на зайнятість, Володимир завжди знаходив час зателефонувати і морально підтримати і підбадьорити Вілія (особливо під час його хвороби). Він активно підтримував наукову школу Вільгельма і його учнів, приїжджав у Київ опонувати докторські і кандидатські дисертації, за що Вільгельм був йому дуже вдячний. В день, коли не стало Вільгельма, Володимир перебував у Парижі, але через секретаря ОІЯД, яким він в той час уже керував, передав свою біль з цього приводу і причину, з якої він не може приїхати. Після повернення з Парижа Володимир спеціально приїхав у Київ, щоб зі мною побувати на могилі Вільгельма. І так буває щоразу, коли він приїжджає в Київ.

Нагороди. Робота Вільгельма відзначена різними нагородами, в тому числі і урядовими. За успіхи в розвитку математичної науки та активну участь у громадському житті указом від 28 грудня 1984 р. Вільгельм нагороджений Грамотою Президії Верховної Ради УРСР. За цикл робіт “Аналітичні методи дослідження динамічних систем” Президія академії наук Української РСР указом від 29 січня 1987 р. присудила Вільгельму премію ім. М.М. Крилова.

Гідним визнанням заслуг Вільгельма перед наукою стало його обрання членом-кореспондентом Національної академії наук України в січні 1988 р. За значний внесок у розвиток математичної фізики і підготовку спеціалістів найвищого гатунку Вільгельм у вересні 1994 р. стає Соросівським професором, одним із перших десяти серед математиків України. За цикл монографій “Функціонально-аналітичні та групові методи сучасної математичної фізики” указом Президента України від 3 грудня 2001 р. Вільгельму присуджена (посмертно) Державна премія України в галузі науки і техніки 2001 р.

В одну із своїх останніх робіт “Що таке швидкість електромагнітного поля?” Вільгельм вклав весь накопичений попередній досвід у царині фізики. Він намагався математично довести, що швидкість електромагнітного поля у вакуумі, яка до цього часу вважається постійною величиною (300000 км/с), насправді є змінною. Час покаже, чи це насправді так. Доля УЧЕНОГО чи його відкриття не завжди є легкими. Адже іноді треба, щоб минули десятиріччя, поки людство усвідомить, що висунута ученим гіпотеза підтверджується в природі, або реалізується на практиці.

Вільгельм у свої 59 років не розгубив юнацького запалу, здатності йти новими, незвіданими шляхами, йти впевнено назустріч невідомому. Він продовжував йти назустріч невідомому так само впевнено, як впевнено виїхав літом 1953 р. з Сільця до Ужгорода, щоб вступити в університет; як студентом вибрав за спеціальність теоретичну фізику після появи на фізичному факультеті молодого кандидата наук, прекрасного фахівця, блискучого педагога і прекрасного ученого, випускника і аспіранта Фізичного інституту ім. Лебедєва АН СРСР Ю.М. Ломсадзе; як у 1960 р. впевнено виїхав у Київ, щоб поступити в аспірантуру, де доля його звела з чудовим, дивовижним, видатним ученим, його учителем О.С. Парасюком, в якого почуття, школа, розум, серце присутні у повному комплексі і з уроків якого Вільгельм виніс щось особливе, унікально особисте, що дозволило йому стати не просто науковим співробітником, а ученим, який дивиться вперед.

Успадкувавши від своїх батьків основні життєві принципи — нікому не робити зла, старатися жити по совісті, багато і чесно працювати, Вільгельм став людиною душевно щедрою, відкритою, чистою і зворушливою, людиною, яка випромінювала потужний потік позитивної енергії. Доля не стелила Вільгельмові під ноги м'який килим, але він, обравши свою дорогу, не зійшов з неї, хоча доводилося долати круті підйоми. І саме завдяки цій наполегливості, працьовитості він і досяг таких висот.

Тяжко хворий, Вільгельм не забув свій колектив, він звернувся листом до почесного і діючого директорів Інституту математики академіків НАН України Ю.О. Митропольського і А.М. Самойленка з проханням зберегти відділ прикладних досліджень та його науковий напрямок і призначити керівником відділу його учня професора А.Г. Нікітіна. Згідно з рішенням вченої ради і дирекції Інституту математики відділ прикладних досліджень збережено, за що їм велика подяка. Під керівництвом Анатолія Гліբовича Нікітіна відділ продовжує успішно працювати за тематикою, започаткованою Вільгельмом, і розвивати своїми науковими працями концепції симетрійного аналізу, запропоновані Вільгельмом. Я вдячна провідним ученим Інституту математики, які завжди підтримували Вільгельма як ученого і надалі підтримують його школу — академікам НАН України І.О. Луковському, І.В. Скрипнику та ін.

Вільгельм любив свою рідну землю, свій народ, свій дім, свою сім'ю, свою роботу і учнів. Він був яскравим ученим як за обдарованістю, так і за людськими якостями, сплавом людяності і відкритості, являв собою унікальну наукову індивідуальність.

Вільгельм витрачав себе безоглядно і щедро, залишаючись вірним своїм ідеалам совісті. Хочеться закінчити цей короткий допис життєвого шляху Вільгельма народною мудрістю: "Не даремно прожив свій вік той, хто збудував будинок, посадив дерево, виховав дітей". Вільгельм збудував і облаштував велику наукову оселю, ім'я якої — українська школа симетрійного аналізу. Він виховав 47 кандидатів, 13 його учнів стали докторами фізико-математичних наук. І зросли, зацвіли та дали свої плоди дерева знань, що посадив їх Вільгельм.

Надіюсь, що учні, виховані Вільгельмом, будуть пам'ятати співпрацю із своїм Учителем, не забудуть його допомогу у вирішенні їхніх проблем, підтримку, любов, добро, працелюбність, вимогливість, принциповість, творче натхнення, талант, завзятість, енергію, які вони черпали від нього, збагачуючись духовно та інтелектуально, і будуть пам'ятати, що роки, проведені поряд із своїм Учителем, були одними з кращих років в їхньому житті.

Як український учений-патріот Вільгельм добре знаний за межами України. Він гідно ніс світом естафету славетних українських учених і примножив своїми працями славу України у світі. Сподіваюсь, що таким щедрим, талановитим, вірним своїм ідеалам, своїй совісті, істинним метром української науки залишиться Вільгельм у нашій пам'яті.

Лютий 2005

Перелік наукових праць В.І. Фущича

Монографії

1. *Фущич В.И., Никитин А.Г.* Симметрии уравнений Максвелла. — Киев: Наук. думка, 1983. — 200 с.
2. *Fushchych W.I., Nikitin A.G.* Symmetries of Maxwell's equations. — Dordrecht: D. Reidel, 1987. — 214 p.
3. *Фущич В.И., Штелень В.М., Серов Н.И.* Симметрийный анализ и точные решения нелинейных уравнений математической физики. — Киев: Наук. думка, 1989. — 336 с.
4. *Фущич В.И., Никитин А.Г.* Симметрия уравнений квантовой механики. — Москва: Наука, 1990. — 400 с.
5. *Фущич В.И., Баранник Л.Ф., Баранник А.Ф.* Подгрупповой анализ групп Галилея, Пуанкаре и редукция нелинейных уравнений. — Киев: Наук. думка, 1991. — 304 с.
6. *Фущич В.И., Жданов Р.З.* Нелинейные спинорные уравнения: симметрия и точные решения. — Киев: Наук. думка, 1992. — 288 с.
7. *Fushchych W.I., Shtelen W.M., Serov N.I.* Symmetry analysis and exact solutions of equations of nonlinear mathematical physics. — Dordrecht: Kluwer Academic Publishers, 1993. — 460 p.
8. *Fushchych W.I., Nikitin A.G.* Symmetries of equations of quantum mechanics. — New York: Allerton Press Inc., 1994. — 480 p.
9. *Fushchych W.I., Zhdanov R.Z.* Symmetries and exact solutions of nonlinear Dirac equations. — Kyiv: Mathematical Ukraina Publisher, 1997. — 384 p.

Наукові статті

1. *Ломсадзе Ю.М., Лендьел В.И., Кривский И.Ю., Фущич В.И., Химич И.В., Лукин Л.П., Эрнст Б.М.* О применении модифицированного метода возмущений к интерпретации нуклон-нуклонных рассеяний // Тр. межвузовской конф. "Проблемы современной теории элементарных частиц", 2–6 октября 1958 г. — Ужгород, 1959. — № 2. — С. 211–215.
2. *Кривський І.Ю., Ломсадзе Ю.М., Фущич В.І., Хіміч І.В.* До проблеми радіаційного розпаду π^--мезона // Укр. фіз. журн. — 1960. — **5**, № 6. — С. 777–780.
3. *Кривский И., Ломсадзе Ю., Фущич В., Химич И.* О двухнейтринной аннигиляции электрона с позитроном // Докл. и сообщения Ужгород. гос. ун-та. Серия физ.-мат. наук. — 1960. — № 3. — С. 23–24.
4. *Фущич В.И.* Аналитические свойства амплитуд рождения в одночастичном приближении как функции двух переменных // Укр. мат. журн. — 1963. — **15**, № 2. — С. 227–232.
5. *Фущич В.И.* Аналитические свойства вклада в амплитуду рождения от простейших диаграмм // Науч. конф. инженеров, аспирантов и младших научных сотрудников. — Киев: Ин-т математики АН УССР, 1963. — С. 94–99.

6. *Фущич В.І.* Аналітичні властивості узагальнених петльових діаграм // Доп. АН УРСР. — 1964. — № 1. — С. 7–9.

7. *Фущич В.І.* Про полюси амплітуди народження для процесу $\pi + d \to d + \pi + \pi$ // Укр. фіз. журн. — 1964. — **9**, № 2. — С. 117–121.

8. *Фущич В.І.* Аналітичні властивості амплітуд народження і метод Чу–Лоу // Укр. фіз. журн. — 1964. — **9**, № 3. — С. 277–282.

9. *Коломыцев В.И., Фущич В.И.* Аналитические свойства амплитуды рассеяния, соответствущей одному классу диаграмм Фейнмана // Укр. мат. журн. — 1964. — **16**, № 5. — С. 610–623.

10. *Фущич В.И.* Об аналитических свойствах некоторых вершинных амплитуд в теории возмущений // Укр. мат. журн. — 1965. — **17**, № 3. — С. 137–141.

11. *Фущич В.И.* Унитарная симметрия и группа Пуанкаре // ЖЭТФ. Письма в редакцию. — 1966. — **2**, № 4. — С. 157–160.

12. *Фущич В.І.* Про представлення групи де Сіттера // Укр. фіз. журн. — 1966. — **11**, № 8. — С. 907–908.

13. *Фущич В.И.* О вложении алгебры Пуанкаре // Укр. фіз. журн. — 1967. — **12**, № 5. — С. 741–746.

14. *Фущич В.И.* Об операторах взаимодействия // Укр. фіз. журн. — 1967. — **12**, № 8. — С. 1331–1338.

15. *Fushchych W.I.* A relativistically invariant mass operator // Ukr. Phys. J. — 1968. — **13**, N 3. — P. 256–262.

16. *Fushchych W.I., Krivsky I.Yu.* On a possible approach to the variable-mass problem // Nuclear Phys. B. — 1968. — **7**. — P. 79–82.

17. *Фущич В.И., Кривский И.Ю.* О волновых уравнениях в 5-пространстве Минковского. — Киев, 1968. — 38 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–68-72).

18. *Fushchych W.I.* Equations of motion in odd-dimensional spaces and T-, C-invariance. — Киев, 1969. — 13 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–69-17).

19. *Fushchych W.I., Krivsky I.Yu.* On representations of the inhomogeneous de Sitter group and equations in five-dimensional Minkowski space // Nuclear Phys. B. — 1969. — **14**. — P. 573–585.

20. *Fushchych W.I., Krivsky I.Yu.* On representations of the inhomogeneous de Sitter group and on equations of the Schrödinger–Foldy type. — Киев, 1969. — 22 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–69-1).

21. *Фущич В.И., Сокур Л.П.* Уравнения Баргмана–Вигнера на неоднородной группе де Ситтера. — Киев, 1969. — 38 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–69-33).

22. *Костирко І.І., Фущич В.І.* Про динамічну алгебру осцилятора в просторі з індефінітною метрикою // Укр. фіз. журн. — 1969. — **14**, № 7. — С. 1113–1115.

23. *Кривский И.Ю., Романенко Г.Д., Фущич В.И.* Уравнения типа Кеммера–Дэффина в пятимерном пространстве Минковского // Теор. мат. физика. — 1969. — **1**, № 2. — С. 242–250.

24. *Fushchych W.I.* Representations of the total inhomogeneous de Sitter group and equations in a five-dimensional approach. I. — Киев, 1970. — 40 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–70-4).
25. *Фущич В.И.* Представления полной неоднородной группы де Ситтера и уравнения в пятимерном подходе. I // Теор. мат. физика. — 1970. — **4**, № 3. — С. 360–382.
26. *Fushchych W.I.* On the P- and T-non-invariant two-component equation for the neutrino // Nuclear Phys. B. — 1970. — **21**. — P. 321–330.
27. *Fushchych W.I.* On the T noninvariant two-component equation for a neutrino. — Киев, 1970. — 14 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–70-29).
28. *Fushchych W.I.* On the additional invariants of the relativistic equations of motion. — Киев, 1970. — 17 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–70-32).
29. *Fushchych W.I., Grishchenko A.L.* On the CP-noninvariant equations for the particle with zero mass and spin $s = \frac{1}{2}$ // Lettere al Nuovo Cimento. — 1970. — **4**, N 20. — P. 927–928.
30. *Fushchych W.I., Grishchenko A.L.* On two-component equations for zero mass particles. — Киев, 1970. — 22 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–70-88Е).
31. *Fushchych W.I., Grishchenko A.L., Nikitin A.G.* On the relativistic equations of motion without “redundant” components. — Киев, 1970. — 28 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–70-89Е).
32. *Fushchych W.I., Sokur L.P.* On equations of motion invariant with respect to the group $P(1,n)$. II. — Киев, 1970. — 17 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–70-32).
33. *Фущич В.И.* О дополнительной инвариантности релятивистских уравнений движения // Теор. мат. физика. — 1971. — **7**, № 1. — С. 3–12.
34. *Фущич В.И.* О PTC-неинвариантных лагранжианах. — Киев, 1971. — 8 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–71-2Р).
35. *Фущич В.И.* О PTC-неинвариантных лагранжианах // Теор. мат. физика. — 1971. — **9**, № 1. — С. 91–93.
36. *Фущич В.И., Грищенко А.Л., Никитин А.Г.* О релятивистских уравнениях движения без “лишних” компонент // Теор. мат. физика. — 1971. — **8**, № 2. — С. 192–205.
37. *Сокур Л.П., Фущич В.И.* Об уравнениях движения, инвариантных относительно группы $P(1,n)$. II // Теор. мат. физика. — 1971. — **6**, № 3. — С. 348–363.
38. *Fushchych W.I.* On the three types of relativistic equations for particles with nonzero mass // Lettere al Nuovo Cimento. — 1972. — **4**, N 9. — P. 344–346.
39. *Fushchych W.I.* P, T, C properties of the Poincaré invariant equations for massive particles // Lettere al Nuovo Cimento. — 1973. — **6**, N 4. — P. 133–137.
40. *Fushchych W.I., Nikitin A.G.* On the possible types of equations for zero-mass particles // Lettere al Nuovo Cimento. — 1973. — **7**, N 11. — P. 439–442.

41. *Fushchych W.I., Nikitin A.G.* On the Poincaré-invariant equations for particles with variable spin and mass. — Киев, 1973. — 19 с. — (Препр./АН УССР. Ин-т теор. физики; ИТФ–73-121Е).
42. *Fushchych W.I.* On the additional invariance of the Dirac and Maxwell equations // Lettere al Nuovo Cimento. — 1974. — **11**, N 10. — P. 508–512.
43. *Fushchych W.I.* On a motion equation for two particles in relativistic quantum mechanics // *Lettere al Nuovo Cimento.* — 1974. — **10**, N 4. — P. 163–167.
44. *Fushchych W.I.* Poincaré-invariant equations with a rising mass spectrum // Lettere al Nuovo Cimento. — 1975. — **14**, N 12. — P. 435–438.
45. *Fushchych W.I., Nikitin A.G.* On the Poincaré-invariant equations for particles with variable spin and mass // Rep. Math. Phys. — 1975. — **8**, N 1. — P. 33–48.
46. *Fushchych W.I., Nikitin A.G., Salogub V.A.* On the equations of motion for particles with arbitrary spin in nonrelativistic mechanics // Lettere al Nuovo Cimento. — 1975. — **14**, N 13. — P. 483–488.
47. *Никитин А.Г., Фущич В.И., Юрик И.И.* Редукция неприводимых унитарных представлений обобщенных групп Пуанкаре по их подгруппах. — Киев, 1975. — 32 с. — (Препр./АН УССР. Ин-т математики; ИМ–75-5).
48. *Фущич В.И.* О дополнительной инвариантности уравнения Клейна–Гордона–Фока // Докл. АН СССР. — 1976. — **230**, № 3. — С. 570–572.
49. *Fushchych W.I., Nikitin A.G.* On the Galilean-invariant equations for particles with arbitrary spin // Lettere al Nuovo Cimento. — 1976. — **16**, N 3. — P. 81–85.
50. *Фущич В.И., Сегеда Ю.Н.* О группах инвариантности некоторых уравнений релятивистской квантовой механики // Укр. мат. журн. — 1976. — **28**, № 6. — С. 844–849.
51. *Владіміров С.А., Фущич В.I.* Максимальна та мінімальна групи симетрії атома водню // Укр. фіз. журн. — 1976. — **21**, № 9. — С. 1460–1462.
52. *Никитин А.Г., Фущич В.И., Юрик И.И.* Редукция неприводимых унитарных представлений обобщенных групп Пуанкаре по их подгруппах // Теор. мат. физика. — 1976. — **26**, № 2. — С. 206–220.
53. *Никитин А.Г., Сегеда Ю.Н., Фущич В.И.* О дополнительной инвариантности уравнений Кеммера–Дэффина и Рариты–Швингера // Теор. мат. физика. — 1976. — **29**, № 1. — С. 82–93.
54. *Фущич В.И.* Групповые свойства дифференциальных уравнений квантовой механики // Проблемы асимптотической теории нелинейных колебаний. — Киев: Наук. думка, 1977. — С. 238–246.
55. *Фущич В.И., Никитин А.Г.* Дифференциальные уравнения движения первого и второго порядка для частиц с произвольным спином. — Киев, 1977. — 48 с. — (Препр./АН УССР. Ин-т математики; ИМ–77-3).
56. *Fushchych W.I., Nikitin A.G.* On the new invariance groups of the Dirac and Kemmer–Duffin–Petiau equations // Lettere al Nuovo Cimento. — 1977. — **19**, N 9. — P. 347–352.
57. *Фущич В.И., Онуфрийчук С.П.* О группах инвариантности одного класса счетной системы уравнений первого порядка с частными производными // Докл. АН СССР. — 1977. — **235**, № 5. — С. 1056–1059.

58. *Фущич В.И., Сегеда Ю.Н.* О новой алгебре инвариантности свободного уравнения Шредингера // Докл. АН СССР. — 1977. — **232**, № 4. — С. 800–801.

59. *Фущич В.И.* О новом методе исследования групповых свойств систем дифференциальных уравнений в частных производных // Теоретико-групповые методы в математической физике. — Киев: Ин-т математики АН УССР, 1978. — С. 5–44.

60. *Фущич В.И., Никитин А.Г.* О группе инвариантности квази-релятивистского уравнения движения // Докл. АН СССР. — 1978. — **238**, № 1. — С. 46–49.

61. *Fushchych W.I., Nikitin A.G., Salogub V.A.* On the non-relativistic motion equations in the Hamiltonian form // Rep. Math. Phys. — 1978. — **13**, N 2. — P. 175–185.

62. *Фущич В.И., Никитин А.Г.* Пуанкаре-инвариантные уравнения движения частиц произвольного спина // Физика элементарных частиц и атомного ядра. — 1978. — **9**, вып. 3. — С. 501–553.

63. *Fushchych W.I., Nikitin A.G.* On the invariance groups of relativistic equations for the spinning particles interacting with external fields // Lettere al Nuovo Cimento. — 1978. — **21**, N 16. — P. 541–546.

64. *Fushchych W.I., Nikitin A.G.* Conformal invariance of relativistic equations for arbitrary spin particles // Lett. Math. Phys. — 1978. — **2**. — P. 471–475.

65. *Фущич В.И., Никитин А.Г.* Групповые свойства уравнений Максвелла // Теоретико-групповые методы в математической физике. — Киев: Ин-т математики АН УССР, 1978. — С. 45–80.

66. *Никитин А.Г., Фущич В.И.* Пуанкаре-инвариантные дифференциальные уравнения для частиц произвольного спина // Теор. мат. физика. — 1978. — **34**, № 3. — С. 319–333.

67. *Фущич В.И.* О новом методе исследования групповых свойств уравнений математической физики // Докл. АН СССР. — 1979. — **246**, № 4. — С. 846–850.

68. *Fushchych W.I., Nikitin A.G.* On the new invariance group of Maxwell equations // Lettere al Nuovo Cimento. — 1979. — **24**, N 7. — P. 220–224.

69. *Fushchych W.I., Nikitin A.G.* On the new invariance algebras of relativistic equations for massless particles // J. Phys. A: Math. Gen. — 1979. — **12**, N 6. — P. 747–757.

70. *Федорчук В.М., Фущич В.И.* О подгруппах обобщенной группы Пуанкаре // Тр. междунар. семинара "Теоретико-групповые методы в физике", Звенигород, 28–30 ноября 1979 г. — М.: Наука, 1980. — Т. 1. — С. 61–66.

71. *Фущич В.И., Наконечный В.В.* Теоретико-алгебраический анализ уравнений Ламе // Укр. мат. журн. — 1980. — **32**, № 2. — С. 267–272.

72. *Fushchych W.I., Nikitin A.G.* Reduction of the representations of the generalised Poincaré algebra by the Galilei algebra // J. Phys. A: Math. Gen. — 1980. — **13**. — P. 2319–2330.

73. *Фущич В.И., Сегеда Ю.Н., Редченко Г.А.* Инвариантные системы уравнений в обобщенной механике // Укр. мат. журн. — 1980. — **32**, № 4. — С. 569–576.
74. *Никитин А.Г., Фущич В.И.* Уравнения движения для частиц произвольного спина, инвариантные относительно группы Галилея // Теор. мат. физика. — 1980. — **44**, № 1. — С. 34–46.
75. *Никитин А.Г., Фущич В.И.* Галилеевски инвариантные уравнения движения со спин-орбитальным взаимодействием // Тр. междунар. семинара "Теоретико-групповые методы в физике", Звенигород, 28–30 ноября 1979 г. — М.: Наука, 1980. — Т. 2. — С. 35–41.
76. *Боголюбов Н.Н., Митропольський Ю.А., Фущич В.И.* Остап Степанович Парасюк (к 60-летию со дня рождения) // Укр. мат. журн. — 1981. — **33**, № 6. — С. 800–801.
77. *Фущич В.И.* Симметрия в задачах математической физики // Теоретико-алгебраические исследования в математической физике. — Киев: Ин-т математики АН УССР, 1981. — С. 6–28.
78. *Фущич В.И.* Об одном способе исследования групповых свойств интегро-дифференциальных уравнений // Укр. мат. журн. — 1981. — **33**, № 6. — С. 834–838.
79. *Fushchych W.I., Moskaliuk S.S.* On some exact solutions of the nonlinear Schrödinger equation in three spatial dimensions // Lettere al Nuovo Cimento. — 1981. — **31**, N 16. — P. 571–576.
80. *Фущич В.И., Никитин А.Г.* Нерелятивистские уравнения движения для частиц с произвольным спином // Физика элементарных частиц и атомного ядра. — 1981. — **12**, вып. 5. — С. 1157–1219.
81. *Фущич В.И., Владимиров В.А.* О дополнительной инвариантности уравнений для векторных полей // Докл. АН СССР. — 1981. — **257**, № 5. — С. 1105–1108.
82. *Fushchych W.I., Nikitin A.G.* On the new symmetries of Maxwell equations // Czechoslovak J. Phys. B. — 1982. — **32**. — P. 476–480.
83. *Фущич В.И., Серов Н.И.* О точных решениях уравнения Борна–Инфельда // Докл. АН СССР. — 1982. — **263**, № 3. — С. 582–586.
84. *Fushchych W.I., Shtelen W.M.* The symmetry and some exact solutions of the relativistic eikonal equation // Lettere al Nuovo Cimento. — 1982. — **34**, N 16. — P. 498–502.
85. *Фущич В.И., Тычинин В.А.* О линеаризации некоторых нелинейных уравнений с помощью нелокальных преобразований. — Киев, 1982. — 54 с. — (Препр./АН УССР. Ин-т математики; № 82.33).
86. *Фущич В.И.* О симметрии и частных решениях некоторых многомерных уравнений математической физики // Теоретико-алгебраические методы в задачах математической физики. – Киев: Ин-т математики АН УССР, 1983. — С. 4–23.
87. *Фущич В.И., Никитин А.Г.* О новых и старых симметриях уравнений Максвелла и Дирака // Физика элементарных частиц и атомного ядра. — 1983. — **14**, вып. 1. — С. 5–57.

88. *Фущич В.И., Селехман Н.А.* Интегро-дифференциальные уравнения, инвариантные относительно групп Галилея, Пуанкаре, Шредингера и конформной группы // Докл. АН УССР. — 1983. — № 5. — С. 21–24.

89. *Fushchych W.I., Serov N.I.* The symmetry and some exact solutions of the nonlinear many-dimensional Liouville, d'Alembert and eikonal equations // J. Phys. A: Math. Gen. — 1983. — **16**. — P. 3645–3656.

90. *Фущич В.И., Серов Н.И.* Симметрия и некоторые точные решения многомерного уравнения Монжа–Ампера // Докл. АН СССР. — 1983. — **273**, № 3. — С. 543–546.

91. *Фущич В.И., Серова М.М.* О точных решениях некоторых нелинейных дифференциальных уравнений, инвариантных относительно групп Евклида и Галилея // Теоретико-алгебраические методы в задачах математической физики. — Киев: Ин-т математики АН УССР, 1983. — С. 24–54.

92. *Фущич В.И., Серова М.М.* О максимальной группе инвариантности и общем решении одномерных уравнений газовой динамики // Докл. АН СССР. — 1983. — **268**, № 5. — С. 1102–1104.

93. *Fushchych W.I., Shtelen W.M.* The symmetry and some exact solutions of the relativistic eikonal equation (Correction) // Lettere al Nuovo Cimento. — 1983. — **36**, N 1. — P. 96–97.

94. *Fushchych W.I., Shtelen W.M.* On some exact solutions of the nonlinear Dirac equation // J. Phys. A: Math. Gen. — 1983. — **16**, N 2. — P. 271–277.

95. *Fushchych W.I., Shtelen W.M.* Conformal symmetry and new exact solutions of SU_2 Yang–Mills theory // Lettere al Nuovo Cimento. — 1983. — **38**, N 2. — P. 37–40.

96. *Фущич В.И., Штелень В.М.* Об инвариантных решениях нелинейного уравнения Дирака // Докл. АН СССР. — 1983. — **269**, № 1. — С. 88–92.

97. *Фущич В.И., Штелень В.М.* О линейных и нелинейных системах дифференциальных уравнений, инвариантных относительно группы Шредингера // Теор. мат. физика. — 1983. — **56**, № 3. — С. 387–394.

98. *Fushchych W.I., Shtelen W.M.* On some exact solutions of the nonlinear equations of quantum electrodynamics // Phys. Lett. B. — 1983. — **128**, N 3–4. — P. 215–217.

99. *Фущич В.И., Серов Н.И., Штелень В.М.* О некоторых точных решениях многомерных нелинейных уравнений Даламбера, Лиувилля, Дирака и уравнения эйконала // Тр. междунар. семинара "Теоретико-групповые методы в физике", Звенигород, 24–26 ноября 1982 г. — М.: Наука, 1983. — Т. 2. — С. 407–413.

100. *Fushchych W.I., Vladimirov V.A.* On the new conservation laws for vector field equations // J. Phys. A: Math. Gen. — 1983. — **16**, N 6. — P. 1921–1925.

101. *Никитин А.Г., Владимиров В.А., Фущич В.И.* О новых симметриях и законах сохранения для электромагнитного поля // Тр. междунар. семинара "Теоретико-групповые методы в физике", Звенигород, 24–26 ноября 1982 г. — М.: Наука, 1983. — Т. 2. — С. 421–428.

102. *Fushchych W.I.* Lie groups and exact solutions of multidimensional nonlinear d'Alembert, Liouville, Schrödinger and Dirac equations // Proc. of the second international conference on operator algebras, ideals, and their applications in theoretical physics (Leipzig, 1983). — Leipzig: Teubner, 1984. — Teubner-Texte Math., **67**. — P. 76–84.

103. *Fushchych W.I.* The symmetry, and exact solutions, of many-dimensional parabolic and hyperbolic nonlinear differential equations // Proc. of the Second International Symposium "Nonlinear and Turbulent Processes in Physics", Kyiv, 1983. — Harwood Academic Publ., 1984. — P. 1341–1345.

104. *Фущич В.И., Москалюк С.С., Серов Н.И.* О точных решениях нелинейных многомерных волновых уравнений // IX междунар. конф. по нелинейным колебаниям, Киев, 30.08–06.09 1981; Т. 1. Аналитические методы теории нелинейных колебаний. — Киев: Наук. думка, 1984. — С. 384–389.

105. *Fushchych W.I., Nikitin A.G.* On one- and two-particle Galilei-invariant wave equations for any spin // Nuovo Cimento A. — 1984. — **81**, N 3. — P. 644–660.

106. *Fushchych W.I., Seheda Yu.N.* Some exact solutions of the many-dimensional sine-Gordon equation // Lettere al Nuovo Cimento. — 1984. — **41**, N 14. — P. 462–464.

107. *Фущич В.И., Серов Н.И.* О симметрии и точных решениях некоторых нелинейных многомерных волновых уравнений // Всесоюз. школа молодых ученых "Комплексные методы в математической физике". — Донецк, 1984. — С. 187.

108. *Фущич В.И., Серов Н.И.* О некоторых точных решениях многомерного уравнения Эйлера–Лагранжа // Докл. АН СССР. — 1984. — **278**, № 4. — С. 847–851.

109. *Баранник Л.Ф., Фущич В.И.* Подалгебры алгебры Ли расширенной группы Пуанкаре $\tilde{P}(1,n)$. — Киев, 1985. — 52 с. — (Препр./АН УССР. Ин-т математики; № 85.90).

110. *Баранник Л.Ф., Лагно В.И., Фущич В.И.* Подалгебры обобщенной алгебры Пуанкаре $AP(2,n)$. — Киев, 1985. — 50 с. — (Препр./АН УССР. Ин-т математики; № 85.89).

111. *Фущич В.И.* О пуанкаре-, галилеево-инвариантных нелинейных уравнениях и методах их решения // Теоретико-групповые исследования уравнений математической физики. — Киев: Ин-т математики АН УССР, 1985. — С. 4–19.

112. *Фущич В.И., Баранник А.Ф., Баранник Л.Ф.* Непрерывные подгруппы обобщенной группы Галилея. I. — Киев, 1985. — 46 с. — (Препр./АН УССР. Ин-т математики; № 85.19).

113. *Fushchych W.I., Barannik A.F., Barannik L.F., Fedorchuk V.M.* Continuous subgroups of the Poincaré group $P(1,4)$ // J. Phys. A: Math. Gen. — 1985. — **18**. — P. 2893–2899.

114. *Fushchych W.I., Cherniha R.M.* The Galilean relativistic principle and nonlinear partial differential equation // J. Phys. A: Math. Gen. — 1985. — **18**. — P. 3491–3503.

115. *Фущич В.И., Корняк В.В.* Реализация на ЭВМ алгоритма вычисления нелокальных симметрий для уравнений типа Дирака. — Киев, 1985. — 18 с. — (Препр./АН УССР. Ин-т математики; № 85.20).

116. *Fushchych W.I., Nikitin A.G., Susloparov W.M.* Relativistic particle of arbitrary spin in the Coulomb and magnetic-monopole field // Nuovo Cimento A. — 1985. — **87**, N 4. — P. 415–424.

117. *Фущич В.И., Штелень В.М.* Конформная симметрия и точные решения нелинейных полевых уравнений // Укр. физ. журн. — 1985. — **30**, № 5. — С. 787–789.

118. *Fushchych W.I., Shtelen W.M.* On nonlocal transformations // Lettere al Nuovo Cimento. — 1985. — **44**, N 1. — P. 40–42.

119. *Fushchych W.I., Shtelen W.M., Serov N.I.* Some exact solutions of multidimensional nonlinear d'Alembert, Liouville, eikonal and Dirac equations // Group Theoretical Methods in Physics, Zvenigorod, 1982. — Harwood Academic Publ., 1985. — P. 489–496.

120. *Fushchych W.I., Shtelen W.M., Zhdanov R.Z.* On the new conformally invariant equations for spinor fields and their exact solutions // Phys. Lett. B. — 1985. — **159**, N 2/3. — P. 189–191.

121. *Фущич В.И., Цифра И.М.* О симметрии нелинейных уравнений электродинамики // Теор. мат. физика. — 1985. — **64**, № 1. — С. 41–50.

122. *Фущич В.И., Тычинин В.А., Жданов Р.З.* Нелокальная линеаризация и точные решения некоторых уравнений Монжа–Ампера, Дирака. — Киев, 1985. — 28 с. — (Препр./АН УССР. Ин-т математики; № 85.88).

123. *Фущич В.И., Жданов Р.З.* Точные решения нелинейных дифференциальных уравнений для спинорного и векторного поля // Теоретико-групповые исследования уравнений математической физики. — Киев: Ин-т математики АН УССР, 1985. — С. 20–30.

124. *Митропольский Ю.А., Ревенко И.В., Фущич В.И.* О симметрии, интеграле движения и некоторых частных решениях пространственной задачи трех тел // Докл. АН СССР. — 1985. — **280**, № 4. — С. 799–804.

125. *Nikitin A.G., Fushchych W.I., Vladimirov V.A.* New symmetries and conservation laws for electromagnetic fields // Group-Theoretical Methods in Physics, Zvenigorod, 1982. — Harwood Academic Publishers, 1985. — P. 497–505.

126. *Баранник А.Ф., Фущич В.И.* О непрерывных подгруппах псевдоортогональных и псевдоунитарных групп. — Киев, 1986. — 48 с. — (Препр./АН УССР. Ин-т математики; № 86.77).

127. *Баранник Л.Ф., Баранник А.Ф., Фущич В.И.* О непрерывных подгруппах обобщенной группы Пуанкаре $P(1,n)$ // Тр. третьего междунар. семинара "Теоретико-групповые методы в физике", Юрмала, 22–24 мая 1985 г. — М.: Наука, 1986. — С. 169–176.

128. *Barannik L.F. Barannik A.F., Fushchych W.I.* On the continuous subgroups of the generalised Poincaré group $P(1,n)$ // Group Theoretical Methods in Physics, Vol. I (Yurmala, 1985). — Utrecht: VNU Sci. Press, 1986. — P. 561–570.

129. *Баранник Л.Ф., Фущич В.И.* Операторы Казимира для обобщенных групп Пуанкаре и группы Галилея // Тр. третьего междунар. семинара "Теоретико-групповые методы в физике", Юрмала, 22–24 мая 1985 г. — М.: Наука, 1986. — С. 176–183.

130. *Barannik L.F., Fushchych W.I.* Casimir operators of the generalised Poincaré and Galilei groups // Group Theoretical Methods in Physics, Vol. II (Yurmala, 1985). — Utrecht: VNU Sci. Press, 1986. — P. 275–282.

131. *Баранник Л.Ф., Фущич В.И.* Инварианты подгрупп обобщенной группы Пуанкаре $P(1,n)$. — Киев, 1986. — 40 с. — (Препр./АН УССР. Ин-т математики; № 86.86).

132. *Фущич В.И.* О симметрии и точных решениях некоторых многомерных уравнений математической физики // Сб. науч. трудов "Исследования по теории функций комплексного переменного с приложениями к механике сплошных сред". — Киев: Наук. думка, 1986. — С. 146–160.

133. *Фущич В.И., Баранник А.Ф., Баранник Л.Ф.* Непрерывные подгруппы обобщенной группы Евклида // Укр. мат. журн. — 1986. — **38**, № 1. — С. 67–72.

134. *Фущич В.И., Чернига Р.М.* О точных решениях двух многомерных нелинейных уравнений шредингеровского типа. — Киев, 1986. — 44 с. — (Препр./АН УССР. Ин-т математики; № 86.85).

135. *Фущич В.И., Федорчук В.М., Федорчук И.М.* Подгрупповая структура обобщенной группы Пуанкаре и точные решения некоторых нелинейных волновых уравнений. — Киев, 1986. — 36 с. — (Препр./АН УССР. Ин-т математики; № 86.27).

136. *Фущич В.И., Штелень В.М., Жданов Р.З.* Конформно-инвариантное обобщение уравнения Дирака–Гейзенберга и его точные решения // Тр. третьего междунар. семинара "Теоретико-групповые методы в физике", Юрмала, 22–24 мая 1985 г. — М.: Наука, 1986. — С. 497–501.

137. *Fushchych W.I., Shtelen W.M., Zhdanov R.Z.* On the conformal generalization of the Dirac–Heisenberg equations for spinor field and some exact solutions // Group Theoretical Methods in Physics, Vol. I (Yurmala, 1985). — Utrecht: VNU Sci. Press, 1986. — 473–477.

138. *Фущич В.И., Славуцкий С.Л.* О нелинейном галилей-инвариантном обобщении уравнений Ламе // Докл. АН СССР. — 1986. — **237**, № 2. — С. 320–323.

139. *Фущич В.И., Славуцкий С.Л.* О симметрии некоторых уравнений идеальной жидкости // Сб. науч. трудов "Исследования по теории функций комплексного переменного с приложениями к механике сплошных сред". — Киев: Наук. думка, 1986. — С. 161–165.

140. *Фущич В.И., Цифра И.М.* Конформно-инвариантные нелинейные уравнения для электромагнитного поля // Тр. третьего междунар. семинара "Теоретико-групповые методы в физике", Юрмала, 22–24 мая 1985 г. — М.: Наука, 1986. — С. 501–505.

141. *Fushchych W.I., Tsyfra I.M.* Conformally invariant nonlinear equations of electromagnetic field // Group theoretical methods in physics, Vol. II (Yurmala, 1985). — Utrecht: VNU Sci. Press, 1986. — P. 111–116.

142. *Баранник Л.Ф., Фущич В.И.* Непрерывные подгруппы обобщенной группы Шредингера. — Киев, 1987. — 48 с. — (Препр./АН УССР. Ин-т математики; № 87.16).

143. *Barannyk L.F., Fushchych W.I.* On subalgebras of the Lie algebra of the extended Poincaré group $\tilde{P}(1,n)$ // J. Math. Phys. — 1987. — **28**, N 7. — P. 1445–1458.

144. *Fushchych W.I.* The symmetry and exact solutions of some multidimensional nonlinear equations of mathematical physics // Proc. of the XV International Conference on "Differential Geometric Methods in Theoretical Physics", Eds. H.D. Doebner, J.D. Hennig. — Singapore; New Jersey; Hong Kong: World Scientific, 1987. — P. 544–549.

145. *Фущич В.И.* Как расширить симметрию дифференциальных уравнений? // Симметрия и решения нелинейных уравнений математической физики. — Киев: Ин-т математики АН УССР, 1987. — С. 4–16.

146. *Фущич В.И.* О симметрии и точных решениях многомерных нелинейных волновых уравнений // Укр. мат. журн. — 1987. — **39**, № 1. — С. 116–123.

147. *Фущич В.И., Кривский И.Ю., Симулик В.М.* О векторных лагранжианах для электромагнитного и спинорного полей. — Киев, 1987. — 39 с. — (Препр./АН УССР. Ин-т математики; № 87.54).

148. *Fushchych W.I., Nikitin A.G.* On the new invariance algebras and superalgebras of relativistic wave equations // J. Phys. A: Math. Gen. — 1987. — **20**. — P. 537–549.

149. *Fushchych W.I., Serov N.I.* On some exact solutions of the three-dimensional non-linear Schrödinger equation // J. Phys. A: Math. Gen. — 1987. — **20**. — L929–L933.

150. *Фущич В.И., Штелень В.М.* О редукции и точных решениях нелинейного уравнения Дирака // Теор. мат. физика. — 1987. — **72**, № 1. — С. 35–44.

151. *Fushchych W.I., Tsyfra I.M.* On a reduction and solutions of non-linear wave equations with broken symmetry // J. Phys. A: Math. Gen. — 1987. — **20**. — L45–L48.

152. *Фущич В.И., Жданов Р.З.* Об одном обобщении метода разделения переменных для линейных систем дифференциальных уравнений // Симметрия и решения нелинейных уравнений математической физики. — Киев: Ин-т математики АН УССР, 1987. — С. 17–22.

153. *Fushchych W.I., Zhdanov R.Z.* On some exact solutions of a system of non-linear differential equations for spinor and vector fields // J. Phys. A: Math. Gen. — 1987. — **20**. — P. 4173–4190.

154. *Баранник Л.Ф., Фущич В.И.* О непрерывных подгруппах конформной группы пространства Минковского $R_{1,n}$. — Киев, 1988. — 48 с. — (Препр./АН УССР. Ин-т математики; № 88.34).

155. *Баранник Л.Ф., Лагно В.И., Фущич В.И.* Подалгебры алгебры Пуанкаре $AP(2,3)$ и симметрийная редукция нелинейного ультрагиперболического уравнения Даламбера. I // Укр. мат. журн. — 1988. — **40**, № 4. — С. 411–416.

156. *Fushchych W.I.* Exact solutions of multidimensional nonlinear Dirac's and Schrödinger's equations. — Minnesota, 1988. — 14 p. — (Prepr./Institute for Mathematics and its Applications, University of Minnesota; N 467).

157. *Фущич В.І., Галіцин А.С., Полубинський А.С.* Нова математична модель дифузійних процесів зі скінченною швидкістю // Доп. АН УРСР. Сер. А. — 1988. — № 8. — С. 21–26.

158. *Fushchych W.I., Krivsky I.Yu., Simulik V.M.* On vector and pseudovector Lagrangians for electromagnetic field — Minnesota, 1988. — 6 p. — (Prepr./Institute for Mathematics and its Applications, University of Minnesota; N 466).

159. *Фущич В.И., Серов Н.И.* Условная инвариантность и точные решения нелинейного уравнения акустики // Докл. АН УССР. Сер. А. — 1988. — № 10. — С. 27–31.

160. *Фущич В.И., Серов Н.И.* Об условной инвариантности нелинейных уравнений Даламбера, Лиувилля, Борна–Инфельда, Монжа–Ампера относительно конформной алгебры // Симметрийный анализ и решения уравнений математической физики. — Киев: Ин-т математики АН УССР, 1988. — С. 98–102.

161. *Фущич В.І., Серов М.І., Чопик В.І.* Умовна інваріантність та нелінійні рівняння теплопровідності // Доп. АН УРСР. Сер. А. — 1988. — № 9. — С. 17–20.

162. *Фущич В.И., Егорченко И.А.* О симметрийных свойствах комплекснозначных нелинейных волновых уравнений // Докл. АН СССР. — 1988. — **298**, № 2. — С. 347–351.

163. *Fushchych W.I., Zhdanov R.Z.* On the reduction and some new exact solutions of the non-linear Dirac and Dirac–Klein–Gordon equations // J. Phys. A: Math. Gen. — 1988. — **21**. — L5–L9.

164. *Fushchych W.I., Zhdanov R.Z.* Non-local ansätze for the Dirac equation // J. Phys. A: Math. Gen. — 1988. — **21**. — L1117–L1121.

165. *Fushchych W.I., Zhdanov R.Z.* On some new exact solutions of nonlinear d'Alembert and Hamilton equations — Minnesota, 1988. — 5 p. — (Prepr./Institute for Mathematics and its Applications, University of Minnesota; N 468).

166. *Фущич В.И., Жданов Р.З.* Симметрия и точные решения нелинейного уравнения Дирака // Физика элементарных частиц и атомного ядра. — 1988. — **19**, вып. 5. — С. 1154–1196.

167. *Баранник А.Ф., Москаленко Ю.Д., Фущич В.И.* Подалгебры афинной алгебры $AIGL(3, R)$. — Киев, 1989. — 32 с. — (Препр./АН УССР. Ин-т математики; № 89.65).

168. *Баранник А.Ф., Москаленко Ю.Д., Фущич В.И.* Подалгебры псевдоортогональной алгебры $AO(3, 3)$. — Киев, 1989. — 40 с. — (Препр./АН УССР. Ин-т математики; № 89.67).

169. *Barannyk L.F., Fushchych W.I.* Continuous subgroups of the generalized Schrödinger groups // J. Math. Phys. — 1989. — **30**, N 2. — P. 280–290.

170. *Баранник Л.Ф., Лагно В.И., Фущич В.И.* Подалгебры алгебры Пуанкаре $AP(2, 3)$ и симметрийная редукция нелинейного ультрагиперболического

уравнения Даламбера. II // Укр. мат. журн. — 1989. — **41**, № 5. — С. 579–584.

171. *Фущич В.И.* Принцип относительности Галилея и нелинейные уравнения в частных производных // Тез. докл. Всесоюз. конф. “Нелинейные проблемы дифференциальных уравнений и математической физики”, 12–15 сентября 1989 г. — Тернополь, 1989. — Ч. II. — С. 444–452.

172. *Фущич В.И., Чернига Р.М.* Галилей-инвариантные нелинейные уравнения шредингеровского типа и их точные решения. I // Укр. мат. журн. — 1989. — **41**, № 10. — С. 1349–1357.

173. *Фущич В.И., Чернига Р.М.* Галилей-инвариантные нелинейные уравнения шредингеровского типа и их точные решения. II // Укр. мат. журн. — 1989. — **41**, № 12. — С. 1687–1694.

174. *Fushchych W.I., Kornyak V.V.* Computer algebra application for determining Lie and Lie–Bäcklund symmetries of differential equations // J. Symbolic Computation. — 1989. — **7**. — P. 611–619.

175. *Fushchych W.I., Krivsky I.Yu., Simulik V.M.* On vector and pseudovector Lagrangians for electromagnetic field // Nuovo Cimento B. — 1989. — **103**, N 4. — P. 423–429.

176. *Фущич В.И., Кривский И.Ю., Симулик В.М.* Нелиевские симметрии и нетеровский анализ законов сохранения для уравнения Дирака. — Киев, 1989. — 39 с. — (Препр./АН УССР. Ин-т математики; № 89.49).

177. *Фущич В.И., Никитин А.Г.* О новых системах и законах сохранения для упругих волн // Докл. АН СССР. — 1989. — **304**, № 2. — С. 333–335.

178. *Фущич В.И., Никитин А.Г.* Исправление: “О новых системах и законах сохранения для упругих волн” // Докл. АН СССР. — 1989. — **306**, № 2. — С. 264.

179. *Фущич В.I., Ревенко I.В.* Про точнi розв’язки рiвнянь Лоренца–Максвелла // Доп. АН УРСР. Сер. А. — 1989. — № 6. — С. 27–31.

180. *Фущич В.I., Ревенко I.В.* Пуанкаре-iнварiантнi рiвняння третього та четвертого порядкiв у механiцi Остроградського // Доп. АН УРСР. Сер. А. — 1989. — № 11. — С. 25–28.

181. *Фущич В.И., Серов Н.И.* Условная инвариатность и точные решения уравнения Буссинеска // Симметрия и решения уравнений математической физики. — Киев: Ин-т математики АН УССР, 1989. — С. 95–102.

182. *Fushchych W.I., Shtelen W.M.* On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter // J. Phys. A: Math. Gen. — 1989. — **22**. — L887–L890.

183. *Fushchych W.I., Shtelen W.M.* On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter // Доп. АН УРСР. Сер. А. — 1989. — № 8. — С. 18–21.

184. *Фущич В.И., Егорченко И.А.* Дифференциальные инварианты алгебры Галилея // Докл. АН УССР. Сер. А. — 1989. — № 4. — С. 29–32.

185. *Фущич В.I., Єгорченко I.А.* Диференцiальнi iнварiанти алгебри Пуанкаре та конформної алгебри // Доп. АН УРСР. Сер. А. — 1989. — № 5. — С. 21–22.

186. *Fushchych W.I., Yegorchenko I.A.* The symmetry and exact solutions of the non-linear d'Alembert equation for complex fields // J. Phys. A: Math. Gen. — 1989. — **22**. — P. 2643–2652.

187. *Fushchych W.I., Zhdanov R.Z.* On some new exact solutions of the nonlinear d'Alembert–Hamilton system // Phys. Lett. A. — 1989. — **141**, N 3-4. — P. 113–115.

188. *Fushchych W.I., Zhdanov R.Z.* Symmetry and exact solutions of nonlinear spinor equations // Phys. Rep. — 1989. — **172**, N 4. — P. 123–174.

189. *Баранник А.Ф., Баранник Л.Ф., Фущич В.И.* Редукция и точные решения уравнения Гамильтона–Якоби. — Киев, 1990. — 40 с. — (Препр./АН УССР. Ин-т математики; № 90.41).

190. *Fushchych W.I.* The symmetries and exact solutions of multi-dimensional nonlinear spinor equations // Proc. of the Annual Seminar of Canadian Mathematical Society on Lie Theory, Differential Equations and Representation Theory, Ed. V. Hussin (Montreal, PQ, 1989). — Montreal: University Montreal, 1990. — P. 161–170.

191. *Фущич В.И.* Об одном обобщении метода С. Ли // Теоретико-алгебраический анализ уравнений математической физики. — Киев: Ин-т математики АН УССР, 1990. — С. 4–9.

192. *Фущич В.И.* О некоторых новых волновых уравнениях математической физики // Теоретико-алгебраический анализ уравнений математической физики. — Киев: Ин-т математики АН УССР, 1990. — С. 9–11.

193. *Фущич В.И., Баранник А.Ф.* О точных решениях нелинейного уравнения д'Аламбера в пространстве Минковского $R_{1,n}$ // Докл. АН УССР. Сер. А. — 1990. — № 6. — С. 31–34.

194. *Фущич В.И., Баранник А.Ф.* Максимальные подалгебры ранга $n-1$ алгебры $AP(1,n)$ и редукция нелинейных волновых уравнений. I // Укр. мат. журн. — 1990. — **42**, № 11. — С. 1552–1559.

195. *Фущич В.И., Баранник А.Ф.* Максимальные подалгебры ранга $n-1$ алгебры $AP(1,n)$ и редукция нелинейных волновых уравнений. II // Укр. мат. журн. 1990. — **42**, № 12. — С. 1693–1700.

196. *Фущич В.И., Баранник А.Ф., Москаленко Ю.Д.* О точных решениях уравнений Даламбера и Лиувилля в псевдоевклидовом пространстве $R_{2,2}$. I // Укр. мат. журн. — 1990. — **42**, № 8. — С. 1122–1128.

197. *Фущич В.И., Баранник А.Ф., Москаленко Ю.Д.* О точных решениях уравнений Даламбера и Лиувилля в псевдоевклидовом пространстве $R_{2,2}$. II // Укр. мат. журн. — 1990. — **42**, № 9. — С. 1237–1244.

198. *Фущич В.И., Чопик В.И.* Условная инвариантность нелинейного уравнения Шредингера // Докл. АН УССР. Сер. А. — 1990. — № 4. — С. 30–33.

199. *Фущич В.И., Чопик В.И., Миронюк П.И.* Условная инвариантность и точные решения трехмерных нелинейных уравнений акустики // Докл. АН УССР. Сер. А. — 1990. — № 9. — С. 25–28.

200. *Фущич В.И., Галицын А.С., Полубинский А.С.* О новой математической модели процессов теплопроводности // Укр. мат. журн. — 1990. — **42**, № 2. — С. 237–245.

201. *Fushchych W.I., Nikitin A.G.* On superalgebras of symmetry operators of relativistic wave equations // Selected Topics in QFT and Mathematical Physics, Proceedings of the 5th International conference (Liblice, Czechoslovakia, June 25–30, 1989), Eds. J. Niederle, J. Fisher. — World Scientific, 1990. — P. 385–391.

202. *Fushchych W.I., Nikitin A.G.* On the new constants of motion for two- and three-particle equations // J. Phys. A: Math. Gen. — 1990. — **23**. — L533–L535.

203. *Фущич В.И., Серов Н.И.* Условная инвариантность и редукция нелинейного уравнения теплопроводности // Докл. АН УССР. Сер. А. — 1990. — № 7. — С. 24–27.

204. *Фущич В.И., Серов Н.И., Амеров Т.К.* Условная инвариантность нелинейного уравнения теплопроводности // Докл. АН УССР. Сер. А. — 1990. — № 11. — С. 16–18.

205. *Fushchych W.I., Shtelen W.M.* Merons and instantons as products of self-interaction of the Dirac–Gürsey spinor field // J. Phys. A: Math. Gen. — 1990. — **23**. — L517–L520.

206. *Фущич В.И., Штелень В.М., Спичак С.В.* О связи между решениями уравнений Дирака и Максвелла. Суперсимметрия уравнения Дирака // Докл. АН УССР. Сер. А. — 1990. — № 3. — С. 36–40.

207. *Фущич В.И., Штелень В.М., Жданов Р.З.* О точных решениях нелинейного уравнения для спинорного поля // Современный групповой анализ: методы и приложения. Некоторые задачи современной физики, под ред. А.В. Флегонтова. — Л.: Ленингр. ин-т информатики и автоматизации АН СССР, 1990. — С. 22–30.

208. *Фущич В.I., Тичинін В.А.* Точні розв'язки та принцип суперпозиції для нелінійного хвильового рівняння // Доп. АН УРСР. Сер. А. — 1990. — № 5. — С. 32–36.

209. *Фущич В.I., Єгорченко I.А.* Про редукцію багатовимірного нелінійного хвильового рівняння до двовимірних рівнянь // Доп. АН УРСР. Сер. А. — 1990. — № 8. — С. 31–33.

210. *Фущич В.И., Жданов Р.З.* Нелиевские анзацы и точные решения нелинейного спинорного уравнения // Укр. мат. журн. — 1990. — **42**, № 7. — С. 958–962.

211. *Фущич В.И., Жданов Р.З., Ревенко И.В.* Совместность и решения нелинейных уравнений Даламбера и Гамильтона. — Киев, 1990. — 65 с. — (Препр./ АН УССР. Ин-т математики; № 90.39).

212. *Парасюк И.О., Фущич В.И.* Качественный анализ семейств ограниченных решений нелинейного трехмерного уравнения Шредингера // Укр. мат. журн. — 1990. — **42**, № 10. — С. 1344–1349.

213. *Баранник А.Ф., Баранник Л.Ф., Фущич В.И.* Редукция и точные решения уравнения эйконала // Укр. мат. журн. — 1991. — **43**, № 4. — С. 461–474.

214. *Баранник А.Ф., Баранник Л.Ф., Фущич В.И.* Связные подгруппы конформной группы $C(1,4)$ // Укр. мат. журн. — 1991. — **43**, № 7–8. — С. 870–884.

215. *Баранник А.Ф., Баранник Л.Ф., Фущич В.И.* Редукция многомерного Пуанкаре-инвариантного нелинейного уравнения к двумерным уравнениям // Укр. мат. журн. — 1991. — **43**, № 10. — С. 1311–1323.

216. *Баранник А.Ф., Марченко В.А., Фущич В.И.* О редукции и точных решениях нелинейных многомерных уравнений Шредингера // Теор. мат. физика. — 1991. — **87**, № 2. — С. 220–234.

217. *Бедрій О., Фущич В.І.* Про електромагнітну структуру мас елементарних частинок // Доп. АН УРСР. — 1991. — № 2. — С. 38–40.

218. *Боголюбов М.М., Митропольський Ю.О., Петрина Д.Я., Самойленко А.М., Фущич В.И.* Остап Степанович Парасюк (до 70-річчя від дня народження) // Укр. мат. журн. — 1991. — **43**, № 11. — С. 1443–1444.

219. *Fushchych W.I.* Construction of solutions of nonlinear d'Alembert, Maxwell, Yang–Mills equations by solutions of nonlinear spinor equations // Nonlinear Fields: Classical, Random, Semiclassical (Karpacz, 1991). — River Edge: World Sci. Publishing, 1991. — P. 544–556.

220. *Фущич В.И.* Условная симметрия уравнений нелинейной математической физики // Укр. мат. журн. — 1991. — **43**, № 11. — С. 1456–1470.

221. *Фущич В.И., Миронюк П.И.* Условная симметрия и точные решения уравнения нелинейной акустики // Докл. АН УССР. — 1991. — № 6. — С. 23–29.

222. *Фущич В.И., Парасюк И.О.* Качественный анализ семейств ограниченных решений многомерного нелинейного уравнения Шредингера // Укр. мат. журн. — 1991. — **43**, № 6. — С. 821–828.

223. *Фущич В.И., Репета В.К.* Точные решения некоторых уравнений газовой динамики и нелинейной акустики // Докл. АН УССР. — 1991. — № 8. — С. 39–45.

224. *Фущич В.И., Серов Н.И.* Условная инвариантность нелинейного волнового уравнения // Укр. мат. журн. — 1991. — **43**, № 3. — С. 394–399.

225. *Фущич В.И., Серов Н.И.* Негрупповая симметрия некоторых нелинейных волновых уравнений // Докл. АН УССР. — 1991. — № 9. — С. 45–49.

226. *Фущич В.И., Серов Н.И., Амеров Т.К.* Об условной симметрии обобщенного уравнения Кортевега–де Фриза // Докл. АН УССР. — 1991. — № 12. — С. 15–18.

227. *Фущич В.И., Серов Н.И., Репета В.К.* Условная симметрия, редукция и точные решения нелинейного волнового уравнения // Докл. АН УССР. — 1991. — № 5. — С. 29–34.

228. *Фущич В.И., Серов Н.И., Репета В.К.* Нелиевская симметрия и точные решения одномерных уравнении газовой динамики // Докл. АН УССР. — 1991. — № 11. — С. 27–33.

229. *Фущич В.И., Штелень В.М.* Чи інваріантні рівняння Максвелла щодо перетворень Галілея? // Доп. АН УРСР. — 1991. — № 3. — С. 22–26.

230. *Fuchchych W.I., Shtelen W.M., Slavutsky S.L.* Reduction and exact solutions of the Navier–Stokes equations // J. Phys. A: Math. Gen. — 1991 — **24**. — P. 971–984.

231. *Fuchchych W.I., Shtelen W.M., Spichak S.V.* On the connection between solutions of Dirac and Maxwell equations, dual Poincaré invariance and superalgebras of invariance and solutions of nonlinear Dirac equations // J. Phys. A: Math. Gen. — 1991. — **24**. — P. 1683–1698.

232. *Фущич В.И., Егорченко И.А.* Редукция и решения нелинейного уравнения для векторного поля // Докл. АН УССР. — 1991. — № 4. — С. 23–25.

233. *Фущич В.И., Егорченко И.А.* Нелиевские анзацы и условная симметрия нелинейного уравнения Шредингера // Укр. мат. журн. — 1991. — **43**, № 12. — С. 1620–1628.

234. *Фущич В.И., Ревенко И.В., Жданов Р.З.* Несимметрийный подход к построению точных решений одного нелинейного волнового уравнения // Докл. АН УССР. — 1991. — № 7. — С. 15–19.

235. *Fushchych W.I., Zhdanov R.Z.* On the non-Lie reduction of the nonlinear Dirac equation // J. Math. Phys. — 1991. — **32**, N 12. — P. 3488–3490.

236. *Фущич В.И., Жданов Р.З., Ревенко И.В.* Общие решения нелинейного волнового уравнения и уравнения эйконала // Укр. мат. журн. — 1991. — **43**, № 11. — С. 1471–1486.

237. *Фущич В.И., Жданов Р.З., Ревенко И.В.* Общие решения нелинейного волнового уравнения и эйконала // Докл. АН УССР. — 1991. — № 10. — С. 29–31.

238. *Fushchych W.I., Zhdanov R.Z., Yegorchenko I.A.* On the reduction of the nonlinear multi-dimensional wave equations and compatibility of the d'Alembert–Hamilton system // J. Math. Anal. and Appl. — 1991. — **161**, N 2. — P. 352–360.

239. *Никитин А.Г., Фущич В.И.* Нелиевские интегралы движения для частиц произвольного спина и для систем взаимодействующих частиц // Теор. мат. физика. — 1991. — **88**, № 3. — 406–415.

240. *Жданов Р.З., Фущич В.И.* О нелиевской симметрии галилеевски-инвариантного уравнения для частицы со спином $s = 1/2$ // Теор. мат. физика. — 1991. — **89**, № 3. — С. 413–419.

241. *Basarab-Horwath P., Fushchych W.I., Serov M.I.* A simple method of finding solutions of the nonlinear d'Alembert equation // J. Phys. A: Math. Gen. — 1992. — **25**. — L871–L877.

242. *Fushchych W.I.* Symmetry analysis. Preface // Symmetry Analysis of Equations of Mathematical Physics. — Kyiv: Institute of Mathematics, 1992. — P. 5–6.

243. *Fushchych W.I.* Conditional symmetry of equations of nonlinear mathematical physics // Symmetry Analysis of Equations of Mathematical Physics.— Kyiv: Institute of Mathematics, 1992. — P. 7–27.

244. *Fushchych W.I.* New nonlinear equations for electromagnetic field having velocity different from c // Доп. АН України. — 1992. — № 4. — С. 24–27.

245. *Fushchych W.I., Nikitin A.G.* The complete sets of conservation laws for the electromagnetic field // J. Phys. A: Math. Gen. — 1992. — **25**. — L231–L233.

246. *Fushchych W.I., Popovych R.O.* Symmetry reduction of the Navier–Stokes equations to linear two-dimensional systems of equations // Доп. АН України. — 1992. — № 8. — С. 29–36.

247. *Фущич В.И., Серов Н.И., Амеров Т.К.* О нелокальных анзацах для одного нелинейного одномерного уравнения теплопроводности // Докл. АН Украины. — 1992. — № 1. — С. 26–30.

248. *Fushchych W.I., Serov N.I., Amerov T.K.* Conditional invariance and exact solutions of gas dynamics equations // Докл. АН Украины. — 1992. — № 5. — С. 35–40.

249. *Fushchych W.I., Serov N.I., Tychinin V.A., Amerov T.K.* On non-local symmetries of nonlinear heat equation // Докл. АН Украины. — 1992. — № 11. — С. 27–33.

250. *Fushchych W.I., Serov N.I., Vorob'eva A.I.* Conditional symmetry and exact solutions of equations of nonstationary filtration // Доп. АН України. — 1992. — № 6. — С. 20–24.

251. *Фущич В.І., Штелень В.М., Попович Р.О.* Про редукцію рівнянь Нав'є–Стокса до лінійних рівнянь теплопровідності // Доп. АН України. — 1992. — № 2. — С. 23–30.

252. *Fushchych W.I., Shtelen W.M., Serov M.I., Popovych R.O.* Q-conditional symmetry of the linear heat equation // Докл. АН Украины. — 1992. — № 12. — С. 28–33.

253. *Fushchych W.I., Tychinin V.A.* Generating solutions for nonlinear equations by the Legendre transformation // Докл. АН Украины. — 1992. — № 7. — С. 20–25.

254. *Fushchych W.I., Tychinin V.A.* Nonlocal symmetry and generating solutions for Harry–Dym type equations // Докл. АН Украины. — 1992. — № 9. — С. 24–30.

255. *Фущич В.И., Тычинин В.А., Серов Н.И.* Формула размножения решений уравнений Кортевега–де Фриза // Укр. мат. журн. — 1992. — **44**, № 5. — С. 716–719.

256. *Fushchych W.I., Yegorchenko I.A.* Second-order differential invariants of the rotation group $O(n)$ and of its extensions: $E(n)$, $P(1,n)$, $G(1,n)$ // Acta Appl. Math. — 1992. — **28**. — P. 69–92.

257. *Fushchych W.I., Zhdanov R.Z.* Conditional symmetry and reduction of partial differential equations // Укр. мат. журн. — 1992. — **44**, № 7. — С. 875–886.

258. *Fushchych W.I., Zhdanov R.Z., Revenko I.V.* On the general solution of the d'Alembert equation with nonlinear eikonal constraint // Symmetry Analysis of Equations of Mathematical Physics. — Kyiv: Institute of Mathematics, 1992. — P. 68–90.

259. *Fushchych W.I., Zhdanov R.Z., Smalij V.F.* On the new exact solutions of the nonlinear Maxwell–Born–Infeld equations // Докл. АН Украины. — 1992. — № 10. — С. 28–33.

260. *Никитин А.Г., Онуфрийчук С.П., Фущич В.И.* Высшие симметрии уравнения Шредингера // Теор. мат. физика. — 1992. — **91**, № 2. — С. 268–278.

261. *Bedrij O., Fushchych W.I.* Fundamental constants of nucleon-meson dynamics // Доп. АН України. — 1993. — № 5. — С. 62–64.

262. *Barannyk A.F., Fushchych W.I.* On maximal subalgebras of the rank $n-1$ of the conformal algebra $AC(1,n)$ // Доп. АН України. — 1993. — № 6. — С. 38–41.

263. *Fushchych W.I.* Conditional symmetries of the equations of mathematical physics // Proc. of the International Workshop “Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics”, Eds. N.H. Ibragimov, M. Torrisi, A. Valenti. — Dordrecht: Kluwer Academic Publishers, 1993. — P. 231–239.

264. *Fushchych W.I., Barannyk L.F., Lagno V.I.* Invariants of one-parameter subgroups of the conformal group $C(1,n)$ // Доп. АН України. — 1993. — № 3. — С. 45–48.

265. *Фущич В.І., Черніга Р.М.* Системи нелінійних еволюційних рівнянь другого порядку, інваріантні відносно алгебри Галілея та її розширень // Доп. АН України. — 1993. — № 8. — С. 44–51.

266. *Фущич В.І., Чопик В.І.* Умовна симетрія та нові зображення алгебри Галілея для нелінійних рівнянь параболічного типу // Укр. мат. журн. — 1993. — **45**, № 10. — С. 1433–1443.

267. *Фущич В.І., Чопик В.І.* Симетрія та неліївська редукція нелінійного рівняння Шредінгера // Укр. мат. журн. — 1993. — **45**, № 4. — С. 539–551.

268. *Fushchych W.I., Chopyk V.I.* Symmetry analysis and ansatzes for the Schrödinger equations with the logarithmic nonlinearity. — Linköping, 1993. — 13 p. — (Prepr./Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-93-08).

269. *Fushchych W.I., Chopyk V.I., Cherkasenko V.P.* Symmetry and exact solutions of multidimensional nonlinear Fokker–Planck equation // Доп. АН України. — 1993. — № 2. — С. 32–42.

270. *Fushchych W.I., Lahno V.I., Zhdanov R.Z.* On nonlinear representation of the conformal algebra $AC(2,2)$ // Доп. АН України. — 1993. — № 9. — С. 44–47.

271. *Фущич В.І., Попович Р.О.* Симетрійна редукція і точні розв’язки рівнянь Нав’є–Стокса. — Київ, 1993. — 56 с. — (Препр./НАН України. Інститут математики; № 93.7).

272. *Фущич В.И., Серов Н.И., Амеров Т.К.* Нелокальные анзацы и решения нелинейной системы уравнений теплопроводности // Укр. мат. журн. — 1993. — **45**, № 2. — С. 293–302.

273. *Fushchych W.I., Serov N.I., Tulupova L.A.* The conditional invariance and exact solutions of the nonlinear diffusion equation // Доп. АН України. — 1993. — № 4. — С. 37–40.

274. *Fushchych W., Shtelen W.M., Basarab-Horwath P.* A new conformal-invariant non-linear spinor equation. — Linköping, 1993. — 8 p. — (Prepr./Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-93-05).

275. *Fushchych W.I., Tychynin V.A.* Generation of solutions for nonlinear equations via the Euler–Amperé transformation // Доп. АН України. — 1993. — № 7. — С. 40–45.

276. *Fushchych W.I., Tychynin V.A.* Hodograph transformations and generating of solutions for nonlinear differential equations // Доп. АН України. — 1993. — № 10. — С. 52–58.

277. *Fushchych W.I., Yegorchenko I.A.* New conditionally invariant solutions for non-linear d'Alembert equation. — Linköping, 1993. — 9 p. — (Prepr./Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-93-07).

278. *Fushchych W.I., Zhdanov R.Z.* Anti-reduction of the nonlinear wave equation // Доп. АН України. — 1993. — № 11. — С. 37–41.

279. *Fushchych W.I., Zhdanov R.Z., Revenko I.V.* On the new approach to variable separation in the wave equation with potential // Доп. АН України. — 1993. — № 1. — С. 27–32.

280. *Zhdanov R.Z., Revenko I.V., Fushchych W.I.* Orthogonal and non-orthogonal separation of variables in the wave equation $u_{tt} - u_{xx} + V(x)u = 0$ // J. Phys. A: Math. Gen. — 1993. — **26**. — P. 5959–5972.

281. *Баранник А.Ф., Баранник Л.Ф., Фущич В.И.* Редукция многомерного уравнения Даламбера к двумерным уравнениям // Укр. мат. журн. — 1994. — **46**, № 6. — С. 651–662.

282. *Basarab-Horwath P., Fushchych W.I., Barannyk L.F.* New solutions of the wave equation by reduction to the heat equation. — Linköping, 1994. — 17 p. — (Prepr./Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-94-35).

283. *Euler N., Köhler A., Fushchych W.I.* Q-symmetry generators and exact solutions for nonlinear heat conduction // Physica Scripta. — 1994. — **49**. — P. 518–524.

284. *Фущич В.І., Черніга Р.М.* Галілей-інваріантні системи нелінійних рівнянь типу Гамільтона–Якобі та реакції-дифузії // Доп. НАН України. — 1994. — № 3. — Р. 31–37.

285. *Фущич В.І., Федорчук В.М., Лейбов О.С.* Симетрійна редукція і деякі точні розв'язки рівняння Монжа–Ампера // Доп. НАН України. — 1994. — № 1. — Р. 47–54.

286. *Fushchych W.I., Popovych R.O.* Symmetry reduction and exact solutions of the Navier–Stokes equations. I // J. Nonlinear Math. Phys. — 1994. — **1**, N 1. — P. 75–113

287. *Fushchych W.I., Popovych R.O.* Symmetry reduction and exact solution of the Navier-Stokes equations. II // J. Nonlinear Math. Phys. — 1994. — **1**, N 2. — P. 156–188.

288. *Fushchych W.I., Popovych R.O., Popovych G.V.* Ansätze of codimension one for the Navier–Stokes field and reduction of the Navier–Stokes equation // Доп. НАН України. — 1994. — № 4. — Р. 37–44.

289. *Fushchych W.I., Zhdanov R.Z.* Antireduction and exact solutions of nonlinear heat equations // J. Nonlinear Math. Phys. — 1994. — **1**, N 1. — P. 60–64.

290. *Fushchych W.I., Zhdanov R.Z.* Conditional symmetry and anti-reduction of nonlinear heat equation // Доп. НАН України. — 1994. — № 5. — Р. 40–43.

291. *Fushchych W.I., Zhdanov R.Z., Lahno V.I.* On linear and non-linear representations of the generalized Poincaré groups in the class of Lie vector fields // J. Nonlinear Math. Phys. — 1994. — **1**, N 3. — P. 295–308.

292. *Fushchych W.I., Tsyfra I.M.* Nonlocal ansatzes for nonlinear wave equation // Доп. НАН України. — 1994. — № 10. — P. 34–39.

293. *Fushchych W.I., Tsyfra I.M., Boyko V.M.* Nonlinear representations for Poincaré and Galilei algebras and nonlinear equations for electromagnetic fields // J. Nonlinear Math. Phys. — 1994. — **1**, N 2. — P. 210–221.

294. *Zhdanov R.Z., Revenko I.V., Fushchych W.I.* On the new approach to variable separation in the two-dimensional Schrödinger equation // Доп. НАН України. — 1994. — № 11. — P. 38–44.

295. *Zhdanov R.Z., Revenko I.V., Fushchych W.I.* Separation of variables in the two-dimensional wave equation with potential // Укр. мат. журн. — 1994. — **46**, № 10. — P. 1343–1361.

296. *Basarab-Horwath P., Barannyk L.F., Fushchych W.I.* New solutions of the wave equation by reduction to the heat equation // J. Phys. A: Math. Gen. — 1995. — **28**. — P. 5291–5304.

297. *Basarab-Horwath P., Euler N., Euler M., Fushchych W.I.* Amplitude-phase representation for solutions of nonlinear d'Alembert equations // J. Phys. A: Math. Gen. — 1995. — **28**. — P. 6193–6201.

298. *Bedrij O., Fushchych W.I.* Planck's constant is not constant in different quantum phenomena // Доп. НАН України. — 1995. — № 12. — C. 26–27.

299. *Fushchych W.I.* Ansatz '95 // J. Nonlinear Math. Phys. — 1995. — **2**, N 3-4. — P. 216–235.

300. *Fushchych W.I.* Galilei invariant nonlinear Schrödinger type equations and their exact solutions // Proc. of the International Symposium on Mathematical Physics "Nonlinear, Deformed and Irreversible Quantum Systems" (15–19 August, 1994, Clausthal, Germany), Eds. H.-D. Doebner, V.K. Dobrev, P. Nattermann. — Singapore; New Jersey; London; Hong Kong: World Scientific, 1995. — P. 214–222.

301. *Fushchych W.I., Barannyk A.F., Moskalenko Yu.D.* On new exact solutions of the multidimensional nonlinear d'Alembert equation // Доп. НАН України. — 1995. — № 2. — P. 33–37.

302. *Fushchych W.I., Boyko V.M.* Symmetry classification of the one-dimensional second order equation of hydrodynamical type. — Linköping, 1995. — 11 p. — (Prepr./Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-95-19).

303. *Fushchych W.I., Cherniha R.M.* Galilei-invariant nonlinear systems of evolution equations. — Linköping, 1995. — 12 p. — (Prepr./Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-95-18).

304. *Fushchych W.I., Cherniha R.M.* Galilei-invariant nonlinear systems of evolution equations // J. Phys. A: Math. Gen. — 1995. — **28**. — P. 5569–5579.

305. *Fushchych W.I., Chopyk V.I., Nattermann P., Scherer W.* Symmetries and reductions of nonlinear Schrödinger equations of Doebner–Goldin type // Rep. Math. Phys. — 1995. — **35**. — P. 129–138.

306. *Fushchych W.I., Roman O.V., Zhdanov R.Z.* Symmetry reduction and exact solutions of nonlinear biwave equations. — Clausthal, 1995. — 14 p. — (Prepr./ Arnold-Sommerfeld Institute for Mathematical Physics, Clausthal, Germany; ASI-TPA/13/95).

307. *Fushchych W.I., Roman O.V., Zhdanov R.Z.* Symmetry and some exact solutions of non-linear polywave equations // Europhys. Lett. — 1995. — **31**, N 2. — P. 75–79.

308. *Fushchych W.I., Zhdanov R.Z., Roman O.V.* Symmetry properties, reduction and exact solutions of biwave equations. — Linköping, 1995. — 16 p. — (Prepr./ Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-95-20).

309. *Lahno V.I., Zhdanov R.Z., Fushchych W.I.* Symmetry reduction and exact solutions of the Yang–Mills equations // J. Nonlinear Math. Phys. — 1995. — **2**, N 1. — P. 51–72.

310. *Zhdanov R.Z., Fushchych W.I.* Conditional symmetry and new classical solutions of the Yang–Mills equations // J. Phys. A: Math. Gen. — 1995. — **28**. — P. 6253–6263.

311. *Zhdanov R.Z., Fushchych W.I.* On non-Lie ansatzes and new exact solutions of the classical Yang–Mills equations // J. Nonlinear Math. Phys. — 1995. — **2**, N 2. — P. 172–181.

312. *Zhdanov R.Z., Fushchych W.I.* Symmetry and reduction of nonlinear Dirac equations // Proc. of the International Symposium on Mathematical Physics "Nonlinear, Deformed and Irreversible Quantum Systems" (15–19 August, 1994, Clausthal, Germany), Eds. H.-D. Doebner, V.K. Dobrev, P. Nattermann. — Singapore; New Jersey; London; Hong Kong: World Scientific, 1995. — P. 223–229.

313. *Zhdanov R.Z., Lahno V.I., Fushchych W.I.* Reduction of the self-dual Yang–Mills equations. I. The Poincaré group // Укр. мат. журн. — 1995. — **47**, № 4. — С. 456–462.

314. *Zhdanov R.Z., Revenko I.V., Fushchych W.I.* On the new approach to variable separation in the time-dependent Schrödinger equation with two space dimensions // J. Math. Phys. — 1995. — **36**, N 10. — P. 5506–5521.

315. *Zhdanov R.Z., Revenko I.V., Fushchych W.I.* On the general solution of the d'Alembert equation with a nonlinear eikonal constraint and its applications // J. Math. Phys. — 1995. — **36**, N 12. — P. 7109–7127.

316. *Фущич В.I., Баранник А.Ф.* Про новий метод побудови розв'язків нелінійних хвильових рівнянь // Доп. НАН України. — 1996. — № 10. — С. 48–51.

317. *Фущич В.I., Баранник Л.Л.* Симетрійна редукція як метод розмноження розв'язків систем лінійних диференціальних рівнянь // Доп. НАН України. — 1996. — № 12. — С. 44–49.

318. *Фущич В.I., Бойко В.М.* Галілей-інваріантні рівняння типу Бюргерса та Кортевега–де Фріза високого порядку // Укр. мат. журн. — 1996. — **48**, № 12. — С. 1589–1601.

319. *Фущич В.I., Бойко В.М.* Пониження порядку та загальні розв'язки деяких класів рівнянь математичної фізики // Доп. НАН України. — 1996. — № 9. — С. 43–48.

320. *Fushchych W.I., Cherniha R.M., Chopyk V.I.* On unique symmetry of two nonlinear generalizations of the Schrödinger equation // J. Nonlinear Math. Phys. — 1996. — **3**, N 3-4. — P. 296–301.

321. *Фущич В.I., Лагно В.I.* Про нові нелінійні рівняння, інваріантні відносно групи Пуанкаре в двовимірному просторі-часі // Доп. НАН України. — 1996. — № 11. — С. 60–65.

322. *Fushchych W.I., Marko P.V., Zhdanov R.Z.* Symmetry classification of multicomponent scale-invariant wave equations. — Clausthal, 1996. — 9 p. — (Prepr./ Arnold-Sommerfeld Institute for Mathematical Physics, Clausthal, Germany; ASI-TPA/8/96).

323. *Fushchych W.I., Nikitin A.G.* Higher symmetries and exact solutions of linear and nonlinear Schrödinger equation. — Clausthal, 1996. — 23 p. — (Prepr./ Arnold-Sommerfeld Institute for Mathematical Physics, Clausthal, Germany; ASI-TPA/9/96).

324. *Fushchych W.I., Roman O.V., Zhdanov R.Z.* Symmetry reduction and exact solutions of nonlinear biwave equations // Rep. Math. Phys. — 1996. — **37**, N 2. — P. 267–281.

325. *Fushchych W.I., Symenoh Z.I., Tsyfra I.M.* Symmetry of the Schrödinger equation with variable potential. — Linköping, 1996. — 15 p. — (Prepr./Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-96-14).

326. *Zhdanov R.Z., Fushchych W.I., Marko P.V.* New scale-invariant nonlinear differential equations for a complex scalar field // Physica D. — 1996. — **95**. — P. 158–162.

327. *Basarab-Horwath P., Barannyk L.L., Fushchych W.I.* Some exact solutions of a conformally invariant nonlinear Schrödinger equation. — Linköping, 1997. — 12 p. — (Prepr./Dept. of Math., Linköping University, Sweden; LiTH-MAT-R-97-11).

328. *Basarab-Horwath P., Fushchych W.I.* Implicit and parabolic ansatzes: some new ansatzes for old equations // Proc. of the Second International Conference "Symmetry in Nonlinear Mathematical Physics. Memorial Prof. W. Fushchych Conference" (July 7–13, 1997, Kyiv), Eds. M.I. Shkil, A.G. Nikitin, V.M. Boyko. — Kyiv: Institute of Mathematics, 1997. — **1**. — P. 34–47.

329. *Basarab-Horwath P., Fushchych W.I., Barannyk L.F.* Solutions of the relativistic nonlinear wave equation by solutions of the nonlinear Schrödinger equation // Rep. Math. Phys. — 1997. — **39**, N 3. — P. 353–374.

330. *Basarab-Horwath P., Fushchych W.I., Roman O.V.* On a new conformal symmetry for a complex scalar field // Phys. Lett. A. — 1997. — **226**. — P. 150–154.

331. *Фущич В.I.* Симетрія рівнянь лінійної та нелінійної квантової механіки // Укр. мат. журн. — 1997. — **49**, № 1. — С. 164–177.

332. *Фущич В.I.* Що таке швидкість електромагнітного поля? // Доп. НАН України. — 1997. — № 1. — С. 51–53.

333. *Fushchych W.I.* Symmetry of equations of nonlinear quantum mechanics // Proc. on the XXI International Colloquium on Group Theoretical Methods in Physics, Group21 "Physical Applications and Mathematical Aspects of Geometry, Groups, and Algebras" (July 15–20, 1996, Goslar, Germany), Eds. H.-D.

Doebner, W. Scherer, P. Natterman. — Singapore: Word Scientific, 1997. — **1**. — P. 439–446.

334. *Фущич В.І., Баранник Л.Л.* Симетрійна редукція по підалгебрах алгебри Пуанкаре однієї нелінійної системи диференціальних рівнянь для векторного поля // Доп. НАН України. — 1997. — № 8. — С. 50–57.

335. *Fushchych W.I., Boyko V.M.* Continuity equation in nonlinear quantum mechanics and the Galilei relativity principle // J. Nonlinear Math. Phys. — 1997. — **4**, N 1-2. — P. 124–128.

336. *Fushchych W.I., Nikitin A.G.* Higher symmetries and exact solutions of linear and nonlinear Schrödinger equation // J. Math. Phys. — 1997. — **38**, N 11. — P. 5944–5959.

337. *Фущич В.І.,. Сєров М.І., Тулупова Л.О.* Умовна симетрія рівнянь Нав'є–Стокса // Укр. мат. журн. — 1997 — **49**, № 6. — С. 806–813.

338. *Fushchych W.I., Symenoh Z.I.* High-order equations of motion in quantum mechanics and Galilean relativity // J. Phys. A: Math. Gen. — 1997. — **30**, N 6. — L131–L135.

339. *Fushchych W.I., Symenoh Z.I.* Symmetry of equations with convection terms // J. Nonlinear Math. Phys. — 1997. — **4**, N 3-4. — P. 470–479.

340. *Fushchych W.I., Tsyfra I.M.* On new Galilei- and Poincaré-invariant nonlinear equations for electromagnetic field // J. Nonlinear Math. Phys. — 1997. — **4**, N 1-2. — P. 44–48.

341. *Lahno V.I., Fushchych W.I.* Reduction of self-dual Yang–Mills equations with respect to subgroups of the extended Poincaré group // Theor. Math. Phys. — 1997. — **110**, N 3. — P. 329–342.

342. *Ma W.X., Bullough R.K., Caudrey P.J., Fushchych W.I.* Time-dependent symmetries of variable-coefficient evolution equations and graded Lie algebras // J. Phys. A: Math. Gen. — 1997. — **30**, N 14. — P. 5141–5149.

343. *Zhdanov R.Z., Fushchych W.I.* On new representations of Galilei groups // J. Nonlinear Math. Phys. — 1997. — **4**, N 3-4. — P. 426–435.

344. *Barannyk L.F., Basarab-Horwath P., Fushchych W.I.* On the classification of subalgebras of the conformal algebra with respect to inner automorphisms // J. Math. Phys. — 1998. — **39**, N 9. — P. 4899–4922.

345. *Boyko V.M., Fushchych W.I.* Lowering of order and general solutions of some classes of partial differential equations // Rep. Math. Phys. — 1998. — **41**, N 3. — P. 311–318.

346. *Fushchych W.I.* What is the velocity of the electromagnetic field? // J. Nonlinear Math. Phys. — 1998. — **5**, N 2. — P. 159–161.

347. *Фущич В.І., Лагно В.І.* Лінійні та нелінійні зображення груп Галілея в двовимірному просторі-часі // Укр. мат. журн. — 1998. — **50**, N 3. — С. 414–423.

348. *Фущич В.І., Сєров М.І., Подошвєлев Ю.Г.* Конформна симетрія нелінійного циліндрично симетричного хвильового рівняння // Доп. НАН України. — 1998. — № 4. — С. 64–68.

349. *Фущич В.І., Сєров М.І., Сєрова М.М., Глеба А.В.* Ліївська та умовна симетрія системи рівнянь Гамільтона–Якобі // Доп. НАН України. — 1998. — № 12. — С. 49–52.

350. *Fushchych W.I., Symenoh Z.I., Tsyfra I.M.* The Schrödinger equation with variable potential // J. Nonlinear Math. Phys. — 1998. — **5**, N 1. — P. 13–22.

351. *Zhdanov R.Z., Revenko I.V., Fushchych W.I.* Stationary mKdV hierarchy and integrability of the Dirac equations by quadratures // Phys. Lett. A. — 1998. — **241**. — P. 155–158.

352. *Фущич В.І., Сєров М.І., Подошвелев Ю.Г.* Конформна інваріантність системи рівнянь ейконалу // Доп. НАН України. — 1999. — № 1. — С. 43–47.

353. *Zhdanov R.Z., Lahno V.I., Fushchych W.I.* On covariant realizations of the Euclid group // Commun. Math. Phys. — 2000. — **212**. — P. 535–556.

О дополнительной инвариантности релятивистских уравнений движения

В.И. ФУЩИЧ

The additional (implicit) symmetry of equations invariant under the full Poincaré group is studied. It is shown that relativistic equations are invariant under the homogeneous de Sitter group $O(1,4)$ (or $O(2,3)$) and the matrix group $O(4)$.

Изучена дополнительная (неявная) симметрия уравнений, инвариантных относительно полной группы Пуанкаре. Показано, что релятивистские уравнения инвариантны относительно однородной группы де Ситтера $O(1,4)$ (или $O(2,3)$) и матричной группы $O(4)$.

Хорошо известно, что некоторые уравнения движения как в нерелятивистской, так и в релятивистской механике обладают дополнительной симметрией (инвариантностью). Например, уравнение Шредингера для атома водорода неявно инвариантно относительно четырехмерной группы вращений [1]; уравнения Максвелла, Дирака (для нулевой массы) инвариантны относительно конформной группы [2].

В настоящей работе показано, что релятивистские уравнения, описывающие свободное движение частиц (и античастиц) с ненулевой и нулевой массами и с произвольным спином s, инвариантны относительно однородной группы де Ситтера $O(1,4)$ и матричной группы $O(4)$. Найден явный вид операторов, являющихся базисными элементами алгебры Ли группы $O(4)$ и коммутирующих с гамильтонианом Дирака.

1. Дополнительная инвариантность уравнений для частицы с ненулевой массой

1. Для установления дополнительной симметрии уравнений, инвариантных относительно группы $P(1,3)$, удобно исходить из уравнений в канонической форме. Релятивистское уравнение, описывающее свободное движение частицы и античастицы со спином s и массой m, в каноническом представлении имеет вид [3, 4]

$$
\begin{gathered}
i\frac{\partial\Phi(t,\boldsymbol{x})}{\partial t} = \mathcal{H}^\Phi\Phi(t,\boldsymbol{x}), \quad \mathcal{H}^\Phi = \gamma_0 E_1, \\
\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_1 = \sqrt{p_1^2+p_2^2+p_3^2+m^2},
\end{gathered}
\tag{1.1}
$$

где Φ — волновая функция частицы, имеющая $2(2s+1)$ компонент; 1 — единичная матрица размерности $(2s+1)\times(2s+1)$. На множестве решений $\{\Phi\}$

Перепечатано из Теор. мат. физика, 1971, **7**, № 1, С. 3–12 по разрешению редколлегии

уравнения (1.1) реализуется неприводимое представление полной группы Пуанкаре $P(1,3)$ (включающей пространственно-временные отражения). Операторы Казимира группы $P(1,3)$ на множестве $\{\Phi\}$ кратны единичному оператору

$$W^2 = W_\alpha W^\alpha = m^2 s(s+1), \quad P^2 = P_\alpha P^\alpha = m^2, \quad W_\alpha = \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta} P^\beta J^{\gamma\delta}, \tag{1.2}$$

где P_α, $J_{\alpha\beta}$ — генераторы группы $P(1,3)$. На множестве $\{\Phi\}$ эти генераторы имеют вид [3, 4]

$$\begin{gathered} P_0 = \mathcal{H}^\Phi = \gamma_0 E_1, \quad P_a = p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \\ J_{ab} = x_a p_b - x_b p_a + \widetilde{S}_{ab}, \quad J_{0a} = x_0 p_a - \frac{1}{2}\left[x_a, \mathcal{H}^\Phi\right]_+ - \gamma_0 \frac{\widetilde{S}_{ab} p_b}{E_1 + m}, \end{gathered} \tag{1.3}$$

$$\widetilde{S}_{ab} = \begin{pmatrix} S_{ab} & 0 \\ 0 & S_{ab} \end{pmatrix}, \tag{1.4}$$

где S_{ab} — $(2s+1)\times(2s+1)$-матрицы, реализующие неприводимое представление алгебры $O(3)$ (группы и их алгебры обозначаются одинаковыми символами).

Инвариантность уравнения (1.1) относительно преобразований из группы $P(1,3)$ была доказана в [3, 4]. Этот факт является следствием того, что для произвольного $\Phi \in \{\Phi\}$ выполняется условие

$$\left[i\frac{\partial}{\partial t} - \mathcal{H}^\Phi, \mathcal{E}\right]_- \Phi = 0, \tag{1.5}$$

где $\mathcal{E}$ — любой элемент из обертывающей алгебры $\mathcal{E}(1,3)$ группы Пуанкаре $P(1,3)$ (относительно обертывающей алгебры $\mathcal{E}(1,3)$ см. [5]).

Теперь докажем следующее утверждение.

Теорема 1. *Уравнение* (1.1) *инвариантно относительно однородной группы де Ситтера* $O(1,4)$.

Доказательство. Рассмотрим оператор

$$R_\mu = \frac{1}{2}(P^\alpha J_{\mu\alpha} + J_{\mu\alpha} P^\alpha), \tag{1.6}$$

принадлежащий обертывающей алгебре $\mathcal{E}(1,3)$. Оператор R_μ удовлетворяет таким коммутационным соотношениям (см., например, [5, 6, 7, 8]):

$$[R_\mu, R_\nu]_- = iP^2 J_{\mu\nu}, \tag{1.7}$$

$$[R_\mu, J_{\alpha\beta}]_- = i(g_{\mu\alpha} R_\beta - g_{\mu\beta} R_\alpha), \tag{1.8}$$

$$[R_\alpha, P_\mu]_- = i(g_{\alpha\mu} P^2 - P_\alpha P_\mu), \tag{1.9}$$

$$[P_\mu, R^2]_- = 2iP^2 R_\mu, \quad R^2 \equiv R_\alpha R^\alpha, \tag{1.10}$$

$$[J_{\mu\nu}, R^2]_- = 0, \quad [R_\mu, R^2]_- = -iP^2(R^\alpha J_{\mu\alpha} + J_{\mu\alpha} R^\alpha). \tag{1.11}$$

Оператор

$$J_{\mu 4} = R_\mu / \sqrt{P^2} \tag{1.12}$$

вместе с операторами $J_{\mu\nu}$ удовлетворяет коммутационным соотношениям алгебры $O(1,4)$, поскольку

$$[J_{\mu 4}, J_{\nu 4}]_- = iJ_{\mu\nu}, \quad \mu,\nu = 0,1,2,3, \tag{1.13}$$

$$[J_{\mu\nu}, J_{\alpha\beta}]_- = i(g_{\mu\beta}J_{\nu\alpha} - g_{\mu\alpha}J_{\nu\beta} + g_{\nu\alpha}J_{\mu\beta} - g_{\nu\beta}J_{\mu\alpha}). \tag{1.14}$$

Так как оператор $J_{\mu 4}$ принадлежит алгебре $\mathcal{E}(1,3)$ ($\sqrt{P^2}$ на решениях уравнения (1.1) кратен единичному оператору), то тем самым теорема доказана.

Замечание 1. Оператор R_μ впервые рассматривал Ю.М. Широков [6]. В настоящее время такой оператор часто используется для получения спектра масс элементарных частиц в теоретико-групповом подходе [7].

Замечание 2. Уравнения вида

$$W^2\Psi(t,\boldsymbol{x}) = m^2 s(s+1)\Psi(t,\boldsymbol{x}), \tag{1.15}$$

$$P^2\Psi(t,\boldsymbol{x}) = m^2\Psi(t,\boldsymbol{x}) \tag{1.16}$$

инварианты, как это следует из теоремы [8], относительно группы $O(1,4)$.

2. В случае, когда $P^2 = -\eta^2$ (η — действительный параметр), группа $P(1,3)$ имеет как унитарные, так и неунитарные представления [3], причем все унитарные представления (по спиновым индексам) бесконечномерны, а значит, и уравнения движения, на множестве решений которых реализуется представление $P(1,3)$, будут бесконечнокомпонентны. Как показано в [9], для представлений класса III ($P^2 < 0$) каноническое уравнение “движения” имеет вид

$$\begin{gathered}
-i\frac{\partial\widetilde{\Phi}(t,\boldsymbol{x})}{\partial x_3} = \widetilde{P}_3\widetilde{\Phi}(t,\boldsymbol{x}), \\
\widetilde{P}_3 = \widetilde{\gamma}_0 E_3, \quad E_3 = \sqrt{p_0^2 - p_1^2 - p_2^2 + \eta^2}, \\
\widetilde{\gamma}_0 = \begin{pmatrix} \hat{1} & 0 \\ 0 & -\hat{1} \end{pmatrix}, \quad p_0 = -i\frac{\partial}{\partial t}.
\end{gathered} \tag{1.17}$$

Здесь $\widetilde{\Phi}(t,\boldsymbol{x})$ — функция, преобразующаяся по неприводимому представлению полной группы $\widetilde{P}(1,3)$, $\hat{1}$ — единичный оператор.

На множестве $\{\widetilde{\Phi}\}$ генераторы группы $P(1,3)$ имеют вид [9]

$$\begin{gathered}
P_0 = p_0, \quad P_a = p_a, \quad P_3 = \widetilde{P}_3 = \widetilde{\gamma}_0 E_3 \quad a = 1,2, \\
J_{ab} = x_a p_b - x_b p_a + \widetilde{S}'_{ab}, \quad b = 1,2, \\
J_{3a} = x_3 p_a - \frac{1}{2}[x_a, \widetilde{P}_3]_+ + \frac{\widetilde{S}'_{ab}p_b + i\widetilde{S}'_{a3}p_0}{E_3 + \eta}, \\
J_{0a} = x_0 p_a - x_a p_0 - i\widetilde{S}'_{3a}, \quad x_0 = t, \\
J_{30} = x_3 p_0 - \frac{1}{2}[x_0, \widetilde{P}_3]_+ - \widetilde{\gamma}_0\frac{i\widetilde{S}'_{3a}p_a}{E_3 + \eta}, \\
\widetilde{S}'_{ab} = \begin{pmatrix} S'_{ab} & 0 \\ 0 & S'_{ab} \end{pmatrix}, \quad \widetilde{S}'_{3a} = \begin{pmatrix} S'_{3a} & 0 \\ 0 & S'_{3a} \end{pmatrix},
\end{gathered} \tag{1.18}$$

где операторы S'_{ab}, iS'_{3a} реализуют неприводимое представление алгебры $O(1,2)$.

Условие типа (1.5) в этом случае имеет вид

$$\left[\widetilde{P}_3 + i\frac{\partial}{\partial x_3}, \mathcal{E}\right]\widetilde{\Phi} = 0. \tag{1.19}$$

Если теперь повторить те же рассуждения, что и при доказательстве теоремы 1, то придем к такому утверждению.

Теорема 2. *Уравнение* (1.17) *инвариантно относительно группы* $O(2,3)$.

Замечание 3. Теоремы 1 и 2 очевидным образом обобщаются и на уравнения, инвариантные относительно группы $P(n,l)$ — вращений и трансляций в $(n+l)$-мерном пространстве Минковского.

3. В этом пункте покажем, что уравнение (1.1), помимо инвариантности относительно групп $P(1,3)$ и $O(1,4)$, инвариантно относительно преобразований (по спиновым индексам, которые не связаны с пространственно-временными преобразованиями)

$$A\Phi = \Phi', \tag{1.20}$$

где A — произвольная матрица размерности $2(2s+1)\times 2(2s+1)$, принадлежащая матричной алгебре $O(4)$.

Прежде всего отметим, что, как следует из представления Фолди–Широкова (1.3), на решениях уравнения (1.1) реализуется прямая сумма двух неприводимых представлений алгебры $O(3)$

$$D(s)\oplus D(s). \tag{1.21}$$

Это означает, что на множестве $\{\Phi\}$ можно реализовать прямую сумму двух неприводимых представлений алгебры $O(4)$

$$D(s,0)\oplus D(0,s). \tag{1.22}$$

На множестве $\{\Phi\}$ базисные элементы алгебры $O(4)$ имеют вид

$$\widetilde{S}_{ab} = \begin{pmatrix} S_{ab} & 0 \\ 0 & S_{ab} \end{pmatrix}, \quad \widetilde{S}_{4a} = \begin{pmatrix} \varepsilon_{abc}S_{bc} & 0 \\ 0 & -\varepsilon_{abc}S_{bc} \end{pmatrix} \tag{1.23}$$

$(a,b,c = 1,2,3)$, причем

$$[\widetilde{S}_{kl}, \widetilde{S}_{rn}]_- = i(g_{kn}\widetilde{S}_{lr} - g_{rk}\widetilde{S}_{ln} + g_{lr}\widetilde{S}_{kn} - g_{ln}\widetilde{S}_{kr}, \quad k,r,n,l = 1,2,3,4. \tag{1.24}$$

Поскольку матрицы $\widetilde{S}_{ab}$, $\widetilde{S}_{4a}$ коммутируют с гамильтонианом $\mathcal{H}^{\Phi}$, уравнение (1.1) инвариантно относительно группы $O(4)$. Таким образом, приходим к следующему утверждению.

Теорема 3. *Уравнение* (1.1) *инвариантно относительно группы* $O(4)$.

Следует подчеркнуть, что из инвариантности уравнения (1.1) относительно группы $O(4)$ вытекает, что, помимо орбитального момента $\boldsymbol{M} = \boldsymbol{x}\times\boldsymbol{p}$ спинового момента $\boldsymbol{S}$, должен сохраняться еще один момент $\boldsymbol{S}'$. Компоненты векторов $\boldsymbol{S}$ и $\boldsymbol{S}'$ определяются через $\widetilde{S}_{kl}$ соотношениями

$$S_a = \frac{1}{2}(\widetilde{S}_{bc} + \widetilde{S}_{4a}), \quad S'_a = \frac{1}{2}(\widetilde{S}_{bc} - \widetilde{S}_{4a}), \tag{1.25}$$

a,b,c — цикл $(1,2,3)$.

Возникновение еще одного момента $\boldsymbol{S}'$ носит, по-видимому, чисто математический характер, связанный с P-, T-, C-инвариантностью уравнения (1.1). Как будет видно ниже, для уравнения Вейля дополнительный момент $\boldsymbol{S}'$ не возникает, в то время как для уравнения Дирака с нулевой массой (без дополнительного условия) он появляется. Не возникает дополнительный момент (по отношению к спину и изоспину) и для четырехкомпонентного уравнения Дирака в пятимерном подходе, которое, как известно [9], C-неинвариантно.

2. Дополнительная инвариантность уравнений для частицы с нулевой массой ($\boldsymbol{P^2 = 0}$, $\boldsymbol{W^2 \neq 0}$)

1. Рассмотрим два типа уравнений, описывающих свободное движение частицы с нулевой массой, "непрерывным" и дискретным спином. В этом случае удобно исходить из следующих уравнений:

$$W_\alpha W^\alpha \Psi(t, \boldsymbol{x}) = \rho^2 \Psi(t, \boldsymbol{x}), \quad P_\alpha P^\alpha \Psi(t, \boldsymbol{x}) = 0, \tag{2.1}$$

где ρ^2 — параметр, характеризующий неприводимое представление группы $P(1,3)$, который (подобно массе для представлений классов I, III, когда $P^2 \neq 0$) может принимать как положительные, так и отрицательные значения. Если $\rho^2 = 0$, то уравнения (2.1) описывают свободное движение частицы с нулевой массой и дискретным спином (нейтрино, фотон и т.д.). Можно, конечно, исходить и из других уравнений движения, но поскольку любые другие уравнения, на решениях которых реализуется неприводимое представление $\widetilde{P}(1,3)$, унитарно эквивалентны системе (2.1), то достаточно установить дополнительную инвариантность для уравнений (2.1).

Для уравнений (2.1) имеет место теорема.

Теорема 4. *Уравнения* (2.1) *для* $\rho^2 > 0$ *инвариантны относительно однородной группы де Ситтера* $O(1,4)$.

Доказательство. В том случае, когда $P^2 = 0$, оператор R_μ удовлетворяет таким коммутационным соотношениям (см. соотношения (1.7)–(1.11)):

$$[R_\mu, R_\nu]_- = 0, \tag{2.2}$$

$$[R_\mu, J_{\alpha\beta}]_- = i(g_{\mu\alpha} R_\beta - g_{\mu\beta} R_\alpha), \tag{2.3}$$

$$[R_\mu, P_\alpha]_- = i P_\alpha P_\mu, \tag{2.4}$$

$$[R^2, P_\mu]_- = [R^2, J_{\alpha\beta}]_- = [R^2, R_\gamma]_- = 0. \tag{2.5}$$

Из соотношений (2.2) и (2.3) видно, что операторы R_μ и $J_{\alpha\beta}$ — базисные элементы алгебры типа Пуанкаре $R(1,3)$. Оператор W^2 в этом случае совпадает с оператором R^2, который, подобно оператору P^2 в алгебре $P(1,3)$, является оператором Казимира алгебры $R(1,3)$. Вектор типа Паули–Любанского алгебры $R(1,3)$ имеет вид

$$\mathcal{V}_\alpha = \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} R^\beta J^{\gamma\delta}. \tag{2.6}$$

Оператор $\mathcal{V}^2 = \mathcal{V}_\alpha \mathcal{V}^\alpha$ — второй оператор Казимира алгебры $R(1,3)$. Рассматривая операторы

$$J_{\mu 4}^R = F_\mu / \sqrt{R^2}, \tag{2.7}$$

где

$$F_\mu = \frac{1}{2}(R^\alpha J_{\mu\alpha} + J_{\mu\alpha} R^\alpha), \tag{2.8}$$

и буквально повторяя рассуждения, приведенные при доказательстве теоремы 1, мы завершаем доказательство теоремы 4.

Проводя аналогичные рассуждения для случая $\rho^2 < 0$, приходим к утверждению.

Теорема 5. *Уравнения* (2.1) *для* $\rho^2 < 0$ *инвариантны относительно группы* $O(2,3)$.

Система уравнений (2.1) в случае $\rho^2 = 0$ инвариантна относительно группы $O(2,4) \supset O(1,4)$. Этот результат следует из теоремы о конформной инвариантности уравнений, описывающих свободное движение частиц с нулевой массой и дискретным спином [2].

2. Тот факт, что при $P^2 = 0$ и $W^2 \neq 0$ операторы R_μ, $J_{\alpha\beta}$ удовлетворяют алгебре типа Пуанкаре $R(1,3)$ (см. (2.3), (2.4)), позволяет рассматривать их как операторы “четырехмерного импульса” в пространстве функций $\{\Phi^R(y_0, y_1, y_2, y_3)\}$, где

$$R_\mu \Phi^R(y_0, y_1, y_2, y_3) = r_\mu \Phi^R(y_0, y_1, y_2, y_3), \tag{2.9}$$

$$r_0 = i\frac{\partial}{\partial y_0}, \quad r_a = -i\frac{\partial}{\partial y_a}, \quad a = 1,2,3. \tag{2.10}$$

Каноническое уравнение движения (для $\rho^2 > 0$), инвариантное относительно алгебры $R(1,3)$, имеет вид

$$i\frac{\partial \Phi^R(y_0, \boldsymbol{y})}{\partial y_0} = \gamma_0 E^R \Phi^R(y_0, \boldsymbol{y}), \quad E^R = \sqrt{r_1^2 + r_2^2 + r_3^2 + \rho^2}, \tag{2.11}$$

где $\Phi^R(y_0, \boldsymbol{y})$ — $2(2s+1)$-компонентная волновая функция.

На множестве решений уравнения (2.11) $\{\Phi^R\}$ операторы Казимира алгебры $R(1,3)$ кратны единичным операторам, т.е.

$$R^2\Phi^R = R^\alpha R_\alpha \Phi^R = \rho^2 \Phi^R, \quad \mathcal{V}^2 \Phi^R = \mathcal{V}^\alpha \mathcal{V}_\alpha \Phi^R = \rho^2 s(s+1)\Phi^R. \tag{2.12}$$

Базисные элементы алгебры $R(1,3)$ на $\{\Phi^R\}$ имеют вид (1.3), где следует совершить замену

$$P_\mu \to R_\mu, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial y_0}, \quad \frac{\partial}{\partial x_a} \to \frac{\partial}{\partial y_a}, \quad E_1 \to E^R.$$

Уравнение (2.11), как и (1.1), инвариантно относительно группы де Ситтера $O(1,4)$ и матричной группы $O(4)$.

Таким образом, параметры ρ и s, характеризующие неприводимые представления алгебры $R(1,3)$, в Φ^R-представлении следует интерпретировать как “массу и спин" частицы. Это означает, что представлениям группы Пуанкаре, для которых $P^2 = 0$ и $W^2 \neq 0$, можно придать вполне ясный смысл, если в качестве полного набора коммутирующих операторов выбрать операторы R_μ и одну из компонент $\mathcal{V}_\mu$, например $\mathcal{V}_3$. Важно отметить, что в пространстве представлений группы $P(1,3)$, где операторы R_μ диагональны, операторы P_μ недиагональны.

3. Об инвариантности уравнения Дирака

1. В этом пункте найдем явный вид операторов, являющихся базисными элементами алгебры Ли группы $O(4)$, коммутирующих с гамильтонианом Дирака.

Уравнение Дирака

$$i\frac{\partial\Psi'(t,\boldsymbol{x})}{\partial t} = (\gamma_0\gamma_a p_a + \gamma_0 m)\Psi'(t,\boldsymbol{x}), \quad a = 1,2,3, \tag{3.1}$$

после преобразования

$$U_1 = \frac{1}{\sqrt{2}}(1-\gamma_4), \tag{3.2}$$

принимает вид

$$\begin{aligned} &i\frac{\partial\Psi(t,\boldsymbol{x})}{\partial t} = \mathcal{H}\Psi(t,\boldsymbol{x}), \quad \mathcal{H} = \gamma_0\gamma_k p_k, \quad k = 1,2,3,4, \\ &\Psi = U_1\Psi', \quad p_4 \equiv m. \end{aligned} \tag{3.3}$$

Для наших целей будет удобно исходить из уравнения Дирака в форме (3.3), что позволит провести одновременно все рассмотрения для $m > 0$ и $m < 0$.

Уравнение (3.3) после преобразования

$$U\left(p, s=\frac{1}{2}\right) = \exp\left\{\frac{\pi}{4}\frac{\gamma_0\mathcal{H}}{E}\right\} \equiv \frac{1}{2}\left(1 + \frac{\gamma_0\mathcal{H}}{E}\right), \quad E = \sqrt{p_k^2} \equiv E_1 \tag{3.4}$$

примет канонический вид

$$i\frac{\partial\Phi(t,\boldsymbol{x})}{\partial t} = \mathcal{H}^\Phi\Phi(t,\boldsymbol{x}), \quad \mathcal{H}^\Phi = \gamma_0 E, \tag{3.5}$$

где γ_0 — четырехрядная матрица Дирака (см. (1.1)).

Генераторы группы $P(1,3)$ на множестве $\{\Phi\}$ выглядят так (представление (3.6) справедливо не только для спина $s = 1/2$, но и для произвольного спина s):

$$\begin{aligned} &P_0 \equiv \mathcal{H}^\Phi = \gamma_0 E, \quad P_a = p_a, \quad J_{ab} = x_a p_b - x_b p_a + \widetilde{S}_{ab}, \\ &J_{0a} = x_0 p_a - \frac{1}{2}[x_a, \mathcal{H}^\Phi]_+ - \gamma_0\frac{\widetilde{S}_{ab}p_b + \widetilde{S}_{a4}p_4}{E}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\widetilde{S}_{kl} = \frac{i}{4}(\gamma_k\gamma_l - \gamma_l\gamma_k), \quad \widetilde{S}_{0k} = \frac{i}{4}(\gamma_0\gamma_k - \gamma_k\gamma_0), \\ &\widetilde{S}_{5k} = -\frac{i}{2}\gamma_k, \quad \widetilde{S}_{05} = \frac{i}{2}\gamma_0. \end{aligned} \tag{3.7}$$

Матрицы $\widetilde{S}_{kl}$ — генераторы группы $O(4)$ и, кроме того, коммутируют с гамильтонианом $\mathcal{H}^\Phi$ в представлении Φ. Это и означает, что уравнение (3.5) дополнительно инвариантно относительно матричной алгебры $O(4)$. С гамильтонианом $\mathcal{H}^\Phi$, очевидно, коммутирует и матрица $\widetilde{S}_{05}$.

Чтобы непосредственно показать, что уравнение (3.3) инвариантно относительно алгебры $O(4)$, достаточно найти операторы типа $\widetilde{S}_{kl}$, которые коммутировали бы с оператором $\mathcal{H}$. Эти операторы нетрудно найти, если воспользоваться унитарным оператором U^{-1}, связывающим представления Φ и Ψ.

Можно непосредственно проверить, что операторы

$$\begin{aligned} &S_{kl}^\Psi = U^{-1}\widetilde{S}_{kl}U = \widetilde{S}_{kl} + \frac{1}{E}(\widetilde{S}_{5k}p_l - \widetilde{S}_{5l}p_k)\left(1 - \frac{2iS_{5r}p_r}{E}\right), \\ &S_{05}^\Psi = U^{-1}\widetilde{S}_{05}U = \frac{i}{2}\frac{\mathcal{H}}{E} \end{aligned} \tag{3.8}$$

коммутируют с оператором $\mathcal{H}$.

Таким образом, уравнение (3.3), а значит, и уравнение (3.1), как для ненулевой, так и для нулевой массы инвариантно относительно алгебры $O(4)$. Этот результат является частным случаем более общего утверждения, доказанного в п. 3, раздела 1.

Следует отметить, что поскольку с γ_0 коммутируют только матрицы $\widetilde{S}_{kl}$ и $\widetilde{S}_{05}$ (матрицы $\gamma_0\widetilde{S}_{rn}$, $\widetilde{S}_{rl}\widetilde{S}_{kn}$ — линейные комбинации $\widetilde{S}_{kl}$ и $\widetilde{S}_{05}$), то алгебра Ли, порожденная ими, является максимальной алгеброй, относительно которой уравнение (3.5) инвариантно.

Дополнительная симметрия уравнения Дирака методами, отличными от наших, исследовалась в работах [10].

Ради полноты изложения приведем явный вид оператора координаты в представлении Ψ

$$X_a^\Psi = U^{-1}x_aU = x_a + \frac{1}{E}\left(\widetilde{S}_{a5} + \frac{\widetilde{S}_{5k}p_k}{E^2}p_a + \frac{\widetilde{S}_{ab}p_b + \widetilde{S}_{a4}p_4}{E}\right). \tag{3.9}$$

2. Двухкомпонентное уравнение Вейля

$$i\frac{\partial\chi(t,\boldsymbol{x})}{\partial t} = \sigma_b p_b \chi(t,\boldsymbol{x}), \tag{3.10}$$

как известно, эквивалентно уравнению Дирака для нулевой массы с дополнительным условием, т.е. эквивалентно системе уравнений

$$i\frac{\partial\Psi(t,\boldsymbol{x})}{\partial t} = \gamma_0\gamma_a p_a \Psi(t,\boldsymbol{x}), \quad a = 1,2,3, \tag{3.11}$$

$$(1 - i\gamma_4)\Psi(t,\boldsymbol{x}) = 0. \tag{3.12}$$

Непосредственной проверкой можно убедиться, что дополнительное условие (3.12) неинвариантно относительно операторов $\widetilde{S}_{kl}^\Psi$, т.е.

$$[\gamma_4, \widetilde{S}_{kl}^\Psi]_- \neq 0.$$

Итак, система уравнений (3.11), (3.12) не обладает дополнительной симметрией относительно группы $O(4)$.

Если над уравнением (3.10) совершить преобразование типа Фолди–Воутхойзена [11], то оно примет канонический вид

$$i\frac{\partial\Phi(t,\boldsymbol{x})}{\partial t} = \sigma_3 E\Phi(t,\boldsymbol{x}), \quad E = \sqrt{p_1^2+p_2^2+p_3^2}. \tag{3.13}$$

Уравнение (3.13) уже явно инвариантно относительно преобразования

$$\Phi \to \sigma_3\Phi, \tag{3.14}$$

следовательно, уравнение Вейля (3.10) дополнительно инвариантно относительно группы $O(2)$.

3. Из предыдущего пункта ясно, что дополнительная инвариантность уравнений движений зависит от компонентности волновой функции. Ниже будет установлена зависимость дополнительной симметрии уравнений от размерности пространства Минковского, в котором они заданы.

Рассмотрим в пятимерном пространстве Минковского два неэквивалентных уравнения типа Дирака, инвариантных относительно неоднородной группы де Ситтера $P(1,4)$:

$$\begin{gathered} i\frac{\partial\Psi_\pm(t,\boldsymbol{x},x_4)}{\partial t} = (\gamma_0\gamma_k p_k + \gamma_0\varkappa)\Psi_\pm(t,\boldsymbol{x},x_4), \\ p_k = -i\frac{\partial}{\partial x_k}, \quad k=1,2,3,4, \end{gathered} \tag{3.15}$$

где $\Psi_\pm$ — четырехкомпонентный спинор, $\varkappa$ — постоянная величина. Проводя для уравнений (3.15) такой же анализ, как и для (3.3) (с некоторыми очевидными изменениями), можно показать, что уравнение (3.15) для функции $\Psi_=$ (или Ψ_-) дополнительно инвариантно относительно группы $O(4)$.

Итак, четырехкомпонентное уравнение Дирака в пятимерном подходе, помимо инвариантности относительно групп $P(1,4)$ и $O(1,5)$, инвариантно относительно матричной группы $O(4)$. Из этого результата, в частности, следует, что спиновый и изоспиновый моменты в пятимерной схеме квантовой механики сохраняются. Это и следовало ожидать, поскольку малой группой группы $P(1,4)$ является группа $O(4)$, которая локально изоморфна группе $SU(2)\times SU(2)$.

Особенностью уравнения (3.15) для функции Ψ_+ (или Ψ_-) является то, что оно в отличие от обычного уравнения Дирака неинвариантно относительно C-преобразований (более детально см. [9]). В пятимерном подходе простейшим спинорным P-, T-, C-инвариантным уравнением является восьмикомпонентное уравнение [9]

$$\begin{gathered} i\frac{\partial\Psi(t,\boldsymbol{x},x_4)}{\partial t} = \left\{\begin{pmatrix}\gamma_0\gamma_k & 0\\ 0 & \gamma_0\gamma_k\end{pmatrix}p_k + \begin{pmatrix}\gamma_0 & 0\\ 0 & -\gamma_0\end{pmatrix}\varkappa\right\}\Psi(t,\boldsymbol{x},x_4), \\ \Psi \equiv \begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}, \end{gathered} \tag{3.16}$$

являющееся объединением двух уравнений (3.15).

Для этого уравнения справедливо следующее утверждение: уравнение (3.16) инвариантно относительно матричной алгебры $O(6)$. Чтобы доказать это утверждение, следует представить уравнение (3.16) в форме

$$i\frac{\partial\Psi(t,\boldsymbol{x},x_4)}{\partial t} = (\Gamma_0\Gamma_k p_k + \Gamma_0\varkappa)\Psi(t,\boldsymbol{x},x_4), \tag{3.17}$$

где восьмирядные матрицы Γ_0, Γ_k и Γ_5, Γ_6 — базисные элементы восьмимерной алгебры Клиффорда, а потом повторить рассуждения, приведенные в пункте 1.

Из приведенного анализа уравнений (3.3), (3.10), (3.15), (3.16) вытекает, что дополнительная инвариантность уравнений движений, инвариантных относительно неоднородных групп типа $P(1,n)$, зависит как от размерности пространства Минковского, так и от компонентности волновых функций.

Замечание 4. Если в уравнении (3.15) положить $\varkappa = 0$, оно будет описывать частицу и античастицу с переменной массой $\sqrt{p_4^2}$ и фиксированным спином $s = 1/2$ [9]. Уравнение (3.5) (для $\varkappa = 0$) инвариантно относительно группы $O(2,5)$, содержащей в качестве подгруппы конформную группу. Следует отметить, что обычное уравнение Дирака с фиксированной массой неинвариантно даже относительно конформной группы.

Автор выражает благодарность А.Л. Грищенко за проверку некоторых формул настоящей работы.

1. Фок В., *Z. Phys.*, 1936, **98**, 145.
2. Lomont J.S., *Nuovo Cim.*, 1957, **6**, 204;
   Gross L., *J. Math. Phys.*, 1964, **5**, 687.
3. Широков Ю.М., *ДАН СССР*, 1954, **94**, 857; 1954, **99**, 737; *ЖЭТФ*, 1957, **33**, 1196.
4. Foldy L.L., *Phys. Rev.*, 1956, **102**, 568.
5. Rosen J., Roman P., *J. Math. Phys.*, 1966, **7**, 2072.
6. Широков Ю.М., *ЖЭТФ*, 1951, **21**, 748.
7. Böhm A., *Phys. Rev.*, 1966, **145**, 1212.
8. Sankaranarajananan A., *Nuovo Cim.*, 1965, **38**, 1441.
9. Фущич В.И., *ТМФ*, 1970, **4**, 360.
10. Pauli W., *Nuovo Cim.*, 1957, **6**, 204;
    Sen Gupta N.D., *Nuovo Cim.*, 1965, **36**, 1181;
    Ибрагимов Н.Х., *ДАН СССР*, 1969, **185**, 1226.
11. Fronsdal C., *Phys. Rev.*, 1959, **113**, 1367.

On the Galilean-invariant equations for particles with arbitrary spin

W.I. FUSHCHYCH, A.G. NIKITIN

In our preceding paper [1] the equations of motion which are invariant under the Galilei group G have been obtained starting with the assumption that the Hamiltonian of a nonrelativistic particle has positive eigenvalues and negative ones. These nonrelativistic equations as well as the relativistic Dirac equation describe the spin-orbit and Darwin interactions after the standard replacement $p_\mu \to \pi_\mu = p_\mu - eA_\mu$. Previously it was generally accepted to think that the spin-orbit and the Darwin interactions are truly relativistic effects [2].

In [1] equations for particles with the lowest spins $s = \frac{1}{2}, 1, \frac{3}{2}$ have been obtained. What puts the equations [1] in a class by themselves is that the transformation properties of a wave function are rather complicated (nonlocal) and it is difficult to establish their invariance under the Galilei transformations after the replacement $p_\mu \to \pi_\mu$.

In the present note equations for arbitrary-spin particles are obtained which possess as good physical properties as the equations [1].

Moreover the related wave functions have simple transformation properties in the case of the equation describing interaction with an external field and in the case of the absence of interaction as well.

We shall start with the assumption that under the Galilei transformation

$$\begin{aligned} &\boldsymbol{x} \to \boldsymbol{x}' = R\boldsymbol{x} + \boldsymbol{V}t + \boldsymbol{a}, \\ &t \to t' = t + b, \end{aligned} \tag{1}$$

the $2(2s+1)$-component wave function $\Psi(t, \boldsymbol{x})$ transforms as

$$\Psi(t, \boldsymbol{x}) \to \Psi'(t', \boldsymbol{x}') = \exp[if(t, \boldsymbol{x})] D^s(R, \boldsymbol{V}) \Psi(t, \boldsymbol{x}), \tag{2}$$

where $D^s(R, \boldsymbol{V})$ is a numerical matrix, depending on the parameters of transformation (1), $\exp[if(t, \boldsymbol{x})]$ is the phase factor [3]

$$f(t, \boldsymbol{x}) = m\boldsymbol{V} \cdot R\boldsymbol{x} + \frac{1}{2} m v^2 t. \tag{3}$$

The generators of Galilei group G, which correspond to transformation (2), have the form

$$\begin{aligned} &P_0 = i\frac{\partial}{\partial t}, \quad P_a = p_a = -i\frac{\partial}{\partial x_a}, \quad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \\ &G_a = t p_a - m x_a + \lambda_a, \quad S_{ab} = \begin{pmatrix} s_{ab} & 0 \\ 0 & s_{ab} \end{pmatrix}, \end{aligned} \tag{4}$$

where s_{ab} are generators of irreducible representation $D(s)$ of group $O(3)$, λ_a are some numerical matrices, which have to be such that the operators (4) satisfy the

Reprinted with permission from Lettere al Nuovo Cimento, 1976, **16**, N 3, P. 81–85

commutation relations of algebra G. It can be shown that the most general (up to equivalence) form of the matrices λ_a satisfying this requirement is

$$\lambda_a = k(\sigma_3 + i\sigma_2)S_a, \quad S_a = \frac{1}{2}\varepsilon_{abc}S_{bc}, \tag{5}$$

where σ_2, σ_3 are the $2(2s+1)$-dimensional Pauli matrices which commute with S_{ab}, k is an arbitrary constant.

To find the motion equations for arbitrary-spin particles

$$i\frac{\partial}{\partial t}\Psi(t, \boldsymbol{x}) = H_s(\boldsymbol{p}, \boldsymbol{s})\Psi(t, \boldsymbol{x}) \tag{6}$$

it is sufficient to construct such operator (Hamiltonian) $H_s(\boldsymbol{p}, \boldsymbol{s})$ that eq. (6) be invariant under the Galilei group G. Equation (6) will be invariant with respect to G, if the following conditions are satisfied:

$$[H_s(\boldsymbol{p}, \boldsymbol{s}), P_a]_- = 0, \quad [H_s(\boldsymbol{p}, \boldsymbol{s}), J_{ab}]_- = 0, \quad [H_s(\boldsymbol{p}, \boldsymbol{s}), G_a]_- = -iP_a. \tag{7}$$

Thus our problem has been reduced to solution of equations (7). The analogous problem has been solved in the relativistic case in [4]. Lately the method of the work [4] has been further developed in works of R.F. Guertin [5].

In order to solve relations (7) we expand H_s in a complete system of the orthoprojectors and Pauli matrices

$$H_s(\boldsymbol{p}, \boldsymbol{s}) = \sum_{\mu, r}\sigma_\mu a_r^\mu \Lambda_r, \quad \mu = 0, 1, 2, 3, \tag{8}$$

where

$$\Lambda_r = \prod_{r\neq r'}\frac{\boldsymbol{s}\cdot\boldsymbol{p}/p - r'}{r - r'}, \quad r, r' = -s, -s+1, \ldots, s,$$

and σ_0 is the $2(2s+1)$-dimensional unit matrix, $a_r^\mu(p)$ are unknown coefficient functions. Substituting (8) into (7), using the relations [4]

$$\begin{aligned}
&[\Lambda_r, x_a] = \frac{S_{ab}p_b}{2p^2}(2\Lambda_r - \Lambda_{r+1} - \Lambda_{r-1}) + \frac{i}{2p}\left(S_a - \frac{p_a}{p}\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\right)(\Lambda_{r+1} - \Lambda_{r-1}),\\
&[\Lambda_r, S_{ab}] = p_a[\Lambda_r, x_b] - p_b[\Lambda_r, x_a],
\end{aligned} \tag{9}$$

and taking into account the completeness and the orthogonality of the orthprojectors, we have found that, up to equivalence, the general form of the Hamiltonian $H_s(\boldsymbol{p}, \boldsymbol{s})$, satisfying (7), is given by the formula

$$H_s = m_0 + \sigma_3\eta m + \frac{p^2}{2m} - \sigma_1 2i\eta h\boldsymbol{S}\cdot\boldsymbol{p} - (\sigma_3 + i\sigma_2)\eta k^2\frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{m}, \tag{10}$$

where η is an arbitrary constant.

Formula (10) gives the free nonrelativistic Hamiltonian for a particle with an arbitrary spin. Equation (6) with the Hamiltonian (10) is invariant under the group G. For the spin $\frac{1}{2}$ particle (when $s = \frac{1}{2}$, $k = -i$, $\eta = 1$) equation (6) can be written in the following compact form

$$(\gamma_\mu p^\mu + m)\Psi = (1 + \gamma_4 - \gamma_0)\frac{p^2}{2m}\Psi, \tag{11}$$

where γ_μ are the Dirac matrices.

The Hamiltonian (10) and the generators (4) are non-Hermitian under the usual scalar product. They are, however, Hermitian under

$$(\Psi_1, \Psi_2) = \int d^3x \ \Psi_1^\dagger M \Psi_2, \tag{12}$$

where M is positive-definite metric operator

$$M = 1 + [i(k - k^*)\sigma_3 - (k + k^*)\sigma_2]\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{m} + 2|k|^2(1 + \sigma_1)\left(\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{m}\right)^2. \tag{13}$$

Besides, if η, k satisfy the condition $\eta k = (\eta k)^*$, the Hamiltonians are Hermitian also in the indefinite metric

$$(\Psi_1, \Psi_2) = \int d^3x \ \Psi_1^\dagger \xi \Psi_2, \tag{14}$$

where

$$\xi = \begin{cases} \sigma_3, & \text{if} \quad \eta^* = \eta, \ k^* = k, \\ \sigma_2, & \text{if} \quad \eta^* = -\eta, \ k^* = -k. \end{cases}$$

With the help of the transformation

$$H_s \to H_s' = V H_s V^{-1}, \quad V = \exp\left[i\frac{\boldsymbol{\lambda}\cdot\boldsymbol{p}}{m}\right], \tag{15}$$

the Hamiltonian (10) can be reduced to the diagonal form

$$H_s' = m_0 + \sigma_3 \eta m + \frac{p^2}{2m}. \tag{16}$$

It is interesting to note that the condition of Galilei invariance admits the possibility to introduce two masses: the rest mass, or the rest energy ($\varepsilon_1 = m_0 + \eta m$, $\varepsilon_2 = m_0 - \eta m$) and the kinetic mass (the coefficient of p^2). Below we consider the case when $m_0 = 0$, $\eta = 1$, i.e. the rest mass is equal to the kinetic mass.

To describe the motion of a charged particle in an external electromagnetic fields we make in (6) and (10) the replacement $p_\mu \to \pi_\mu$ (symmetrizing preliminarily the Hamiltonian in p_a [1]). This leads to the equation

$$i\frac{\partial}{\partial t}\Psi(t, \boldsymbol{x}) = H_s(\boldsymbol{\pi})\Psi(t, \boldsymbol{x}), \tag{17}$$

$$H_s(\boldsymbol{\pi}) = \sigma_3 m + \frac{\boldsymbol{\pi}^2}{2m} + \sigma_1 2ik(\boldsymbol{S}\cdot\boldsymbol{p}) + \frac{2k^2}{m}(\sigma_3 + i\sigma_2)\left[(\boldsymbol{S}\cdot\boldsymbol{\pi})^2 + \frac{1}{2}(\boldsymbol{S}\cdot\boldsymbol{M})\right], \tag{18}$$

where $H_a = i\varepsilon_{abc}[\pi_b, \pi_c]_-$ are components of the magnetic field vector.

It is important to note that eq. (17) is still invariant with respect to the Galilei transformations (1) and (2), if the vector potential is transformed according to [2]

$$\boldsymbol{A} \to \boldsymbol{A}' = R\boldsymbol{A}, \quad A_0 \to A_0' = A_0 + \boldsymbol{V}R\boldsymbol{A}. \tag{19}$$

To prove this statement it is sufficient to use the exact form of the matrix $D^s(R,\boldsymbol{V})$ in (2)

$$D^s(R,\boldsymbol{V}) = (1 + i\boldsymbol{\lambda}\cdot\boldsymbol{V})\cdot\begin{pmatrix} D^s(R) & 0 \\ 0 & D^s(R)\end{pmatrix}, \qquad (20)$$

where $D^s(R)$ the matrices from the representation $D(s)$ of the group $O(3)$.

As in the case of the Dirac equation [6] the Hamiltonian (18) cannot be diagonalized exactly. We shall make the approximate diagonalization of the operator (18) up to the terms of power $1/m^2$ with using of the operator

$$V(\boldsymbol{\pi}) = \exp[iB_3^s]\exp[iB_2^s]\exp[iB_1^s], \qquad (21)$$

where

$$\begin{aligned} &B_1^s = i\sigma_2 k\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{m}, \quad E_a = -\frac{\partial A_a}{\partial x_a} - \frac{\partial A_a}{\partial t}, \\ &B_2^s = -\sigma_1 k\frac{[\boldsymbol{S}\cdot\boldsymbol{\pi},\boldsymbol{\pi}^2]_-}{4m^2} - i\sigma_1 k^2\frac{(\boldsymbol{S}\cdot\boldsymbol{\pi})^2 - \frac12\boldsymbol{S}\cdot\boldsymbol{H}}{m^2} - i\sigma_1 k\frac{\boldsymbol{S}\cdot\boldsymbol{E}}{2m^2}, \\ &B_3^s = -\frac23 ik^3\sigma_2\left(\frac{\boldsymbol{S}\cdot\boldsymbol{\pi}}{m}\right)^3 + ik^3\frac{[\boldsymbol{S}\cdot\boldsymbol{\pi},\boldsymbol{S}\cdot\boldsymbol{H}]_+}{m^3}\sigma_2 + \sigma_2\frac{k^2[(\boldsymbol{S}\cdot\boldsymbol{\pi})^2, eA_0]}{m^3}. \end{aligned} \qquad (22)$$

As a result we obtain

$$\begin{aligned} V(\boldsymbol{\pi})H^s(\boldsymbol{\pi})V^{-1}(\boldsymbol{\pi}) &= \sigma_3 m + \frac{\boldsymbol{\pi}^2}{2m} + eA_0 + k^2\sigma_3\frac{\boldsymbol{S}\cdot\boldsymbol{H}}{m} - \\ &- \frac{k^2}{4m^2}\boldsymbol{S}\cdot(\boldsymbol{\pi}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{\pi}) + \frac{k^2}{6m^2}s(s+1)\,\mathrm{div}\,\boldsymbol{E} + \frac{k^2}{12m^2}Q_{ab}\frac{\partial E_b}{\partial x_a} + \\ &+ \frac{k^3}{m^2}\boldsymbol{S}\cdot(\boldsymbol{\pi}\times\boldsymbol{H} - \boldsymbol{H}\times\boldsymbol{\pi}) - \frac13\frac{k^3}{m^2}Q_{ab}\frac{\partial H_a}{\partial x_b} + o\left(\frac{1}{m^3}\right), \end{aligned} \qquad (23)$$

where Q_{ab} is the tensor of quadrupole interaction

$$Q_{ab} = 3[S_a, S_b]_+ - 2\delta_{ab}s(s+1). \qquad (24)$$

It is readily seen from (23) that $-k^2$ can be interpreted as the dipole magnetic moment of the particle. If $s = \frac12$, $-k^2 = 1$ (it corresponds to the "normal" dipole moment), the first seven constituents of the approximate Hamiltonian coincide on the set $\Phi^+ = \frac12(1+\sigma_3)\Phi$ with the Foldy–Wouthuysen Hamiltonian, which had been obtained from the relativistic Dirac equation. The last two terms in (23) can be interpreted as the magnetic spin-orbit and the magnetic quadrupole interactions of the particle with the field.

In conclusion we note that we have not required the invariance with respect to the time reflection for eq. (6). This invariance has been ensured if one doubles (brings to $4(2s+1)$) the number of the components of the wave function and assumes that the particle energy can take both positive and negative values. An analogous situation takes place in the relativistic theory [7].

As in the relativistic theory, it is possible to construct for the particle with spin s the nonrelativistic wave equations with another (different from $2(2s+1)$ or $4(2s+1)$)

number of components. For instance, the spin-one and spin-zero particles can bo described by the Galilean-invariant equations

$$(\beta_\mu p^\mu - m)\Psi = \left[\beta_0 \frac{p^2}{2m} + \beta_0^2 \frac{(\boldsymbol{\beta}\cdot\boldsymbol{p})^2}{m}\right]\Psi, \qquad (25)$$

where β_μ are the 10×10- or 5×5-dimensional Kemmer–Duffin–Petiau matrices. These equations will be considered in another work.

Note. The equations obtained in [1] and in the present paper can be considered as those with the broken Lorentz symmetry. Actually, equations (12) from [1] and (11) from the present work have the form of the Dirac equation with the additional term which is noninvariant under the Poincaré group, but is Galilean invariant. The second-order equations with this broken symmetry have the form

$$(p_\mu p^\mu - m^2)\Psi = B\Psi, \qquad (26)$$

where $B = p^4/4m^2$ for the equations of ref. [1] and $B = m(1+2\sigma_3) + p^2\sigma_3 + p^4/4m^2$ for the equations from the present paper (if $m_0 = m$, $\eta = 1$).

1. Fushchych W.I., Nikitin A.G., Salogub V.A., *Lett. Nuovo Cimento*, 1975, **14**, № 13, 483–488.
2. Levi-Leblond J.-M., *Comm. Math. Phys.*, 1967, **6**, 286.
3. Bargman V., *Ann. Math.*, 1954, **59**, 1;
Hamermesh M., *Am. of Phys.*, 1960, **9**, 518.
4. Fushchych W.I., Grishchenko A.L., Nikitin A.G., *Teor. Mat. Fiz.*, 1971, **8**, № 2, 192–205 (in Russian); *Theor. Math. Phys*, 1971, **8**, 766 (in English).
5. Guertin R.F., *Ann. of Phys.*, 1974, **88**, 504; 1975, **91**, 386.
6. Foldy L.L., Wouthuysen S.A., *Phys. Rev.*, 1950, **68**, 29.
7. Fushchych W.I., *Lett. Nuovo Cimento*, 1973, **6**, № 4, 133–137.

*It was apparently the first paper where it was demonstrated that spin orbit coupling can be described in the framework of Galilei-invariant approach. *Editors' Remark.*

On the new invariance groups of the Dirac and Kemmer–Duffin–Petiau equations

W.I. FUSHCHYCH, A.G. NIKITIN

In works [1–6] the canonical-transformation method has been proposed for investigation of group properties of differential equations of quantum mechanics. This method essence in that the system of differential equation is first transformed to the diagonal or Jordan form and then the invariance algebra of the transformed equation is established. The explicit form of this algebra basis elements for the starting equations is found by the inverse transformation.

The main distinguishing feature of this method from the classical Lie approach [7, 8] is that the basis elements of invariance algebra of the corresponding equations do not belong to the class of differential operators, but are as a rule integrodifferential operators. The new invariance algebras of the Dirac [1, 2] (the results of the work [2] have been generalized by Jayaraman (*J. Phys. A*, 1976, **9**, 1181) to the case of the equation without redundant components for any spin, see also [1]), Maxwell [2], Klein–Gordon [3], Kemmer–Duffin–Petiau (KDP) and Rarita–Schwinger [4] equations have been found just in the class of integrodifferential operators.

The aim of this note is to describe invariance algebras of the Dirac and KDP equations in the class of differential operators. The theorems given below (which establish new group properties of the Dirac and KDP equations) are proved using the canonical-transformation method.

To establish an invariance of the equation

$$L(p_0,p_1,p_2,p_3)\Psi(x_0,\boldsymbol{x}) \equiv L\Psi = 0, \quad p_\mu = i\frac{\partial}{\partial x^\mu} \tag{1}$$

under the set of transformations $\Psi \to \Psi'_A = Q_A\Psi$ is to found a set of operators $Q \equiv \{Q_A\}$ such that

$$[L,Q_A]_-\Psi = 0, \quad \forall\, Q_A \in Q, \tag{2}$$

where Ψ is a function which satisfies eq. (1). Condition (2) can be written in the operator form

$$[L,Q_A]_- = F\cdot L, \tag{3}$$

where F is some set of operators, which are defined in the space of equation (1) solutions.

Theorem 1. *The Dirac equation*

$$L_{\frac{1}{2}}\Psi \equiv (\gamma_\mu p^\mu + m)\Psi = 0 \tag{4}$$

is invariant under the 16*-dimensional Lie algebra* A_{16}*, whose basis elements are*

$$P_\mu = p_\mu = i\frac{p}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{2}\gamma_\mu\gamma_\nu, \tag{5}$$

Reprinted with permission from Lettere al Nuovo Cimento, 1977, **19**, N 9, P. 347–352

$$Q_{\mu\nu} = i\gamma_\mu\gamma_\nu + \frac{i}{m}(1 + i\gamma_4)(\gamma_\mu p_\nu - \gamma_\nu p_\mu), \quad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{6}$$

Proof. The theorem validity can be established by direct verification. Indeed, one obtains by direct calculation that $Q_{\mu\nu}$ satisfies the invariance condition (3)

$$[Q_{\mu\nu}, L_{\frac{1}{2}}]_- = F^{\frac{1}{2}}_{\mu\nu} L_{\frac{1}{2}}, \quad F^{\frac{1}{2}}_{\mu\nu} = \frac{i}{m}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) \tag{7}$$

and form together with P_μ, $J_{\mu\nu}$ the Lie algebra

$$\begin{aligned}
&[P_\mu, P_\nu]_- = 0, \quad [P_\lambda, J_{\mu\nu}]_- = i(g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu), \quad [P_\lambda, Q_{\mu\nu}]_- = 0,\\
&[J_{\mu\nu}, J_{\lambda\sigma}]_- = i(g_{\mu\lambda}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\lambda} - g_{\mu\sigma}J_{\nu\lambda} - g_{\nu\lambda}J_{\mu\sigma}),\\
&[Q_{\mu\nu}, J_{\lambda\sigma}]_- = \frac{1}{2}[Q_{\mu\nu}, Q_{\lambda\sigma}]_- = i(g_{\mu\lambda}Q_{\nu\sigma} + g_{\nu\sigma}Q_{\mu\lambda} - g_{\mu\sigma}Q_{\nu\lambda} - g_{\nu\lambda}Q_{\mu\sigma}).
\end{aligned} \tag{8}$$

A more elegant and constructive way, which shows the method to obtain operators (6) is to transform eq. (4) to the diagonal form. After such a transformation the theorem statements become obvious.

Such transformation can be carried out in two steps. First, eq. (4) is multiplied by the invertible differential operator

$$\begin{aligned}
&W = 1 - \frac{1}{m}\gamma_\mu p^\mu - \frac{1}{2m^2}(1 + i\gamma_4)p_\mu p^\mu,\\
&W = 1 + \frac{1}{m}\gamma_\mu p^\mu - \frac{1}{2m^2}(1 - i\gamma_4)p_\mu p^\mu.
\end{aligned} \tag{9}$$

As a result we obtain the equation

$$WL_{\frac{1}{2}}\Psi = 0, \tag{10}$$

which is equivalent to the starting eq. (4). Then using operator

$$V = \exp\left[\frac{1}{2m}(1 + i\gamma_4)\gamma_\mu p^\mu\right] \equiv 1 + \frac{1}{2m}(1 + i\gamma_4)\gamma_\mu p^\mu \tag{11}$$

we reduce eq. (10) to the diagonal form

$$L'\Phi \equiv V(WL_{\frac{1}{2}})V^{-1}\Phi = \left[\lambda^+ m + \frac{\lambda^-}{m}(p_\mu p^\mu - m^2)\right]\Phi = 0, \tag{12}$$

where $\Phi = V\Psi$, $\lambda^\pm = \frac{1}{2}(1 \pm i\gamma_4)$.

Equation (12) is equivalent to the starting eq. (4) and contains the only matrix γ_4. So it is evident that the matrices $Q'_{\mu\nu} = i\gamma_\mu\gamma_\nu$ commute with the operator $L_{\frac{1}{2}}$. These matrices satisfy the commutation relations of the Lie algebra of the $SU_2 \otimes SU_2$ group and satisfy the relations (8) together with the generators $P'_\mu = VP_\mu V^{-1} = P_\mu$ and $J'_{\mu\nu} = VJ_{\mu\nu}V^{-1} = J_{\mu\nu}$.

To complete the proof it is sufficient to find the explicit form of the matrices $Q'_{\mu\nu}$ in the starting Ψ-representation. Calculating $Q_{\mu\nu} = V^{-1}Q'_{\mu\nu}V$, one obtains the operators (6). The theorem is proved.

Corollary 1. If one makes in (4), (9)–(12) the substitution

$$\gamma_\mu p^\mu \to \gamma_\mu = (p_\mu - eA_\mu)\gamma_\mu, \quad p_\mu p^\mu \to \pi_\mu\pi^\mu - \frac{ie}{2}\gamma_\mu\gamma_\nu F_{\mu\nu},$$

where A_μ is the vector potential, and $F_{\mu\nu}$ is the tensor of the electromagnetic field, the transformations (9)–(12) establish the one-to-one correspondence between the solutions of the Dirac and of the Zaitsev–Gell–Mann equations [9].

Corollary 2. The above founded operators $Q_{\mu\nu}$ can be used to find the constants of motion for the particle interacting with external field. For instance the operator the $Q = \varepsilon_{abc} Q_{bc}(\boldsymbol{\pi})(H_a - iE_a)$ is a constant of motion for a particle moving in the homogeneous constant magnetic field $\boldsymbol{H}$ and the electric field $\boldsymbol{E}(Q_{bc}(\boldsymbol{\pi}))$ is obtained from (6) by the change $p_\mu \to \pi_\mu$.

Corollary 3. In theorem 1 the invariance condition of eq. (4) is formulated in terms of Lie algebras. A natural question arises: what kind of group transformations is generated by $Q_{\mu\nu}$? Using the explicit form of the generators (6), one obtains these transformations in the form

$$
\begin{gathered}
\Psi(x) \to \Psi'(x) = \exp[iQ_{ab}\theta_{ab}]\Psi(x) = (\cos\theta_{ab} - \gamma_a\gamma_b \sin\theta_{ab})\Psi(x) + \\
+ \frac{1}{m}(1 + i\gamma_4)\sin\theta_{ab}\left(\gamma_a \frac{\partial\Psi(x)}{\partial x_b} - \gamma_b \frac{\partial\Psi(x)}{\partial x_a}\right), \\
\Psi(x) \to \Psi'(x) = \exp[iQ_{0a}\theta_{ab}]\Psi(x) = (\cosh\theta_{0a} - i\sinh\theta_{0a}\gamma_0\gamma_a)\Psi(x) + \\
+ \frac{i}{m}(1 + i\gamma_4)\sinh\theta_{0a}\left(\gamma_0 \frac{\partial\Psi(x)}{\partial x_a} - \gamma_a \frac{\partial\Psi(x)}{\partial x_0}\right), \\
x_\mu \to x'_\mu = \exp[iQ_{ab}\theta_{ab}]x_\mu \exp[-iQ_{ab}\theta_{ab}] = x_\mu + \frac{1}{m}(1 + i\gamma_4)\sin\theta_{ab} \times \\
\times (\gamma_a g_{\mu b} - \gamma_b g_{\mu a})(\cos\theta_{ab} - \gamma_a\gamma_b \sin\theta_{ab}), \\
x_\mu \to x'_\mu = \exp[iQ_{0a}\theta_{0a}]x_\mu \exp[-iQ_{0a}\theta_{0a}] = x_\mu + \frac{i}{m}(1 + i\gamma_4)\sinh\theta_{0a} \times \\
\times (\gamma_0 g_{\mu a} - \gamma_a g_{\mu 0})(\cosh\theta_{0a} - i\gamma_0\gamma_a \sinh\theta_{0a}),
\end{gathered}
\tag{13}
$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ are transformation parameters (there is no sum over a, b). Transformations (13) together with the Lorentz transformations form the 16-parameter invariance group of the Dirac equation.

In quantum field theory not only the Dirac equation (4) but the system of two four-component equations for functions Ψ and $\bar{\Psi}$ is considered. Such system is equivalent to one eight-component Dirac equation

$$
(\Gamma_\mu p^\mu + m)\Psi(x_0, \boldsymbol{x}) = 0, \tag{14}
$$

where Γ_μ are (8×8)-dimensional matrices, which satisfy together with Γ_4, Γ_5, Γ_6 the Clifford algebra (for details see, e.g., [5]).

The system of eq. (14) has a more extended symmetry than the four-component Dirac equation. It is shown in [5] that the additional invariance algebra of eq. (4) is the Lie algebra of the group $O(6)$. This result admits the following strengthening:

Theorem 2. *Equation* (14) *is invariant under the* 40*-dimensional Lie algebra* A_{40}*. The basis elements of this algebra have the form*

$$
P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{2}\Gamma_\mu\Gamma_\nu,
$$

$$\tilde{Q}_{mn} = i\Gamma_m\Gamma_n + \frac{i}{m}(1+i\Gamma_6)(\Gamma_m\Gamma_n - \Gamma_n\Gamma_m), \quad m,n = 1,2,\ldots 5,$$
$$\tilde{\tilde{Q}}_{mn} = \left[\Gamma_6 + \frac{i}{m}(1+i\Gamma_6)\Gamma_\mu p^\mu\right]\tilde{Q}_{mn}, \tag{15}$$

where, by definition,

$$p_{a+3}\Psi(x_0,\boldsymbol{x}) = -i\frac{\partial\Psi(x_0,\boldsymbol{x})}{\partial x_{a+3}} \equiv 0.$$

Proof can be carried out in full analogy with the proof of theorem 1. We only draw attention to the fact, that Q_{mn} satisfies the Lie algebra of the group SU_4.

Let us now consider the group properties of the KDP equation, which describes particles with spin $s=1$. This equation has the form

$$L_1\Psi(x_0,\boldsymbol{x}) = 0, \quad L_1 = \beta_\mu p^\mu + m, \tag{16}$$

where β_μ are the ten-row KDP matrices.

It follows from the above that the KDP equation has to possess a more extended symmetry than eq. (4). This conclusion is supported by the following

Theorem 3. *The KDP equation is invariant with respect to the* 26*-dimensional Lie algebra* A_{26}*, whose basis elements are differential operators and have the form*

$$P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + i[\beta_\mu,\beta_\nu]_-,$$
$$\lambda_a = [c_{ab}, c_{ac}]_+, \quad \lambda_{a+3} = c_{bc}, \quad \lambda_7 = -i[c_{12}c_{23}c_{31} - c_{23}c_{31}c_{12}],$$
$$\lambda_8 = -\frac{i}{\sqrt{3}}(c_{12}c_{23}c_{31} + c_{23}c_{31}c_{12} - 2c_{31}c_{12}c_{23}), \quad \lambda_{8+a} = c_{ab}c_{0b}, \tag{17}$$
$$\lambda_{11+a} = ic_{0a}, \quad \lambda_{15} = (c_{12}c_{23}c_{02} - c_{23}c_{31}c_{03}),$$
$$\lambda_{16} = \frac{1}{\sqrt{3}}(c_{12}c_{23}c_{02} + c_{23}c_{31}c_{03} - 2c_{31}c_{12}c_{01}),$$

where

$$c_{\mu\nu} = i[\beta_\mu,\beta_\nu]_- + \frac{1}{m}(a_\mu p_\nu - a_\nu p_\mu), \quad (a,b,c) \text{ — cycl } (1,2,3),$$
$$a_\mu = i[\beta_5,\beta_\mu]_- + i\beta_\mu, \quad \beta_5 = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma. \tag{18}$$

Proof. First we shall show, that the operators λ_f satisfy the invariance condition (3). By direct verification one obtains

$$[c_{\mu\nu}, L_1]_- = F^1_{\mu\nu}L_1, \quad F^1_{\mu\nu} = (L_1 - 2m)\frac{i}{m^2}(\beta_\mu p_\nu - \beta_\nu p_\mu). \tag{19}$$

It follows from eq. (19) that the operators $c_{\mu\nu}$ (and hence all λ_f) satisfy eq. (3).

The operators (17b) satisfy the commutation relations of the Lie algebra of the $SU_3 \otimes SU_3$ group. This fact can be verified directly, but a more simple way is to make previously the transformation $\lambda \to V\lambda_f V^{-1} = \lambda'_f$, where

$$V = \exp\left[\frac{i}{m}a_\mu p^\mu\right], \quad c_{\mu\nu} \to c'_{\mu\nu} = Vc_{\mu\nu}V^{-1} = i[\beta_\mu,\beta_\nu]_-. \tag{20}$$

Using eq. (20) it is not difficult to make sure that the operators λ'_f and $p'_\mu = Vp_\mu V^{-1} = p_\mu$, $J'_{\mu\nu} = VJ_{\mu\nu}V^{-1} = J_{\mu\nu}$ form a Lie algebra. The theorem in proved.

In conclusion let us note that the main part of the theorems 1, 2, 3 (i.e. the invariance of eqs. (4), (14), (16) under the corresponding algebras) can be proved also by the transformation $L_s \to \tilde{V}L_s\tilde{V}^{-1}$, where $\tilde{V}$ is the integrodifferential operator

$$\tilde{V} = \exp\left[i\frac{S_{4a}p_a}{p}\operatorname{arctg}\frac{p}{m}\right]\exp\left[\frac{S_{ab}p_c}{p}\operatorname{arctgh}\frac{p}{E}\right]. \tag{21}$$

The preference of this transformation is that it can be easily generalized for the case of an arbitrary spin, but the basis elements $Q_{\mu\nu}$ of the new invariance algebra have to be integrodifferential operators (as like as (21)). Thus, for the Dirac equation one obtains

$$Q_{ab} = i\gamma_a\gamma_b + \frac{i}{m}(\gamma_a p_b - \gamma_b p_a)(1 + i\gamma_4\hat{\varepsilon}), \quad Q_{0a} = i\hat{\varepsilon}Q_{bc},$$

where $\hat{\varepsilon}$ is the integrodifferential operator of energy sign

$$\hat{\varepsilon} = \frac{H^D}{|H^D|} = (\gamma_0\gamma_a p_a + \gamma_0 m)(m^2 + p^2)^{-1/2}.$$

1. Fushchych W.I., *Theor. Math. Fiz.*, 1971, **7**, № 1, 3–12 (in Russian); Preprint ITF-70-32E, Kiev, 1970 (in English).
2. Fushchych W.I., *Lett. Nuovo Cimento*, 1974, **11**, № 10, 508–512.
3. Fushchych W.I., *DAN USSR*, 1976, **230**, № 3, 570–573 (in Russian).
4. Nikitin A.G., Segeda Yu.N., Fushchych W.I., *Theor. Math. Fiz.*, 1976, **29**, № 1, 82–94 (in Russian).
5. Fushchych W.I., *Lett. Nuovo Cimento*, 1973, **6**, № 4, 133–137.
6. Fushchych W.I., Segeda Yu.N., *Ukr. Math. J.*, 1976, **28**, № 6, 844–849 (in Russian).
7. Ovsiannikov L.V., The group properties of equations, Novosibirsk, 1962;
   Ibrahimov N.H., *DAN USSR*, 1969, **185**, 1226 (in Russian).
8. Niederer U., *Helv. Phys. Acta*, 1972, **45**, 802;
   Anderson R.L., Kumei S., Wulfam C.E., *Phys. Rev. Lett.*, 1972, **28**, 988;
   Boyer C.P., Kalnins E.G., Miller W., *J. Math. Phys.*, 1975, **16**, 499.
9. Zaitsev G.A., *JETP*, 1955, **28**, 524 (in Russian); *DAN USSR*, 1957, **113**, 1248 (in Russian);
   Feynman R.P., Gell-Mann M., *Phys. Rev.*, 1958, **109**, 193.

Пуанкаре-инвариантные дифференциальные уравнения для частиц произвольного спина

А.Г. НИКИТИН, В.И. ФУЩИЧ

The first and the second order differential equations have been deduced which describe the motion of relativistic particle with arbitrary spin. On the basis of these equations, the problem of the motion of the arbitrary spin particle in the homogeneous magnetic field has been solved exactly. The covariant position and spin operators have been obtained which are distinct from the Newton–Wigner and the Foldy–Wouthuysen operators. The approximate diagonalization of the Hamiltonian of the particle interacting with the external electromagnetic field has been carried out.

Выведены дифференциальные уравнения первого и второго порядка, описывающие движение релятивистской частицы с произвольным спином. На основе этих уравнений точно решена задача о движении частицы произвольного спина в однородном магнитном поле. Найдены ковариантные операторы координаты и спина частицы, отличные от известных операторов Ньютона–Вигнера и Фолди–Ваутхойзена. Осуществлена приближенная диагонализация гамильтониана частицы, взаимодействующей с внешним электромагнитным полем.

Введение

Во всех явно ковариантных релятивистских уравнениях первого порядка, описывающих движение частиц со спином $s > 1/2$, волновая функция имеет больше компонент, чем число возможных $2(2s+1)$ состояний свободной системы частица–античастица. Это "излишество" является, видимо, одной из причин появления в уравнениях Кеммера–Дэффина [1] ($s = 1$), Рариты–Швингера [2] ($s = 3/2$), описывающих поведение частиц во внешних электромагнитных полях, решений, соответствующих движению частиц с ненулевой массой со скоростью большей, чем скорость света в вакууме. К настоящему времени только уравнение Дирака, не имеющее лишних компонент, не приводит к указанным нефизическим следствиям.

Такое исключительное положение уравнения Дирака послужило стимулом для построения уравнений движения вида

$$i\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = H(\boldsymbol{p},s)\Psi(t,\boldsymbol{x}), \quad p_a = -i\frac{\partial}{\partial x_a} \tag{0.1}$$

для частицы с произвольным спином, где волновая функция Ψ имеет только $2(2s+1)$ компонент [3, 4]. Особенность уравнений (0.1) состоит в том, что гамильтониан $H(\boldsymbol{p},s)$ при $s > 1/2$ является интегро-дифференциальным оператором. Требование отсутствия лишних компонент у волновой функции и условие

Перепечатано из Теор. мат. физика, 1978, **34**, № 3, С. 319–333 по разрешению редколлегии

эрмитовости гамильтониана и других генераторов группы Пуанкаре относительно обычного скалярного произведения

$$(\Psi_1, \Psi_2) = \int d^3x \ \Psi_1^\dagger(t, \boldsymbol{x})\Psi_2(t, \boldsymbol{x}) \qquad (0.2)$$

приводят к нелокальным уравнениям движения (0.1) в конфигурационном пространстве. Это обстоятельство (нелокальность соответствующих гамильтонианов) сильно затрудняет применение уравнений вида (0.1) для описания поведения частиц со спином $s > 1/2$ во внешних электромагнитных полях. В [4] на основании уравнений (0.1) решена задача о взаимодействии частицы произвольного спина с внешним полем в предположении, что импульс частицы мал по сравнению с ее массой покоя, т.е. получено квазирелятивистское описание частицы во внешнем поле.

К аналогичным трудностям приводят уравнения, полученные Вивером, Хаммером, Гудом [6] и Мэтьюзом с соавторами [7]. Основное отличие этих уравнений от уравнений, полученных в [3, 4], состоит в том, что уравнения [6, 7] определены в пространстве со скалярным произведением

$$(\Psi_1, \Psi_2) = \int d^3x\Psi_1^\dagger(t, \boldsymbol{x})M\Psi_2(t, \boldsymbol{x}), \qquad (0.3)$$

где M — некоторый интегро-дифференциальный метрический оператор, зависящий от импульса и спиновых матриц.

Гуертин [8], развивая подход [3, 4], вывел уравнения вида (0.1), используя индефинитную метрику. Эти уравнения для $s > 1$ также являются интегро-дифференциальными.

Настоящая работа является продолжением статей [3, 4]. Исходя из требования, чтобы гамильтониан $H(\boldsymbol{p}, s)$ в (0.1) был дифференциальным оператором первого или второго порядка, найдены все возможные (с точностью до преобразований эквивалентности) пуанкаре-инвариантные уравнения для релятивистской частицы произвольного спина, допускающие, как и уравнение Дирака, стандартное введение взаимодействия с внешним полем. Волновая функция в дифференциальных уравнениях второго порядка имеет только $2(2s+1)$ компонент. Для нижайших целых спинов ($s = 0, 1$) эти уравнения совпадают с известными уравнениями Тамма–Сакаты–Такетани (ТСТ) [9]. При этом, как и в формализме ТСТ, гамильтониан $H(\boldsymbol{p}, s)$ не эрмитов относительно (0.2), но эрмитов в пространстве с индефинитной метрикой. Таким образом, индефинитность метрики — это цена, которую приходится платить за то, что гамильтониан $H(\boldsymbol{p}, s)$ в уравнении (0.1) является дифференциальным оператором, а волновая функция $\Psi(t, \boldsymbol{x})$ не имеет лишних компонент.

С использованием полученных уравнений точно решена задача о движении релятивистской частицы произвольного спина в однородном магнитном поле. Показано, что найденные уравнения не приводят к парадоксу нарушения причинности, свойственному, например, уравнению Рариты–Швингера [2].

1. Постановка задачи

Дифференциальные уравнения движения частицы произвольного спина получим, исходя из следующего представления генераторов P_μ, $J_{\mu\nu}$ группы $P(1,3)$ [5]:

$$\begin{aligned} &P_0 = H_s, \quad P_a = p_a = -i\frac{\partial}{\partial x_a}, \\ &J_{ab} = x_a p_b - x_b p_a + S_{ab}, \quad J_{0a} = x_0 p_a - \frac{1}{2}[x_a, H_s]_+ + \lambda_a, \quad x_0 = t, \end{aligned} \tag{1.1}$$

где $[A,B]_+ = AB + BA$, H_s — неизвестный пока дифференциальный оператор, включающий производные по $\partial/\partial x_a$ не выше второго порядка,

$$S_{ab} = S_c = \begin{pmatrix} \hat{S}_c & 0 \\ 0 & \hat{S}_c \end{pmatrix}, \quad (a,b,c) \text{ цикл } (1,2,3), \tag{1.2}$$

$\hat{S}_c$ — генераторы неприводимого представления $D(s)$ группы $O(3)$, λ_α — некоторые операторы, явный вид которых определяется требованием, чтобы генераторы (1.1) удовлетворяли алгебре Пуанкаре $P(1,3)$.

Формулы (1.1) задают самый общий вид генераторов группы Пуанкаре, соответствующих локальным преобразованиям $2(2s+1)$-компонентной волновой функции системы “частица + античастица” при повороте системы координат. Представления вида (1.1), где H_s при $s > 1$ принадлежит классу интегро-дифференциальных операторов, рассматривались ранее в [8].

Определение. *Будем говорить, что уравнение* (0.1) *пуанкаре-инвариантно и описывает свободное движение частицы с массой m и спином s, если операторы P_a, $J_{\mu\nu}$* (1.1) *и гамильтониан H_s удовлетворяют коммутационным соотношениям алгебры $P(1,3)$:*

$$[P_\mu, P_\nu]_- = 0, \quad [P_\mu, J_{\mu\nu}]_- = i(g_{\mu\nu}P_\lambda - g_{\mu\lambda}P_\nu), \tag{1.3а}$$

$$[J_{\mu\nu}, J_{\lambda\sigma}]_- = i(g_{\mu\sigma}J_{\nu\lambda} + g_{\nu\lambda}J_{\mu\sigma} - g_{\mu\lambda}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\lambda}), \tag{1.3б}$$

$$P_\mu P^\mu \equiv H_s^2 - p_a^2 = m^2, \tag{1.3в}$$

$$W_\mu W^\mu \Psi = m^2 s(s+1)\Psi, \tag{1.3г}$$

где $[A,B]_- = AB - BA$, $g_{\mu\nu}$ — метрический тензор, $g_{\nu\nu} = (-1,1,1,1)$, W_μ — вектор Любанского–Паули

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\sigma\lambda}J_{\nu\sigma}P_\lambda. \tag{1.4}$$

Из сказанного следует, что если найдем все такие операторы H_s и λ_a, для которых будут удовлетворяться соотношения (1.3), то тем самым будет решена задача о построении пуанкаре-инвариантных уравнений вида (0.1). Действительно, если удовлетворяются соотношения (1.3), то выполняются условия инвариантности уравнения (0.1) относительно алгебры Пуанкаре $P(1,3)$

$$\left[i\frac{\partial}{\partial t} - H_s, Q_A\right]_- \Psi = 0, \tag{1.5}$$

где Q_A — произвольный генератор группы $P(1,3)$.

2. Дифференциальные операторы $\boldsymbol{H}_s$ второго порядка

Решение нашей задачи приведем в виде следующей теоремы.

Теорема. *Все возможные (с точностью до преобразований эквивалентности, осуществляемых числовыми матрицами) дифференциальные операторы второго порядка H_s, удовлетворяющие алгебре $P(1,3)$ (1.3), задаются формулами*

$$H_s = \sigma_1 m + \sigma_3 k_1 \boldsymbol{S}\cdot\boldsymbol{p} + \frac{1}{2m}(\sigma_1 - i\sigma_2)\left[p^2 - (k_1\boldsymbol{S}\cdot\boldsymbol{p})^2\right], \tag{2.1}$$

$$H_1 = \sigma_1\left(m + \frac{p^2}{2m}\right) - \frac{i}{2m}\sigma_2[p^2 + 2k_2(\boldsymbol{S}\cdot\boldsymbol{p})^2] + \frac{1}{m}\sigma_3\sqrt{k_2(k_2-1)}\,(\boldsymbol{S}\cdot\boldsymbol{p})^2, \tag{2.2}$$
$$p^2 = p_1^2 + p_2^2 + p_3^2,$$

$$H_1 = \sigma_1\left[m + \frac{p^2}{2m} - \frac{(k_3\boldsymbol{S}\cdot\boldsymbol{p})^2}{2m}\right] + \sigma_3 k_3 \boldsymbol{S}\cdot\boldsymbol{p} - \frac{i}{2m}\sigma_2\left[p^2 + (k_3-2)(\boldsymbol{S}\cdot\boldsymbol{p})^2\right], \tag{2.3}$$

$$H_{3/2} = \sigma_1\left(m + \frac{p^2}{2m}\right) + \frac{ik_4}{2m}\sigma_2\left[(\boldsymbol{S}\cdot\boldsymbol{p})^2 - \frac{5}{4}p^2\right] + \frac{1}{2m}\sqrt{k_4^2-1}\,\sigma_3 p^2, \tag{2.4}$$

$$\begin{aligned} H_{3/2} = \sigma_1\left[m + \frac{p^2}{2m} - \frac{(k_5\boldsymbol{S}\cdot\boldsymbol{p})^2}{2m}\right] + \sigma_3 k_5 \boldsymbol{S}\cdot\boldsymbol{p} - \\ - \frac{i}{8m}\sigma_2\left[\left(5k_5^2 - 4\right)(\boldsymbol{S}\cdot\boldsymbol{p})^2 - \left(9k_5^2 - 5\right)p^2\right], \end{aligned} \tag{2.5}$$

где σ_a — $2(2s+1)$-рядные матрицы Паули, коммутирующие с S_a, k_l ($l = 1, 2, \ldots, 5$) — произвольные комплексные параметры.

Доказательство может быть проведено по схеме, подробно описанной в [3–5]. Ради краткости мы его опускаем. Приведем только явный вид операторов λ_a, при которых генераторы (1.1), (2.1)–(2.5) удовлетворяют соотношениям (1.3) (в чем можно убедиться непосредственной проверкой).

В случае, когда гамильтониан H_s имеет вид (2.1), получим

$$\lambda_a = \left(1 - \frac{k_1}{2}\right)\left[i\sigma_3 S_a - \frac{1}{2m}(\sigma_1 - i\sigma_2)(\boldsymbol{p}\times\boldsymbol{S})_a\right]. \tag{2.6}$$

В случае, когда H_s задается одной из формул (2.2)–(2.5), имеем

$$\begin{aligned} \lambda_a = \frac{i}{2EB_s}\left\{p_a\left(2 + \left[\frac{H_s}{E}, \sigma_1\right]_-\right) - 2\dot{x}_a H_s - E[\dot{x}_a, \sigma_1]_-\right\} + \\ + \frac{H_s}{E(E+m)}\left[S_{ab}p_b - \frac{i}{EB_s}S_{ab}p_b(\sigma_1 E + H_s)\right], \end{aligned} \tag{2.7}$$

где $B_s = 2E + [H_s, \sigma_1]_+$, $E = \left(p^2 + m^2\right)^{1/2}$, $p = \left(p_1^2 + p_2^2 + p_3^2\right)^{1/2}$, $\dot{A} = i[H_s, A]_-$.

Замечание 1. Из формул (2.1)–(2.5) видно, что соотношения (1.3) определяют гамильтонианы релятивистской частицы с точностью до постоянных комплексных чисел k_l ($l = 1, 2, \ldots, 5$). Уравнение (0.1) с такими гамильтонианами инвариантно относительно преобразования “сильного отражения” $\Theta = CPT$, но, вообще говоря, не инвариантно относительно P-, C- и T-преобразований. Инвариантность уравнения (0.1) относительно любого из этих преобразований может быть обеспечена специальным выбором чисел k_i. Например, если в формуле (2.1) для спина $s = 1/2$ положить $k_1 = 1/s$, в формулах (2.2)–(2.5) положить $k_2 = 1$, $k_3 = 0$, $k_4 = 1$, $k_5 = 0$, то получим P-, C-, T-инвариантные гамильтонианы вида

$$H_0 = \sigma_1\left(m + \frac{p^2}{2m}\right) - i\sigma_2\frac{p^2}{2m}, \tag{2.8}$$

$$H_{1/2} = \sigma_1 m + 2\sigma_3 \boldsymbol{S}\cdot\boldsymbol{p}, \tag{2.9}$$

$$H_1 = \sigma_1\left(m + \frac{p^2}{2m}\right) + i\sigma_2\left(\frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{m} - \frac{p^2}{2m}\right), \tag{2.10}$$

$$H_{3/2} = \sigma_1\left(m + \frac{p^2}{2m}\right) + i\sigma_2\left[\frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{2m} - \frac{5p^2}{8m}\right]. \tag{2.11}$$

Оператор (2.9) совпадает с гамильтонианом Дирака, а операторы (2.8), (2.10) — с гамильтонианами ТСТ [9] для частиц со спином $s = 0, 1$. Оператор (2.1) для спина $s = 1/2$ рассматривался ранее в [10].

Замечание 2. Все генераторы группы $P(1,3)$, определяемые формулами (1.1), (2.1), (2.6), принадлежат классу дифференциальных операторов. При $k_1 = 2$ генераторы J_{0a} (1.1), (2.6) принимают особо простой вид [3, 4]

$$J_{0a} = x_0 p_a - \frac{1}{2}[x_a, H_s]_+. \tag{2.12}$$

Замечание 3. Гамильтонианы (2.1)–(2.5) и остальные генераторы (1.1), (1.2), (2.6), (2.7) группы $P(1,3)$ могут быть приведены к канонической форме Фолди–Широкова [11, 12] и [3, 4]. Это достигается посредством изометрического преобразования

$$\begin{aligned}
&P_0 \to P_0^k = VP_0V^{-1} = \sigma_1 E, \quad P_a \to P_a^k = VP_aV^{-1} = p_a,\\
&J_{ab} \to J_{ab}^k = VJ_{ab}V^{-1} = x_a p_b - x_b p_a + S_{ab},\\
&J_{0a} \to J_{0a}^k = VJ_{0a}V^{-1} = x_0 p_a - \frac{1}{2}[x_a, P_0^k]_+ - \sigma_1\frac{S_{ab}p_b}{E+m},\\
&E = (m^2 + p^2)^{1/2},
\end{aligned} \tag{2.13}$$

где операторы V имеют вид

$$V = V_1 V_2 V_3, \quad V_1 = \exp\left(\sigma_1 \frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\,\mathrm{arth}\,\frac{p}{E}\right),$$

$$V_2 = \frac{1}{2m}[E\lambda^+ + m\lambda^- - 2\sigma_1\lambda^-\boldsymbol{S}\cdot\boldsymbol{p}],$$

$$V_3 = \exp\left[\frac{1}{2m}\sigma_1\lambda^+(k_1 - 2)\boldsymbol{S}\cdot\boldsymbol{p}\right], \quad \lambda^\pm = \frac{1}{2}(1 \pm \sigma_3),$$

для гамильтонианов (2.1) и

$$V = (E + \sigma_1 H_s)\left(2E^2 + E(H_s, \sigma_1]_+\right)^{1/2}$$

для гамильтонианов (2.2)–(2.5).

3. Дифференциальные гамильтоновы уравнения первого порядка

По аналогии с теорией Дирака для электрона постулируем, что в уравнении (0.1) гамильтониан $\hat{H}_s$ релятивистской частицы с произвольным спином является дифференциальным оператором, включающим в себя производные по пространственным переменным не выше первого порядка

$$\hat{H}_s = \hat{\Gamma}_a^{(s)} p_a + \hat{\Gamma}_0^{(s)} m, \tag{3.1}$$

где $\hat{\Gamma}_\mu^{(s)}$ — некоторые числовые матрицы.

Генераторы представления группы Пуанкаре, которое реализуется на решениях уравнения (0.1) с гамильтонианом (2.1), выберем в виде

$$P_0 = \hat{H}_s, \quad P_a = p_a = -i\frac{\partial}{\partial x_a}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{3.2}$$

где $S_{\mu\nu}$ — матрицы, образующие конечномерное представление (в общем случае приводимое) алгебры $O(1,3)$. Формулы (3.2) задают самый общий вид генераторов группы $P(1,3)$, соответствующий локальным преобразованиям волновой функции.

Определить все возможные гамильтонианы вида (3.1) означает найти все такие матрицы $\hat{\Gamma}_\mu^{(s)}$ и $S_{\mu\nu}$, что операторы (3.1), (3.2) удовлетворяют алгебре Пуанкаре (1.3).

Покажем, что искомые уравнения движения частицы со спином s и массой m имеют вид

$$\hat{H}_s\Psi = i\frac{\partial}{\partial t}\Psi, \quad \hat{H}_s = \Gamma_0^{(s)}\Gamma_a^{(s)} p_a + \Gamma_0^{(s)} m, \tag{3.3а}$$

$$\hat{P}_s\Psi = 0, \quad \hat{P}_s = P_s + \frac{1}{2m}\left(1 - \Gamma_4^{(s)}\right)\left[\Gamma_\mu^{(s)} p^\mu, P_s\right]_-, \tag{3.3б}$$

$$P_s = \frac{1}{4s}\left[S_{ab}^2 - 2s(s-1)\right], \quad S_{ab}^2 = \sum_{a,b} S_{ab}S_{ab}, \tag{3.3в}$$

где $\Gamma_\mu^{(s)}$, S_{ab} — $8s$-рядные матрицы, задаваемые соотношениями

$$\begin{gathered}
[\Gamma_\mu^{(s)}, \Gamma_\nu^{(s)}]_+ = 2g_{\mu\nu}, \quad \Gamma_4^{(s)} = i\Gamma_0^{(s)}\Gamma_1^{(s)}\Gamma_2^{(s)}\Gamma_3^{(s)}, \quad S_{\mu\nu} = \tau_{\mu\nu} + j_{\mu\nu}, \\
[\tau_{\mu\nu}, j_{\lambda\sigma}]_- = 0, \quad \tau_{\mu\nu} = \frac{i}{2}\Gamma_\mu^{(s)}\Gamma_\nu^{(s)}, \quad j_{ab} = j_c, \quad j_{0a} = ij_a, \\
[j_a, j_b]_- = ij_c, \quad \sum_a j_a^2 = j(j+1) = s(s-1),
\end{gathered} \tag{3.4}$$

т.е. матрицы $\Gamma_\mu^{(s)}$, как и в случае уравнения Дирака, удовлетворяют алгебре Клиффорда, а матрицы $S_{\mu\nu}$ являются генераторами представления $\left[D\left(\frac{1}{2},0\right) \oplus D\left(0,\frac{1}{2}\right)\right] \oplus D\left(s-\frac{1}{2},0\right)$ группы $O(1,3)$. Действительно, используя (3.4), нетрудно убедиться, что гамильтониан (3.3а) и генераторы (3.2) удовлетворяют условиям

(1.3а), (1.3б). Что же касается условия (1.3г), то согласно (1.4), (3.2)–(3.4) его можно записать в виде (3.3б)

$$\frac{1}{2s}\left[\frac{1}{m^2}W_\mu W^\mu - s(s-1)\right]\Psi \equiv \hat{P}_s\Psi = \Psi,$$

где $\hat{P}_s$ — оператор проектирования на подпространство, соответствующее фиксированному спину s [5].

Используя тождество

$$\left(1+\Gamma_4^{(s)}\right)P_s = \frac{1}{8s}\left[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)\right]\left(1+\Gamma_4^{(s)}\right),$$

уравнения (2.3) можно записать в явно ковариантной форме

$$\left(\Gamma_\mu^{(s)}p^\mu - m\right)\Psi = 0, \tag{3.5а}$$

$$\left(\Gamma_\mu^{(s)}p^\mu + m\right)\left(1+\Gamma_4^{(s)}\right)\left[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)\right]\Psi = 16ms\Psi. \tag{3.5б}$$

В силу изложенного выше уравнения (3.5) пуанкаре-инвариантны и описывают свободное движение частицы с фиксированным спином s и массой m.

Замечание 1. Уравнения (2.5) определены и для случая $m = 0$. Налагая при этом на волновую функцию Ψ пуанкаре-инвариантное дополнительное условие $\left(1-\Gamma_4^{(s)}\right)\Psi = 0$, получаем из (3.5) уравнения движения для безмассовых частиц произвольного спина, которые при $s = 1/2$ эквивалентны уравнению Вейля для нейтрино, а при $s = 1$ — уравнениям Максвелла для электромагнитного поля в вакууме [13].

Замечание 2. Посредством преобразования $\Psi \to \Phi = W\Psi$, где

$$W = \exp\left(\frac{\Gamma_a^{(s)}p_a}{p}\operatorname{arctg}\frac{p}{m}\right)\exp\left(\Gamma_0^{(s)}\frac{j_a p_a}{p}\operatorname{arth}\frac{p}{E}\right),$$

уравнения (2.3), (3.5) могут быть приведены к диагональной форме

$$i\frac{\partial}{\partial t}\Phi = \Gamma_0^{(s)}E\Phi, \quad P_s\Phi = \Phi.$$

На решениях уравнений (3.6) генераторы группы $P(1,3)$ имеют каноническую форму (2.1).

Отметим, что в [14] также предлагались $8s$-компонентные дифференциальные уравнения первого порядка, описывающие движение свободной частицы с произвольным спином s. Эти уравнения, в отличие от (5.1), (5.2), становятся несовместными при учете взаимодействия частицы с внешним полем.

4. Ковариантные операторы координаты и спина

При переходе к новой инерциальной системе отсчета операторы физических величин N_i (координаты, спина и т.д.) преобразуются следующим образом:

$$N_i \to N_i' = \exp(iQ_l\theta_l)N_i\exp(-iQ_l\theta_l),$$

где Q_l $(l = 1, 2, \ldots, 10)$ — генераторы группы Пуанкаре, θ_l — параметры преобразования.

Одна из трудностей, с которой приходится сталкиваться в представлениях типа (1.1) (когда генераторы J_{0a} нельзя записать в виде суммы коммутирующих "спиновой" и "орбитальной" частей), состоит в том, что оператор x_μ имеет нековариантный закон преобразования, при котором не сохраняется величина интервала, $x_0^2 - x_a^2 \neq (x_0')^2 - (x_a')^2$. Следовательно, x_μ нельзя интерпретировать как ковариантный оператор координаты.

Ниже определим ковариантный оператор координаты в представлении (1.1), (2.1). Тем самым в принципе будет решена задача для произвольного представления (1.1), (2.6), поскольку генераторы J_{0a} (2.12) и (1.1), (2.6) связаны преобразованием эквивалентности $J_{0a} \to VJ_{0a}V^{-1}$, где

$$V = \exp\left[(\sigma_1 - i\sigma_2)(2 - k_1)\frac{1}{2m}\boldsymbol{S}\cdot\boldsymbol{p}\right].$$

Перейдем к представлению, в котором генераторы J_{0a} (2.12) имеют локально-ковариантную форму

$$\hat{J}_{0a} = x_0 p_a - x_a p_0 + S_{0a}, \quad S_{0a} = i\sigma_3 S_a, \quad p_0 = i\frac{\partial}{\partial x_0}. \tag{4.1}$$

Это достигается посредством преобразования

$$\hat{J}_{0a} = VJ_{0a}V^{-1}, \quad V = \exp\left[-\frac{i}{2m}(\sigma_2 + i\sigma_2)(2\boldsymbol{S}\cdot\boldsymbol{p} - p_0)\right]. \tag{4.2}$$

В представлении (4.1) ковариантный оператор координаты $\hat{X}_\mu$ можно выбрать в виде $\hat{X}_\mu = x_\mu$. С помощью преобразования, обратного (3.2), получаем явный вид этих операторов в исходном представлении (2.12)

$$\hat{X}_\mu = \hat{V}^{-1}X_\mu\hat{V} = x_\mu + \frac{1}{m}(i\sigma_1 + \sigma_2)\xi_\mu, \quad \xi_a = S_a, \quad \xi_0 = \frac{1}{2}\sigma_3. \tag{4.3}$$

При переходе к новой инерциальной системе координат операторы X_μ преобразуются как компоненты четырехвектора и удовлетворяют каноническим перестановочным соотношениям

$$[p_\mu, X_\nu]_- = ig_{\mu\nu}, \quad [X_\mu, X_\nu]_- = 0. \tag{4.4}$$

Все это позволяет сделать вывод, что X_μ (4.3) можно интерпретировать как ковариантный оператор координаты частицы.

В случае $s = 1/2$ операторы (4.3) принимают явно ковариантную форму

$$X_\mu = x_\mu + \frac{i}{2m}(1 + \gamma_4)\gamma_\mu, \tag{4.5}$$

где $\gamma_4 = \sigma_3$, $\gamma_0 = \sigma_1$, $\gamma_a = -2i\sigma_2 S_a$ — матрицы Дирака. В силу изложенного выше оператор (4.5) может быть выбран в качестве ковариантного оператора координаты дираковской частицы. Интересно отметить, что при таком определении координаты оператор скорости

$$\dot{X}_a = -i[H_{1/2}, X_a]_- = (1 + \gamma_4)\gamma_0\frac{p_a}{m}$$

(где $H_{1/2}$ — гамильтониан Дирака (2.9)) имеет сплошной спектр и удовлетворяет соотношению $[\dot{X}_a, \dot{X}_b] = 0$. При этом, однако, $[H_{1/2}, \dot{X}_a]_- \neq 0$.

Подчеркнем, что оператор (4.5) существенно отличается от операторов координаты, предложенных ранее Ньютоном и Вигнером [15], Фолди и Ваутхойзеном [16] и многими другими [17]. Это отличие состоит в том, что оператор (4.5) локален и преобразуется как ковариантный четырехвектор, в то время как операторы координаты, предложенные в [15–17], принадлежат классу нелокальных интегро-дифференциальных операторов с нековариантным законом преобразования.

Приведем явный вид ковариантного оператора спина $\Sigma_{\mu\nu}$ частицы, описываемой уравнением (0.1) с гамильтонианом (2.1):

$$\Sigma_{ab} = S_{ab} + \frac{1}{2m}(i\sigma_1 + \sigma_2)S_{cd}p_d, \quad (a,b,c) = (1,2,3),$$
$$\Sigma_{0a} = i\sigma_3 S_{bc} - \frac{1}{m}(i\sigma_1 + \sigma_2)[2\boldsymbol{S}\cdot\boldsymbol{p} - p_0, S_{bc}]_+.$$

По аналогии с (4.1)–(4.3) можно показать, что операторы $\Sigma_{\mu\nu}$ преобразуются как ковариантный тензор второго ранга, а оператор Σ_{ab} коммутирует с гамильтонианом и является интегралом движения.

Отметим еще, что оператор координаты частицы, описываемой уравнениями (3.5), может быть получен из (4.5) с помощью замены $\gamma_k \to \Gamma_k^{(s)}$.

5. Уравнение для заряженной частицы во внешнем электромагнитном поле

Можно показать, что введение минимального электромагнитного взаимодействия непосредственно в уравнения (3.3) или (3.5) приводит к тому, что как уравнения (3.3), так и уравнения (3.5) становятся несовместными. Чтобы преодолеть эту трудность, запишем (3.3) в виде одного уравнения

$$\left[\hat{P}_s\left(i\frac{\partial}{\partial t} - \hat{H}_s\right) + \varkappa\left(1 - \hat{P}_s\right)\right]\Psi = 0, \tag{5.1}$$

где $\varkappa$ — произвольный параметр. Эквивалентность (5.1) и (3.3) следует из соотношений

$$\left[i\frac{\partial}{\partial t} - \hat{H}_s, \hat{P}_s\right]_- = 0, \quad \hat{P}_s\hat{P}_s = \hat{P}_s.$$

Явно ковариантная система (3.5), в свою очередь, может быть записана в виде

$$\begin{gathered}\left[B_s\left(\Gamma_\mu^{(s)}p^\mu - m\right) - \varkappa(1 - B_s)\right]\Psi = 0,\\ B_s = \frac{1}{16ms}\left(\Gamma_\mu^{(s)}p^\mu + m\right)\left(1 + \Gamma_4^{(s)}\right)[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)],\end{gathered} \tag{5.2}$$

поскольку

$$\left[B_s, \Gamma_\mu^{(s)}p^\mu - m\right]_- \Psi = 0, \quad B_sB_s = B_s.$$

Сделаем в (5.1), (5.2) замену $p_\mu \to \pi_\mu = p_\mu - eA_\mu$, где A_μ — вектор-потенциал электромагнитного поля, и покажем, что в результате (5.1) и (5.2) сводятся к системе явно ковариантных дифференциальных уравнений первого порядка, описывающих причинное движение заряженной частицы произвольного спина во внешнем поле. Поскольку уравнения (5.1) и (5.2) в конечном итоге приводят к одинаковым результатам, рассмотрим только уравнение (5.1), которое принимает вид

$$\left\{\hat{P}_s(\boldsymbol{\pi})[\pi_0 - \hat{H}_s(\boldsymbol{\pi})] + \varkappa[1 - \hat{P}_s(\boldsymbol{\pi})]\right\}\Psi = 0, \tag{5.3}$$

$$\hat{H}_s(\boldsymbol{\pi}) = \Gamma_0^{(s)}\Gamma_a^{(s)}\pi_a + \Gamma_0^{(s)}m, \quad \hat{P}_s(\boldsymbol{\pi}) = P_s + \frac{1}{2m}\left(1 - \Gamma_4^{(s)}\right)\left[\Gamma_\mu^{(s)}\pi^\mu, P_s\right]_-. \tag{5.4}$$

Умножая (5.3) на $\hat{P}_s(\boldsymbol{\pi})$ и $\left[1 - \hat{P}_s(\boldsymbol{\pi})\right]$ и используя тождества

$$\left[\pi_0 - \hat{H}_s(\boldsymbol{\pi}), \hat{P}_s(\boldsymbol{\pi})\right]_-\hat{P}_s(\boldsymbol{\pi}) \equiv \frac{1}{4m}\Gamma_0^{(s)}\Big(1 - \Gamma_4^{(s)}\Big)\left(\frac{1}{s}S_{\mu\nu} - i\Gamma_\mu^{(s)}\Gamma_\nu^{(s)}\right)F_{\mu\nu}\hat{P}_s(\boldsymbol{\pi}),$$

$$\hat{P}_s(\boldsymbol{\pi})\hat{P}_s(\boldsymbol{\pi}) = \hat{P}_s(\boldsymbol{\pi}), \quad F_{\mu\nu} = -[\pi_\mu, \pi_\nu]_-,$$

приходим к системе уравнений

$$\begin{aligned} & i\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = \hat{H}_s(\boldsymbol{\pi}, A_0)\Psi(t,\boldsymbol{x}), \\ & \hat{H}_s(\boldsymbol{\pi}, A_0) = \Gamma_0^{(s)}\Gamma_a^{(s)}\pi_a + \Gamma_0^{(s)}m + eA_0 + \\ & \qquad + \frac{1}{4m}\Gamma_0^{(s)}\left(1 - \Gamma_4^{(s)}\right)\left[\frac{1}{s}S_{\mu\nu} - i\Gamma_\mu^{(s)}\Gamma_\mu^{(s)}\right]F_{\mu\nu}, \end{aligned} \tag{5.5}$$

$$\left\{P_s + \frac{1}{2m}\left(1 - \Gamma_4^{(s)}\right)\left[\Gamma_\mu^{(s)}\pi^\mu, P_s\right]_-\right\}\Psi = 0, \tag{5.6}$$

которая, как и (3.3), может быть записана в явно ковариантной форме

$$\left[\left(\Gamma_\mu^{(s)}\pi^\mu - m\right) + \frac{1}{4m}\left(1 - \Gamma_4^{(s)}\right)\left(\frac{1}{s}S_{\mu\nu} - i\Gamma_\mu^{(s)}\Gamma_\nu^{(s)}\right)F_{\mu\nu}\right]\Psi = 0, \tag{5.7}$$

$$\left(m + \Gamma_\mu^{(s)}\pi^\mu\right)\left(1 - \Gamma_4^{(s)}\right)\left[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)\right]\Psi = 16ms\Psi. \tag{5.8}$$

Покажем, что уравнения (5.7), (5.8) не приводят к нарушению причинности. Для этого сделаем замену

$$\Psi(t,\boldsymbol{x}) = V\Psi(t,\boldsymbol{x}), \quad V = \exp\left[\left(1 - \Gamma_4^{(s)}\right)\frac{1}{2m}\Gamma_\mu^{(s)}\pi^\mu\right]. \tag{5.9}$$

Подставив (5.9) в (5.7) и умножив результат слева на оператор

$$F = m + \frac{1}{2}\left(\Gamma_\mu^{(s)}\pi^\mu - \frac{1}{sm}\tilde{S}_{\mu\nu}F_{\mu\nu} - \frac{1}{m^2}\pi_\mu\pi^\mu\right)\left(1 - \Gamma_4^{(s)}\right),$$

где $\tilde{S}_{ab} = S_{ab}$, $\tilde{S}_{0a} = iS_{bc}$, прийдем к уравнению

$$\left(\pi_\mu \pi^\mu - m^2 - \frac{1}{2s}\tilde{S}_{\mu\nu}F_{\mu\nu}\right)\Phi(t, \boldsymbol{x}) = 0. \tag{5.10}$$

Из (5.8), (5.9) получаем дополнительное условие для Φ в виде

$$P_s\Phi = \Phi \quad \text{или} \quad \frac{1}{2}S_{ab}^2\Phi = s(s+1)\Phi. \tag{5.11}$$

Формулы (5.10), (5.11) обобщают уравнение Зайцева–Фейнмана–Гелл-Манна [18] для $s = 1/2$ на случай частицы произвольного спина. Решения $\Phi(t, \boldsymbol{x})$ этого уравнения, как известно [19], описывают причинное распространение волн (с досветовой скоростью). Таковы же, очевидно, и свойства решений $\Psi(t, \boldsymbol{x})$ уравнений (5.7), (5.8), связанных с $\Phi(t, \boldsymbol{x})$ преобразованием эквивалентности (5.9).

Таким образом, мы показали, что уравнения (5.7), (5.8) описывают движение заряженной релятивистской частицы с произвольным спином во внешнем электромагнитном поле и не приводят к нарушению принципа причинности. Отметим еще, что уравнения (5.7), (5.8) допускают лагранжеву формулировку. Действительно, выберем плотность лагранжиана $L(x)$ в виде

$$\begin{aligned} L(x) = \left(m\bar{\Psi}' + i\frac{\partial\bar{\Psi}'}{\partial x_\mu}\check{\Gamma}_\mu^{(s)}\right)\left(1 + \check{\Gamma}_4^{(s)}\right)[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)]i\check{\Gamma}_\lambda^{(s)}\frac{\partial\Psi'}{\partial x_\lambda} + \\ + i\frac{\partial\bar{\Psi}'}{\partial x_\lambda}\check{\Gamma}_\lambda^{(s)}[S_{\mu\nu}S^{\mu\nu} - 4s(s-1)]\left(m\Psi' + i\check{\Gamma}_\mu^{(s)}\frac{\partial\Psi'}{\partial x_\mu}\right) + 16ms\bar{\Psi}'\Psi', \end{aligned} \tag{5.12}$$

где

$$\Psi' = \begin{pmatrix} \Psi \\ \chi \end{pmatrix}, \quad \bar{\Psi}' = \Psi'^\dagger\check{\Gamma}_0^{(s)}\check{\Gamma}_5^{(s)},$$

Ψ и χ — $8s$-компонентные функции, а $\check{\Gamma}_\mu^{(s)}$, $\check{S}_{\mu\nu}$ — матрицы размерности $16s\times16s$:

$$\check{\Gamma}_k^{(s)} = \begin{pmatrix} \Gamma_k^{(s)} & 0 \\ 0 & \Gamma_k^{(s)} \end{pmatrix}, \quad \check{\Gamma}_0^{(s)} = \begin{pmatrix} \Gamma_0^{(s)} & 0 \\ 0 & -\Gamma_0^{(s)} \end{pmatrix},$$

$$\check{\Gamma}_5^{(s)} = \begin{pmatrix} 0 & \Gamma_0^{(s)} \\ \Gamma_0^{(s)} & 0 \end{pmatrix}, \quad \check{S}_{\mu\nu} = \begin{pmatrix} S_{\mu\nu} & 0 \\ 0 & S_{\mu\nu} \end{pmatrix}.$$

Используя принцип минимального действия, получаем из (5.12) уравнения (3.5) для функции Ψ и уравнения, комплексно-сопряженные (3.5) для функции χ. Сделав в (5.12) минимальную замену $\dfrac{\partial}{\partial x_\mu} \to \dfrac{\partial}{\partial x_\mu} + ieA_\mu$, прийдем к уравнениям (5.7), (5.8).

6. Разложение по степеням $1/m$

Гамильтониан (5.5) может иметь как положительные, так и отрицательные собственные значения. С помощью серии последовательных преобразований получим из (5.5) уравнение для состояний с положительной энергией подобно тому, как это было сделано Фолди и Ваутхойзеном [16] для уравнения Дирака. При этом оператор $\hat{H}_s(\pi, A_0)$ будет представлен в виде ряда по степеням $1/m$, удобном для вычислений по теории возмущений.

Основная трудность при диагонализации уравнений (5.5), (5.6) состоит в том, что необходимо найти преобразования, одновременно приводящие к диагональной форме два различных уравнения. Сначала диагонализируем дополнительное условие (5.6), а затем, используя операторы, коммутирующие с преобразованным уравнением (5.6), приводим к диагональной форме уравнение (5.7).

Подвергнем волновую функцию $\Psi(t, \boldsymbol{x})$ преобразованию $\Psi \to \widetilde{\Psi} = V\Psi$, где

$$V = \exp\left[\frac{1}{2m}\left(1 - \Gamma_4^{(s)}\right)\left(\Gamma_a^{(s)}\pi_a - k_1\Gamma_0^{(s)}S_a\pi_a\right)\right]. \tag{6.1}$$

Подействовав оператором (6.1) слева на (5.5), (5.6), получим уравнение для $\widetilde{\Psi}$

$$\begin{gathered} H_s(\boldsymbol{\pi}, A_0)\widetilde{\Psi} = i\frac{\partial}{\partial t}\widetilde{\Psi}, \quad H_s(\boldsymbol{\pi}, A_0) = \Gamma_0^{(s)}m + k_1\Gamma_4^{(s)}\boldsymbol{S}\cdot\boldsymbol{\pi} + \\ + \frac{1}{2m}\Gamma_0^{(s)}\left(1 - \Gamma_4^{(s)}\right)\left[\pi^2 - (k_1\boldsymbol{S}\cdot\boldsymbol{p})^2 + \frac{1}{s}\boldsymbol{S}(\boldsymbol{H} - i\boldsymbol{E} + ik_1\boldsymbol{E})\right], \end{gathered} \tag{6.2}$$

$$P_s\widetilde{\Psi} = \widetilde{\Psi} \quad \text{или} \quad \frac{1}{2}S_{ab}\widetilde{\Psi} = s(s+1)\widetilde{\Psi}, \tag{6.3}$$

где $H_a = -i[\pi_b, \pi_c]_-$ и $E_a = -i[\pi_0, \pi_a]_-$ — напряженности магнитного и электрического полей, P_s — проектор (3.3в).

Из (6.3), (3.4) заключаем, что волновая функция $\widetilde{\Psi}$ имеет $2(2s+1)$ отличных от нуля компонент. Матрицы S_{ab} и коммутирующие с ними матрицы $\Gamma_0^{(s)}$, $\Gamma_4^{(s)}$ на множестве таких функций можно представить в виде

$$S_{ab} \sim S_c = \begin{pmatrix} s_c & 0 \\ 0 & s_c \end{pmatrix}, \ \Gamma_0^{(s)} \sim \sigma_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \Gamma_4^{(s)} \sim \sigma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{6.4}$$

где s_c — генераторы представления $D(s)$ группы $O(3)$, I и 0 — $(2s+1)$-рядные единичная и нулевая матрицы. Подставив (6.4) в (6.2), получим гамильтониан $H_s(\boldsymbol{\pi}, A_0)$ в форме

$$\begin{gathered} H_s(\boldsymbol{\pi}, A_0) = \sigma_1 m + k_1\sigma_3\boldsymbol{S}\cdot\boldsymbol{\pi} + \frac{1}{2m}(\sigma_1 - i\sigma_2)\times \\ \times\left\{\pi^2 - (k_1\boldsymbol{S}\cdot\boldsymbol{\pi})^2 + \frac{e}{s}\boldsymbol{S}\cdot[\boldsymbol{H} - i(1 - k_1 s)\boldsymbol{E}]\right\} + eA_0. \end{gathered} \tag{6.5}$$

Формула (6.5) обобщает гамильтониан свободной частицы произвольного спина (2.1) на случай взаимодействия с внешним электромагнитным полем. Таким образом, исходя из явно ковариантных уравнений (5.7), (5.8), мы получили рецепт введения взаимодействия в пуанкаре-инвариантные уравнения без лишних компонент, найденные в разделе 1.

Преобразуем (6.5) к диагональной форме. Как и в случае уравнения Дирака [16], это можно осуществить только приближенно для $\pi_\mu \ll m$. Совершая серию последовательных преобразований

$$\begin{aligned}
&H_s(\boldsymbol{\pi}, A_0) \to V_3V_2V_1H_s(\boldsymbol{\pi}, A_0)V_1^{-1}V_2^{-1}V_3^{-1} = H_s'(\boldsymbol{\pi}, A_0),\\
&V_1 = \exp\left(-i\sigma_2\frac{k_1\boldsymbol{S}\cdot\boldsymbol{\pi}}{m}\right),\\
&V_2 = \exp\left\{\frac{1}{4m^2}\sigma_3\left[\pi^2 - (k_1\boldsymbol{S}\cdot\boldsymbol{\pi})^2 - \frac{e}{s}\boldsymbol{S}\cdot\boldsymbol{H} + ie\left(\frac{1}{s} - k_1\right)\boldsymbol{S}\cdot\boldsymbol{E}\right]\right\},\\
&V_3 = \exp\left\{-\frac{i}{m^3}\left[\frac{1}{12}(k_1\boldsymbol{S}\cdot\boldsymbol{\pi})^3 + \right.\right.\\
&\qquad\left.\left. + \frac{1}{8}\left[\pi^2 - (k_1\boldsymbol{S}\cdot\boldsymbol{\pi})^2 - \frac{e}{s}\boldsymbol{S}\cdot\boldsymbol{H} + \frac{ie}{s}(1 - sk_1)\boldsymbol{S}\cdot\boldsymbol{E}, \pi_0\right]_-\right]\right\}
\end{aligned} \tag{6.6}$$

и пренебрегая членами порядка $1/m^3$, получаем

$$\begin{aligned}
H_s'(\boldsymbol{\pi}, A_0) = \sigma_1&\left(m + \frac{\pi^2}{2m} - \frac{e\boldsymbol{S}\cdot\boldsymbol{H}}{2sm}\right) + eA_0 - \frac{e}{16m^2s^2}(\boldsymbol{S}\cdot\boldsymbol{E}\times\boldsymbol{\pi} - \boldsymbol{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E}) -\\
&- \frac{e}{24m^2s^2}\left[\frac{1}{2}Q_{ab}\frac{\partial E_a}{\partial x_b} + s(s+1)\,\mathrm{div}\,\boldsymbol{E}\right] +\\
&+ \frac{ie(2s-1)}{8m^2s^2}(\boldsymbol{S}\cdot\boldsymbol{\pi}\times\boldsymbol{H} - \boldsymbol{S}\cdot\boldsymbol{H}\times\boldsymbol{\pi}) + \frac{e}{24m^2s^2}Q_{ab}\frac{\partial H_a}{\partial x_b},
\end{aligned} \tag{6.7}$$

$$Q_{ab} = 3[S_a, S_b]_+ - 2\delta_{ab}s(s+1).$$

На множестве функций, удовлетворяющих дополнительному условию $\sigma_1\Phi = \Phi$, гамильтониан (6.7) положительно определен и содержит слагаемые соответствующие дипольному $\left(-\frac{e}{2sm}\boldsymbol{S}\cdot\boldsymbol{H}\right)$, квадрупольному $\left(-\frac{e}{48s^2m^2}Q_{ab}\frac{\partial E_a}{\partial x_b}\right)$, спин-орбитальному $\left(-\frac{e}{16m^2s^2}(\boldsymbol{S}\cdot\boldsymbol{E}\times\boldsymbol{\pi} - \boldsymbol{S}\cdot\boldsymbol{\pi}\times\boldsymbol{E})\right)$ и дарвиновскому $\left(-\frac{e(s+1)}{24sm^2}\,\mathrm{div}\,\boldsymbol{E}\right)$ взаимодействиям частицы с полем. Два последних члена в (6.7) можно интерпретировать как магнитное спин-орбитальное и магнитное квадрупольное взаимодействия.

Приближенный гамильтониан (6.6) совпадает с полученным в работе [20], в которой в качестве исходного использовалось уравнение Зайцева–Фейнмана–Гелл-Манна (5.10). В случае $s = 1/2$ (6.7) совпадает с гамильтонианом Фолди–Ваутхойзена [16], полученным из уравнения Дирака.

7. Точное решение уравнений движения частиц произвольного спина в однородном магнитном поле

Рассмотрим систему уравнений (5.5), (5.6) для случая частицы в однородном магнитном поле. Не умаляя общности, можно считать, что вектор напряженности этого поля $\boldsymbol{H}$ параллелен третьей проекции импульса частицы p_3. Тогда компоненты тензора электромагнитного поля $F_{\mu\nu}$ равны

$$F_{0a} = E_a = 0, \quad F_{23} = H_1 = 0, \quad F_{31} = H_2 = 0, \quad F_{12} = H_3 = H. \tag{7.1}$$

Из (7.1) следует, что π_μ можно выбрать в виде

$$\pi_1 = p_1 - eHx_2, \quad \pi_2 = p_2, \quad \pi_3 = p_3, \quad \pi_0 = i\frac{\partial}{\partial t}. \tag{7.2}$$

Подставив (7.1), (7.2) в (5.8), получим $H_s(\boldsymbol{\pi})$ в форме

$$H_s(\boldsymbol{\pi}) = \Gamma_0^{(s)}\Gamma_a^{(s)}\pi_a + \Gamma_0^{(s)}m + \frac{H}{2m}\Gamma_0^{(s)}\left(1 - \Gamma_4^{(s)}\right)\left(i\Gamma_1^{(s)}\Gamma_2^{(s)} - \frac{1}{s}S_{12}\right). \tag{7.3}$$

Преобразуем $H_s(\boldsymbol{\pi})$ к такому виду, чтобы он содержал только коммутирующие величины. Это позволит нам, не решая уравнений движения (5.5), (5.6), определить спектр собственных значений гамильтониана (7.3). Действительно, в результате преобразования

$$H_s(\boldsymbol{\pi}) \to H_s'(\boldsymbol{\pi}) = VH_sV^{-1}, \quad \hat{P}_s(\boldsymbol{\pi}) \to \hat{P}_s(\boldsymbol{\pi}) = V\hat{P}_s(\boldsymbol{\pi})V^{-1}, \tag{7.4}$$

где

$$V = \lambda^+ + \mathcal{E}^{-1}\lambda^{-1}\Gamma_0^{(s)}H_s(\boldsymbol{\pi}), \quad \mathcal{E} = \left(\pi^2 - \frac{1}{s}S_{12}H + m^2\right)^{1/2},$$

$$V^{-1} = \frac{1}{m}\left(\lambda^-\mathcal{E} + H_s(\boldsymbol{\pi})\lambda^-\Gamma_0^{(s)}\right), \quad \lambda^\pm = \frac{1}{2}\left(1 \pm \Gamma_4^{(s)}\right),$$

получаем

$$H_s'(\boldsymbol{\pi}) = \Gamma_0^{(s)}\left(m^2 + \pi^2 - \frac{1}{s}S_{12}H\right)^{1/2}, \tag{7.5}$$

$$P_s\Phi = \Phi \quad \text{или} \quad \frac{1}{2}S_{ab}^2\Phi = s(s+1)\Phi, \quad \Phi = V\Psi. \tag{7.6}$$

Операторы $\Gamma_0^{(s)}$, S_{12} и π^2 коммутируют друг с другом и имеют следующие собственные значения:

$$\Gamma_0^{(s)}\Phi = \varepsilon\Phi, \quad \varepsilon = \pm 1, \quad S_{12}\Phi = s_3\Phi, \quad s_3 = -s, -s+1, \ldots, s, \tag{7.7}$$

$$\pi^2\Phi = \left[(2n+1)H + p_3^2\right]\Phi, \quad n = 0, 1, 2, \ldots. \tag{7.8}$$

Формулы (7.7) следуют непосредственно из (3.4), (7.6), а соотношение (7.8) приведено, например, в [21].

Квадрат гамильтониана (7.5) и операторы (7.7), (7.8) имеют общую систему собственных функций $\Phi_{\varepsilon n s_3 p_3}$. Отсюда и из (7.7), (7.8) заключаем, что собственные значения гамильтониана (7.5) равны

$$E_{\varepsilon n s_3 p_3} = \varepsilon\left[m^2 + (2n + 1 - s_3/s)eH + p_3^2\right]^{1/2}. \tag{7.9}$$

Соотношение (7.9) обобщает известную формулу [21] для уровней энергии электрона в однородном магнитном поле на случай частицы с произвольным спином s. Как видно из (7.9), значения энергии такой частицы действительны при любых s, в то время как уравнения Рариты–Швингера для $s = 3/2$ при решении аналогичной задачи приводят к комплексным значениям энергии [2].

Приведем для полноты вид собственных функций $\Phi_{\varepsilon n s_3 p_3}$:

$$\Phi_{\varepsilon n s_3 p_3} = \Phi_\varepsilon \Phi_{s_3} \Phi_{n p_3}, \tag{7.10}$$

где Φ_{np_3} — собственные функции оператора π^2 [21]:

$$\Phi_{np_3} = \exp(ip_1x_1 + ip_3x_3)\exp\left[-\frac{H}{2}\left(x_2 + \frac{p_1}{H}\right)\right] H_n\left[\sqrt{H}\left(x_2 + \frac{p_1}{H}\right)\right], \tag{7.11}$$

H_n — полиномы Эрмита, а Φ_ε, Φ_{s_3} — собственные функции операторов $\Gamma_0^{(s)}$ и S_{12}, явный вид которых может быть легко найден для любого конкретного представления матриц $\Gamma_0^{(s)}$ и S_{12}.

1. Tsai W., Yildis A., *Phys. Rev. D*, 1971, **4**, 3643.
2. Velo G., Zwanzinger D., *Phys. Rev.*, 1969, **186**, 2218;
   Seetharaman M., Prabhakaran I., Mathews P.M., *Phys. Rev. D*, 1975, **12**, 458.
3. Фущич В.И., Грищенко А.Л., Никитин А.Г., *ТМФ*, 1971, **8**, 192; Preprint ITF-70-89E, Kiev, 1970.
4. Fushchych W.I., Nikitin A.G., *Rep. Math. Phys.*, 1975, **8**, 33;
   Никитин А.Г., *УФЖ*, 1973, **18**, 1605; 1974, **19**, 1000.
5. Фущич В.И., Никитин А.Г., Дифференциальные уравнения движения первого и второго порядка для частиц с произвольным спином, Препринт ИМ-77-3, Киев, Институт математики АН УССР, 1977, 48 с.
6. Weaver D.L., Hammer C.L., Good R.H., *Phys. Rev. B*, 1964, **135**, 241.
7. Mathews P.M., *Phys. Rev.*, 1966, **143**, 978;
   Jayaraman I., *Nuovo Cim. A*, 1973, **13**, 877; 1973, **14**, 343.
8. Guertin R.F., *Ann. Phys.*, 1974, **88**, 504; 1975, **91**, 386.
9. Тамм И.Е., *ДАН СССР*, 1940, **29**, 551;
   Taketani M., Sakata S., *Proc. Phys. Math. Soc. (Japan)*, 1940, **22**, 757.
10. Guertin R.F., Preprint Rice University, Houston, 1975.
11. Foldy L.L., *Phys. Rev.*, 1956, **102**, 568.
12. Широков Ю.М., *ЖЭТФ*, 1957, **33**, 1196.
13. Lomont I.S., *Phys. Rev.*, 1958, **111**, 1710;
    Moses H.E., *Nuovo Cim. Suppl.*, 1958, **7**, 1.
14. Lomont I.S., Moses H.E., *Phys. Rev.*, 1960, **118**, 337;
    Dowker I.S., *Proc. Roy. Soc. A*, 1967, **293**, 351.
15. Швебер С., Введение в релятивистскую квантовую теорию поля, ИЛ, 1963, с. 69.
16. Foldy L.L., Wouthuysen S.A., *Phys. Rev.*, 1950, **78**, 29.
17. Durand B., *Phys. Rev. D*, 1976, **14**, 1554.
18. Зайцев Г.А., *ЖЭТФ*, 1955, **28**, 524; *ДАН СССР*, 1957, **113**, 1248;
    Feynman R.P., Gell-Mann M., *Phys. Rev.*, 1958, **109**, 193.
19. Hurley W.I., *Phys. Rev. D*, 1971, **4**, 3605.
20. James K.R., *Proc. Phys. Soc. (London)*, 1968, **1**, 334.
21. Ахиезер А.И., Берестецкий В.Б., Квантовая электродинамика, Наука, 1969, С. 142.

*For further progress in description of relativistic particles with higher spins refer to [Niederle J., Nikitin A.G., Relativistic wave equations for interacting, massive particles with arbitrary half-integer spins, *Phys. Rev. D*, 2001, **64**, N 12, 125013; hep-th/0412213]. *Editors' Remark.*

On the new invariance algebras of relativistic equations for massless particles

W.I. FUSHCHYCH, A.G. NIKITIN

We show that the massless Dirac equation and Maxwell equations are invariant under a 23-dimensional Lie algebra, which is isomorphic to the Lie algebra of the group $C_4 \otimes U(2) \otimes U(2)$. It is also demonstrated that any Poincaré-invariant equation for a particle of zero mass and of discrete spin provide a unitary representation of the conformal group and that the conformal group generators can be expressed via the generators of the Poincaré group.

1. Introduction

Bateman [1] and Cunningham [3] discovered that Maxwell's equations for a free electromagnetic field are invariant under conformal transformations. Nearly fifty years ago the conformal invariance of an arbitrary relativistic equation, for a massless particle with discrete spin was established by Dirac [4] for a spin-$\frac{1}{2}$ particle and by McLennan [20] for a particle of any spin.

Until now the question of whether the conformal group is the maximally extensive symmetry group for the equations of motion for massless particles remained unsettled. A positive answer to this question has been obtained only in the frame of the classical Sophus Lie approach [24], but as has been found recently, Lie methods do not permit the possibility to obtain all possible symmetry groups of differential equations.

The restriction of the Lie method is that it applies only to those symmetry groups whose generators belong to the class of differential operators of first order. Using the non-Lie approach, in which the group generators can be differential operators of any order and even integro-differential operators, the new invariance groups of relativistic wave equations have been found [6–9]. It was demonstrated that any Poincaré-invariant equation for a free particle of spin $s \geq \frac{1}{2}$ possess additional invariance under the group $SU(2) \otimes SU(2)$ [6, 7]; that the Kemmer–Duffin–Petiau equation is invariant under the group $SU(3) \otimes SU(3)$ [23], and that the Rarita–Schwinger equation was invariant under the group $O(6) \otimes O(6)$ [10]. The non-Lie approach was also used to find symmetry groups of the Dirac and Kemmer–Duffin–Petiau equations describing the particles in an external electromagnetic field [12]. Other examples of symmetries which cannot be obtained in the classical Lie approach are symmetry groups of the non-relativistic oscillator [16] and of the Hydrogen atom [5].

In the present paper, using a non-Lie approach we find new symmetry groups of the massless Dirac and Maxwell's equations. These groups are generated not by transformations of coordinates, but by transformations of the Dirac wave function Ψ and the vectors of electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{H}$ of the type

$$\Psi \to \Psi' = f\left(\Psi, \frac{\partial \Psi}{\partial x_a}, \frac{\partial^2 \Psi}{\partial x_a \partial x_b}, \dots\right), \tag{1.1}$$

Reprinted with permission from J. Phys. A: Math. Gen., 1979, **12**, P. 747–757

Original article is available at http://www.iop.org/EJ/journal/JPhysA

$$\boldsymbol{E} \to \boldsymbol{E}' = \boldsymbol{g}\left(\boldsymbol{E}, \boldsymbol{H}, \frac{\partial \boldsymbol{E}}{\partial x_a}, \frac{\partial \boldsymbol{H}}{\partial x_a}, \frac{\partial^2 \boldsymbol{E}}{\partial x_a \partial x_b}, \frac{\partial^2 \boldsymbol{H}}{\partial x_a \partial x_b}, \dots\right),$$
$$\boldsymbol{H} \to \boldsymbol{H}' = \boldsymbol{h}\left(\boldsymbol{E}, \boldsymbol{H}, \frac{\partial \boldsymbol{E}}{\partial x_a}, \frac{\partial \boldsymbol{H}}{\partial x_a}, \frac{\partial^2 \boldsymbol{E}}{\partial x_a \partial x_b}, \frac{\partial^2 \boldsymbol{H}}{\partial x_a \partial x_b}, \dots\right), \tag{1.2}$$

where functions f and $\boldsymbol{g}$, $\boldsymbol{h}$ can depend on any order derivatives of Ψ and $\boldsymbol{E}$, $\boldsymbol{H}$ respectively.

It is demonstrated that Maxwell's equations are invariant under the group $U(2) \otimes U(2)$; the explicit forms of functions $\boldsymbol{g}$ and $\boldsymbol{h}$ in (1.2), which generate the transformations of this group, are found. It is also shown that the Dirac equation (with $m = 0$) and Maxwell's equations are invariant under a 23-parametrical Lie group, which is isomorphic to the group $C_4 \otimes U(2) \otimes U(2)$. The obtained results admit direct generalisation to the relativistic wave equations for massless particles of any spin. The generators of conformal group which keep the Weyl equation and the massless Dirac equation invariant are expressed in a form which is transparently Hermitian. It is demonstrated that any (generally speaking, reducible) representation of Poincaré group, which corresponds to zero mass and discrete spin, can be extended to the conformal group representation. The explicit expression for generators of the conformal group C_4 via generators of the Poincaré group $P(1,3)$ has been found. We therefore present a constructive proof of the statement that any relativistic equation for a discrete spin and zero-mass particle provides the unitary representation of the conformal group (for Maxwell and Bargman–Wigner equations this has been demonstrated by Gross [13]).

2. The Hermitian representation of generators of conformal group for any spin

The conformal invariance properties of any relativistic equation of motion for a particle of zero mass and discrete spin can be formulated as the following statement.

Theorem 1. *Any Poincaré-invariant equation for zero-mass and discrete spin particle is invariant under the conformal algebra C_4*, basis elements of which are given by operators P_μ, $J_{\mu\nu}$ and*

$$D = -\frac{1}{2}[P_0 P_a/P^2, J_{0a}]_+,$$
$$K_\mu = \frac{1}{2}\left([P_0/P^2, [J_{0b}, J_{\mu b}]_+]_+ - [P_\mu/P^2, J_{0b}J_{0b}]_+\right) + \left(P_\mu/P^2\right)\left(\Lambda^2 - \frac{1}{2}\right), \tag{2.1}$$

where P_μ and $J_{\mu\nu}$ are basis elements of algebra $P(1,3)$,

$$[A, B]_+ = AB + BA, \quad P^2 = P_1^2 + P_2^2 + P_3^2, \quad \Lambda = \frac{1}{2}\varepsilon_{abc} J_{ab} P_c P_0^{-1}$$

and D, K_μ are operators which extend algebra $P(1,3)$ to algebra C_4.

Proof. Inasmuch as operators P_μ and $J_{\mu\nu}$ by definition satisfy the algebra

$$[P_\mu, P_\nu]_- = 0, \quad [J_{\mu\nu}, P_\lambda]_- = i(g_{\nu\lambda} P_\mu - g_{\mu\lambda} P_\nu),$$
$$[J_{\mu\nu}, J_{\lambda\sigma}]_- = i(g_{\nu\lambda} J_{\mu\sigma} + g_{\mu\sigma} J_{\nu\lambda} - g_{\mu\lambda} J_{\nu\sigma} - g_{\nu\sigma} J_{\mu\lambda}), \tag{2.2}$$

*We use the same notation for the groups and for the corresponding Lie algebras.

the theorem proof is reduced to the verification of correctness of the following commutation relations:

$$[J_{\mu\nu}, K_\lambda]_- = i(g_{\nu\lambda}K_\mu - g_{\mu\lambda}K_\nu), \quad [K_\mu, P_\nu]_- = 2i(g_{\mu\nu}D - J_{\mu\nu}),$$
$$[D, P_\mu]_- = iP_\mu, \quad [D, K_\mu]_- = -iK_\mu, \quad [K_\mu, K_\nu]_- = 0, \quad [J_{\mu\nu}, D]_- = 0, \tag{2.3}$$

which determine together with (2.2) the algebra C_4 (see, e.g., [19]). It is not difficult to carry out such a verification, bearing in mind that for the set of solutions of any relativistic equation for a particle of zero mass and of discrete spin the following relations are satisfied:

$$P_\mu P^\mu = 0, \quad W_\mu W^\mu = 0, \quad W_\mu = \Lambda P_\mu, \tag{2.4}$$

where W_μ is the Lubansky–Pauli vector

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma.$$

So the formulae (2.1) have determined the explicit form of the conformal group generators via the given generators P_μ, $J_{\mu\nu}$ of the group $P(1,3)$. The theorem is proved.

We note that the generators K_μ and D are written in a transparently Hermitian form, and hence they generate the unitary representation of the conformal group. The constructive character of theorem 1 will be demonstrated in the next section.

3. Manifestly Hermitian representation of generators of conformal group for Dirac and Weyl equations

The results given above can be used to find the explicit form of generators of the conformal group representation, which are realised on the set of solutions of any relativistic equation for a massless particle. In this section we shall demonstrate it for the massless Dirac and Weyl equations.

The Dirac equation for a massless particle of spin $\frac{1}{2}$ can be written in the form

$$L\Psi = 0, \quad L = i\frac{\partial}{\partial t} - \gamma_0\gamma_a p_a, \quad p_a = -i\frac{\partial}{\partial x_a}, \tag{3.1}$$

where γ_μ are the four-row Dirac matrices.

Let $\{Q_A\}$ be a set of the generators of a Lie group G. Equation (3.1) is by definition invariant under G if operators Q_A satisfy the relations

$$[L, Q_A]_- = F_A L, \tag{3.2}$$

where F_A are some operators which are defined on the set of the solutions of equation (3.1).

A well known example of such operators is the set of Poincaré group generators

$$P_0 = H = \gamma_0\gamma_a p_a, \quad P_a = p_a,$$
$$J_{ab} = x_a p_b - x_b p_a + S_{ab}, \quad J_{0a} = x_0 p_a - \frac{1}{2}[x_a, H]_+, \tag{3.3}$$

where

$$x_0 = t, \quad S_{ab} = \frac{1}{4}i(\gamma_a\gamma_b - \gamma_b\gamma_a).$$

According to theorem 1, representation (3.3) can be extended to the representation of Lie algebra of the conformal group. Substituting (3.3) into (2.4), we obtain the operators

$$D = \frac{1}{2}[x_\mu, P_\mu], \quad K_\mu = [J_{\mu\nu}, x^\nu]_+ + \frac{1}{2}[P_\mu, x_\nu x^\nu]_+ \tag{3.4}$$

which satisfy the invariance condition (3.2) (where $F_A \equiv 0$) and the commutation relations (2.5). Operators (3.3) and (3.4) are transparently Hermitian under the usual scalar product

$$(\Psi_1, \Psi_2) = \int d^3x \; \Psi_1^\dagger \Psi_2 \tag{3.5}$$

and therefore generate the unitary representation of the conformal group.

Let us note that on the set of solutions of equation (3.1) generators (3.3) and (3.4) can also be written in the usual form (see, e.g., [19])

$$\begin{aligned}
&P_\mu = p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \quad D = x_\mu p^\mu + \frac{3}{2}i,\\
&J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{1}{4}i[\gamma_\mu, \gamma_\nu]_-,\\
&K_\nu = 2x_\nu D - x_\mu x^\mu p_\nu - \frac{1}{2}x^\mu[\gamma_\nu, \gamma_\mu]_-
\end{aligned} \tag{3.6}$$

which is not manifestly Hermitian.

The Weyl equation for the neutrino,

$$i\frac{\partial \phi}{\partial t} = \sigma_a p_a \phi, \tag{3.7}$$

where σ_a are Pauli matrices, is equivalent to the system including equation (3.1) and the Poincaré-invariant subsidiary condition

$$(1 + i\gamma_4)\Psi = 0, \quad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{3.8}$$

The exact form of the Hermitian generators of the conformal group which are provided by equation (3.7) can be obtained from (3.3) and (3.4) by the substitution

$$p_0 \to \sigma_a p_a, \quad S_{ab} \to \frac{1}{4}i(\sigma_a\sigma_b - \sigma_b\sigma_a). \tag{3.9}$$

Finally, if P_μ and $J_{\mu\nu}$ are generators of the irreducible representation of Poincaré group in the Lomont–Moses form [18], then the formulae (2.1) give generators of the conformal group in the form of Bose and Parker [2].

4. Additional symmetry of the Dirac equation with $m = 0$

Some years ago a new invariance algebra of equation (3.1) was found [6, 7]; which differs from the algebra of conformal group generators. The basis elements of this algebra have the form

$$
\begin{aligned}
&\Sigma_{ab} = S_{ab} - \frac{1}{2}\left(\gamma_a \hat{p}_b - \gamma_b \hat{p}_a\right)\left(1 + \gamma_a \hat{p}_a\right),\\
&\Sigma_{4a} = \frac{1}{2}\gamma_4\gamma_a + \frac{1}{2}\gamma_4 \hat{p}_a \left(1 + \gamma_b \hat{p}_b\right),
\end{aligned}
\tag{4.1}
$$

where

$$
\hat{p}_a = p_a p^{-1}, \quad p = \left(p_1^2 + p_2^2 + p_3^2\right)^{1/2}, \quad a, b = 1, 2, 3.
$$

Operators (4.1) realise the representation $D\left(\frac{1}{2}, 0\right) \otimes D\left(0, \frac{1}{2}\right)$ of the Lie algebra of the group $O(4) \sim SU(2) \otimes SU(2)$, but do not form closed algebra together with (3.3), (3.4) or (3.8). Below we will obtain 23-dimensional invariance algebra of equation (3.1), which includes the Lie algebras of groups C_4 and $U(2) \otimes U(2)$.

Theorem 2. *The Dirac equation* (3.1) *is invariant under the* 23*-dimensional Lie algebra, which is isomorphic to algebra of generators of group* $C_4 \otimes U(2) \otimes U(2)$*. Basis elements of this algebra have the form*

$$
\begin{aligned}
&P_0 = p_0 = i\frac{\partial}{\partial t}, \quad P_a = p_a = -i\frac{\partial}{\partial x_a}, \quad J_{ab} = x_a p_b - x_b p_a + S_{ab},\\
&J_{0a} = x_0 p_a - x_a p_0 - \frac{iH}{2p}(1 - i\gamma_4)\gamma_a\gamma_b\hat{p}_b + \hat{\Sigma}_{0a}, \quad D = x_\mu p^\mu + i,\\
&K_\mu = \left(-x_\nu x^\nu + J_{ab}S_{ab}p^{-2} + p^{-2}\right)p_\mu + 2\left[x_\mu + (1 - \delta_{\mu 0})(1 - \gamma_0)S_{\mu b}\hat{p}_b\right]D,\\
&\hat{\Sigma}_{0c} = \frac{1}{2}\gamma_4\left(\hat{p}_a + \gamma_0 S_{ab}\hat{p}_b\right), \quad \hat{\Sigma}_5 = \frac{H}{p},\\
&\hat{\Sigma}_{ab} = \frac{1}{2}\varepsilon_{abc}\frac{H}{p}\hat{\Sigma}_{0c}, \quad \hat{\Sigma}_6 = 1, \quad a, b, c = 1, 2, 3.
\end{aligned}
\tag{4.2}
$$

Proof. Let us transform equation (3.1) and generators (4.2) to a representation in which the theorem statements can be easily verified. Using for this purpose the operator

$$
V = V^{-1} = \frac{1}{2}\left[1 + \gamma_0 + (1 - \gamma_0)\varepsilon_{abc}S_{ab}\hat{p}_c\right]
\tag{4.3}
$$

we obtain

$$
L'\Psi' = 0, \quad \Psi' = V\Psi, \quad L' = VLV^{-1} = i\frac{\partial}{\partial t} - i\gamma_4 p,
\tag{4.4}
$$

$$
\begin{aligned}
&P'_\mu = VP_\mu V^{-1} = P_\mu, \quad J'_{ab} = VJ_{ab}V^{-1} = J_{ab},\\
&J'_{0a} = VJ_{0a}V^{-1} = x_0 p_a - x_a p_0 + \frac{i}{2}\gamma_0\gamma_a,\\
&D' = VDV^{-1} = D = x_\mu p^\mu + i,
\end{aligned}
$$

$$K'_\mu = VK_\mu V^{-1} = -x_\nu x^\nu p_\mu + 2x_\mu D',$$
$$\hat{\Sigma}'_{ab} = V\hat{\Sigma}_{ab}V^{-1} = S_{ab}, \quad \hat{\Sigma}'_{0a} = V\hat{\Sigma}_{ab}V^{-1} = \frac{i}{2}\gamma_0\gamma_a, \tag{4.5}$$
$$\hat{\Sigma}'_5 = V\hat{\Sigma}_5V^{-1} = i\gamma_4, \quad \hat{\Sigma}'_6 = V\hat{\Sigma}_6V^{-1} = \hat{\Sigma}_6,$$

It is not difficult to make sure that the operators (4.4) and (4.5) satisfy the invariance condition (3.2):

$$[L', P'_\mu]_- = [L', J'_{ab}]_- = [L', \hat{\Sigma}_{\mu\nu}]_- = [L', \hat{\Sigma}'_\alpha]_- = 0,$$
$$[L', K'_0]_- = 2i\left[x_0 + (x_a p_a - i)i\gamma_4 p^{-1}\right] L', \quad [L', K'_a]_- = 2i(x_a + i\hat{p}_a x_0\gamma_4)L',$$
$$[L', D]_- = iL', \quad [L', J'_{0a}]_- = \gamma_4\hat{p}_a L'$$

and the commutation relations

$$[P'_\mu, P'_\nu]_- = 0, \quad [P'_\mu, J'_{\nu\lambda}]_- = i(g_{\mu\lambda}P'_\nu - g_{\nu\lambda}P'_\mu),$$
$$[J'_{\mu\nu}, J'_{\lambda\sigma}]_- = i(g_{\mu\sigma}J'_{\nu\lambda} + g_{\nu\lambda}J'_{\mu\sigma} - g_{\mu\lambda}J'_{\nu\sigma} - g_{\nu\sigma}J'_{\mu\lambda}),$$
$$[P'_\mu, D']_- = -iP'_\mu, \quad [K'_\mu, D']_- = iK'_\mu, \quad [J'_{\mu\nu}, D']_- = 0,$$
$$[P'_\mu, K'_\nu]_- = 2i(J'_{\mu\nu} - \hat{\Sigma}'_{\mu\nu} - g_{\mu\nu}D'),$$
$$[J'_{\mu\nu}, \hat{\Sigma}'_{\lambda\sigma}]_- = [\hat{\Sigma}'_{\mu\nu}, \hat{\Sigma}'_{\lambda\sigma}]_- = i(g_{\mu\sigma}\hat{\Sigma}'_{\nu\lambda} + g_{\nu\lambda}\hat{\Sigma}'_{\mu\sigma} - g_{\mu\lambda}\hat{\Sigma}'_{\nu\sigma} - g_{\nu\sigma}\hat{\Sigma}'_{\mu\lambda}),$$
$$[\hat{\Sigma}'_{\mu\nu}, P'_\lambda]_- = [\hat{\Sigma}'_{\mu\nu}, D']_- = [\hat{\Sigma}'_{\mu\nu}, K'_\lambda]_- = [\hat{\Sigma}'_\alpha, Q'_A]_- = 0.$$

Algebra (4.6) is isomorphic to the algebra of generators of the group $C_4 \otimes U(2) \otimes U(2)$. The theorem is proved.

We note that the subsidiary condition (3.8) is not invariant under the transformations which are generated by operators $\hat{\Sigma}_{\mu\nu}$. Therefore the Weyl equation (3.7) is not invariant with respect to the whole algebra (4.2), but is invariant with respect to its subalgebra C_4.

It should be emphasised that the generators (4.2) belong to the class of nonlocal integro-differential operators, and therefore cannot be obtain in the classical Lie approach.

5. Symmetry of Maxwell's equations

The Maxwell equations for a free electromagnetic field have the form

$$\boldsymbol{p} \times \boldsymbol{E} = i\frac{\partial \boldsymbol{H}}{\partial t}, \quad \boldsymbol{p} \times \boldsymbol{H} = -i\frac{\partial \boldsymbol{E}}{\partial t},$$
$$\boldsymbol{p} \cdot \boldsymbol{E} = 0, \quad \boldsymbol{p} \cdot \boldsymbol{H} = 0, \tag{5.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are vectors of the electric and magnetic field strengths.

Equations (5.1) are invariant under the conformal group. It is well known that these equations are also invariant under the transformations [14,15]

$$E_a \to H_a, \quad H_a \to -E_a \tag{5.2}$$

and under the more general ones [25]

$$
\begin{aligned}
&E_a \to E_a \cos\theta + H_a \sin\theta,\\
&H_a \to H_a \cos\theta - E_a \sin\theta.
\end{aligned}
\tag{5.3}
$$

We now demonstrate that the symmetry of the Maxwell equations is more extensive, namely, equations (5.1) are invariant under the set of transformations which realise a representation of the group $U(2)\otimes U(2)$ and include (5.3) as a one-parameter subgroup. The theorem about such an invariance of the Maxwell equations in the class of transformations of kind (1.1) and (1.2) had been formulated by one of us [9] without showing the exact form of functions $\boldsymbol{g}$ and $\boldsymbol{h}$. Below we give the explicit transformation laws for E_a and H_a.

Theorem 3. *Maxwell equations* (5.1) *are invariant under the transformations*

$$
\begin{aligned}
&H_a \to H_a' = H_a\cos\theta + [iD_{ab}E_b\theta_1 - \varepsilon_{abc}\hat{p}_b(H_c\theta_3 + iD_{cd}E_d\theta_2)]\,\frac{\sin\theta}{\theta},\\
&E_a \to E_a' = E_a\cos\theta + [iD_{ab}H_b\theta_1 - \varepsilon_{abc}\hat{p}_b(E_c\theta_3 + iD_{cd}H_d\theta_2)]\,\frac{\sin\theta}{\theta};
\end{aligned}
\tag{5.4a}
$$

$$
\begin{aligned}
&H_a \to H_a'' = H_a\cos\lambda - [i\varepsilon_{abc}\hat{p}_bD_{cd}H_d\lambda_1 + D_{ad}H_d\lambda_2 - E_a\lambda_3]\,\frac{\sin\lambda}{\lambda},\\
&E_a \to E_a'' = E_a\cos\lambda + [i\varepsilon_{abc}\hat{p}_bD_{cd}E_d\lambda_1 + D_{ad}E_d\lambda_2 - H_a\lambda_3]\,\frac{\sin\lambda}{\lambda};
\end{aligned}
\tag{5.4b}
$$

$$
\begin{aligned}
&H_a \to H_a''' = H_a\cos\eta - \varepsilon_{abc}\hat{p}_bE_c\sin\eta,\\
&E_a \to E_a''' = E_a\cos\eta + \varepsilon_{abc}\hat{p}_bH_c\sin\eta;
\end{aligned}
\tag{5.4c}
$$

$$
\begin{aligned}
&H_a \to H_a'''' = \exp(i\phi)H_a,\\
&E_a \to E_a'''' = \exp(i\phi)E_a,
\end{aligned}
\tag{5.4d}
$$

where

$$
\begin{aligned}
&D_{ad} = \left[\left(p_a^2p_c^2 + p_a^2p_b^2 - p_b^2p_c^2\right)\delta_{ad} + p_1p_2p_3\left(p_b\delta_{cd} + p_c\delta_{bd} - p_a\hat{p}_d\right)\right]L^{-1},\\
&L = \frac{1}{2}\sqrt{2}\left[\left(p_1^2 - p_2^2\right)p_3^4 + \left(p_1^2 - p_3^2\right)p_2^4 + \left(p_2^2 - p_3^2\right)p_1^4\right]^{1/2},
\end{aligned}
$$

and where (a,b,c) *is a cyclic permutation of* $(1,2,3)$*;*

$$
\lambda = \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)^{1/2}, \quad \theta = \left(\theta_1^2 + \theta_2^2 + \theta_3^2\right)^{1/2},
$$

θ_a*,* λ_a*,* η *and* ϕ *are real parameters. Transformations* (5.4) *realise a representation of group* $U(2)\otimes U(2)$.

Proof. One can be convinced by direct verification that E_a', H_a', E_a'', H_a'', E_a''', H_a''', E_a'''', H_a'''' satisfy equation (5.1). But a more elegant and constructive way, which shows the method of obtaining the group (5.4) is to transform the equations to a form for which the theorem statements become obvious.

Let us write equations (5.1) in the matrix form [10, 11, 22]

$$
i\frac{\partial}{\partial t}\Psi = \alpha_ap_a\Psi, \quad \sigma_3S_{4a}p_a\Psi = 0,
\tag{5.5}
$$

where Ψ is an eight-component wavefunction

$$\Psi = \text{column}(H_1, H_2, H_3, \phi_1, E_1, E_2, E_3, \phi_2) \tag{5.6}$$

and α_a, S_{4a} are matrices of the form

$$\alpha_a = 2\sigma_2\tau_a,$$

$$\sigma_2 = i\begin{pmatrix} \hat{0} & -\hat{I} \\ \hat{I} & \hat{0} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \hat{I} & \hat{0} \\ \hat{0} & -\hat{I} \end{pmatrix}, \quad \tau_a = \begin{pmatrix} \hat{\tau}_a & 0 \\ 0 & \hat{\tau}_a \end{pmatrix}, \tag{5.7}$$

$$\hat{\tau}_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\tau}_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\hat{\tau}_3 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad S_{4a} = \begin{pmatrix} \hat{S}_{4a} & \hat{0} \\ \hat{0} & -\hat{S}_{4a} \end{pmatrix},$$

$$\hat{S}_{41} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \hat{S}_{42} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

$$\hat{S}_{43} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

$\hat{0}$ and $\hat{I}$ are four-row square zero and unit matrices. The matrices $\hat{S}_{4a}$ and

$$\hat{S}_{ab} = \frac{1}{2}\left(\hat{S}_{4c} + 2\hat{\tau}_c\right)\varepsilon_{abc}$$

realize the representation $D\left(\frac{1}{2}, \frac{1}{2}\right)$ of algebra $O(4)$. Writing equations (5.5) componentwize we obtain the usual form (5.1) for the Maxwell equation and the conditions for ϕ_1 and ϕ_2:

$$\phi_1 = C_1, \quad \phi_2 = C_2,$$

where C_1 and C_2 are constants which can be reduced to zero without loss of generality*.

Using the unitary operator

$$U = \exp\left(-i\frac{S_a\tilde{p}_a}{\tilde{p}}\tan^{-1}\frac{\tilde{p}}{p_1 + p_2 + p_3}\right), \tag{5.8}$$

*The analogous "Dirac-like" formulation of the Maxwell equations (but using a four-component wave function and subsidiary condition different from (5.5b) has been proposed previously by Lomont [17] and Moses [21].

where

$$\tilde{p}_a = p_b - p_c, \quad \tilde{p} = \left(\tilde{p}_1^2 + \tilde{p}_2^2 + \tilde{p}_3^2\right)^{1/2}, \quad S_a = \begin{pmatrix} \hat{S}_{bc} & \hat{0} \\ \hat{0} & \hat{S}_{bc} \end{pmatrix},$$

one reduces the equations (5.5) to the symmetrical form

$$\begin{aligned} &L_1'\Phi = 0, \quad L_1' = UL_1U^\dagger = i\frac{\partial}{\partial t} - \frac{1}{\sqrt{3}}(\alpha_1 + \alpha_2 + \alpha_3)p; \\ &L_2'\Phi = 0, \quad L_2' = UL_2U^\dagger = \frac{1}{\sqrt{3}}(S_{41} + S_{42} + S_{43}), \quad \Phi = U\Psi. \end{aligned} \tag{5.9}$$

Operator (5.8) also transforms the helicity operator $S_p = S_a p_a p^{-1}$ to the symmetrical matrix form:

$$US_pU^\dagger = (S_1 + S_2 + S_3)/\sqrt{3}.$$

The invariance condition (3.2) for equations (5.9) takes the form

$$[L_1', Q_A']_- = f_A^1 L_1' + f_A^2 L_2', \quad [L_2', Q_A']_- = \tilde{f}_A^1 L_1' + \tilde{f}_A^2 L_2'. \tag{5.10}$$

The conditions (5.10) are obviously satisfied by any operator which commutes with the matrices

$$A = (\alpha_1 + \alpha_2 + \alpha_3)/\sqrt{3} \quad \text{and} \quad B = (S_{41} + S_{42} + S_{43})/\sqrt{3}. \tag{5.11}$$

We choose the complete set of such operators in the form

$$\begin{aligned} &Q_{12}' = (S_1 + S_2 + S_3)/\sqrt{3}, \quad Q_{23}' = iQ_{12}'Q_{31}', \\ &Q_{31}' = \sum_a (S_b - S_c)p_a^2\left(p_b^2 - p_c^2\right)L^{-1}/\sqrt{3}, \\ &Q_{4a}' = AQ_{bc}', \quad Q_5' = A, \quad Q_6' = \sigma_0 = \begin{pmatrix} \hat{I} & \hat{0} \\ \hat{0} & \hat{I} \end{pmatrix}. \end{aligned} \tag{5.12}$$

Operators (5.12) are invariant under the permutation

$$S_a \to S_b, \quad p_a \to p_b, \quad a, b = 1, 2, 3.$$

Operators (5.12) satisfy the invariance condition (5.10) (with $f_A^1 = f_A^2 = \tilde{f}_A^1 = \tilde{f}_A^2 = 0$) and the commutation relations

$$\begin{aligned} &[Q_{kl}', Q_{mn}']_- = 2i(\delta_{km}Q_{ln}' + \delta_{ln}Q_{km}' - \delta_{kn}Q_{lm}' - \delta_{lm}Q_{kn}'), \\ &[Q_5', Q_{kl}']_- = [Q_6', Q_{kl}']_- = [Q_5', Q_6']_- = 0. \end{aligned} \tag{5.13}$$

These operators also satisfy the conditions

$$(Q_{kl}')^2\Phi = (Q_5')^2\Phi = (Q_6')^2\Phi = \Phi,$$

i.e. they realise a representation of the Lie algebra of group $U(2)\otimes U(2)$ and Q_{kl}' form the representation $D\left(0, \frac{1}{2}\right) \otimes D\left(\frac{1}{2}, 0\right)$ of the group $SU(2) \otimes SU(2)$.

It follows from the above that equations (5.9) are invariant under an arbitrary transformation from the group $U(2)\otimes U(2)$:

$$
\begin{aligned}
&\Phi\to\Phi'=\exp\left(\frac{1}{2}i\varepsilon_{abc}Q'_{ab}\theta_c\right)\Phi=\left(\cos\theta+\frac{1}{2}i\theta^{-1}\varepsilon_{abc}Q'_{ab}\theta_c\right)\Phi,\\
&\Phi\to\Phi''=\exp(iQ'_{4a}\lambda_a)\Phi=\left(\cos\lambda+iS_{4a}\lambda_a\frac{\sin\lambda}{\lambda}\right)\Phi,\\
&\Phi\to\Phi'''=\exp(iQ'_5\phi)\Phi=(\cos\phi+iQ'_5\sin\phi)\Phi,\\
&\Phi\to\Phi''''=\exp(iQ'_6\eta)\Phi=\exp(i\eta)\Phi.
\end{aligned}
\tag{5.14}
$$

Returning with the help of operator (5.8) to the starting function Ψ one obtains from (5.14) the following transformation laws:

$$
\begin{aligned}
&\Psi\to\Psi'=\left(\cos\theta+\frac{1}{2\theta}\varepsilon_{abc}Q_{ab}\sin\theta\right)\Psi,\\
&\Psi\to\Psi''=\left(\cos\lambda+\frac{i}{\lambda}Q_{4a}\lambda_a\sin\lambda\right)\Psi,\\
&\Psi\to\Psi'''=(\cos\phi+iQ_5\sin\phi)\Psi,\\
&\Psi\to\Psi''''=\exp(i\eta)\Psi,
\end{aligned}
\tag{5.15}
$$

where

$$
\begin{aligned}
&Q_{kl}=W^{-1}Q_{kl}W,\quad Q_\lambda=W^{-1}Q_\lambda W,\quad \lambda=5,6,\\
&Q_{12}=S_a\hat{p}_a,\quad Q_{23}=\sigma_1F,\quad Q_{31}=i\sigma_1S_a\hat{p}_aF,\\
&Q_{4a}=\frac{1}{2}\sigma_2S_b\hat{p}_b\varepsilon_{abc}Q_{bc},\quad Q_5=\sigma_2S_b\hat{p}_b,\quad Q_6=1,\\
&F=L^{-1}\Bigg(\sum_{a\neq b\neq c}\left[\left(p_a^2p_c^2+p_a^2p_b^2-p_b^2p_c^2\right)\left(1-S_a^2\right)+p_1p_2p_3p_aS_bS_c\right]-\\
&\qquad -pp_1p_2p_3\left[1-\left(S_a\hat{p}_a\right)^2\right]\Bigg).
\end{aligned}
\tag{5.16}
$$

Substituting (5.6) and (5.16) into (5.15), we come to the formulae (5.4). The theorem is proved.

So we have found a new eight-parameter symmetry group of the Maxwell equations which is given by transformations (5.4). The main property of such transformations is that they are carried out by the nonlocal (integro-differential) operators.

It is necessary to emphasise that transformations (5.4) have nothing to do with the Lorentz ones, inasmuch as they realise the unitary finite-dimensional representation of the compact group $U(2)\otimes U(2)$. If $\lambda_1=\lambda_2=0$, then formulae (5.4b) give the Heaviside–Larmor–Rainich transformation (5.3).

Transformations (5.4) are unitary under the usual scalar product (3.5). Substituting (5.6) into (3.5), we discover that the transformations (5.4) do not change the quantity

$$
\mathcal{E}=\int d^3x\,\left(\boldsymbol{E}^2+\boldsymbol{H}^2\right),
$$

which is associated with the full energy of the electromagnetic field.

If the parameters θ_a, λ_a, η and ϕ in (5.4) are the complex ones, the transformations (5.4) realise the representation of the group $GL(2)\otimes GL(2)$. Such transformations also keep equations (5.1) invariant, but are, non-unitary.

Using theorem 1, we can show that equations (5.5) provide a Hermitian representation of the Lie algebra of the conformal group. The basis elements of this algebra have the form

$$\begin{aligned} &P_0 = \boldsymbol{\alpha}\cdot\boldsymbol{p}, \quad P_a = p_a, \\ &J_{ab} = x_a p_b - x_b p_a + S_{ab} = X_a p_b - X_b p_a + \hat{p}_c\Lambda, \\ &J_{0a} = tp_a - \frac{1}{2}[X_a, P_0]_+, \quad D = \frac{1}{2}[x_a, p_a]_+ - tP_0 \equiv -\frac{1}{2}[X_\mu, P^\mu]_+, \\ &K_\mu = -[J_{\mu\nu}, X^\nu]_+ + \frac{1}{2}[P_\mu, X_\nu X^\nu]_+ - P_\mu\left(\Lambda^2 + \frac{1}{4}\right)/p^2, \end{aligned} \tag{5.17}$$

where

$$X_0 = x_0 = t, \quad \Lambda = \frac{1}{2}\varepsilon_{abc}S_{ab}\hat{p}_c p^{-1}, \quad X_a = x_a + S_{ab}p_b p^{-2}.$$

But generators (5.17) together with (5.16) do not form a closed algebra. The symmetry of equations (5.5) under the 23-dimensional Lie algebra, which includes the subalgebras C_4 and $U(2)\otimes U(2)$, is established in the following theorem.

Theorem 4. *Equations* (5.5) *are invariant under the* 23*-dimensional Lie algebra, spanned on operators* (5.16) *and the generators*

$$\begin{aligned} &\hat{p}_\mu = p_\mu, \quad \hat{J}_{\mu\nu} = x'_\mu p_\nu - x'_\nu p_\mu, \\ &\hat{D} = x'_\mu p^\mu + i, \quad \hat{K}'_\mu = -x'_\nu x'^\mu p_\mu + 2x'_\mu \hat{D}, \end{aligned} \tag{5.18}$$

where

$$\begin{aligned} &x'_0 = x_0, \\ &x'_a = x_a + (S_b - S_c)(\sqrt{3}p - p_1 - p_2 - p_3) + S_d\tilde{p}_d(\sqrt{3}\hat{p}_a + 1) + \\ &\qquad + (p_b - p_c)(S_1 + S_2 + S_3)\{p[3p + \sqrt{3}(p_1 + p_2 + p_3)]\}^{-1}. \end{aligned}$$

The proof can be carried out in full analogy with the proof of theorem 2 (but with using operator (5.8) instead of (3.3)). The operators (5.18) satisfy algebra (2.2) and (2.3) and commute with (5.16).

It is not difficult to generalise the statements of theorem 4 to the case of "Dirac-like" equations for massless particles of any spin [11, 22].

We note that generators (5.16) and (5.17) are nonlocal (integro-differential) ones. This means that the invariance algebra of the Maxwell equations which we have obtained in principle cannot be obtained in the classical Lie approach, where the group generators always belong to the class of first-order differential operators.

1. Bateman H., *Proc. London Math. Soc.*, 1909, **8**, 223–264.
2. Bose S.K., Parker R., *J. Math. Phys.*, 1969, **10**, 812–813.
3. Cunningham E., *Proc. Lond. Math. Soc.*, 1909, **8**, 77–97.
4. Dirac P.A.M., *Ann. Math.*, 1936, **37**, 429–435.

5. Fock V.A., *Z. Phys.*, 1935, **98**, 145–149.
6. Fushchych W.I., Preprint E-70-32, Kiev, Institute for Theoretical Physics, 1970.
7. Fushchych W.I., *Teor. Mat. Fiz.* 1971, **7**, 3–12 (transl. *Theor. Math. Phys.*, 1971, **7**, 3–11).
8. Fushchych W.I., *Nuovo Cim. Lett.*, 1973, **6**, 133–138.
9. Fushchych W.I., *Nuovo Cim. Lett.*, 1974, **11**, 508–512.
10. Fushchych W.I., Nikitin A.G., *Nuovo Cim. Lett.*, 1977, **19**, 347–352.
11. Fushchych W.I., Nikitin A.G., Preprint 77-3, Mathematical Institute, Kiev, 1977.
12. Fushchych W.I., Nikitin A.G., *Nuovo Cim. Lett.*, 1978, **21**, 541–546.
13. Gross L., *J. Math. Phys.*, 1964, **5**, 687–695.
14. Heaviside O., Electromagnetic Theory, London, 1893.
15. Larmor, Collected papers London, 1928.
16. Levi-Leblond, *Am. J. Phys.*, 1971, **39**, 502–506.
17. Lomont I.S., *Phys. Rev.*, 1958, **111**, 1700–1709.
18. Lomont I.S., Moses H.E., *J. Math. Phys.*, 1962, **3**, 405–408.
19. Mack G., Salam A., *Ann. Phys., NY*, 1969, **53**, 174–202.
20. McLennan A., *Nuovo Cim.*, 1956, **3**, 1360–1380.
21. Moses H.E., *Nuovo Cim. Suppl.*, 1958, **7**, 1–18.
22. Nikitin A.G., Fushchych W.I., *Teor. Mat. Fiz.*, 1978, **34**, 319–333.
23. Nitikin A.G., Segeda Yu.N., Fushchych W.I., *Teor. Mat. Fiz.*, 1976, **29**, 82–94 (transl. *Theor. Math. Phys.*, 1976, **29**, 943–954).
24. Ovsjannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
25. Rainich G.Y., *Trans. Am. Math. Soc.*, 1925, **27**, 106–125.

Уравнения движения для частиц произвольного спина, инвариантные относительно группы Галилея

А.Г. НИКИТИН, В.И. ФУЩИЧ

The first and second order differential equations are derived which are invariant under the Galilei group and describe a particle with arbitrary spin. These equations admit the Lagrangian formulation and describe the dipole, spin-orbital and Darwin couplings of a particle with external electromagnetic field which are considered traditionally as pure relativistic effects. The problem of the motion of spin 1/2 nonrelativistic particle in homogeneous and constant magnetic field is exactly solved.

Выведены системы дифференциальных уравнений первого и второго порядков, инвариантные относительно группы Галилея и описывающие движение частицы с произвольным спином. Эти уравнения допускают лагранжеву формулировку и описывают дипольное, спин-орбитальное и дарвиновское взаимодействия частицы с внешним электромагнитным полем, которые традиционно считались чисто релятивистскими эффектами. Приведены примеры бесконечнокомпонентных уравнений, инвариантных относительно группы Галилея. Точно решена задача о движении нерелятивистской частицы со спином $s = 1/2$ в однородном магнитном поле.

Введение

Релятивистские уравнения движения для частиц с произвольным спином вызывают большой и устойчивый интерес физиков и математиков (см. [1] и цитируемую там литературу). И в то же время имеется удивительно мало публикаций, посвященных уравнениям, инвариантным относительно группы Галилея. Между тем еще в 1954 г. Баргман [2] показал, что с помощью центрального расширения группы Галилея понятие спина частицы может быть последовательно введено и в нерелятивистскую квантовую механику.

В [3, 4] получены галилеевски-инвариантные дифференциальные уравнения первого порядка, описывающие движение нерелятивистской частицы произвольного спина. Эти уравнения описывают дипольное взаимодействие частицы с внешним полем, но не учитывают такие хорошо известные физические эффекты, как спин-орбитальное и дарвиновское взаимодействия.

В настоящей работе с использованием методики, разработанной в [1, 5, 6] для вывода пуанкаре-инвариантных уравнений, получены галилеевски-инвариантные уравнения движения для частицы с произвольным спином s, позволяющие описать указанные взаимодействия. Это достигнуто с помощью расширения группы Галилея G до группы G^*, включающей преобразование одновременного отражения координат и времени. Полученные уравнения имеют шредингерову форму

$$i\frac{\partial\Psi(t,\boldsymbol{x})}{\partial t} = H_s(\boldsymbol{p})\Psi(t,\boldsymbol{x}), \quad p_a = -i\frac{\partial}{\partial x_a} \tag{0.1}$$

Перепечатано из Теор. мат. физика, 1980, **44**, № 1, С. 34–46 по разрешению редколлегии

(где $H_s(\boldsymbol{p})$ — некоторый дифференциальный оператор второго порядка, Ψ — $2(2s+1)$-компонентная волновая функция), допускают лагранжеву формулировку и описывают дипольное, спин-орбитальное, дарвиновское и квадрупольное взаимодействия частицы спина s с внешним электромагнитным полем. Это означает, в частности, что перечисленные взаимодействия, которые обычно вводятся как релятивистские поправки, могут последовательно рассматриваться в рамках нерелятивистской квантовой механики.

В работе получены также галилеевски-инвариантные дифференциальные уравнения первого порядка, описывающие движение частицы с произвольным спином. После минимальной замены $p_\mu \to P_\mu - eA_\mu$ эти уравнения также описывают спин-орбитальное и дарвиновское взаимодействия частицы с полем. Приведен пример бесконечно компонентных уравнений, инвариантных относительно группы Галилея.

1. Основные определения и постановка задачи

Группой Галилея G называется совокупность преобразований координат x_a $(a = 1, 2, 3)$ и времени t следующего вида:

$$x_a \to x'_a = R_{ab}x_b + V_a t + b_a, \quad t \to t' = t + b_0, \tag{1.1}$$

где R_{ab} — оператор трехмерного поворота, V_a и b_μ — произвольные действительные параметры.

Представление группы G однозначно определяется заданием явного вида инфинитезимальных операторов P_μ, J_a и G_a, соответствующих сдвигам, поворотам и собственно галилеевским преобразованиям координат.

Определение. *Будем говорить, что уравнение* (0.1) *инвариантно относительно группы Галилея, если гамильтониан* $H_s = P_0$ *и генераторы* P_a, J_a, G_a *удовлетворяют коммутационным соотношениям*

$$[P_a, P_b] = 0, \quad [P_a, J_b] = i\varepsilon_{abc}P_c, \tag{1.2а}$$

$$[G_a, G_b] = 0, \quad [G_a, J_b] = i\varepsilon_{abc}G_c, \tag{1.2б}$$

$$[P_a, G_b] = i\delta_{ab}M, \quad [M, P_\mu] = [M, J_a] = [M, G_a] = 0, \tag{1.2в}$$

$$[H_s, P_a] = [H_s, J_a] = 0, \tag{1.2г}$$

$$[H_s, G_a] = iP_a, \quad a, b = 1, 2, 3, \quad \mu = 0, 1, 2, 3. \tag{1.2д}$$

Соотношения (1.2) определяют алгебру Ли группы Галилея, которая имеет три инвариантных оператора (оператора Казимира)

$$\begin{aligned} &2MC_1 = 2MP_0 - P_aP_a, \quad C_2 = M, \\ &C_3 = (MJ_a - \varepsilon_{abc}P_bG_c)(MJ_a - \varepsilon_{ade}P_dG_e). \end{aligned} \tag{1.3}$$

Собственные значения операторов C_1, C_2 и C_3 ассоциируются с внутренней энергией, спином и массой частицы, описываемой инвариантным уравнением (0.1).

Задачу нахождения всех возможных (с точностью до эквивалентности) галилеевски-инвариантных уравнений вида (0.1) решим в двух, вообще говоря, неэквивалентных подходах. В подходе I задача формулируется следующим образом: найти все такие гамильтонианы H_s^{I}, чтобы операторы

$$P_0^{\mathrm{I}} = H_s^{\mathrm{I}}, \quad P_a^{\mathrm{I}} = p_a = -i\frac{\partial}{\partial x_a},$$
$$J_a^{\mathrm{I}} = (\boldsymbol{x}\times\boldsymbol{p})_a + S_a, \quad G_a^{\mathrm{I}} = tp_a - mx_a + \lambda_a^{\mathrm{I}} \tag{1.4}$$

удовлетворяли алгебре Ли расширенной группы Галилея (1.2). Здесь

$$S_c = \begin{pmatrix} s_c & 0 \\ 0 & s_c \end{pmatrix}, \quad (a,b,c) \ \text{— цикл } (1,2,3); \tag{1.5}$$

s_c — генераторы неприводимого представления $D(s)$ группы $O(3)$, m — параметр, задающий массу частицы, λ_a^{I} — некоторые числовые матрицы, явный вид которых определим ниже.

Формулы (1.4) задают общий вид генераторов группы Галилея, соответствующих локальным преобразованиям $2(2s+1)$-компонентной волновой функции при переходе к новой системе координат (1.1),

$$\Psi(t,\boldsymbol{x}) \to \Psi'(t',\boldsymbol{x}') = \exp[if(t,\boldsymbol{x})]D^s(R_{ab}, v_a)\Psi(t,\boldsymbol{x}), \tag{1.6}$$

где $D^s(R_{ab}, v_a)$ — некоторая числовая матрица, зависящая от параметров преобразования (1.1), $f(t,\boldsymbol{x})$ — фазовый множитель [2]:

$$f(t,\boldsymbol{x}) = mv_aR_{ab}x_b + \frac{1}{2}mv_av_a. \tag{1.7}$$

Ниже убедимся, что операторы H_s^{I} всегда могут быть выбраны такими, чтобы уравнение (0.1) было инвариантно также относительно антиунитарного преобразования отражения координат и времени:

$$\Psi(t,\boldsymbol{x}) \to r_1\Psi^*(-t,-\boldsymbol{x}), \quad r_1^2 = 1, \tag{1.8}$$

где r_1 — некоторая матрица.

В подходе II задача сводится к определению всех возможных дифференциальных операторов H_s^{II}, таких, чтобы генераторы

$$P_0^{\mathrm{II}} = H_s^{\mathrm{II}}, \quad P_a^{\mathrm{II}} = p_a = -i\frac{\partial}{\partial x_a},$$
$$J_a^{\mathrm{II}} = (\boldsymbol{x}\times\boldsymbol{p})_a + S_a, \quad G_a^{\mathrm{II}} = tp_a - \sigma_3 mx_a + \lambda_a^{\mathrm{II}} \tag{1.9}$$

удовлетворяли алгебре (1.2). Здесь σ_3 — одна из матриц Паули

$$\sigma_0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

I и 0 — $(2s+1)$-рядные квадратные единичная и нулевая матрицы, λ_a^{II} — некоторые операторы (в общем случае зависящие от p_a), которые нам также предстоит найти. Можно показать, что формулы (1.9) задают общий вид генераторов группы G, при котором уравнение (0.1) инвариантно относительно унитарного преобразования $\Psi(t,\boldsymbol{x}) \to r_2\Psi(-t,-\boldsymbol{x})$, $r_2 = \sigma_2$.

Потребуем, чтобы генераторы (1.9) были эрмитовы относительно обычного принятого в квантовой механике скалярного произведения

$$(\Psi_1,\Psi_2) = \int d^3x\ \Psi_1^\dagger\Psi_2. \tag{1.10}$$

Существенное отличие представления (1.4) от (1.9) состоит в том, что генераторы $H_s^{\rm I}$, $G_a^{\rm I}$ неэрмитовы относительно (1.10), но эрмитовы в гильбертовом пространстве со скалярным произведением

$$(\Psi_1,\Psi_2) = \int d^3x\ \Psi_1^\dagger\hat{M}\Psi_2, \tag{1.11}$$

где $\hat{M}$ — некоторый положительно-определенный дифференциальный оператор, или относительно индефинитной метрики, когда $\hat{M}$ в (1.11) — некоторая числовая положительно-неопределенная матрица. Явный вид $\hat{M}$ будет найден ниже. Таким образом, усложнение метрики — это та цена, которую приходится платить за локальные преобразования (1.6) волновой функции. Аналогичная ситуация имеет место и для релятивистских уравнений [1].

Потребуем, чтобы $H_s^{\rm II}$ удовлетворял условию

$$\left(H_s^{\rm II}\right) = \left(m + p^2/2m\right)^2. \tag{1.12}$$

Это эквивалентно требованию, чтобы внутренняя энергия частицы совпадала с ее массой.

Таким образом, задача нахождения галилеевски-инвариантных уравнений вида (0.1) сводится к решению системы соотношений (1.2) для операторов (1.4) и (1.9).

2. Явный вид гамильтонианов $H_s^{\rm I}$

Решение задачи I приведем в виде следующей теоремы.

Теорема 1. *Все возможные (с точностью до эквивалентности) гамильтонианы $H_s^{\rm I}$, удовлетворяющие совместно с генераторами* (1.4) *коммутационным соотношениям* (1.2), (1.4)*, задаются формулами*

$$H_s^{\rm I} = \sigma_3\eta m - 2i\eta k\sigma_1\boldsymbol{S}\cdot\boldsymbol{p} + \frac{1}{2m}C_{ab}p_ap_b, \quad a,b = 1,2,3, \tag{2.1а}$$

$$\tilde{H}_s^{\rm I} = \sigma_1\tilde{\eta}m + \frac{p^2}{2m} - 2\eta k(\sigma_2 - i\sigma_3)\boldsymbol{S}\cdot\boldsymbol{p}, \tag{2.1б}$$

где $C_{ab} = \delta_{ab} - 2\eta k^2(\sigma_3 + i\sigma_2)(S_aS_b + S_bS_a)$, η, k и $\tilde{k}$ — произвольные параметры.

Доказательство. Определим сначала явный вид матриц $\lambda_a^{\rm I}$ из (1.4). Из (1.2б) получаем для $\lambda_a^{\rm I}$ следующие уравнения:

$$[\lambda_a^{\rm I},\lambda_b^{\rm I}] = 0, \quad [\lambda_a^{\rm I}, S_b] = i\varepsilon_{abc}\lambda_c^{\rm I}, \quad [S_a,S_b] = i\varepsilon_{abc}S_c. \tag{2.2}$$

Из (1.5), (2.2) заключаем, что матрицы $\lambda_a^{\rm I}$, не умаляя общности, можно представить в форме

$$\lambda_a^{\rm I} = k(\sigma_3 + i\sigma_2)S_a, \tag{2.3}$$

где k — произвольный коэффициент.

Найдем общий вид гамильтониана $H_s^{\rm I}$ в представлении, где $\lambda_a^{\rm I} = 0$. Переход к такому представлению осуществляется с помощью оператора [7]

$$V = \exp\left(i\boldsymbol{\lambda}^{\rm I}\cdot\boldsymbol{p}/m\right) = 1 + i\boldsymbol{\lambda}^{\rm I}\cdot\boldsymbol{p}/m. \tag{2.4}$$

Используя (2.4), получаем

$$\begin{aligned}&\left(H_s^{\rm I}\right)' = VH_s^{\rm I}V^{-1}, \quad \left(P_a^{\rm I}\right) = VP_a^{\rm I}V^{-1} = p_a,\\ &J_a' = VJ_aV^{-1} = J_a, \quad \left(G_a^{\rm I}\right) = VG_a^{\rm I}V^{-1} = tp_a - mx_a.\end{aligned} \tag{2.5}$$

Из (2.5), (1.2) заключаем, что общий вид оператора $\left(H_s^{\rm I}\right)'$ задается формулой

$$\left(H_s^{\rm I}\right)' = p^2/2m + A, \quad A = \sigma_\mu a^\mu m, \tag{2.6}$$

где a_μ — произвольные коэффициенты, причем, не умаляя общности, можно положить $a_0 = 0$.

Можно показать, что с помощью преобразований, не изменяющих общего вида $\lambda_a^{\rm I}$ (2.3), матрица A (2.6) сводится к одной из следующих форм:

$$A = \sigma_3\eta m \quad \text{или} \quad A = \sigma_1\tilde{\eta}m. \tag{2.7}$$

Подставив (2.7) в (2.6), с помощью преобразования, обратного (2.5), прийдем к формулам (2.1). Теорема доказана.

Формулы (2.1) задают нерелятивистские гамильтонианы для частиц с произвольным спином. В случае $s = 1/2$, $k = -i$, $\eta = 1$ уравнение (0.1), (2.1a) может быть записано в компактной форме

$$(\gamma_\mu p^\mu - m)\,\Psi = (1 + \gamma_4 - \gamma_0)\,\frac{p^2}{2m}\Psi, \tag{2.8}$$

где $\gamma_0 = \sigma_3$, $\gamma_a = -2i\sigma_2 S_a$, $\gamma_4 = i\gamma_0\gamma_1\gamma_2\gamma_3$ — матрицы Дирака.

Отметим, что все гамильтонианы (2.1) принадлежат классу дифференциальных операторов второго порядка, что априори не требовалось. В рамках группы Пуанкаре гамильтонианы частицы с произвольным спином бывают, как правило, интегро-дифференциальными операторами [1, 5].

Параметры k, η и $\tilde{\eta}$ всегда можно выбрать такими, чтобы уравнения (0.1), (2.1) были инвариантны относительно антиунитарной операции отражения координат и времени (1.8). Необходимым и достаточным условием такой инвариантности является одновременное выполнение соотношений

$$\eta^* = \pm\eta, \quad k^* = \pm k \quad \text{или} \quad \tilde{\eta}^* = \tilde{\eta}, \quad k^* = k, \tag{2.9}$$

при этом $r_1 = \sigma_1\Delta$, если $\eta^* = -\eta$, $k^* = -k$ или $\tilde{\eta}^* = \eta$, $k^* = k$, $r_1 = \Delta$, если $\eta^* = \eta$, $k^* = k$, $\Delta = \begin{pmatrix}\Delta' & 0\\ 0 & \Delta'\end{pmatrix}$, где Δ' — матрицы, определяемые с точностью до фазы соотношениями [8]

$$\Delta' s_a = -s_a^*\Delta', \quad (\Delta')^2 = (-1)^{2s}.$$

Таким образом, при ограничениях на параметры η, $\tilde{\eta}$ и k, задаваемых формулами (2.9), уравнения (0.1), (2.1) инвариантны относительно расширенной группы Галилея, включающей преобразования (1.8).

Гамильтонианы (2.1) и операторы (1.4), (2.3) неэрмитовы в скалярном произведении (1.10). Однако эти операторы эрмитовы в метрике (1.11), где $\hat{M}$ — положительно-определенный оператор

$$\begin{aligned}\hat{M} = \left(V^{-1}\right)^{+} V^{-1} = 1 + [i(k-k^*)\sigma_3 - (k+k^*)\sigma_2]\boldsymbol{S}\cdot\boldsymbol{p}/m +\\ + 2(k^*k)(1+\sigma_1)(\boldsymbol{S}\cdot\boldsymbol{p})^2/m^2.\end{aligned} \tag{2.10}$$

Кроме того, если η, k и $\tilde{\eta}$ удовлетворяют условиям (2.9), гамильтонианы (2.1) эрмитовы в индефинитной метрике вида (1.11), когда

$$\hat{M} = \xi = \begin{cases} \sigma_3, & \text{если } \eta^* = \eta,\ k^* = k,\ \tilde{\eta}^* = -\eta, \\ \sigma_2, & \text{если } \eta^* = -\eta,\ k^* = -k,\ \tilde{\eta}^* = -\tilde{\eta}. \end{cases} \tag{2.11}$$

Если выполняется (2.11), то уравнения (0.1), (2.1) могут быть получены с помощью вариационного принципа. Соответствующие лагранжианы имеют вид

$$\begin{aligned}L(t,\boldsymbol{x}) = \frac{i}{2}\left(\bar{\Psi}\frac{\partial\Psi}{\partial t} - \frac{\partial\bar{\Psi}}{\partial t}\Psi\right) - \eta m\bar{\Psi}\sigma_3\Psi -\\ - \eta k\left(\bar{\Psi}\sigma_1 S_a\frac{\partial\Psi}{\partial x_a} - \frac{\partial\bar{\Psi}}{\partial x_a}\sigma_1 S_a\Psi\right) - \frac{1}{2m}\frac{\partial\bar{\Psi}}{\partial x_a}C_{ab}\frac{\partial\Psi}{\partial x_b},\end{aligned} \tag{2.12а}$$

когда H_s^{I} задается формулой (2.1а), и

$$\begin{aligned}L(t,\boldsymbol{x}) = \frac{i}{2}\left\{\bar{\Psi}\frac{\partial\Psi}{\partial t} - \frac{\partial\bar{\Psi}}{\partial t}\Psi - 2i\tilde{\eta}m\bar{\Psi}\sigma_1\Psi +\right.\\ \left. + 2\tilde{\eta}k\left[\bar{\Psi}(\sigma_2 - i\sigma_3)S_a\frac{\partial\Psi}{\partial x_a} - \frac{\partial\bar{\Psi}}{\partial x_a}(\sigma_2 - i\sigma_3)S_a\Psi\right]\right\} - \frac{1}{2m}\frac{\partial\bar{\Psi}}{\partial x_a}\frac{\partial\Psi}{\partial x_b},\end{aligned} \tag{2.12б}$$

если гамильтониан имеет вид (2.16). Здесь $\bar{\Psi} = \Psi^\dagger\xi$.

Лагранжианы (2.12) являются скалярами относительно преобразований (1.1), (1.6), где

$$D^s(R_{ab}, V_a) = \left(1 + i\boldsymbol{\lambda}^{\mathrm{I}}\cdot\boldsymbol{v}\right) D^s(R_{ab}), \tag{2.13}$$

где $D^s(R_{ab})$ — матрицы, реализующие прямую сумму двух неприводимых представлений $D(s)\oplus D(s)$ группы $SO(3)$.

3. Явный вид гамильтонианов H_s^{II}

Решим задачу II, т.е. найдем дифференциальные операторы, удовлетворяющие совместно с (1.9) соотношениям (1.2), (1.12).

Теорема 2. *Все возможные (с точностью до преобразований эквивалентности) дифференциальные операторы* $H_s^{\rm II}$*, которые эрмитовы в метрике* (1.10) *и удовлетворяют условиям* (1.2), (1.9), (1.12), *задаются формулами*

$$H_s^{\rm II}=\sigma_3\left[m+\frac{p^2}{m}-\frac{(S_aS_b+S_bS_a)p_ap_b}{2ms^2}\sin^2\theta_s\right]+ \\ +\sigma_2\sqrt{2}\sin\theta_s\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{s}+\sigma_1\left[a_s\frac{p^2}{2m}+\frac{b_s}{4ms^2}(S_aS_b+S_bS_a)p_ap_b\right], \tag{3.1}$$

где

$$a_{1/2}=\sin 2\theta_{1/2},\quad b_{1/2}=0,\quad a_1=1,\quad b_1=\sin 2\theta_1,$$
$$a_{3/2}=b_{3/2}-\frac{5}{4}\sin 2\theta_{3/2}=-\frac{1}{8}\sin 2\theta_{3/2}-\frac{3}{4}\sin\theta_{3/2}\left(1-\frac{1}{9}\sin^2\theta_{3/2}\right)^{1/2},$$
$$a_s=b_s=\theta_s=0,\quad s>3/2,$$

а $\theta_{1/2}$, θ_1, $\theta_{3/2}$ *— произвольные действительные параметры.*

Доказательство. Прежде всего покажем, что операторы $H_s^{\rm II}$ могут включать производные не выше второго порядка. Действительно, пусть $H_s^{\rm II}=\sum\limits_{i=0}^{N}H_i$, где H_i содержит производные только i-го порядка, тогда из (1.12) получаем

$$H_NH_N=H_N^+H_N=0\quad\text{или}\quad H_N=0,\quad\text{если}\quad 2<N<\infty. \tag{3.2}$$

Представим искомые дифференциальные операторы $H_s^{\rm II}$ в виде разложения по спиновым матрицам и $2(2s+1)$-рядным матрицам Паули (1.9):

$$H_s^{\rm II}=\sum_{\mu=0}^{s}\left[a_\mu^s m+b_\mu^s\frac{p^2}{2m}+c_\mu^s\boldsymbol{S}\cdot\boldsymbol{p}+d_\mu^s\frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{2m}\right]\sigma_\mu, \tag{3.3}$$

где a_μ^s, b_μ^s, c_μ^s, d_μ^s — произвольные действительные коэффициенты. Используя операторы ортогонального проектирования [1, 5]

$$\Lambda_r=\prod_{r'\neq r}\frac{(\boldsymbol{S}\cdot\boldsymbol{p})p^{-1}-r'}{r-r'},\quad r,r'=-s,-s+1,\dots,s,$$
$$\Lambda_r\cdot\Lambda_{r'}=\delta_{rr'},\quad \sum_r\Lambda_r=1,\quad \sum_r r^l\Lambda_r=\left(\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p}\right)^l,$$

$H_s^{\rm II}$ можно переписать в виде

$$H_s^{\rm II}=\sum_{\mu=0}^{3}\sum_{r=-s}^{s}\left[a_\mu^s m+\left(b_\mu^s+r^2d_\mu^s\right)\frac{p^2}{2m}+rpc_\mu^2\right]\sigma_\mu\Lambda_r. \tag{3.4}$$

Операторы (3.4), очевидно, удовлетворяют условиям (1.2г), (1.10). Потребуем, чтобы выполнялось условие (1.12). Подставив (3.4) в (1.12), используя ортогональность операторов Λ_r и приравнивая независимые слагаемые, получаем, что a^s_μ, b^s_μ, c^s_μ, d^s_μ должны удовлетворять одной из следующих систем уравнений:

$$\begin{aligned}
&\sum_{i=1}^{3}(a_i^s)^2=0, \quad \sum_{i=1}^{3}\left[r^2\left(c_i^s\right)^2+a_i^s\left(b_i^s+r^2d_i^s\right)\right]=1,\\
&\sum_{i=1}^{r}rc_i^s\left(b_i^s+r^2d_i^s\right)=0, \quad \sum_{i=1}^{3}rc_i^sa_i^s=0, \quad \sum_{i=1}^{3}\left(b_i^s+r^2d_i^s\right)^2=1,\\
&a_0^s=b_0^s=d_0^s=c_0^s=0
\end{aligned} \tag{3.5}$$

или

$$a_0^s=b_0^s=1, \quad d_0^s=c_0^s=a_i^s=c_i^s=d_i^s=0, \quad i=1,2,3. \tag{3.6}$$

Общее решение уравнений (3.5), (3.4) (с точностью до преобразований эквивалентности, осуществляемых числовыми матрицами) и задается формулами (3.1). Можно показать, что решение (3.6) несовместно с (1.2а), (1.2б) и (1.2д).

Для завершения доказательства теоремы достаточно теперь указать явный вид операторов λ_a^{II}, при котором операторы (1.8) удовлетворяют соотношениям (1.2б), (1.2д). Нетрудно убедиться, что λ_a^{II} можно выбрать в форме

$$\lambda_a^{\mathrm{II}}=[U,\sigma_3x_a]U^+, \tag{3.7}$$

где

$$U=\left(E+\sigma_3H_s^{\mathrm{II}}\right)\Big/\sqrt{2E\left(E+\frac{1}{2}H_s^{\mathrm{II}}\sigma_3+\frac{1}{2}\sigma_3H_s^{\mathrm{II}}\right)}, \quad E=m+p^2/2m, \tag{3.8}$$

— оператор, диагонализующий гамильтонианы (3.1) и генераторы (1.8):

$$U^\dagger H_s^{\mathrm{II}}U=\sigma_3E, \quad U^\dagger G_aU=tp_a-\sigma_3mx_a. \tag{3.9}$$

Теорема доказана.

В случае $\theta_{1/2}=\pi/4$ уравнение (0.1), (3.1а) принимает особо простой вид (ср. (2.8)):

$$\left(\gamma_\mu p^\mu+m\right)\Psi=i\gamma_4\frac{p^2}{2m}\Psi. \tag{3.10}$$

Уравнение (3.10) отличается от релятивистского уравнения Дирака только наличием слагаемого в правой части, которое, очевидно, нарушает инвариантность относительно группы Пуанкаре, но сохраняет инвариантность относительно группы Галилея.

4. Нерелятивистская частица во внешнем электромагнитном поле

Для того, чтобы перейти к описанию движения заряженной частицы во внешнем электромагнитном поле, сделаем в уравнении (0.1) обычную замену

$$p_\mu\to\pi_\mu=p_\mu-eA_\mu, \tag{4.1}$$

где A_μ — четырехвектор-потенциал внешнего поля. В результате приходим к уравнениям

$$i\frac{\partial}{\partial t}\Psi(t,\boldsymbol{x}) = H_s^\alpha(\boldsymbol{\pi}, A_0)\Psi(t,\boldsymbol{x}), \quad \alpha = \mathrm{I}, \mathrm{II}, \tag{4.2}$$

где $H_s^\alpha(\boldsymbol{\pi}, A_0)$ — один из гамильтонианов, полученных из (2.1), (3.1) заменой (4.1):

$$\begin{aligned} H_s^{\mathrm{I}}(\boldsymbol{\pi}, A_0) &= \sigma_3\eta m + \frac{\boldsymbol{\pi}^2}{2m} - 2i\eta k\sigma_1\boldsymbol{S}\cdot\boldsymbol{\pi} + eA_0 - \\ &- (\sigma_3 + i\sigma_2)\frac{\eta k^2}{m}\left[(\boldsymbol{S}\cdot\boldsymbol{\pi})^2 + \frac{1}{2}e\boldsymbol{S}\cdot\boldsymbol{H}\right], \end{aligned} \tag{4.3а}$$

$$\tilde{H}_s^{\mathrm{I}}(\boldsymbol{\pi}, A_0) = \sigma_3\tilde{\eta}m + \frac{\boldsymbol{\pi}^2}{2m} - 2\eta k(\sigma_2 - i\sigma_3)\boldsymbol{S}\cdot\boldsymbol{\pi} + eA_0, \tag{4.3б}$$

$$\begin{aligned} H_s^{\mathrm{II}}(\boldsymbol{\pi}, A_0) &= \sigma_3\left[m + \frac{\boldsymbol{\pi}^2}{2m} - \frac{(\boldsymbol{S}\cdot\boldsymbol{\pi})^2}{ms^2}\sin^2\theta_s - e\frac{\boldsymbol{S}\cdot\boldsymbol{H}}{2ms^2}\sin^2\theta_s\right] + \\ &+ \sigma_1\left[a_s\frac{\boldsymbol{\pi}^2}{2m} + b_s\frac{(\boldsymbol{S}\cdot\boldsymbol{\pi})^2}{2ms^2} + eb_s\frac{\boldsymbol{S}\cdot\boldsymbol{H}}{4ms^2}\right] + \sigma_2\sqrt{2}\frac{\boldsymbol{S}\cdot\boldsymbol{\pi}}{s}\sin\theta_s + eA_0. \end{aligned} \tag{4.3в}$$

В этих формулах $H_a = -i\varepsilon_{abc}\pi_b\pi_c$ — напряженность магнитного поля.

Уравнения (4.2), (4.3), очевидно, инвариантны относительно калибровочных преобразований. Кроме того, как и до введения взаимодействия уравнения (4.3) с гамильтонианами (4.3а), (4.3б) инвариантны относительно преобразований из группы Галилея (1.6), (2.13), если вектор-потенциал преобразуется по закону [3]

$$A_b \to A'_b = R_{bc}A_c, \quad A_0 \to A'_0 = A_0 + v_aA_a. \tag{4.4}$$

Анализ уравнений (4.2) удобно производить в представлении, в котором операторы (4.3) квазидиагональны (т.е. коммутируют с одной из σ-матриц). Как и в случае уравнения Дирака, гамильтонианы (4.3) могут быть диагонализованы только приближенно. Ниже осуществим такую диагонализацию и представим гамильтониан частицы с произвольным спином в виде ряда по степеням $1/m$, удобном для вычислений с использованием теории возмущений.

Диагонализация гамильтонианов (4.3) с точностью до членов порядка $1/m^2$ осуществляется с помощью операторов

$$\begin{aligned} V^\alpha &= \exp\left(iC_s^\alpha + \sigma_3\frac{1}{2\eta^\alpha m}\frac{\partial B_s^\alpha}{\partial t}\right)\exp\left(iB_s^\alpha\right)\exp\left(iA_s^\alpha\right), \quad \alpha = \mathrm{I},\ \mathrm{II}, \\ \tilde{V}^\alpha &= \exp\left(i\tilde{C}_s^\alpha\right)\exp\left(i\tilde{B}_s^\alpha\right)\exp\left(i\tilde{A}_s^\alpha\right), \end{aligned} \tag{4.5}$$

где

$$A_s^{\mathrm{I}} = -i\sigma_2 k\frac{\boldsymbol{S}\cdot\boldsymbol{\pi}}{m}, \quad A_s^{\mathrm{II}} = -\sigma_1\frac{\sqrt{2}\sin\theta_s}{2ms}\boldsymbol{S}\cdot\boldsymbol{\pi}, \quad \eta^{\mathrm{I}} = \eta, \quad \eta^{\mathrm{II}} = 1,$$

$$B_s^{\mathrm{I}} = \sigma_1\frac{k}{2m^2}\left\{\frac{1}{2\eta}[\boldsymbol{S}\cdot\boldsymbol{\pi}, \boldsymbol{\pi}^2] + ik[2(\boldsymbol{S}\cdot\boldsymbol{\pi})^2 + e\boldsymbol{S}\cdot\boldsymbol{H}] + \frac{e}{\eta}\boldsymbol{S}\cdot\boldsymbol{E}\right\},$$

$$C_s^{\mathrm{I}} = \sigma_2 \frac{k^2}{m^2} \left\{ -\frac{2ik}{3} (\boldsymbol{S} \cdot \boldsymbol{\pi})^2 + iek[\boldsymbol{S} \cdot \boldsymbol{\pi}, \boldsymbol{S} \cdot \boldsymbol{H}]_+ + [(\boldsymbol{S} \cdot \boldsymbol{\pi})^2, eA_0] \right\},$$

$$B_s^{\mathrm{II}} = \sigma_2 \frac{1}{4m^2} \left\{ a_s \boldsymbol{\pi}^2 + \frac{b_s}{2s^2} [2(\boldsymbol{S} \cdot \boldsymbol{\pi})^2 + e\boldsymbol{S} \cdot \boldsymbol{H}] + \frac{e\sqrt{2} \sin\theta_s}{s} \boldsymbol{S} \cdot \boldsymbol{E} \right\},$$

$$C_s^{\mathrm{II}} = \sigma_1 \frac{1}{8m^3} \left\{ \frac{\sqrt{2}\sin\theta_s}{s} \left[\boldsymbol{S} \cdot \boldsymbol{\pi}, \boldsymbol{\pi}^2 - \frac{e \sin^2\theta_s}{s^2} \boldsymbol{S} \cdot \boldsymbol{H} \right]_+ - \right.$$

$$\left. - \frac{4\sqrt{2}\sin^3\theta_s}{s^3} (\boldsymbol{S} \cdot \boldsymbol{\pi})^3 - iea_s[\boldsymbol{\pi}^2, A_0] - \frac{ieb_s}{s^2} [(\boldsymbol{S} \cdot \boldsymbol{\pi})^2, A_0] \right\},$$

$$\tilde{A}_s^{\mathrm{I}} = -ik(\sigma_2 - i\sigma_3) \frac{\boldsymbol{S} \cdot \boldsymbol{\pi}}{m}, \quad \tilde{B}_s^{\mathrm{I}} = \frac{k}{2\eta m^2} (\sigma_2 - i\sigma_3) \boldsymbol{S} \cdot \boldsymbol{E},$$

$$\tilde{C}_s^{\mathrm{I}} = -\frac{ik}{4\tilde{\eta} m^3} (\sigma_2 - i\sigma_3)[\boldsymbol{\pi}^2, \boldsymbol{S} \cdot \boldsymbol{\pi}] - \frac{i}{2\tilde{\eta} m} \frac{\partial \tilde{B}_s^{\mathrm{I}}}{\partial t}.$$

Непосредственным вычислением получаем

$$\begin{gathered} [H_s^\alpha(\boldsymbol{\pi}, A_0)]' = V^\alpha H_s^\alpha(\boldsymbol{\pi}, A_0)(V^\alpha)^{-1} = A^\alpha m + B^\alpha \left(\frac{\boldsymbol{\pi}^2}{2m} + eA_0 \right) + \\ + \sigma_3 e C^\alpha \frac{\boldsymbol{S} \cdot \boldsymbol{H}}{m} + \frac{e}{4m^2} D^\alpha \boldsymbol{S} \cdot (\boldsymbol{\pi} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{\pi}) + \frac{e}{6m^2} F^\alpha s(s+1) \operatorname{div} \boldsymbol{E} + \\ + \frac{1}{12m^2} G^\alpha Q_{ab} \frac{\partial E_a}{\partial x_b} + \frac{n^\alpha}{m^2} \boldsymbol{S} \cdot (\boldsymbol{\pi} \times \boldsymbol{H} - \boldsymbol{H} \times \boldsymbol{\pi}) + \frac{L^\alpha e}{m^2} Q_{ab} \frac{\partial H_a}{\partial x_b} + o\left(\frac{1}{m^3} \right), \\ \left[\tilde{H}_s^{\mathrm{I}}(\boldsymbol{\pi}, A_0) \right]' = \tilde{V}^{\mathrm{I}} \tilde{H}_s^{\mathrm{I}} \left(\tilde{V}^{\mathrm{I}} \right)^{-1} = \sigma_3 \tilde{\eta} m + \frac{\boldsymbol{\pi}^2}{2m} + eA_0 + o\left(\frac{1}{m^3} \right). \end{gathered} \tag{4.6}$$

где

$$\begin{gathered} A^{\mathrm{I}} = \sigma_3 \eta, \quad B^{\mathrm{I}} = 1, \quad C^{\mathrm{I}} = -\eta k^2, \\ -D^{\mathrm{I}} = F^{\mathrm{I}} = G^{\mathrm{I}} = k^2, \quad n^{\mathrm{I}} = -3L^{\mathrm{I}} = \eta k^3, \end{gathered} \tag{4.7а}$$

$$\begin{gathered} A^{\mathrm{II}} = B^{\mathrm{II}} = \sigma_3, \quad -C^{\mathrm{II}} = D^{\mathrm{II}} = -F^{\mathrm{II}} = -G^{\mathrm{II}} = \frac{\sin^2\theta_s}{2s^2}, \\ n^{\mathrm{II}} = \frac{\sqrt{2}\sin\theta_s}{2s} \left(-a_s + \frac{b_s}{4s^2} \right), \quad L^{\mathrm{II}} = \frac{\sqrt{2} b_s \sin\theta_s}{24s^2}, \\ Q_{ab} = (e/2)\{3[S_a, S_b]_+ - 2\delta_{ab} s(s+1)\}. \end{gathered} \tag{4.7б}$$

Операторы (4.6), (4.7) содержат слагаемые, соответствующие дипольному ($\sim \boldsymbol{S} \cdot \boldsymbol{H}$), спин-орбитальному ($\sim\ \boldsymbol{S} \cdot (\boldsymbol{\pi} \times \boldsymbol{E} - \boldsymbol{E} \times \boldsymbol{\pi})$), квадрупольному ($\sim\ Q_{ab}\partial E_a/\partial x_b$) и дарвиновскому ($\sim\ \operatorname{div} \boldsymbol{E}$) взаимодействиям частицы с полем. Два последних слагаемых в (4.6), (4.7) можно интерпретировать как магнитное спин-орбитальное и магнитное квадрупольное взаимодействия. Аналогичную структуру имеют приближенные гамильтонианы, полученные из релятивистских уравнений [5, 6]. В случае $s = 1/2$, $\eta = 1$, $k^2 = -1$, $\theta_s = \pi/4$ семь первых слагаемых в (4.6), (4.7) совпадают с гамильтонианом Фолди–Вуйтхойзена [9], полученным при диагонализации уравнения Дирака. Таким образом, в приближении $1/m^2$ нерелятивистские уравнения (4.2), (4.6), (4.7) описывают движение частицы со спином $s = 1/2$ во внешнем электромагнитном поле с той же точностью, что и релятивистское уравнение Дирака.

Отметим, что для некоторых классов внешних полей уравнения (4.2) могут быть решены точно. Приведем без доказательств собственные значения гамильтониана (4.3б) для частицы со спином, взаимодействующей с постоянным однородным магнитным полем [10]

$$H^{\rm II}_{1/2}(\boldsymbol{\pi}, A_0)\Psi_{\varepsilon s_3 n p_3} = E_{\varepsilon s_3 n p_3}\Psi_{\varepsilon s_3 n p_3},$$
$$E_{\varepsilon s_3 n p_3} = \varepsilon\left\{m^2 + \xi^2 + p_3^2 + \frac{\left(\xi^2 + p_3^2\right)^2}{4m^2} + \left(\frac{eH_3}{2m}\right)^2 + \right.$$
$$\left. + \varepsilon\frac{eH_3}{m}\left[m^2\cos^2 2\theta_{1/2} + \xi^2 + \frac{\left(\xi^2 + p_3^2\right)^2}{4m^2}\right]^{1/2}\right\}^{1/2},$$

где $\xi^2 = (2n+1)eH_3$, $H_1 = H_2 = 0$, $n = 0, 1, 2, \ldots$, $\varepsilon = \pm 1$, $s_3 = \pm 1/2$.

5. Уравнения первого порядка

Остановимся вкратце на задаче описания галилеевски-инвариантных дифференциальных уравнений вида

$$F\Psi = 0, \quad F = \beta_\mu p^\mu + \beta_5 m, \quad p_\mu - i\partial/\partial x_\mu, \tag{5.1}$$

где β_μ, β_5 — некоторые числовые матрицы.

Уравнение (5.1) по определению инвариантно относительно группы Галилея, если выполняются соотношения

$$[F, Q_A] = f_A F, \quad A = 1, 2, \ldots, 10, \tag{5.2}$$

где через Q_A обозначен произвольный генератор группы $G : \{Q_A\} = \{P_0, P_a, G_a, J_a\}$, f_A — некоторые операторы, определенные на множестве решений уравнения (5.1).

Полагая $f_A \equiv 0$ и выбирая генераторы P_μ, J_a, G_a в форме (1.4), где S_a и λ_a — произвольные матрицы (что соответствует локальным преобразованиям Галилея (1.6) для функции Ψ), получаем из (5.2) следующую систему перестановочных соотношений для матриц β_μ, β_4, λ_a, S_a:

$$\begin{gathered} [S_a, \beta_5] = [S_a, \beta_0] = 0, \quad [S_a, \beta_b] = i\varepsilon_{abc}\beta_c, \\ [\lambda_a, \beta_5] = i\beta_a, \quad [\lambda_a, \beta_b] = i\delta_{ab}\beta_0, \quad [\lambda_a, \beta_0] = 0, \end{gathered} \tag{5.3}$$

где λ_a, S_a — матрицы, удовлетворяющие соотношениям (2.2).

Таким образом, задача описания галилеевски-инвариантных уравнений вида (5.1) сводится в нашей постановке к нахождению матриц S_a, λ_a, β_5, β_a, удовлетворяющих условиям (2.2), (5.3).

Приведем частное решение системы (2.2), (5.3), позволяющее получить уравнения вида (5.1) для нерелятивистских частиц произвольного спина. Обозначим через S_{kl}, $k,l = 1,2,3,4,5,6$, генераторы неприводимого представления группы $SO(6)$. Тогда матрицы

$$S_a = \frac{1}{2}\varepsilon_{abc}S_{bc}, \quad \lambda_a = \frac{1}{2}(iS_{6a} + S_{5a}), \quad a = 1,2,3,$$
$$\beta_a = 2S_{4a}, \quad \beta_0 = iS_{46} + S_{45}, \quad \beta_5 = 2(I + iS_{46} - S_{45}) \tag{5.4}$$

удовлетворяют коммутационным соотношениям (2.2), (5.3), т.е. формулы (5.4) дают решение поставленной задачи.

Полагая в (5.1), (5.4) $S_{kl} = (i/4)[\gamma_k, \gamma_l]$, $S_{6k} = \frac{1}{2}\gamma_k$, где γ_k — эрмитовы четырехрядные матрицы Дирака, получаем уравнение, эквивалентное уравнению Леви–Леблонда [3] для частицы со спином $s = 1/2$. Выбирая иные представления алгебры Ли группы $SO(6)$, получаем из (5.1), (5.4) уравнения для частиц с другими значениями спина.

Уравнения (5.1), (5.4), как и уравнения второго порядка, рассмотренные выше, позволяют описать спин-орбитальное взаимодействие частицы с внешним полем. Например, полагая $S_{kl} = i[\hat{\beta}_k, \hat{\beta}_l]$, $S_{6k} = \hat{\beta}_k$, где $\hat{\beta}_k$ — десятирядные матрицы Кеммера–Деффина–Петье (которые могут быть выбраны, скажем, в форме, приведенной в монографии [12]), и делая в (5.1) замену $p_\mu \to \pi_\mu$, где $\pi_a = p_a$, $\pi_0 = p_0 - eA_0$, получаем после несложных, но несколько громоздких вычислений, уравнение для трехкомпонентной волновой функции $\Psi^{(3)}$

$$i\frac{\partial}{\partial t}\Psi^{(3)} = H\Psi^{(3)}, \quad H = m + \frac{\boldsymbol{\pi}^2}{2m} + eA_0 + e\frac{\boldsymbol{S}\cdot\boldsymbol{E}}{2m}, \tag{5.5}$$

где S_a — спиновые матрицы для $s = 1$. Посредством преобразования $H \to H' = VHV^{-1}$, где $V = \exp(i\boldsymbol{S}\cdot\boldsymbol{\pi}/m)$, гамильтониан (5.5) приводится к форме, аналогичной (4.6),

$$H' = \frac{\boldsymbol{\pi}^2}{2m} + eA_0 - \frac{1}{8m^2}\Big[\boldsymbol{S}\cdot(\boldsymbol{\pi}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{\pi}) - \\ - \frac{1}{3}Q_{ab}\frac{\partial E_a}{\partial x_b} - \frac{4}{3}\,\mathrm{div}\,\boldsymbol{E}\Big] + o\left(\frac{1}{m^3}\right). \tag{5.6}$$

Оператор (5.6), как и (4.6), (4.7), содержит слагаемые, описывающие дарвиновское, спин-орбитальное и квадрупольное взаимодействия частицы с внешним электрическим полем.

6. Заключительные замечания

1. Выше получены системы дифференциальных уравнений первого и второго порядков, которые инвариантны относительно преобразований Галилея и калибровочных преобразований и описывают дипольное, квадрупольное, спин-орбитальное и дарвиновское взаимодействия частицы произвольного спина с внешним электромагнитным полем. Перечисленные взаимодействия, таким образом, не являются чисто релятивистскими эффектами и могут последовательно рассматриваться в рамках нерелятивистской квантовой механики (см. также [10, 11]).

2. Уравнения (2.8), (3.10) имеют такую структуру, что левая часть их совпадает с релятивистским уравнением Дирака, а в правой части содержатся члены, которые нарушают симметрию относительно группы Пуанкаре и обеспечивают инвариантность уравнения относительно группы Галилея. Такой способ нарушения пуанкаре-симметрии является одним из возможных подходов для получения галилеевски-инвариантных уравнений движения частиц с произвольным спином. Так, исходя из релятивистских уравнений без лишних компонент, выведенных в [1, 6], с помощью добавления членов, нарушающих пуанкаре-инвариантность, но сохраняющих симметрию относительно группы $E(3)$, можно получить уравнения (1.2), (2.1а).

3. Уравнения вида (0.1) и (5.1), конечно, не исчерпывают всех возможных линейных дифференциальных уравнений, инвариантных относительно группы Галилея. Например, для описания движения нерелятивистской частицы со спином $s = 1$ можно использовать галилеевски-инвариантный аналог уравнений Прока

$$\left(2mp_0 - p^2\right)\Psi_\nu = 0, \quad \nu = 0, 1, 2, 3, \quad m\Psi_0 - p_a\Psi_a, \quad a = 1, 2, 3.$$

4. Неэрмитовость генераторов (1.4) относительно обычного скалярного произведения (1.10) обусловлена неэрмитовостью конечномерных представлений алгебры (2.2) (которая изоморфна алгебре Ли группы Евклида $E(3)$). Аналогичная ситуация имеет место и в релятивистской теории, где на решениях конечномерных по спиновым индексам уравнений движения всегда реализуются неунитарные представления однородной группы Лоренца, а требование унитарности этих представлений приводит к бесконечно компонентным уравнениям. Поэтому представляет интерес рассмотреть бесконечно компонентные уравнения, инвариантные относительно группы Галилея. Приведем пример таких уравнений.

Обозначим через $\tilde{S}_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3, 4, 5)$ генераторы унитарного бесконечномерного представления группы $O(1, 5)$. Тогда уравнение в форме (5.1), (5.4), где $S_{kl} = \tilde{S}_{kl}$ $(k, l = 1, 2, 3, 4, 5)$, $S_{6k} = i\tilde{S}_{0k}$, инвариантно относительно группы Галилея.

1. Фущич В.И., Никитин А.Г., *ЭЧАЯ*, 1978, **9**, 501.
2. Bargman V., *Ann. Math.*, 1954, **59**, 1;
   Hammermesh M., *Ann. Phys.*, 1960, **9**, 518.
3. Levi-Leblond J.-M., *Commun. Math. Phys.*, 1967, **6**, 286.
4. Hurley W.J., *Phys. Rev. D*, 1974, **7**, 1185.
5. Фущич В.И., Грищенко А.Л., Никитин А.Г., *ТМФ*, 1971, **8**, 192.
6. Никитин А.Г., Фущич В.И., *ТМФ*, 1978, **34**, 319.
7. Никитин А.Г., Салогуб В.А., *УФЖ*, 1975, **20**, 1730.
8. Foldy L.L., *Phys. Rev.*, 1956, **102**, 568.
9. Foldy L.L., Wouthuysen S.A., *Phys. Rev.*, 1950, **68**, 29.
10. Fushchych W.I., Nikitin A.G., Salogub V.A., *Lett. Nuovo Cim.*, 1975, **14**, 483;
    Fushchych W.I., Nikitin A.G., *Lett. Nuovo Cim.*, 1976, **16**, 81.
11. Fushchych W.I., Nikitin A.G., Salogub V.A., *Rep. Math. Phys.*, 1978, **13**, 175.
12. Ахиезер А.И., Берестецкий В.Б., Квантовая электродинамика, Наука, 1969, 176.

*The complete description of a class of first order wave equations for Galilean particle with arbitrary spin was presented in [Fushchych W.I., Nikitin A.G., Symmetries of equations of quantum mechanics, New York, Allerton Press Inc., 1994, 480 p.]. For the related latest results refer to [Santos E.S., de Montigny M., Khanna F.C., Santana A.E., Galilean covariant Lagrangian models, *J. Phys. A*, 2004, **37**, N 41, 9771–9789]. *Editors' Remark.*

The Galilean relativistic principle and nonlinear partial differential equations

W.I. FUSHCHYCH, R.M. CHERNIHA

The second-order partial differential equations invariant under transformations of Galilei, rotation, scale and projection are described.

1. Introduction

From the mathematical point of view the Galilean relativistic principle (in a restricted sense) is nothing other than the requirement of the equations of motion to be invariant under the linear transformations

$$t \to t' = t, \quad x_a \to x'_a = x_a + v_a t, \quad a = 1, 2, 3,$$

v_a being transformation parameters (the inertial reference system velocity $\boldsymbol{v}$ component). These transformations form a three-parameter Lie group. In order to construct linear and nonlinear partial differential equations (PDE)

$$\mathcal{L}U(t, \boldsymbol{x}) = 0, \quad \boldsymbol{x} = (x_1, \ldots, x_n)$$

(where $\mathcal{L}$ is a linear or nonlinear operator, which is invariant under the Galilean transformations) it is also necessary to give the law of transformation for the dependent variable of $U(t, \boldsymbol{x})$. Under different transformation laws of the function $U(t, \boldsymbol{x})$ we obtain different classes of PDE.

As is well known, the linear heat equation in the $(n + 1)$-dimensional space

$$\begin{aligned} &\Delta U = \lambda U_0, \quad \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2, \quad U = U(t, \boldsymbol{x}), \\ &U_0 = U_t = \partial U/\partial t, \quad \lambda = \text{const} \end{aligned} \tag{1.1}$$

is invariant under the following transformations:

$$t \to t' = t, \quad x_a \to x'_a = x_a + v_a t, \quad a = \overline{1, n}, \tag{1.2}$$

$$U \to U' = \exp\left[-\frac{\lambda}{2} v_a \left(x_a + \frac{1}{2} v_a t\right)\right] U, \tag{1.3}$$

v_a being the transformation parameters.

(1.3) defines the transformation law for the dependent function $U(t, \boldsymbol{x})$ under the Galilean transformations (1.2).

The $\frac{1}{2}(n^2 + 3n + 6)$-dimensional algebra with the basic elements

$$G_a = t\partial_a - \frac{1}{2}\lambda x_a U \partial_U, \quad \partial_a = \partial/\partial x_a, \quad \partial_U = \partial/\partial U, \quad a = \overline{1, n}, \tag{1.4a}$$

$$J_{ab} = x_a \partial_b - x_b \partial_a, \quad a \neq b, \quad a, b = \overline{1, n}, \tag{1.4b}$$

Reprinted with permission from J. Phys. A: Math. Gen., 1985, **18**, P. 3491–3503

Original article is available at http://www.iop.org/EJ/journal/JPhysA

$$\Pi = t^2\partial_t + tx_a\partial_a - \left(\frac{1}{4}\lambda|x|^2 + \frac{1}{2}nt\right)U\partial_U, \quad |x|^2 = x_ax_a, \tag{1.4c}$$

$$D = 2t\partial_t + x_a\partial_a + kU\partial_U, \quad k = \text{const}, \tag{1.4d}$$

$$P_0 = \partial_t, \quad P_a = \partial_a \tag{1.4e}$$

(where the repeated indices imply summation) is maximal in the Lie restriction invariance algebra (IA) of (1.1).

The set of operators (1.4) forms a Lie algebra, which will be noted by the symbol $SLi(1,n)$, i.e. the special Lie algebra. This name is natural because in the previous century Lie [10] (see also [13]) was the first to calculate the maximal IA of the two-dimensional $U(t,x_1)$ heat equation. The maximal IA of the $(3+1)$-dimensional Schrödinger equation, which coincides with (1.1) (differing only by constant coefficients), was calculated by Niederer [11]. For some more details on this, see, for example, [6, 7].

From the group-theoretical point of view (1.3) defines the projective representation of the group (1.2). Apart from the projective representation (1.3) the group (1.2) has another representation, the infinitesimal operator of which

$$\widetilde{G}_a = t\partial_a, \quad a = \overline{1,n} \tag{1.5}$$

being different from the G_a operators (1.4a).

The operators (1.5) generate the following transformations:

$$t \to t' = t, \quad x_a \to x'_a = x_a + v_at, \quad U \to U' = U. \tag{1.6}$$

We call (1.2) and (1.3) the projective Galilean transformations (PGT) and (1.6) the Galilean transformations (GT).

Equation (1.1) admits operators (1.4a) but does not admit operators (1.5).

In § 2 we describe the nonlinear second-order PDE

$$F(t,x,U,U_0,\underset{I}{U},\underset{II}{U}) \equiv -\Delta U + A(t,x,U)U_t + B(t,x,U,\underset{I}{U}) = 0, \tag{1.7}$$

where

$$\underset{I}{U} = (U_1,\dots,U_n), \quad \underset{II}{U} = (U_{11},U_{12},\dots,U_{nn}),$$
$$U_a = \partial U/\partial x_a, \quad U_{ab} = \partial^2 U/\partial x_a\partial x_b, \quad a,b = \overline{1,n},$$

F, A, B being arbitrary differentiable functions, invariant under the PGT (1.2) and (1.3) as well as projective and scale transformations generated by operators (1.4c) and (1.4d).

In § 3 we construct the most general nonlinear PDE of the form

$$F(t,x,U,U_0,\underset{I}{U},U_{00},U_{01},\dots,U_{0n},\underset{II}{U}) = 0, \quad \partial^2U/\partial t\partial x_a = U_{0a} \tag{1.8}$$

which are invariant under the GT (1.6) and the translation group generated by the operators (1.4e). In particular, it is established that a set of equations of the form (1.8) does not contain linear equations (except, obviously $U_0 = 0$, $U_{00} = 0$) invariant under the GT (1.6) and the group of time and space translations.

In the final part of § 3 we give several examples of Galilei invariant equations in independent variables (t, x_1) space, for which general solutions are constructed.

It should be noted that equations of the class (1.7) are widely used to describe nonlinear diffusion, heat and other processes. In particular, this class includes diffusion equation of the form

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x_a}\left(C(U)\frac{\partial U}{\partial x_a}\right) \tag{1.9}$$

as well as nonlinear Schödinger equations (if U is a complex function) and Hamilton–Jacobi equations. The group classification of (1.9) for the one-dimensional case was carried out by Ovsyannikov [12] and for the three-dimensional case by Dorodnitsyn et al [3] and Fushchych [4].

2. Equation invariant under the projective Galilean transformations

First of all in this section we are going to find the conditions to be imposed on the functions A and B under which (1.7) is invariant under the PGT (1.2) and (1.3). The complete solution of this problem is given by the following theorem.

Theorem 1. *Equation* (1.7) *is invariant under the PGT if and only if*

$$A(t, x, U) = f(t, w), \tag{2.1}$$

$$B(t, x, U, \underset{I}{U}) = Ug(t, w, w_1, \ldots, w_n) + (f(t, w) - \lambda)\left(\frac{x_a U_a}{t} + \frac{\lambda |x|^2}{4t^2} U\right), \tag{2.2}$$

where

$$w = U \exp\left(\frac{\lambda |x|^2}{4t}\right), \quad |x|^2 = x_a x_a, \tag{2.3}$$

$$w_a = \left(U_a + \frac{\lambda x_a}{2t} U\right) \exp\left(\frac{\lambda |x|^2}{4t}\right), \quad a = \overline{1, n}, \tag{2.4}$$

and f, g are arbitrary differentiable functions.

Proof. To prove the theorem let us use the Lie method (for a modern account, see [1, 13]). According to Lie's approach, (1.7) is considered as a manifold in the space of the following variables: t, x, U, $\underset{I}{U}$, $\underset{II}{U}$. (1.7) is invariant under the transformations generated by an infinitesimal operator

$$X = \xi^\mu(t, x, U)\frac{\partial}{\partial x_\mu} + \eta(t, x, U)\frac{\partial}{\partial U}, \quad \mu = \overline{0, n}$$

when the following invariance condition is fulfilled:

$$\overset{2}{X} F = \overset{2}{X}(-\Delta U + A U_t + B)|_{F=0} = 0, \tag{2.5}$$

where $\overset{2}{X}$ is the second prolongation of the infinitesimal operator X, i.e.

$$\overset{2}{X} = X + \rho^\mu(t,x,U,\underset{I}{U})\frac{\partial}{\partial U_\mu} + \sigma^{\mu\nu}(t,x,U,\underset{I}{U},\underset{II}{U})\frac{\partial}{\partial U_{\mu\nu}}, \quad \mu,\nu=\overline{0,n}, \tag{2.6}$$

$$\rho^\mu = \eta_\mu + U_\mu\eta_U - U_i(\xi^i_\mu + U_\mu\xi^i_U), \quad i=\overline{0,n},$$

$$\begin{aligned}\sigma^{\mu\nu} &= \eta_{\mu\nu} + U_\nu\eta_{\mu U} + U_\mu\eta_{\nu U} + U_\mu U_\nu\eta_{UU} + U_{\mu\nu}\eta_U - U_i(\xi^i_{\mu\nu} + U_\nu\xi^i_{\mu U}) - \\ &- U_\mu U_i(\xi^i_{\nu U} + U_\nu\xi^i_{UU}) - U_{\mu i}(\xi^i_\nu + U_\nu\xi^i_U) - \\ &- U_{i\nu}(\xi^i_\mu + U_\mu\xi^i_U) - U_{\mu\nu}U_i\xi^i_U, \quad i=\overline{0,n}.\end{aligned}$$

Substituting (2.6) into (2.5), we obtain

$$\begin{aligned}&\left[-(\sigma^{11}+\ldots+\sigma^{nn}) + \xi^\mu\left(\frac{\partial A}{\partial x_\mu}U_0 + \frac{\partial B}{\partial x_\mu}\right) + \eta\left(\frac{\partial A}{\partial U}U_0 + \frac{\partial B}{\partial U}\right) + \right. \\ &\left.\left. + \rho^0 A + \rho^a\frac{\partial B}{\partial x_a}\right]\right|_{F=0} = 0, \quad a=\overline{1,n}.\end{aligned} \tag{2.7}$$

After explicit expressions for ρ^μ, $\sigma^{\mu\nu}$ have been substituted into (2.7) and the obtained relation being split into separate parts for coefficients at U_{0a} and U_{ab}, $a\neq b$, the conditions for ξ^μ are found:

$$\begin{aligned}&\xi^0_a \equiv \partial\xi^0/\partial x_a = 0, \quad \xi^\mu_U \equiv \partial\xi^\mu/\partial U = 0, \quad \xi^a_b + \xi^b_a = 0, \\ &a\neq b, \quad a,b=\overline{1,n}, \quad \mu=\overline{0,n}.\end{aligned} \tag{2.8}$$

After taking into account (2.8) the invariance condition, written in its complete form, is given by

$$\begin{aligned}&\left[\xi^\mu\left(\frac{\partial A}{\partial x_\mu}U_0 + \frac{\partial B}{\partial x_\mu}\right) + \eta\left(\frac{\partial A}{\partial U}U_0 + \frac{\partial B}{\partial U}\right) + (\eta_0 + \eta_U U_0 - U_\mu\xi^\mu_0)A + \right. \\ &+ (\eta_a + \eta_U U_a - U_b\xi^b_a)\frac{\partial B}{\partial U_a} - \Delta\eta - U_aU_a\eta_{UU} - \\ &\left.\left. - 2U_a\eta_{aU} - \eta_u\Delta U + 2U_{aa}\xi^a_a + U_a\Delta\xi^\mu\right]\right|_{F=0} = 0.\end{aligned} \tag{2.9}$$

In our case, taking into consideration the explicit form of the operators (1.4a), the coefficient functions ξ^μ, η of the operator X are written in the form

$$\xi^0 = 0, \quad \xi^a = g_a t, \quad \eta = -\frac{1}{2}\lambda g_a x_a U,$$

where g_a, $a=\overline{1,n}$ are arbitrary parameters.

Having used the explicit form of ξ^μ and η as well as the arbitrary nature and independence of the parameters g_a, (2.9) is reduced to the following linear differential equation system, which enables one to find the functions $A(t,x,U)$ and $B(t,x,U,\underset{I}{U})$:

$$t\frac{\partial A}{\partial x_a} - \frac{1}{2}\lambda x_a U\frac{\partial A}{\partial U} = 0, \quad a=\overline{1,n}, \tag{2.10}$$

$$\frac{2}{\lambda}t\frac{\partial B}{\partial x_a} - x_a U\frac{\partial B}{\partial U} - U\frac{\partial B}{\partial U_a} - x_a U_1\frac{\partial B}{\partial U_1} - \cdots - x_a U_n\frac{\partial B}{\partial U_n} + \\ + x_a B - \frac{2}{\lambda}U_a(A-\lambda) = 0, \quad a = \overline{1,n}. \tag{2.11}$$

Thus, the proof of the theorem is reduced to the construction of the general solution of the strongly overdetermined system (2.10) and (2.11) consisting of $2n$ equations for the functions A and B.

Now let us proceed in using the standard method to find the solutions of the first-order PDE (see, e.g., Courant and Hilbert [2]).

Let us write the system of characteristic ordinary differential equations (ODE) corresponding to the system (2.10)

$$\frac{dx_a}{t} = \frac{dU}{-\frac{1}{2}\lambda x_a U}, \quad a = \overline{1,n}. \tag{2.12}$$

From (2.12) we obtain two invariants necessary for the construction of the general solution of the system (2.10):

$$w = U\exp\left(\frac{\lambda|x|^2}{4t}\right), \quad w_0 = t. \tag{2.13}$$

Consequently, the general solution of (2.10) is determined by invariants (2.13) and has the form

$$A(t,x,U) = f(w,w_0), \tag{2.14}$$

where f is an arbitrary differentiable function.

Now let us write the characteristic system of ODE (2.11):

$$-\frac{dx_a}{(2/\lambda)t} = \frac{dU}{x_a U} = \frac{dU_a}{U + x_a U_a} = \frac{dU_1}{x_a U_1} = \cdots = \frac{dU_{a-1}}{x_a U_{a-1}} = \\ = \frac{dU_{a+1}}{x_a U_{a+1}} = \cdots = \frac{dU_n}{x_a U_n} = \frac{dB}{x_a B + (2/\lambda)(\lambda - f(w,w_0))U_a}, \quad a = \overline{1,n}. \tag{2.15}$$

In (2.15), contrary to all the previous ones, the repeated indices do not mean summation.

Having solved the system (2.15) we obtain the following system of invariants necessary for the determination of the function B:

$$w = U\exp\left(\frac{\lambda|x|^2}{4t}\right), \quad w_0 = t, \\ w_a = \left(U_a + \frac{\lambda x_a}{2t}U\right)\exp\left(\frac{\lambda|x|^2}{4t}\right), \quad a = \overline{1,n}, \\ I = \left[B + (\lambda - f(w,w_0))\left(\frac{x_a U_a}{t} + \frac{\lambda|x|^2}{4t^2}U\right)\right]\exp\left(\frac{\lambda|x|^2}{4t}\right). \tag{2.16}$$

The function B is, consequently, determined from the functional equation

$$\phi(w,w_0,w_1,\ldots,w_n,I) = 0 \tag{2.17}$$

which gives us the general solution of (2.11):

$$B = Ug(w, w_0, w_1, \ldots, w_n) + (f(w, w_0) - \lambda)\left(\frac{x_a U_a}{t} + \frac{\lambda |x|^2}{4t^2} U\right), \tag{2.18}$$

where g is an arbitrary differentiable function.

Thus, we are able to construct all the equations of the form (1.7), which are invariant under PGT, completing by this the proof of the theorem.

Consequence 1. If one supposes the coefficient B in (1.7) to be independent of the derivatives $\underset{I}{U}$, then

$$\Delta U = \lambda U_0 + Ug(w, t) \tag{2.19}$$

is the most general equation, invariant under the PGT, g being here an arbitrary differentiable function.

A class of equations (1.7) with coefficients (2.1) and (2.2) contains as a subclass a set of equations which are invariant under the operators (1.4b) of the rotation group. The complete description of (1.7) which admits both operators (1.4a) and (1.4b) is given by the following theorem.

Theorem 2. *Equations from the class* (1.7) *are invariant under the operators* (1.4a) *and* (1.4b) *if and only if they have the form*

$$\Delta U = f(w, t) U_t + Ug(w, w_a w_a, t) + (f(w, t) - \lambda)\left(\frac{x_a U_a}{t} + \frac{\lambda |x|^2}{4t^2} U\right), \tag{2.20}$$

where

$$w_a w_a = \left[U_a U_a + \lambda x_a U_a \frac{U}{t} + \left(\frac{\lambda |x| U}{2t}\right)^2\right] \exp\left(\frac{\lambda |x|^2}{2t}\right).$$

This theorem is proved in the same way as the first one. The only difference is that one should substitute into the invariance condition (2.9) the coefficients A and B from (2.1) and (2.2) and the values of ξ^μ, η from (1.4b).

It should be noted that equations of the form (2.19) are obtained as a particular case of (2.20), i.e. when the function B in (1.7) is independent on the derivatives $\underset{I}{U}$. Invariance under PGT automatically implies here invariance under the rotation group.

The further restriction of the class of equations (2.19) is achieved by the requirement for the equations to be invariant under the projective operator Π (1.4c) and the operator of scale transformations D (1.4d). The two following theorems are proved in quite a similar way to the ones above.

Theorem 3. *Among equations* (2.19) *only equations*

$$\Delta U = \lambda U_t + \frac{U}{t^2} g\left(t^{n/2} w\right), \tag{2.21}$$

where g is an arbitrary differentiable function, admit the operator Π (1.4c).

Theorem 4. *Among equations* (2.19) *only equations*

$$\Delta U = \lambda U_t + \lambda_1 \frac{U}{t^2}\left(\frac{U}{\varepsilon(t,x)}\right)^\beta, \quad t^{n/2} w \equiv \frac{U}{\varepsilon} \times \text{const}, \qquad \lambda_1 = \text{const}, \quad \beta = \text{const}, \tag{2.22}$$

where

$$\varepsilon(t,x) = \left[\frac{1}{2}\left(\frac{\lambda}{\pi t}\right)^{1/2}\right]^n \exp\left(-\frac{\lambda |x|^2}{4t}\right) \tag{2.23}$$

is a fundamental solution of (1.1), *admit the operator* Π (1.4c) *and the operator*

$$D = 2t\partial_t + x_a \partial_{x_a} + (2/\beta - n) U \partial_U. \tag{2.24}$$

Note 1. If one implies $\beta = 0$ in (2.22), the obtained equation has the form

$$\Delta U = \lambda U_t + \lambda_1 U/t^2 \tag{2.25}$$

which may be reduced to (1.1) by means of the local substitution

$$U = W(t,x) \exp\left(\frac{\lambda_1}{\lambda t}\right), \quad \lambda \neq 0.$$

Note 2. The coefficients of all classes of equations constructed above contain (explicitly or implicitly) the fundamental solution $\varepsilon(t,x)$ of (1.1). This is apparently due to the fact that $\varepsilon(t,x)$ (with an approximation to an arbitrary constant) is the complete solution of the system

$$\Delta = \lambda U_t, \qquad G_a(U) \equiv t U_a + \frac{1}{2}\lambda x_a U = 0, \quad a = \overline{1,n}. \tag{2.26}$$

Note 3. The above theorems may be generalised for the systems of equations of the form*

$$\Delta U^{(k)} = A^{(k)}\left(t, x, U^{(1)}, \ldots, U^{(m)}\right) U_t^{(k)} + B^{(k)}\left(t, x, U^{(1)}, \ldots, U^{(m)}\right), \quad k = 1, 2, \ldots, m. \tag{2.27}$$

In particular, amongst the equations (2.27) only equations

$$\Delta U^{(k)} = \lambda U_t^{(k)} + U^{(k)} g^{(k)}\left(t, w^{(1)}, \ldots, w^{(m)}\right), \quad k = 1, 2, \ldots, m,$$

where $w^{(k)} = U^{(k)} \exp\left(\lambda |x|^2/4t\right)$, $g^{(k)}$ are arbitrary differentiable functions, are invariant under the Galilean transformations with the infinitesimal operators

$$G_a = t\frac{\partial}{\partial x_a} - \frac{1}{2}\lambda x_a \left(U^{(1)}\frac{\partial}{\partial U^{(1)}} + \cdots + U^{(m)}\frac{\partial}{\partial U^{(m)}}\right), \quad a = \overline{1,n}.$$

*For more details see Fushchych W.I., Cherniha R.M., *Ukr. Math. J.*, 1989, **41**, № 10, 1349–1357. *Editors' Remark.*

3. The second-order equations, invariant under the Galilean transformations

In this section we shall construct all the equations of the form

$$U_t = C(t,x,U)\Delta U + K(t,x,U,\underset{I}{U}), \tag{3.1}$$

where $C(t,x,U)$, $K(t,x,U,\underset{I}{U})$ are arbitrary differentiable functions, invariant under the operators $\widetilde{G}_a$ (1.5), generating the GT (1.6). Also we shall distinguish all the second-order equations of the form (1.8) which admit the following operators:

$$\widetilde{G}_a = tP_a, \quad P_a = \partial_a, \quad P_0 = \partial_t, \quad a = \overline{1,n}. \tag{3.2}$$

These operators satisfy the commutational relations

$$[\widetilde{G}_a, P_b] = 0, \quad [P_\mu, P_\nu] = 0, \quad [\widetilde{G}_a, P_0] = -P_a. \tag{3.3}$$

It turns out that the class of such equations is rather broad. In particular, it contains the many-dimensional Monge–Ampère equation (see Fushchych and Serov [8]) and the non-relativistic analogue of the latter. All these equations are considerably nonlinear, and as a rule they cannot be reduced to the form containing a linear plus a nonlinear term.

The following statement gives the solution of the first problem, which was posed at the beginning of this section.

Theorem 5. (3.1) *is invariant under the GT* (1.6) *if and only if*

$$C(t,x,U) = f(t,U), \tag{3.4}$$

$$K(t,x,U,\underset{I}{U}) = g(t,U,\underset{I}{U}) - x_aU_a/t, \tag{3.5}$$

where f, g are arbitrary differentiable functions.

To prove this theorem one should repeat the same procedures used in proving theorem 1, with the only obvious difference that the coefficient functions of the $\widetilde{G}_a$ operator, i.e.

$$\xi^0 = 0, \quad \xi^a = g_at, \quad a = \overline{1,n}, \quad \eta = 0$$

should be substituted into (2.9).

Now let us formulate several more statements, giving the complete description of the equations of class (3.1), invariant under $\widetilde{G}_a$, J_{ab} and the operators

$$\widetilde{\Pi} = t^2\partial_t + tx_a\partial_{x_a}, \tag{3.6}$$

$$\widetilde{D} = 2t\partial_t + x_a\partial_{x_a}. \tag{3.7}$$

Theorem 6. *Among the set of equations* (3.1) *only the equations given by*

$$\begin{aligned} &U_t = f(t,U)\Delta U + g(t,U,w_{n+1}) - x_aU_a/t, \\ &w_{n+1} = U_aU_a \end{aligned} \tag{3.8}$$

are invariant under the operators $\widetilde{G}_a$ and J_{ab}, $a,b = \overline{1,n}$.

Theorem 7. (3.8) *is invariant under the projective transformations generated by the operator* (3.6) *if and only if*

$$f(t,U) = \widetilde{f}(U), \quad g(t,U,w_{n+1}) = t^{-2}\widetilde{g}(U,t^2 w_{n+1}), \tag{3.9}$$

where $\widetilde{f}$, $\widetilde{g}$ are arbitrary differentiable functions.

Theorem 8. *Amongst equations of the form* (3.8) *only equations*

$$U_t = \widetilde{f}(U)\Delta U + U_a U_a \widetilde{g}(U) - x_a U_a/t \tag{3.10}$$

are invariant under the projective and scale transformations generated by the operators (3.6) *and* (3.7).

Theorem 9. *The maximal IA of the simplest linear equation from the class* (3.10)*:*

$$U_t = \lambda\Delta U - x_a U_a/t, \quad \lambda = \text{const} \tag{3.11}$$

is an algebra $SLi(1,n)$ with basic operators:

$$\widetilde{G}_a = t\partial_a, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad \widetilde{\Pi} = t^2\partial_t + tx_a\partial_{x_a}, \quad I = U\partial_U,$$

$$\widetilde{D} = 2t\partial_t + x_a\partial_{x_a}, \quad \widetilde{P}_a = \partial_{x_a} + \frac{x_a}{2\lambda t}I, \quad \widetilde{P}_t = \partial_t + \left(\frac{n}{2t} - \frac{|x|^2}{4\lambda t^2}\right) I.$$

Note 4. (3.11), by means of the local substitution

$$U = W(t,x)t^{n/2}\exp\left(\frac{\lambda|x|^2}{4t}\right)$$

or, in the equivalent notation,

$$U = \frac{W(t,x)}{\varepsilon(t,x)}, \quad \varepsilon(t,x) = \left[\frac{1}{2}\left(\frac{\lambda}{\pi t}\right)^{1/2}\right]^n \exp\left(-\frac{\lambda|x|^2}{4t}\right)$$

may be reduced to (1.1) for the function $W(t,x)$.

Note 5. The classes of equations given in theorems 5 and 6 can be obtained from the equations given in theorems 1 and 2. For this purpose it would be enough to apply the above substitution from note 4.

Note 6. Equations invariant under GT (1.6) (see theorem 5) can be transformed by means of the substitution of the independent variables

$$t = \theta(t'), \quad x_a = \theta(t')x_a + \theta^{(a)}(t'), \quad a = \overline{1,n},$$

where $\theta(t') \neq \text{const}$, $\theta^{(a)}$, $a = \overline{1,n}$ being arbitrary differentiable functions, to the equations given by

$$U'_{t'} = f'(t',U')\Delta U' + g'(t',U',\underset{I}{U'}),$$

where

$$U'(t',x') = U(t,x), \quad f'(t',U') = \frac{d\theta}{dt'}(\theta(t'))^{-2}f(\theta(t'),U'),$$

$$g'(t,,U',\underset{I}{U'}) = \frac{d\theta}{dt'} g(\theta(t'),U',\underset{I}{U'}(\theta(t'))^{-1}) + \\ + \left(\frac{d\theta^{(a)}(t')}{dt'}(\theta(t'))^{-1} - \frac{d\theta}{dt'}\theta^{(a)}(t')(\theta(t'))^{-2} \right) U'_a.$$

In particular if

$$\theta(t') = t', \quad \theta^{(a)}(t') = 0, \quad a = \overline{1,n}$$

one obtains the equations

$$U'_{t'} = t'^{-2} f(t',U')\Delta U' + g(t',U',\underset{I}{U'}t^{-1}).$$

Consequence 2. It follows from the theorems given in §§ 2 and 3 that the nonlinear diffusion equation (1.9) is invariant neither under PGT (1.2) and (1.3), nor under GT (1.6). It means that the Galilean principle of invariance is not satisfied by (1.9). Nonlinear equations, invariant under PGT and x and t translations, are obtained by Fushchych [5].

Now let us proceed in solving the second problem: to describe all the second-order equations

$$F(x_0, x_1, U, U_0, U_1, U_{00}, U_{01}, U_{11}) = 0 \tag{3.12}$$

in the two-dimensional space (x_0, x_1), which are invariant under GT and translations generated by operators (3.2).

Theorem 10. *Amongst the set of equations* (3.12) *only the equations given by*

$$F_1(w^{(I)}, w^{(II)}, U, U_1, U_{11}) = 0 \tag{3.13}$$

are invariant under GT (1.6) *and translations.* (3.13) *contains the following notation:*

$$w^{(I)} = \det \begin{pmatrix} U_0 & U_1 \\ U_{01} & U_{11} \end{pmatrix}, \quad w^{(II)} = \det \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} \tag{3.14}$$

of the determinant of matrices, the elements of which are the first- and second-order derivatives of the function U. Here F_1 is an arbitrary differentiable function.

Proof. The invariance of (3.12) under translations, i.e. operators P_0, P_1, is equivalent to the requirement

$$\frac{\partial F}{\partial x_0} = \frac{\partial F}{\partial x_1} = 0. \tag{3.15}$$

Taking into account (3.15) we obtain the following expression for the action of the twice prolonged operator $\overset{2}{X}$ on the manifold (3.12) (see (2.6))

$$\left(\eta \frac{\partial F}{\partial U} + \rho^{\mu} \frac{\partial F}{\partial U_{\mu}} + \sigma^{\mu\nu} \frac{\partial F}{\partial U_{\mu\nu}} \right) \Bigg|_{F=0} = 0, \quad \mu, \nu = 0, 1. \tag{3.16}$$

The coefficient functions of operators $\{G_a\}$ are given by

$$\xi^0 = \eta = 0, \quad \xi^1 = t. \tag{3.17}$$

The coefficient functions $\{\rho^\mu\} = \{\rho^0, \rho^1\}$, $\{\sigma^{\mu\nu}\} = \{\sigma^{00}, \sigma^{01}, \sigma^{10}, \sigma^{11}\}$ are determined from the formulae given in § 2. Taking into account (3.17) we obtain

$$\rho^0 = -U_1, \quad \rho^1 = 0, \quad \sigma^{00} = -2U_{01}, \quad \sigma^{01} = \sigma^{10} = -U_{11}, \quad \sigma^{11} = 0. \tag{3.18}$$

With the help of formulae (3.17) and (3.18) the invariance condition (3.16) can easily be reduced to the following linear PDE for the function F:

$$U_1 \frac{\partial F}{\partial U_0} + 2U_{01} \frac{\partial F}{\partial U_{00}} + U_{11} \frac{\partial F}{\partial U_{01}} = 0, \tag{3.19}$$

which can be readily solved. The general solution of (3.19) is an arbitrary differentiable function

$$F = F_1(w^{(I)}, w^{(II)}, U, U_1, U_{11})$$

which depends on five variables. The theorem is proved.

Theorem 10, without any substantial complications, is generalised for the case of $(n+1)$-dimensional space

$$\begin{gathered} F(x_0, x_1, \ldots, x_n, U, U_0, \underset{I}{U}, U_{00}, U_{01}, \ldots, U_{0n}, \underset{II}{U}) = 0, \\ \underset{I}{U} = (U_1, \ldots, U_n), \quad \underset{II}{U} = (U_{11}, U_{12}, \ldots, U_{nn}) \end{gathered} \tag{3.20}$$

i.e. we have the following theorem.

Theorem 11. *Amongst equations of the class* (3.20) *only equations given by*

$$F_1(w^{(I)}, w^{(II)}, U, \underset{I}{U}, \underset{II}{U}) = 0 \tag{3.21}$$

are invariant under GT (1.6) *and $x_0, x_1, \ldots, x_n$ coordinate translations, where*

$$w^{(I)} = \det \begin{pmatrix} U_0 & U_1 & \cdots & U_n \\ U_{10} & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ U_{n0} & U_{n1} & \cdots & U_{nn} \end{pmatrix}, \quad w^{(II)} = \det \begin{pmatrix} U_{00} & U_{01} & \cdots & U_{0n} \\ U_{10} & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ U_{n0} & U_{n1} & \cdots & U_{nn} \end{pmatrix}. \tag{3.22}$$

Note 7. In the specific case when

$$F_1 \equiv w^{(II)} = \det(U_{\mu\nu}) = 0, \quad U_{\mu\nu} = \partial^2 U/\partial x_\mu \partial x_\nu$$

a many-dimensional Monge–Ampère equation is obtained, the group properties of which have been studied by Fushchych and Serov [8].

Note 8. In the case

$$F_1 = w^{(I)} - \lambda = 0, \quad \lambda = \text{const} \tag{3.23}$$

the maximal IA of this equation is generated by an operator

$$\begin{gathered} X = \xi^\mu \frac{\partial}{\partial x_\mu} + \eta \frac{\partial}{\partial U}, \\ \xi^0 = C_{00} t + d_0, \quad \xi^a = C_{ab} x_b + f_a(t), \quad a, b = \overline{1, n}, \\ \eta = CU + d, \quad C = \frac{C_{00} + 2(C_{11} + \cdots + C_{nn})}{n+1}, \end{gathered} \tag{3.24}$$

where C_{00}, C_{ab}, d_0, d are arbitrary constants, and $f_a(t)$, $a=\overline{1,n}$ are arbitrary differentiable functions.

It means that the maximal IA of (3.23) is infinitely dimensional. In particular, this algebra contains operators of the form

$$\partial_{x_0},\ \partial_{x_a},\ \partial_U,\ x_b\partial_{x_a},\quad a\neq b,\quad a,b=\overline{1,n}, \tag{3.25a}$$

$$\begin{aligned} &D_0 = x_0\partial_{x_0} + \frac{U}{n+1}\partial_U,\\ &D_1 = x_1\partial_{x_1} + \frac{2U}{n+1}\partial_U,\quad \ldots,\quad D_n = x_n\partial_{x_n} + \frac{2U}{n+1}\partial_U, \end{aligned} \tag{3.25b}$$

$$X_1 = f_1(t)\partial_{x_1},\quad \ldots,\quad X_n = f_n(t)\partial_{x_n}. \tag{3.25c}$$

Note 9. It is possible to construct a general solution for the two-dimensional equation

$$w^{(I)} = \det\begin{pmatrix} U_0 & U_1 \\ U_{01} & U_{11}\end{pmatrix} = 0. \tag{3.26}$$

To prove this, we represent (3.26) as follows:

$$\frac{\partial}{\partial x_1}\left(\frac{U_1}{U_0}\right) = 0$$

and then we obtain the general solution

$$U = F(x_1 + G(x_0)),$$

where F and G are arbitrary differentiable functions. Direct verification shows that

$$U = F(\mathcal{L}_a x_a + G(x_0)),\quad a=\overline{1,n},\quad \mathcal{L}_a = \text{const}$$

is a particular solution of $(n+1)$-dimensional equation (3.23) under $\lambda=0$.

Note 10. Equations

$$w^{(I)} \equiv \begin{vmatrix} U_0 & \cdots & U_n \\ U_{10} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots \\ U_{n0} & \cdots & U_{nn}\end{vmatrix} = F(U), \tag{3.27}$$

where $F(U)$ is an arbitrary twice differentiable function, can be reduced to (3.23) at $\lambda=1$ for the function $W(x_0,\ldots,x_n)$ by the substitution

$$W = \int [F(U)]^{-1/(n+1)} dU.$$

Note 11. Maximal IA of the equation

$$w^{(I)} = F(U_aU_a),\quad U_aU_a = U_1^2 + \cdots + U_n^2 \tag{3.28}$$

is generated by the basis operators (3.25c) and

$$\begin{aligned} &\partial_{x_0},\ \partial_{x_a}, \partial_U,\ x_a\partial_{x_b} - x_b\partial_{x_a},\quad a\neq b,\quad a,b=\overline{1,n},\\ &D = (1-n)\partial_{x_0} + x_a\partial_{x_a} + U\partial_U. \end{aligned}$$

In particular, in the case $n = 1$ for equations of the class (3.28)

$$\begin{vmatrix} U_0 & U_1 \\ U_{10} & U_{11} \end{vmatrix} = U_1^2, \tag{3.29}$$

$$\begin{vmatrix} U_0 & U_1 \\ U_{10} & U_{11} \end{vmatrix} = U_1^3 \tag{3.30}$$

one can obtain the general solutions, namely $U = F(x_1 e^{-x_0} + G(x_0))$ is the general solution of (3.29) and $\phi(U, x_0 U + G(x_0) - x_1) = 0$ is the general solution of (3.30) written in an implicit form, F, G, ϕ being arbitrary differentiable functions.

In conclusion, we note that among the Galilei invariant equations (3.21) one can distinguish a class of equations

$$U_0 = \lambda(U, \underset{I}{U})\Delta U + Q(U, \underset{I}{U}) - w^{(III)}/w^{(II)}, \tag{3.31}$$

$$w^{(III)} = \begin{vmatrix} 0 & U_1 & \cdots & U_n \\ U_{10} & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ U_{n0} & U_{n1} & \cdots & U_{nn} \end{vmatrix}, \quad w^{(II)} = \begin{vmatrix} U_{11} & \cdots & U_{1n} \\ U_{21} & \cdots & U_{2n} \\ \cdots & \cdots & \cdots \\ U_{n1} & \cdots & U_{nn} \end{vmatrix},$$

λ, Q being arbitrary functions.

As to the structure, equations of the form (3.31) are diffusive type nonlinear equations with a strongly nonlinear addition. The properties of (3.31) will be studied by us in a further paper*.

1. Bluman A., Cole B., Similarity methods for differential equations, Berlin, Springer, 1974.
2. Courant R., Hilbert D., The methods of mathematical physics, Moscow, Nauka, 1951.
3. Dorodnitsyn V.A., Knyazeva I.V., Svirshchevsky S.R., *Diff. Uravn.* (*Differential equations*), 1983, **19**, 1215–1223.
4. Fushchych W.I., On symmetry and particular solutions of some multi-dimensional equations of mathematical physics, in Algebraic-Theoretical Methods in Mathematical Physics Problems, Kyiv, Institute of Mathematics, 1983, 4–23.
5. Fushchych W.I., The symmetry and exact solutions of many dimensional parabolic and hyperbolic nonlinear differential equations, Kyiv, Institute of Mathematics, 1984.
6. Fushchych W.I., Nikitin A.G., *Fiz. elem. chastits i atom. yadra* (*Physics of elementary particles and atomic nucleus*), 1981, **12**, 1157–1219.
7. Fushchych W.I., Nikitin A.G., The symmetry of Maxwell equations, Kyiv, Naukova Dumka, 1983.
8. Fushchych W.I., Serov N.I., *Dokl. Akad. Nauk USSR*, 1983, **273**, 543–546.
9. Ibragimov N.H., The group of transformations in mathematical physics, Moscow, Nauka, 1983.
10. Lie S., Über die Integration durch bestimente Integrale von einer Klasse lineare partiellen Differentialgleichungen, 1881, **6**, 328–368.
11. Niederer U., *Helv. Phys. Acta*, 1972, **45**, 808–816.
12. Ovsyannikov L.V., *Dokl. Akad. Nauk USSR*, 1959, **125**, 492–495.
13. Ovsyannikov L.V., The group analysis of differential equations, Moscow, Nauka, 1978.

*See Cherniha R.M., *J. Math. Anal. Appl.*, 2001, **253**, 126–141. *Editors' Remark.*

Как расширить симметрию дифференциальных уравнений?

В.И. ФУЩИЧ

Предложен простой способ расширения симметрии дифференциальных уравнений.

1. Лиевский критерий инвариантности. Рассмотрим в четырехмерном пространстве $R(1,3)$ систему нелинейных дифференциальных уравнений (ДУ) в частных производных

$$L(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{n}{u}) = 0, \tag{1}$$

где вектор $u \equiv (u_1, u_1, \ldots, u_n)$, $x \in R(1,3)$, $\underset{1}{u} \equiv \left(\frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)$, $\underset{k}{u}$, $k = \overline{1, r}$ — совокупность всевозможных производных r-го порядка.

Согласно Ли уравнение (1) инвариантно относительно оператора

$$Q = \xi^\mu(x, u)\frac{\partial}{\partial x_\mu} + \eta^k(x, u)\frac{\partial}{\partial u_k}, \tag{2}$$

если выполняется следующее условие:

$$\tilde{Q}L = \lambda_0(x, u, \underset{1}{u}, \ldots, \underset{k}{u})L \quad \text{или} \quad \tilde{Q}L\Big|_{L=0} = 0, \tag{3}$$

где $\tilde{Q}$ — соответствующее число раз продолженный оператор Q, λ_0 — произвольная дифференциальная функция (более подробно см., например, [1–3]). Условие (3) назовем лиевским критерием инвариантности уравнения (1). Более общее определение инвариантности введено в [4, 5], которое дало возможность обнаружить новые симметрии уравнений Максвелла, Дирака, Ламе [6].

Хорошо известно, что если уравнение обладает нетривиальной симметрией, то это свойство существенно для явного построения широких классов точных решений нелинейных дифференциальных уравнений в частных производных (ДУЧП). Многие ДУЧП имеют довольно узкую группу инвариантности. Поэтому весьма существенно указать конструктивные способы расширения симметрии уравнений.

В настоящее время интенсивно развиваются два направления решения этой проблемы. Одно из них состоит в разработке новых методов исследования симметрийных свойств ДУЧП (см. библиографию в [6]), позволяющих обнаружить как локальные, так и нелокальные симметрии. Другое направление наметилось в работах [3, 6–10], где изучается симметрия не всех решений ДУ, а только некоторых подмножеств решений. В неявном виде, как теперь стало ясно, эта идея заложена, в частности, в методе разделений переменных и, конечно, использовалось без привлечения теоретико-алгебраических методов многими исследователями прошлого века. Ниже именно это второе направление будет обсуждаться.

Передруковано з “Симметрия и решения нелинейных уравнений математической физики”, Киев, Институт математики АН УССР, 1987, С. 4–16 за дозволом

На конкретных примерах будет указан способ расширения симметрии ДУЧП. Как будет видно из дальнейшего, он очевидным образом обобщается и на другие ДУ.

2. Уравнение Максвелла. Рассмотрим систему уравнений Максвелла

$$\frac{\partial \vec{E}}{\partial t} = \mathrm{rot}\, \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\mathrm{rot}\, \vec{E}, \tag{4}$$

$\vec{E}$, $\vec{H}$ — векторы напряженностей электромагнитного поля.

Операторы, порождающие преобразования Лоренца, имеют вид

$$J_{0a} = x_0 p_a - x_a p_0 + S_{0a}, \quad P_0 = i\frac{\partial}{\partial x_0}, \quad P_a = -i\frac{\partial}{\partial x_a}, \tag{5}$$

$S_{0a} = iS_a$ — 6×6-матрицы, реализующие соответствующее представление алгебры Ли группы $SU(2)$ [6].

Записав матрицы S_a через E_k, H_l и $\frac{\partial}{\partial E_k}$, $\frac{\partial}{\partial H_l}$ и представив (4) в виде (1) [6]

$$L\Psi = 0, \quad L = \frac{\partial}{\partial t} - i\sigma_2 S_a P_a, \tag{6}$$

можно убедиться, что

$$\tilde{J}_{0a} L \neq \lambda_a L \quad \text{или} \quad \tilde{J}_{0a} L\Big|_L \neq 0, \quad a = 1, 2, 3. \tag{7}$$

В (6) вектор-столбец $\Psi = (E_1, E_2, E_3, H_1, H_2, H_3)$. Для уравнения (4) $\tilde{J}_{0a} = J_{0a}$. Условие (7) означает, что система (4) неинвариантна относительно операторов $\{J_{0a}\}$, а следовательно, уравнение (2) не инвариантно относительно группы Лоренца $O(1,3)$. Действие операторов $\{J_{0a}\}$ на L можно записать в виде

$$\tilde{Q}L = \lambda_0(x, u, u_1, \ldots, u_r)L + \lambda_1(x, u, u_1, \ldots, u_r)L_1, \quad \lambda_1 \neq 0, \tag{8}$$

где $\tilde{Q}$ — любой из операторов $\{\tilde{J}_{01}, \tilde{J}_{02}, \tilde{J}_{03}\}$. Отсюда видно, что если на множество решений наложить дополнительное условие

$$L_1(x, u, u_1, \ldots, u_r) = 0, \tag{9}$$

то система (4) будет инвариантна относительно операторов $\{J_{0a}\}$. Для системы (4) эти дополнительные условия имеют вид

$$\mathrm{div}\, \vec{E} = 0, \quad \mathrm{div}\, \vec{H} = 0. \tag{10}$$

Таким образом, уравнения (4) в совокупности с дополнительными условиями (10) инвариантны относительно алгебры Ли $AO(1,3)$ группы $O(1,3)$. Обобщая понятие инвариантности, введенное в [6–10] и приведенные только что рассуждения, Н.И. Серов и автор ввели понятие условной инвариантности ДУ.

Определение. *Систему уравнений* (1) *назовем условно инвариантной, если она инвариантна относительно оператора* Q *при дополнительном условии* (9) *и*

$$\tilde{Q}L_1 = \lambda_2(x, u, u_1, \ldots, u_k)L + \lambda_3(x, u, u_1, \ldots, u_k)L_1, \tag{11}$$

где λ_1, λ_2 *— произвольные дифференцируемые функции.*

В данном определении, конечно, предполагается, что система (1), (9) совместна. Очевидно, что не всякое дополнительное условие (уравнение) расширяет симметрию исходного уравнения. Поэтому важно научиться строить такие дополнительные условия, чтобы симметрия всей системы была шире, чем симметрия исходного уравнения (1).

3. Условная инвариантность систем гиперболического и параболического типов. Система гиперболических уравнений второго порядка

$$\begin{aligned} &\Box\vec{E}=\vec{0}, \quad \vec{E}=\{E_1,E_2,E_3\}, \quad \Box=\frac{\partial^2}{\partial t^2}-\Delta,\\ &\Box\vec{H}=\vec{0}, \quad \vec{H}=\{H_1,H_2,H_3\} \end{aligned} \tag{12}$$

инвариантна относительно конформных операторов

$$K_\mu=2x_\mu D-x_\nu x^\nu P_\mu+2x^\nu S_{\mu\nu}, \quad D=x_\mu P^\mu+2i, \tag{13}$$

где $S_{\mu\nu}$ — матрицы, реализующие представление алгебры $AO(1,3)$.

Однако система (12) условно инвариантна относительно операторов (13). В этом случае дополнительное условие (9) является системой уравнений Максвелла (4), (10). Подробное доказательство этого факта дано в [6].

Рассмотрим систему линейных уравнений параболического типа

$$\begin{aligned} &L\Psi=0, \quad L=p_0-\frac{p_ap_a}{2m},\\ &p_0=i\frac{\partial}{\partial x_0}, \quad p_a=-i\frac{\partial}{\partial x_a}, \quad a=1,2,3. \end{aligned} \tag{14}$$

$\Psi=\{\Psi_1,\Psi_2,\ldots,\Psi_n\}$ — вектор-функция, m — параметр.

Уравнения (14) условно инвариантны относительно операторов из расширенной алгебры Галилея $AG(1,3)$

$$\begin{aligned} &G_a=tp_a-mx_a+q_a,\\ &A=tD-t^2p_0+\frac{1}{2}m\vec{x}^{\,2}-\vec{q}\,\vec{x}, \quad D=2rp_0-\vec{x}\vec{p}+q_0, \end{aligned} \tag{15}$$

если на решения Ψ положить дополнительные условия

$$\begin{aligned} &L_3\Psi=0, \quad L_4\Psi=0,\\ &L_3=q_0-\frac{3}{2}i-\frac{\vec{q}\vec{p}}{m}, \quad L_4=q_1^2+q_2^2+q_3^2. \end{aligned} \tag{16}$$

В (15), (16) матрицы q_0, $\vec{q}$ удовлетворяют коммутационным соотношениям

$$[q_a,q_b]=0, \quad [q_0,q_a]=iq_a.$$

В [7] доказано, что уравнения (16) являются необходимыми и достаточными условиями того, чтобы система (14) была инвариантна относительно операторов (15).

4. Расширение симметрии уравнения Даламбера. Хорошо известно, что максимальной (в смысле С. Ли) локальной группой инвариантности линейного волнового уравнения

$$\Box u(x)=0, \quad x=(x_0,x_1,\ldots,x_n), \tag{17}$$

является конформная группа $C(1, n)$. В [11] доказано, что если на решения $u(x)$ наложить условия

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = 0, \tag{18}$$

то переопределенная система (17), (18) инвариантна относительно бесконечномерной алгебры с операторами

$$\begin{gathered} Q = \xi^\mu(x,u)\frac{\partial}{\partial x_\mu} + \eta(x,u)\frac{\partial}{\partial u}, \\ \xi^\mu = c_{00}x_\mu + c_{\mu\nu}(u)x^\nu + d_\mu(u), \quad \eta(x,u) = \eta(u), \end{gathered} \tag{19}$$

где $c_{00}(u)$, $c_{\mu\nu}(u)$, $\eta(u)$ — произвольные гладкие функции от зависимой переменной $u(x)$.

Итак, уравнение Даламбера условно инвариантно относительно бесконечномерной алгебры (19). Такое существенное расширение симметрии волнового уравнения приводит к уникальному свойству нелинейной системы (17), (18): если u_1 — решение (18), (19), то и произвольная гладкая функция от этого решения $\Phi(u_1) = u_2$ является решением (17), (18).

5. Условная инвариантность уравнения четвертого порядка. Рассмотрим уравнение

$$\left(\frac{\partial}{\partial x_0} + \Delta\right)\left(\frac{\partial}{\partial x_0} - \Delta\right)u = 0. \tag{20}$$

Применяя метод Ли к уравнению (20), можно показать, что оно неинвариантно относительно алгебры Галилея $AG(1,3)$. Уравнение (20) является дифференциальным следствием уравнения теплопроводности

$$\left(\frac{\partial}{\partial x_0} - \Delta\right)u = 0, \quad u \equiv u(x), \tag{21}$$

которое, как известно, инвариантно относительно преобразований Галилея. Причина сужения симметрии уравнения (20), по сравнению с уравнением (21), связана с тем, что множество решений уравнения (20) шире, чем множество решений уравнения (21). Однако, если на $u(x)$ наложить дополнительное условие в виде уравнения Гамильтона–Якоби

$$\frac{\partial u}{\partial x_0} + \frac{\partial u}{\partial x_a}\frac{\partial u}{\partial x_a} = 0, \quad a = 1,2,3, \tag{22}$$

то система (21), (22) будет инвариантна относительно галилеевских операторов вида

$$G_a = up_a - \frac{1}{2}x_a p_0.$$

Отметим, что эти операторы порождают необычные преобразования Галилея.

Итак, уравнение (20) условно инвариантно относительно алгебры Галилея. Более подробно этот вопрос изучен в [12].

6. Расширение симметрии нелинейного уравнения теплопроводности. Нелинейное уравнение

$$\frac{\partial u}{\partial x_0} + \frac{\partial}{\partial x_a}\left\{c(u)\frac{\partial u}{\partial x_a}\right\} = 0, \quad c(u) \neq \text{const}, \tag{23}$$

неинвариантно относительно преобразований Галилея, а следовательно, для него не выполняется принцип относительности Галилея [9], т.е. уравнение (23) неинвариантно относительно операторов

$$G_a = x_0\frac{\partial}{\partial x_a} + \mu(u)x_a\frac{\partial}{\partial u}, \quad a = 1, 2, 3, \tag{24}$$

где $\mu(u)$ — произвольная гладкая функция от $u(x)$.

Чтобы расширить симметрию нелинейного уравнения теплопроводности до группы Галилея, достаточно дополнить (24) уравнением типа Гамильтона–Якоби

$$\frac{\partial u}{\partial x_0} + \frac{1}{2\mu(u)}\frac{\partial u}{\partial x_a}\frac{\partial u}{\partial x_a} = 0, \tag{25}$$

причем

$$\mu(u) = \frac{u}{2c(u)}. \tag{26}$$

Аналогичным способом можно расширить симметрию уравнений

$$\frac{\partial^2 u}{\partial x_0^2} = C(x, u, \underset{1}{u})\Delta u, \tag{27}$$

которое широко применяется в нелинейной акустике, в теории нелинейных волн.

Более подробно эти результаты будут обсуждаться и опубликованы в работе Н.И. Серова и автора.

7. О некоторых нерешенных задачах. В этом пункте укажем несколько задач, которые представляются автору важными для развития и применения теоретико-алгебраических методов.

1. Описать дифференциальные уравнения (дополнительные условия) первого и второго порядка

$$F_1(x, u, \underset{1}{u}, \underset{2}{u}, a_{\mu\nu}, F_0), \quad u = u(x_0, x_1, x_2, x_3), \tag{28}$$

которые расширяют симметрию уравнения

$$a_{\mu\nu}(x, u, \underset{1}{u})\frac{\partial^2 u}{\partial x_\mu \partial x_\nu} + F_0(x, u, \underset{1}{u}) = 0 \tag{29}$$

до групп $O(1,3)$, $P(1,3)$, $C(1,3)$, $P(1,4)$, $C(1,4)$. F_0, F_1, $a_{\mu\nu}$ — гладкие функции.

Рассмотреть отдельно случай двумерных уравнений $\{x = (x_0, x_1)\}$ и описать все уравнения (28), (29), инвариантные относительно бесконечномерной алгебры с оператором

$$Q = \{f(x_0 + x_1) + g(x_0 - x_1)\}\frac{\partial}{\partial x_0} + \{f(x_0 + x_1) - g(x_0 - x_1)\}\frac{\partial}{\partial x_1},$$

где f и g — произвольные функции.

2. Исследовать групповые свойства и построить решения следующих уравнений:

$$\begin{gathered}\Box u + F_0(x, u, \underset{1}{u}) = 0, \quad (K_\mu u)(K^\mu u) = \lambda, \\ K_\mu = 2x_\mu D - x_\nu x^\nu p_\mu + \lambda_1, \quad D = \frac{1}{2}(x_\alpha p^\alpha + p^\alpha x_\alpha) + \lambda_2;\end{gathered} \tag{30}$$

$$\begin{gathered}\Box u + F_0(x, u, \underset{1}{u}) = 0, \\ (J_{\mu\nu} u)(J^{\mu\nu} u) = \lambda_3, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu,\end{gathered} \tag{31}$$

где λ_1, λ_2, λ_3 — произвольные константы. Рассмотреть волновое уравнение (30) с дополнительным условием $D^2 u(x) = \lambda$.

3. Описать системы дополнительных условий (уравнений) первого порядка, расширяющих симметрию уравнений параболического типа

$$\frac{\partial u}{\partial x_0} + C_{lk}(u, \underset{1}{u}) \frac{\partial^2 u}{\partial x_l \partial x_k} + F_0(u, \underset{1}{u}) = 0. \tag{32}$$

Рассмотреть в качестве дополнительного условия уравнение первого порядка

$$a_0(x, u) \frac{\partial u}{\partial x_0} + a_{kl}(x, u) \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_l} + b_k(x, u) \frac{\partial u}{\partial x_k} = 0.$$

4. Исследовать групповые свойства и построить семейства частных решений нелинейного уравнения Дирака

$$\gamma_\mu p^\mu \Psi = F(\bar{\Psi}\Psi)\Psi \tag{33}$$

совместно с одним из следующих дополнительных условий:

$$a\bar{\Psi}\Psi + b\bar{\Psi}\gamma_4\Psi = 0, \tag{34}$$

$$a(\bar{\Psi}\gamma_\mu\Psi)^2 + b(\bar{\Psi}\gamma_4\gamma_\mu\Psi)^2 = 0, \tag{35}$$

$$a\frac{\partial(\bar{\Psi}\Psi)}{\partial x_\mu}\frac{\partial(\bar{\Psi}\Psi)}{\partial x^\mu} + b\frac{\partial(\bar{\Psi}\gamma_4\Psi)}{\partial x_\mu}\frac{\partial(\bar{\Psi}\gamma_4\Psi)}{\partial x^\mu} = 0, \tag{36}$$

a, b — произвольные постоянные.

Рассмотреть случаи: $F(\bar{\Psi}\Psi) = m = \text{const}$, $F(\bar{\Psi}\Psi) = 0$, Ψ — четырехкомпонентный спинор.

5. Исследовать симметрию и построить точные решения уравнений

$$\Box\Psi + \left(F(\bar{\Psi}\Psi), \frac{\partial\Psi}{\partial x_\mu}\right)\Psi = 0 \tag{37}$$

с дополнительными условиями

$$a\frac{\partial(\bar{\Psi}\gamma_\mu\Psi)}{\partial x_\mu} + b\frac{\partial(\bar{\Psi}\gamma_4\gamma_\mu\Psi)}{\partial x_\mu} = 0, \tag{38}$$

$$a(\bar{\Psi}\Psi) + b(\bar{\Psi}\gamma_4\Psi) = 0. \tag{39}$$

6. Провести теоретико-алгебраический анализ системы уравнений

$$
\begin{aligned}
&(\gamma_\mu w^\mu + w^\mu \gamma_\mu)\Psi + F(\bar{\Psi}\Psi)\Psi = 0,\\
&w_\mu = \{w_0, \vec{w}\} = \{w_0, w_1, w_2, w_3\}, \quad w_0 = \vec{p}\vec{J} = p_1 J_1 + p_2 J_2 + p_3 J_3,\\
&J_i = \varepsilon_{ikl} J_{kl}, \quad \vec{w} = p_0 \vec{J} - (\vec{p} \times \vec{N}), \quad \vec{N} = (J_{01}, J_{02}, J_{03}),\\
&J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu).
\end{aligned}
\tag{40}
$$

7. Описать пуанкаре-инвариантные и конформно-инвариантные первого и второго для спинора Ψ, предполагая, что ток

$$
j_\mu = a(\bar{\Psi}\gamma_\mu\Psi) + b(\bar{\Psi}\gamma_4\gamma_\mu\Psi) + c(\bar{\Psi}p_\mu\Psi) + d(\bar{\Psi}w_\mu\Psi)
$$

удовлетворяет уравнению непрерывности $\frac{\partial j_\mu}{\partial x_\mu} = 0$, a, b, c, d — произвольные константы.

8. Исследовать групповые свойства и построить частные решения систем четырех дифференциальных уравнений первого порядка

$$
\begin{aligned}
&\gamma_\mu\gamma_\nu J^{\mu\nu}\Psi + \lambda(\bar{\Psi}\Psi)^k\Psi = 0,\\
&J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu].
\end{aligned}
$$

9. Построить семейства точных решений уравнений второго порядка

$$
\Box\Psi = F\left(\bar{\Psi}\Psi, \frac{\partial\bar{\Psi}}{\partial x_\alpha}, \frac{\partial\Psi}{\partial x_\beta}\right)\Psi
$$

с дополнительным условием

$$
\begin{aligned}
&\bar{\Psi}\gamma_\mu p^\mu\Psi = a(\bar{\Psi}\Psi) + b(\bar{\Psi}\gamma_\mu\Psi)^2 + c(\bar{\Psi}\gamma_4\gamma_\mu\Psi)^2,\\
&(\bar{\Psi}w_\mu\Psi)(\bar{\Psi}w^\mu\Psi) = \lambda(\bar{\Psi}\Psi).
\end{aligned}
$$

Рассмотреть случаи: $F = -m^2$, $F = (\bar{\Psi}\Psi)^r$, m, r, b, c — произвольные константы.

10. С помощью следующих потенциалов (B_μ, φ):

$$
\begin{aligned}
&F_{\mu\nu} = K_\mu B_\nu - K_\nu B_\mu, \quad K_\mu = 2x_\mu D - x_\nu x^\nu p_\mu + \lambda_1,\\
&F_{\mu\nu} = J_{\mu\nu}\varphi, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad u_i = \varepsilon_{ikl}J_{kl}\varphi,
\end{aligned}
$$

построить семейства точных решений уравнений для электромагнитного поля и для поля Эйлера–Навье–Стокса

$$
\frac{\partial u_i}{\partial t} + u_l\frac{\partial u_i}{\partial x_l} + \lambda\Delta u_i = 0, \quad i, k, l = 1, 2, 3.
$$

11. Описать анзацы вида

$$
u = f(x)\varphi(\omega) + g(x),
$$

которые редуцируют уравнения второго порядка

$$
a_{\mu\nu}(x, u, \underset{1}{u})\frac{\partial^2 u}{\partial x_\mu \partial x_\nu} + F(x, u, \underset{1}{u}) = 0
\tag{41}
$$

к обыкновенным ДУ. Важно рассмотреть случаи, когда уравнение (41) не инвариантно ни относительно групп $P(1,3)$, $C(1,3)$, ни относительно подгрупп этих групп. Нетривиальные примеры таких уравнений приведены в [9, 10].

12. Исследовать симметрию и построить классы точных решений следующих систем уравнений:

$$D_t\vec{E} = \mathrm{rot}\ \vec{H}, \quad D_t\vec{H} = -\mathrm{rot}\ \vec{E},$$

$$D_t \equiv \frac{\partial}{\partial t} + \lambda_1 E_k \frac{\partial}{\partial x_k} + \lambda_2 H_k \frac{\partial}{\partial x_k};$$

$$D_\nu F_{\mu\nu} = 0, \quad D_\nu = \frac{\partial}{\partial x_\nu} + F_{\nu\alpha}\frac{\partial}{\partial x_\alpha}, \quad \mu = \overline{0,3};$$

$$D_\alpha F_{\mu\nu} + D_\mu F_{\nu\alpha} + D_\nu F_{\alpha\mu} = 0.$$

Рассмотреть случаи, когда $\lambda_1 = \lambda_2 = 1$; $\lambda_1 = 1$, $\lambda_2 = 0$; $\lambda_2 = 1$, $\lambda_1 = 0$. Приведенные уравнения можно рассматривать как нелинейное обобщение уравнений Максвелла. При этом, конечно, следует добавить к первой системе уравнений условие неразрывности: $\mathrm{div}\ \vec{E} = 0$, $\mathrm{div}\ \vec{H} = 0$.

13. Провести подробно теоретико-алгебраический анализ переопределенных уравнений

$$\Box u + F_1(x, u, \underset{1}{u}) = 0, \tag{42}$$

$$\{b_{\mu\nu}(x,u)J_{\mu\nu} + c_\mu(x,u)P_\mu + d_\mu(x,u)K_\mu + e(x,u)D\}F_2(x,u) = 0, \tag{43}$$

$$\Box u + F_3(x, u, \underset{1}{u}) = 0, \tag{44}$$

$$a_{\mu\nu}(x,u)\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x_\nu} = F_4(x,u), \tag{45}$$

$$J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad P_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu},$$

$$K_\mu = 2x_\mu - x_\nu x^\nu P_\mu, \quad D = \frac{1}{2}(x_\mu P^\mu + P_\mu x^\mu).$$

Описать функции F_1, F_2, F_3, F_4, $a_{\mu\nu}$, $b_{\mu\nu}$, c_μ, d_μ, e, при которых уравнения (42)–(45) инвариантны относительньно групп $C(1,3)$, $C(1,4)$, $P(1,3)$, $P(1,4)$ и их подгрупп. Если удастся при некоторых конкретных функциях F_2, $b_{\mu\nu}$, ... решить уравнение (43), то это даст нам анзацы для решения нелинейного волнового уравнения (42), которые не могут быть получены с помощью метода С. Ли. В том случае, когда уравнение (42) инвариантно относительно операторов P_μ, $J_{\mu\nu}$, K_μ, D, а функции $b_{\mu\nu}$, c_μ, d_μ, e являются постоянными, уравнение (43) дает нам лиевские анзацы для нахождения инвариантных решений уравнения (42). Решения уравнения (43) приводят к нелиевским анзацам для волнового уравнения (42). При этом, конечно, необходимо, чтобы система (42), (43) была совместной.

14. Исследовать симметрию и построить первые интегралы для обыкновенной системы дифференциальных уравнений

$$\frac{dx_\mu}{d\tau} = x_\mu F_1(x, \bar{\Psi}\Psi) + (\bar{\Psi}\gamma_\mu\Psi)F_2(x, \dot{x}),$$

$$\gamma_\mu P^\mu \Psi = F_3(\bar{\Psi}\Psi)\Psi.$$

Приведенная система ОДУ описывает движение классической частицы в спинорном поле Ψ . Рассмотреть случай, когда $F_3(\bar{\Psi}\Psi) = m = \text{const}$.

15. Описать все системы ОДУ вида

$$\begin{gathered}m(\vec{v},\vec{E},\vec{H})\frac{d\vec{v}}{dt} = \vec{x}F_1(x,\vec{v},\vec{E},\vec{H}) + \vec{v}F_2(x,\vec{v},\vec{E},\vec{H}) + \\ + \vec{E}F_3(x,\vec{v},\vec{E},\vec{H}) + \vec{H}F_4(x,\vec{v},\vec{E},\vec{H}),\end{gathered} \tag{46}$$

инвариантные относительно групп $P(1,3)$, $G(1,3)$ и их расширений ($C(1,3)$, $P(1,4)$, $C(1,4)$, $G(1,4)$). В (46) $\vec{v} = \frac{d\vec{x}}{dt}$, $x = (t, x_1, x_2, x_3)$, $\vec{v} = (v_1, v_2, v_3)$, $\vec{E}$, $\vec{H}$ — векторы электромагнитного поля.

16. Существуют ли нетривиальные решения для спинорного поля

$$p_\mu p^\mu \Psi + F(\bar{\Psi}\Psi, \bar{\Psi}\gamma_\mu p^\mu\Psi)\Psi = 0,$$

для которых

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \lambda\bar{\Psi}\gamma_\mu\Psi, \quad F_{\mu\nu} = \lambda_1\bar{\Psi}[\gamma_\mu,\gamma_\nu]\Psi,$$
$$\frac{\partial F_{\mu\nu}}{\partial x_\alpha} + \frac{\partial F_{\nu\alpha}}{\partial x_\mu} + \frac{\partial F_{\alpha\mu}}{\partial x_\nu} = 0$$

или

$$p_\alpha p^\alpha A_\mu + p_\mu(p_\nu A^\nu) = m^2 A_\mu + A_\mu F(\bar{\Psi}\Psi), \quad A_\mu = \lambda\bar{\Psi}\gamma_\mu\Psi,$$

или

$$p_\alpha p^\alpha u = m^2 u + F(u), \quad u = \lambda(\bar{\Psi}\Psi),$$

где λ, λ_1 — произвольные параметры.

17. Исследовать симметрийные свойства и построить решения интегро-дифференциального уравнения для спинора

$$p_0\Psi = \left(p_1^2 + p_2^2 + p_3^2 + m^2\right)^{1/2}\Psi + F(\bar{\Psi}\Psi)\Psi \tag{47}$$

с дополнительными нелинейными условиями:

$$\bar{\Psi}\gamma_\mu p^\mu\Psi = \lambda\bar{\Psi}\Psi, \quad \bar{\Psi}(1-\gamma_4)\Psi = 0. \tag{48}$$

Рассмотреть отдельно случай $F = 0$, $\lambda = m$. В этом случае решения линейного уравнения Дирака (с положительной энергией) удовлетворяют уравнению (47) и первому нелинейному условию (48).

18. Исследовать пространства с такими метриками:

$$\left(x_\mu + \lambda_1\frac{\partial u}{\partial x_\mu} + \lambda_2 x_\nu\frac{\partial^2 u}{\partial x_\nu\partial x_\mu}\right)\left(x^\mu + \lambda_1\frac{\partial u}{\partial x^\mu} + \lambda_2 x_\alpha\frac{\partial^2 u}{\partial x_\alpha\partial x^\mu}\right) = F_1(x,u),$$

u — скалярная функция,

$$\left\{x_\mu + \lambda_1\bar{\Psi}\gamma_\mu\Psi + \lambda_2\frac{\partial(\bar{\Psi}\Psi)}{\partial x_\mu}\right\}\left\{x^\mu + \lambda_1\bar{\Psi}\gamma^\mu\Psi + \lambda_2\frac{\partial(\bar{\Psi}\Psi)}{\partial x^\mu}\right\} = F_2(x,\Psi),$$
$$(x_\mu + \lambda_1\gamma_\mu\Psi + \lambda_2 p_\mu\Psi)(x^\mu + \lambda_1\gamma^\mu\Psi + \lambda_2 p^\mu\Psi) = F_3(x,\bar{\Psi}\Psi).$$

Рассмотреть случаи: $F_1 = \mathrm{const}$, $F_2 = \mathrm{const}$; $F_1 = x^2 \pm u^2$, $F_2 = x^2 \pm (\bar{\Psi}\Psi)$, $F_3 = x^2 \pm (\bar{\Psi}\Psi)$.

19. Исследовать симметрию и построить классы точных решений систем:

$$p_\mu p^\mu u_1 = F_1(u_1, u_2),$$
$$p_\mu p^\mu u_2 = F_2(u_1, u_2),$$
$$(p_\mu u_1)(p^\mu u_2) = \mathrm{const},$$
$$p_\mu u_1 p^\mu u_1 = m_1, \quad p_\mu u_2 p^\mu u_2 = m_2, \quad p_\mu u_1 p^\mu u_2 = m_3.$$

20. Провести детальный теоретико-алгебраический анализ уравнений

$$\frac{1}{2}(\gamma_\mu w^\mu + w^\mu \gamma_\mu)\Psi = \lambda\Psi, \quad w_\mu = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}P^\nu J^{\alpha\beta},$$
$$\{\gamma_\mu P^\mu + \lambda\gamma_\mu(\bar{\Psi}w^\mu\Psi)\}\Psi = 0,$$

Проанализировать случай, когда Ψ матрица 4×4. Обычно Ψ — столбец из 4 функций.

21. Исследовать симметрию и построить решения дифференциальных неравенств:

$$(p_0 u)^2 - (p_a u)(p_a u) > 0,$$
$$p_0 u > \left\{(p_1 u)^2 + (p_2 u)^2 + (p_3 u)^2\right\}^{1/2}, \quad p_0 u > 0.$$

1. *Овсянников Л.В.* Групповой анализ дифференциальных уравнений. — М.: Наука, 1978. — 400 с.
2. *Ибрагимов Н.Х.* Группы преобразований в математической физике. — М.: Наука, 1983. — 278 с.
3. *Olver P.J.* Applications of Lie groups to differential equations. — New York: Springer Verlag, 1986.
4. *Фущич В.И.* О дополнительной инвариантности релятивистских уравнений движения // Теор. мат. физика. — 1971. — **7**, № 1. — С. 3–12.
5. *Фущич В.И.* О новом методе исследования групповых свойств уравнений математической физики // Докл. АН СССР. — 1979. — **246**, № 4. — С. 846–850.
6. *Фущич В.И., Никитин А.Г.* Симметрии уравнений Максвелла. — Киев: Наук. думка, 1983. — 200 с.
7. *Фущич В.И., Штелень В.М.* О линейных и нелинейных системах дифференциальных уравнений, инвариантных относительно группы Шредингера // Теор. мат. физика. — 1983. — **56**, № 3. — С. 387–394.
8. *Olver P., Rosenau Ph.* The construction of special solutions to partial differential equations // Phys. Lett. A. — 1986. — **114**, N 3. — P. 107–112.
9. *Фущич В.И.* О симметрии и точных решениях многомерных нелинейных волновых уравнений // Укр. мат. журн. — 1987. — **39**, № 1. — С. 116–123.
10. *Fushchych W.I., Tsyfra I.M.* On a reduction and solutions of non-linear wave equations with broken symmetry // J. Phys. A: Math. Gen. — 1987. — **20**. — L45–L48.
11. *Шульга М.В.* Симметрия и некоторые частные решения уравнения Даламбера с нелинейным условием // Теоретико-групповые исследования уравнений математической-физики. — Киев: Ин-т математики АН УССР, 1985. — С. 36–38.
12. *Чопик В.И.* О групповых свойствах линейных уравнений параболического типа с нелинейными дополнительными условиями // Симметрия и решения нелинейных уравнений математической физики. — Киев: Ин-т математики АН УССР, 1987. — С. 63–66.

On the new invariance algebras and superalgebras of relativistic wave equations

W.I. FUSHCHYCH, A.G. NIKITIN

We show that any relativistic wave equation for a particle with mass $m > 0$ and arbitrary spin s is invariant under the Lie algebra of the group $GL(2s+1, C)$. The explicit form of basis elements of this algebra is given for any s. The complete sets of symmetry operators of the Dirac and Maxwell equations are obtained, which belong to the classes of the first- and second-order differential operators with matrix coefficients. Corresponding new conservation laws and constants of motion are found.

1. Introduction

The classical Lie approach is the main mathematical apparatus used for the analysis of symmetry of partial differential equations [1, 30]. This approach was used to prove that the Poincaré group is the maximal symmetry group of the Dirac equation [2, 22] and that the maximal symmetry of Maxwell's equations is determined by the conformal group replenished by the Heaviside–Larmor–Rainich transformation. However, in spite of its power and universality, the Lie approach does not make it possible to find all symmetry operators of the given equation. Actually it gives possibility of finding only such symmetry operators which are the first-order differential operators.

Using the non-Lie approach [5, 6, 8, 9], in which the group generators can be differential operators of any order and even integro-differential operators, the new invariance groups of a number of relativistic wave equations have been found. It has been demonstrated that the Dirac equation is invariant under the group $SU(2)\times SU(2)$ [5, 6, 12] and that the Kemmer–Duffin–Petiau equation for vector field is invariant under the group $SU(3)\times SU(3)$ [29, 12]. The non-Lie approach enables to find an additional symmetry of the Dirac and Kemmer–Duffin–Petiau equations describing the particles interacting external electromagnetic field [13, 27]. The hidden symmetry of Maxwell's equations has also been found which is described by the eightparameter transformation group including the subgroup of Heaviside–Larmor–Rainich transformations [13, 14, 15, 17].

In this paper we continue to study symmetries of the Dirac, Weyl and Maxwell equations and of relativistic wave equations for any spin particles. The main results obtained here can be formulated as follows.

(i) We found that any Poincaré-invariant wave equation for a particle of arbitrary spin s and mass $m = 0$ is additionally invariant under the $2(2s+1)(2s+1)$-dimensional Lie algebra which is isomorphic to the Lie algebra of the group $GL(2s+1, C)$. The explicit form of basis elements of this invariance algebra is found for any value of s.

Reprinted with permission from J. Phys. A: Math. Gen., 1987, **20**, P. 537–549

Original article is available at http://www.iop.org/EJ/journal/JPhysA

Thus the additional symmetry of *an arbitrary* relativistic wave equation is described whereas previously one studied, as a rule, the symmetry properties of specific equations.

(ii) In our earlier work we restricted ourselves to studying symmetry operators of relativistic wave equations which belong to a finite-dimensional Lie algebra [17]. Here we also consider the symmetry operators belonging to the classes of first and second-order differential operators with matrix coefficients which, generally speaking, are not basis elements of any finite-dimensional Lie algebra, but are closely connected with conservation laws. The complete set of symmetry operators of the Dirac equation in the class of first-order differential operators with matrix coefficients (class $\mathfrak{M}_1$) is found. We also obtain the symmetry operators of the Weyl and Maxwell equations which form a basis of the Lie superalgebra.

(iii) The new conservation laws and motion constants, which are connected with hidden symmetry of the Dirac and Maxwell equations, are found.

The results of this paper supplement and in some sense complete those ones obtained by us and expanded by other authors [3, 31, 24, 32] by studying the additional symmetry of Poincaré-invariant wave equations.

2. Additional symmetry of Poincaré-invariant wave equations for arbitrary spin particles

In this section we demonstrate that any relativistic wave equation for a particle of non-zero mass and spin $s = 0$ has more extensive symmetry than Poincaré invariance, and describe this additional symmetry exactly.

Let us write an arbitrary linear (differential or integro-differential) equation in the following symbolic form

$$L\psi = 0, \tag{2.1}$$

where L is a linear operator defined on a vector space H, $\psi \in H$.

Let Q be an operator defined on H. We say that Q is the symmetry operator of the equation (2.1), if

$$L(Q\psi) = 0 \tag{2.2}$$

for any ψ satisfying (2.1).

Definition. *Equation* (2.1) *is Poincaré-invariant and describes a particle of mass m and spin s if it has* 10 *symmetry operators P_μ, $J_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$, which form a basis of the Lie algebra of Poincaré group, and any solution ψ satisfies the conditions*

$$P_\mu P^\mu \psi = m^2\psi, \quad W_\mu W^\mu = -m^2 s(s+1)\psi, \tag{2.3}$$

where W_μ is the Lubansky–Pauli vector

$$W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma. \tag{2.4}$$

Below we consider only such equations (2.1) which satisfy the given definition and so can be interpreted as equations for a relastivistic particle of spin s and mass m. The symmetry operators P_μ, $J_{\mu\nu}$ of such a equation satisfy the commutation relations

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu),$$
$$[J_{\mu\nu}, J_{\lambda\sigma}] = i(g_{\mu\sigma}J_{\nu\lambda} + g_{\nu\lambda}J_{\mu\sigma} - g_{\mu\lambda}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\lambda}) \tag{2.5}$$

which characterise the Lie algebra of the Poincaré group $P(1,3)$. The eigenvalues of the corresponding Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ are fixed and given by the relations (2.3). Let us emphasise that we do not make any supposition with regards to the explicit form of the operators P_μ and $J_{\mu\nu}$ — they can be differential operators of first order and non-local (integro-differential) operators as well.

Theorem 1. *Any Poincaré-invariant equation for a particle of mass m and spin s is invariant under the algebra*[1] *$GL(2s+1, C)$.*

Proof. Let P_μ, $J_{\mu\nu}$ be symmetry operators of the equation (2.1) satisfying the commutation relations (2.5). Then in accordance with definition (2.3) the following combinations

$$Q^\pm_{\mu\nu} = \frac{1}{m^2}[\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma \pm i(P_\mu W_\nu - P_\nu W_\mu)] \tag{2.6}$$

are also the symmetry operators of these equations.

Using (2.5) and the relations

$$[W_\mu, P_\nu] = 0, \quad [W_\mu, W_\nu] = i\varepsilon_{\mu\nu\rho\sigma}P^\rho W^\sigma \tag{2.7}$$

can make sure that the operators (2.6) satisfy the conditions

$$[Q^\pm_{\mu\nu}, Q^\pm_{\lambda\sigma}] = i(g_{\mu\sigma}Q^\pm_{\nu\lambda} + g_{\nu\lambda}Q^\pm_{\mu\sigma} - g_{\mu\lambda}Q^\pm_{\nu\sigma} - g_{\nu\sigma}Q^\pm_{\mu\lambda})m^{-4}(P_\mu P^\mu)^2, \tag{2.8}$$

$$C_1 = \frac{1}{4}Q^\pm_{\mu\nu}Q^{\pm\mu\nu} = -m^4 W_\lambda W^\lambda P_\sigma P^\sigma,$$
$$C_2 = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}Q^{\pm\mu\nu}Q^{\pm\rho\sigma} = \mp i m^{-4}W_\mu W^\mu P_\sigma P^\sigma. \tag{2.9}$$

It follows from (2.3) and (2.8) that on the set of solutions of equation (2.1) the operators (2.6) satisfy the commutation relations

$$[Q^\pm_{\mu\nu}, Q^\pm_{\lambda\sigma}]\psi = i(g_{\mu\sigma}Q^\pm_{\nu\lambda} + g_{\nu\lambda}Q^\pm_{\mu\sigma} - g_{\mu\lambda}Q^\pm_{\nu\sigma} - g_{\nu\sigma}Q^\pm_{\mu\lambda})\psi, \tag{2.10}$$

which characterise the Lie algebra of group $SL(2,C)$. From (2.3) and (2.9) one obtains the eigenvalues of corresponding Casimir operators

$$C_1\psi = \frac{1}{2}\left(l_0^2 + l_1^2 - 1\right)\psi, \quad C_2\psi = il_0 l_1\psi, \tag{2.11}$$

where $l_0 = s$, $l_1 = \pm(s+1)$.

So we have demonstrated that any Poincaré-invariant equation for a particle of non-zero mass and spin $s \neq 0$ is additionally invariant under the algebra $SL(2,C)$, the basis elements of which belong to the enveloping algebra of the $P(1,3)$ and are given exactly by the relations (2.6). According to (2.11) operators (2.6) realise the

[1] We use the same notation for the groups and for the corresponding Lie algebras.

representation $D(l_0,l_1)=D(s,\pm(s+1))$ of the algebra $GL(2,C)$. Now we see that this invariance algebra can be extended to a $2(2s+1)$-dimensional Lie algebra isomorphic to $GL(2s+1,C)$. Namely, basis elements of the algebra $GL(2s+1,C)$ have the following form

$$\begin{aligned} &\lambda_{n+k\,n}=a_{nk}(Q^+_{23}-Q^+_{02})P^s_n, \quad \lambda_{n\,n+k}=a_{kn}P^s_n(Q^+_{23}+Q^+_{02}),\\ &\tilde{\lambda}_{mn}=Q_1\lambda_{mn}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} &P^s_n=\prod_{n'\neq n}\frac{Q_{12}-s-1+n'}{n'-n}, \quad Q_1=\frac{\varepsilon_{abc}}{2s(s+1)}Q^+_{0a}Q^+_{bc},\\ &m,n=1,2,\ldots,2s+1, \quad k=0,1,\ldots,2s-n \end{aligned}$$

and a_{kn} are the coefficients determined by the recurrent relations

$$\begin{aligned} &a_{0n}=1, \quad a_{1n}=[n(2s+1-n)]^{-1/2},\\ &a_{\lambda n}=a_{\lambda-1\,n}a_{\lambda-1\,n+\lambda-1}, \quad \lambda=2,3,\ldots,2s-n. \end{aligned}$$

Actually the polynomials of the symmetry operators $Q^+_{\mu\nu}$ given by the relations (2.12) manifestly are symmetry operators of equation (2.1). Operators (2.11) form the basis algebra $GL(2s+1,C)$ inasmuch as they satisfy the following commutation relations

$$\begin{aligned} &[\lambda_{ab},\lambda_{cd}]=-[\tilde{\lambda}_{ab},\lambda_{cd}]=\delta_{bc}\lambda_{ad}-\delta_{ad}\lambda_{bc},\\ &[\lambda_{ab},\tilde{\lambda}_{cd}]=\delta_{bc}\tilde{\lambda}_{ad}-\delta_{ad}\tilde{\lambda}_{bc}, \quad a,b,c,d=1,2,\ldots,2s+1 \end{aligned} \tag{2.13}$$

which characterise the algebra $GL(2s+1,C)$. The relations (2.13) are satisfied on the set of solutions of the equation (2.1).It can be verified by direct calculation using the equivalent matrix representation for the basis elements of the algebra $SL(2,C)$ (which is evaluated according to (2.11))

$$Q^+_{ab}=\varepsilon_{abc}S_c, \quad Q^+_{0a}=-S_a.$$

Here S_a are the matrices which realise the representation $D(s)$ of the $SO(3)$ algebra in the Gelfand–Zetlin basis [21]. Thus the theorem is proved.

So if equation (2.1) is Poincaré invariant and describes a particle of spin s and mass $m>0$, it is invariant also under the algebra $GL(2s+1,C)$, the basis elements of which belong to the enveloping algebra of algebra $P(1,3)$. Operators (2.12) together with the Poincaré generators P_μ and $J_{\mu\nu}$ form a basis of the $10+2(2s+1)$-dimensional Lie algebra isomorphic to the algebra $P(1,3)\oplus GL(2s+1,C)$. The last statement can be easily verified by moving to the new basis $P_\mu\to P_\mu$, $J_{\mu\nu}\to J_{\mu\nu}-Q_{\mu\nu}$, $\lambda_{mn}\to\lambda_{mn}$, $\tilde{\lambda}_{mn}\to\tilde{\lambda}_{mn}$, where

$$\begin{aligned} &Q_{12}=\sum_n(s-n+1)\lambda_{mn}, \quad Q_{03}=\sum_n(s-n+1)\tilde{\lambda}_{mn},\\ &Q_{23}=\sum_n\frac{1}{2a_{1n}}(\lambda_{n\,n+1}+\lambda_{n+1\,n}), \quad Q_{31}=-i[Q_{12},Q_{23}],\\ &Q_{02}=i[Q_{23},Q_{03}], \quad Q_{01}=-i[Q_{31},Q_{03}]. \end{aligned}$$

The theorem proved has a constructive character since as it gives the explicit form of basis elements of the additional invariance algebra via the Poincaré generators. Starting, for example, from the Poincaré generators for the Dirac equation

$$P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{4}[\gamma_\mu, \gamma_\nu], \tag{2.14}$$

where γ_μ are the Dirac matrices, one obtains from (2.6) the additional symmetry operators of this equation found earlier by Fushchych and Nikitin [12]. In an analogous way to formulae (2.6) and (2.12), the additional invariance algebras of the Kemmer–Duffin–Petiau and Proca equations can be obtained (see [12, 17, 19, 20]) and even the invariance algebra of infinite-component wave equations [18] can be found.

Let us note that relativistic wave equations for a particle of spin $s > 0$ also possess such additional invariance algebras which belong to the class of integro-differential operators [5, 8, 9, 16, 17, 29, 27] and, generally speaking, are not numbered among the enveloping algebras of the algebra $P(1,3)$.

3. Symmetry operators of the Dirac equation in the class $\mathfrak{M}_1$

Here we consider in detail the symmetry properties of the Dirac equation

$$L\psi \equiv (\gamma^\mu p_\mu - m)\psi = 0. \tag{3.1}$$

It is well known that the symmetry of equation (3.1) which can be found in the classical Lie approach is exhausted by invariance under the algebra $P(1,3)$, the basis elements of which are given in (2.14), and under a corresponding group of transformations, i.e. the Poincaré group.

Theorem 1 gives the possibility of extending the set of symmetry operators of the Dirac equation. Actually, using formulae (2.6), (2.14) and (3.1) one obtains the additional symmetry operators [12, 17]

$$Q^\pm_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu] + \frac{i}{2m}(\gamma_\mu p_\nu - \gamma_\nu p_\mu)(1 \pm i\gamma_4). \tag{3.2}$$

The operators (3.2) are the first-order differential operators with matrix coefficients (i.e. belong to the class $\mathfrak{M}_1$) and so they cannot be found in the frames of classical Lie approach. But these operators (with fixed sign $\pm$) form the basis of 16-dimensional Lie algebra together with the Poincaré generators (2.14). It follows from the above that the Dirac equation is invariant under the 16-parameter group including the Lorentz transformations (generated by P_μ, $J_{\mu\nu}$) and the transformations which are generated by the operators (3.2). Specifically these transformations have the form

$$\psi \to \psi' = \exp(2i\theta Q)\psi = (\cos\theta - \gamma_1\gamma_2\sin\theta)\psi\frac{i}{m}(1 \mp i\gamma_4)\sin\theta\left(\gamma_1\frac{\partial\psi}{\partial x_2} - \gamma_2\frac{\partial\psi}{\partial x_1}\right)$$

if $Q = Q^\pm_{12}$ etc [12].

It can be interesting to know whether the operators (2.14) and (3.3) exhaust all symmetry operators of the Dirac equation in the class $\mathfrak{M}_1$. It turns out that this is not so.

Here we find the complete set of symmetry operators $Q \in \mathfrak{M}_1$ for equation (3.1) which, however, do not form the basis of Lie algebra.

Theorem 2. *The Dirac equation has* 26 *linearly independent symmetry operators* $Q \in \mathfrak{M}_1$. *These operators include the Poincaré generators* (2.14), *identity operator and fifteen operators given below*

$$\begin{aligned}
&\eta_\mu = \frac{1}{4} i\gamma_4 (p_\mu - m\gamma_\mu), \quad \omega_{\mu\nu} = mS_{\mu\nu} + \frac{1}{2} i(\gamma_\mu p_\nu - \gamma_\nu p_\mu),\\
&A_\mu = \omega_{\mu\nu} x^\nu + x^\nu \omega_{\mu\nu} - i\gamma_\mu, \quad B = i\gamma_4 (D - m\gamma_\mu x^\mu),
\end{aligned} \tag{3.3}$$

where

$$D = x^\mu p_\mu + \frac{3}{2} i, \quad S_{\mu\nu} = \frac{1}{4} i[\gamma_\mu, \gamma_\nu], \quad \mu, \nu = 0, 1, 2, 3. \tag{3.3$'$}$$

Proof. To find all linearly independent symmetry operators of the Dirac equation in the class $\mathfrak{M}_1$ it is necessary to obtain the general solution of the following operator equations

$$[L, Q] = f_Q L, \tag{3.4}$$

where $L = \gamma^\mu p_\mu - m$, Q and f_Q are unknown operators belonging to $\mathfrak{M}_1$:

$$Q = \tilde{A}^\mu p_\mu + \tilde{B}, \quad f_Q = \tilde{C}^\mu p_\mu + \tilde{D},$$

$\tilde{A}_\mu$, $\tilde{B}_\mu$, $\tilde{C}_\mu$ and $\tilde{D}$ are 4×4 matrices depending on $x = (x_0, \boldsymbol{x})$.

Relations (3.4) mean that the operators on the RHS and LHS give the same result acting on arbitrary solutions of the Dirac equation. On the set of these solutions operator p_0 can be expressed via the operators p_a with matrix coefficients: $p_0\psi = H\psi \equiv (\gamma_0 m + \gamma_0\gamma_a p_a)\psi$. In other words it is sufficient to restrict ourself by considering symmetry operators of a form such that

$$Q = \boldsymbol{B} \cdot \boldsymbol{p} + G, \tag{3.5}$$

where $\boldsymbol{B}$ and G are 4×4 matrices depending on x. For the operators (3.5) the invariance condition (3.4) reduces to the following form:

$$[p_0 - H, Q] = f_Q (p_0 - H), \tag{3.6}$$

where $f_Q \equiv 0$ insofar as the commutator on the LHS cannot depend on p_0.

An unknown operator (3.5) can be expanded via a complete set of the Dirac matrices

$$\begin{aligned}
&\boldsymbol{B} = I\boldsymbol{d}^0 + i\gamma_4 \boldsymbol{d}^1 + \gamma_\nu \boldsymbol{n}^\nu + S_{\mu\nu} \boldsymbol{m}^{\mu\nu} + \gamma_4\gamma_\nu \boldsymbol{b}^\nu,\\
&G = Ia^0 + i\gamma_4 a^1 + \gamma_\nu c^\nu + S_{\mu\nu} f^{\mu\nu} + \gamma_4\gamma_\nu g^\nu,
\end{aligned} \tag{3.7}$$

where $\boldsymbol{d}^0$, $\boldsymbol{d}^1$, $\boldsymbol{n}^\nu$, $\boldsymbol{m}^{\mu\nu}$, $\boldsymbol{b}^\nu$, a^0, a^1, c^ν, $f^{\mu\nu}$, g^ν are unknown functions on x.

Substituting (3.5) and (3.7) into (3.6) and equating coefficients by the linearly independent matrices and differential operators one comes to the following system of partial differential equations:

$$\begin{aligned}
&\boldsymbol{n}^0 = \boldsymbol{b}^0 = 0, \quad n^a_b = i\varepsilon_{abc} d^2_c, \quad b^a_b = i\varepsilon_{abc} d^3_c,\\
&m^{0a}_b = i\delta_{ab} \boldsymbol{A}^0, \quad m^{ab}_c = \varepsilon_{abc} \boldsymbol{A}^1, \quad a, b, c = 1, 2, 3,
\end{aligned} \tag{3.8}$$

$$
\begin{gathered}
\frac{\partial d_a^\mu}{\partial x_b}=-\frac{\partial d_b^\mu}{\partial x_a},\quad \frac{\partial d_a^\mu}{\partial x_a}=\frac{\partial d_b^\mu}{\partial x_b},\quad a\neq b,\quad m\,\mathrm{div}\,\boldsymbol{d}^0=0,\quad m\,\mathrm{div}\,\boldsymbol{d}^1=2ima^1,\\
\dot{\boldsymbol{d}}^2=-\frac{1}{2}\,\mathrm{rot}\,\boldsymbol{d}^3,\quad \dot{\boldsymbol{d}}^3=\frac{1}{2}\,\mathrm{rot}\,\boldsymbol{d}^2,\quad \dot{\boldsymbol{d}}^i=-\mathrm{grad}\,\boldsymbol{A}^i,\quad \mathrm{div}\,\boldsymbol{d}^i=-3\boldsymbol{A}^i,\ i=0,1,\\
c^a=-\frac{1}{2}(\mathrm{rot}\,\boldsymbol{d}^2)_a,\quad c^0=-\frac{1}{3}\,\mathrm{div}\,\boldsymbol{d}^3+m\boldsymbol{A}^0,\quad g^0=\frac{1}{3}\,\mathrm{div}\,\boldsymbol{d}^2,\\
g^a=-\frac{1}{2}(\mathrm{rot}\,\boldsymbol{d}^3)_a-imd_a^1,\quad \dot{a}^0=-\frac{1}{2}i\,\mathrm{div}\,\boldsymbol{d}^0,\quad \mathrm{grad}\,a^0=-\frac{3}{2}i\ddot{\boldsymbol{d}}^0,\\
a^1=-\frac{1}{2}i\,\mathrm{div}\,\dot{\boldsymbol{d}}^1+\frac{1}{3}m\,\mathrm{div}\,\boldsymbol{d}^2,\quad \mathrm{grad}\,a^1=-m\dot{\boldsymbol{d}}^2-\frac{3}{2}i\ddot{\boldsymbol{d}}^1,\\
f^{0a}=\frac{1}{2}\dot{d}_a^0-\frac{1}{4}i(\mathrm{rot}\,\boldsymbol{d}^1)_a,\quad f^{ab}=\varepsilon_{abc}\left[\frac{1}{2}i\dot{d}_c^1+\frac{1}{4}(\mathrm{rot}\,\boldsymbol{d}^0)_c+md_c^2\right],
\end{gathered}
\tag{3.9}
$$

where the dot denotes the derivative on x_0 and there is no sum by the repeated indices. The symbol $\boldsymbol{d}^\mu$ denotes a vector with components $(d_1^\mu, d_2^\mu, d_3^\mu)$ (analogous notation is used for other vector quantities).

The first line in (3.9) gives the equations in the Killing form. Using this circumstance it is not difficult to obtain the general solution of the system (3.9) for $m\neq 0$:

$$
\begin{gathered}
\boldsymbol{d}^0=\boldsymbol{x}\times\boldsymbol{\eta}+\boldsymbol{\rho}x_0+\boldsymbol{\nu},\quad \boldsymbol{d}^1=\boldsymbol{\xi}+\lambda\boldsymbol{x},\quad \boldsymbol{d}^2=\boldsymbol{x}\times\boldsymbol{\varepsilon}+\boldsymbol{\zeta},\\
\boldsymbol{d}^3=\boldsymbol{\varepsilon}x_0+\mu\boldsymbol{x}+\boldsymbol{\sigma},\quad g^0=0,\quad g^a=-im(\xi_a+\lambda x_a),\\
f^{0a}=\frac{1}{2}\rho_a,\quad f^{ab}=\frac{1}{2}\varepsilon_{abc}(2m\zeta_c-\eta_c)+m(x_a\varepsilon_b-x_b\varepsilon_a),\\
c^0=-\mu-m(\boldsymbol{\rho}\cdot\boldsymbol{x}+\varkappa),\quad c^a=\varepsilon^a,\quad A^0=-\boldsymbol{\rho}\cdot\boldsymbol{x}-\varkappa,\\
A^1=-\lambda x_0+\omega,\quad a^0=\Omega,\quad a^1=-\frac{3}{2}i\lambda.
\end{gathered}
\tag{3.10}
$$

Here the Greek letters denote arbitrary constants.

So the general solution of the system (3.9) depends on 26 arbitrary numerical parameters. Substituting (3.7), (3.8) and (3.10) into (3.5) and using equation (3.1), one obtains a general expression for the symmetry operator $Q\in\mathfrak{M}_1$ for the Dirac equation as a linear combination of the Poincaré group generators (2.14), identity operator and the operators (3.3). The theorem is proved.

So we have obtained the complete set of the symmetry operators $Q\in\mathfrak{M}_1$ for the Dirac equation with $m\neq 0$. Besides the Poincaré group generators (2.14) this set includes four operators which coincide on the set of the equation (3.1) solutions with Lubansky–Pauli vector (2.4), six operators $\omega_{\mu\nu}=\frac{1}{2}(Q^+_{\mu\nu}+Q^-_{\mu\nu})$, trivial identity operator and five symmetry operators B and A_μ, $\mu=0,1,2,3$, which belong to the enveloping algebra generated by the Poincaré generators.

The operators (3.3) satisfy the following commutation relations

$$
[B,P_\mu]=-2i\eta_\mu,\quad [B,\eta_\mu]=-\frac{1}{2}i(P_\mu+mA_\mu),\quad [A_\mu,P_\nu]=\frac{1}{m}[\eta_\mu,\eta_\nu]=-2i\omega_{\mu\nu}.
$$

However these operators do not form the basis of the Lie algebra inasmuch as the commutators $[\omega_{\mu\nu},\omega_{\lambda\sigma}]$ do not belong to the class $\mathfrak{M}_1$.

One of the most interesting consequences of the symmetry described in theorem 2 is the existence of new conservation laws for the Dirac equation. Corresponding new conserved currents have the form

$$\eta_{\nu\mu} = \frac{1}{4}\left(\bar{\psi}\gamma_4\gamma_\nu\frac{\partial\Psi}{\partial x^\mu} - \frac{\partial\psi}{\partial x^\mu}\gamma_\nu\gamma_4\psi\right) + m\bar{\psi}\gamma_4 S_{\mu\nu}\psi,$$

$$\omega_{\mu\rho\nu} = \frac{1}{4}i\left(\frac{\partial\bar{\psi}}{\partial x^0}S_{\nu\mu}\psi + \bar{\psi}S_{\nu\rho}\frac{\partial\psi}{\partial x^\mu} - \bar{\psi}S_{\nu\mu}\frac{\partial\psi}{\partial x^\rho} - \frac{\partial\bar{\psi}}{\partial x^\mu}S_{\mu\nu}\psi\right) + \tag{3.11}$$

$$+\frac{1}{2}m\bar{\psi}[S_{\mu\nu},\gamma_\lambda]_+\psi, \quad B_\nu = 2x^\mu\eta_{\mu\nu}, \quad A_{\mu\nu} = 2x^\lambda\omega_{\mu\lambda\nu}.$$

The tensors $\eta_{\mu\nu}$, $\omega_{\mu\rho\nu}$, $A_{\mu\nu}$ and the vector B_ν correspond to the symmetry operators η_μ, $\omega_{\mu\rho}$, $\boldsymbol{A}_\mu$ and $\boldsymbol{B}$. All quantities (3.11) satisfy the continuity equations

$$p^\nu\eta_{\mu\nu} = 0, \quad p^\nu\omega_{\mu\rho\nu} = 0, \quad p^\nu A_{\mu\nu} = 0, \quad p^\nu B_\nu = 0$$

and so generate conservation laws.

4. Additional symmetry of the Weyl and massless Dirac equations

Here we study the symmetry of the Weyl equation

$$\sigma^\mu p_\mu\varphi = 0, \tag{4.1}$$

where φ is the two-component spinor and σ^μ the Pauli matrices. Putting

$$\psi = \begin{pmatrix} \varphi + \varphi^* \\ i(\varphi^* - \varphi) \end{pmatrix} \tag{4.2}$$

one can rewrite this equation in the Dirac form

$$\gamma^\mu p_\mu\psi = 0, \tag{4.3}$$

where γ^μ are the Dirac matrices in the Majorana representation. So we consider the symmetry properties of equation (4.3) in order to obtain the results which are valued as for the Weyl equation as for the massless Dirac one.

Theorem 3. *The massless Dirac equation has* 46 *symmetry operators* $Q \in \mathfrak{M}_1$. *These operators are*

$$P_\mu,\ J_{\mu\nu},\ K_\mu,\ D,\ F,\ FP_\mu,\ FJ_{\mu\nu},\ FK_\mu,\ FD,\ I, \tag{4.4a}$$

$$\hat{A}_\mu = \hat{\omega}_{\mu\nu}x^\mu + x^\nu\hat{\omega}_{\mu\nu} - \gamma_\mu, \quad \hat{\omega}_{\mu\nu} = \gamma_\mu p_\nu - \gamma_\nu p_\mu, \quad F\hat{A}_\mu, \tag{4.4b}$$

where $K_\mu = 2x_\mu D - p_\mu x_\nu x^\nu + 2S_{\mu\nu}x^\nu$, $F = i\gamma_4$, P_μ, $J_{\mu\nu}$ *and* D *are given in* (2.14) *and* (3.5′).

Proof. This can be carried out in full analogy with the proof of theorem 2. The general solution of the system (3.8) for the case $m = 0$ has the form

$$\boldsymbol{d}^\alpha = \boldsymbol{x}\boldsymbol{x}\cdot\boldsymbol{\mu}^\alpha + \frac{1}{2}\mu^\alpha\left(x_0^2 - \boldsymbol{x}^2\right) + \boldsymbol{x}\times\boldsymbol{\eta}^\alpha + \nu^\alpha\boldsymbol{x} + \rho^\alpha x_0\boldsymbol{x} + \lambda^\alpha + \omega^\alpha x_0,$$

$$\alpha=0,1,\quad \boldsymbol{d}^2=\boldsymbol{x}\times\boldsymbol{\varepsilon}+\xi\boldsymbol{x}-\boldsymbol{\zeta}x_0+\boldsymbol{\varphi},\quad \boldsymbol{d}^3=\boldsymbol{x}\times\boldsymbol{\xi}+\sigma\boldsymbol{x}+\boldsymbol{\varepsilon}x_0+\boldsymbol{\varkappa},$$
$$A^\alpha=-\left[\boldsymbol{x}\cdot\boldsymbol{\mu}^\alpha x_0+\frac{1}{2}\rho^\alpha\left(x_0^2+\boldsymbol{x}^2\right)+\nu^\alpha x_0+\boldsymbol{\omega}^\alpha\boldsymbol{x}+\chi^\alpha\right],$$
$$a^\alpha=-\frac{3}{2}i\left(\boldsymbol{x}\cdot\boldsymbol{\mu}^\alpha+\rho^\alpha x_0+\delta^\alpha\right),\quad c^0=\sigma,\quad x^a=-\varepsilon_a,\quad g^0=\xi,\tag{4.5}$$
$$g^a=-\zeta_a,\quad f^{0a}=\frac{1}{2}(-\eta_a^1+\rho^0x_a+\mu_a^0x_0+\omega_a^0),$$
$$f^{ab}=-\frac{1}{2}\varepsilon_{abc}(\mu_c^1x_0+\rho_1^1x_c+\omega_c^1+\eta_c^0$$

and includes 46 independent parameters denoted by the Greek letters. Substituting (3.5) and (4.5) into (3.7) and using equation (4.3) one obtains a general expression for the symmetry operator of the massless Dirac equation in the form of a linear combination of the operators (4.4). Thus the theorem is proved.

Among the operators (4.4) there are exactly fourteen symmetry operators, which do not belong to the enveloping algebra generated by the conformal group generators P_μ, $J_{\mu\nu}$, K_μ, D and by the operator $F=i\gamma_4$. These essentially new symmetry operators are given in (4.4b).

Operators (4.4) transform the real wave function ψ (4.2) into real wave function $\psi'=Q\psi$ and so they are also the symmetry operators for the Weyl equation (4.1). Incidentally the linear transformations of ψ (4.2) generate linear and antilinear transformations of a two-component Weyl spinor.

The operators (4.4) do not form a basis of the Lie algebra. However, one can consider different subsets of the operators (4.4) which have the structure of the Lie algebra or superalgebra. Thus the operators (4.4a) form the basis of 32-dimensional Lie algebra including the Lie algebra of the conformal group. The operators $J_{\mu\nu}$, $\hat{\omega}_{\mu\nu}$, F and $\lambda_\mu=FP_\mu$ satisfy the following commutation and anticommutation relations:

$$[\hat{\omega}_{\mu\nu},\hat{\omega}_{\lambda\sigma}]_+=-2i[J_{\mu\nu},p_\lambda p_\sigma]=2(g_{\mu\lambda}p_\nu p_\sigma+g_{\nu\sigma}p_\mu p_\lambda-g_{\mu\sigma}p_\nu p_\lambda-g_{\nu\lambda}p_\mu p_\sigma),$$
$$[J_{\mu\nu},\hat{\omega}_{\lambda\sigma}]=i(g_{\mu\sigma}\hat{\omega}_{\nu\lambda}+g_{\nu\lambda}\hat{\omega}_{\mu\sigma}-g_{\mu\lambda}\hat{\omega}_{\nu\sigma}-g_{\nu\sigma}\hat{\omega}_{\mu\lambda}),\quad F^2=I,\tag{4.6}$$
$$[\hat{\omega}_{\mu\nu},\lambda_\rho]_+=[\omega_{\mu\nu},F]_+=0,\quad [\lambda_\mu,\lambda_\nu]_+=2p_\mu p_\nu,\quad [\lambda_\mu,F]_+=2P_\mu,$$

where the symbol $[A,B]_+$ denotes the anticommutator $[A,B]_+=AB+BA$.

It follows from (2.5) and (4.6) that the set of symmetry operators $\{P_\mu, J_{\mu\nu}, p_\mu p_\nu, I, F, \lambda_\mu, \hat{\omega}_{\mu\nu}\}$ form the basis of the Lie superalgebra (which includes ten symmetry operators $p_\mu p_\nu$ not belonging to the class $\mathfrak{M}_1$).

5. The symmetry and supersymmetry of Maxwell's equations

We shall write Maxwell's equations for the electromagnetic field in vacuum in the following form [17]:

$$L_1\psi\equiv(i\partial/\partial x_0+\sigma_2\boldsymbol{S}\cdot\boldsymbol{p})\psi=0,$$
$$L_2^a\psi\equiv(p_a-\boldsymbol{S}\cdot\boldsymbol{p}p_a)\psi=0.\tag{5.1}$$

Here

$$\sigma_2=i\begin{pmatrix}0&-1\\1&0\end{pmatrix},\quad S_a=i\begin{pmatrix}s_a&0\\0&s_a\end{pmatrix},\tag{5.2}$$

where 1 and 0 are unit and zero 3×3 matrices, s_a are the generators of irreducible representation $D(1)$ of the group $SO(3)$ with the matrix elements $(s_a)_{bc} = i\varepsilon_{abc}$. The symbol ψ denotes the six-component function, $\psi =$ column $(E_1, E_2, E_3, H_1, H_2, H_3)$, where E_a and H_a are the components of the vectors of electric and magnetic fields strengths.

It is well known that the Maxwell equations are invariant under the conformal group $C(1,3)$ and under the group H of Heaviside–Larmor–Rainich transformations. Moreover it was found [14, 15, 16, 17] that these equations also have the additional hidden symmetry in the class of integro-differential operators which is determined by the algebra $GL(2,C)$. It was demonstrated also that $GL(2,C)$ is the most extensive invariance algebra of the Maxwell equations if one supposes the symmetry operators do not depend on x.

Here we study the symmetry of the Maxwell equations in quite another aspect. The requirement that the symmetry operators belong to a finite-dimensional Lie algebra is very important if one is interested in studying the symmetry groups of the equations considered. However for many applications (e.g. for constructing conservation laws) this requirement is not essential. So we do not require that the symmetry operators of Maxwell's equations should belong to a finite-dimensional Lie algebra but restrict the class of operators considered by the second-order differential operators with constant matrix coefficients. In other words we consider the symmetry operators of a form such that

$$Q = d_{ab}p_a p_b + c_b p_b + g, \quad a, b = 1, 2, 3, \tag{5.3}$$

where d_{ab}, c_b and g are 6×6 numerical matrices. The operators (5.3) do not depend on po inasmuch as one can always $p_0\psi$ via $\sigma_2 \boldsymbol{S} \cdot \boldsymbol{p}\psi$ according to equation (5.1). Let us denote the class of the operators (5.3) by the symbol $\mathfrak{M}_2$.

We shall see that the Maxwell equations have non-trivial symmetry operators in the class $\mathfrak{M}_2$ which do not belong to the enveloping algebra of the conformal group generators. On the other hand the analysis of more extensive classes of the Maxwell equation symmetry operators is very complicated and cannot be carried out within the framework of the present paper.

The invariance condition for equation (5.1) under the operators (5.3) can be written in the following general form [17]

$$[L_1, Q] = \alpha_Q^1 L_1 + \beta_Q^{1a} L_2^a, \quad [L_2^a, Q] = \alpha_Q^{2a} L_1 + \beta_Q^{2a,d} L_2^d, \tag{5.4}$$

where in our case $\alpha_Q^1 = \alpha_Q^{2a} \equiv 0$ since the commutators on the LHS cannot depend on p_0, and

$$\beta_Q^{1a} = g_{bc}^a p_b p_c + f_b^a p_b + h^a, \quad \beta_Q^{2a,d} = g_{bc}^{a,d} p_b p_c + f_c^{a,d} p_c + h^{a,d}, \tag{5.5}$$

where g_{bc}^k, f_b^k, h^k are numerical matrices, $k = a$ or $k = a, d$.

Any of the matrices in (5.3) and (5.5) can be represented as a linear combination of the matrices D_c^ν and G_{cd}^ν, where

$$D_c^\nu = \sigma_\nu S_c, \quad G_{cd}^\nu = \sigma_\nu(\delta_{cd} - S_c S_d - S_d S_c).$$

Here σ_ν are the 6×6 Pauli matrices commuting with S_a of (5.2). Calculating the commutators in (5.4) and equating the coefficients by the linearly independent matrices and differential operators one can prove the following statement.

Theorem 4. *The Maxwell equations* (5.1) *have ten linearly independent symmetry operators* $Q \in \mathfrak{M}_2$ *which do not belong to the enveloping algebra of the Lie algebra of the group* $C(1,3) \otimes H$*. These operators have the form*

$$Q_{ab} = \sigma_1 q_{ab}, \quad \tilde{Q}_{ab} = \sigma_3 q_{ab}, \tag{5.6}$$

where

$$q_{ab} = [(\boldsymbol{S} \times \boldsymbol{p})_a, (\boldsymbol{S} \times \boldsymbol{p})_b] - p^2\delta_{ab}, \quad p^2 = p_1^2 + p_2^2 + p_3^2.$$

Proof. The proof can be carried out in full analogy with the proofs of theorems 2 and 3 and so can be omitted. We note only that equations (5.3)–(5.5) are satisfied by the 46 linearly independent operators given below:

$$\begin{gathered} \sigma_0, \quad i\sigma_0 p_a, \quad \sigma_0 p_a p_b, \quad \sigma_0 \boldsymbol{S}\cdot\boldsymbol{p}, \quad ip_a \boldsymbol{S}\cdot\boldsymbol{p}, \qquad i\sigma_2, \\ \sigma^2 p_a, \quad i\sigma_2 p_a p_b, \quad \sigma_2 \boldsymbol{S}\cdot\boldsymbol{p}, \quad i\sigma_2 p_a \boldsymbol{S}\cdot\boldsymbol{p}, \quad Q_{ab}, \quad \tilde{Q}_{ab}, \end{gathered} \tag{5.7}$$

where Q_{ab} and $\tilde{Q}_{ab}$ are given in (5.6). All operators of (5.7) with the exception of Q_{ab} and $\tilde{Q}_{ab}$ can be expressed via P_a, $\boldsymbol{S}\cdot\boldsymbol{p} = \frac{1}{2}\varepsilon_{abc}J_{ab}P_c$ and σ_2, where J_{ab} and P_a are the Poincaré generators given by the formulae (2.14) with $\frac{1}{4}i[\gamma_a, \gamma_b] \to \varepsilon_{abc}S_c$, σ_2 is the matrix of (5.2) (which is the generator of the Heaviside–Larmor–Rainich transformations).

Note 1. From twenty operators (5.6) exactly ten are linearly independent in so far as

$$(Q_{11} + Q_{22} + Q_{33})\psi = (\tilde{Q}_{11} + \tilde{Q}_{22} + \tilde{Q}_{33})\psi = 0,$$

where ψ is an arbitrary solution of equations (5.1).

Note 2. The operators α_Q^1, β_Q^{1a} and α_Q^{2a} from (5.4) which correspond to the symmetry operators (5.6) are zero matrices. For $\beta_{Q_{ab}}^{2c,d}$ and $\beta_{Q_{ab}}^{2c,d}$ one obtains by direct calculation

$$\beta_{Q_{bc}}^{2a,d} = i\sigma_2\beta_{Q_{bc}}^{2a,d} = -\sigma_1\delta_{ad}[(\boldsymbol{S}\times\boldsymbol{p})_a, (\boldsymbol{S}\times\boldsymbol{p})_b]_+.$$

So we have determined the complete set of the Maxwell equation symmetry operators in the class $\mathfrak{M}_2$. Using the notation given in (5.2) and below formula (5.2), it is not difficult to represent the transformations $\psi \to Q_{ab}\psi$ and $\psi \to \tilde{Q}_{ab}\psi$ generated by operators (5.6), in the terms of the electromagnetic field strengths

$$E_c \to g_{ab}^{cd} H_d, \quad H_c \to g_{ab}^{cd} E_d, \tag{5.8}$$

$$E_c \to g_{ab}^{cd} E_d, \quad H_c \to -q_{ab}^{cd} H_d, \tag{5.9}$$

where

$$g_{ab}^{cd} = p_a p_b \delta_{cd} - p_a p_c \delta_{bd} - p_b p_c \delta_{ad} + p^2(\delta_{ac}\delta_{bd} + \delta_{bc}\delta_{ad} - \delta_{ab}\delta_{cd}).$$

The invariance of Maxwell's equations under transformations (5.8) and (5.9) can be easily verified by direct calculation.

The operators (5.6) do not form a basis of a Lie algebra. However, one can consider subsets of the operators (5.6) which can be extended to the Lie superalgebras. One of these subsets includes the following operators:

$$
\begin{aligned}
&Q^1 = \frac{1}{2}\varepsilon_{abc} c_a Q_{bc}, \quad Q^2 = \frac{1}{2}\varepsilon_{abc} c_a \tilde{Q}_{bc}, \\
&Q^3 = \boldsymbol{S}\cdot\boldsymbol{p}, \quad Q^4 = \frac{1}{2} c_a c_b p_a p_b, \quad Q^5 = p^2,
\end{aligned} \tag{5.10}
$$

where c_a are arbitrary numbers satisfying the condition $c_a c_a = 1$. The operators (5.10) satisfy the relations

$$
\begin{aligned}
&[Q^a, Q^b]_+ = 2\delta_{ab}\left(Q^a\right)^2, \quad \left(Q^1\right)^2 = \left(Q^2\right)^2 = Q^6 \equiv \left(Q^4 - Q^5\right)^2, \\
&\left(Q^3\right)^2 \psi = Q^5\psi, \quad [Q^a, Q^4] = [Q^a, Q^5] = [Q^4, Q^5] = 0
\end{aligned}
$$

and so form the basis of the Lie superalgebra together with the operator Q^6. This superalgebra can be extended by adding the operators $Q^{6+a} = i\sigma^2 \boldsymbol{S}\cdot\boldsymbol{p}Q^a$, $Q^{9+a} = i\sigma^2\boldsymbol{S}\cdot\boldsymbol{p}(Q^a)^2$ and $Q^{12+a} = p^2(Q^a)^2$, $a = 1,2,3$, which satisfy the relations

$$
\begin{aligned}
&[Q^{6+a}, Q^{6+b}]_+ = 2\delta_{ab} Q^{12+b}, \quad [Q^{6+a}, Q^b]_+ = 2\delta_{ab} Q^{6+b}, \\
&[Q^{9+a}, Q^B] = [Q^{12+a}, Q^B] = 0, \quad B = 1, 2, \dots, 15.
\end{aligned}
$$

In conclusion let us give the explicit form of the motion constants of the electromagnetic field which correspond to the symmetry operators (5.6). Due to the Maxwell equations the following bilinear combinations do not depend on x_0 and so are conserved in time

$$
\begin{aligned}
I_{ab} &= \int d^3x\, \psi^T Q_{ab}\psi = \int d^3x\, [(\text{rot}\,\boldsymbol{H})_a(\text{rot}\,\boldsymbol{H})_b - (\text{rot}\,\boldsymbol{E})_a(\text{rot}\,\boldsymbol{E})_b + \\
&\quad + E_a p^2 E_b - H_a p^2 H_b\big], \\
\tilde{I}_{ab} &= \int d^3x\, \psi^T \tilde{Q}_{ab}\psi = \int d^3x\, \big[E_a p^2 H_b + H_a p^2 E_b - \\
&\quad - (\text{rot}\,\boldsymbol{E})_a(\text{rot}\,\boldsymbol{H})_b - (\text{rot}\,\boldsymbol{H})_a(\text{rot}\,\boldsymbol{E})_b].
\end{aligned} \tag{5.11}
$$

In contrast with the classical motion constants (energy, momentum, etc) the integral combinations (5.11) depend not only on $\boldsymbol{E}$ and $\boldsymbol{H}$ but also on the derivatives of these vectors.

So starting from the symmetry operators (5.6) found above we obtain ten new constants of motion for the electromagnetic field in vacuum given by relations (5.11). These motion constants, in contrast to the Lipkin ones [25, 4, 23, 26], have nothing to do with the Lorentz or conformal invariance of the Maxwell equations inasmuch as the corresponding symmetry operators (5.6) do not belong to the enveloping algebra of the algebra $C(1,3) \oplus H$.

Acknowledgment. We would like to thank the referee for his useful comments.

Note added in proof. In the formulation of theorem 3 we have omitted six symmetry operators of the massless Dirac equation, which have the form $Q_{\mu\nu} = -Q_{\nu\mu} = [K_\nu, A_\mu]$. So this equation has 52 linearly independent symmetry operators $Q \in \mathfrak{M}_1$ All the symmetry operators $Q \in \mathfrak{M}_1$ for the Dirac equation with $m \neq 0$ belong to the enveloping algebra of algebra $P(1,3)$ inasmuch as operator B (3.3) can be represented as $D\psi = \frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma}\psi$ on the set of the Dirac equation solutions.

1. Ames W.F., Nonlinear partial differential equations in engineering, Vol. 1, New York, Academic, 1965.
2. Danilov Yu.A., Preprint IAE-1976, Kurchalov Institute of Atomic Energy, 1968.
3. Da Silveira, *Nuovo Cimento A*, 1980, **56**, 385–395.
4. Fradkin D.M., *J. Math. Phys.*, 1965, **6**, 879–890.
5. Fushchych W.I., Preprint ITP-E-70-32, Kyiv, Institute for Theoretical Physics, 1970.
6. Fushchych W.I., *Teor. Mat. Fiz.*, 1971, **7**, 3–12 (*Theor. Math. Phys.*, **7**, 3–11).
7. Fushchych W.I., *Nuovo Cimento Lett.*, 1973, **6**, 133–138.
8. Fushchych W.I., *Nuovo Cimento Lett.*, 1974, **11**, 508–512.
9. Fushchych W.I., On the new method of investigation of the group properties of systems of partial differential equations, in Group-Theoretical Methods in Mathematical Physics, Kyiv, Institute of Mathematics, 1978, 5–44.
10. Fushchych W.I., *Dokl. Akad. Nauk SSSR*, 1979, **246**, 846–850.
11. Fushchych W.I., *Ukr. Mat. Zurn.*, 1981, **33**, 834–837 (*Ukr. Math. J.*, **33**, 632–635).
12. Fushchych W.I., Nikitin A.G., *Nuovo Cimento Lett.*, 1977, **19**, 347–352.
13. Fushchych W.I., Nikitin A.G., *Nuovo Cimento Lett.*, 1978, **21**, 541–545.
14. Fushchych W.I., Nikitin A.G., *Nuovo Cimento Lett.*, 1979, **24**, 220–224.
15. Fushchych W.I., Nikitin A.G., *J. Phys. A: Math. Gen.*, 1979, **12**, 747–757.
16. Fushchych W.I., Nikitin A.G., *Czech. J. Phys. B*, 1982, **32**, 476–480.
17. Fushchych W.I., Nikitin A.G., Symmetries of Maxwell's equations, Kyiv, Naukova Dumka, 1983 (Engl. transl., Dordrecht, Reidel, 1986).
18. Fushchych W.I., Onufrijchuck S.P., *Dokl. Akad. Nauk SSSR*, 1977, **235**, 1056–1059.
19. Fushchych W.I., Vladimirov V.A., *Dokl. Akad. Nauk SSSR*, 1981, **287**, 1105–1108.
20. Fushchych W.I., Vladimirov V.A., *J. Phys. A: Math. Gen.*, 1983, **16**, 1921–1925.
21. Gelfand I.M., Zetlin M.L., *Dokl. Akad. Nauk SSSR*, 1950, **71**, 1017–1020.
22. Ibragimov N.H., *Dokl. Akad. Nauk SSSR*, 1969, **185**, 1226–1228.
23. Kibble T.W.B., *J. Math. Phys.*, 1965, **6**, 1022–1025.
24. Kotelnikov G.A., *Nuovo Cimento B*, 1982, **72**, 68–78.
25. Lipkin D.M., *J. Math. Phys.*, 1964, **5**, 696–700 .
26. Michelsson J., Niederle J., *Lett. Math. Phys.*, 1984, **8**, 195–205.
27. Nikitin A.G., On the invariance group of Kemmer–Duffin–Petiau equations for particle with anomalous, in Group-Theoretical Methods in Mathematical Physics, Kyiv, Institute of Mathematics, 1978, 81–95.
28. Nikitin A.G., Symmetries of the Weyl equation, in Group-Theoretical Studies of Mathematical Physics equations, Kyiv, Institute of Mathematics, 1985, 31–35.
29. Nikitin A.G., Segeda Yu.N., Fushchych W.I., *Teor. Mat. Fiz.*, 1976, **29**, 82–94 (*Theor. Math. Phys.*, **29**, 943–954).
30. Ovsjannikov L.V., The group analysis of differential equations, Moscow, Nauka, 1978.
31. Stražev V.I., *Vest. Akad. Nauk BSSR*, 1981, **5**, 75–80.
32. Stražev V.I., Shkol'nikov P.L., *Izv. Vuzov Fiz.*, 1984, **4**, 81–88.

*In [Beckers J., Debergh N., Nikitin A.G., Lie extended symmetries and relativistic particles, *J. Phys. A*, 1992, **25**, 6145–6154] the results of this paper were generalized to Kemmer–Duffin–Petiau equation. *Editors' Remark.*

On a reduction and solutions of non-linear wave equations with broken symmetry

W.I. FUSHCHYCH, I.M. TSYFRA

A generalised definition for invariance of partial differential equations is proposed. Exact solutions of the equations with broken symmetry are obtained.

Let us consider the non-linear wave equation

$$\begin{gathered}\Box u + F_1(u) = 0, \quad u = u(x_0, x_1, x_2, x_3), \\ \Box = \partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2, \quad \partial_\mu = \partial/\partial x_\mu, \qquad \mu = 0, 1, 2, 3,\end{gathered} \tag{1}$$

where $F_1(u)$ is an arbitrary smooth function. The ansatz

$$u = f(x)\varphi(\omega) + g(x) \tag{2}$$

suggested by Fushchych [5] was used to construct the family of exact solutions of equations (1). $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is the function to be determined and $\omega = (\omega_1, \omega_2, \omega_3)$ are new invariant variable. Wide classes of exact solutions of equation (1) have been constructed by Fushchych and Serov [7, 8], Fushchych et al [10] and Fushchych and Shtelen [9]. It is important to note that Poincaré invariance of equation (1) was used.

The possibility of using an ansatz of type (2) to find exact solutions of the non-linear wave equations with broken symmetry naturally arises in connection with the fact that many equations of theoretical physics are not invariant with respect to the Poincaré, Galilei and Euclidean groups. A more specific formulation of this problem is as follows: are we able to construct the solutions of wave equations not invariant with respect to the Lorentz group, for example, but nevertheless with the help of the Lorentz-invariant ansatz?

The present letter suggests an affirmative answer to this question, i.e. we construct the many-dimensional non-linear wave equations with broken symmetry. The multi-parametrical exact solutions of these equations are found with the help of ansatz (2), previously used to find exact solutions of Poincaré- and Galilei-invariant equations only. It is obvious that ansatz (2) cannot be applied to the equations with arbitrary breakdown of symmetry, which is why the equation with the breakdown of symmetry should have some hidden symmetry. The set of equations with such symmetry was considered by Fushchych and Nikitin [6]. We do not deal with the symmetry of all the solutions of the equations but only with a definite subset of solutions, which may be much wider that the symmetry of the equation itself. This idea will be used below.

Let us consider the wave equation with broken symmetry

$$Lu \equiv \Box u + F(x, \underset{1}{u}, u) = 0, \tag{3}$$

Reprinted with permission from J. Phys. A: Math. Gen., 1987, **20**, L45–L48

Original article is available at http://www.iop.org/EJ/journal/JPhysA

where $F(x,\underset{1}{u},u)$ is an arbitrary smooth function, depending on $x=(x_0,x_1,\,x_2,x_3)$, $\underset{1}{u}\equiv(\partial u/\partial x_0,\partial u/\partial x_1,\partial u/\partial x_2,\partial u/\partial x_3)$. Following Fushchych [4] we generalise the Lie definition of invariance of equation (3).

Definition. *We shall say that equation* (3) *is invariant with respect to some set of operators* $\hat{Q}=\{\hat{Q}_A\}$, $A=1,2,\ldots,N$, *a number of linearly independent operators, if the following condition is fulfilled:*

$$\hat{Q}_A Lu\Big|_{\substack{Lu=0,\\ \{\hat{Q}_A u\}=0}}=0, \tag{4}$$

where $\{\hat{Q}_A u\}=0$ *is a set of equations*

$$\hat{Q}_A u=0,\quad D\hat{Q}_A u=0,\quad D^2\hat{Q}_A u=0,\quad \ldots,\quad D^n\hat{Q}_A u=0, \tag{5}$$

where D *is an operator of total differentiation. Condition* (4) *is a necessary condition for reduction of differential equations.*

Definition (4) is a generalisation of the Lie definition (see, e.g., Ovsyannikov [12])

$$\hat{Q}_A Lu\Big|_{Lu=0}=0, \tag{6}$$

where $\hat{Q}_A$ are a number of first-order differential operators forming a Lie algebra.

To demonstrate the efficacy of definition (4) and to find exact solutions of equation (3) we choose the function F in a form

$$F=-\left(\frac{\lambda_0}{x_0}\right)^2\left(\frac{\partial u}{\partial x_0}\right)^2+\left(\frac{\lambda_1}{x_1}\right)^2\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\lambda_2}{x_2}\right)^2\left(\frac{\partial u}{\partial x_2}\right)^2+\left(\frac{\lambda_3}{x_3}\right)^2\left(\frac{\partial u}{\partial x_3}\right)^2, \tag{7}$$

where λ_μ are arbitrary parameters and $x_\mu\neq 0$.

Theorem. *The maximal local (in the Lie sense) invariance group of equations* (3) *and* (7) *is the two-parametrical group of the transformations*

$$x_\mu=e^a x_\mu,\quad u'=e^{2a}u\quad \textit{and}\quad u'=u+c,\quad c=\text{const}, \tag{8}$$

where a *is real parameter.*

The proof of the theorem is reduced to application of the well known Lie algorithm and we do not present it here. One can make sure that non-linearity breaks the rotational and translational symmetry.

Now we show that the Lorentz-non-invariant equations (3) and (7) are reduced to an ordinary differential equation with the help of the Lorentz-invariant ansatz

$$u=\varphi(\omega),\quad \omega=x_\mu x^\mu=x_0^2-x_1^2-x_2^2-x_3^2. \tag{9}$$

Substituting (9) into (3) and (7) we obtain the ordinary differential equation

$$\omega\frac{d^2\varphi}{d\omega^2}+2\frac{d\varphi}{d\omega}=\lambda^2\left(\frac{d\varphi}{d\omega}\right)^2,\quad \lambda^2=\lambda_0^2-\lambda_1^2-\lambda_2^2-\lambda_3^2. \tag{10}$$

Solving equation (10), we obtain

$$\varphi(\omega)=-2\left(-\lambda^2\right)^{-1/2}\operatorname{arctg}\left[\omega\left(-\lambda^2\right)^{-1/2}\right],\quad -\lambda^2>0, \tag{11}$$

$$\varphi(\omega) = \left(\lambda^2\right)^{-1/2} \ln\left(\frac{\left(\lambda^2\right)^{1/2} + \omega}{\left(\lambda^2\right)^{1/2} - \omega}\right), \quad -\lambda^2 < 0. \tag{12}$$

Thus the Lorentz-non-invariant (in the Lie sense) equations (3) and (7) are reduced to an ordinary differential equation.

Formulae (11) and (12) give a Lorentz-invariant family of solutions of equations (3) and (7). It means that the following set of conditions is fulfilled:

$$J_{\mu\nu}u(x) = 0, \quad \mu, \nu = 0, 1, 2, 3, \tag{13}$$

$$J_{0a} = x_0 \frac{\partial}{\partial x_a} + x_a \frac{\partial}{\partial x_0}, \quad J_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a}, \quad a, b = 1, 2, 3 \tag{14}$$

for the set of solutions (11) and (12).

The operators (14) generate Lorentz transformations. Equations (13) are the concrete realisation of the first equation of (5). In this case the index A varies from 1 to 6.

Thus, equations (13) pick out a Lorentz-invariant subset of the set of all solutions of equations (3) and (7). In other words, equations (3) and (7) are Lorentz-invariant in the sense of definition (4).

Now let us consider the equation

$$\frac{\partial^2 u}{\partial t^2} = \lambda \Delta u (\nabla u)^2, \quad \lambda = \frac{1}{3} m^2. \tag{15}$$

It is simple to verify that equation (15) is not invariant with respect to Galilean transformations, generated by operators

$$G_a = t \frac{\partial}{\partial x_a} + m x_a, \quad a = 1, 2, 3. \tag{16}$$

In this case equations $\{\hat{Q}_A u\} = 0$ are

$$G_a u = t \frac{\partial u}{\partial x_a} - m x_a u = 0, \tag{17}$$

$$\frac{\partial}{\partial t}(G_a u) = 0. \tag{18}$$

Thus equation (15) is invariant under transformations generated by operators (16) in the sense of definition (4). It means that the subset of solutions of equations (15) picked out by means of conditions (17) and (18) is invariant under Galilean transformations while equation (15) is not invariant under these transformations.

The Galilean-invariant ansatz has the form

$$u = \varphi(t) + m\left(x_1^2 + x_2^2 + x_3^2\right)/2t, \quad \omega = t, \quad f = 1. \tag{19}$$

Substituting (19) into (15), we obtain

$$\frac{d^2\varphi}{\partial t^2} = 0 \quad \leftrightarrow \quad u = m\left(x_1^2 + x_2^2 + x_3^2\right)/2t + At + C, \tag{20}$$

where A and C are arbitrary constants.

A generalised definition of the invariance (4) can be applied to the system of partial differential equations.

Let us consider, for example, a non-linear Dirac system of equations:

$$\begin{aligned} &\gamma_\mu\partial^\mu\psi + g\left[2\bar\psi(x_\mu\partial^\mu)\psi - \left(x^2/c_\alpha x^\alpha\right)\bar\psi(c_\mu\partial^\mu)\psi\right]M^{-1}(x)(\bar\psi\psi)^{1/3}\psi = 0,\\ &M(x) = 2(c_\alpha x^\alpha)^{-1}\bar\psi S_{\mu\nu}c^\mu\beta^\nu\psi + \bar\psi\psi,\\ &S_{\mu\nu} = \frac{1}{4}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \quad \mu,\nu,\alpha = 0,1,2,3, \end{aligned} \tag{21}$$

where g, β_μ, c_α are arbitrary parameters.

Equation (21) is not invariant under conformal transformations. Nevertheless, it is reduced to the system of ordinary differential equations

$$i\gamma_\mu\beta^\mu\frac{d\varphi}{d\omega} + g(\bar\varphi\varphi)^{1/3}\varphi = 0 \tag{22}$$

with the help of the conformally invariant ansatz (4)

$$\psi(x) = \left[\gamma_\mu x^\mu/(x^2)^2\right]\varphi(\omega), \quad \omega = \beta_\mu x^\mu/x^2, \quad \beta^2 \neq 0, \quad x^2 = x_\mu x^\mu \neq 0, \tag{23}$$

where $\varphi(\omega)$ is the four-component spinor depending on a variable ω. The general solution of equation (22) is the vector function

$$\varphi = \exp\left[-i\frac{\gamma_\mu\beta^\mu}{\beta^2}g(\bar\chi\chi)^{1/3}\omega\right]\chi, \tag{24}$$

where χ is a constant spinor.

Equation (21) is invariant under the transformations generated by the operator $c_\mu K^\mu$ on a set of solutions of the equations

$$c_\mu K^\mu\psi = 0, \quad c^\mu K_\mu = 2(cx)(x\partial) - x^2(c\partial) + 2(cx) - (\gamma c)(\gamma x). \tag{25}$$

In conclusion we note that an idea like the one set forward here was used by Bluman and Cole [2], Ames [1], Fokas [3] and Olver and Rosenau [11], as was kindly indicated by the referee. We are grateful to the referee for his valuable remarks.

1. Ames W.F., Nonlinear partial differential equations in engineering, Vol. 2, New York, Academic, 1972.
2. Bluman G.W., Cole I.D., *J. Math. Mech.*, 1969, **18**, 1025–1042.
3. Fokas A.S., PhD thesis, California Institute of Technology, 1979.
4. Fushchych W.I., *Dokl. Akad. Nauk*, 1979, **246**, 846–850.
5. Fushchych W.I., The symmetry of mathematical physics problems, in Algebraic-Theoretical Studies in Mathematical Physics, Kyiv, Institute of Mathematics, 1981, 6–28.
6. Fushchych W.I., Nikitin A.G., Symmetry of Maxwell equations, Kyiv, Naukova Dumka, 1983.
7. Fushchych W.I., Serov N.I., *J. Phys. A: Math Gen.*, 1983, **16**, 3645–3656.
8. Fushchych W.I., Serov N.I., *Dokl. Akad. Nauk*, 1983, **273**, 543.
9. Fushchych W.I., Shtelen W.M., *J. Phys. A: Math Gen.*, 1983, **16**, 271–277.
10. Fushchych W.I., Shtelen W.M., Zhdanov R.Z., *Phys. Lett. B*, 1985, **159**, 189–191.
11. Olver P.J., Rosenau P., *Phys. Lett. A*, 1986, **114**, 107–112.
12. Ovsyannikov L.V., The group analysis of differential equations, Moscow, Nauka, 1978.

On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter

W.I. FUSHCHYCH, W.M. SHTELEN

The concept of approximate symmetry is introduced. We describe all nonlinearities $F(u)$ with which the non-linear wave equation $\Box u + \lambda u^3 + \varepsilon F(u) = 0$ with a small parameter ε is approximately scale and conformally invariant. Some approximate solutions of wave equations in question are obtained using the approximate symmetry.

Let us consider the non-linear wave equation

$$\Box u + \lambda u^3 + \varepsilon F(u) = 0, \tag{1}$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alembertian, $\mu = \overline{0,3}$; λ is an arbitrary constant; $\varepsilon \ll 1$ is a small parameter; $u = u(x)$, $x \in R(1,3)$; $F(u)$ is an arbitrary smooth function. By means of Lie's method (see [5, 4]) one can make sure that when $F(u) \neq 0$ and $F(u) \neq u^3$, equation (1) is invariant under the Poincaré group $P(1,3)$ only, because the term $\varepsilon F(u)$ breaks down the scale and conformal symmetry of the equation $\Box u + \lambda u^3 = 0$.

Below we describe all functions $F(u)$ with which equation (1) is approximately invariant under the scale and conformal transformations.

Let us represent an arbitrary solution, analytic in ε, of equation (1) in the form

$$u = w + \varepsilon v, \tag{2}$$

where w and v are some smooth functions of x. After substitution of (2) into (1) and equating to zero the coefficients of zero and first power of ε we get the following system of partial differential equations (PDE):

$$\begin{aligned} &\Box u + \lambda w^3 = 0, \\ &\Box v + 3\lambda w^2 v + F(w) = 0. \end{aligned} \tag{3}$$

Definition. *We shall call the approximate symmetry of equation* (1) *the (exact) symmetry of the system* (3).

Theorem 1. *Equation* (1) *is approximately scale invariant (in the sense of the above definition) if and only if*

$$F(u) = \begin{cases} \dfrac{2\lambda b}{k+1} u^3 + \dfrac{3\lambda c}{k} u^2 + a u^{2-k}, & k \neq 0, -1, \\ 2\lambda b u^3 + 3\lambda c u^2 \ln u + a u^2, & k = 0, \\ 2\lambda b u^3 \ln u - 3\lambda c u^2 + a u^3, & k = -1 \end{cases} \tag{4}$$

(k, a, b, c are arbitrary constants), with the generator of scale transformations having the form

$$D = x\partial - w\partial_w + (kv + bw + c)\partial_v. \tag{5}$$

Reprinted with permission from J. Phys. A: Math. Gen., 1989, **22**, L887–L890

Original article is available at http://www.iop.org/EJ/journal/JPhysA

Proof. Using Lie's algorithm [5, 4] we find from the condition of invariance that the generator of scale transformations should have the form

$$D = x\partial - w\partial_w + \eta^2(v,w)\partial_v$$

provided (following from the invariance of the second equation of system (3)) that

$$\eta^2_{vv} = \eta^2_{ww} = \eta^2_{wv} = 0 \quad \Rightarrow \quad \eta^2 = kv + bw + c,$$
$$2\lambda bw^3 + 3\lambda cw^2 + (2-k)F - w\frac{dF}{dw} = 0. \tag{6}$$

The general solution of equations (6) is given in (4). Thus the theorem is proved.

In particular, as follows from Theorem 1, the equation

$$\Box u + \lambda u^3 + \varepsilon u = 0 \tag{7}$$

is approximately scale invariant and the corresponding generator has the form $D = x\partial - w\partial_w + v\partial_v$. This statement holds true even if $\lambda = 0$.

Theorem 2. *Equation* (1) *is approximately conformally invariant if and only if*

$$F(u) = -3\lambda\beta u^2 + au^3 \tag{8}$$

with the generator of conformal transformations having the form

$$K = 2cx[x\partial - w\partial_w - (v-\beta)\partial_v] - x^2 c\partial, \tag{9}$$

where β, a, c_μ are arbitrary constants.

The proof of Theorem 2 is performed in the same spirit as that of Theorem 1.

Suppose that in (2)

$$v = f(w), \tag{10}$$

where f is an arbitrary differentiable function. In this case the system (3) takes the form

$$\Box w + \lambda w^3 = 0, \tag{11}$$

$$w_\mu w^\mu \ddot{f} + \Box w\dot{f} + 3\lambda w^2 f + F(w) = 0, \quad w_\mu \equiv \partial w/\partial x^\mu. \tag{12}$$

From the condition of splitting of equation (12) one has to put

$$w_\mu w^\mu = A(w), \tag{13}$$

where A is some function of w. Equation (13) is compatible with (11) if $A(w) = \lambda w^4$, i.e.

$$w_\mu w^\mu = \lambda w^4. \tag{14}$$

(For more details see [1, 2].) Taking account of (11) and (14) we rewrite (12) as

$$\lambda(w^2\ddot{f} - w\dot{f} + 3f) + w^{-2}F(w) = 0. \tag{15}$$

So, if we find function $f(w)$ as a solution of equation (15), we thereby obtain by means of expressions (2) and (10) approximate solutions of equation (1). It will be noted that a subset of such solutions of equation (1) is approximately conformally invariant since the corresponding approximate system (11) and (14) is conformally invariant [1, 2]. Solutions of equation (15) for functions $F(w)$ given in (4) have the form

$$f(w) = \begin{cases} -\dfrac{a}{\lambda[k(k+2)+3]}w^{-k} - \dfrac{b}{k+1}w - \dfrac{c}{k}, & k \neq 0, -1, \\ -c\ln w - bw - \dfrac{1}{3}(2c + a/\lambda), & k = 0, \\ -w(a/2\lambda + b\ln w) + c, & k = -1. \end{cases} \tag{16}$$

The solution of the system (11) and (14) is the function

$$w = \pm[\lambda(x_\nu + a_\nu)(x^\nu + a^\nu)]^{-1/2}, \tag{17}$$

where a_ν are arbitrary constants.

When $\lambda = 0$, the non-trivial condition of splitting of equation (12) compatible with the equation $\Box w = 0$ is

$$w_\mu w^\mu = 1. \tag{18}$$

So, in this case we find approximate solutions of equation (1) by means of expressions (2) and (10), where function $f(w)$ is determined from the equation

$$\ddot{f} + F(w) = 0 \tag{19}$$

and w, in turn, is determined from the system

$$\Box w = 0, \quad w_\mu w^\mu = 1. \tag{20}$$

The system (20) is invariant under the extended Poincaré group $\tilde{P}(1,4)$ and has solution [1]

$$w = \alpha x + a, \quad \alpha_\nu \alpha^\nu = 1, \tag{21}$$

where a, α_ν are arbitrary constants.

In particular, equation

$$\Box u + \varepsilon u = 0 \tag{22}$$

is approximately invariant under the group $\tilde{P}(1,4)$ on the subset of solutions

$$u = w - \varepsilon\left(\frac{1}{6}w^3 + a_1 w + a_2\right), \tag{23}$$

where w is given in (21) and a_1, a_2 are arbitrary constants.

In conclusion, let us note some generalisations of the concept of approximate symmetry studied in this paper. First of all, obviously, one can consider higher orders of approximation of u in ε, i.e. $u = w + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \cdots$, and can study the symmetry of the corresponding approximate system of PDE for functions w, $v^{(1)}$, $v^{(2)}$, and so on. Secondly, one can expand in ε-series not only dependent variables, but also independent ones, e.g. $x_0 \equiv t = x + \varepsilon z^{(1)} + \varepsilon^2 z^{(2)} + \cdots$, and can construct in this way the corresponding approximate system and then study its symmetry. Another approach to the study of approximate symmetry it to use some special approximations, say the two-point Padé approximants

$$u = \sum_{k=0}^{m} \varepsilon^k f_k \left(\sum_{j=0}^{n} \varepsilon^j g_j \right)^{-1}, \quad m, n < \infty, \tag{24}$$

where functions f_k, g_j are determined from the condition: when $\varepsilon \to 0$ expression (24) coincides with the expansion

$$u = v^{(0)} + \varepsilon v^{(1)} + \varepsilon^2 v^{(2)} + \cdots, \quad \varepsilon \ll 1$$

and when $\varepsilon \to \infty$ (24) coincides with the expansion

$$u = w^{(0)} + \varepsilon^{-1} w^{(1)} + \varepsilon^{-2} w^{(2)} + \cdots, \quad \varepsilon \gg 1.$$

We also note that the symmetry of a system of PDE which approximates the non-linear wave equation was studied by Shulga [6]. Using symmetry properties, Mitropolsky and Shulga [3] obtained some asymptotic solutions of the non-linear wave equation.

Note added. Readers who are less well acquitted with work in this might refer to the related work of Winternitz et al [7] which is also concerned with this type of non-linear wave equation from a symmetry point of view.

1. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989.
2. Fushchych W.I., Zhdanov R.Z., Preprint N 468, University of Minnesota, 1988.
3. Mitropolsky Yu.A., Shulga M.W., *Dokl. Akad. Nauk*, 1987, **295**, № 1, 30–33.
4. Olver P., Applications of Lie groups to differential equations, Berlin, Springer, 1986.
5. Ovsyannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
6. Shulga M.W., in Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Kyiv, Institute of Mathematics, 1987, 96–99.
7. Winternitz P., Grundland A.M., Tuszynski J.A., *J. Math. Phys.*, 1987, **28**, 2194–2212.

Условная инвариантность и редукция нелинейного уравнения теплопроводности

В.И. ФУЩИЧ, Н.И. СЕРОВ

The conditional symmetry of the nonlinear heat conduction equation has been studied. Some exact solutions of the equations are obtained.

Рассмотрим нелинейное уравнение теплопроводности

$$u_0 + u_{11} = F(u), \tag{1}$$

где $u = u(x) \in \mathbb{R}_1$, $x = (x_0, x_1) \in \mathbb{R}_2$, $u_0 = \partial u/\partial x_0$, $u_{11} = \partial^2 u/\partial x_1^2$, $F(u)$ — гладкая функция, нелинейно зависящая от u.

В работах [1, 2] при помощи метода С. Ли [3] исследована инвариантность нелинейного уравнения теплопроводности. Из результатов этих работ следует, что уравнение (1) может быть инвариантно только относительно следующих операторов:

$$\partial_0,\ \ \partial_1,\ \ G = e^{x_0}(\partial_1 + m x_1 u \partial_u),\ \ D = 2x_0\partial_0 + x_1\partial_1 + M(u)\partial_u,\ \ X = e^{x_0} u \partial_u, \tag{2}$$

где $m = \text{const}$, $M(u)$ — некоторая заданная функция.

В настоящей работе исследована условная инвариантность (более подробно см. [4]) уравнения (1). Операторы условной инвариантности использованы для редукции исходного уравнения к обыкновенным дифференциальным уравнениям, а также для нахождения его точных решений.

Пусть

$$Q = A(x,u)\partial_0 + B(x,u)\partial_1 + C(x,u)\partial_u, \tag{3}$$

где A, B, C — гладкие функции своих аргументов, дифференциальный оператор первого порядка, действующий на многообразии (x, u).

Теорема 1. *Уравнение* (1) *Q-условно инвариантно* (*см.* [4]) *относительно оператора* (3), *если функции A, B, C удовлетворяют следующей системе дифференциальных уравнений.*

Случай I. $A \neq 0$ (*не умаляя общности, можно положить* $A = 1$) :

$$\begin{aligned} &B_{uu} = 0,\ \ C_{uu} = 2(B_{1u} + BB_u),\ \ 3B_u F = 2(C_{1u} + B_u C) - (B_0 + B_{11} + 2BB_1),\\ &CF_u - (C_u - 2B_1)F = C_0 + C_{11} + 2CB_1. \end{aligned} \tag{4}$$

Здесь и везде ниже индекс внизу возле функции означает дифференцирование по соответствующему аргументу.

Случай II. $A = 0$, $B = 1$:

$$CF_u - C_u F = C_0 + C_{11} + 2CC_{1u} + C^2 C_{uu}. \tag{5}$$

Передруковано з Докл. АН УССР. Сер. А, 1990, № 7, С. 24–27 за дозволом

Теорема 2. *Уравнение* (1) *Q-условно инвариантно относительно оператора* (3) *в предположении, что* $A = 1$, $B_u \neq 0$ *тогда и только тогда, когда оно локально эквивалентно уравнению*

$$u_0 + u_{11} = \lambda u^3 + \lambda_1 u + \lambda_2, \quad \lambda, \lambda_1, \lambda_2 = \mathrm{const}. \tag{6}$$

При этом оператор (3) *имеет вид*

$$Q = \partial_0 + \frac{3}{2}\sqrt{2\lambda}u\partial_1 + \frac{3}{2}(\lambda u^3 + \lambda_1 u + \lambda_2)\partial_u. \tag{7}$$

Доказательство теоремы 1 аналогично доказательству теоремы 5.7.2 из [4], а теорема 2 является результатом решения системы (4) при $B_u \neq 0$.

Используем оператор (7) для нахождения анзацев, редуцирующих уравнение (6) к обыкновенным дифференциальным уравнениям.

Рассмотрим типичные, существенно различные относительно корней правой части, представители уравнения (6):

$$\begin{array}{llll} 1) & u_0 + u_{11} = \lambda(u^3 - u); & 2) & u_0 + u_{11} = \lambda(u^3 - 3u + 2); \\ 3) & u_0 + u_{11} = \lambda u^3; & 4) & u_0 + u_{11} = \lambda(u^3 + u). \end{array} \tag{8}$$

Анзацы, полученные с помощью оператора (7), для каждого из уравнений (8) соответственно имеют вид:

$$\begin{array}{ll} 1) & 2\,\mathrm{arcth}\,u + \sqrt{2\lambda}x_1 = \varphi(\omega), \quad \omega = -\ln(1 - u^{-2}) + 3\lambda x_0; \\ 2) & -\dfrac{4}{9}\ln\dfrac{u+2}{u-1} - \dfrac{2}{3}(u-1)^{-1} - \sqrt{2\lambda}x_1 = \varphi(\omega), \\ & \omega = \dfrac{2}{9}\ln\dfrac{u+2}{u-1} - \dfrac{2}{3}(u-1)^{-1} - 3\lambda x_0; \\ 3) & \dfrac{2}{u} + \sqrt{2\lambda}x_1 = \varphi(\omega), \quad \omega = -\dfrac{1}{u^2} - 3\lambda x_0; \\ 4) & 2\,\mathrm{arctg}\,u - \sqrt{2\lambda}x_1 = \varphi(\omega), \quad \omega = -\ln(1 + u^{-2}) - 3\lambda x_0. \end{array} \tag{9}$$

Анзацы (9) редуцируют соответствующие уравнения (8) к следующим обыкновенным дифференциальным уравнениям:

$$1)\ 2\ddot\varphi = \dot\varphi^3 - \dot\varphi; \quad 2)\ 2\ddot\varphi = \dot\varphi^3 - 3\dot\varphi + 2; \quad 3)\ 2\ddot\varphi = \dot\varphi^3; \quad 4)\ 2\ddot\varphi = \dot\varphi^3 + \dot\varphi. \tag{10}$$

Обратим внимание на нелинейности в правых частях уравнений (10) и сравним их с нелинейностями исходных уравнений (8). Видим, что анзацы (9) позволили не только редуцировать уравнения (8), но и существенно изменили их нелинейные правые части, когда вместо функции u появилась функция $\dot\varphi$. Это позволяет проинтегрировать уравнения (10) и представить их общие решения с помощью элементарных функций:

$$\begin{array}{ll} 1)\ \varphi(\omega) = -2\,\mathrm{arcth}\sqrt{c_1 e^{\omega} + 1} + c_2; & 2)\ \ln\left[c_1 - \dfrac{3}{2}(\varphi + 2\omega)\right] = \ln c_2 - \dfrac{3}{2}(\varphi - \omega); \\ 3)\ \varphi(\omega) = 2\sqrt{c_1 - \omega} + c_2; & 4)\ \varphi(\omega) = 2\,\mathrm{arctg}\sqrt{c_1 e^{\omega} - 1} + c_2, \end{array} \tag{11}$$

где c_1, c_2 — постоянные интегрирования.

Используя формулы (9) и (11), находим решения уравнений (8) соответственно:

$$
\begin{aligned}
&1)\ \operatorname{arcth} u + \operatorname{arcth}\sqrt{\frac{u^2}{u^2-1}c_1 e^{3\lambda x_0}+1} = \frac{1}{2}(c_2 - \sqrt{2\lambda}x_1);\\
&2)\ u = -\frac{2c_2\exp\left(-\frac{9}{2}\lambda x_0 + \frac{3}{2}\sqrt{2\lambda}x_1\right) + 9\lambda x_0 + \frac{3}{2}\sqrt{2\lambda}x_1 + c_1 - 3}{c_2\exp\left(-\frac{9}{2}\lambda x_0 + \frac{3}{2}\sqrt{2\lambda}x_1\right) - 9\lambda x_0 - \frac{3}{2}\sqrt{2\lambda}x_1 - c_1};\\
&3)\ u = \frac{\sqrt{2/\lambda}(x_1+c_1)}{3(x_0+c_2) - \frac{1}{2}(x_1+c_1)^2};\\
&4)\ \operatorname{arctg} u - \operatorname{arctg}\sqrt{\frac{u^2}{u^2-1}c_1 e^{-3\lambda x_0}-1} = \frac{1}{2}(c_2 + \sqrt{2\lambda}x_1).
\end{aligned}
\tag{12}
$$

Отметим также, что и в предположении $B_u = 0$ для уравнения (1) можно найти операторы вида (3), не входящие в алгебру (2). Эти результаты представим в виде таблицы.

Таблица

Вид функции $F(u)$	F — решение уравнения $F''F = 2$	F — решение уравнения $F''F = 2(F'-1)$	$F(u) = \lambda u^3$
Оператор Q	$2\sqrt{x_0}\partial_1 + F(u)\partial_u$	$x_1\partial_1 + F(u)\partial_u$	$x_1^2\partial_0 + 3x_1\partial_1 + 3u\partial_u$
Анзац	$F'(u) = \varphi(x_0) + \frac{x_1}{\sqrt{x_0}}$	$F'(u) = x_1^2\varphi(x_0) + 1$	$u = x_1\varphi(\omega)$, $\omega = x_0 - \frac{x_1^2}{6}$
Редуцированное уравнение	$\varphi' + \frac{1}{2x_0}\varphi = 2$	$\varphi' - 2\varphi + 2\varphi^2 = 0$	$\varphi'' = 9\lambda\varphi^3$
Решение редуцированного уравнения	$\varphi = \frac{c_1}{\sqrt{x_0}} + \frac{4}{3}x_0$	$\varphi = \frac{1}{1+c_1e^{-2x_0}}$	$\int_0^{\omega}\frac{d\tau}{\sqrt{c_1+\tau^4}} = \frac{3}{2}\sqrt{2\lambda}(\omega + c_2)$
Решение уравнения (1)	$F'(u) = \frac{x_1+c_1}{\sqrt{x_0}} + \frac{4}{3}x_0$	$F'(u) = \frac{x_1^2}{1+c_1e^{-2x_0}} + 1$	$\int_0^{u/x_1}\frac{d\tau}{\sqrt{c_1+\tau^4}} = \frac{3}{2}\sqrt{2\lambda}\left(x_0 - \frac{x_1^2}{6} + c_2\right)$

Замечание. Полученные результаты легко переносятся на случай произвольного количества переменных $x = (x_0, \boldsymbol{x}) \in \mathbb{R}_{1+n}$ в уравнении (1).

В заключение приведем некоторые результаты, полученные нами для уравнения

$$
u_0 + u_{11} = F(u, u_1). \tag{13}
$$

Теорема 3. *Уравнение*

$$
u_0 + uu_1 + u_{11} = \lambda(u)u_1^3, \tag{14}
$$

где $\lambda(u)$ — произвольная дифференцируемая функция, Q-условно инвариантно относительно оператора

$$
Q = \partial_0 + u\partial_1. \tag{15}
$$

Теорема 4. *Уравнение*

$$u_0 + u_{11} = uu_1(1 - uu_1)(2 - uu_1) \tag{16}$$

Q-условно инвариантно относительно оператора

$$Q = \partial_0 + u\partial_1 + \partial_u. \tag{17}$$

При $\lambda(u) = 0$ уравнение (14) является уравнением Бюргерса. Анзац

$$x_0u - x_1 = \varphi(u), \tag{18}$$

получаемый с помощью оператора (15), редуцирует уравнение (14) к уравнению

$$\ddot{\varphi} = \lambda(u). \tag{19}$$

Анзац

$$\frac{1}{2}u^2 - x_1 = \varphi(\omega), \quad \omega = u - x_0, \tag{20}$$

полученный с помощью оператора (17), редуцирует уравнение (16) к уравнению

$$\ddot{\varphi} = \dot{\varphi}^3 + 1. \tag{21}$$

Общее решение уравнения (21) имеет вид

$$\ln\left[\sin\frac{\sqrt{3}}{2}(\varphi + \omega + c_2)\right] = -\frac{3}{2}(\varphi - \omega - c_1). \tag{22}$$

Из формул (20) и (22) находим решение уравнения (16)

$$\ln\left\{\sin\frac{\sqrt{3}}{4}\left[(u+1)^2 - 2(x_0+x_1) + c_2\right]\right\} = -\frac{3}{4}\left[(u-1)^2 + 2(x_0-x_1) + c_1\right]. \tag{23}$$

1. *Овсянников Л.В.* Групповые свойства уравнения нелинейной теплопроводности // Докл. АН СССР. — 1959. — **125**, № 3. — С. 492–495.

2. *Дородницын В.А., Князева И.В., Свирщевский С.Р.* Групповые свойства уравнения теплопроводности с источником в двумерном и трехмерном случаях // Дифференц. уравнения. — 1983. — **19**, № 7. — С. 1215–1224.

3. *Овсянников Л.В.* Групповой анализ дифференциальных уравнений. — М.: Наука, 1978. — 400 с.

4. *Фущич В.И., Штелень В.М., Серов Н.И.* Симметрийный анализ и точные решения нелинейных уравнений математической физики. — Киев: Наук. думка, 1989. — 336 с.

Условная симметрия уравнений нелинейной математической физики

В.И. ФУЩИЧ

Представлен обзор результатов по исследованию условной симметрии нелинейных уравнений математической и теоретической физики: волнового уравнения, уравнений Шредингера, Буссинеска, Кортевега–де Фриза, Максвелла, Дирака. Построены семейства точных решений, которые не могут быть получены в классическом подходе Ли.

1. Введение. В настоящей статье будут представлены некоторые результаты по исследованию условной симметрии нелинейных уравнений математической и теоретической физики, полученные в Институте математики АН Украины.

Термин и концепция "условная симметрия уравнения" или "условная инвариантность" введены в [1–10]. Под условной симметрией уравнения понимаем симметрию некоторого подмножества решений. Очевидно, такое общее определение условной симметрии требует детализации, в противном случае оно неэффективно. Конкретизация этого понятия означает следующее: аналитически описать условия на решения уравнения, при которых некоторое подмножество решений имеет более широкие (или другие) симметрийные свойства, чем все множество решений. Если такое описание осуществлено, то можем получить такие решения уравнения, которые невозможно получить в классическом подходе Ли, в котором, как известно, редукция многомерного дифференциального уравнения в частных производных (ДУЧП) к уравнениям с меньшим числом переменных проводится с использованием симметрии всего множества решений.

Эйлер, Ли, Бейтмен (1914), В. Смирнов и Л. Соболев (1932) и многие другие классики использовали в неявном виде симметрию подмножеств решений линейных уравнений Даламбера, Лапласа для построения точных решений.

Сравнительно недавно Блумен, Коул [11] предложили "неклассический метод решений, инвариантных относительно группы" для линейного теплового уравнения.

Олвер и Розенау [12] построили решения одномерного нелинейного уравнения акустики

$$u_{00} = uu_{11}, \quad u_{00} = \frac{\partial^2 u}{\partial t^2}, \quad u_{11} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

которые не могут быть получены с помощью метода Ли. Кларксон и Крускал [13] предложили "новый метод инвариантной редукции уравнения Буссинеска"

$$u_{00} + \frac{1}{2}(u^2)_{xx} + u_{xxxx} = 0. \tag{2}$$

Вывод 1. Если воспользоваться концепцией "условная симметрия ДУЧП", то все перечисленные результаты получаются с помощью единого симметрийного подхода.

Передруковано з Укр. мат. журн., 1991, **43**, № 11, С. 1456–1470 за дозволом

Вывод 2. Большинство линейных и нелинейных уравнений математической и теоретической физики: Даламбера, Максвелла, Шредингера, Дирака, Буссинеска, нелинейной теплопроводности и акустики обладают условной симметрией.

Замечание 1. Все решения уравнения Буссинеска (2), построенные Кларксоном и Крускалом, получены на основе концепции условной симметрии независимо в работах Леви и Винтерница [14] и В. Фущича и Н. Серова [10].

Рассмотрим некоторую систему ДУЧП

$$L(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{s}{u}) = 0, \tag{3}$$

$u = u(x)$, $x \in \mathbb{R}(n+1)$, $u \in \mathbb{R}$; $\underset{n}{u}$ — совокупность всевозможных производных n-го порядка.

Согласно Ли уравнение (3) инвариантно относительно оператора первого порядка

$$X = \xi^{\mu}(x, u)\frac{\partial}{\partial x_{\mu}} + \eta(x, u)\frac{\partial}{\partial u}, \tag{4}$$

если X — s-раз продолженный оператор удовлетворяет условию

$$\underset{s}{X} L = \lambda L, \quad \text{или} \quad \underset{s}{X} L\Big|_{L=0} = 0, \tag{5}$$

где $\lambda = \lambda(x, u, \underset{1}{u}, \ldots)$ — некоторое дифференциальное выражение.

Обозначим через символ $Q = \{Q_1, \ldots, Q_k\}$ совокупность операторов, не принадлежащих алгебре инвариантности (AI) уравнения (3), т.е. $Q_l \notin AI$, $l = 1, 2, \ldots, k$.

Определение 1 [2, 5]. *Уравнение* (3) *назовем условно инвариантным относительно оператора Q, если существует нетривиальное дополнительное условие на решение уравнения*

$$L_1(x, u, \underset{1}{u}, \ldots, \underset{s}{u}) = 0, \tag{6}$$

при котором уравнение (3) *вместе с уравнением* (6) *инвариантно относительно операторов Q. При этом предполагается, что уравнения* (3) *и* (6) *совместны.*

Дополнительное условие (6) выделяет из всего множества решений уравнения (3) некоторое подмножество. Оказывается, что для многих важных нелинейных уравнений математической физики эти подмножества имеют симметрию более широкую, чем все множество решений. Именно такие подмножества необходимо научиться выделять.

Пусть действие оператора Q на уравнение (3) задается формулой

$$\underset{s}{Q} L = \lambda_0 L + \lambda_1 L_1, \tag{7}$$

или

$$\underset{s}{Q} L\Big|_{\substack{Lu = 0 \\ L_1 u = 0}} = 0,$$

где $\lambda_0, \lambda_1 \neq 0$ — некоторые дифференциальные выражения, зависящие от x, u, $\underset{1}{u}$, $\ldots$, $\underset{s}{u}$, $\underset{s}{Q}$ — s-раз продолженный оператор из Q. В наиболее простейшем случае условие инвариантности уравнений (3) и (6) означает, что

$$\underset{s}{Q} L_1 = \lambda_2 L + \lambda_3 L_1, \tag{8}$$

где λ_2, λ_3 — некоторые дифференциальные выражения.

Главная проблема нашего подхода — описать в явном виде дополнительные уравнения вида (6), которые расширяют симметрию уравнения (3).

Эта общая и трудная проблема существенно упрощается, если в качестве дополнительного условия (6) выбрать такое нелинейное уравнение первого порядка:

$$Qu = 0, \tag{9}$$

где

$$Q = J^\mu(x,u)\partial_\mu + Z(x,u)\partial_u, \quad \partial_\mu \equiv \frac{\partial}{\partial x_\mu}, \quad \partial_u \equiv \frac{\partial}{\partial u}. \tag{10}$$

При этом условие инвариантности уравнений (3), (9) имеют вид

$$\underset{s}{Q} L = \lambda_0 L + \lambda_1 (Qu). \tag{11}$$

Определение 2. *Будем говорить, что уравнение* (3) *Q-условно инвариантно, если система* (3), (9) *инвариантна относительно оператора* (10).

Остановимся теперь на простейшем одномерном нелинейном уравнении акустики (1).

2. Условная симметрия уравнения (2).

Теорема 1 [8]. *Уравнение* (1) *Q-условно инвариантно относительно оператора* (10)*, если коэффициентные функции*

$$J^0 \equiv A(x), \quad J^1 \equiv B(x), \quad Z = h(x)u + q(x), \quad x = (x_0, x_1),$$

удовлетворяют следующим дифференциальным уравнениям.

Случай 1*:* $A \neq 0$, $B \neq 0$*;* $h = 2\left(B_1 - A_0 + \frac{B}{A}A_1\right)$, $q = 2\frac{B}{A}B_0$*;*

$$\begin{aligned}
&h_{00} + \frac{2h}{A}h_0 - \left[\frac{h}{A}A_{00} + \frac{2h}{A}A_{00} + 2\left(\frac{h}{A}\right)_1 B_0\right] = q_{11} - \left[\frac{q}{A}A_{11} + 2\left(\frac{q}{A}\right)_1 A_1\right],\\
&h_{11} = \frac{h}{A}A_{11} + 2\left(\frac{h}{A}\right)_1 A_1,\\
&h_{00} + 2\frac{q}{A}q_0 - - \left[\frac{q}{A}A_{00} + 2\left(\frac{q}{A}\right)_1 B_0\right] = 0,\\
&B_{11} - 2h_1 - \left[\frac{B}{A}A_{11} + 2\left(\frac{B}{A}\right)_1 A_1 + 2\frac{h}{A}A_1\right] = 0,\\
&B_{00} + 2\frac{B}{A}h_0 - \left[\frac{B}{A}A_{00} + 2\left(\frac{B}{A}\right)_1 B_0\right] = 0.
\end{aligned} \tag{12}$$

Индексы внизу означают соответствующую производную.

Случай 2: $A = 0$, $B \neq 0$ (не умаляя общности, можно положить $B = 1$);

$$h_0 = 0, \quad h_{11} + 3hh_1 + h^3 = 0,$$
$$q_{11} + hg_1 + (3h_1 + 2h^2)q = 0, \quad q_{00} - qq_1 - hq^2 = 0. \tag{13}$$

Случай 3: $A_1 = 1$, $B = 0$;

$$h_1 = 0, \quad h_{00} + hh_0 - h^3 = q_{11},$$
$$q(q_0 + hq) = 0, \quad q_{00} + h_0 q - h^2 q = 0. \tag{14}$$

Итак, задача об Q-условной симметрии уравнения (2) свелась к построению частных или общих решений уравнений (12)–(14). Подчеркнем, что коэффициентные функции $J^\mu(x,u)$, $Z(x,u)$ оператора Q, в отличие от коэффициентных функций ξ^μ, η (4), являются решениями нелинейных уравнений. Это обстоятельство существенно затрудняет задачу об описании условной симметрии заданных уравнений. Однако широкие классы частных решений таких уравнений можно построить.

Решая систему (12)–(14), мы нашли 12 типов неэквивалентных операторов условной симметрии уравнения (2). Два из них имеют вид

$$Q_1 = x_0^2 x_1 \partial_1 + \left(x_0^2 u + 3x_1^2 + b_5 x_0^5 + b_6\right)\partial_u, \tag{15}$$

$$Q_2 = \partial_1 + [W(x_0)x_1 + f(x_0)\partial_u], \quad W'' = W^2, \quad f'' = Wf, \tag{16}$$

W — функция Вейерштрасса.

Оператор (15) порождает анзац

$$U = x_1\varphi(x_0) + 3x_0^{-2}x_1 - b_5 x_0^3 + b_6 x_0^{-2}. \tag{17}$$

Анзац (17) редуцирует нелинейное уравнение (2) к линейному ОДУ

$$x_0^2\varphi''(x_0) = 6\varphi. \tag{18}$$

Оператор (16) порождает анзац

$$u = \frac{1}{2}W(x_0)x_1^2 + f(x_0)x_1 + \varphi(x_0). \tag{19}$$

Анзац (19) редуцирует уравнение (2) к линейному ОДУ с потенциалом Вейерштрасса W

$$\varphi''(x_0) = W\varphi(x_0). \tag{20}$$

Замечание 2. Аналогичным методом построены семейства точных решений многомерного уравнения [8]

$$u_{00} = u\Delta u. \tag{21}$$

Вывод 3. Анзацы, порождаемые операторами условной симметрии, во многих случаях редуцируют исходное нелинейное уравнение к линейному уравнению. Лиевская редукция, как правило, не меняет нелинейную структуру уравнения.

3. Условная симметрия уравнения Даламбера. Рассмотрим нелинейное уравнение

$$\Box u = F_1(u), \quad u = u(x_0, x_1, x_2, x_3), \tag{22}$$

lt $F_1(u)$ — произвольная гладкая функция. Максимально широкой симметрией уравнения (22) является конформная группа $C(1,3)$ в том и только в том случае, когда $F_1(u) = 0$ или $F_1(u) = \lambda u^3$. Наложим на решение (22) пуанкаре-инвариантное условие эйконального типа

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = F_2(u), \tag{23}$$

где $F_2(u)$ — гладкая функция.

Теорема 2 [14]. *В том случае, когда $F_1 = F_2 = 0$, уравнение* (22) *при условии* (23) *инвариантно относительно бесконечномерной алгебры, коэффициенты оператора* (4) *имеют вид*

$$\xi^\mu(x,u) = c^{00}(u)x^\mu + c^{\mu\nu}(u)x^\nu + d^\mu(u), \quad \eta(x,u) = \eta(u),$$

где $c^{00}(u)$, $c^{\mu\nu}(u)$, $\eta(u)$ — произвольные гладкие функции, зависящие только от u.

Из этой теоремы видно, что дополнительное условие (23) ($F_2 = 0$) выделяет из множества всех решений линейного уравнения Даламбера ($F_1 = 0$) подмножество с уникальными симметрийными свойствами. Кроме того, система (22), (23) ($F_1 = F_2 = 0$) обладает тем свойством, что произвольная гладкая функция от решения будет снова решением.

Теорема 3 [9]. *Система* (22), (23) *инвариантна относительно конформной группы $C(1,3)$ тогда и только тогда, когда*

$$F_1 = 3\lambda(u+c)^{-1}, \quad F_2 = \lambda, \tag{24}$$

где $\lambda, c = \text{const}$.

Итак, дополнительное условие эйконального типа (23) расширяет класс нелинейных волновых уравнений, инвариантных относительно конформной группы. Это означает, что мы можем построить широкие классы точных решений уравнения (22), используя подгруппы конформной группы.

Замечание 3. Система (22), (23) [15] полностью проинтегрирована.

Рассмотрим лоренц-неинтегрированное волновое уравнение [4]

$$Lu \equiv \Box u + F(x, u, \underset{1}{u}) = 0 \tag{25}$$

$$\begin{aligned} F = -\left(\frac{\lambda_0}{x_0}\right)^2\left(\frac{\partial u}{\partial x_0}\right)^2 + \left(\frac{\lambda_1}{x_1}\right)^2\left(\frac{\partial u}{\partial x_1}\right)^2 + \\ + \left(\frac{\lambda_2}{x_2}\right)^2\left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\lambda_3}{x_3}\right)^2\left(\frac{\partial u}{\partial x_3}\right)^2, \quad x_\mu \neq 0. \end{aligned} \tag{26}$$

Максимальной группой инвариантности уравнения (25), (26) является двухпараметрическая группа

$$x_\mu \to x_\mu = e^a x_\mu, \quad u \to u' = u + b,$$

где a и b — произвольные параметры группы.

Дополнительное условие типа (6) к уравнению (25) выберем в виде

$$I_{\mu\nu}u(x) = 0, \quad I_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad \nu,\mu = 0,1,2,3. \tag{27}$$

Непосредственной проверкой условий инвариантности (7) можно убедиться, что уравнения (25), (27) инвариантны относительно группы Лоренца $O(1,3)$. Это означает, что лоренц-инвариантный анзац

$$u = \varphi(\omega), \quad \omega = x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{28}$$

редуцирует нелинейное волновое уравнение (25) к ОДУ

$$\omega\frac{d^2\varphi}{d\omega^2} + 2\frac{d\varphi}{d\omega} + \lambda^2\left(\frac{d\varphi}{d\omega}\right)^2 = 0, \quad \lambda^2 = \lambda_\mu\lambda^\mu = \lambda_0^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2.$$

Решением этого уравнения являются функции

$$\varphi(\omega) = 2(-\lambda^2)^{-1/2}\operatorname{arctg}[\omega(-\lambda^2)^{-1/2}], \quad \lambda^2 < 0,$$

$$\varphi(\omega) = (\lambda^2)^{-1/2}\ln\left\{\frac{(\lambda^2)^{1/2}+\omega}{(\lambda^2)^{1/2}-\omega}\right\}, \quad \lambda^2 > 0,$$

$$\varphi(\omega) = \frac{c_1}{\omega} + c^2, \quad \lambda^2 = 0,$$

где c_1, c_2 — константы.

Таким образом, условие (27) выделяет из множества решений лоренц-неинвариантного уравнения (25) подмножество, которое инвариантно относительно шестипараметрической группы Лоренца. Такое существенное расширение симметрии дает возможность построить широкие классы точных решений нелинейного волнового уравнения (25).

4. Условная симметрия нелинейного уравнения Шредингера. Рассмотрим нелинейное уравнение вида

$$Su + F(|u|)u = 0, \quad S \equiv i\frac{\partial}{\partial x_0} + \lambda_1\Delta. \tag{29}$$

Уравнение (29) при произвольной функции $F(|u|)$ инвариантно относительно алгебры Галилея $AG(1,n)$ с базисными элементами

$$\begin{aligned} &P_0 = \partial_0, \quad P_a = \partial_a, \quad J_{ab} = x_aP_b - x_bP_a, \quad a,b = \overline{1,n}, \\ &G_a = x_0P_a + \frac{1}{2\lambda_1}x_aR_1, \end{aligned} \tag{30}$$

где

$$R_1 = i\left(u\frac{\partial}{\partial u} + u^*\frac{\partial}{\partial u^*}\right).$$

Среди множества нелинейных уравнений (29) только два уравнения имеют более широкую симметрию, чем уравнение (29) [16, 17]:

$$Su + \lambda_2|u|^r u = 0, \tag{31}$$

$$Su + \lambda_3|u|^{4/n}u = 0, \tag{32}$$

где λ_2, λ_3, r — произвольные действительные параметры, n — число пространственных переменных в уравнении (29).

Уравнение (31) инвариантно относительно расширений алгебры Галилея $AG_1(1,n) = \langle AG(1,n), D\rangle$ с базисными элементами $AG(1,n)$ (30) и оператора масштабных преобразований

$$D = 2x_0 P_0 + x_a P_a + \frac{2}{r} R_2, \tag{33}$$

где единичный оператор имеет вид

$$R_2 = u\frac{\partial}{\partial u} + u^* \frac{\partial}{\partial u^*}.$$

Уравнение (32) инвариантно относительно обобщенной алгебры Галилея $AG_2(1,n) = \langle AG_1(1,n), A\rangle$ с базисными элементами (30), (33) и оператора проективных преобразований

$$A = x_0^2 P_0 + x_0 x_a P_a + \frac{x^2}{4\lambda_3} R_1 - \frac{n}{2} x_0 R_2.$$

Теорема 4 [18]. *Уравнение Шредингера* (23) *условно инвариантно относительно оператора*

$$Q_1 = \ln\left(\frac{u}{u^*}\right) R_1 + x_a P_a - cR_2, \quad c = \text{const}, \tag{34}$$

если

$$F(|u|) = \lambda_4 |u|^{-4/r} + \lambda_5 |u|^{4/r},$$

где λ_4, λ_5, r — произвольные параметры, а модуль функции u удовлетворяет уравнению

$$\lambda_1 \Delta |u| + \lambda_6 |u|^{\frac{r+4}{r}} = 0. \tag{35}$$

Теорема 5 [18]. *Уравнение* (32) *вместе с уравнением* (35) *инвариантно относительно алгебры $AG_2(1,n)$ и оператора Q_1* (34).

Итак, налагая на решения линейного уравнения (29) дополнительные условия (35), мы расширили его симметрию.

5. Условная симметрия нелинейных уравнений теплопроводности. Для описания нелинейных процессов тепломассопереноса широко используются одномерные уравнения вида

$$u_0 + u_{11} = F(u), \tag{36}$$

$$u_0 + u u_{11} = 0, \tag{37}$$

где $F(u)$ — гладкая функция.

Будем искать оператор условной симметрии в виде

$$Q = A(x,u)\partial_0 + B(x,u)\partial_1 + C(x,u)\partial_u, \tag{38}$$

где A, B, C — гладкие функции.

Теорема 6 [19]. *Уравнение* (36) *Q-условно инвариантно относительно оператора* (38)*, если функции A, B, C удовлетворяют следующей системе дифференциальных уравнений.*

Случай 1*: $A = 1$;*

$$\begin{aligned} &B_{uu} = 0, \quad C_{uu} = 2(B_{1u} + BB_u), \\ &3B_u F = 2(C_{1u} + B_u C) - (B_0 + B_{11} + 2BB_1), \\ &CF_u - (C_u - 2B_1)F = C_0 + C_{11} + 2CB_1. \end{aligned} \tag{39}$$

Здесь и ниже индекс внизу возле функции означает дифференцирование по соответствующему аргументу (x_0, x_1, u)*.*

Случай 2*: $A = 0, B = 0$;*

$$CF_u - C_u F = C_0 + C_{11} + 2CC_{1u} + C^2 C_{uu}. \tag{40}$$

Если построить общие решения нелинейных систем (39), (40)*, тогда мы опишем Q-условную симметрию уравнения* (36)*.*

Теорема 7 [19]. *Уравнение* (36) *Q-условно инвариантно относительно оператора* (38) $(A = 1,\ B_u \neq 0)$ *тогда и только тогда, когда оно локально эквивалентно уравнению*

$$u_0 + u_{11} = b_3 u^3 + b_1 u + b_0, \quad b_0, b_1, b_3 = \mathrm{const}, \tag{41}$$

и оператор (38) *имеет вид*

$$Q = \partial_0 + \frac{3}{2}\sqrt{2b_3}\,u\partial_1 + \frac{3}{2}(b_3 u^3 + b_1 u - b_0)\partial_u. \tag{42}$$

Уравнение (41) можно свести к одному из четырех канонических уравнений

$$u_0 + u_{11} = \lambda u(u^2 - 1), \tag{43}$$

$$u_0 + uu_{11} = \lambda(u^3 - 3u + 2), \tag{44}$$

$$u_0 + u_{11} = \lambda u^3, \tag{45}$$

$$u_0 + uu_{11} = \lambda u(u^2 + 1). \tag{46}$$

Анзацы, построенные с помощью оператора (42) для уравнений (43)–(46), соответственно имеют вид

$$\varphi(\omega) = 2\,\mathrm{arctg}\,u + \sqrt{2\lambda}x_1, \quad \omega = -\ln(1 - u^{-2}) + 2\lambda x_0; \tag{47}$$

$$\begin{aligned} &\varphi(\omega) = -\frac{4}{9}\ln\frac{u+2}{u-1} - \frac{2}{3}(u-1)^{-1} - \sqrt{2\lambda}x_1, \\ &\omega = \frac{2}{9}\ln\frac{u+2}{u-1} - \frac{2}{3}(u-1)^{-1} - 3\lambda x_0; \end{aligned} \tag{48}$$

$$\varphi(\omega) = 2u^{-1} + \sqrt{2\lambda}x_1, \quad \omega = -u^{-2} - 3\lambda x_0; \tag{49}$$

$$\varphi(\omega) = 2\,\mathrm{arctg}\,u - \sqrt{2\lambda}x_1, \quad \omega = -\ln(1 + u^{-2}) - 3\lambda x_0. \tag{50}$$

Анзацы (47)–(50) редуцируют уравнения (43)–(46) к ОДУ

$$2\ddot{\varphi} = (\dot{\varphi}^2 - 1)\dot{\varphi}, \quad 2\ddot{\varphi} = \dot{\varphi}^3 - 3\dot{\varphi} + 2, \tag{51}$$

$$2\ddot{\varphi} = \dot{\varphi}^3, \quad 2\ddot{\varphi} = \dot{\varphi}(\dot{\varphi}^2 + 1). \tag{52}$$

Из редуцированных уравнений (51), (52) видно, что анзацы, порожденные оператором условной инвариантности (42), существенно изменили нелинейные правые части. Это позволило построить общие решения (51), (52) в элементарных функциях

$$\varphi(\omega) = -2\,\mathrm{arctg}\left(\sqrt{c_1 \exp\omega + 1}\right) + c_2, \tag{53}$$

$$\ln\left[c_1 - \frac{3}{2}(\varphi + 2\omega)\right] = \ln c_2 - \frac{3}{2}(\varphi - \omega), \tag{54}$$

$$\varphi(\omega) = 2\sqrt{c_1 - \omega} + c_2, \tag{55}$$

$$\varphi(\omega) = 2\,\mathrm{arctg}\left(\sqrt{c_1 \exp\omega - 1}\right) + c_2, \tag{56}$$

где $c_1, c_2 = \mathrm{const}$.

Итак, подставляя (53)—(56) в (47)—(50), получаем семейство точных решений уравнений (43)—(46). Эти решения не могут быть получены с помощью метода Ли.

Теорема 8 [20]. *Уравнение* (37) *условно инвариантно относительно оператора* (38) $A = 1$, *если коэффициентные функции* B, C *удовлетворяют следующей системе уравнений:*

$$uC_{uu} = 2(BB_u + uB_{u1}), \quad B_{uu} = 0, \tag{57}$$

$$B_0 + uB_{11} - CBU^{-1} - 2uC_{u1} + 2BB_1 - 2B_uC = 0, \tag{58}$$

$$C_0 + uC_{11} - C^2u^{-1} + 2B_1C = 0. \tag{59}$$

Решая систему уравнений (57)–(59), находим явный вид оператора (38)

$$\begin{gathered} Q = b_1Q_1 + b_2Q_2 + b_3D_1 + b_4D_2 + b_5\partial_0 + b_6\partial_1, \\ Q_1 = x_1\partial_0 + u\partial_1, \quad Q_2 = x_1^2\partial_0 + 2x_1u\partial_1 + 2u^2\partial_u, \end{gathered} \tag{60}$$

$$D_1 = 2_0\partial_0 + x_1\partial_1, \quad D_2 = x_1\partial_1 + 2u\partial u, \quad b_i = \mathrm{const}, \quad i = \overline{1,6}. \tag{61}$$

Теорема 9 [20]. *Уравнение* (37) Q-*условно инвариантно относительно оператора*

$$Q = \partial_1 + C(x,u)\partial_u, \tag{62}$$

если $C(x,u)$ *удовлетворяет условию*

$$C_0 + u\left(C_{11} + 2CC_{1u} + C^2C_{uu}\right) + C_1C + C^2C_u = 0. \tag{63}$$

Построив частные или общие решения уравнения (63), получим явные выражения для операторов условной симметрии. Некоторые из таких операторов (62) имеют вид

$$Q_3 = \sqrt{x_0}\partial_1 + \sqrt{2u}\partial_u, \tag{64}$$

$$Q_4 = \sqrt{2x_0}\partial_1 + R(u)\partial_u, \tag{65}$$

$$Q_5 = \partial_1 + \ln u\partial_u, \tag{66}$$

$$Q_6 = x_0\partial_1 + x_1\partial_u, \tag{67}$$

где $R(u)$ — решения дифференциального уравнения

$$u\ddot{R}(u) + \dot{R}(u) = R^{-1}.$$

Приведем несколько анзацев, которые порождают операторы Q_1, Q_2, Q_3:

$$x_0 u - \frac{1}{2}x_1^2 = \varphi(u), \tag{68}$$

$$\frac{2ux_0}{x_1} - x_1 = \varphi\left(\frac{u}{x_1}\right), \tag{69}$$

$$u = \frac{1}{2}\left(\frac{x_1}{\sqrt{x_0}} + \varphi(x_0)\right)^2. \tag{70}$$

Редуцированные уравнения имеют весьма простой вид:

$$\ddot{\varphi}(u) = 0 \quad \text{для анзаца (68)},$$

$$\ddot{\varphi}\left(\frac{u}{x_1}\right) = 0 \quad \text{для анзаца (69)},$$

$$2x_0\dot{\varphi}(x_0) + \varphi = 0 \quad \text{для анзаца (70)}, \quad x_0 \neq 0.$$

Итак, анзацы (68)–(70) редуцируют нелинейное уравнение теплопроводности к линейным ОДУ.

6. Уравнение типа Кортевега–де Фриза. Рассмотрим нелинейное уравнение

$$u_0 + F(u)u_1^k + u_{111} = 0, \tag{71}$$

$u_{111} = \frac{\partial^3 u}{\partial x^3}$, k — произвольный действительный параметр. При $F(u) = u$, $k = 1$ уравнение (1) совпадает с классическим уравнением КдФ.

Теорема [23]. *Уравнение Q-условно инвариантно относительно оператора галилеевского типа*

$$Q = x_0^r\partial_1 + H(x,u)\partial_u, \tag{72}$$

где r — произвольный действительный параметр, если

$$1) \quad F(u) = \lambda_1 u^{\frac{2-k}{k}} + \lambda_2 u^{\frac{1-k}{2}}, \quad H(x,u) = \left(\frac{k\lambda_1}{2}\right)^{-1/k} u^{1/2}; \tag{73}$$

$$2)\quad F(u)=(\lambda_1\ln u)^{1-k},\quad H(x,u)=(k\lambda_1)^{-1/k}u; \tag{74}$$

$$\begin{aligned}3)\quad &F(u)=(\lambda_1\arcsin u+\lambda_2)(1-u^2)^{\frac{1-k}{2}},\\ &H(x_1u)=(k\lambda_1)^{-1/k}(1-u^2)^{1/2};\end{aligned} \tag{75}$$

$$\begin{aligned}4)\quad &F(u)=(\lambda_1\operatorname{Arsh}u+\lambda_2)(1-u^2)^{\frac{1-k}{2}},\\ &H(x,u)=(k\lambda_1)^{-1/k}(1+u^2)^{1/2};\end{aligned} \tag{76}$$

$$5)\quad F(u)=\lambda_1 u,\quad H(x,u)=(k\lambda_1)^{-1/k}; \tag{77}$$

где $r\neq k^{-1}$, $k\neq 0$, λ_1, λ_2 *— произвольные постоянные.*

С помощью операторов условной инвариантности (72) редуцируем (71) к ОДУ и построим следующие точные решения:

$$u=\left\{\frac{x_1}{2}\left(\frac{k\lambda_1x_0}{2}\right)^{-1/k}+\lambda x_0^{-1/k}-\frac{\lambda_2}{\lambda_1}\right\}^2,$$

когда $F(u)$ имеет вид (73);

$$u=\exp\left\{-\frac{k(k\lambda_1)^{-3/k}}{k-2}x_0^{-\frac{3}{k}+1}+\lambda x_0^{-1/k}+(k\lambda_1x_0)^{-1/k}x_1-\frac{\lambda_2}{\lambda_1}\right\},$$

при $k\neq -2$, $F(u)$ имеет вид (74); когда $k=2$

$$u=\exp\left\{-(2\lambda_1)^{-3/2}x_0^{-1/2}\ln x_0+\lambda x_0^{-1/2}+(2\lambda_1x_0)^{-1/2}x_1-\frac{\lambda_2}{\lambda_1}\right\},$$

$$u=\sin\left\{\frac{k(k\lambda_1)^{-3/k}}{k-2}x_0^{-\frac{3}{k}+1}+\lambda x_0^{-1/k}+(k\lambda_1x_0)^{-1/k}x_1-\frac{\lambda_2}{\lambda_1}\right\},\quad k\neq 2,$$

$$u=\sin\left\{(2\lambda_1)^{-3/2}\frac{\ln x_0}{\sqrt{x_0}}+\lambda x_0^{-1/2}+(2\lambda_1x_0)^{-1/2}x_1-\frac{\lambda_2}{\lambda_1}\right\},\quad k=2,$$

когда $F(u)$ имеет вид (75);

$$u=\operatorname{sh}\left\{-\frac{k(k\lambda_1)^{-3/k}}{k-2}x_0^{-3/k+1}+\lambda x_0^{-1/k}+(k\lambda_1x_0)^{-1/k}x_1\right\},\quad k\neq 2,$$

$$u=\operatorname{sh}\left\{-(2\lambda_1)^{-3/2}x_0^{-1/2}\ln x_0+\lambda x_0^{-1/2}+(2\lambda_1x_0)^{-1/2}x_1\right\},\quad k=2,$$

когда $F(u)$ имеет вид (76). Во всех формулах λ — произвольный параметр. Итак, изучив условную симметрию уравнения (1), мы построим нетривиальные классы точных решений.

7. Нелинейное волновое уравнение. Уравнение вида

$$u_{00}-(F(u)u_1)_1=0 \tag{78}$$

широко применяется для описания нелинейных волновых процессов. Групповые свойства (78) методом Ли детально исследованы в [24]. В зависимости от явного вида функции $F(u)$ уравнение (78) обладает широкой условной симметрией.

Теорема [25]. *Уравнение* (78) Q-*условно инвариантно относительно оператора*

$$Q = A(x,u)\partial_0 + B(x,u)\partial_1 + H(x,u)\partial_u,$$

если функции $A(x,u)$, $B(x,u)$, $H(x,u)$, $F(u)$ *удовлетворяют следующей системе уравнений.*

Случай 1*:* $A = 1$, $D = F - B^2$*;*

$$\begin{aligned}
&(B_u D^{-1})_u = 0,\\
&F(H_1 D^{-1})_1 - (H_0 D^{-1})_0 - H^2(H_u D^{-1})_u - H(H_0 D^{-1})_u - H(H_u D^{-1})_0 +\\
&\qquad + D^2\{2F(B_0 D_1 - B_1 H_0 + H[B_u H_1 - B_1 H_u]) - BHH_1 F\} = 0,\\
&D^2 H_{uu} + D\{(H\dot{F})_u + 2B(B_u H_u - B_{uu} H) - 2FB_{1u} - 2BB_{0u}\} -\\
&\qquad - HD_4^2 + 2BB_0 D_u + 2BB_1(B\dot{F} - 2B_u F) = 0;\\
&D\{B_{00} + 2(B_0 H)_u - 2(BH_{0u} - B_u H_0) + 2(H_1 F)_u -\\
&\qquad - B_{11} F + B_{uu} H^2 + 2BHH_{uu}\} - D_u\{B_0 H + B_u H^2 + 2BHH_u\} +\\
&\qquad + B\{B_1 H\dot{F} + 2B_0^2 + 2B_0 B_u H + 4BB_0 H_u + 4B_1 H_u F - 2B_1^2 F\} = 0.
\end{aligned}$$

Случай 2*:* $A = 1, B = F^{1/2}$*;*

$$\begin{aligned}
&1)\quad \dot{B}H + 2BH_u = 0, \quad H_0 + HH_u - BH_1 = 0;\\
&2)\quad \dot{B}H + 2BH_u \neq 0, \quad H_0 + HH_u - BH_1 = 0;\\
&[\ddot{B}H^2 + 2\dot{B}(BH_1 + HH_u) + 2B(H_{0u} + HH_{uu} + BH_{1u})] =\\
&\qquad = (H_0 + HH_u - HH_1) - [H_{00} + H^2 H_{uu} - B^2 H_{11} + 2HH_{0u} - 2\dot{B}HH_1] \times\\
&\qquad \times (\dot{B}H + 2BH_u) = 0.
\end{aligned}$$

Случай 3*:* $A = 0, B = 1$

$$H_{00} - H^3\dot{F} - (3HH_1 + 2H^2 H_u)\dot{F} - (H_{11} + 2HH_{1u})F = 0.$$

Решая эти системы, при конкретных выборах функции $F(u)$ построены явные виды операторов Q. Приведем только некоторые из полученных операторов и анзацев:

$$\begin{aligned}
&F(u) = \exp u, && Q_1 = x_1\partial_1 + \partial_u, \quad u = \ln x_1 + \varphi(x_0),\\
& && Q_2 = \partial_0 + 2\,\mathrm{tg}\,x_0\partial_u, \quad \exp u = \varphi(x_1)\cos^{-2} x_0;\\
&F(u) = u^k, && Q_1 = \partial_0 + \exp\left(\frac{u}{2}\right)\partial_1 - 4x_0^{-1}\partial_u,\\
& && Q_2 = (k+1)x_1\partial_1 + u\partial_u;\\
& && x_0\exp\left(\frac{u}{2}\right) + x_1 + \varphi\left(x_0^2\exp\frac{u}{2}\right) = 0;\\
& && u^{k+1} = x_1\varphi^{k+1}(x_0);\\
&F(u) = u^{-1/2}, && Q_1 = \partial_0 + x_1 u^{1/2}\partial_u,\\
& && Q_2 = x_1^2\partial_0 + (4x_0 + a_1 x_1^5)u^{1/2}\partial_u;\\
& && 2u^{1/2} = x_0 x_1 + \varphi(x_1),\\
& && u^{1/2} = x_0^2 x_1^{-2} + \frac{a_1}{2}x_0 x_1^3 + \varphi(x_1),
\end{aligned}$$

где a_1, a_2, a_3 — постоянные.

Наиболее простые решения уравнения (78), построенные с помощью анзацев, имеют вид

$$\begin{aligned}
&\exp u = (x_1^2 + a_1)\cos^{-2} x_0, \quad \exp u = x_1 \exp x_0, \quad \text{если} \quad F(u) = \exp u;\\
&u^{k+1} = x_0^{k+1} x_1, \quad \text{если} \quad F(u) = u^k;\\
&u = x_0 x_1 + \frac{x_0^4}{12} + a_1, \quad u = W(x_0) x_1^2, \quad \text{если} \quad F(u) = u;\\
&u^{1/2} = W(x_1) x_0^2, \quad 2u^{1/2} = x_0 x_1 + \frac{x_1^4}{24} + a_1,\\
&u^{1/2} = x_0^2 x_1^{-2} + 3a_1 x_0 x_1^3 + \frac{a_1^2}{6} x_1^8 + a_2 x_1^{-1} + a_3 x_1^2, \quad \text{если} \quad F(u) = u^{-1/2}.
\end{aligned}$$

Итак, нами проведена классификация и редукция нелинейных волновых уравнений (78), обладающих условной симметрией.

8. Трехмерное нелинейное уравнение акустики. Ограниченные звуковые пучки описывают нелинейным уравнением вида

$$u_{00} - (F(u)u_1)_1 - u_{22} - u_{33} = 0. \tag{79}$$

В том случае, когда $F(u) = u$, оно совпадает с уравнением Хохлова–Заболотской

$$u_{01} - (uu_1)_1 - u_{22} - u_{33} = 0. \tag{80}$$

Положим на решение (79) дополнительное условие в виде нелинейного уравнения первого порядка

$$u_0 u_1 - F(u) u_1^2 - u_2^2 - u_3^2 = 0. \tag{81}$$

Теорема [26]. *Уравнение* (80) *при условии* (81) *инвариантно относительно бесконечномерной алгебры с оператором*

$$X = a_i(u) R_i, \quad i = \overline{1,12}, \tag{82}$$

где $a_i(u)$ *— произвольные гладкие функции зависимой переменной* u,

$$\begin{aligned}
&R_{\mu+1} = \partial_\mu, \quad \mu = \overline{0,3}, \quad R_5 = x_3\partial_2 - x_2\partial_3,\\
&R_6 = x_2\partial_1 + 2x_0\partial_2, \quad R_7 = x_3\partial_1 + 2x_0\partial_3, \quad R_8 = x^\mu \partial_\mu,\\
&R_9 = 4x_0\partial_0 + 2x_1\partial_1 + 3x_2\partial_2 + 3x_3\partial_3 - 2\frac{F(u)}{F'(u)}\partial_u, \quad R_{10} = F'(u) x_0 \partial_1 - \partial_u,\\
&R_{11} = x_2\partial_0 + 2(x_1 + F(u)x_0)\partial_2, \quad R_{12} = x_3\partial_0 + 2(x_1 + 2F(u)x_0)\partial_3.
\end{aligned}$$

Операторы $\langle R_1, \ldots, R_8\rangle$ являются лиевскими операторами симметрии уравнения (80), $\langle R_9, \ldots, R_{12}\rangle$ операторы условной симметрии уравнения (79). Воспользовавшись операторами условной симметрии уравнения (79) $\langle R_9, \ldots, R_{12}\rangle$, можно построить широкие классы точных решений. Например, оператор $X = \partial_0 + a(u)\partial_1$ порождает следующие анзацы:

$$u = \varphi(\omega_1, \omega_2, \omega_3), \quad \omega_1 = a(u)x_0 + x_3, \quad \omega_2 = x_2, \quad \omega_3 = x_3. \tag{83}$$

Анзац (83) редуцирует четырехмерное уравнение (79), (81) к трехмерному

$$
\begin{aligned}
&(a(\varphi)-\varphi)\varphi_{11}-\varphi_{22}-\varphi_{33}+\left(\frac{da(\varphi)}{d\varphi}-1\right)\varphi_1^2=0,\\
&(a(\varphi)-\varphi)\varphi_1^2-\varphi_2^2-\varphi_3^2=0 \quad \varphi_i=\frac{\partial\varphi}{\partial\omega_i}, \quad i=\overline{1,3}.
\end{aligned}
\tag{84}
$$

Конкретизируя функцию $a(u)$, в некоторых случаях можно построить общее решение (84). Пусть $a(u)=u+1$. Тогда имеем систему

$$
\varphi_{11}-\varphi_{22}-\varphi_{33}=0, \tag{85}
$$

$$
\varphi_1^2-\varphi_2^2-\varphi_3^2=0. \tag{86}
$$

Систему (85) естественно назвать уравнением Бейтмена (1914 г.) — Соболева — Смирнова (1932–1933 гг.), поскольку именно они детально изучали ее. Уравнение (85) имеет общее решение и задается формулой Соболева–Смирнова

$$
\varphi=c_1(\varphi)\omega_1+c_2(\varphi)\omega_2+c_3(\varphi)\omega_3, \tag{87}
$$

где c_1, c_2, c_3 — произвольные функции, удовлетворяющие условиям

$$
c_1^2-c_2^2-c_3^2=0, \quad c_2^2+c_3^2\neq 0.
$$

Таким образом, формула (87) задает класс точных решений трехмерных нелинейных уравнений (85), (86).

Итак, анзацы (68)–(70) редуцируют нелинейное уравнение теплопроводности (37) к линейным ОДУ.

9. Условная симметрия уравнения Дирака. Рассмотрим нелинейное уравнение Дирака

$$
\{\gamma_\mu p^\mu-\lambda(\bar\Psi\Psi)\}\Psi(x)=0 \tag{88}
$$

и наложим на его решение условие $\bar\Psi\Psi=1$. Тогда (71) становится линейным уравнением с нелинейным дополнительным условием

$$
(\gamma_\mu p^\mu-\lambda)\Psi=0, \quad \bar\Psi\Psi=1. \tag{89}
$$

Система (72) условно инвариантна относительно операторов [9]

$$
Q_1=p_0-\lambda\gamma_0, \quad Q_2=p_3-\lambda\gamma_3. \tag{90}
$$

В рассматриваемом случае уравнение типа (6) имеет вид

$$
Q_1\Psi=0 \quad \text{и} \quad Q_2\Psi=0. \tag{91}
$$

Оператор Q_1 порождает анзац

$$
\Psi(x)=\exp(-i\lambda\gamma_0x_0)\varphi(x_1,x_2,x_3), \tag{92}
$$

где $\varphi(x_1,x_2,x_3)$ — четырехкомпонентная вектор-функция, зависящая только от трех переменных.

10. Условная симметрия уравнений Максвелла. Рассмотрим линейную систему

$$\frac{\partial \vec{E}}{\partial t} = \mathrm{rot}\ \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\mathrm{rot}\ \vec{E}. \tag{93}$$

Можно непосредственно проверить, что система (93) не инвариантна относительно преобразований Лоренца. Однако, если добавить к системе (93) известные дополнительные условия

$$\mathrm{div}\ \vec{E} = 0, \quad \mathrm{div}\ \vec{H} = 0,$$

то система (93), (94) становится лоренц-инвариантной. Приведенная точка зрения на уравнения Максвелла [1–10, 21] указывает на естественность термина “условная симметрия” и физическую важность этой концепции для широкого класса уравнений математической физики [22].

Заключение. Исследование условий симметрии ДУЧП только началось. Приведенные результаты свидетельствуют о том, что на этом пути следует ожидать качественно нового понимания симметрии уравнения, симметрийной классификации ДУЧП, редукции многомерных нелинейных уравнений к уравнениям с меньшим числом переменных, процесса линеаризации нелинейных уравнений.

Одним из наиболее фундаментальных законов физики, механики, гидромеханики, биофизики является принцип относительности, т.е. равноправие всех инерциальных систем отсчета. На математическом языке этот принцип означает инвариантность уравнения движения относительно либо преобразований Галилея, либо преобразований Лоренца. ДУЧП, не удовлетворяющие этому принципу, обычно не рассматриваются в физических теориях, поскольку они несовместимы с принципом относительности. Такие уравнения не могут быть использованы для математического описания движения реальных физических систем.

Понятие условной инвариантности дает возможность существенно расширить классы уравнений, удовлетворяющих принципу относительности. Уравнения, которые не совместимы, в обычном смысле, с принципом относительности могут условно удовлетворять ему, т.е. существуют нетривиальные условия на решения таких уравнений, выделяющие подмножества решений исходного уравнения, инвариантные относительно либо преобразований Галилея, либо преобразований Лоренца. Описание и детальное изучение классов уравнений, условно инвариантных относительно групп Галилея, Пуанкаре и их подгрупп, представляется автору весьма важной задачей математической физики.

Условная симметрия, например, скалярного уравнения дает возможность строить такие анзацы, которые увеличивают (антиредукция) число зависимых переменных. Она позволяет провести не только редукцию по числу независимых переменных, но при этом увеличить число зависимых переменных. Подчеркнем, что такие анзацы существенно меняют структуру нелинейностей исходного уравнения. И, конечно, они не могут быть построены в рамках классической схемы Ли. Процесс линеаризации, например, нелинейной системы Навье–Стокса в нашем подходе следует рассматривать как замену нелинейного уравнения на линейную систему

$$\frac{\partial \vec{u}}{\partial t} + \Delta \vec{u} + \vec{\nabla} p = 0, \quad \mathrm{div}\ \vec{u} = 0, \tag{94}$$

при нелинейном дополнительном условии

$$(\vec{u}\vec{\nabla})\vec{u} = 0, \quad \text{или} \quad \{(\vec{u}\vec{\nabla})\vec{u}\}^2 = 0. \tag{95}$$

Линейное уравнение Навье–Стокса при нелинейном дополнительном условии обладает нетривиальной условной симметрией. Очевидно, в качестве дополнительного условия к нелинейному уравнению Навье–Стокса можно выбрать и такие уравнения:

$$(\vec{u}\vec{\nabla})\vec{u} + \vec{\nabla}p = 0.$$

Детальному изучению условной линеаризации нелинейных ДУЧП будут посвящены отдельные публикации.

1. *Фущич В.И.* О симметрии и частных решениях некоторых многомерных уравнений математической физики // Теоретико-алгебраические методы в задачах математической физики. – Киев: Ин-т математики АН УССР, 1983. — С. 4–23.
2. *Фущич В.И.* Как расширить симметрию дифференциальных уравнений? // Симметрия и решения нелинейных уравнений математической физики. — Киев: Ин-т математики АН УССР, 1987. — С. 4–16.
3. *Фущич В.И.* О симметрии и точных решениях многомерных нелинейных волновых уравнений // Укр. мат. журн. — 1987. — **39**, № 1. — С. 116–123.
4. *Fushchych W.I., Tsyfra I.M.* On a reduction and solutions of non-linear wave equations with broken symmetry // J. Phys. A: Math. Gen. — 1987. — **20**. — L45–L48.
5. *Fushchych W.I., Nikitin A.G.* Symmetries of Maxwell's equations. — Dordrecht: D. Reidel, 1987. — 214 p.
6. *Fushchych W.I., Zhdanov R.Z.* On some new exact solutions of nonlinear d'Allembert and Hamilton equations — Minnesota, 1988. — 5 p. — (Prepr./Institute for Mathematics and its Applications, University of Minnesota; N 468).
7. *Фущич В.I., Серов М.I., Чопик В.I.* Умовна інваріантність та нелінійні рівняння теплопровідності // Доп. АН УРСР. Сер. А. — 1988. — № 9. — С. 17–20.
8. *Фущич В.И., Серов Н.И.* Условная инвариантность и точные решения нелинейного уравнения акустики // Докл. АН УССР. Сер. А. — 1988. — № 10. — С. 27–31.
9. *Fushchych W.I., Zhdanov R.Z.* Symmetry and exact solutions of nonlinear spinor equations // Phys. Rep. — 1989. — **172**, N 4. — P. 123–174.
10. *Фущич В.И., Серов Н.И.* Условная инвариатность и точные решения уравнения Буссинеска // Симметрия и решения уравнений математической физики. — Киев: Ин-т математики АН УССР, 1989. — С. 95–102.
11. *Bluman G.W., Cole J.D.* The general similarity solution of the heat equation // J. Math. Mech. — 1969. — **18**, N 11. — P. 1025–1042.
12. *Olver P., Rosenau Ph.* The construction of special solutions to partial differential equations // Phys. Lett. A. — 1986. — **114**, N 3. — P. 107–112.
13. *Clarkson P., Kruskal M.* New similarity reductions of the Boussinesq equation // J. Math. Phys. — 1989. — **30**, N 10. — P. 2201–2213.
14. *Levi D., Winternitz P.* Nonclassical symmetry reduction: example of the Boussinesq equation // J. Phys. A. — 1989. — **22**. — P. 2915–2924.
15. *Шульга М.В.* Симметрия и некоторые частные решения уравнения Даламбера с нелинейным условием // Теоретико-групповые исследования уравнений математической физики. — Киев: Ин-т математики АН УССР, 1985. — С. 36–38.
16. *Фущич В.И., Жданов Р.З., Ревенко И.В.* Совместность и решения нелинейных уравнений Даламбера и Гамильтона. — Киев, 1990. — 65 с. — (Препр./ АН УССР. Ин-т математики; № 90.39).

17. *Fushchych W.I., Serov N.I.* On some exact solutions of the three-dimensional non-linear Schrödinger equation // J. Phys. A: Math. Gen. — 1987. — **20**. — L929–L933.

18. *Фущич В.И., Чопик В.И.* Условная инвариантность нелинейного уравнения Шредингера // Докл. АН УССР. Сер. А. — 1990. — № 4. — С. 30–33.

19. *Фущич В.И., Серов Н.И.* Условная инвариантность и редукция нелинейного уравнения теплопроводности // Докл. АН УССР. Сер. А. — 1990. — № 7. — С. 24–27.

20. *Фущич В.И., Серов Н.И., Амеров Т.К.* Условная инвариантность нелинейного уравнения теплопроводности // Докл. АН УССР. Сер. А. — 1990. — № 11. — С. 16–18.

21. *Фущич В.И.* Об одном обобщении метода С. Ли // Теоретико-алгебраический анализ уравнений математической физики. — Киев: Ин-т математики АН УССР, 1990. — С. 4–9.

22. *Фущич В.И., Штелень В.М., Серов Н.И.* Симметрийный анализ и точные решения нелинейных уравнений математической физики. — Киев: Наук. думка, 1989. — 336 с.

23. *Фущич В.И., Серов Н.И., Амеров Т.К.* Об условной симметрии обобщенного уравнения Кортевега–де Фриза // Докл. АН УССР. — 1991. — № 12. — С. 15–18.

24. *Ames W.F., Lohner R.I., Adams E.* Group properties of $u_{tt} = (f(u)u_x)_x$ // Intern. J. Non-Linear Mech. — 1981. — **16**, N 5/6. — P. 439–447.

25. *Фущич В.И., Серов Н.И., Репета В.К.* Условная симметрия, редукция и точные решения нелинейного волнового уравнения // Докл. АН УССР. — 1991. — № 5. — С. 29–34.

26. *Фущич В.И., Чопик В.И., Миронюк П.И.* Условная инвариантность и точные решения трехмерных нелинейных уравнений акустики // Докл. АН УССР. Сер. А. — 1990. — № 9. — С. 25–28.

The complete sets of conservation laws for the electromagnetic field

W.I. FUSHCHYCH, A.G. NIKITIN

We present a compact and simple formulation of zero- and first-order conserved currents for the electromagnetic field and give the number of independent n-order currents.

New conservation laws for the electromagnetic field, discovered by Lipkin [1], had obtained an adequate mathematical and physical interpretation long ago, see e.g. [2–6]. It happens that these conservation laws are nothing but a small part of the infinite series of conserved quantities which exist for any self-adjoint linear system of differential equations; among their number are Maxwell's equations [7]. As to the physical interpretation of Lipkin's zilch tensor it can be connected with conservation of polarization of the electromagnetic field [5, 6].

The aim of the present letter is to establish certain rules in the bewildering complexity of the conservation laws and to describe complete sets of them for the electromagnetic field.

We say that an arbitrary bilinear function $j_\mu^{(m)} = f_\mu^{(m)}(D^n F, D^k F)$ is a conserved current if it satisfies the continuity equation

$$\partial^\mu j_\mu^{(m)} = 0, \quad \mu = 0,1,2,3. \tag{1}$$

Here $F = F_{\mu\nu}$ is the tensor of the electromagnetic field,

$$D^n = \prod_{\lambda=0}^{n} \partial^{\mu_\lambda}, \quad \mu_\lambda = 0,1,2,3, \quad m = \max(n+k).$$

It follows from (1) according to the Ostrogradskii–Gauss theorem that the following quantity is conserved in time:

$$\langle j_0^{(m)} \rangle = \int d^3x j_0^{(m)}.$$

We say conserved currents $j_\mu^{(m)}$ and $j_\mu^{\prime(m)}$ are equivalent if

$$\langle j_0^{(m)} \rangle = \langle j_0^{\prime(m)} \rangle.$$

Proposition 1. *There exist exactly* 15 *non-equivalent conserved currents of zero order for Maxwell's equation. All these currents can be represented in the form*

$$j_\mu^{(0)} = T_{\mu\nu} K^\nu, \tag{2}$$

where $T_{\mu\nu}$ is the traceless energy-momentum tensor of the electromagnetic field and K^ν is a Killing vector satisfying the equations

$$\partial^\nu K^\mu + \partial^\mu K^\nu - \frac{1}{2} g^{\mu\nu} \partial_\lambda K^\lambda = 0. \tag{3}$$

Reprinted with permission from J. Phys. A: Math. Gen., 1992, **25**, L231–L233

Original article is available at http://www.iop.org/EJ/journal/JPhysA

Proof. This reduces to finding the general solution of the equation

$$\partial^0 \langle j_0^{(0)} \rangle = \partial^0 \int d^3x j_0^{(0)}(F, F) = 0, \tag{4}$$

where $j_0^{(0)}(F, F)$ is a bilinear combination of components of the tensor of the electromagnetic field. It is not difficult to find such a solution, decomposing $j_0^{(0)}$ by the complete set of symmetric matrices of dimension 6×6

$$\begin{gathered}
j_0^{(0)} = \varphi^T Q \varphi, \quad \varphi = \text{column}(F_{01}, F_{02}, F_{03}, F_{23}, F_{31}, F_{12}),\\
Q = (\sigma_0 A_0^{ab} + \sigma_1 A_1^{ab} + \sigma_3 A_3^{ab}) Z_{ab} + \delta_2 S_a K^a,\\
Z_{ab} = 2\delta_{ab} + S_a S_b + S_b S_a, \quad a, b = 1, 2, 3,\\
S_a = \begin{pmatrix} S_a & \hat{0} \\ \hat{0} & S_a \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} I & \hat{0} \\ \hat{0} & I \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} \hat{0} & I \\ I & \hat{0} \end{pmatrix},\\
\sigma_2 = \begin{pmatrix} \hat{0} & -I \\ I & \hat{0} \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} I & \hat{0} \\ \hat{0} & -I \end{pmatrix},\\
S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{gathered}$$

where $\hat{0}$ and I are the zero and unit matrices of dimension 3×3, A_λ^{ab}, K^a are unknown functions of x_μ. Indeed substituting (5) into (4) and using the Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial^\mu \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = 0$$

we come to the relations $A_1^{ab} = A_3^{ab} = 0$, $A_2^{ab} = -\delta^{ab} K^0$ and to the equations (3) for K^0 and K^a.

Thus we have found all non-equivalent $j_0^{(0)}$ satisfying (4). The corresponding expressions for $j_\mu^{(0)}$ with $\mu \neq 0$ can be obtained using Lorentz transformations.

Formula (2) gives an elegant formulation of the classical conservation laws of Bessel–Hagen [8]. We present a direct (and simple) proof that there are not another conserved bilinear combination of the electromagnetic field strengths.

In an analogous way it is possible to prove the following assertion.

Proposition 2. *There exist exactly* 84 *conserved currents of first order for the electromagnetic field. All these currents can be represented in the form*

$$j_\mu^{(1)} = K^{\sigma\nu} Z_{\sigma\nu,\mu} + 2\varepsilon_{\mu\nu\lambda\sigma} (\partial^\lambda K^{\rho\nu}) T^{\sigma\rho}, \tag{5}$$

where $T^{\sigma\rho}$ *is the energy-momentum tensor,* $Z_{\sigma\nu,\mu}$ *is Lipkin's zilch tensor,* $\varepsilon_{\mu\nu\rho\sigma}$ *is the completely antisymmetric unit tensor,* $K^{\sigma\nu}$ *is a conformal Killing tensor of valence* 2*, satisfying the equations*

$$\partial^{(\mu} K^{\sigma\nu)} = \frac{1}{3} \partial_\lambda K^{\lambda(\mu} g^{\sigma\nu)}, \quad K^{\sigma\nu} = K^{\nu\sigma}, \quad K_\mu^\mu = 0, \tag{6}$$

where symmetrization is imposed over the indices in brackets.

Using the relations

$$\begin{gathered}
\partial^\mu Z_{\lambda\sigma,\mu} = 0, \quad Z_{\mu\nu,}{}^\nu = 0, \quad \partial_\lambda T^{\lambda\mu} = 0, \quad T^\lambda{}_\lambda = 0,\\
\partial^\rho (\varepsilon_{\rho\lambda\nu\sigma} T^{\sigma\mu} + \varepsilon_{\rho\mu\nu\sigma} T^{\sigma\lambda}) = Z_{\lambda\nu,}{}^\mu + Z_{\mu\nu,}{}^\lambda
\end{gathered}$$

and the equations (7) we can ensure that the currents (6) really satisfy the continuity equation (1).

Thus all non-equivalent conserved currents of first order are given by formula (6). The general solution of the equation (7) is a fourth-order polynomial of x_μ depending on 84 parameters; for the explicit expression of $K^{\sigma\nu}$ see e.g. [9]. Formula (6) describes well known and also 'new' conserved currents; the latter depend on the fourth degree of x_μ.

In conclusion we note that in an analogous way it is possible to describe conserved currents for the electromagnetic field of an arbitrary order m. For $m > 1$ such currents are defined by two fundamental quantities i.e. by the conformal Killing tensor of valence $m + 1$ and the Floyd–Penrose tensor of valence $R_1 + 2R_2$ where $R_1 = m - 1$, $R_2 = 2$. The higher order conserved currents will be considered in a separate paper; here we present only the number of linearly independent currents of order m:

$$N_m = \frac{1}{2}(2m+5)\left[2m(m+1)(m+4)(m+5) + (m+2)^2(m+3)^2\right], \quad m > 1.$$

For the details about generalized Killing and Floyd–Penrose tensors in application to higher symmetries of Poincaré- and Galilei-invariant wave equations see the extended version of our book [10]. Non-Lie symmetries and conservation laws for Maxwell's equations are discussed in [11].

1. Lipkin D.M., *J. Math. Phys.*, 1964, **5**, 696.
2. Klibble T.W., *J. Math. Phys.*, 1964, **5**, 1022.
3. Morgan T.A., *J. Math. Phys.*, 1964, **5**, 1659.
4. Michelson J., Niederle J., *Lett. Math. Phys.*, 1984, **8**, 195.
5. O'Connel R.F., Tompkins D.R., *Nuovo Cimento*, 1965, **39**, 391.
6. Candlin P.I., *Nuovo Cimento*, 1965, **37** 1390.
7. Olver P., Applications of Lie groups to differential equations, New York, Springer, 1986.
8. Bessel-Hagen E., *Math. Ann.*, 1921, **84**, 258.
9. Nikitin A.G., *Ukr. Math. J.*, 1991, **43**, 786.
10. Fushchych W.I., Nikitin A.G., Symmetries of the equations of quantum mechanics, Moscow, Nauka, 1990 (in Russian); New York, Allerton Press, 1994.
11. Fuschych W.I., Nikitin A.G., Symmetries of Maxwell's equations, Dordrecht, Reidel, 1987.

*For further development of the results of this paper see [Anco S.C., Pohjanpelto J., Conserved currents of massless fields of spin $s \geq \frac{1}{2}$, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 2003, **459**, N 2033, 1215–1239] and references therein. *Editors' Remark.*

Q-conditional symmetry of the linear heat equation

W.I. FUSHCHYCH, W.M. SHTELEN, M.I. SEROV, R.O. POPOVYCH

Исследована Q-условная симметрия одномерного линейного уравнения теплопроводности. Получены определяющие уравнения для коэффициентов оператора Q-условной симметрии, изучена их лиевская симметрия, получены некоторые их точные решения. Найдены нелокальные замены, сводящие определяющие уравнения к исходному уравнению теплопроводности. Показано, как можно использовать операторы Q-условной симметрии для линеаризации нелинейных ДУЧП и размножения решений уравнения теплопроводности.

In this article we consider in full detail, as a simple but non-trivial example, how to find and use Q-conditional symmetry of the one-dimensional linear heat equation

$$u_0 = u_{11} \tag{1}$$

($u = u(x_0, x_1)$, $u_0 = \partial u/\partial x_0$, $u_1 = \partial u/\partial x_1$ and so on).

It is known [1] that the maximal in Lie sense invariance algebra of equation (1) is an algebra with the basis elements

$$\begin{gathered}\partial_0 = \frac{\partial}{\partial x_0}, \quad \partial_1 = \frac{\partial}{\partial x_1}, \quad G = x_0\partial_1 - \frac{1}{2}x_1 u\partial_u, \quad I = u\partial_u, \\ D = 2x_0\partial_0 + x_1\partial_1, \quad \Pi = x_0\left(x_0\partial_0 + x_1\partial_1 - \frac{1}{2}u\partial_u\right) - \frac{x_1^2}{4}u\partial_u, \\ L = f(x_0, x_1)\partial_u \quad (f_0 = f_{11}).\end{gathered} \tag{2}$$

The problem of finding non-classical symmetry (in our terminology Q-conditional symmetry) was firstly put forward by Bluman and Cole [5]. However, in this important paper the authors did not give explicitly none of operators which would be different from those of (2). Below we will present quite complete investigation of this problem. All notions used without explanations are defined in [1–4].

Definition 1 [2, 4]. *A differential equation of order m*

$$S_1(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = 0 \tag{3}$$

for a function $u = u(x)$ where $\underset{k}{u}$ denotes all partial derivatives of order k is called conditionally invariant under an operator Q if there is an additional condition of the form

$$S_2(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = 0 \tag{4}$$

compatible with (3), *that*

$$\tilde{Q}S_\alpha\Big|_{\substack{S_1=0\\ S_2=0}} = 0, \quad \alpha = 1, 2, \tag{5}$$

In the formula (5) $\tilde{Q}$ *is the standard prolongation of* Q.

Передруковано з Докл. АН Украины, 1992, № 12, С. 28–33 за дозволом

In that particular case when equation (4) has the form

$$Qu = 0 \tag{6}$$

equation (3) is called Q-conditionally invariant under the operator Q. The notion of Q-conditional invariance coincides with the notion of "non-classical" invariance introduced by Bluman and Cole in the work [5].

The general form of a first-order operator is

$$Q = A(x_0, x_1, u)\partial_0 + B(x_0, x_1, u)\partial_1 + C(x_0, x_1, u)\partial_u, \tag{7}$$

where A, B, C are some differentiable functions of x_0, x_1, u to be determined from the invariance condition (5). It will be noted that because of the imposed condition (6)

$$Qu = 0 \ \Leftrightarrow \ Au_0 + Bu_1 = C \tag{8}$$

there are really only two independent cases of operator (7).

Theorem 1. *The heat equation* (1) *is Q-conditionally invariant under operator* (7) *if and only if its coordinates are as follows:*

Case 1.

$$A = 1, \quad B = W^1(x_0, x_1), \quad C = W^2(x_0, x_1)u + W^3(x_0, x_1) \tag{9}$$

and functions $\vec{W} = \vec{W}(x_0, x_1) = \{W^1, W^2, W^3\}$ satisfy equations

$$(\partial_0 + 2W^1_1 - \partial_{11})\vec{W} = \vec{F}, \quad \vec{F} = \{-2W^2_1, 0, 0\}. \tag{10}$$

Case 2.

$$A = 0, \quad B = 1, \quad C = v(x_0, x_1, u) \tag{11}$$

and function $v = v(x_0, x_1, u)$ satisfies the PDE

$$v_0 = v_{11} + 2vv_{1u} + v^2 v_{uu}. \tag{12}$$

Proof. From the criterion of invariance

$$\underset{2}{Q}(u_0 - u_{11})\Big|_{\substack{u_0 = u_{11}, \\ Qu = 0}} = 0, \tag{13}$$

absolutely analogously to the standard Lie's algorithm one finds the defining equations for the coordinates of operator (7) which can be reduced to (9)–(12). It is to be pointed out that unlike Lie's algorithm, in the cases considered above the defining equations (10), (12) are nonlinear ones and it is a typical feature of Q-conditional invariance.

It goes without saying that Q-conditional invariance includes Lie's invariance in particular. So, in our case of the heat equation, we obtain infinitesimals (2) as simplest solutions of (10), (12):

$$\begin{aligned}
&A = 1, \quad \vec{W} = 0 \quad \Rightarrow \quad Q = \partial_0, \\
&A = v = 0, \quad B = 1 \quad \Rightarrow \quad Q = \partial_1, \\
&A = 0, \quad B = 1, \quad v = -\frac{x_1 u}{2x_0} \quad \Rightarrow \quad Q = G, \\
&A = 1, \quad W^1 = \frac{x_1}{2x_0}, \quad W^2 = W^3 = 0 \quad \Rightarrow \quad Q = D, \\
&A = 1, \quad W^1 = \frac{x_1}{x_0}, \quad W^2 = -(2x_0 + x_1^2)/4x_0^2, \quad W^3 = 0 \quad \Rightarrow \quad Q = \Pi.
\end{aligned} \tag{14}$$

Remark 1. System of defining equations (10) was firstly obtained by Bluman and Cole [5]. Further investigation of system (10) was continued in [6], where the question of linearization of the first two equations of (10) had been studied. The general solution of the problem of linearization of equations (10), (12) will be given after a while.

Now let us list some concrete operators (7) of Q-conditional invariance of equation (1) obtained as partial solutions of the defining equations (10), (12). In the following Table we also give corresponding invariant ansätze and the reduced equations.

Of course, operators 1–10 from Table do not exhaust all possible operators of Q-conditional invariance.

N	Operator Q	Ansatz	Reduced equation
1	$-x_1\partial_0+\partial_1$	$u=\varphi\left(x_0+\frac{x_1^2}{2}\right)$	$\varphi''=0$
2	$-x_1\partial_0+\partial_1+x_1^3\partial_u$	$u=\varphi\left(x_0+\frac{x_1^2}{2}\right)+\frac{x_1^4}{4}$	$\varphi''=-3$
3	$x_1^2\partial_0-3x_1\partial_1-3u\partial_u$	$u=x_1\varphi\left(x_0+\frac{x_1^2}{6}\right)$	$\varphi''=0$
4	$x_1^2\partial_0-3x_1\partial_1-(3u+x_1^5)\partial_u$	$u=x_1\varphi\left(x_0+\frac{x_1^2}{6}\right)+\frac{x_1^5}{12}$	$\varphi''=-15$
5	$x_1\partial_1+u\partial_u$	$u=x_1\varphi(x_0)$	$\varphi'=0$
6	$\operatorname{cth}x_1\partial_1+u\partial_u$	$u=\varphi(x_0)\operatorname{ch}x_1$	$\varphi'-\varphi=0$
7	$-\operatorname{ctg}x_1\partial_1+u\partial_u$	$u=\varphi(x_0)\cos x_1$	$\varphi'+\varphi=0$
8	$\partial_1-u\partial_u-\frac{u}{2x_0-x_1}\partial_u$	$u=(2x_0-x_1)e^{-x_1}\varphi(x_0)$	$\varphi'-\varphi=0$
9	$\partial_1-\sqrt{-2(x_0+u)}\partial_u$	$u=-x_0-\frac{1}{2}[x_1+\varphi(x_0)]^2$	$\varphi'=0$
10	$\left(x_0+\frac{x_1^2}{2}\right)\partial_0-x_1\partial_1$	$u=\varphi\left(x_0x_1+\frac{x_1^2}{3!}\right)$	$\varphi''=0$

Next we study Lie symmetry of the defining equations (10), (12).

Theorem 2. *The Lie maximal invariance algebra of system* (10) *is given by the operators*

$$\begin{aligned}
&\partial_0,\quad \partial_1,\quad G^{(1)}=x_0\partial_1+\partial_{W^1}-\frac{1}{2}W^1\partial_{W^2}-\frac{1}{2}x_1W^3\partial_{W^3},\\
&D^{(1)}=2x_0\partial_0+x_1\partial_1-W^1\partial_{W^1}-2W^2\partial_{W^2},\quad I^{(1)}=W^3\partial_{W^3},\\
&\Pi^{(1)}=x_0\left(x_0\partial_0+x_1\partial_1-W^1\partial_{W^1}-2W^2\partial_{W^2}-\frac{5}{2}W^3\partial_{W^3}\right)+\\
&\qquad +x_1\left(\partial_{W^1}-\frac{1}{2}W^1\partial_{W^2}\right)-\frac{1}{2}\partial_{W^2}-\frac{x_1^2}{4}W^3\partial_{W^3},\\
&X=(f_0+f_1W^1-fW^2)\partial_{W^3}.
\end{aligned}\tag{15}$$

where $f=f(x_0,x_1)$ is an arbitrary solution of (1), *that is $f_0=f_{11}$.*

Theorem 3. *The Lie maximal invariance algebra of equation* (12) *is given by the operators*

$$\begin{aligned}
&\partial_0,\quad \partial_1,\quad D^{(2)}=2x_0\partial_0+x_1\partial_1+u\partial_u,\quad D^{(3)}=u\partial_u+v\partial_v,\\
&G^{(2)}=x_0\partial_1-\frac{1}{2}x_1(u\partial_u+v\partial_v)-\frac{1}{2}u\partial_v,\\
&\Pi^{(2)}=x_0\left(x_0\partial_0+x_1\partial_1-\frac{1}{2}u\partial_u-\frac{3}{2}v\partial_v\right)-\frac{x_1^2}{4}(u\partial_u+v\partial_v)-\frac{x_1}{2}u\partial_v,\\
&R=f\partial_u+f_1\partial_v\quad (f_0=f_{11}).
\end{aligned}\tag{16}$$

One can get the proofs of these two theorems by means of the standard Lie's algorithm.

Operators (15), (16) can be used to find exact solutions of equations (10), (12). In particular, using the formula of generating solutions at the expense of invariance under $\Pi^{(2)}$

$$v^{\rm II}(x_0,x_1,u)=(1-\theta x_0)^{-3/2}\exp\left\{\frac{\theta x_1^2}{4(1-\theta x_0)}\right\}v^1(x_0',x_1',u')+\frac{\theta}{1-\theta x_0}\frac{x_1u}{2},$$
$$x_0'=\frac{x_0}{1-\theta x_0},\quad x_1'=\frac{x_1}{1-\theta x_0}, \tag{17}$$
$$u'=(1-\theta x_0)^{1/2}\exp\left\{-\frac{1}{4}\frac{\theta x_1^2}{1-\theta x_0}\right\}u\quad(\theta=\mathrm{const})$$

one can construct new solutions of equations (12) starting from known ones.

Solutions of equations (10), (12) can be obtained by the use of reduction on subalgebras of the invariance algebras (15), (16). For example, using the subalgebra $\langle\partial_0+a_i^{(1)}\rangle$ of the algebra (15) we find the following solution of the system (10)

$$W^1=\frac{C_1^2-C_3^2}{-C_1\,\mathrm{tg}(C_1x_1+C_2)+C_3\,\mathrm{tg}(C_3x_1+C_4)},$$
$$W^2=-C_1C_3\frac{C_1\,\mathrm{tg}(C_3x_1+C_4)-C_3\,\mathrm{tg}(C_1x_1+C_2)}{-C_1\,\mathrm{tg}(C_1x_1+C_2)+C_3\,\mathrm{tg}(C_3x_1+C_4)}, \tag{18}$$
$$W^3=(\varphi_{11}-W^1\varphi_1-W^2\varphi)e^{ax_0},$$

where $C_1,\dots,C_4$ are arbitrary constants, $\varphi=\varphi(x_1)$, $\varphi_{11}=a\varphi$.

Theorem 4. *The system* (10) *is reduced to the system of disconnected heat equations*

$$\vec z_0=\vec z_{11}\quad(\vec z=\vec z(x_0,x_1)=\{z^1,z^2,z^3\}) \tag{19}$$

with the help of the nonlocal transformation

$$W^1=-\frac{z_{11}^1z^2-z^1z_{11}^2}{z_1^1z^2-z^1z_1^2},\quad W^2=-\frac{z_{11}^1z_1^2-z_1^1z_{11}^2}{z_1^1z^2-z^1z_1^2},$$
$$W^3=z_{11}^3+W^1z_1^3-W^2z^3. \tag{20}$$

Expressions (20) result in (after using the corresponding operator (7), (9)) the ansatz

$$u=z^1\varphi(\omega)+z^3,\quad\omega=\frac{z^2}{z^1} \tag{21}$$

(z^1, z^2, z^3 are solutions of (19)), and the reduced equation is $\varphi''=0$. This means that

$$u=C_1z^1+C_2z^2+z^3. \tag{22}$$

So, we get just the well-known superposition principle for the heat equation.

Letting $W^2=W^3=0$ we get from (10) the Burgers equation

$$W_0^1+2W^1W_1^1=W_{11}^1. \tag{23}$$

Using Hopf–Cole transformation one obtains solutions of equation (23) in the form

$$W^1 = -\partial_1 \ln f = -\frac{f_1}{f} \quad (f_0 = f_{11}). \tag{24}$$

This result in the operator

$$Q = f\partial_0 - f_1\partial_1. \tag{25}$$

Q-conditional symmetry of equation (1) under the operator (25) lead to the following statement.

Theorem 5. *If function f is an arbitrary solution of the heat equation* (1) *and u is the general integral of the ODE*

$$f_1 dx_0 + f dx_1 = 0, \tag{26}$$

then u satisfies equation (1).

Proof. We note that equation (26) is a perfect differential equation and therefore its general solution $u(x_0, x_1) = C$ possesses the following property

$$u_0 = f_1, \quad u_1 = f. \tag{27}$$

Having used (27) we obtain

$$u_0 - u_{11} = f_1 - f_1 = 0$$

and the theorem is proved.

Theorem 5 may be considered as another algorithm of generating solutions of equation (1). Indeed, even starting from a rather trivial solution of the heat equation $u = 1$ we get the chain of quite interesting solutions

$$1 \to x_1 \ \to \ x_0 + \frac{x_1^2}{2!} \ \to \ x_0x_1 + \frac{x_1^3}{3!} \ \to \ \cdots, \tag{28}$$

and among them the solutions

$$\frac{x_1^{2m}}{(2m)!} + \frac{x_0}{1!}\frac{x_1^{2m-2}}{(2m-2)!} + \frac{x_0^2}{2!}\frac{x_1^{2m-4}}{(2m-4)!} + \cdots + \frac{x_0^{m-1}}{(m-1)!}\frac{x_1^2}{2!} + \frac{x_0^m}{m!}, \tag{29}$$

$$\frac{x_1^{2m+1}}{(2m+1)!} + \frac{x_0}{1!}\frac{x^{2m-1}}{(2m-1)!} + \frac{x_0^2}{2!}\frac{x_1^{2m-3}}{(2m-3)!} + \cdots + \frac{x_0^{m-1}}{(m-1)!}\frac{x_1^3}{3!} + \frac{x_0^m}{m!}\frac{x_1}{1!}. \tag{30}$$

It will be also noted that supposing function v in (12) to be independent on x_1 and denoting

$$v = \frac{1}{w(x_0, u)} \tag{31}$$

we get instead of (12) the following remarkable nonlinear heat equation

$$w_0 = \partial_u(w^{-2}w_u). \tag{32}$$

One easily sees that the operator

$$Q = w(x_0, u)\partial_1 + \partial_u \tag{33}$$

sets the connection between equations (32) and (1):

$$
\begin{aligned}
&w_0 - \partial_u(w^{-2}w_u) = \frac{1}{u_1}\partial_1\left(\frac{u_0 - u_{11}}{u_1}\right),\\
&u_0 - u_{11} = \frac{1}{w}\int[w_0 - \partial_u(w^{-2}w_u)]du
\end{aligned}
\tag{34}
$$

by means of the change of variables

$$
w(x_0, u) = \frac{\partial x_1(x_0, u)}{\partial u}, \qquad \frac{\partial u(x_0, x_1)}{\partial x_1} = \frac{1}{w(x_0, u)}. \tag{35}
$$

This result has been obtained differently in [7, 8].

If v from (12) has the form

$$
v = \varphi(x_0, x_1)u \tag{36}
$$

then (12) is reduced to the Burgers equation for φ

$$
\varphi_0 = 2\varphi\varphi_1 + \varphi_{11} \tag{37}
$$

and one may say that operator

$$
Q = \partial_1 + \varphi u \partial_u \tag{38}
$$

sets the connection between equation (37) and (1) via the substitution

$$
\varphi = \frac{f_1}{f}. \tag{39}
$$

Letting

$$
v = \varphi(x_0, x_1)u + h(x_0, x_1) \tag{40}
$$

and substituting it into (12) one finds the Burgers equation (37) for function φ and the following equation for h

$$
h_0 = 2h\varphi_1 + h_{11}. \tag{41}
$$

System of equation (37), (41) was also obtained in [6] when considering the system (10). Having made the change of variables

$$
\varphi = \frac{f_1}{f}, \quad h = \frac{f_1}{f}g - g_1 \tag{42}
$$

we reduced (37), (41) to two disconnected heat equations

$$
f_0 = f_{11}, \quad g_0 = g_{11}. \tag{43}
$$

Now we consider how to linearise the equation (12) in general case. Let us introduce the notations

$$
S^1(x_0, x_1, u, v) = v_0 - (v_{11} + 2vv_{1u} + v^2 v_{uu}). \tag{44}
$$

After changing the variables

$$
v = -\frac{z_1}{z_u}, \quad z = z(x_0, x_1, u) \tag{45}
$$

we get

$$S^1(x_0,x_1,u,v) = -\frac{1}{z_u}(\partial_1 + v\partial_u)S^2(x_0,x_1,u,z), \tag{46}$$

where

$$S^2(x_0,x_1,u,v) = z_0 - z_{11} + 2\frac{z_1}{z_u}z_{1u} - \frac{z_1^2}{z_u^2}z_{uu}. \tag{47}$$

Having applied the hodograph transformation

$$y_0 = x_0, \quad y_1 = x_1, \quad y_2 = z, \quad R = u \tag{48}$$

we get

$$S^2(x_0,x_1,u,z) = -\frac{1}{R_2}(R_0 - R_{11}), \tag{49}$$

where $R = R(y_0, y_1, y_2)$.

1. *Фущич В.И., Штелень В.М., Серов Н.И.* Симметрийный анализ и точные решения нелинейных уравнений математической физики. — Киев: Наук. думка, 1989. — 336 с.
2. *Фущич В.И.* Как расширить симметрию дифференциальных уравнений? // Симметрия и решения нелинейных уравнений математической физики. — Киев: Ин-т математики АН УССР, 1987. — С. 4–16.
3. *Фущич В.И.* Условная симметрия уравнений нелинейной математической физики // Укр. мат. журн. — 1991. — **43**, № 11. — С. 1456–1470.
4. *Фущич В.И., Серов Н.И.* Условная инвариантность и редукция нелинейного уравнения теплопроводности // Докл. АН УССР. Сер. А. — 1990. — № 7. — С. 24–27.
5. *Bluman G.W., Cole J.D.* The general similarity solution of the heat equation // J. Math. Mech. — 1969. — **18**, N 11. — P. 1025–1042.
6. *Webb G.M.* Lie symmetries of a coupled nonlinear Burgers–heat equation system // J. Phys. A. — 1990. — **23**, N 17. — P. 3885–3894.
7. *Rosen G.* Nonlinear heat condition in solid H_2 // Phys. Rev. B. — 1979. — **19**, N 4. — P. 2398–2399.
8. *Bluman G.W., Kumei S.* On the remarkable nonlinear diffusion equation $\frac{\partial}{\partial x}\left[a(u+b)^{-2}\frac{\partial u}{\partial x}\right] - \frac{\partial u}{\partial t} = 0$ // J. Math. Phys. — 1980. — **21**, N 5. — P. 1019–1023.

*A more precise definition of Q-conditional symmetries was given by Zhdanov R.Z., Tsyfra I.M. and Popovych R.O [A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, **238**, N 1, 101–123; math-ph/0207023] with using equivalence Q-conditional symmetries and reductions.

Results on connection of initial equations and systems of determining equations for Q-conditional symmetries have been generalized to linear transfer equations [Popovych R.O., On the symmetry and exact solutions of a transport equation, *Ukr. Math. J.*, 1995, **47**, N 1, 121–125], (1+1)-dimensional evolution equations [Zhdanov R.Z., Lahno V.I., Conditional symmetry of a porous medium equation, *Phys. D*, 1998, **122**, 178–186] evolution equations of arbitrary dimensions [Popovych R.O., On a class of Q-conditional symmetries and solutions of evolution equations, in Symmetry and Analytic Methods in Mathematical Physics, *Proceedings of Institute of Mathematics*, Kyiv, 1998, **19**, 194–199] and systems of such equations [Vasilenko O.F., Popovych R.O., On class of reducing operators and solutions of evolution equations, *Vestnik PGTU*, 1999, N 8, 269–273]. *Editors' Remark.*

Second-order differential invariants of the rotation group $O(n)$ and of its extensions: $E(n)$, $P(1,n)$, $G(1,n)$

W.I. FUSHCHYCH, I.A. YEGORCHENKO

Functional bases of second-order differential invariants of the Euclid, Poincaré, Galilei, conformal, and projective algebras are constructed. The results obtained allow us to describe new classes of nonlinear many-dimensional invariant equations.

0. Introduction

The concept of the invariant is widely used in various domains of mathematics. In this paper, we investigate the differential invariants within the framework of symmetry analysis of differential equations.

Differential invariants and construction of invariant equations were considered by S. Lie [1] and his followers [2, 3]. Tresse [2] had proved the theorem on the existence and finiteness of a functional basis of differential invariants. However, there exist few papers devoted to the construction in explicit form of differential invariants for specific groups involved in mechanics and mathematical physics.

Knowledge of differential invariants of a certain algebra or group facilitates classification of equations invariant with respect to this algebra or group. There are also some general methods for the investigation of differential equations which need the explicit form of differential invariants for these equations' symmetry groups (see, e.g., [3, 4]).

A brief review of our investigation of second-order differential invariants for the Poincaré and Galilei groups is given in [5, 6]. Our results on functional bases of differential invariants are founded on the Lemma about functionally independent invariants for the proper orthogonal group and two n-dimensional symmetric tensors of the order 2.

We should like to stress that we consider functionally independent invariants of but not irreducible ones, as in the classical theory of invariants.

Bases of irreducible invariants for the group $O(3)$ and three-dimensional symmetric tensors and vectors are adduced in [7].

The definitions of differential invariants differ in various domains of mathematics, e.g. in differential geometry and symmetry analysis of differential equations. Thus, we believe that some preliminary notes are necessary, though these formulae and definitions can be found in [8, 9, 10].

We deal with Lie algebras consisting of the infinitesimal operators

$$X = \xi^i(x,u)\partial_{x_i} + \eta^r(x,u)\partial_{u^r}. \tag{0.1}$$

Here $x = (x^1, x^2, \dots, x^n)$, $u = (u^1, \dots, u^m)$. We usually mean the summation over the repeating indices.

Reprinted with permission from Acta Appl. Math., 1992, **28**, P. 69–92

Definition 1. *The function*

$$F = F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}),$$

where $\underset{k}{u}$ is the set of all k^{th}-order partial derivatives of the function u is called a differential invariant for the Lie algebra L with basis elements X_i of the form (0.1) *($L = \langle X_i \rangle$) if it is an invariant of the lth prolongation of this algebra:*

$$\overset{l}{X}_s F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = \lambda_s(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) F, \tag{0.2}$$

where the λ_s are some functions; when $\lambda_i = 0$, F is called an absolute invariant; when $\lambda_i \neq 0$, it is a relative invariant.

Further, we deal mostly with absolute differential invariants and when writing 'differential invariant' we mean 'absolute differential invariant'.

Definition 2. *A maximal set of functionally independent invariants of order $r \leq l$ of the Lie algebra L is called a functional basis of the l^{th}-order differential invariants for the algebra L.*

We consider invariants of order 1 and 2, and need the first and second prolongations of the operator X (0.1) (see, e.g., [8–11])

$$\overset{1}{X} = X + \eta^r_i \partial_{u^r_i}, \quad \overset{2}{X} = \overset{1}{X} + \eta^r_{ij} \partial_{u^r_{ij}}$$

the coefficients η^r_i and η^r_{ij} taking the form

$$\eta^r_i = (\partial_{x_i} + u^s_i \partial_{u^s}) \eta^r - u^r_k (\partial_{x_i} + u^s_i \partial_{u^s}) \xi^k,$$
$$\eta^r_{ij} = (\partial_{x_i} + u^s_j \partial_{u^s} + u^s_{jk} \partial_{u^s_k}) \eta^r_i - u^r_{ik} (\partial_{x_j} + u^s_j \partial_{u^s}) \xi^k.$$

While writing out lists of invariants, we shall use the following designations

$$\begin{gathered}
u_a \equiv \frac{\partial u}{\partial x_a}, \quad u_{ab} \equiv \frac{\partial^2 u}{\partial x_a \partial x_b}, \\
S_k(u_{ab}) \equiv u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k} u_{a_k a_1}, \\
S_{jk}(u_{ab}, v_{ab}) \equiv u_{a_1 a_2} \cdots u_{a_{j-1} a_j} v_{a_j a_{j+1}} \cdots v_{a_k a_1}, \\
R_k(u_a, u_{ab}) \equiv u_{a_1} u_{a_k} u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k}.
\end{gathered} \tag{0.3}$$

Here and further we mean summation over the repeated indices from 1 to n. In all the lists of invariants, k takes on the values from 1 to n and j takes the values from 0 to k. We shall not discern the upper and lower indices with respect to summation: for all Latin indices

$$x_a x_a \equiv x_a x^a \equiv x^a x_a = x_1^2 + x_2^2 + \cdots + x_n^2.$$

1. Differential invariants for the Euclid algebra

The Euclid algebra $AE(n)$ is defined by basis operators

$$\partial_a \equiv \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a \partial_b - x_b \partial_a. \tag{1.1}$$

Here and further, the letters a, b, c, d, when used as indices, take on the values from 1 to n, n being the number of space variables ($n \geq 3$).

The algebra $AE(n)$ is an invariance algebra for a wide class of many-dimensional scalar equations involved in mathematical physics — the Schrödinger, heat, d'Alembert equations, etc.

In this section, we shall explain in detail how to construct a functional basis of the second-order differential invariants for the algebra $AE(n)$. This basis will be further used to find invariant bases for various algebras containing the Euclid algebra as a subalgebra — the Poincaré, Galilei, conformal, projective algebras, etc.

1.1. The main results. Let us first formulate the main results of the section in the form of theorems.

Theorem 1. *There is a functional basis of second-order differential invariants for the Euclid algebra $AE(n)$ with the basis operators* (1.1) *for the scalar function $u = u(x_1, \ldots, x_n)$ consisting of these $2n + 1$ invariants*

$$u, \quad S_k(u_{ab}), \quad R_k(u_a, u_{ab}). \tag{1.2}$$

Theorem 2. *The second-order differential invariants of the algebra $AE(n)$* (1.1) *for the set of scalar functions u^r, $r = 1, \ldots, m$, can be represented as functions of the following expressions:*

$$u^r, \quad S_{jk}(u^1_{ab}, u^r_{ab}), \quad R_k(u^r_a, u^1_{ab}). \tag{1.3}$$

1.2. Proofs of the theorems. Absolute differential invariants are obtained as solutions of a linear system of first-order partial differential equations (PDE). Thus, the number of elements of a functional basis is equal to the number of independent integrals of this system. This number is equal to the difference between the number of variables on which the functions being sought depend, and the rank of the corresponding system of PDE (in our case, this rank is equal to the generic rank of the prolonged operator algebra [8, 9].

To prove the fact that N invariants which have been found, $F^i = F^i(x, u, \underset{1}{u}, \ldots, \underset{l}{u})$, form a functional basis, it is necessary and sufficient to prove the following statements:

(1) the F^i are invariants;

(2) the F^i are functionally independent;

(3) the set of invariants F^i is complete or N is equal to the difference of the number of variables $(x, u, \underset{1}{u}, \ldots, \underset{l}{u})$ and the rank of the system of defining operators.

We seek second-order differential invariants in the form

$$F = F(x, u, \underset{1}{u}, \underset{2}{u}).$$

It follows from the condition of invariance with respect to translation operators ∂_a that F does not depend on x_a; evidently, u is an invariant of the operators (1.1). Thus, it is sufficient to seek invariants depending on $\underset{1}{u}$ and $\underset{2}{u}$ only. The criterion of the absolute invariance (0.1) in this case has the form

$$\hat{J}_{ab} F(\underset{1}{u}, \underset{2}{u}) = 0, \tag{1.4}$$

where

$$\hat{J}_{ab} = u^r_a \partial_{u^r_b} - u^r_b \partial_{u^r_a} + 2(u^r_{ac}\partial_{u^r_{bc}} - u^r_{bc}\partial_{u^r_{ac}}), \tag{1.5}$$

the summation over r from 1 to m being implied.

In that way, the problem of finding the second-order differential invariants of the algebra $AE(n)$ is reduced to the construction of a functional basis for the rotational algebra $AO(n)$ with the basis operators (1.5) for m vectors and m symmetric tensors of order 2.

Lemma 1. *The rank of the algebra $AO(n)$ is equal to $(n(n-1))/2$.*

Proof. It is sufficient to prove the lemma for $m = 1$. The basis of the algebra (1.5) consists of $(n(n-l))/2$ operators. According to definition [8], its rank is equal to the generic rank of the coefficient matrix of these operators. Let us put $u_{ab} = 0$ when $a \neq b$ and write down the coefficient columns by $\partial_{u_{ab}}$ of the operators (1.5):

$$\begin{pmatrix} u_{11} - u_{22} & 0 & \cdots & 0 \\ 0 & u_{11} - u_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & u_{n-1,n-1} - u_{nn} \end{pmatrix}. \tag{1.6}$$

When $u_{aa} \neq u_{bb}$ for $a \neq b$ and all $u_{aa} \neq 0$, the determinant of the matrix (1.6) does not vanish, therefore its generic rank (that is, the generic rank the algebra being considered) cannot be less than $(n(n-1))/2$. The lemma is proved. ■

Lemma 2. *The expressions*

$$S_k(u_{ab}), \quad R_k(u_a, u_{ab}) \tag{1.7}$$

are functionally independent.

Proof. To establish independence of expressions (1.7), it is sufficient to consider the case when $u_{ab} = 0$ if $a \neq b$ and $u_{aa} \neq 0$. Let us write down the Jacobian of the invariants

$$\left|\begin{array}{ccc|ccc} 1 & \cdots & 1 & & & \\ 2u_{11} & \cdots & 2u_{nn} & & \mathbf{0} & \\ \cdots & \cdots & \cdots & & & \\ nu_{11}^{n-1} & \cdots & nu_{rr}^{n-1} & & & \\ \hline & & & 2u_1 & \cdots & 2u_n \\ & \cdots & & \cdots & \cdots & \cdots \\ & & & 2u_1u_{11}^{n-1} & \cdots & 2u_nu_{nn}^{n-1} \end{array}\right| \tag{1.8}$$

The Jacobian (1.8) is equal up to a coefficient to the product of two Vandermonde determinants and is not equal to zero if $u_{aa} \neq u_{bb}$ whenever $a \neq b$. Thus, the expressions (1.17) are functionally independent. ■

Proof of Theorem 1. The fact that expressions (1.2) are invariants of $AO(n)$ can be easily proved by direct substitution of these expressions into the invariance conditions. Nevertheless, it is useful to note that $S_k(u_{ab})$ are traces of the symmetric matrix $(u_{ab}) = U$ and its powers, $R_k(u_a, u_{ab})$ are the scalar products of the vector $(u_a) = (u_1, \ldots, u_n)$, the matrix U^{k-1} and the vector $(u_a)^T$.

The invariants for the vector (u_a) and the symmetric tensor (u_{ab}) depend on their $(n(n+3))/2$ elements. Thus, it follows from Lemma 1 that a functional basis of the algebra $AO(n)$ for (u_a) and (u_{ab}) must consist of

$$\frac{n(n+3)}{2} - \frac{n(n-1)}{2} = 2n$$

invariants.

Therefore the set (1.7) is a complete set of functionally independent invariants of the form $F = F(\underset{1}{u}, \underset{2}{u})$ and (1.2) represents a functional basis of the second-order invariants for the algebra $AE(n)$. The theorem is proved. ■

Let us consider the case of two vectors (u_a), (v_a) and two symmetric tensors of the second order (u_{ab}), (v_{ab}). The operators of the rotation algebra have the form (1.5), $u \equiv u^1$, $v \equiv u^2$.

In this case, a functional basis of invariants contains

$$2\left(\frac{n(n-1)}{2} + 2n\right) - \frac{n(n-1)}{2} = \frac{n(n+7)}{2}$$

elements for which we take the following expressions

$$R_k(u_a, u_{ab}), \quad R_k(v_a, u_{ab}), \quad S_{jk}(u_{ab}, v_{ab}). \tag{1.9}$$

The invariance of expressions (1.9) with respect to the operators (1.5) can be easily proved by their direct substitution to (1.4). To establish their functional independence, we shall use the following lemma.

Lemma 3. *Let*

$$U = (u_{ab})_{a,b=1,\ldots,n}, \quad V = (v_{ab})_{a,b=1,\ldots,n}$$

be symmetric matrices. Then the expressions

$$S_{jk}(u_{ab}, v_{ab}) = \operatorname{tr} U^j V^{k-j}, \quad j = 0, \ldots, k; \quad k = 1, \ldots, n, \tag{1.10}$$

are functionally independent.

Proof. To prove Lemma 3, it is sufficient to show that the generic rank of the Jacobi matrix of expressions (1.10) is equal to $(n(n+3))/2$ that is the difference between the number of independent elements of U and V and the rank of the operators (1.5). We shall limit ourselves to the case when $u_{ab} = 0$ if $a \neq b$. Then equations (1.10) depend on $(n(n+3))/2$ variables and their independence is equivalent to the nonvanishing of the Jacobian.

Let us write down the elements of the Jacobian which are needed for further reasoning

$$\left|\begin{array}{ccc|ccccccc} 1 & \cdots & 1 & & & & & & & \\ 2u_{11} & \cdots & 2u_{n} & & & & \mathbf{0} & & & \\ \cdots & \cdots & \cdots & & & & & & & \\ nu_{11}^{n-1} & \cdots & nu_{nn}^{n-1} & & & & & & & \\ \hline & & & 1 & 0 & \cdots & 0 & 1 & \cdots & 1 \\ & \cdots & & 2v_{11} & 4v_{12} & \cdots & 4v_{1n} & 2v_{22} & \cdots & 2v_{nn} \\ & & & & & \cdots & & & & \end{array}\right|. \tag{1.11}$$

Since, in the first n rows, all the elements besides the first n columns are equal to j zero, the Jacobian (1.11) is equal to the product of the Jacobian of the elements $\operatorname{tr} U^k$, $k = 1, \dots, n$, and the Jacobian of all other elements. According to Lemma 2, the expressions $\operatorname{tr} U^k$, $k = 1, \dots, n$, are independent and their Jacobian is not equal to zero; thus, it remains to show the nonvanishing of the Jacobian and the functional independence only for the elements

$$\operatorname{tr} U^j V^{k-j}, \quad j = 0, \dots, k-1; \quad k = 1, \dots, n.$$

It follows from (1.11) that it is sufficient to show the nonvanishing of this Jacobian without the $(n+1)$th rows and columns. Thus, to prove the lemma, it is enough to show that the following expressions are independent

$$\operatorname{tr} U^j V^{k-j} V, \quad j = 0, \dots, k; \quad k = 1, \dots, n-1. \tag{1.12}$$

The above reasoning allows us to make use of the principle of mathematical induction.

When $n = 1$, u_{11} and v_{11} are independent and the lemma is true. Let us suppose that it is true for $n-1$ and then prove from this that it is valid for n. Let the expressions

$$\operatorname{tr} U^j V^{k-j}, \quad j = 0, \dots, k; \quad k = 1, \dots, n-1, \tag{1.13}$$

where U, V are symmetric $(n-1) \times (n-1)$ matrices and are independent. Then, we shall prove the independence of (1.12) for the same matrices. The sets (1.12) and (1.13) coincide with the exception of the following subsets

$$\operatorname{tr} U^j V^{n-j}, \quad j = 0, \dots, n-1 \tag{1.14}$$

belong only to (1.12) and

$$\operatorname{tr} U^j, \quad j = 1, \dots, n-1 \tag{1.15}$$

belong only to (1.13).

The assumption of validity of the lemma for $n-1$ means that for two symmetric tensors of order 2, the set (1.13) is a functional basis of invariants of the rotation algebra. Thus, all the invariants of this algebra can be represented as functions of (1.13). To prove the functional independence of (1.12), it is sufficient to prove the nondegeneracy of the Jacobi matrix of the functions expressing the invariants (1.12) with (1.13). This matrix has the form

$$\left(\begin{array}{cccc|c|c}
1 & & \mathbf{0} & & & \\
 & 1 & & & & \cdots \\
 & & \ddots & & & \\
 & \mathbf{0} & & 1 & & \\
\hline
 & & & & W & \\
\hline
 & & \mathbf{0} & & & \dfrac{\partial(\operatorname{tr} U^j V^{n-j})}{\partial(\operatorname{tr} U^j)}
\end{array}\right), \tag{1.16}$$

W being the derivative by $\operatorname{tr} V$ of the expression

$$\operatorname{tr} V^n = F(\operatorname{tr} V^k,\ k = 1, \dots, n-1).$$

(We know that from the Hamilton–Cayley theorem); $W \neq 0$.

We have only to prove the nonvanishing of the Jacobian of the expressions

$$\operatorname{tr}(U^j V^{n-j}) = F(\operatorname{tr} U^k,\ k = 1, \dots, n-1, \dots). \tag{1.17}$$

When $V = E$, the corresponding quadrant of the matrix (1.16) is the unit matrix and its determinant does not vanish identically. This fact proves the nondegeneracy of the matrix (1.16). The expressions (1.17) can be obtained from the Hamilton–Cayley theorem. They are polynomials and, thus, continuous functions of their arguments.

The functional independence of the expressions (1.12) for $(n-1)\times(n-1)$ matrices implies their independence for $n \times n$ matrices. From the above, it follows that the expressions (1.10) are independent, thus Lemma 3 is proved. ■

Proof of Theorem 2. It is easy to see from the structure of the set (1.3) that the invariants involving $(u_a^1), \dots, (u_a^m)$, $(u_{ab}^2), \dots, (u_{ab}^m)$ depend on the components of (u_{ab}^1) and of the corresponding vector or tensor, thus it is sufficient to prove the functional independence of each of the following sets:

$$R_k(u_a^r, u_{ab}^1) \qquad \text{for every } r = 1, \dots, m;$$
$$S_{jk}(u_{ab}^1, u_{ab}^r) \qquad \text{for every } r = 2, \dots, m;$$

The functional independence of each set of $R_k(u_a^r, u_{ab}^1)$ can be proved similarly to the proof of Lemma 2. The functional independence of the set $S_{jk}(u_{ab}^1, u_{ab}^r)$ easily follows from Lemma 3, u^r are evidently independent of other elements of (1.3).

To make sure that expressions (1.3) are invariants of $AO(n)$, it is sufficient to substitute them into the condition (1.4).

The set (1.3) consists of

$$2mn + m + (m-1)\frac{n(n-1)}{2} = m\left(\frac{n(n+1)}{2} + n + 1\right) - \frac{n(n-1)}{2}$$

elements and, thus, it is complete.

So we have proved that this set forms a basis of invariants for the algebra $AE(1.n)$ (1.1).

1.3. Bases of invariants for the extended Euclid algebra and for the conformal algebra. The extended Euclid algebra $AE_1(n)$ for one scalar function is defined by the basis operators ∂_a, J_{ab} (1.1) and D depending on a parameter λ:

$$D = x_a\partial_a + \lambda u \partial_u \quad (\partial_u = \partial/\partial u). \tag{1.18}$$

The basis of the conformal algebra $AC(n)$ consists of the operators ∂_a, J_{ab} (1.1) and D (1.18) and

$$K_a = 2x_a D - x_a x_b \partial_b. \tag{1.19}$$

Theorem 3. *There is a functional basis for the extended Euclid algebra that has the following form*

(1) *when* $\lambda \neq 0$*:*

$$\frac{R_k(u_a, u_{ab})}{u^{k(1-2/\lambda)+1}}, \quad \frac{S_k(u_{ab})}{u^{k(1-2/\lambda)}}; \tag{1.20}$$

(2) *when* $\lambda = 0$*:*

$$u, \quad \frac{R_k(u_a, u_{ab})}{(u_{aa})^k}, \quad \frac{S_k(u_{ab})}{(u_{aa})^k} \quad (k \neq 1); \tag{1.21}$$

a functional basis for the conformal algebra has the following form:
(1) *when* $\lambda \neq 0$*:*

$$S_k(\theta_{ab})u^{k(2/\lambda-1)}; \tag{1.22}$$

(2) *when* $\lambda = 0$*:*

$$u, \quad S_k(w_{ab})(u_a u_a)^{-2k} \quad (k \neq n), \tag{1.23}$$

where

$$\begin{aligned} \theta_{ab} &= \lambda u_{ab} + (1-\lambda)\frac{u_a u_b}{u} - \delta_{ab}\frac{u_c u_c}{2u}, \\ w_{ab} &= u_c u_c\left(u_{ab} + \frac{\delta_{ab}}{2-n}u_{dd}\right) - u_c(u_a u_{bc} + u_b u_{ac}), \end{aligned} \tag{1.24}$$

δ_{ab} *being the Kronecker symbol.*

Proof. To find absolute differential invariants of the algebra $AE_1(n)$, it is necessary to add to (1.4) the following condition

$$\overset{2}{D} F \equiv x_a F_{x_a} + \lambda u F_u + (\lambda - 1)u_a F_{u_a} + (\lambda - 2)u_{ab}F_{u_{ab}} = 0. \tag{1.25}$$

Solving equation (1.25) for

$$F = F(u, R_k(u_a, u_{ab}), S_k(u_{ab})),$$

we obtain functional bases (1.20), (1.21) for the extended Euclid algebra.

The second-order differential invariants of the algebra $AC(n)$ are defined by the conditions (1.4), (1.25) and

$$k_a \overset{2}{K}_a F = 0, \tag{1.26}$$

where k_a are arbitrary real numbers, $\overset{2}{K}_a$ are the second prolongations of the operators K_a (1.19):

$$\overset{2}{K}_a = 2x_a \overset{2}{D} + x_b \overset{2}{J}_{ab} + 2\lambda[u\partial_{u_a} + 2u_b\partial_{u_{ab}}] + 2u_a\partial_{u_{cc}} - 4u_b\partial_{u_{ab}}.$$

Solving this system for an arbitrary n requires a lot of cumbersome computations. It is simpler to construct conformally covariant tensors from u, u_a, u_{ab} and then to construct invariants of the rotation algebra.

Definition 3. *Tensors* θ_a *and* θ_{ab} *of order* 1 *and* 2 *are called covariant with respect to some algebra* $L = \langle J_{ab}, X_i \rangle$ *if*

$$\begin{aligned} X_i\theta_a &= \sigma^i_{ab}\theta_b + \sigma^i\theta_a, \\ X_i\theta_{ab} &= \rho^i_{ac}\theta_{cb} + \rho^i_{bc}\theta_{ac} + \rho^i\theta_{ab}, \end{aligned} \tag{1.27}$$

X_i *are operators of the form* (0.1), ρ^i, σ^i *are some functions,* σ^i_{ab}, ρ^i_{ab} *are some skew-symmetric tensors.*

It is easy to show that the expressions $S_k(\theta_{ab})$, $R_k(\theta_a, \theta_{ab})$, where θ_a, θ_{ab} are tensors covariant with respect to the algebra L are relative invariants of this algebra.

The fact that θ_{ab} and w_{ab} (1.24) are covariant with respect to the conformal algebra $AC(n)$ can be verified by direct substitution of these tensors into the conditions (1.27) for the operators $\overset{2}{D}$ and $\overset{2}{K}_a$.

The rank of the second prolongation of the algebra $AC(n)$ is equal to the number of its operators

$$\frac{n(n-1)}{2} + n + n + 1 = \frac{n(n+3)}{2} + 1$$

and, therefore, a functional basis of second-order differential invariants must contain n invariants.

The functional independence of the expressions (1.22) follows from Lemma 2 if we notice that the transformation $u_{ab} \to \theta_{ab}$ is nondegenerated. The same is true for the set (1.23).

The expressions (1.22) and (1.23) satisfy (1.25) and (1.26) for the corresponding λ and they are invariants of the conformal algebra.

All that is stated above leads to the conclusion that (1.22) and (1.23) form functional bases for the conformal algebra $AC(n)$ with $\lambda \neq 0$ and $\lambda = 0$, respectively.

Note 1. Using condition (1.26), it is easy to show that when $\lambda \neq 0$ covariant tensors exist for $AC(n)$ of order 2 only; when $\lambda = 0$, the tensors w_{ab} (l.24) and u_a are conformally covariant but $S_k(w_{ab})$ and $R_k(u_a, w_{ab})$ are dependent.

Theorem 4. *The second-order differential invariants for a vector function $u = (u^1, \ldots, u^m)$ and for the algebra $AE_1(n) = \langle \partial_a, J_{ab}, D \rangle$, the operator D having the form*

$$D = x_a \partial_a + \lambda u^r \partial_{u^r} \tag{1.28}$$

with a summation over r from 1 to m, can be represented as the functions of the following expressions:

(1) *when $\lambda \neq 0$:*

$$\frac{u^r}{u^1}\ (r = 2, \ldots, m), \quad \frac{S_{jk}(u^1_{ab}, u^r_{ab})}{(u^1)^{k(1-2/\lambda)}}, \quad \frac{R_k(u^r_a, u^1_{ab})}{(u^1)^{k(1-2/\lambda)+1}};$$

(2) *when $\lambda = 0$:*

$$u^r, \quad R_k(u^r_a, u^1_{ab})(u^1_{aa})^{-k}, \quad S_{jk}(u^1_{ab}, u^r_{ab})(u^1_{aa})^{-k}$$

(when $r = 1$ then $k \neq 1$);

the corresponding basis for the conformal algebra $AC(n) = \langle \partial_a, J_{ab}, D, K_a \rangle$ ($K_a = 2x_a D - x_b x_b \partial_a$) has the following form:

(1) *when $\lambda \neq 0$:*

$$\begin{aligned} &S_{jk}(\theta^r_{ab}, \theta^1_{ab})(u^1)^{k(2/\lambda-1)}, \quad \frac{u^r}{u^1}, \\ &R_k(\theta^r_a, \theta^1_{ab})^{k(2/\lambda-1)-1} \quad (r = 2, \ldots, m); \end{aligned} \tag{1.29a}$$

(2) *when* $\lambda = 0$*:*

$$\begin{aligned} &u^r \ (r = 1, \ldots, m), \quad (u^1_d u^1_d)^{-2k} S_{jk}(w^1_{ab}, w^r_{ab}), \\ &(u^1_d u^1_d)^{1-2k} R_k(u^r_a, w^1_{ab}) \quad (r = 2, \ldots, m) \end{aligned} \tag{1.29b}$$

(for the set of invariants $(u^1_d u^1_d)^{-2k} S_k(w_{ab})$*,* k *does not take the value* n*); the tensors* θ^r_{ab}*,* w^r_{ab} *are constructed similarly to* (1.24) *and*

$$\theta^r_a = \frac{u^r_a}{u^r} - \frac{u^1_a}{u^1}.$$

Theorem 4 is proved similarly to Theorem 3.

The functional independence of the sets of invariants follows from Lemmas 2 and 3 taking into account the fact that transformations $u^r_{ab} \to \theta^r_{ab}$, $u^r_{ab} \to w^r_{ab}$ $(r = 1, \ldots, m)$ and $u^r_a \to \theta^r_a$ $(r = 2, \ldots, m)$ are nondegenerate.

1.4. Differential invariants of the rotation algebra. The rotation algebra is defined by the basis operators J_{ab} (1.1).

The second-order invariants of this algebra for m scalar functions u^r are constructed with x_a, u^r, u^r_a, w^r_{ab} similarly to invariants of the Euclid algebra.

Theorem 5. *There is a functional basis of the second-order differential invariants for the algebra* $AO(n)$ *that has the form*

$$u^r, \quad S_{jk}(u^1_{ab}, u^r_{ab}), \quad R_k(u^r_a, u^1_{ab}), \quad R_k(x_a, u^1_{ab}), \quad r = 1, \ldots, m;$$

the corresponding basis of invariants for the algebra $\langle J_{ab}, D\rangle$*, where* D *is defined by* (1.28)*, consists of the expressions*

$$\begin{aligned} &\frac{u^r}{u^1} \ (r = 2, \ldots, m), \quad \frac{S_{jk}(u^1_{ab}, u^r_{ab})}{(u^1)^{k(1-2/\lambda)}}, \quad R_k(u^r_a, u^1_{ab})(u^1)^{2k/\lambda-1-k}, \\ &R_k(x_a, u^1_{ab})(u^1)^{2/\lambda(k-2)-k+1}, \quad \text{when } \lambda \neq 0; \\ &u^r, \quad R_k(u^r_a, u^1_{ab})(u^1_{aa})^{-k}, \quad S_{jk}(u^1_{ab}, u^r_{ab})(u^1_{aa})^{-k} \ (k \neq 1 \text{ when } r = 1), \\ &R_k(x_a, u^1_{ab})(u^1_{aa})^{2-k} \quad \text{when } \lambda = 0. \end{aligned}$$

A basis of invariants for the algebra $\langle J_{ab}, D, K_a\rangle$ *when* $\lambda \neq 0$*, consists of the expressions* (1.29a) *and*

$$\frac{R_k(x_a, \theta^1_{ab})}{x^2(u^1)^{(k-1)(1-2/\lambda)}}, \quad k = 2, \ldots, n+1;$$

when $\lambda = 0$ *it consists of the expressions* (1.29b) *and*

$$\frac{R_k(x_a, w^1_{ab})}{x^2(w^1_{aa})^{k-1}} \quad (x^2 = x_a x_a).$$

The proof of this theorem is similar to the proofs of Theorems 2 and 3; notice that (x_a) is a co variant tensor with respect to the conformal operators.

2. Differential invariants of the Poincaré and conformal algebra

In this section, we consider differential invariants of the second order for a set of m scalar functions

$$u^r = u^r(x_0, x_1, \dots, x_n), \quad n \geq 3. \tag{2.1}$$

The Poincaré algebra $AP(1,n)$ is defined by the basis operators

$$p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\mu}, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{2.2}$$

where μ, ν take the values $0, 1, \dots, n$; the summation is implied over the repeated indices (if they are small Greek letters) in the following way:

$$x_\nu x^\nu \equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - \dots - x_n^2, \quad g_{\mu\nu} = \operatorname{diag}(1, -1, \dots, -1). \tag{2.3}$$

We consider x_ν and x^ν equal with respect to summation not to mix signs of derivatives and numbers of functions.

The quasilinear second-order invariants of the Poincaré algebra were described in [12].

Theorem 6. *There is a functional basis of the second-order differential invariants of the Poincaré algebra $AP(l,n)$ for a set of m scalar functions u^r consisting of*

$$m(2n+3) + (m-1)\frac{n(n+1)}{2}$$

invariants

$$u^r, \quad R_k(u^r_\mu, u^1_{\mu\nu}), \quad S_{jk}(u^r_{\mu\nu}, u^1_{\mu\nu}).$$

In this section, everywhere $k = 1, \dots, n+1$; $j = 0, \dots, k$; $r = 1, \dots, m$.

For the extended Poincaré algebra $A\tilde{P}(l,n) = \langle p_\mu, J_{\mu\nu}, D\rangle$, where

$$D = x_\mu p_\mu + \lambda u^r p_{u^r} \tag{2.4}$$

($p_{u^r} = i(\partial/\partial u^r)$, the summation over r from 1 to m is implied) the corresponding basis has the following form:

(1) when $\lambda = 0$:

$$u^r, \quad S_{jk}(u^r_{\mu\nu}, u^1_{\mu\nu})(u^1_{\alpha\alpha})^{-k}, \quad R_k(u^r_\mu, u^1_{\mu\nu})(u^1_{\alpha\alpha})^{-k};$$

(2) when $\lambda \neq 0$:

$$\frac{u^r}{u^1}, \quad S_{jk}(u^r_{\mu\nu}, u^1_{\mu\nu})(u^1)^{k(2/\lambda-1)}, \quad R_k(u^r_\mu, u^1_{\mu\nu})(u^1)^{2k/\lambda-k-1},$$

where S_{jk}, R_k are defined similarly to (0.3) and the summation over small Greek indices is of the type (2.2).

For the conformal algebra $AC(1,n) = \langle p_\mu, J_{\mu\nu}, D, K_\mu\rangle$, where

$$K_\mu = 2x_\mu D - x_\nu x_\nu p_\mu$$

(D being the dilation operator (2.3)), the corresponding basis consists of the expressions

$$S_{jk}(\theta^r_{\mu\nu},\theta^1_{\mu\nu})(u^1)^{k(2/\lambda-1)},\quad \frac{u^r}{u^1},\quad R_k(\theta^r_\mu,\theta^1_{\mu\nu})(u^1)^{k(2/\lambda-1)-1};$$

when $\lambda\neq 0$; $r=2,\ldots,m$, there is no summation over r; the conformally covariant tensors have the form

$$\theta^r_\mu=\frac{u^r_\mu}{u^r}-\frac{u^1_\mu}{u^1},\quad \theta^r_{\mu\nu}=\lambda u^r_{\mu\nu}+(1-\lambda)\frac{u^r_\mu u^r_\nu}{u^r}-g_{\mu\nu}\frac{u^r_\beta u^r_\beta}{2u^r}.$$

When $\lambda=0$, the corresponding basis of invariants for the conformal algebra has the form

$$u^r,\quad S_{jk}(w_{\mu\nu},w^1_{\mu\nu})(u^1_\alpha u^1_\alpha)^{-2k},\quad R_k(u^r_\mu,w^1_{\mu\nu})(u^1_\alpha u^1_\alpha)^{1-2k},\quad r=2,\ldots,m;$$

the tensors $(w^r_{\mu\nu})$,

$$w^r_{\mu\nu}=u^r_\alpha u^r_\alpha\left(u^r_{\mu\nu}-\frac{g_{\mu\nu}}{1-n}u^r_{\beta\beta}\right)-u^r_\beta(u^r_\mu u^r_{\beta\nu}+u^r_\nu u^r_{\beta\mu})$$

are conformally invariant (there is no summation over r).

The proof of Theorem 6 follows from those of Theorems 2, 3 for $x=(x_1,\ldots,x_{n+1})$ if we substitute ix_0 instead of x_{n+1}.

Similarly to the results of Paragraph 1.4, it is possible to construct the invariants of the algebras $\langle J_{\mu\nu}\rangle$, $\langle J_{\mu\nu},D\rangle$, $\langle J_{\mu\nu},D,K_\mu\rangle$.

The obtained results allow us to construct new nonlinear many-dimensional equations, e.g. the equation

$$\frac{u_\alpha u_\alpha}{1-n}u_{\nu\nu}-u_\mu u_\nu u_{\mu\nu}=(u_\nu u_\nu)^2F(u),$$

where F is an arbitrary function, is invariant under the algebra $AC(1,n)$, $\lambda=0$. The left member of the above equation is equal to $w_{\mu\mu}$.

There is another quasi-linear relativistic equation with rich symmetry properties

$$(1-u_\alpha u_\alpha)u_{\mu\mu}-u_\alpha u_\mu u_{\alpha\mu}=0,$$

that is, the Born–Infeld equation. The symmetry and solutions of this equation were investigated in [10, 13]. This equation is invariant under the algebra $AP(1,n+1)$ with the basis operators

$$J_{AB}=x_Ap_B-x_Bp_A,$$

$A,B=1,\ldots,n+1$, $x_{n+1}\equiv u$.

Let us consider the class of equations

$$u_{\mu\nu}u_{\mu\nu}=F(u_{\mu\mu},u_\mu u_\nu u_{\mu\nu},u_\mu u_\mu,u).$$

It is evident that they are invariant with respect to the Poincaré algebra $AP(1,n)$ out the straightforward search the conformally invariant equations from this class with the standard Lie technique requires a lot of cumbersome calculations. The use of differential invariants turns this problem into one of elementary algebra, e.g. if $\lambda \neq 0$

$$F - u_{\mu\nu}u_{\mu\nu} = -\frac{1}{\lambda}S_2(\theta_{\mu\nu}) + u^{2(1-2/\lambda)}\phi(S_1(\theta_{\mu\nu})u^{2/\lambda-1}),$$

where $\theta_{\mu\nu}$ is of the form (1.24) and ϕ is an arbitrary function. Whence

$$F = u^{2(1-2/\lambda)}\phi\left(u^{2/\lambda-1}\left(u_{\mu\mu} - \frac{\lambda+n}{\lambda}\frac{u_\alpha u_\alpha}{u}\right)\right) - \\ - \frac{1}{\lambda^2 u^2}(\lambda^2+n^2)(u_\alpha u_\alpha)^2 - \frac{2(1-\lambda)}{\lambda u}u_\mu u_\nu u_{\mu\nu} + \frac{2u_{\mu\mu}u_\alpha u_\alpha}{\lambda u}.$$

It is useful to note that besides the traces of matrix powers (0.3), one can utilize all possible invariants of covariant tensors $\theta^r_{\mu\nu}$, $w^r_{\mu\nu}$ to construct conformally invariant equations.

3. Differential invariants of an infinite-dimensional algebra

It is well-known that the simplest first-order relativistic equation — the eikonal or Hamilton equation

$$u_\alpha u_\alpha \equiv u_0^2 - u_1^2 - \cdots - u_n^2 = 0 \tag{3.1}$$

is invariant under the infinite-dimensional algebra $AP^\infty(1,n)$ generated by the operators [10, 14]

$$X = (b^{\mu\nu}x_\nu + a^\mu)\partial_\mu + \eta(u)\partial_u, \tag{3.2}$$

$-b^{\mu\nu} = b^{\nu\mu}$, a^μ, η being arbitrary differentiate functions on u. Equation (3.1) is widely used in geometrical optics.

In this section, we describe a class of second-order equations invariant under the algebra (3.2).

It is easy to show that the tensor of the rank 2

$$\theta_{\mu\nu} = u_\mu u_{\lambda\nu}u_\lambda + u_\nu u_{\lambda\mu}u_\lambda - u_\mu u_\nu u_{\lambda\lambda} - u_\lambda u_\lambda u_{\mu\nu} \tag{3.3}$$

is covariant under the algebra $AP^\infty(1,n)$ (3.2).

Theorem 7. *The equations of the form*

$$S_k(\theta_{\mu\nu}) = 0, \quad k = 1, 2, \ldots, \tag{3.4}$$

S_k *being defined as* (0.3)*, are invariant with respect to the algebra* $AP^\infty(1,n)$ (3.2)*.*

The problem of the description of all such equations is more difficult and we do not consider it here.

Let us investigate in more detail the quasi-linear second-order equation of the form

$$u_\mu u_{\mu\nu}u_\nu - u_\mu u_\mu u_{\alpha\alpha} = 0. \tag{3.5}$$

Theorem 8. *When $n \geq 2$, equation* (3.5) *is invariant with respect to the algebra $A\tilde{P}^{\infty}(1,n)$ with generators of the form*

$$X + d(u)x_\mu\partial_\mu,$$

X is of the form (3.2)*, $d(u)$ is an arbitrary function on u.*

The proofs of Theorems 7 and 8 can be easily obtained with the Lie technique using the criterion of invariance

$$\overset{2}{X} S_k(\theta_{\mu\nu})\Big|_{S_k(\theta_{\mu\nu})=0} = 0,$$

where $\overset{2}{X}$ is the second prolongation of the operator X [8–10].

4. Differential invariants of the Galilei algebra

4.1. It is well-known that the heat equation

$$\begin{aligned} &2\mu u_t + \Delta u = 0, \quad \Delta u \equiv u_{aa}, \\ &u = u(t,\boldsymbol{x}), \quad \boldsymbol{x} = (x_1,\ldots,x_n), \quad n \geq 3 \end{aligned} \tag{4.1}$$

is invariant under the generalized Galilei algebra $AG_2^I(1,n)$ with the basis operators

$$\begin{aligned} &\partial_t = \frac{\partial}{\partial t}, \quad \partial_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \\ &G_a = t\partial_a + \mu x_a u\partial_u \quad \left(\partial_u = \frac{\partial}{\partial u}\right), \quad u\partial_u, \quad D = 2t\partial_t + x_a\partial_a + \lambda u\partial_u, \\ &A = tD - t^2\partial_t + \frac{\mu\boldsymbol{x}^2}{2}u\partial_u \quad \left(\lambda = -\frac{n}{2}\right). \end{aligned} \tag{4.2}$$

The Schrödinger equation

$$2im\psi_t + \psi_{aa} = 0, \tag{4.3}$$

$\psi = \psi(t,\boldsymbol{x})$ being a complex-valued function, is also invariant [16] under the generalized Galilei algebra with the basis operators

$$\begin{aligned} &p_0 = i\frac{\partial}{\partial t}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad J_{ab} = x_a p_b - x_b p_a, \quad J = i(\psi\partial_\psi - \psi^*\partial_{\psi^*}), \\ &G_a = tp_a - mx_a J, \quad D = 2tp_0 - x_a p_a + \lambda I \quad (I = \psi\partial_\psi + \psi^*\partial_{\psi^*}), \\ &A = t^2 p_0 - tx_a p_a + \lambda t I + \frac{m\boldsymbol{x}^2}{2}J \quad \left(\lambda = -\frac{n}{2}\right). \end{aligned} \tag{4.4}$$

The asterisk means the complex conjugation.

We shall designate the algebra (4.4) with the symbol $AG_2^{II}(1,n)$. Besides,

$$AG^I(1,n) = \langle \partial_t, \partial_a, u\partial_u, G_a, J_{ab} \rangle,$$

the operators being of the form (4.2). A basis of the algebra $AG_1^I(1,n)$ consists of the basis operators or $AG^I(1,n)$ and of the operator D. Furthermore $AG^{II}(1,n) =$

$\langle p_0, p_a, J, J_{ab}, G_a\rangle$ (4.4). A basis of the algebra $AG_1^{II}(1,n)$ consists of the previous operators and also D (4.4).

To simplify the form of invariants, we introduce the following change of dependent variables:

$$u = \exp\varphi, \quad \psi = \exp\phi \quad \left(\operatorname{Im}\phi = \operatorname{arctg}\frac{\operatorname{Im}\psi}{\operatorname{Re}\psi}\right). \tag{4.5}$$

All the indices k in the expressions of the type (0.3) here will take on values from 1 to n, the indices j will take on values from 0 to k.

We seek invariants of the algebra $AG_2^I(1,n)$ in the form

$$F = F(\varphi_t, \varphi_a, \varphi_{tt}, \varphi_{at}, \varphi_{ab}). \tag{4.6}$$

Obviously, they do not include φ, x_a, and t because the basis (4.2) contains operators ∂_φ, ∂_a, ∂_t.

Using the definition of an absolute differential invariant (0.2) we get the following conditions on the function F (4.6):

$$\overset{2}{J}_{ab}F = \varphi_a F_{\varphi_b} - \varphi_b F_{\varphi_a} + F_{\varphi_{bt}}\varphi_{at} - \varphi_{bt}F_{\varphi_{at}} + 2\varphi_{ac}F_{\varphi_{bc}} - 2\varphi_{bc}F_{\varphi_{ac}} = 0, \tag{4.7}$$

$$\overset{2}{G}_a F = -\varphi_a F_{\varphi_t} + \mu F_{\varphi_a} - 2\varphi_{at}F_{\varphi_{tt}} - \varphi_{ab}F_{\varphi_{bt}} = 0, \tag{4.8}$$

$$\overset{2}{D} F = -2\varphi_t F_{\varphi_t} - \varphi_a F_{\varphi_a} - 4\varphi_{tt}F_{\varphi_{tt}} - 3\varphi_{at}F_{\varphi_{at}} - 2\varphi_{ab}F_{\varphi_{ab}} = 0, \tag{4.9}$$

$$\overset{2}{A} F = t\overset{2}{D} F + x_a\overset{2}{G}_a F - \lambda F_{\varphi_t} - 2\varphi_t F_{\varphi_{tt}} - \varphi_a F_{\varphi_{at}} + \mu\delta_{ab}F_{\varphi_{ab}} = 0. \tag{4.10}$$

From equations (4.8), we can see that the tensors

$$\theta_a = \mu\varphi_{at} + \varphi_b\varphi_{ab}, \quad \varphi_{ab} \tag{4.11}$$

are covariant with respect to the algebra $AG^I(1,n)$ $(\mu\neq 0)$.

Theorem 9. *There is a functional basis of absolute differential invariants for the algebra $AG^I(1,n)$, when $\mu\neq 0$, consisting of these $2n+2$ invariants:*

$$\begin{gathered} M_1 = 2\mu\varphi_t + \varphi_a\varphi_a, \quad M_2 = \mu^2\varphi_{tt} + 2\mu\varphi_a\varphi_{at} + \varphi_a\varphi_b\varphi_{ab}, \\ R_k = R_k(\theta_a, \theta_{ab}), \quad S_k = S_k(\varphi_{ab}). \end{gathered} \tag{4.12}$$

For the algebra $AG_1^I(1,n)$ $(\mu\neq 0)$ such a basis has the form

$$\frac{M_2}{M_1^2}, \quad \frac{R_k}{M_1^{2+k}}, \quad \frac{S_k}{M_1^k}. \tag{4.13}$$

For the algebra $AG_2^I(1,n)$ $(\mu\neq 0)$, there is a basis of the form

$$\frac{N_2}{N_1^2}, \quad \frac{\hat{R}_k}{N_1^{2+k}}, \quad \frac{\hat{S}_k}{N_1^k} \quad (k = 2,\ldots,n), \tag{4.14}$$

where

$$N_1 = 2\mu\varphi_t + \varphi_a\varphi_a + \varphi_{aa},$$
$$N_2 = \mu^2\varphi_{tt} + 2\mu\left(\frac{1}{n}\varphi_t\varphi_{aa} + \varphi_a\varphi_{at}\right) + \varphi_a\varphi_b\varphi_{ab} + \frac{1}{n}\varphi_a\varphi_a\varphi_{bb} + \frac{1}{n}\varphi_{bb}^2,$$
$$\hat{R}_k = \sum_{l=0}^{k} R_l(\varphi_{aa})^{k-1}\frac{(-n)^l k!}{l!(k-l)!}, \tag{4.15}$$
$$\hat{S}_k = \sum_{l=0}^{k} \frac{(-n)^l(k-1)!(k+1)}{(l+1)!(k-l)!} S_l(\varphi_{aa})^{k-l},$$

S_k, R_k *are defined by* (4.12) *and* θ_a *has the form* (4.11).

The proof of this theorem is similar to the proof of Theorems 2 and 3. We shall present here only some hints to the proof.

It is evident that the function F must depend on the invariants of the Euclid algebra

$$F = F(\varphi_t, \varphi_{tt}, R_k(\varphi_a, \varphi_{ab}), R_k(\varphi_{at}, \varphi_{ab}), S_{\varphi_{ab}})).$$

First we construct two invariants of $AG^I(1,n)$ M_1 and M_2 (4.12) which depend on φ_t and φ_{tt} respectively. The other invariants of the adduced basis (4.12) do not depend on φ_t or φ_{tt} and the sets $\{M_1, M_2\}$ and $\{R_k, S_k\}$ are independent. The invariants R_k, S_k are constructed with the covariant tensors θ_a, φ_{ab} (4.11) similarly to invariants of the conformal algebra investigated above, and it is easy to see that they are independent.

The generic ranks of the prolonged algebras $AG^I(1,n)$, $AG_1^I(1,n)$, $AG_2^I(1,n)$ are equal to the numbers of their operators and from this fact we can compute the number of elements in the bases for these algebras.

Adding to (4.7) and (4.8) the condition (4.9), we obtain from the invariants (4.12) the basis (4.13) for the algebra $AG_1^I(1,n)$.

Relative invariants $\hat{R}_k$, $\hat{S}_k$ (4.15) of the algebra $AG_2^I(1,n)$ were found from the equation

$$\lambda F_{\varphi_t} - 2\varphi_t F_{\varphi_{tt}} - \varphi_a F_{\varphi_{at}} + \mu\delta_{ab}F_{\varphi_{ab}} = 0,$$

$F = F(R_k, S_k)$, and then we constructed absolute invariants using (4.9). Besides, it is possible to construct analogues to $\hat{R}_k$, $\hat{S}_k$ with $AG_2^I(1,n)$-covariant tensors θ_a (4.11) and

$$\theta_{ab} = \varphi_{ab} - \frac{2\delta_{ab}}{n}(\varphi_c\varphi_c + \mu\varphi_t).$$

Considering (φ_{at}), (φ_a), (φ_{ab}) as independent vectors and tensors and putting $\varphi_{ab} = 0$ whenever $a \neq b$, $\varphi_a = 0$, we see from Lemma 2 that the adduced sets of invariants are independent.

Note 2. A basis of invariants for the Galilei algebra without translations contains expressions (4.12) and

$$R_k(h_a, \phi_{ab}), \quad \frac{1}{2}\mu\boldsymbol{x}^2 - \varphi t,$$

the Galilei-covariant vector h_a having the form

$$h_a = \mu x_a - t\varphi_a.$$

Let us also adduce an A-covariant tensor

$$\hat{h}_a = \frac{\mu x_a}{t} - \varphi_a$$

depending on x_a, and a relative invariant of the operators A and D (4.2)

$$\exp\left\{\varphi - \frac{\mu \boldsymbol{x}^2}{2t}\right\}$$

with which it is possible to construct a basis of invariants for the algebra $\langle G_a, J_{ab}, D, A\rangle$.

We have presented a method to find the bases of invariants for Lie algebras for which J_{ab} (1.1) are basis operators. Further, we shall adduce functional bases for the algebras $AG_2^I(1,n)$ where $\mu = 0$ and $AG_2^{II}(1,n)$ where $\mu = 0$ or $\mu \neq 0$. We omit proofs because they are similar to proofs of the previous theorems.

It is evident from the conditions (4.7)–(4.10) that the case $\mu = 0$ for the algebra $AG_2^I(1,n)$ has to be specially considered. The tensors (φ_a) and (φ_{ab}) are covariant with respect to this algebra; the tensor (θ_a) involved in invariants is defined by an implicit correlation

$$\varphi_{bt} = \theta_a \varphi_{ab}. \tag{4.16}$$

Theorem 10. *There is a functional basis of the second-order differential invariants for the algebra $AG^I(1,n)$, where $\mu = 0$, that has the form*

$$M_1 = \varphi_t - \varphi_a\theta_a, \quad M_2 = \varphi_{tt} - \varphi_{at}\theta_a, \quad R_k = R_k(\varphi_a, \varphi_{ab}), \quad S_k = S_k(\varphi_{ab}). \tag{4.17}$$

The corresponding basis for the algebra $AG_1^I(1,n)$, where $\mu = 0$ has the form

$$\frac{M_1^2}{M_2}, \quad \frac{R_k}{M_1^k}, \quad \frac{S_k}{M_1^k};$$

for the algebra $AG_2^I(1,n)$, when $\mu = 0$, it has the form

$$\frac{R_k}{M^{1/2k}}, \quad \frac{S_k}{M^{1/2k}},$$

where R_k, S_k are defined by (4.17) *and*

$$M = (\varphi_t - \theta_a\varphi_a)^2 + (\varphi_{tt} - \varphi_{at}\theta_a)(\lambda + \varphi_a\varphi_b r_{ab}).$$

Here, the matrix $\{r_{ab}\} = \{\varphi_{ab}\}^{-1}$; $\theta_a = r_{ab}\varphi_{bt}$ are the same as in (4.16).

Note 3. It is possible to use, instead of M_1, M_2, the invariants

$$\hat{M}_1 = \begin{vmatrix} \varphi_t & \varphi_1 & \cdots & \varphi_n \\ \varphi_{1t} & \varphi_{11} & \cdots & \varphi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{nt} & \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix}, \quad \hat{M}_2 = \begin{vmatrix} \varphi_{tt} & \varphi_{1t} & \cdots & \varphi_{nt} \\ \varphi_{1t} & \varphi_{11} & \cdots & \varphi_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_{nt} & \varphi_{n1} & \cdots & \varphi_{nn} \end{vmatrix},$$

which have been found in [17] as the solution of the problem of finding the equations invariant under the Galilei algebra when $\mu = 0$.

Note 4. The invariants for the algebra $\langle J_{ab}, G_a, J, D, A\rangle$ (4.2), where $\mu = 0$, which depend on x_a, t, can be constructed with φ_a, φ_{ab} and the following covariant vector

$$\hat{h}_a = \frac{h_a}{t} + \frac{2}{n}t\varphi_a\varphi_t + \frac{4}{n}\frac{x_b\varphi_b\varphi_a}{t},$$

where $h_a = x_b\varphi_{ab} + t\varphi_{at}$ is covariant with respect to the operators G_a when $\mu = 0$.

4.2. Let us proceed to describe the basis of the invariants for the algebra $AG_2^{II}(1,n)$.

Theorem 11. *Any absolute differential invariant of order ≤ 2 for the algebras listed below is a function of the following expressions:*

(1) $AG^{II}(1,n)$, $m \neq 0$:

$$\begin{aligned}
&\phi + \phi^*, \quad M_1 = 2im\phi_t + \phi_a\phi_a, \quad M_1^*,\\
&M_2 = -m^2\phi_{tt} + 2im\phi_a\phi_{at} + \phi_a\phi_b\phi_{ab}, \quad M_2^*,\\
&S_{jk} = S_{jk}(\phi_{ab}, \phi^*_{ab}), \quad R_k^1 = R_k(\theta_a, \phi_{ab}),\\
&R_k^2 = R_k(\theta_a^*, \phi_{ab}), \quad R_k^3 = R_k(\phi_a + \phi_a^*, \phi_{ab}),
\end{aligned}$$

the covariant tensors being $\theta_a = -im\phi_{at} + \phi_b\phi_{ab}$*;*

(2) $AG_1^{II}(1,n)$, $m \neq 0$:

$$\begin{aligned}
&\frac{M_1^*}{M_1}, \quad \frac{M_2}{M_1^2}, \quad \frac{M_2^*}{M_1^2}, \quad \frac{R_k^l}{M_1^{2+k}}\ (l = 1,2), \quad \frac{R_k^3}{M_1^k}, \quad \frac{S_{jk}}{M_1^k},\\
&\phi + \phi^* \ \ \text{when } \lambda = 0, \quad M_1 e^{(2/\lambda)(\phi+\phi^*)} \ \ \text{when } \lambda \neq 0;
\end{aligned}$$

(3) $AG_2^{II}(1,n)$, $m \neq 0$, $\lambda = -\frac{n}{2}$:

$$N_1 e^{(-4/n)(\phi+\phi^*)}, \quad \frac{N_1}{N_1^*}, \quad \frac{N_2}{N_1^2}, \quad \frac{N_2^*}{N_1^2}, \quad \frac{\hat{R}_k^l}{N_1^{2+k}}\ (l = 1,2), \quad \frac{\hat{R}_k^3}{N_1^k}, \quad \frac{\hat{S}_{jk}}{N_1^k},$$

where

$$\begin{aligned}
&N_1 = 2im\phi_t + \phi_{aa} + \phi_a\phi_a,\\
&N_2 = -m^2\phi_{tt} + 2im\left(\phi_a\phi_{at} + \frac{1}{n}\phi_t\phi_{aa}\right) + \phi_a\phi_b\phi_{ab} + \frac{1}{n}\phi_a\phi_a\phi_{bb} + \frac{1}{n}\phi_{aa}^2,\\
&\hat{S}_{jk} = \sum_{l=0}^{k}\sum_{r=0}^{j} S_{rl}(-n)^l C_j^r C_k^{l+1-r}(\phi_{aa})^{j-r}(\phi^*_{aa})^{k-l-j+r} + k(\phi_{aa})^j(\phi^*_{aa})^{k-j-1},\\
&\hat{R}_k^l = \sum_{j=0}^{k} R_j^l(\phi_{aa})^{k-j}\frac{(-n)^j k!}{j!(k-j)!} \quad (l = 1,2,3).
\end{aligned}$$

The invariants for the algebras $AG^{II}(1,n)$, $AG_1^{II}(1,n)$ $(m = 0)$ can be constructed similarly to the case of real function. Let us adduce a functional basis for the algebra $AG_2^{II}(1,n)$.

(1) *when $\lambda = 0$, then there is a basis consisting of the following expressions:*

$$\phi + \phi^*, \quad \frac{N_1^2}{N_2^2}, \quad \frac{N_1^{*2}}{N_2}, \quad \frac{(S_{jk})^2}{N_1^k}, \quad (R_k^l)^2 N_1^{-k-1}\ (l = 1,2,4);$$

(2) $\lambda \neq 0$:

$$N_1 e^{(4/\lambda)(\phi+\phi^*)}, \quad \frac{N_1^*}{N_1}, \quad N_3 e^{(3/\lambda)(\phi+\phi^*)}, \quad \frac{(R_k^l)^2}{N_1^k}\ (l=1,2,3), \quad \frac{(S_{jk})^2}{N_1^k},$$

where

$$N_1 = (\phi_t - \theta_a\phi_a)^2 + (\phi_{tt} - \theta_a\phi_{at})(\lambda + \phi_a\phi_{ab}r_{ab})$$
$$(with\ \{r_{ab}\} = \{\phi_{ab}\}^{-1}\ and\ \theta_a = r_{ab}\phi_{bt}),$$
$$N_2 = (\phi_t - \phi_c\theta_c)\phi_a^*\phi_b^*r_{ab}^* - (\phi_t^* - \phi_c^*\theta_c^*)\phi_a\phi_b r_{ab},$$
$$N_3 = (\phi_t - \phi_t^*) - \tau_a(\phi_a - \phi_a^*) \quad (\tau_a(\lambda\phi_{ab} + \phi_a\phi_b) = \phi_b\phi_t + \lambda\phi_{bt}),$$
$$R_k^1 = R_k(\phi_a, \phi_{ab}), \quad R_k^2 = R_k(\phi_a^*, \phi_{ab}), \quad R_k^3 = R_k(\theta_a - \theta_a^*, \phi_{ab}),$$
$$R_k^4 = R_k(\rho_a, \phi_{ab}) \quad (\rho_a = (\phi_t - \theta_b\phi_b)(\phi_c^* r_{ac} - \phi_c r_{ac}^*) - \phi_b\phi_d r_{bd}(\theta_a - \theta_a^*)).$$

The proof of this theorem will be easier if we notice that by putting $\mu = im$ in (4.4), we obtain operators similar to the operators (4.2).

The change of variables (4.5) in the adduced invariants allows us to obtain bases for the algebras AG_2^I and AG_2^{II} in the representations (4.2) and (4.4). These results can also be generalized for the case of several scalar functions.

4.3. Let us present some examples of new invariant equations

$$\phi_{tt} + \frac{1}{\mu^2}\left\{2\mu\left(\frac{1}{n}\phi_t\phi_{aa} + \phi_a\phi_t\right) + \phi_a\phi_b\phi_{ab} + \frac{1}{n}\phi_a\phi_a\phi_{bb} + \frac{1}{n}\phi_{bb}^2\right\} = \\ = (2\mu\phi_t + \phi_a\phi_a + \phi_{aa})^2 F, \tag{4.18}$$

$$-m^2\phi_{tt} + 2im\left(\phi_a\phi_{at} + \frac{1}{n}\phi_t\phi_{aa}\right) + \phi_a\phi_b\phi_{ab} + \frac{1}{n}\phi_a\phi_a\phi_{bb} + \frac{1}{n}\phi_{aa}^2 = \\ = (2im\phi_t + \phi_a\phi_a + \phi_{aa})^2 F. \tag{4.19}$$

Equations (4.18) and (4.19) are invariant, respectively, under the algebras $AG_2^I(1,n)$, $\mu \neq 0$ (4.2), and $AG_2^{II}(1,n)$, $m \neq 0$ (4.4). The F's are arbitrary functions of the invariants for corresponding algebras.

Evidently, wide classes of invariant equations can be constructed with the adduced invariants.

5. Conclusion

It is well-known that a mathematical model of physical or some other phenomena must obey one of the relativity principles of Galilei or Poincaré. Speaking the language of mathematics, it means that the equations of the model must be invariant under the Galilei or the Poincaré groups. Having bases of differential invariants for these groups (or for the corresponding algebras), we can describe all the invariant scalar equations, or sort the invariant ones out of a set of equations.

The construction of differential invariants for vector and spinor fields presents more complicated problems. The first-order invariants for a four-dimensional vector potential had been found in [18]. The cases of spinor and many-dimensional vector Poincaré-invariant equations and corresponding bases of invariants are still to be investigated.

Note 5. After having prepared the present paper, we became acquainted with the article [19] where realizations of the Poincaré group $P(1,1)$ and the corresponding conformal group were investigated, and all second-order scalar differential equations invariant under these groups were obtained. Reference [19] contains bases of absolute differential invariants of the order 2 for the Poincaré, the similitude, and the conformal groups in $(1+1)$-dimensional Minkowski space for various realizations of the corresponding Lie algebras.

Note 6. It was noticed by the referee that an essential misunderstanding arose in the calculation of second prolongations for differential operators, e.g. in formulae (1.5) and (1.25).

When we calculate such prolongations with the usual Lie technique (see, e.g., [8]), we imply that action of an operator of the form $X^{ab}\partial_{u_{ab}}$, where X^{ab} are some functions, is as follows

$$X^{ab}\partial_{u_{ab}}(u_{cd}u_{cd}) = 2X^{ab}u_{ab}, \quad \partial_{u_{ab}}u_{cd} = \delta_{ac}\delta_{bd}.$$

With this assumption, $\partial_{u_{ab}}u_{ba} = 0$, $a \neq b$.

Otherwise, the second prolongation of the operator J_{ab} (1.1) will be of the form

$$\overset{2}{J}_{ab} = J_{ab} + \hat{J}_{ab}, \quad \hat{J}_{ab} = u_a\partial_{u_b} - u_b\partial_{u_a} + u_{ac}\partial_{u_{bc}} - u_{bc}\partial_{u_{ac}} + u_{ab}(\partial_{u_{bb}} - \partial_{u_{aa}}).$$

Note 7. The equations which are conditionally invariant with respect to the Poincaré and Galilei algebras were investigated in [20, 21].

Acknowledgement. Authors would like to thank the referees for valuable comments.

1. Lie S., *Math. Ann.*, 1884, **24**, 52–89.
2. Tresse A., *Acta Math.*, 1894, **18**, 1–88.
3. Vessiot E., *Acta Math.*, 1904, **28**, 307–349.
4. Michal A.D., *Proc. Nat. Acad. Sci.*, 1951, **37**, 623–627.
5. Fushchych W.I., Yegorchenko I.A., *Dokl AN Ukr. SSR. Ser. A*, 1989, № 4, 29–32.
6. Fuschchych W.I., Yegorchenko I.A., *Dokl. AN Ukr. SSR. Ser. A*, 1989, № 5, 21–22.
7. Spencer A.J.M., Theory of invariants, New York, London, Academic Press, 1971.
8. Ovsyannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.
9. Olver P., Application of Lie groups to differential equations, New York, Springer-Verlag, 1987.
10. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989 (in Russian); Kluwer Publishers, 1993 (in English).
11. Bluman G.W., Kumei S., Symmetries and differential equations, New York, Springer Verlag, 1989.
12. Fuschchych W.I., Yegorchenko I.A., *Dokl. AN SSSR*, 1988, **298**, 347–351.
13. Fuschchych W.I., Serov N.I., *Dokl. AN SSSR*, 1984, **278**, 847.
14. Fushchych W.I., Shtelen W.M., *Lett. Nuovo Cimento*, 1982, **34**, 498.
15. Goff J.A., *Amer. J. Math.*, 1927, **49**, 117–122.
16. Niederer U., *Helv. Phys. Acta*, 1972, **45**, 802–810.
17. Fushchych W.I., Cherniha R.M., *J. Phys. A*, 1985, **18**, 3491–3503.
18. Yegorchenko I.A., Symmetry properties of nonlinear equations for complex vector fields, Preprint 89.48, Institute of Mathematics of the Ukr. Acad. Sci, 1989.
19. Rideau G., Winternitz P., *J. Math. Phys.*, 1990, **31**, 1095–1105.
20. Fushchych W.I., Nikitin A.G., Symmetries of Maxwell's equations, Dordrecht, D. Reidel, 1987.
21. Fushchych W.I., *Ukrain. Mat. Zh.*, 1991, **43**, 1456.

Conditional symmetry and reduction of partial differential equations

W.I. FUSHCHYCH, R.Z. ZHDANOV

Sufficient reduction conditions for partial differential equations possessing nontrivial conditional symmetry are established. The results obtained generalize the classical reduction conditions of differential equations by means of group-invariant solutions. A number of examples illustrating the reduction in the number of independent and dependent variables of systems of partial differential equations are considered.

An analysis of well-known methods for the construction of exact solutions of nonlinear partial differential equations (PDE) (e.g., method of group-theoretic reduction [1, 2], method of differential constraints [3], method of ansatz [4–6]) led us to conclude that most of these methods involve narrowing the set of solutions, i.e., out of the whole set of solutions of the particular equations specific subsets are selected that admit analytic description. In order to implement this approach, certain additional constraints (expressed in the form of equations) that enable us to distinguish these subsets must be imposed on the solution set. For obvious reasons, these additional equations are assumed to be simpler than the initial equations. By complementing the initial equation with additional constraints, we are usually led to an over-determined system of PDE. Consequently, there arises the problem of investigating the consistency of a system of PDE. A second restriction on the choice of these additional constraints is that the resulting system of PDE possesses broader symmetry than the initial system of PDE (or simply a different type of symmetry).

In the present paper we establish sufficient conditions for the reduction of differential equations that generalize the classical reduction conditions of PDE possessing a nontrivial Lie transformation group. Our concern will be with the following:

$$U_A(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \quad A = \overline{1, M}, \tag{1}$$

$$\xi_{a\mu}(x,u)u^\alpha_{x_\mu} - \eta^\alpha_a(x,u) = 0, \quad a = \overline{1, N}, \tag{2}$$

where $x = (x_0, x_1, \ldots, x_{n-1})$, $u(x) = (u^0(x), \ldots, u^{m-1}(x))$, $\underset{s}{u} = \{\partial^s u^\alpha/\partial x_{\mu_1}\ldots\partial x_{\mu_s}$, $0 \le \mu_i \le n-1\}$, $s = \overline{1, r}$, U_A, $\xi_{a\mu}$, η^α_a are sufficiently smooth functions, $N \le n-1$.

Below summation over repeated indices is understood. Let us introduce the notation

$$R_1 = \operatorname{rank} \|\xi_{a\mu}(x,u)\|^{N\ n-1}_{a=1\ \mu=0}, \quad R_2 = \operatorname{rank} \|\xi_{a\mu}(x,u),\ \eta^\alpha_a(x,u)\|^{N\ n-1\ m-1}_{a=1\ \mu=0\ \alpha=0}.$$

It is self-evident that $R_1 \le R_2$. We shall prove that the case $R_1 = R_2$ leads to a reduction in the number of independent variables of the PDE (1), while the case $R_1 < R_2$ leads to a reduction in the number of independent and the number of dependent variables of the PDE (1).

Передруковано з Укр. мат. журн., 1992, **44**, № 7, С. 875–886 за дозволом

1. Reduction of number of independent variables of PDE. In this section we assume that $R_1 = R_2$.

Definition 1. *The set of first-order differential equations*

$$Q_a = \xi_{a\mu}(x,u)\partial_{x_\mu} + \eta^\alpha_a(x,u)\partial_{u^\alpha}, \tag{3}$$

where $\partial_{x_\mu} = \partial/\partial_{x_\mu}$, $\partial_{u^\alpha} = \partial/\partial u^\alpha$; $\xi_{a\mu}$, η^α_a *are smooth functions, is said to be involutive if there exist function* $f^c_{ab}(x,u)$ *such that:*

$$[Q_a, Q_b] = f^c_{ab} Q_c, \quad a, b = \overline{1, N}. \tag{4}$$

Here $[Q_1, Q_2] = Q_1Q_2 - Q_2Q_1$.

The simplest example of an involutive set of operators is a Lie algebra.

It is well-known that conditions (4) ensure that the over-determined system of PDE (2) is consistent (Frobenius theorem [7]). The general solution of the system (2) is given by the formulas

$$F^\alpha(\omega_1, \omega_2, \ldots, \omega_{n+m-R_1}) = 0, \quad \alpha = \overline{0, m-1}, \tag{5}$$

where $\omega_j = \omega_j(x,u)$ are functionally independent first integrals of the system of PDE (2) and F_α are arbitrary smooth functions.

By virtue of the condition $R_1 = R_2$, first integrals (say, $\omega_1, \ldots, \omega_m$) may be chosen that satisfy the condition

$$\det \|\partial\omega_j/\partial u^\alpha\|^{m\ m-1}_{j=1\ \alpha=0} \neq 0. \tag{6}$$

By solving (5) with respect to ω_j, $j = 1, \ldots, m$, we have

$$\omega_j = \varphi_j(\omega_{m+1}, \omega_{m+2}, \ldots, \omega_{m+n-R_1}), \quad j = \overline{1, m}, \tag{7}$$

where φ_j are arbitrary smooth functions

Definition 2. *Formula* (7) *is called the ansatz of the field* $u^\alpha = u^\alpha(x)$ *invariant with respect to the involutive set of operators* (3) *provided* (6) *is satisfied.*

Formula (7) become especially simple and self-evident if

$$\begin{aligned} &\partial\xi_{a\mu}/\partial u^\alpha = 0, \quad \eta^\alpha_a = f^{\alpha\beta}_a(x)u^\beta, \\ &a = \overline{1, N}, \quad \mu = \overline{0, n-1}, \quad \alpha, \beta, \gamma = \overline{0, m-1}. \end{aligned} \tag{8}$$

Under conditions (8) the operators in (3) may be rewritten in the following non-Lie form [8]:

$$Q_a = \xi_{a\mu}(x)\partial x_\mu + \eta_a(x), \quad a = \overline{1, N}, \tag{9}$$

where $\eta_a = \| - \partial\eta^\alpha_a/\partial u^\beta\|^{m-1}_{\alpha,\beta=0}$ are $(m \times m)$ matrices and the system (2) takes the form

$$\xi_{a\mu}(x)u_{x_\mu} + \eta_a(x)u = 0, \quad a = \overline{1, N}. \tag{10}$$

Here $u = (u^0, u^1, \ldots, u^{m-1})^T$ is a column function.

In this case, the set of functionally independent first integrals of the system (2) with $R_1 = R_2$ may be chosen as follows [7]:

$$\begin{aligned} &\omega_j = b_{j\alpha}(x)u^\alpha, \quad j = \overline{1, m}, \\ &\omega_i = \omega_i(x), \quad i = \overline{m+1, m+n-R_1} \end{aligned} \tag{11}$$

and, moreover, $\det \|b_{j\alpha}(x)\|^{m\ m-1}_{i=1\ \alpha=0} \neq 0$.

Substituting (11) in (7) and solving for the variables u^α, $\alpha = 0, \ldots, m-1$, we have

$$u^\alpha = A^{\alpha\beta}(x)\varphi^\beta(\omega_{m+1}, \omega_{m+2}, \ldots, \omega_{m+n-R_1})$$

or (in matrix notation)

$$u = A(x)\varphi(\omega_{m+1}, \omega_{m+2}, \ldots, \omega_{m+n-R_1}). \tag{12}$$

It is easily verified that the matrix

$$(x) = \left(\|b_{j\alpha}(x)\|_{j=1}^{m}{}_{\alpha=0}^{m-1}\right)^{-1}$$

satisfies the following system of PDE:

$$Q_a A \equiv \xi_{a\mu}(x)A_{x_\mu} + \eta_a(x)A = 0, \quad a = \overline{1, N}, \tag{13}$$

and that the functions $\omega_{m+1}(x), \omega_{m+2}(x), \ldots, \omega_{m+n-R_1}(x)$ form a complete set of functionally independent first integrals of the system of PDE

$$\xi_{a\mu}(x)\omega_{x_\mu} = 0, \quad a = \overline{1, N}. \tag{14}$$

The ansatz (7) is said to *reduce* the system of PDE (1) if substitution of (7) in (1) yields a system of PDE for the functions $\varphi^0, \varphi^1, \ldots, \varphi^{m-1}$ that contains only the new independent variables $\omega_{m+1}, \omega_{m+2}, \ldots, \omega_{m+1-R_1}$.

Definition 3. *The system of PDE* (1) *is conditionally invariant with respect to the involutive set of differential operators* (3) *if the over-determined system of PDE* (1)*,* (2) *is Lie invariant with respect to a one-parameter transformation group with generators* Q_a*,* $a = 1, \ldots, N$*.*

Before stating the reduction theorem, we prove several auxiliary assertions.

Lemma 1. *Suppose that the operators* (3) *form an involutive set. Then the set of differential operators*

$$Q'_a = \lambda_{ab}(x)Q_b, \quad a = \overline{1, N} \tag{15}$$

with $\det \|\lambda_{ab}(x,u)\|_{a,b=1}^N \neq 0$ *is also involutive.*

We prove the assertion by direct computation. In fact,

$$[Q'_a, Q'_b] = [\lambda_{ac}Q_c, \lambda_{bd}Q_d] = \lambda_{ac}(Q_c\lambda_{bd})Q_d - \lambda_{bd}(Q_d\lambda_{ac})Q_c + \lambda_{ac}\lambda_{bd}f_{cd}^{d_1}Q_{d_1} = \\ = \tilde{f}_{ab}^c Q_c = \tilde{f}_{ab}^c \lambda_{cd}^{-1} Q'_d.$$

Here λ_{cd}^{-1} are the elements of the inverse of the matrix $\|\lambda_{ab}(x,u)\|_{a,b=1}^N$.

Lemma 2. *Suppose that the differential operators* (3) *satisfy the condition* $R_1 = R_2$ *and that the conditions*

$$[Q_a, Q_b] = 0, \quad a, b = \overline{1, N} \tag{16}$$

are satisfied. Then there exists a change of variables

$$x'_\mu = f_\mu(x,u), \quad \mu = \overline{0, n-1}, \quad u'^\alpha = g^\alpha(x,u), \quad \alpha = \overline{0, m-1} \tag{17}$$

that reduces the operators Q_a *to the form* $Q'_a = \partial_{x'_{a-1}}$*.*

Proof. It is known that for any first-order differential operator

$$Q = \xi_\mu(x,u)\partial_{x_\mu} + \eta^\alpha(x,u)\partial_{u^\alpha},$$

where ξ_μ and η^α are sufficiently smooth functions, there exists a change of variables (17) that reduces the operator Q to the form $Q' = \partial_{x'_0}$ (cf. [1]). Consequently, the operator Q_1 from the set (3) is reduced to the form $Q'_1 = \partial_{x'_0}$ by means of the change of variables (17). From the condition $[Q_1, Q_a] = 0$, $a = 2, \ldots, N$, it follows that the coefficients of the operators $Q'_2, Q'_3, \ldots, Q'_N$ do not depend on the variable x'_0, whence the operator Q'_2 reduces to the operator $Q''_2 = \partial_{x''_1}$ under the change of variables

$$x''_0 = x'_0, \quad x''_\mu = f'_\mu(x'_1, \ldots, x'_{n-1}, u'), \quad \mu = \overline{1, n-1},$$
$$u''_\alpha = g'^\alpha(x'_1, \ldots, x'_{n-1}, u'), \quad \alpha = \overline{0, m-1},$$

without the form of the operator Q'_1 changing.

Repeating the above procedure $N-2$ times completes the proof.

Lemma 3. *A system of PDE of the form* (1) *that is conditionally invariant with respect to a set of differential operators $\partial_{x'_\mu}$, $\mu = 0, N-1$, possesses the structure*

$$U_A = F_{AB}W_B(x_N, x_{N+1}, \ldots, x_{n-1}, u, \underset{1}{u}, \ldots, \underset{r}{u}) + F^\alpha_{A\mu}u^\alpha_{x_\mu},$$
$$A = \overline{1, M}, \quad \alpha = \overline{0, m-1}, \quad \mu = \overline{0, N-1}, \tag{18}$$

where F_{AB} and $F^\alpha_{A\mu}$ are arbitrary smooth functions of x and $u, \underset{1}{u}, \ldots, \underset{r}{u}$, W_B are arbitrary smooth functions, and, moreover, $\|F_{AB}\|^M_{A,B=1} \neq 0$.

We shall prove the lemma with $N = 1$. By Definition 3, the system (1) is conditionally invariant under the operator $Q = \partial_{x_0}$ if the system

$$U_A(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \quad A = \overline{1, M},$$
$$u^\alpha_{x_0} = 0, \quad \alpha = \overline{1, m-1} \tag{19}$$

is Lie invariant with respect to a one-parameter translation group with respect to the variable x_0. Denoting by $\tilde{Q}$ the r-th extension of Q, the Lie invariant criteria for the system of PDE (19) under this group assume the form (cf. [1, 2])

$$\tilde{Q}U_A\Big|_{\substack{U_B = 0\\ u^\alpha_{x_0} = 0}} = 0, \quad A, B = \overline{1, N}, \quad \alpha = \overline{0, m-1}, \tag{19a}$$

$$\tilde{Q}u^\alpha_{x_0}\Big|_{\substack{U_B = 0\\ u^\beta_{x_0} = 0}} = 0, \quad B = \overline{1, N}, \quad \alpha, \beta = \overline{0, m-1}. \tag{19b}$$

Direct computation shows that the relations

$$\tilde{Q} \equiv \partial_{x_0}, \quad \tilde{Q}u^\alpha_{x_0} \equiv \partial_{x_0}(u^\alpha_{x_0}) = 0$$

hold (recall that in the extended space of the variables $x, u, \underset{1}{u}, \ldots, \underset{r}{u}$ variables x_0 and $u^\alpha_{x_0}$ are independent), whence, using the method of undetermined coefficients, we may rewrite (19a) and (19b) in the form

$$\partial U_A/\partial x_0 = R_{AB}U_B + P^\alpha_A u^\alpha_{x_0}, \quad A = \overline{1, M}, \tag{19c}$$

where R_{AB} and P^α_A are arbitrary smooth functions of $x, u, \underset{1}{u}, \ldots, \underset{r}{u}$.

The system (19c) may be considered a system of inhomogeneous ordinary differential equations for the functions U_A, $A = 1, \ldots, M$. Integrating (19c) with respect to $P^\alpha_A = 0$, we have

$$U^{(0)}_A = F_{AB} W_B, \quad A = \overline{1, M},$$

where W_B, $B = 1, \ldots, M$, are arbitrary smooth functions of the variables $x_1, x_2, \ldots, x_{n-1}, u, \underset{1}{u}, \ldots, \underset{r}{u}$; $F = \|F_{AB}\|^M_{A,B=1}$ is the fundamental matrix of the system (19c) (which is known to satisfy the condition $\det F \neq 0$).

Further, by applying the method of variation of an arbitrary parameter, we deduce (18) with $N = 1$, where

$$F^\alpha_{A0} = F_{AB} \int (F)^{-1}_{BC} P^\alpha_c \, dx_0, \quad A = \overline{1, M}, \quad \alpha = \overline{0, m-1}.$$

The lemma is proved.

Theorem 1. *Suppose that the system of PDE* (1) *is conditionally invariant with respect to the involutive set of operators* (3)*. Then the ansatz invariant with respect to the set of operators* (3) *reduces this system.*

Proof. By the definition of the quantity $R_1, R_1 \leq N$. We denote by δ the difference $N - R_1$. Then R_1 equations of the system (2) are linearly independent (without loss of generality, we may assume that it is the first R_1 equations which are linearly independent), and the other δ equations are linear combinations of these first R_1 equations.

By the condition that $R_1 = R_2$, there exists a nonsingular $(R_1 \times R_1)$ matrix $||\lambda_{ab}(x, u)||^{R_1}_{a,b=1}$ such that

$$\lambda_{ab}(\xi_{b\mu} u^\alpha_{x_\mu} - \eta^\alpha_b) = u^\alpha_{x_{a-1}} + \sum_{\mu=R_1}^{n-1} \tilde{\xi}_{a\mu} u^\alpha_{x_\mu} - \tilde{\eta}^\alpha_a, \quad a = \overline{1, R_1} \quad \alpha = \overline{0, m-1}.$$

By the definition of conditional invariance, the system of PDE (1), (2) is invariant with respect to one-parameter transformation groups with generators (3), whence the equivalent system of PDE

$$\begin{aligned} &U_A(x, u, \underset{1}{u}, \ldots, \underset{r}{u}) = 0, \quad A = \overline{1, M}, \\ &u^\alpha_{x_{a-1}} + \sum_{\mu=R_1}^{n-1} \tilde{\xi}_{a\mu} u^\alpha_{x_\mu} - \tilde{\eta}^\alpha_a = 0, \quad a = \overline{1, R_1}, \quad \alpha = \overline{0, m-1} \end{aligned} \tag{20}$$

is invariant with respect to a one-parameter group with generators

$$Q'_a = \lambda_{ab} Q_b = \partial_{x_{a-1}} + \sum_{\mu=R_1}^{n-1} \tilde{\xi}_{a\mu} \partial_{x_\mu} + \tilde{\eta}^\alpha_a \partial_{u^\alpha}. \tag{21}$$

In fact, the action of a one-parameter transformation group with infinitesimal operator Q_a on the solution manifold of the system (20) is equivalent to an identity transformation.

Since the set of operators (21) is involutive (Lemma 1), there exist functions $f^c_{ab}(x,u)$ such that

$$[Q'_a, Q'_b] = f^c_{ab} Q'_c, \quad a,b,c = \overline{1,R_1}. \tag{22}$$

Computing the commutators on the left side of (22) and equating the coefficients of the linearly independent operators ∂_{x_0}, ∂_{x_1}, $\partial_{x_{R_1-1}}$ gives us $f^c_{ab} = 0$, with $a,b,c = 1,\ldots,R_1$. Consequently, the operators Q'_a commute. Hence, by Lemma 2, there exists a change of variables (17) that reduces these operators to the form $Q''_a = \partial/\partial x'_{a-1}$.

Expressed in terms of the new variables x' and $u'(x')$, the system (20) takes the form

$$\begin{aligned}
&U'_A(x', u', \underset{1}{u'}, \ldots, \underset{r}{u'}) = 0, \quad A = \overline{1,M}, \\
&u'^{\alpha}_{x_{a-1}} = 0, \quad \alpha = \overline{0,m-1}, \quad a = \overline{1,R_1}.
\end{aligned} \tag{23}$$

Moreover, the system of PDE (23) is conditionally invariant with respect to the set of operators $Q''_a = \partial'_{x_{a-1}}$, $a = 1,\ldots,R_1$, whence, by Lemma 3, the system (23) may be rewritten in the form

$$\begin{aligned}
&U'_A = F_{AB} W_B(x'_{R_1}, \ldots, x'_{n-1}, u', \underset{1}{u'}, \ldots, \underset{r}{u'}) + F^{\alpha}_{A\mu} u'^{\alpha}_{x'_\mu}, \\
&A = \overline{1,M}, \quad \alpha = \overline{0,m-1}, \quad \mu = \overline{0,R_1-1}, \\
&u'^{\alpha}_{x'_{a-1}} = 0, \quad \alpha = \overline{0,m-1}, \quad a = \overline{1,R_1},
\end{aligned}$$

where $\det \|F_{AB}\|^{R_1}_{A,B=1} \neq 0$, whence

$$\begin{aligned}
&W_A(x'_{R_1}, \ldots, x'_{n-1}, u', \underset{1}{u'}, \ldots, \underset{r}{u'}) = 0, \\
&u'^{\alpha}_{x'_{a-1}} = 0, \quad A = \overline{1,R_1}, \quad \alpha = \overline{0,m-1}, \quad a = \overline{1,R_1}.
\end{aligned} \tag{24}$$

The ansatz of the field $u'^{\alpha} = u'^{\alpha}(x')$ invariant under the involutive set of operators $Q''_c = \partial_{x'_{a-1}}$, $a = 1,\ldots,R_1$, is given by the formulas

$$u'^{\alpha} = \varphi^{\alpha}(x'_{R_1}, x'_{R_1+1}, \ldots, x'_{n-1}), \quad \alpha = \overline{0,m-1}. \tag{25}$$

Here φ^{α} are arbitrary sufficiently smooth functions.

Substituting (25) in (24), we obtain

$$W_A(x'_{R_1}, \ldots, x'_{n-1}, u', \underset{1}{u'}, \ldots, \underset{r}{u'}) \equiv W'_A(x'_{R_1}, \ldots, x'_{n-1}, \varphi, \underset{1}{\varphi}, \ldots, \underset{r}{\varphi}) = 0, \tag{26}$$

where $\underset{s}{\varphi}$ is the set of partial derivatives of the functions $\varphi^{\alpha} = \varphi^{\alpha}(x'_{R_1}, \ldots, x'_{n-1})$ of order s.

Rewriting ansatz (25) in terms of the initial variables x and $u(x)$

$$g^{\alpha}(x,u) = \varphi^{\alpha}(f_{R},(x,u), \ldots, f_{n-1}(x,u)), \quad \alpha = \overline{0,m-1}, \tag{27}$$

yields the ansatz for the field $u^{\alpha} = u^{\alpha}(x), \alpha = 0,\ldots,m-1$, invariant with respect to the involutive set of operators (3) that reduces the system (1) to a system of PDE with $n - R_1$ independent variables. The theorem is proved.

Corollary. *Suppose that the operators*

$$Q_a = \xi_{a\mu}(x,u)\partial_{x_\mu} + \eta^{\alpha}_a(x,u)\partial_{u^{\alpha}}, \quad a = \overline{1,N}, \quad N \leq n-1$$

are the basis elements of a subalgebra of the invariance algebra of the system of equations (1) *and, moreover, that* $R_1 = R_2$*. Then the ansatz invariant in the Lie algebra* $\langle Q_1, Q_2, \dots, Q_N\rangle$ *reduces the system* (1) *to a system of PDE having* $n-N$ *independent variables.*

Proof. From the definition of a Lie algebra it follows that the operators Q_a satisfy (4) with $f^c_{ab} = \text{const}$. Consequently, they form an involutive set of first-order differential operators, which renders the above assertion a direct consequence of Theorem 1.

By the above assertion, the classical reduction theorem for differential equations by means of group-invariant solutions [1, 2, 9] is a special case of Theorem 1. If any one of the operators Q_a does not belong to the invariance algebra of the given equation and if the conditions of Theorem 1 hold, a reduction via Q_a-conditionally invariant ansätzes is obtained (numerous examples of conditionally invariant solutions are constructed in [4–6, 10–14]).

We shall now consider several examples.

Example 1. The Lie-maximal invariance algebra of the Schrödinger equation

$$\Delta_3 u + U(\vec{x}^{\,2})u = 0 \tag{28}$$

with arbitrary function U is the Lie algebra of the rotation group having basis elements

$$J_{ab} = x_a\partial_{x_b} - x_b\partial x_a, \quad a, b = \overline{1, 3}. \tag{29}$$

To obtain the ansatz invariant relative to the set of operators (29), the complete set of first integrals of the following system of PDE must be constructed:

$$x_a u_{x_b} - x_b u_{x_a} = 0, \quad a, b = \overline{1, 3}. \tag{30}$$

This set contains $3 - R_1$ functionally invariant first integrals, where

$$R_1 = \text{rank}\, \|\xi_{ab}(x)\|^3_{a,b=1} = \text{rank} \left\| \begin{matrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{matrix} \right\| = 2.$$

Consequently, the ansatz for the field $u = u(\vec{x})$ invariant with respect to a Lie algebra having basis elements (29) has the form

$$u(\vec{x}) = \varphi(\omega), \tag{31}$$

where $\varphi \in C^2(\mathbb{R}^1, \mathbb{C}^1)$ is an arbitrary smooth function and $\omega = \omega(\vec{x})$ is the first integral of the system of PDE (30). It is not hard to see that $\omega = \vec{x}^{\,2}$ satisfies (30) and, consequently, is the first integral. Substitution of (31) in (28) yields an ordinary differential equation for the function $\varphi(\omega)$:

$$4\omega\ddot{\varphi} + 6\dot{\varphi} + U(\omega)\varphi = 0.$$

Thus, the ansatz for the field $u = u(\vec{x})$ invariant with respect to a three-dimensional Lie algebra with basis elements (29) reduces (28) to a $(3 - R_1)$-dimensional PDE (in this case, to an ordinary differential equation).

Example 2. Consider the nonlinear eikonal equation

$$u^2_{x_0} - u^2_{x_1} - u^2_{x_2} - u^2_{x_3} + 1 = 0. \tag{32}$$

As shown in [15], the maximal invariance algebra of (32) is the 21-parameter conformal algebra $AC(2,3)$. This algebra contains, in particular, a one-dimensional subalgebra generated by the operator $Q = x_0\partial_u - u\partial_{x_0}$.

To obtain the ansatz invariant under the operator Q, the complete set of first integrals of the following PDE must be constructed:

$$uu_{x_0} + x_0 = 0. \tag{33}$$

The solution of (33) is sought for in the implicit form $f(x,u) = 0$, whence $uf_{x_0} - x_0 f_u = 0$.

The complete set of first integrals of the latter PDE is $\omega_0 = u^2 + x_0^2$, $\omega_1 = x_1$, $\omega_2 = x_2$, $\omega_3 = x_3$. Solving $f(\omega_0, \omega_1, \omega_2, \omega_3) = 0$ with respect to ω_0, we have

$$u^2 + x_0^2 = \varphi(\omega_1, \omega_2, \omega_3) \tag{34}$$

Consequently, (34) gives the ansatz of the field $u^\alpha = u^\alpha(x)$ invariant under the operator Q. Solving (34) for u yields

$$u = \{-x_0^2 + \varphi(\omega_1, \omega_2, \omega_3)\}^{1/2}. \tag{35}$$

Let us emphasize that ansatz (34) cannot be represented in the form (12), since the coefficients of Q do not satisfy condition (8).

Substituting (35) in (32) gives us a three-dimensional PDE for the function $\varphi = \varphi(\vec{\omega})$:

$$\varphi_{\omega_1}^2 + \varphi_{\omega_2}^2 + \varphi_{\omega_3}^2 - \varphi^2 = 0.$$

Example 3. A detailed group-theoretic analysis of the nonlinear wave equation

$$u_{tt} = (a^2(u)u_x)_x, \tag{36}$$

where $a(u)$ is some smooth function, was performed in [16]. It was established that the maximal invariance algebra of (36) has the basis operators

$$Q_1 = \partial_t, \quad Q_2 = \partial_x, \quad Q_3 = t\partial_t + x\partial_x, \tag{37}$$

whence the most general group-invariant ansatz for the PDE (36) is given by the formula $u = \varphi(\omega)$, where $\omega = \omega(t,x)$ is the first integral of the PDE

$$\{\alpha\partial_t + \beta\partial_x + \delta(t\partial_t + x\partial_x)\}\omega(t,x) = 0. \tag{38}$$

Here α, β, and δ are arbitrary real constants. Using transformations from the group G with generators of the form (37), Eq. (38) may be reduced to either one of the following equations:

1) $\alpha\omega_t + \beta\omega_x = 0 \quad$ (under $\delta = 0$);
2) $t\omega_t + x\omega_x = 0 \quad$ (under $\delta \neq 0$).

The first integrals of these equations are given by the formulas $\omega = \alpha x - \beta t$ and $\omega = xt^{-1}$, respectively.

Thus, there are two distinct group-invariant ansätzes of the PDE (36) with arbitrary function $a(u)$:

$$\begin{aligned} &1) \quad u(t,x) = \varphi(\alpha x - \beta t); \\ &2) \quad u(t,x) = \varphi(xt^{-1}). \end{aligned} \tag{39}$$

Substitution of the above ansätzes in (36) yields the ordinary differential equations

$$1)\quad (\beta^2 - \alpha^2 a^2(\varphi))\ddot{\varphi} - 2\alpha^2 a(\varphi)\dot{a}(\varphi)\dot{\varphi}^2 = 0;$$
$$2)\quad (\omega^2 - a^2(\varphi))\ddot{\varphi} - 2\omega\dot{\varphi} - 2a(\varphi)\dot{a}(\varphi)\dot{\varphi}^2 = 0.$$

It was established recently [17] that ansätzes (39) do not exhaust the complete set of ansätzes reducing the PDE (36) to ordinary differential equations. This result is a consequence of conditional symmetry, a property that is not found within the framework of the infinitesimal Lie method.

Let us show, following [17], that (36) is conditionally invariant under the operator

$$Q = \partial_t - \varepsilon a(u)\partial_x, \tag{40}$$

where $\varepsilon = \pm 1$.

Proceeding on the basis of the second extension of Q in (36), we have

$$\tilde{Q}\{u_{tt} - (a^2(u)u_x)_x\} = \varepsilon\dot{a}u_x\{u_{tt} - (a^2u_x)_x\} + \varepsilon(\dot{a}\dot{u}_x + \dot{a}\partial_x)(u_t^2 - a^2u_x^2), \tag{41}$$

whence it follows that the PDE (36) is Lie-noninvariant with respect to a group with infinitesimal operator (40). But if the additional constraint

$$Q_u \equiv u_t - \varepsilon a(u)u_x = 0 \tag{42}$$

is imposed on $u(t,x)$, the right side of (41) vanishes. Consequently, the system (36), (42) is Lie-invariant with respect to a group with generator (40), whence we conclude that the initial PDE (36) is conditionally invariant under the operator Q.

The complete set of functionally independent first integrals of (42) may be chosen in the form $\omega_1 = u$, $\omega_2 = x + \varepsilon a(u)t$.

Consequently, the ansatz invariant under the operator Q is given by the formula $\omega_2 = \varphi(\omega^1)$, or

$$x + \varepsilon a(u)t = \varphi(u), \tag{43}$$

where $\varphi(u)$ is an arbitrary sufficiently smooth function.

Substituting (43) in (36) leads us to conclude that the PDE (36) is satisfied identically. Put differently, (43) gives a solution of the nonlinear equation (36) for an arbitrary function $\varphi(u)$. Recall that solutions that are obtained by means of the group-invariant ansätzes (39) contain two arbitrary constants of integration, and cannot, in theory, contain arbitrary functions.

Thus, the conditional symmetry of PDE enlarges the range of possibilities for reduction of PDE in an essential way.

Example 4. Consider the system of nonlinear Dirac equations

$$\{i\gamma_\mu\partial_\mu - \lambda(\bar{\psi}\psi)^{1/2k}\}\psi = 0, \tag{44}$$

where γ_μ, $\mu = 0,\ldots,3$, are (4×4) Dirac matrices, $\psi = \psi(x_0,x_1,x_2,x_3)$ a four-dimensional complex column function, $\bar{\psi} = (\psi^*)^T\gamma_0$, λ, k real constants, and $\partial_\mu = \partial/\partial x_\mu$, $\mu = 0,\ldots,3$.

It is well known (cf. [5]) that the Lie-maximal invariance group of the system of PDE (44) is the 11-parameter extended Poincaré group complemented with the 3-parameter group of linear transformations in the space ψ^α, $\psi^{*\alpha}$. In [5, 10] it is established that the conditional symmetry of the nonlinear Dirac equation is essentially broader. From [10], it follows that the system: (44) is conditionally invariant with respect to the involutive set of operators

$$Q_1 = \frac{1}{2}(\partial_0 - \partial_3), \quad Q_2 = \omega_1\partial_2 - \{B_1\psi\}^\alpha\partial_{\psi^\alpha},$$
$$Q_3 = \frac{1}{2}(\partial_0 + \partial_3) - \dot{\omega}_1(x_1\partial_1 + x_2\partial_2) - \dot{\omega}_2\partial_1 - \{B_2\psi\}^\alpha\partial_{\psi^\alpha}, \tag{45}$$

where B_1 and B_2 are (4×4) matrices of the form

$$B_1 = \frac{1}{2}(1-2k)\dot{\omega}_1\gamma_2(\gamma_0+\gamma_3),$$
$$B_2 = -k\dot{\omega}_1 + (2\omega_1)(2\dot{\omega}_1^2 - \omega_1\ddot{\omega}_1)(\gamma_1x_1 + 2(k-1)\gamma_2x_2)(\gamma_0+\gamma_3) + (2\omega_1)^{-1}\times$$
$$\times\,((2\dot{\omega}_1\dot{\omega}_2 - \omega_1\ddot{\omega}_2)\gamma_1 + 2(\omega_3\dot{\omega}_1 - \omega_1\dot{\omega}_3)\gamma_2)(\gamma_0+\gamma_3),$$

ω_1, ω_2, and ω_3 are arbitrary smooth functions of $x_0 + x_3$, and $\{\psi\}^\alpha$ denotes the α-th component of the function ψ. Since the coefficients of the operators (45) satisfy conditions (8), they may be rewritten in non-Lie form:

$$Q_1 = \frac{1}{2}(\partial_0 - \partial_3), \quad Q_2 = \omega_1\partial_2 + B_1,$$
$$Q_3 = \tfrac{1}{2}(\partial_0 + \partial_3) - \dot{\omega}_1(x_1\partial_1 + x_2\partial_2) - \dot{\omega}_2\partial_1 + B_2.$$

Consequently, the ansatz of the field $\psi(x)$ invariant with respect to the set of operators Q_1, Q_2, Q_3 must be found in the form (12), where $A(x)$ is a (4×4) matrix and $\omega = \omega(x)$ a real function satisfying the following system of PDE

$$\frac{1}{2}(A_{x_0} - A_{x_2}) = 0, \quad \omega_1A_{x_2} + B_1A = 0,$$
$$\frac{1}{2}(A_{x_0} + A_{x_3}) - (\dot{\omega}_1x_1 + \dot{\omega}_2)A_{x_1} - \dot{\omega}_1x_2A_{x_2} - B_2A = 0,$$
$$\omega_{x_0} - \omega_{x_3} = 0, \quad \omega_{x_2} = 0,$$
$$\omega_{x_0} + \omega_{x_3} - 2(\dot{\omega}_1x_1 + \dot{\omega}_2)\omega_{x_1} - 2\dot{\omega}_1x_2\omega_{x_2} = 0.$$

Omitting the steps in integration of the above system, let us write down the final result, the ansatz for the field $\psi = \psi(x)$ invariant with respect to the involutive set of operators (45):

$$\psi(x) = \omega_1^k\exp\{(2\omega_1)^{-1}(\dot{\omega}_1x_1 + \dot{\omega}_2)\gamma_1(\gamma_0+\gamma_3) + $$
$$+\,(2\omega_1)^{-1}((2k-1)\dot{\omega}_1x_2 + \omega_3)\gamma_2(\gamma_0+\gamma_3)\}\varphi(\omega_1x_1 + \omega_2). \tag{46}$$

This ansatz reduces the system of PDE (44) to a system of ordinary differential equations for the 4-component function $\varphi = \varphi(\omega)$,

$$i\gamma_1\dot{\varphi} - \lambda(\bar{\varphi}\varphi)^{1/2k}\varphi = 0. \tag{47}$$

The general solution of the system (47) has the form [5]

$$\varphi = \exp\{i\lambda\gamma_1(\bar{\chi}\chi)^{1/2k}\omega\}\chi,$$

where χ is an arbitrary constant 4-component column. Substituting the resulting expression for $\varphi = \varphi(\omega)$ in (46) gives us the class of exact solutions of the nonlinear Dirac equation containing three arbitrary functions.

Nonlinear equations of mathematical and theoretical physics that admit nontrivial conditional symmetry have been analyzed in [14].

3. Reduction of number of independent and number of dependent variables of PDE. Suppose (3) is an involutive set of operators that satisfy the condition $R_2 - R_1 = \delta > 0$. In this case we have to modify somewhat the above technique of reducing PDE by means of ansätzes invariant with respect to the involutive set (3). Note that the case in which (3) are basis operators of a subalgebra of the Lie invariance algebra of a given equation satisfying the condition $R_1 < R_2$ leads to "partially invariant" solutions [18].

We wish to solve the initial system of PDE in implicit form:

$$\omega^\alpha(x,u) = 0, \quad \alpha = \overline{0, m-1}, \tag{48}$$

where ω^α are smooth functions satisfying the condition

$$\det \|\partial\omega^\alpha/\partial u^\beta\|_{\alpha,\beta=0}^{m-1} \neq 0. \tag{49}$$

As a result, (1) and (2) assume the form

$$H_A(x, u, \omega, \underset{1}{\omega}, \ldots, \underset{r}{\omega}) = 0, \quad A = \overline{1, M}, \tag{50}$$

$$\xi_{a\mu}(x,u)\omega^\alpha_{x_\mu} + \eta^\beta_a(x,u)\omega^\alpha_{u^\beta} = 0, \quad a = \overline{1, N}, \tag{51}$$

where $\underset{s}{\omega} = \{\partial^s\omega/\partial x_{\mu_1}\cdots\partial_{x_{\mu_p}}\partial u^{\alpha_1}\cdots\partial u^{\alpha_q},\ p+q=s\}$.

It is clear that, as they are defined in the space of the variables x, u, $\omega(x,u)$, the operators (3) satisfy the condition $R_1' = R_2'$ (since the coefficients of ∂_{ω^α} are all zero). By means of the same reasoning as in the proof of Theorem 1, we may establish the following result. There exists a change of variables (17) that reduces the system (51) to the form

$$\omega^\alpha_{x'_\mu} = 0, \quad \mu = \overline{0, R_1 - 1}, \quad \omega^\alpha_{u'^\beta} = 0, \quad \beta = \overline{0, \delta - 1}. \tag{52}$$

If the system (48), (50) is conditionally invariant with respect to the set of operators (3) and if condition (52) holds, it may be rewritten as follows:

$$\begin{aligned} &\omega^a(x', u') = 0, \quad \alpha = \overline{0, m-1}, \\ &H_A'(x'_{R_1}, \ldots, x'_{n-1}, u'^\delta, \ldots, u'^{m-1}, \omega, \underset{1}{\omega}, \ldots, \underset{r}{\omega}) = 0, \end{aligned} \tag{53}$$

where the symbol $\underset{s}{\omega}$ denotes the collection of partial derivatives of the function ω of order s with respect to the variables $x'_{R_1}, \ldots, x'_{n-1}, u'^\delta, \ldots, u'^{m-1}$.

Integrating (52) yields the ansatz of the field w^α:

$$\omega^\alpha = F^\alpha(x'_{R_1}, \ldots, x'_{n-1}, u'^\delta, \ldots, u'^{m-1}), \quad \alpha = \overline{0, m-1}, \tag{54}$$

where F^α are arbitrary smooth functions. But the ansatz of the field $u'^\alpha(x')$ cannot be obtained by substituting (54) in the relations $\omega^\alpha(x', u'(x')) = 0$, $\alpha = 0, \ldots, m-1$, since the inequality $R_2 - R_1 = \delta > 0$ violates the condition (49) (if $\delta > 0$, the matrix $\|\partial\omega^\alpha/\partial u^{i\beta}\|_{\alpha,\beta=0}^{m-1}$ has null columns).

To overcome this problem, we shall, by definition, let the expressions

$$\begin{aligned} &F^\alpha(x'_{R_1}, \ldots, x'_{n-1}, u'^\delta, \ldots, u'^{m-1}) = 0, \quad \alpha = \overline{\delta, m-1}, \\ &u'^j = C_j, \quad j = \overline{0, \delta - 1} \end{aligned}$$

be the ansatz of the field $u'^{\alpha} = u'^{\alpha}(x')$ invariant with respect to the set of operators

$$Q_j = \partial_{x'_{j-1}}, \quad j = \overline{1, R_1}, \quad X_i = \partial_{u'^{i-1}}, \quad i = \overline{1, \delta}. \tag{55}$$

The latter ansatz may be rewritten in the form

$$\begin{aligned} &u'^{\alpha} = C_\alpha, \quad \alpha = \overline{0, \delta - 1}, \\ &u'^{\alpha+\beta} = \varphi^\beta(x'_{R_1}, \ldots, x'_{n-1}), \quad \beta = \overline{0, m - \delta - 1}, \end{aligned} \tag{56}$$

where φ^β are arbitrary smooth functions and C_α are arbitrary constants.

Rewriting (56) in terms of the initial variables gives us

$$\begin{aligned} &g^{\alpha}(x, u) = C_\alpha, \quad \alpha = \overline{0, \delta - 1}, \\ &g^{\beta+\delta}(x, u) = \varphi^\beta(f_{R_1}(x, u), \ldots, f_{n-1}(x, u)), \quad \beta = \overline{0, m - \delta - 1}. \end{aligned} \tag{57}$$

Moreover, substituting (57) in the initial system of PDE (1) or, equivalently, substituting the expressions $\omega^\alpha = g^\alpha - C_\alpha$, $\alpha = 0, \ldots, \delta - 1$, $\omega^\beta = g^{\beta+\delta} - \varphi^\beta$, $0 \le \beta \le m - \delta - 1$ in the PDE (50) yields a system of M differential equations for $m - \delta$ functions. Consequently, the dimension of the system (1) decreases by R_1 independent and δ dependent variables.

Let us rewrite (57) in a form more convenient in applications. For this purpose, note that, without loss of generality, we may renumber the operators (3) satisfying the condition $R_2 - R_1 = \delta > 0$ in such a way that the first R_1 operators satisfy the condition

$$\operatorname{rank} \|\xi_{a\mu}\|_{a=1}^{R_1}{}_{\mu=0}^{n-1} = \operatorname{rank} \|\xi_{a\mu}, \eta_a^\alpha\|_{a=1}^{R_1}{}_{\alpha=0}^{m-1}{}_{\mu=0}^{n-1}$$

and the last $N - R_2$ operators are linear combinations of the previous R_2 operators.

Let $\omega_j(x, u)$, $j = 1, \ldots, m+n-R_2$, be the complete set of functionally independent first integrals of the system (51) and, moreover,

$$\operatorname{rank} \|\partial\omega_j / \partial u^\alpha\|_{j=1}^{m-\delta}{}_{\alpha=0}^{m-1} = m - \delta$$

and let $\rho_j(x, u)$ be the solutions of the equations $Q_{1+R_1}\rho(x, u) = 1$ with $i = 1, 2, \ldots, \delta$. Then (57) may be expressed in the following equivalent form:

$$\begin{aligned} &\rho_i(x, u) = C_i, \quad i = \overline{1, \delta}, \\ &\omega_j(x, u) = \varphi^j(\omega_{R_1}(x, u), \ldots, \omega_{n-1}(x, u)), \quad j = \overline{1, m - \delta}. \end{aligned} \tag{58}$$

Definition 4. *Expressions* (58) *are called the ansatz of the field* $u^\alpha = u^\alpha(x)$ *invariant with respect to the involutive set of operators* (3) *provided* $R_2 - R_1 \equiv \delta > 0$.

The above reasoning may be summarized in the form of a theorem.

Theorem 2. *Suppose that the system of PDE* (1) *is conditionally invariant with respect to the involutive system of operators* (3) *and, moreover, that* $R_1 < R_2$. *Then the system* (1) *is reduced by the ansatz invariant with respect to the set of operators* (3).

Example 1. The system of two wave equations

$$\Box u = 0, \quad \Box v = 0 \tag{59}$$

is invariant with respect to a one-parameter group with infinitesimal operator $Q=\partial_v$. Since $R_1=0$ and $R_2=1$, the parameter δ is equal to 1. The complete set of first integrals of the equation $\partial\omega(x,u,v)/\partial v=0$ is given by the functions

$$\omega_\mu=x_\mu,\quad \mu=\overline{0,3},\quad \omega_4=u,$$

whence the ansatz for the field $u(x)$, $v(x)$ invariant under the operator Q has the form (58)

$$u=\varphi(\omega_0,\omega_1,\omega_2,\omega_3),\quad v=C,\quad C=\text{const}.$$

Substituting the above expressions in (59) yields

$$\varphi_{\omega_0\omega_0}-\varphi_{\omega_1\omega_1}-\varphi_{\omega_2\omega_2}-\varphi_{\omega_3\omega_3}=0$$

i.e., the number of dependent variables of the initial system (59) is reduced.

Example 2. Consider the system of nonlinear Thirring equations

$$iv_x=mu+\lambda_1|u|^2v,\quad iu_y=mv+\lambda_2|v|^2u, \tag{60}$$

where u, v are complex functions of x, y and λ_1, λ_2 are real constants.

The above system admits a one-parameter transformation group with generator

$$Q=iu\partial_u+iv\partial_v-iu^*\partial_{u^*}-iv^*\partial_{v^*}.$$

Following the change of variables

$$u(x,y)=H_1(x,y)\exp\{iZ_1(x,y)+iZ_2(x,y)\},$$
$$v(x,y)=H_2(x,y)\exp\{iZ_1(x,y)-iZ_2(x,y)\},$$

where H_j and Z_j are the new dependent variables, Q assumes the form $Q'=\partial_{Z_1}$. Consequently, the ansatz invariant under Q has the form

$$\begin{aligned}u(x,y)&=H_1(x,y)\exp\{iC+iZ_2(x,y)\},\\ v(x,y)&=H_2(x,y)\exp\{iC-iZ_2(x,y)\}.\end{aligned} \tag{61}$$

Substitution of (61) in (60) yields a system of four PDE for the three functions H_1, H_2, and Z_2,

$$H_{2x}=mH_{1x}\sin 2Z_2,\quad H_{1y}=-mH_2\sin 2Z_2,$$
$$H_2Z_{2x}=mH_1\cos 2Z_2+\lambda_1H_1H_2^2,\quad -H_1Z_{2y}=mH_2\cos 2Z_2+\lambda_2H_2H_1^2.$$

Example 3. A group analysis of the one-dimensional gas dynamics equations

$$u_t+uu_x+\rho^{-1}p_x=0,\quad \rho_t+(u\rho)_x=0,\quad p_t+(up)_x+(\gamma-1)pu_x=0 \tag{62}$$

has been carried out by Ovsyannikov [1], who established, in particular, that the invariance algebra of the system of PDE (62) contains the basis element

$$Q=p\partial_p+\rho\partial_\rho. \tag{63}$$

The complete set of functionally independent first integrals of the equation $Qw(t,x,u,p,\rho)=0$ is: $\omega_1=u$, $\omega_2=p\rho^{-1}$, $\omega_3=t$, and $\omega_4=x$. Consequently, the ansatz invariant under Q (63) may be chosen in the form

$$u=\varphi^1(t,x),\quad p\rho^{-1}=\varphi^2(t,x),\quad \ln\rho+F(p\rho^{-1})=C, \tag{64}$$

where $C=\text{const}$ and F is some smooth function.

Substituting the ansatz (64) in the system of PDE (62) yields a system of three differential equations for the two unknown functions $\varphi^1(t,x)$ and $\varphi^2(t,x)$:

$$\begin{aligned}&\varphi^1_t + \varphi^1\varphi^1_x - \varphi^2\dot F(\varphi^2)\varphi^2_x = 0, \quad \varphi^2_t + \varphi^1\varphi^2_x + (\gamma-1)\varphi^2\varphi^1_x = 0,\\ &\varphi^1_x((1-\gamma)\varphi^2\dot F(\varphi^2) - 1) = 0,\end{aligned} \tag{65}$$

Thus we have achieved a reduction of the number of dependent variables of the gas dynamics equations.

It is of interest that if $\varphi^1_x \neq 0$, it follows from the third equation of the system (65) that $F = \lambda + (1-\gamma)^{-1}\ln(\rho^{-1}p)$. Substituting this expression in (62) yields $p = k\rho^\gamma$, $k \in \mathbb{R}^1$, which is the relation that characterizes a polytropic gas.

1. *Ovsyannikov L.V.* Group analysis of differential equations. — M.: Nauka, 1978 (in Russian).
2. *Olver P.* Applications of Lie groups to differential equations. — New York: Springer, 1986.
3. *Sidorov A.F., Shapeev V.P., Yanenko N.N.* Method of differential constraints and its applications in gas dynamics. — Novosibirsk: Nauka, 1984 (in Russian).
4. *Fushchych W.I., Shtelen V.M., Serov N.I.* Symmetry analysis and exact solutions of nonlinear equations of mathematical physics. — Kyiv: Naukova Dumka, 1989 (in Russian).
5. *Fushchych W.I., Zhdanov R.Z.* Symmetry and exact solutions of nonlinear spinor equations // Phys. Rep. — 1989. — **172**, N 4. — P. 123–174.
6. *Fushchych W.I., Zhdanov R.Z.* On some new exact solutions of the nonlinear d'Alembert–Hamilton system // Phys. Lett. A. — 1989. — **141**, N 3–4. — P. 113–115.
7. *Courant R., Gilbert D.* Methods of mathematical physics, Vols. 1 and 2. — M.: Gostekhizdat, 1951 (in Russian).
8. *Fushchych W.I., Nikitin A.G.* Symmetry of equations of quantum mechanics. — M.: Nauka, 1989 (in Russian).
9. *Morgan A.* The reduction by one of the number of independent variables in some systems of partial differential equations // Quart. J. Math. — 1952. — **3**, N 12. — P. 250–259.
10. *Fushchych W.I., Zhdanov R.Z.* Non-Lie ansätzes and exact solutions of the nonlinear spinor equation // Ukr. Math. J. — 1990. — **42**, N 7. — P. 958–962.
11. *Zhdanov R.Z., Andreitsev A.Yu.* On non-Lie reduction of Galilei-invariant spinor equations // Dokl. Akad. Nauk UkrSSR. Ser. A. — 1990. — N 7. — P. 8–11.
12. *Olver P., Rosenau P.* The construction of special solutions to partial differential equations // Phys. Lett. A. — 1986. — **114**, N 3. — P. 107–112.
13. *Clarkson P., Kruskal M.* New similarity solutions for the Boussinesq equation // J. Math. Phys. — 1989. — **30**, N 10. — P. 2201–2213.
14. *Fushchych W.I.* Conditional symmetry of nonlinear equations of mathematical physics // Ukr. Math. J. — 1991. — **43**, N 11. — P. 1456–1471.
15. *Fushchych W.I., Shtelen W.M.* The symmetry and some exact solutions of the relativistic eikonal equation // Lettere al Nuovo Cimento. — 1982. — **34**, N 16. — P. 498–502.
16. *Ames W.F., Lohner R., Adams E.* Group properties of $u_{tt} = [f(u)u_x]_x$ // Nonlinear Phenomena in Mathematical Science. — New York: Academic, 1982. — P. 1–6.
17. *Fushchych W.I., Revenko I.V., Zhdanov R.Z.* Non-symmetry approach to the construction of exact solutions of some nonlinear wave equations // Dokl. Akad. Nauk UkrSSR. Ser. A. — 1991. — N 7. — P. 15–16.
18. *Ovsyannikov L.V.* Partial invariance // Dokl. Akad. Nauk SSSR. — 1969. — **186**, N 1. — P. 22–25.

Hodograph transformations and generating of solutions for nonlinear differential equations

W.I. FUSHCHYCH, V.A. TYCHYNIN

Перетворення годографа однієї скалярної функції в $\mathbb{R}(1,1)$ та $\mathbb{R}(1,3)$, а також двох скалярних функцій в $\mathbb{R}(1,1)$ використані для розмноження розв'язків нелінійних рівнянь; побудовані класи годограф-інваріантних рівнянь другого порядку.

The results of using the hodograph transformations for solution of applied problems are well-known. One can find them for example in [1, 2, 3]. We note also the paper [4], in which a number of invariants for hodograph transformation as well as hodograph-invariant equations were constructed.

1. Hodograph-invariant and -linearizable equations in $\mathbb{R}(1,1)$. Let us consider the hodograph transformation for one scalar function ($M = 1$) of two independent variables $x = (x_0, x_1)$, $n = 2$:

$$\begin{aligned} &u(x) = y_1, \quad x_0 = y_0, \quad x_1 = v(y), \\ &\delta = v_1 = \partial_1 v = \frac{\partial v}{\partial y_1} \neq 0, \quad y = (y_0, y_1). \end{aligned} \tag{1}$$

Differential prolongations of the transformation (1) generate such expressions for the first and second order derivatives:

$$u_1 = v_1^{-1}, \quad u_0 = -v_0 v_1^{-1}, \tag{2}$$

$$\begin{aligned} &u_{11} = -v_1^{-3} v_{11}, \quad u_{10} = -v_1^{-3}(v_1 v_{10} - v_0 v_{11}), \\ &u_{00} = -v_1^{-3}[v_0^2 v_{11} - 2v_0 v_1 v_{10} + v_1^2 v_{00}]. \end{aligned} \tag{3}$$

It is clear that (1) is an involutory transformation. This allows to write a set of differential expressions of order ≤ 2, which are absolutely invariant under the transformation (1):

$$\begin{aligned} &f^0(x_0), \quad f^1(x_1, u), \quad f^2(u_1, u_1^{-1}), \quad f^3(u_0, -u_0 u_1^{-1}), \quad f^4(u_{11}, -u_1^{-3} u_{11}), \\ &f^5(u_{10}, -u_1^{-3}(u_1 u_{10} - u_0 u_{11})), \quad f^6(u_{00}, -u_1^{-3}[u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00}]). \end{aligned} \tag{4}$$

Here f^0 is an arbitrary smooth function, f^i, $i = \overline{1,6}$ are arbitrary functions symmetric on arguments, i.e. $f^i(x,z) = f^i(z,x)$. So, the second order PDE invariant under the transformation (1) has the form

$$F(\{f^\sigma\}) = 0, \quad \{f^\sigma\} = \{f^0, f^1, \ldots, f^6\}, \quad \sigma = \overline{0,6}, \tag{5}$$

F is an arbitrary smooth function.

Such well-known equations are contained in the class (5):

1) $u_0^2 - u_1^2 - 1 = 0$ — the eikonal equation; (6)

Передруковано з Доп. АН України, 1993, № 10, С. 52–58 за дозволом

$$2)\quad u_{11} - u_{00}[u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00}] = 0 \text{ — the Born–Infeld equation;} \tag{7}$$

$$3)\quad u_{00}u_{11} - u_{10}^2 = 0 \text{ — the Monge–Ampeŕe equation;} \tag{8}$$

$$4)\quad u_0 = f(u_1)u_{11}, \quad f(u_1) = f(u_1^{-1})u_1^{-2} \text{ — the nonlinear heat equation [5].} \tag{9}$$

Particularly, such equation as

$$u_0 - u_1^{-1}u_{11} = 0 \tag{10}$$

is contained in the last class (9).

Let $\overset{(1)}{u}(x_0, x_1)$ be a known solution of Eq. (5). To construct a new solution $\overset{(2)}{u}(x_0, x_1)$ let us write the first solution replacing in it an argument x_1 for parameter τ: $\overset{(1)}{u}(x_0, \tau)$ and substitute it to the hodograph transformation formula (1). So, we obtain the solutions generating formula for Eq. (5).

$$\overset{(2)}{u}(x_0, x_1) = \tau, \quad x_1 = \overset{(1)}{u}(x_0, \tau). \tag{11}$$

Let us now describe some class of (1)-linearizable equations. Making use of formulae (1) to transform general linear second order PDE

$$b^{\mu\nu}(y)v_{\mu\nu} + b^{\mu}(y)v_{\mu} + b(y)v + c(y) = 0, \quad y = (y_0, y_1), \quad \mu, \nu = 0, 1, \tag{12}$$

we obtain

$$\begin{aligned} & b^{00}(x_0, u)u_1^{-3}(u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00}) - \\ & \quad - 2b^{10}(x_0, u)u_1^{-3}(u_1 u_{10} - u_0 u_{11}) + b^{11}(x_0, u)u_1^{-3}u_{11} + \\ & \quad + b^0(x_0, u)u_1^{-1}u_0 + b^1(x_0, u)u_1^{-1} - b(x_0, u)x_1 - c(x_0, u) = 0. \end{aligned} \tag{13}$$

$b^{\mu\nu}$, b^{μ}, c are arbitrary smooth functions, $b^{10} = b^{01}$. Summation over repeated indices is understood in the space $\mathbb{R}(1, 1)$ with the metric $g_{\mu\nu} = \operatorname{diag}(1, -1)$. The repeated use of this transformation to Eq. (12) turn us again to the Eq. (11).

For any equation of the class (12) the principle of nonlinear superposition is satisfied

$$\overset{(3)}{u}(x_0, x_1) = \overset{(1)}{u}(x_0, \tau), \quad \overset{(1)}{u}(x_0, x_1) = \overset{(2)}{u}(x_0, x_1 - \tau), \tag{14}$$

Here $\overset{(k)}{u}(x_0, x_1)$, $k = 1, 2$ are known solutions of Eq. (12), $\overset{(3)}{u}(x_0, x_1)$ is a new solution of this equation. Parameter τ must be eliminated due to second equality of the system (13). For example, such equations important for applications are contained in this class (12):

$$\begin{aligned} & u_0 - u_1^{-2}u_{11} = 0, \quad u_0 u_{11} - u_1 u_{10} = 0, \\ & u_0^2 u_{11} - 2u_0 u_1 u_{10} + u_1^2 u_{00} = 0, \quad u_0 - c(x_0, u)u_1 = 0. \end{aligned}$$

Let us consider now an example of constructing new solutions from two known ones by means of solutions superposition formula (13).

Example 1. A nonlinear heat equation

$$u_0 - u_1^{-2}u_{11} = 0$$

is reduced to the linear equation

$$v_0 - v_{11} = 0 \tag{15}$$

Therefore, the formula (13) is true for (14). The functions

$$\overset{(1)}{u} = x_1, \quad \overset{(2)}{u} = \sqrt{x_1 - 2x_0} \tag{16}$$

are both partial solutions of Eq. (14). We construct a new solution $\overset{(3)}{u}$ of this Eq. (14) via $\overset{(1)}{u}$ and $\overset{(2)}{u}$. It has the form

$$\overset{(3)}{u}(x_0, x_1) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + x_1 - 2x_0}, \tag{17}$$

2. Hodograph-invariant and -linearizable equations in $\mathbb{R}(1,3)$. The hodograph transformation of a scalar function $u(x)$ of four independent variables $x = (x_0, x_1, x_2, x_3)$ has the form

$$v(x) = y_1, \quad x_1 = v(y), \quad x_\theta = y_\theta, \quad \theta = 0, 2, 3. \tag{18}$$

Prolongation formulae for (18) are obtained via calculations [6, 7]:

$$\begin{gathered} u_1 = v_1^{-1}, \quad u_\theta = -v_1^{-1} v_\theta, \quad u_{11} = -v_1^{-3} v_{11}, \\ u_{1\theta} = -v_1^{-3}(v_1 v_{1\theta} - v_\theta v_{11}), \quad v_{\theta\theta} = -v_1^{-3}(v_1^2 v_{\theta\theta} - 2v_\theta v_1 v_{1\theta} + v_\theta^2 v_{11}), \\ u_{\theta\gamma} = -v_1^{-3}[v_1(v_1 v_{\theta\gamma} - v_\gamma v_{1\theta}) - v_\theta(v_1 v_{1\gamma} - v_\gamma v_{11})]. \end{gathered} \tag{19}$$

Here $\theta, \gamma = 0, 2, 3$, $\theta \neq \gamma$. Making use of involutivity of the transformation (18) we list for it a such set of absolute differential invariant expressions of order ≤ 2:

$$\begin{gathered} f^0(x_0, x_2, x_3), \quad f^1(x_1, u), \quad f^2(u_1, u_1^{-1}), \quad f^3(u_\theta, -u_1^{-1} u_\theta), \\ f^4(u_{11}, -u_1^{-3} u_{11}), \quad f^5(u_{1\theta}, -u_1^{-3}(u_1 u_{1\theta} - u_\theta u_{11})), \\ f^6(u_{\theta\theta}, -u_1^{-3}(u_1^2 u_{\theta\theta} - 2u_1 u_\theta u_{1\theta} + u_\theta^2 u_{11})). \\ f^7(u_{\theta\gamma}, -u_1^{-3}[u_1(u_1 u_{\gamma\theta} - u_\gamma u_{1\theta}) - u_\theta(u_1 u_{1\gamma} - u_\gamma u_{11})]). \end{gathered} \tag{20}$$

There is no summation over θ here, as before, f^0 is an arbitrary smooth function, f^j, $j = \overline{1,7}$ are arbitrary symmetric.

An equation invariant under transformation (18) has the form

$$F(\{f^\lambda\}) = 0 \quad (\lambda = \overline{0,7}). \tag{21}$$

The solutions generating formula has the same form as (10)

$$\overset{(2)}{u}(x_0, x_1, x_2, x_3) = \tau, \quad x_1 = \overset{(1)}{u}(x_0, \tau, x_2, x_3). \tag{22}$$

Here $\overset{(1)}{u}(x)$ is a known solution of Eq. (21), $\overset{(2)}{u}(x)$ is its new solution. The following well-known equations are contained in this class (21):

1) $u_0^2 - u_a u_a - 1 = 0$, $\quad a = \overline{1,3}$, the eikonal equation;
2) $(1 - u_\nu u^\nu)\Box u - u^\mu u^\nu u_{\mu\nu} = 0$, $\quad \mu, \nu = \overline{0,3}$, the Born–Infeld equation [8];
3) $\det(u_{\mu\nu}) = 0$ the Monge–Amperé equation.

Here summation over repeated indices is understood in the space $\mathbb{R}(1,3)$ with the metric $g_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$.

$$\Box u = \partial_\mu \partial^\mu u = u_{00} - u_{11} - u_{22} - u_{33}$$

is the d'Alembert operator,

$$u_a u_a = u_1^2 + u_2^2 + u_2^2 + u_3^2 = (\nabla u)^2.$$

The class of hodograph-linearizable equations in $\mathbb{R}(1,3)$ is constructed analogously as above. Making use of transformation (18) for linear equation (11), written in $\mathbb{R}(1,3)$, we get

$$\begin{aligned} & b^{11}(x_\delta,u)u_1^{-3}u_{11} + b^{\theta\theta}(x_\delta,u)u_1^{-3}(u_1^2 u_{\theta\theta} - 2u_1 u_\theta u_{10} + u_\theta^2 u_{11}) + \\ & \quad + b^{\gamma\theta}(x_\delta,u)u_1^{-3}[u_1(u_1 u_{\gamma\theta} - u_\gamma u_{10}) - u_\theta(u_1 u_{1\gamma} - u_\gamma u_{11})] + \\ & \quad + b^1(x_\delta,u)u_1^{-1}u_\theta - b(x_\delta,u)x_1 - c(x_\delta,u) = 0, \quad x_\delta = (x_0,x_2,x_3). \end{aligned} \tag{23}$$

Here $\delta, \theta = 0,2,3$ and summation over θ is understood in the space $\mathbb{R}(1,2)$ with metric $\tilde{g}_{\theta\gamma} = \mathrm{diag}\,(1,-1,-1)$.

Note, that multidimensional nonlinear heat equation

$$u_0 - u_1^{-2}(1 + u_2^2 + u_3^2)u_{11} - u_{22} - u_{33} + 2u_1^{-1}(u_2 u_{12} + u_3 u_{13}) = 0 \tag{24}$$

reduces due to transformation (18) to linear equation $v_0 = \Delta_{(3)}v$, where $\Delta_{(3)} \equiv \partial_1^2 + \partial_2^2 + \partial_3^2$ is the Laplace operator.

So, the solutions superposition formula for the equations (23) and (24) is

$$\overset{(3)}{u}(x_0,x_1,x_2,x_3) = \overset{(1)}{u}(x_0,\tau,x_2,x_3), \tag{25}$$

$$\overset{(1)}{u}(x_0,\tau,x_2,x_3) = \overset{(2)}{u}(x_0,x_1-\tau,x_2,x_3). \tag{26}$$

Example 2. Let partial solutions of Eq. (24)

$$\overset{(1)}{u} = x_0 - x_2 - x_3 - \ln\frac{x_1 - c_2}{c_1}, \qquad \overset{(2)}{u} = \left[\frac{9}{4}c_3^2(x_1 - c_4)^2 - x_2^2 - x_3^2\right]^{\frac{1}{2}}$$

be initial for generating a new solution $\overset{(3)}{u}$. Then this new solution of Eq. (24) is determined via (25), (26) by the equality

$$\begin{aligned} & \overset{(3)}{u}{}^2(x) + x_2^2 + x_3^2 = c_3\left[x_1 - c_2 - c_1\exp\{x_0 - x_2 - x_3 - \overset{(3)}{u}(x)\}\right]^2, \\ & c_3 = \frac{9}{4}c_3^2, \quad c_2 = c_4 + c_2. \end{aligned} \tag{27}$$

Thus, the formula (27) gives us a new solution of Eq. (24) in the implicit form.

3. Hodograph-invariant and -linearizable systems of PDE in $\mathbb{R}(1,1)$. Let us consider two functions $u^\mu(x_0,x_1)$, $\mu = 0,1$ of independent variables x_0, x_1. The hodograph transformation in this case, as is known [2], has the form

$$\begin{aligned} & u^0(x_0,x_1) = y_0, \quad u^1(x_0,x_1) = y_1, \quad x_0 = v^0(y_0,y_1), \quad x_1 = v^1(y_0,y_1), \\ & \delta = u_1^1 u_0^0 - u_0^1 u_1^0 \neq 0, \quad \delta^* = v_1^1 v_0^0 - v_0^1 v_1^0 \neq 0. \end{aligned} \tag{28}$$

The first and second order derivatives are changing as

$$u^1_1=\delta^{*-1}v^0_0,\quad u^1_0=-\delta^{*-1}v^1_0,\quad u^0_1=-\delta^{*-1}v^0_1,\quad u^0_0=\delta^{*-1}v^1_1, \tag{29}$$

$$\begin{aligned}
u^1_{11}&=-\delta^{*-3}[(v^0_0)^2(v^1_0v^0_{11}-v^0_0v^1_{11})+(v^0_1)^2(v^1_0v^0_{00}-v^0_0v^1_{00})-\\
&\quad-2v^0_1v^0_0(u^1_0v^0_{10}-v^0_0v^1_{10})],\\
u^1_{00}&=-\delta^{*-3}[(v^1_0)^2(v^1_0v^0_{11}-v^0_0v^1_{11}+(v^1_1)^2(v^1_0v^0_{00}-v^0_0v^1_{00})-\\
&\quad-2v^1_0v^1_1(v^1_0v^0_{10}-v^0_0v^1_{10})],\\
u^1_{10}&=\delta^{*-3}[v^0_0v^1_0(v^1_0v^0_{11}-v^0_0v^1_{11})+v^0_1v^1_1(v^1_0v^0_{00}-v^0_0v^1_{00})-\\
&\quad-(v^1_0v^0_{10}-v^0_1v^1_{10})(v^1_1v^0_0+v^1_0v^0_1)],\\
u^0_{11}&=-\delta^{*-3}[(v^0_0)^2(v^0_1v^1_{11}-v^1_1v^0_{11})+(v^0_1)^2(v^0_1v^1_{00}-v^1_1v^0_{00})-\\
&\quad-2v^0_1v^0_0(v^0_1v^1_{10}-v^1_1v^0_{10})],\\
u^0_{00}&=-\delta^{*-3}[(v^1_0)^2(v^0_1v^1_{11}-v^1_1v^0_{11})+(v^1_1)^2(v^0_1v^1_{00}-v^1_1v^0_{00})-\\
&\quad-2v^1_1v^1_0(v^0_1v^1_{10}-v^1_1v^0_{10})],\\
u^0_{10}&=-\delta^{*-3}[v^0_0v^1_0(v^0_1v^1_{11}-v^1_1v^0_{11})+v^0_1v^1_1(v^0_1v^1_{00}-v^1_1v^0_{00})-\\
&\quad-(v^0_1v^1_{10}-v^1_1v^0_{10})(v^1_1v^0_0+v^1_0v^0_1)].
\end{aligned} \tag{30}$$

Let us now construct the absolute differential invariants with respect to (28)–(30) of order ≤ 2. Making use of involutivity of this transformation we get

$$f^1(x_\mu,u^\mu),\quad \mu=0,1,\quad f^2(u^\mu_\mu,\delta u^\nu_\nu),\quad \mu\neq\nu,\ \ \mu,\nu=0,1,$$

there is no summation over repeated indices here,

$$\begin{aligned}
&f^3(u^\mu_\nu,-\delta^{-1}u^\mu_\nu),\quad \mu\neq\nu,\ \ \mu,\nu=0,1;\\
&f^4(u^1_{11},-\delta^{-3}[(u^0_0)^2(u^1_0v^0_{11}-u^0_0u^1_{11})+(u^0_1)^2(u^1_0u^0_{00}-u^0_0u^1_{00})-\\
&\quad-2u^0_1u^0_0(u^1_0u^0_{10}-u^0_0u^1_{10})]),\\
&f^5(u^1_{00},-\delta^{-3}[(u^1_0)^2(u^1_0u^0_{11}-u^0_0u^1_{11})+(u^1_1)^2(u^1_0u^0_{00}-u^0_0u^1_{00})-\\
&\quad-2u^1_0u^1_1(u^1_0v^0_{10}-u^0_0u^1_{10})]),\\
&f^6(u^1_{10},-\delta^{-3}[u^0_0u^1_0(u^1_0u^0_{11}-u^0_0u^1_{11})+u^0_1u^1_1(u^1_0u^0_{00}-u^0_0u^1_{00}-\\
&\quad-(u^1_0u^0_{10}-u^0_0u^1_{10})(u^1_1u^0_0+u^1_0u^0_1)]),\\
&f^7(u^0_{11},-\delta^{-3}[(u^0_0)^2(u^0_1u^1_{11}-u^1_1u^0_{11})+(u^0_1)^2(u^0_1u^1_{00}-u^1_1u^0_{00})-\\
&\quad-2u^0_1u^0_0(u^0_1u^1_{10}-u^1_1u^0_{10})]),\\
&f^8(u^0_{00},-\delta^{-3}[(u^1_0)^2(u^0_1u^1_{11}-u^1_1u^0_{11})+(u^1_1)^2(u^0_1u^1_{00}-u^1_1u^0_{00})-\\
&\quad-2u^1_1u^1_0(u^0_1u^1_{10}-u^1_1u^0_{10})]),\\
&f^9(u^0_{10},-\delta^{-3}[u^0_0u^1_0(u^0_1u^1_{11}-u^1_1u^0_{11})+u^0_1u^1_1(u^0_1u^1_{00}-u^1_1u^0_{00})-\\
&\quad-(u^0_1u^1_{10}-u^1_1u^0_{10})(u^1_1u^0_0+u^1_0u^0_1)]).
\end{aligned} \tag{31}$$

All functions f^k, $k=\overline{1,9}$ are arbitrary smooth and symmetric.

So, we now are able to construct the hodograph-invariant system of second order PDEs

$$F^\sigma(\{f^k\})=0,\quad k=\overline{1,9},\quad \sigma=1,2,\ldots,N. \tag{32}$$

We construct a new solution $\overset{(2)}{u} = (\overset{(2)}{u}{}^0, \overset{(2)}{u}{}^1)$ of system (32) via known solution $\overset{(1)}{u} = (\overset{(1)}{u}{}^0, \overset{(1)}{u}{}^1)$ according to the formula

$$\overset{(2)}{u}(x) = \tau, \quad x = \overset{(1)}{u}(\tau). \tag{33}$$

Here $x = (x_0, x_1)$, $\tau = (\tau^0, \tau^1)$, τ^μ are parameters to be eliminated out of system (33).

Example 3. Let us consider the simplest hodograph-invariant system of first order PDE

$$u_0^1 - u_1^0 = 0, \quad u_1^1 - u_0^0 = 0. \tag{34}$$

It is easily to verify, that pair of functions

$$\overset{(1)}{u}{}^0 = 2x_0x_1 + c, \quad \overset{(1)}{u}{}^1 - x_0^2 + x_1^2$$

is the solution of system (34). Making use of formula (33) one obtains the new solution of this system

$$\begin{aligned} \overset{(2)}{u}{}^1 &= \pm\frac{1}{\sqrt{2}}\left[x_1 \pm \sqrt{x_1^2 + (x_0 - c)^2}\right]^{\frac{1}{2}}, \\ \overset{(2)}{u}{}^0 &= \pm\frac{x_0 - c}{\sqrt{2}}\left[x_1 \pm \sqrt{x_1^2 + (x_0 - c)^2}\right]^{-\frac{1}{2}}. \end{aligned} \tag{35}$$

Let us consider the linear system of first order PDEs

$$b_\mu^{\sigma\nu}(y)v_\mu^\nu + b^{\sigma\nu}(y)v^\nu + c^\sigma(y) = 0. \tag{36}$$

Here $b_\mu^{\sigma\nu}$, $b^{\sigma\nu}$, c^σ are arbitrary smooth functions of $y = (y_0, y_1)$, summation over repeated indices is understood in the space with metric $g_{\mu\nu}^* = \operatorname{diag}(1, 1)$. This system (36) under transformation (28) reduces into system of nonlinear PDEs

$$\begin{aligned} & b^{\sigma 0}(u)\delta^{-1}u_1^1 - b_1^{\sigma 0}(u)\delta^{-1}u_1^0 - b_0^{\sigma 1}(u)\delta^{-1}u_0^1 + \\ & \quad + b_1^{\sigma 1}(u)\delta^{-1}u_0^0 + b^{\sigma 0}(u)x_0 + b^{\sigma 1}(u)x_1 + c^\sigma(u) = 0. \end{aligned} \tag{37}$$

The solutions superposition formula for the system (37) has the form

$$\begin{aligned} \overset{(3)}{u}{}^0(x_0, x_1) &= \overset{(1)}{u}{}^0(\tau^0, \tau^1), \quad \overset{(1)}{u}{}^0(\tau^0, \tau^1) = \overset{(2)}{u}{}^0(x_0 - \tau^0, x_1 - \tau^1), \\ \overset{(3)}{u}{}^1(x_0, x_1) &= \overset{(1)}{u}{}^1(\tau^0, \tau^1), \quad \overset{(1)}{u}{}^1(\tau^0, \tau^1) = \overset{(2)}{u}{}^1(x_0 - \tau^0, x_1 - \tau^1). \end{aligned} \tag{38}$$

Making use of designations $u = (u^0, u^1)$, $x = (x_0, x_1)$, $\tau = (\tau^0, \tau^1)$, one can rewrite the formula (38) in another way:

$$\overset{(3)}{u}(x) = \overset{(1)}{u}(\tau), \quad \overset{(1)}{u}(\tau) = \overset{(2)}{u}(x - \tau). \tag{38a}$$

Example 4. It is obvious, that two pairs of functions

$$\begin{aligned} & \overset{(1)}{u} = \frac{1}{2}x_0, \quad \overset{(1)}{\rho} = (2\lambda)^{-1}\sqrt{\frac{1}{4}x_0^2 - x_1}, \\ & \overset{(2)}{u} = x_0^{-1}\left[\frac{1}{2}c_1 + x_1\right], \quad \overset{(2)}{\rho} = (2\lambda x_0)^{-1}c_0 \end{aligned} \tag{39}$$

give two partial solutions of the system

$$
\begin{aligned}
&u_0 + uu_1 + 4\lambda^2\rho\rho_1 = 0,\\
&\rho_0 + u_1\rho + u\rho_1 = 0.
\end{aligned}
\tag{40}
$$

Let us apply the formula (38) to construct a new solution $\overset{(3)}{u}$, $\overset{(3)}{\rho}$ via (39). Finally we get

$$
\begin{aligned}
&\overset{(3)}{u}{}^2(x_0,x_1) - c_2^2(x_0 - 2\overset{(3)}{u}(x_0,x_1))^{-2} - x_0\overset{(3)}{u}(x_0,x_1) + x_1 + \frac{1}{2}c_1 = 0,\\
&\overset{(3)}{\rho}(x_0,x_1) = (2\lambda)^{-1}\left[x_0\overset{(3)}{u}(x_0,x_1) - \overset{(3)}{u}{}^2(x_0,x_1) - x_1 - \frac{1}{2}c_1\right]^{\frac{1}{2}}.
\end{aligned}
$$

1. *Forsyth A.R.* Theory of differential equations. – New York: Dover Publication, 1959. — Vol. 5. — 478 p.; Vol. 6. — 596 p.
2. *Ames W.F.* Nonlinear partial differential equations in engineering. — New York: Academic Press, 1965. — Vol. 1. — 511 p.; 1972. — Vol. 2. — 301 p.
3. *Курант Р.* Уравнения в частных производных. — М.: Мир, 1964. — 830 с.
4. *Фущич В.И., Серов Н.И.* Негрупповая симметрия некоторых нелинейных волновых уравнений // Докл. АН УССР. — 1991. — № 9. — С. 45–49.
5. *Fushchych W.I., Serov N.I., Tychinin V.A., Amerov T.K.* On non-local symmetries of nonlinear heat equation // Докл. АН Украины. — 1992. — № 11. — С. 27–33.
6. *Фущич В.И., Тычинин В.А.* О линеаризации некоторых нелинейных уравнений с помощью нелокальных преобразований. — Киев, 1982. — 54 с. — (Препр./АН УССР. Ин-т математики; № 82.33).
7. *Фущич В.И., Тычинин В.А., Жданов Р.З.* Нелокальная линеаризация и точные решения некоторых уравнений Монжа–Ампера, Дирака. — Киев, 1985. — 28 с. — (Препр./АН УССР. Ин-т математики; № 85.88).
8. *Тычинин В.А.* Нелокальная линеаризация и точные решения уравнения Борна–Инфельда и некоторых его обобщений // Теоретико-групповые исследования уравнений математической физики. — Киев: Ин-т математики АН УССР, 1986. — С. 54–60.

Orthogonal and non-orthogonal separation of variables in the wave equation $u_{tt} - u_{xx} + V(x)u = 0$

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

We develop a direct approach to the separation of variables in partial differential equations. Within the framework of this approach, the problem of the separation of variables in the wave equation with time-independent potential reduces to solving an over-determined system of nonlinear differential equations. We have succeeded in constructing its general solution and, as a result, all potentials $V(x)$ permitting variable separation have been found. For each of them we have constructed all inequivalent coordinate systems providing separability of the equation under study. It should be noted that the above approach yields both orthogonal and non-orthogonal systems of coordinates.

1. Introduction

Separation of variables (SV) in two- and three-dimensional Laplace, Helmholtz, d'Alembert and Klein–Gordon–Fock equations has been carried out in classical works by Bocher [1], Darboux [2], Eisenhart [3], Stepvanov [4], Olevsky [5], and Kalnins and Miller (see [6] and references therein). Nevertheless, a complete solution to the problem of sv in a two-dimensional wave equation with time-independent potential

$$(\Box + V(x))u \equiv u_{tt} - u_{xx} + V(x)u = 0 \tag{1}$$

has not been obtained yet. In (1) $u = u(t,x) \in C^2(\mathbb{R}^2, \mathbb{R}^1)$, $V(x) \in C(\mathbb{R}^1, \mathbb{R}^1)$.

Equations belonging to the class (1) are widely used in modern mathematical physics and can be related to other important linear and nonlinear partial differential equations (PDE). First, we mention the Lorentz-invariant wave equation

$$u_{y_0y_0} - u_{y_1y_1} + U(y_0^2 - y_1^2)u = 0. \tag{2}$$

The above equation can be reduced to the form (1) with the change of variables [7]

$$t = \exp(y_1/2)\cosh y_0, \quad x = \exp(y_1/2)\sinh y_0$$

and what is more, potentials $V(\tau)$, $U(\tau)$ are connected by the following relation:

$$U(\tau) = (4\tau)^{-1}V(\tau).$$

Another related equation is the hyperbolic type equation

$$v_{x_0x_0} - c^2(x_1)v_{x_1x_1} = 0 \tag{3}$$

that is widely used in various areas of mathematical physics.

Reprinted with permission from J. Phys. A: Math. Gen., 1993, **26**, P. 5959–5972

Original article is available at http://www.iop.org/EJ/journal/JPhysA

Equation (3) is reduced to the form (1) by the change of variables

$$u(t,x) = [c(x_0)]^{-1/2} v(x_0, x_1), \quad t = x_0, \quad x = \int [c(x_1)]^{-1} dx_1$$

and what is more

$$V(x) = -c^{3/2}(x_1)[c^{1/2}(x_1)], \tag{4}$$

where $x = \int [c(x_1)]^{-1} dx_1$.

The third related equation is the nonlinear wave equation

$$W_{tt} - [c^{-2}(W) W_x]_x = 0. \tag{5}$$

By substitution $W = R_x$, equation (5) is reduced to the form

$$R_{tt} - c^{-2}(R_x) R_{xx} = 0.$$

Applying to the above equation the Legendre transformation

$$x_0 = R_t, \quad x_1 = R_x, \quad v_{x_0} = t, \quad v_{x_1} = x, \quad v = tR_t + xR_x - R,$$

we obtain (3). Consequently, the method of SV in the linear equation (1) makes it possible to construct exact solutions of the nonlinear wave equation (5).

Let us also mention the Euler–Poisson–Darboux equation

$$v_{tt} - v_{xx} - x^{-1} v_x + m^2 x^{-2} v = 0 \tag{6}$$

that is reduced to an equation of the form (1)

$$u_{tt} - u_{xx} + (m^2 - 1/4) x^{-2} u = 0$$

by the change of dependent variable $v(t,x) = x^{-1/2} u(t,x)$.

For the solution of (1) with separated variables $\omega_1(t,x)$, $\omega_2(t,x)$, we use the ansatz

$$u(t,x) = Q(t,x) \varphi_1(\omega_1) \varphi_2(\omega_2) \tag{7}$$

which reduces PDE (1) to two ordinary differential equations (ODE) for functions φ_1, φ_2.

There exist three possibilities for SV in (1). The first is to separate it into two second-order ODE. The second possibility is to separate (1) into first-order and second-order ODE, and the third possibility is to separate (1) into two first-order ODE. In the present paper we shall investigate in detail the first two possibilities. The third possibility requires special separate consideration and will be the topic of future publications.

Consider the following ODE:

$$\ddot{\varphi}_i = A_i(\omega_i, \lambda) \dot{\varphi}_i + B_i(\omega_i, \lambda) \varphi_i, \quad i = 1, 2, \tag{8}$$

where $A_i, B_i \subset C^2(\mathbb{R}^1 \times \Lambda, \mathbb{R}^1)$ are some unknown functions, $\lambda \in \Lambda \subset \mathbb{R}^1$ is a real parameter (separation constant).

Definition 1 [7, 8]. *Equation* (1) *separates into two ODE if substitution of the ansatz* (7) *into* (1) *with subsequent exclusion of the second derivatives* $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ *according to* (8) *yields an identity with respect to the variables* $\dot{\varphi}_i$, φ_i, λ *(considered as independent).*

On the basis of the above definition one can formulate a constructive procedure of SV in (1), suggested for the first time in [7]. At the first step, one has to substitute expression (7) into (1) and to express the second derivatives $\ddot{\varphi}_1$, $\ddot{\varphi}_2$ via functions $\dot{\varphi}_i$, φ_i according to (8). At the second step, the equality obtained is split with respect to the independent variables $\dot{\varphi}_i$, φ_i, λ. As a result, one obtains an over-determined system of partial differential equations for functions Q, ω_1 and ω_2 with undefined coefficients. The general solution of this system gives rise to all systems of coordinates providing separability of (1).

Definition 2. *Equation* (1) *separates into first- and second-order ODE*

$$\begin{aligned} &\dot{\varphi}_1 = A(\omega_1, \lambda)\varphi_1, \\ &\ddot{\varphi}_2 = B_1(\omega_2, \lambda)\dot{\varphi}_2 + B_2(\omega_2, \lambda)\varphi_2 \end{aligned} \tag{9}$$

if substitution of the ansatz (7) *into* (1) *with subsequent exclusion of derivatives* $\dot{\varphi}_1$, $\ddot{\varphi}_2$ *according to* (9) *yields an identity with respect to the variables* φ_1, $\dot{\varphi}_2$, φ_2, λ *(considered as independent).*

Let us emphasize that the above approach to SV in (1) has much in common with the non-Lie method of reduction of nonlinear PDE suggested in [9–11]. It is also important to note that the idea to represent solutions of linear differential equations in the "separated" form (7) goes as far as the classical works by Fourier and Euler (for a modern exposition of the problem of SV, see Miller [12] and Koornwinder [13]).

2. Orthogonal separation of variables in equation (1)

It is evident that (1) admits SV in Cartesian coordinates $\omega_1 = t$, $\omega_2 = x$ under arbitrary $V = V(x)$.

Definition 3. *Equation* (1) *admits non-trivial SV if there exist at least one coordinate system* $\omega_1(t, x)$, $\omega_2(t, x)$ *different from the Cartesian system providing its separability.*

Next, if one makes in (1) the following transformations:

$$\begin{aligned} &t \to C_1 t, \quad x \to C_1 x, \\ &t \to t, \quad x \to x + C_2, \quad C_i \in \mathbb{R}^1 \end{aligned} \tag{10}$$

then the class of equations (1) transforms into itself and what is more

$$\begin{aligned} &V(x) \to V'(x) = C_1^2 V(C_1 x), \\ &V(x) \to V'(x) = V(x + C_2). \end{aligned} \tag{10a}$$

That is why potentials $V(x)$ and $V'(x)$, connected by one of the above relations, are considered as equivalent ones.

When separating variables in (1) one has to solve an intermediate problem of description of all inequivalent potentials such that the equation admits non-trivial SV (classification problem). The next step is to obtain a complete description of the coordinate systems providing SV in (1) with these potentials.

First, we adduce the principal results on separation of (1) into two second-order ODE and then give an outline of the proof of the corresponding theorems.

Theorem 1. *Equation* (1) *admits non-trivial SV in the sense of Definition 1 iff the function $V(x)$ is given, up to equivalence relations* (10a)*, by one of the following formulae:*

$$
\begin{array}{ll}
(1) & V = mx; \\
(2) & V = mx^{-2}; \\
(3) & V = m\sin^{-2}x; \\
(4) & V = m\sinh^{-2}x; \\
(5) & V = m\cosh^{-2}x; \\
(6) & V = m\exp x; \\
(7) & V = \cos^{-2}x(m_1 + m_2\sin x); \\
(8) & V = \cosh^{-2}x(m_1 + m_2\sinh x); \\
(9) & V = \sinh^{-2}x(m_1 + m_2\cosh x); \\
(10) & V = m_1\exp x + m_2\exp 2x; \\
(11) & V = m_1 + m_2x^{-2}; \\
(12) & V = m.
\end{array}
\qquad (11)
$$

Here m, m_1, m_2 are arbitrary real parameters, $m_2 \neq 0$.

Note 1. Equation (1) having the potential (6) from (11) is transformed with the change of variables [7]

$$x' = \exp(x/2)\cosh t, \quad t' = \exp(x/2)\sinh t$$

into (1) with $V(x) = m$.

Note 2. Equations (1) having the potentials (3), (4), (5) from (11) are transformed into (1) with $V(x) = mx^{-2}$ by means of changes of variables [7]

$$
\begin{array}{l}
x' = \tan\xi + \tan\eta, \quad t' = \tan\xi - \tan\eta, \\
x' = \tanh\xi + \tanh\eta, \quad t' = \tanh\xi - \tanh\eta, \\
x' = \coth\xi + \tanh\eta, \quad t' = \coth\xi - \tanh\eta.
\end{array}
$$

Hereafter $\xi = \frac{1}{2}(x+t)$, $\eta = \frac{1}{2}(x-t)$ are cone variables.

By virtue of the above remarks, the validity of the assertion follows from Theorem 1.

Theorem 2. *Provided equation* (1) *admits non-trivial SV in the sense of Definition 1, it is locally equivalent to one of the following equations:*

$$
\begin{array}{ll}
(1) & \Box u + mxu = 0; \\
(2) & \Box u + mx^{-2}u = 0; \\
(3) & \Box u + \cos^{-2}x(m_1 + m_2\sin x)u = 0; \\
(4) & \Box u + \cosh^{-2}x(m_1 + m_2\sinh x) = 0; \\
(5) & \Box u + \sinh^{-2}x(m_1 + m_2\cosh x) = 0; \\
(6) & \Box u + \exp x(m_1 + m_2\exp x)u = 0; \\
(7) & \Box u + (m_1 + m_2x^{-2})u = 0; \\
(8) & \Box u + mu = 0.
\end{array}
\qquad (12)
$$

Thus, there exist eight inequivalent types of equations of the form (1) admitting non-trivial SV.

It is well known that there are 11 coordinate systems providing separability of the Klein–Gordon–Fock equation $\Box u + mu = 0$ into two second-order ODE [6]. Besides that, in [14] it was established that the Euler–Poisson–Darboux equation (6), which is equivalent to the second equation of (12), separates in nine coordinate systems. That is why cases $V(x) = m$ and $V(x) = mx^{-2}$ are not considered here.

As is shown below, the general form of solution with separated variables of (12) is as follows:

$$u(t,x) = \varphi_1(\omega_1(t,x))\varphi_2(\omega_2(t,x)), \tag{13}$$

where $\varphi_1(\omega_1)$, $\varphi_2(\omega_2)$ are arbitrary solutions of the separated ODE

$$\ddot{\varphi}_i = (\lambda + g_i(\omega_i))\varphi_i, \quad i = 1,2 \tag{14}$$

and explicit forms of the functions $\omega_i(t,x)$, $g_i(\omega_i)$ are given below.

Theorem 3. *Equation* $\Box u + mxu = 0$ *separates in two coordinate systems*

$$\begin{aligned}
&(1) \quad \omega_1 = t \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m\omega_2;\\
&(2) \quad \omega_1 = (x+t)^{1/2} + (x-t)^{1/2}, \quad \omega_2 = (x+t)^{1/2} - (x-t)^{1/2},\\
&\qquad g_1 = -\frac{1}{4}m\omega_1^4, \quad g_2 = -\frac{1}{4}m\omega_2^4.
\end{aligned} \tag{15}$$

Theorem 4. *Equation* $\Box u + \sin^{-2} x(m_1 + m_2\cos x)u = 0$ *separates in four coordinate systems*

$$\begin{aligned}
&(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \cos^{-2}\omega_2(m_1 + m_2\sin\omega_2);\\
&(2) \quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \arctan\sinh(\omega_1+\omega_2) \pm \arctan\sinh(\omega_1-\omega_2),\\
&\qquad g_1 = (m_1+m_2)\sinh^{-2}\omega_1, \quad g_2 = -(m_1-m_2)\cosh^{-2}\omega_2;\\
&(3) \quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \arctan\frac{\mathrm{sn}\,(\omega_1+\omega_2)}{\mathrm{cn}\,(\omega_1+\omega_2)} \pm \arctan\frac{\mathrm{sn}\,(\omega_1-\omega_2)}{\mathrm{cn}\,(\omega_1-\omega_2)},\\
&\qquad g_1 = m_1\,\mathrm{dn}^2\omega_1\,\mathrm{cn}^{-2}\omega_1\,\mathrm{sn}^{-2}\omega_1 + m_2[\mathrm{cn}^{-2}\omega_1 - \mathrm{dn}^2\omega_1\,\mathrm{cn}^{-2}\omega_1],\\
&\qquad g_2 = m_1k^4\,\mathrm{sn}^2\omega_2\,\mathrm{cn}^2\omega_2\,\mathrm{dn}^{-2}\omega_2 + m_2k^2[\mathrm{cn}^2\omega_2\,\mathrm{dn}^{-2}\omega_2 - \mathrm{sn}^2\omega_2];\\
&(4) \quad \begin{Bmatrix} x \\ t \end{Bmatrix} = \arctan\left(\frac{k}{k'}\right)^{1/2}\mathrm{cn}\,(\omega_1+\omega_2) \pm \arctan\left(\frac{k}{k'}\right)^{1/2}\mathrm{cn}\,(\omega_1-\omega_2),\\
&\qquad g_1 = m_1[\mathrm{dn}^2\omega_1\,\mathrm{cn}^{-2}\omega_1 + k^2\,\mathrm{sn}^2\omega_1] + m_2[(k')^2\,\mathrm{cn}^{-2}\omega_1 + k^2\,\mathrm{cn}^2\omega_1],\\
&\qquad g_2 = m_1[\mathrm{dn}^2\omega_2\,\mathrm{cn}^{-2}\omega_2 + k^2\,\mathrm{sn}^2\omega_2] + m_2[(k')^2\,\mathrm{cn}^{-2}\omega_2 + k^2\,\mathrm{cn}^2\omega_2].
\end{aligned} \tag{16}$$

In the above formulae (16) k, $k' = (1-k^2)^{1/2}$ *are the moduli of corresponding elliptic Jacobi functions, and* k *is an arbitrary constant satisfying the inequality* $0 < k < 1$.

Theorem 5. *Equation* $\Box u + \cosh^{-2} x(m_1 + m_2\sinh x)u = 0$ *separates in four coordinate systems*

$$(1) \quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \cosh^{-2}\omega_2(m_1 + m_2\sinh\omega_2);$$

$$
\begin{aligned}
&(2)\quad \left\{\begin{matrix} t \\ x \end{matrix}\right\} = -\ln\left[\left(\frac{k'}{k}\right)^{1/2} \operatorname{cn}(\omega_1+\omega_2)\right] \pm \ln\left[\left(\frac{k'}{k}\right)^{1/2} \operatorname{cn}(\omega_1-\omega_2)\right],\\
&\quad g_1 = m_1(k')^2 \operatorname{dn}^{-2}2\omega_1 + m_2 \operatorname{cn}2\omega_1 \operatorname{dn}^{-2}2\omega_1,\\
&\quad g_2 = m_1(k')^2 \operatorname{dn}^{-2}2\omega_2 + m_2 \operatorname{cn}2\omega_2 \operatorname{dn}^{-2}2\omega_2;\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\sinh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\cosh\frac{1}{2}(\omega_1-\omega_2),\\
&\quad g_1 = \cosh^{-2}\omega_1(m_1 - m_2\sinh\omega_1), \quad g_2 = \cosh^{-2}\omega_2(m_1 - m_2\sinh\omega_2);\\
&(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\frac{\operatorname{sn}\frac{1}{2}(\omega_1+\omega_2)}{\operatorname{cn}\frac{1}{2}(\omega_1+\omega_2)} \pm \ln\, \operatorname{dn}\frac{1}{2}(\omega_1+\omega_2),\\
&\quad g_1 = -m_1k^2\operatorname{sn}^2\omega_1 + k^2m_2\operatorname{sn}\omega_1\operatorname{cn}\omega_1,\\
&\quad g_2 = -m_1k^2\operatorname{sn}^2\omega_2 + k^2m_2\operatorname{sn}\omega_2\operatorname{cn}\omega_2.
\end{aligned}
\tag{17}
$$

Here k, $k' = (1-k^2)^{1/2}$ are the moduli of corresponding elliptic functions, $0 \le k \le 1$.

Theorem 6. *Equation $\Box u + \sinh^{-2} x(m_1 + m_2\cosh x)u = 0$ separates in eleven coordinate systems:*

$$
\begin{aligned}
&(1)\quad \omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \sinh^{-2}\omega_2(m_1 + m_2\cosh\omega_2);\\
&(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\frac{1}{2}(\omega_1+\omega_2) \pm \ln\frac{1}{2}(\omega_1-\omega_2),\\
&\quad g_1 = (m_1-m_2)\omega_1^{-2}, \quad g_2 = (m_1+m_2)\omega_2^{-2};\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = -\ln\sin\frac{1}{2}(\omega_1+\omega_2) \pm \ln\sin\frac{1}{2}(\omega_1-\omega_2),\\
&\quad g_1 = (m_1-m_2)\sin^{-2}\omega_1, \quad g_2 = (m_1+m_2)\sin^{-2}\omega_2;\\
&(4)\quad \left\{\begin{matrix} t \\ x \end{matrix}\right\} = -\ln\sinh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\sinh\frac{1}{2}(\omega_1-\omega_2),\\
&\quad g_1 = \sinh^{-2}\omega_1(m_1+m_2)\cosh\omega_1), \quad g_2 = \sinh^{-2}\omega_2(m_1 - m_2\cosh\omega_2);\\
&(5)\quad \left\{\begin{matrix} t \\ x \end{matrix}\right\} = -\ln\cosh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\cosh\frac{1}{2}(\omega_1-\omega_2),\\
&\quad g_1 = \sinh^{-2}\omega_1(m_1 - m_2\cosh\omega_1), \quad g_2 = \sinh^{-2}\omega_2(m_1 - m_2\cosh\omega_2);\\
&(6)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\tanh\frac{1}{2}(\omega_1+\omega_2) \pm \ln\tanh\frac{1}{2}(\omega_1-\omega_2),\\
&\quad g_1 = \cosh^{-2}\omega_1(m_1-m_2), \quad g_2 = -\cosh^{-2}\omega_2(m_1+m_2);\\
&(7)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \ln\tan\frac{1}{2}(\omega_1+\omega_2) \pm \ln\tan\frac{1}{2}(\omega_1-\omega_2),\\
&\quad g_1 = \cos^{-2}\omega_1(m_1+m_2), \quad g_2 = \cos^{-2}\omega_2(m_1-m_2);\\
&(8)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctanh}\operatorname{cn}(\omega_1+\omega_2) \pm \operatorname{arctanh}\operatorname{cn}(\omega_1-\omega_2),\\
&\quad g_1 = (m_1+m_2)\operatorname{dn}^2\omega_1\operatorname{cn}^{-2}\omega_1 + (m_1-m_2)k^2\operatorname{sn}^2\omega_1,\\
&\quad g_2 = (m_1-m_2)\operatorname{dn}^2\omega_2\operatorname{cn}^{-2}\omega_2 + (m_1+m_2)k^2\operatorname{sn}^2\omega_2;\\
&(9)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \operatorname{arctanh}\operatorname{dn}(\omega_1+\omega_2) \pm \operatorname{arctanh}\operatorname{dn}(\omega_1-\omega_2),
\end{aligned}
$$

$$g_1 = (m_1 + m_2)k^2 \operatorname{cn}^2\omega_1 \operatorname{dn}^{-2}\omega_1 + (m - m_2)k^2 \operatorname{sn}^2\omega_1,$$
$$g_2 = (m_1 - m_2)k^2 \operatorname{cn}^2\omega_2 \operatorname{cn}^{-2}\omega_2 + (m_1 + m_2)k^2 \operatorname{sn}^2\omega_2;$$

(10) $$\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \operatorname{arctanh} \operatorname{sn}(\omega_1 + \omega_2) \pm \operatorname{arctanh} \operatorname{sn}(\omega_1 - \omega_2),$$
$$g_1 = (m_1 + m_2) \operatorname{sn}^{-2}\omega_1 + (m_1 - m_2)k^2 \operatorname{sn}^2\omega_1,$$
$$g_2 = (m_1 + m_2)k^2 \operatorname{cn}^2\omega_2 \operatorname{dn}^{-2}\omega_2 + (m_1 - m_2)k^2 \operatorname{dn}^2\omega_2 \operatorname{cn}^{-2}\omega_2; \qquad (18)$$

(11) $$\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \pm \ln \operatorname{cn}(\omega_1 + \omega_2) \pm \ln \operatorname{cn}(\omega_1 - \omega_2),$$
$$g_1 = -m_1 \operatorname{sn}^{-2}\omega_1 - m_2 \operatorname{cn}\omega_1 \operatorname{sn}^{-2}\omega_1,$$
$$g_2 = -m_1 \operatorname{sn}^{-2}\omega_2 - m_2 \operatorname{cn}\omega_2 \operatorname{sn}^{-2}\omega_2.$$

Here k are the moduli of corresponding elliptic functions, $0 < k < 1$.

Theorem 7. *Equation $\Box u + \exp x(m_1 + m_2 \exp x)u = 0$ separates in six coordinate systems:*

(1) $\omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = \exp\omega_2(m_1 + m_2 \exp\omega_2);$

(2) $$\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = -\ln\cos(\omega_1 + \omega_2) \pm \ln\cos(\omega_1 - \omega_2),$$
$$g_1 = -2m_1 \cos 2\omega_1 - \frac{1}{2} m_2 \cos 4\omega_1,$$
$$g_2 = -2m_1 \cos 2\omega_2 - \frac{1}{2} m_2 \cos 4\omega_2;$$

(3) $$\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\sinh(\omega_1 + \omega_2) \pm \ln\sinh(\omega_1 - \omega_2),$$
$$g_1 = -2m_1 \cosh 2\omega_1 - \frac{1}{2} m_2 \cosh 4\omega_1,$$
$$g_2 = -2m_1 \cosh 2\omega_2 - \frac{1}{2} m_2 \cosh 4\omega_2;$$

(4) $$\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\cosh(\omega_1 + \omega_2) \pm \ln\cosh(\omega_1 - \omega_2), \qquad (19)$$
$$g_1 = -2m_1 \cosh 2\omega_1 - \frac{1}{2} m_2 \cosh 4\omega_1,$$
$$g_2 = -2m_1 \cosh 2\omega_2 - \frac{1}{2} m_2 \cosh 4\omega_2;$$

(5) $$\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln\cosh(\omega_1 + \omega_2) \pm \ln\sinh(\omega_1 - \omega_2),$$
$$g_1 = -2m_1 \sinh 2\omega_1 - \frac{1}{2} m_2 \cosh 4\omega_1,$$
$$g_2 = -2m_1 \sinh 2\omega_2 - \frac{1}{2} m_2 \cosh 4\omega_2;$$

(6) $$\left\{ \begin{matrix} x \\ t \end{matrix} \right\} = \ln(\omega_1 + \omega_2) \pm \ln(\omega_1 - \omega_2),$$
$$g_1 = 2m_1 + 2m_2\omega_1^2, \quad g_2 = -2m_1 + 2m_2\omega_2^2.$$

Theorem 8. *Equation $\Box u + (m_1 + m_2 x^{-2})u = 0$ separates in six coordinate systems:*

(1) $\omega_1 = t, \quad \omega_2 = x, \quad g_1 = 0, \quad g_2 = m_1 + m_2\omega_2^{-2};$

$$\begin{aligned}
&(2)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \exp(\omega_1+\omega_2) \pm \exp(\omega_1-\omega_2),\\
&\qquad g_1 = 4m_1 \exp 2\omega_1, \quad g_2 = m_2 \cosh^{-2}\omega_2;\\
&(3)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \sin(\omega_1+\omega_2) \pm \sin(\omega_1-\omega_2),\\
&\qquad g_1 = 2m_1 \cos 2\omega_1 + m_2 \sin^{-2}\omega_1, \quad g_2 = -2m_1\cos 2\omega_2 + m_2\cos^{-2}\omega_2;\\
&(4)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \sinh(\omega_1+\omega_2) \pm \sinh(\omega_1-\omega_2),\\
&\qquad g_1 = 2m_1 \sinh 2\omega_1 + m_2 \sinh^{-2}\omega_1,\\
&\qquad g_2 = -2m_1 \sinh 2\omega_2 - m_2 \sinh^{-2}\omega_2;\\
&(5)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = \cosh(\omega_1+\omega_2) \pm \cosh(\omega_1-\omega_2),\\
&\qquad g_1 = 2m_1 \cosh 2\omega_1 - m_2 \cosh^{-2}\omega_1, \quad g_2 = 2m_1\cosh 2\omega_2 - m_2\cosh^{-2}\omega_2;\\
&(6)\quad \left\{\begin{matrix} x \\ t \end{matrix}\right\} = (\omega_1+\omega_2)^2 \pm (\omega_1-\omega_2)^2,\\
&\qquad g_1 = -16m_1\omega_1^2 + m_2\omega_1^{-2}, \quad g_2 = -16m_1\omega_2^2 + m_2\omega_2^{-2}.
\end{aligned} \tag{20}$$

We now give a sketch of the proof of the above assertions. Substituting ansatz (7) into (1), expressing functions $\ddot{\varphi}_i$ via functions $\dot{\varphi}_1$, φ_i by means of equalities (8) and splitting the equation obtained with respect to independent variables $\dot{\varphi}_i$, φ_i we obtain the following system of nonlinear PDE:

$$(1)\quad Q\Box\omega_i + 2(Q_t\omega_{it} - Q_x\omega_{ix}) + QA_i(\omega_i,\lambda)(\omega_{it}^2 - \omega_{ix}^2) = 0, \quad i=1,2; \tag{21}$$

$$(2)\quad \Box Q + Q[B_1(\omega_1,\lambda)(\omega_{1t}^2-\omega_{1x}^2) + B_2(\omega_2,\lambda)(\omega_{2t}^2-\omega_{2x}^2)] + QV(x) = 0; \tag{22}$$

$$(3)\quad \omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0. \tag{23}$$

Here $\Box = \partial_t^2 - \partial_x^2$.

Thus, to separate variables in the linear PDE (1) one has to construct the general solution of the system of nonlinear equations (21)–(23). The same assertion holds true for any general linear differential equation, i.e. the problem of SV is an essentially nonlinear one.

It is not difficult to become convinced of the fact that, from (23), it follows that

$$(\omega_{1t}^2-\omega_{1x}^2)(\omega_{2t}^2-\omega_{2x}^2) \neq 0. \tag{24}$$

Differentiating (21) with respect to λ and using (24) we obtain

$$A_{1\lambda} = A_{2\lambda} = 0,$$

whence $B_{1\lambda}B_{2\lambda} \neq 0$. Differentiating (22) with respect to λ, we have

$$B_{1\lambda}(\omega_{1t}^2-\omega_{1x}^2) + B_{2\lambda}(\omega_{2t}^2-\omega_{2x}^2) = 0$$

or

$$\frac{B_{1\lambda}}{B_{2\lambda}} = -\frac{\omega_{2t}^2-\omega_{2x}^2}{\omega_{1t}^2-\omega_{1x}^2}.$$

Differentiation of the above equality with respect to λ yields

$$B_{1\lambda\lambda}B_{2\lambda} - B_{1\lambda}B_{2\lambda\lambda} = 0$$

or

$$\frac{B_{1\lambda\lambda}}{B_{1\lambda}} = \frac{B_{2\lambda\lambda}}{B_{2\lambda}}.$$

Since functions $B_1 = B_1(\omega_1)$, $B_2 = B_2(\omega_2)$ are independent, there exists a function $\varkappa(\lambda)$ such that

$$B_{i\lambda\lambda} = \varkappa(\lambda)B_{i\lambda}, \quad i = 1, 2.$$

Integrating the above differential equation with respect to λ we obtain

$$B_i(\omega_i) = \Lambda(\lambda)f_i(\omega_i) + g_i(\omega_i), \quad i = 1, 2,$$

where f_i, g_i are arbitrary smooth functions.

On redefining the parameter $\lambda \to \Lambda(\lambda)$, we have

$$B_i(\omega_i) = \lambda f_i(\omega_i) + g_i(\omega_i). \tag{25}$$

Substitution of (25) into (22) with subsequent splitting with respect to λ yields the following equations:

$$\Box Q + Q[g_1(\omega_{1t}^2 - \omega_{1x}^2) + g_2(\omega_{1t}^2 - \omega_{1x}^2)] + V(x)Q = 0, \tag{26}$$

$$f_1(\omega_{1t}^2 - \omega_{1x}^2) + f_2(\omega_{2t}^2 - \omega_{2x}^2) = 0. \tag{27}$$

Thus, system (21)–(23) is equivalent to the system of equations (21), (23), (26), (27). Before integrating, we make a remark: it is evident that the structure of ansatz (7) is not altered by transformation

$$Q \to Q' = Qh_1(\omega_1)h_2(\omega_2), \quad \omega_i \to \omega_i' = R_i(\omega_i), \quad i = 1, 2, \tag{28}$$

where h_i, R_i are smooth-enough functions. This is why solutions of the system under study connected by relations (28) are considered to be equivalent.

Choosing the functions h_i, R_i in a proper way, we can put in (21) and (27)

$$f_1 = f_2 = 1, \quad A_1 = A_2 = 0.$$

Consequently, functions ω_1, ω_2 satisfy equations of the form

$$\omega_{1t}\omega_{2t} - \omega_{1x}\omega_{2x} = 0, \quad \omega_{1t}^2 - \omega_{1x}^2 + \omega_{2t}^2 - \omega_{2x}^2 = 0,$$

whence

$$(\omega_1 \pm \omega_2)_t^2 - (\omega_1 \pm \omega_2)_x^2 = 0.$$

Integrating the above equations, we obtain

$$\omega_1 = F(\xi) + G(\eta), \quad \omega_2 = F(\xi) - G(\eta), \tag{29}$$

where $F, G \subset C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions, $\xi = (x+t)/2$, $\eta = (x-t)/2$.

Substitution of (29) into (21) with $A_1 = A_2 = 0$ yields the following equations:

$$(\ln Q)_t = 0, \quad (\ln Q)_x = 0,$$

whence $Q = 1$.

Finally, substituting the results obtained into (26), we have

$$V(x) = [g_1(F+G) - g_2(F-G)]\frac{dF}{d\xi}\frac{dG}{d\eta}. \tag{30}$$

Thus, the problem of integrating an over-determined system of nonlinear differential equations (21)–(23) is reduced to integration of the functional-differential equation (30).

Let us summarize the results obtained. The general form of solution of (1) with separated variables is as follows

$$u = \varphi(F(\xi)+G(\eta))\varphi_2(F(\xi)-G(\eta)) \tag{31}$$

where φ_i are arbitrary solutions of (14), functions $F(\xi)$, $G(\eta)$, $g_1(F+G)$, $g_2(F-G)$ being determined by (30).

To integrate Eq. (31) we make the hodograph transformation

$$\xi = P(F), \quad \eta = R(G), \tag{32}$$

where $\dot P \not\equiv 0$, $\dot R \not\equiv 0$.

After making the transformation (32), we obtain

$$g_1(F+G) - g_2(F-G) = \dot P(F)\dot R(G)V(P+R). \tag{33}$$

Evidently, equation (33) is equivalent to the following equation:

$$(\partial_F^2 - \partial_G^2)[\dot P(F)\dot R(G)V(P+R)] = 0$$

or

$$(\dddot P\dot P^{-1} - \dddot R\dot R^{-1})V + 3(\ddot P - \ddot R)\dot V + (\dot P^2 - \dot R^2)\ddot V = 0. \tag{34}$$

Thus, to integrate (30) it is enough to construct all functions $P(F)$, $R(G)$, $V(P+R)$ satisfying (34) and to substitute them into (33).

In [8] we have proved the following assertion:

Lemma. *The general solution of* (34) *determined up to transformation* (10) *is given by one of the following formulae:*

(1) $V = V(x)$ *is an arbitrary function,* $\dot P = \alpha$, $\dot R = \alpha$;

(2) $V = mx$, $\dot P^2 = \alpha P + \beta$, $\dot R^2 = \alpha R + \gamma$;

(3) $V = mx^{-2}$, $P = Q_1(F)$, $R = Q_2(G)$,

$$\begin{aligned} \dot Q_1^2 &= \alpha Q_1^4 + \beta Q_1^3 + \gamma Q_1^2 + \delta Q_1 + \rho, \\ \dot Q_2^2 &= \alpha Q_2^4 - \beta Q_2^3 + \gamma Q_2^2 - \delta Q_2 + \rho; \end{aligned} \tag{35}$$

(4) $V = m\sinh^{-2} x$, $P = \operatorname{arctanh} Q_1(F)$, $R = \tan Q_2(G)$

and Q_1, Q_2 *are determined by* (35);

$$(5)\quad V = m\sinh^{-2} x, \quad P = \operatorname{arctanh} Q_1(F), \quad R = \operatorname{arctanh} Q_2(G)$$

and Q_1, Q_2 *are determined by* (35);

$$(6)\quad V = m\cosh^{-2} x, \quad P = \operatorname{arccoth} Q_1(F), \quad R = \operatorname{arctanh} Q_2(G)$$

and Q_1, Q_2 *are determined by* (35);

$$\begin{array}{ll}
(7) & V = m\exp x, \\
 & \dot{P}^2 = \alpha\exp 2P + \beta\exp P + \gamma, \quad \dot{R}^2 = \alpha\exp 2R + \delta\exp R + \rho; \\
(8) & V = \cos^{-2} x(m_1 + m_2\sin x), \\
 & \dot{P}^2 = \alpha\sin 2P + \beta\cos 2P + \gamma, \quad \dot{R}^2 = \alpha\sin 2R + \beta\cos 2R + \gamma; \\
(9) & V = \cosh^{-2} x(m_1 + m_2\sinh x), \\
 & \dot{P}^2 = \alpha\sinh 2P + \beta\cosh 2P + \gamma, \quad \dot{R}^2 = \alpha\sinh 2R - \beta\cosh 2R + \gamma; \\
(10) & V = \sinh^{-2} x(m_1 + m_2\cosh x), \\
 & \dot{P}^2 = \alpha\sinh 2P + \beta\cosh 2P + \gamma, \quad \dot{R}^2 = -\alpha\sinh 2R + \beta\cosh 2R + \gamma; \\
(11) & V = (m_1 + m_2\exp x)\exp x, \\
 & \ddot{P} = -\dot{P}^2 + \beta, \quad \ddot{R} = -\dot{R}^2 + \beta; \\
(12) & V = m_1 + m_2 x^{-2}, \\
 & \dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \ddot{R}^2 = \alpha R^2 - \beta R + \gamma, \\
(13) & V = m, \\
 & \dot{P}^2 = \alpha P^2 + \beta P + \gamma, \quad \dot{R}^2 = \alpha R^2 + \delta R + \rho.
\end{array}$$

Here α, β, γ, δ, ρ, m_1, m_2, m *are arbitrary real parameters;* $x = \xi + \eta = P + R$.

Theorems 1 and 2 are direct consequences of the above Lemma. To prove Theorems 3–8 one has to integrate the ODE for $P(F)$, $R(G)$ and substitute the expressions obtained into formulae (32)

$$\frac{1}{2}(x+t) = P(F) \equiv P((\omega_1+\omega_2)/2), \quad \frac{1}{2}(x-t) = R(G) \equiv R((\omega_1-\omega_2)/2)$$

and into (33).

Thus, the problem of separation of the wave equation (1) into two second-order differential equations is completely solved.

Since all coordinate systems ω_1, ω_2 satisfy equation (23), we have orthogonal separation of variables. To obtain non-orthogonal coordinate systems providing separability of (1) one has to carry out SV following Definition 2.

3. Non-orthogonal separation of variables in equation (1)

Utilizing the SV procedure in (1) determined by Definition 2, we come to the following assertions (corresponding computations are omitted).

Theorem 9. *Equation* (1) *admits SV in the sense of Definition 2 iff it is locally-equivalent to one of the following equations:*

$$(1)\ \Box u + mu = 0; \quad (2)\ \Box u + mx^{-2}u = 0,$$

where m is an arbitrary real constant.

Theorem 10. *Equation $\Box u + mu = 0$ separates in two coordinate systems*

$$\begin{aligned}
(1)\quad & \omega_1 = \xi, \quad \omega_2 = \xi + \eta, \\
& \dot\varphi_1 = -\lambda\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\varphi_2; \\
(2)\quad & \omega_1 = \xi, \quad \omega_2 = \ln\xi + \ln\eta, \\
& \dot\varphi_1 = -\lambda\omega_1^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\exp(\omega_2)\varphi_2.
\end{aligned}$$

Theorem 11. *Equation $\Box u + mx^{-2}u = 0$ separates in eight coordinate systems*

$$\begin{aligned}
(1)\quad & \omega_1 = \xi, \quad \omega_2 = \xi + \eta, \\
& \dot\varphi_1 = -\lambda\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\omega_2^{-2}\varphi_2; \\
(2)\quad & \omega_1 = \xi, \quad \omega_2 = \arctan\xi + \arctan\eta, \\
& \dot\varphi_1 = -\lambda(1+\omega_1^2)\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sin^{-2}\omega_2\varphi_2; \\
(3)\quad & \omega_1 = \xi, \quad \omega_2 = \operatorname{arctanh}\xi + \operatorname{arctanh}\eta, \\
& \dot\varphi_1 = -\lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sinh^{-2}\omega_2\varphi_2; \\
(4)\quad & \omega_1 = \xi, \quad \omega_2 = \operatorname{arccoth}\xi + \operatorname{arccoth}, \eta, \\
& \dot\varphi_1 = \lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\sinh^{-2}\omega_2\varphi_2; \\
(5)\quad & \omega_1 = \xi, \quad \omega_2 = \operatorname{arctanh}\xi + \operatorname{arctanh}\eta, \\
& \dot\varphi_1 = -\lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2, \\
(6)\quad & \omega_1 = \xi, \quad \omega_2 = \operatorname{arccoth}\xi + \operatorname{arccoth}\eta, \\
& \dot\varphi_1 = \lambda(1-\omega_1^2)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2; \\
(7)\quad & \omega_1 = \xi, \quad \omega_2 = \frac{1}{2}(\ln\xi - \ln\eta), \\
& \dot\varphi_1 = -\lambda(2\omega_1)^{-1}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 - m\cosh^{-2}\omega_2\varphi_2; \\
(8)\quad & \omega_1 = \xi, \quad \omega_2 = \xi^{-1} + \eta^{-1}, \\
& \dot\varphi_1 = \lambda\omega_1^{-2}\varphi_1, \quad \ddot\varphi_2 = \lambda\dot\varphi_2 + m\omega_2^{-2}\varphi_2.
\end{aligned}$$

In the above formulae λ is a separation constant, $\xi = \frac{1}{2}(x+t)$, $\eta = \frac{1}{2}(x-t)$.

As a direct check shows, the above coordinate systems do not satisfy (23). Consequently, they are non-orthogonal.

4. Conclusion

Let us say a few words about the intrinsic characterization of SV in (1). It is well known that the solution of the second-order linear PDE with separated variables is a joint eigenfunction of mutually-commuting symmetry operators of the equation under study (for more detail, see [13, 14]). Below, we construct the second-order symmetry operator of (1) such that solution with separated variables is its eigenfunction and parameter λ is an eigenvalue.

Making in (1) the change of variables (29), we obtain

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = V(\xi+\eta)[\dot{F}(\xi)\dot{G}(\eta)]^{-1}u.$$

Provided (1) admits SV, by virtue of (33) there exist functions $g_1(F+G)$, $g_2(F-G)$ such that

$$V(\xi+\eta)[\dot{F}(\xi)\dot{G}(\eta)]^{-1} = g_1(F+G) - g_2(F-G).$$

Since $F+G=\omega_1, F-G=\omega_2$, equation (36) takes the form

$$u_{\omega_1\omega_1} - u_{\omega_2\omega_2} = [g_1(\omega_1) - g_2(\omega_2)]u$$

or

$$Xu = 0, \quad X = \partial^2_{\omega_1} - \partial^2_{\omega_2} - g_1(\omega_1) + g_2(\omega_2).$$

Clearly, the operators $Q_i = \partial^2_{\omega_i} - g_i(\omega_i)$, $i=1,2$ commute with the operator X, i.e. they are symmetry operators of (1) and, what is more, the relations

$$Q_i u = Q_i\varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda\varphi_1(\omega_1)\varphi_2(\omega_2) = \lambda u, \quad i=1,2$$

hold.

It should be noted that V.N. Shapovalov carried out classification of potentials $V(x)$ such that (1) admitted a non-trivial second-order symmetry operator [15] but he lost cases (4) and (9) from Theorem 1.

It was shown by Osborne and Stuart [16] that the method of SV could be applied to nonlinear PDE. In [8] we suggested a regular approach to SV in nonlinear partial differential equations. In future publications we intend to apply this approach to separate variables in the nonlinear wave equation $u_{tt} - u_{xx} = F(u)$.

1. Bocher M., Die Reihentwickelungen der Potentialtheorie, Leipzig, Teubner, 1894.
2. Darboux G., Lecons sur les Systemes Orhogonaux et les Coordonnees Curvilignes, Paris, Hermann, 1910.
3. Eisenhart L.P., *Ann. Math.*, 1934, **35**, 284.
4. Stepanov V.V., *Matemat. Sbornik*, 1942, **11(53)**, 204.
5. Olevsky M.N., *Matemat. Sbornik*, 1950, **27(69)**, 379.
6. Kalnins E., Miller W., *J. Math. Phys.*, 1974, **15**, 1025.
7. Zhdanov R.Z, Revenko I.V. and Fushchych W.I., *Doklady AN Ukrainy. Ser. A*, 1993, № 1, 27.
8. Zhdanov R.Z., Revenko I.V., Fushchych W.I., *Ukrain. Mat. Zh.*, 1994, **46**, 1343.
9. Fushchych W.I., Serov N.I., *J. Phys. A: Math. Gen.*, 1983, **16**, 3645.
10. Fushchych W.I., Zhdanov R.Z., *Phys. Lett. A*, 1989, **141**, 113.
11. Fushchych W.I., Zhdanov R.Z., Yegorchenko I.A., *Math. Anal. Appl.*, 1991, **161**, 352.
12. Miller W., Symmetry and separation of variables, Addison-Wesley, 1977.
13. Koornwinder T.H., *Lecture Notes in Math.*, 1980, **810**, 240.
14. Kalnins E., Miller W., *J. Math. Phys.*, 1976, **17**, 369.
15. Shapovalov V.N., *Izv. Vuzov. Fizika*, 1969, **9**, 64.
16. Osborn A., Stuart A., *J. Math. Phys.*, 1978, **19**, 1573.

Symmetry reduction and exact solutions of the Navier–Stokes equations

W.I. FUSHCHYCH, R.O. POPOVYCH

Ansatzes for the Navier–Stokes field are described. These ansatzes reduce the Navier–Stokes equations to system of differential equations in three, two, and one independent variables. The large sets of exact solutions of the Navier–Stokes equations are constructed.

1. Introduction

The Navier–Stokes equations (NSEs)

$$\begin{aligned} &\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p = \vec{0}, \\ &\operatorname{div}\vec{u} = 0 \end{aligned} \tag{1.1}$$

which describe the motion of an incompressible viscous fluid are the basic equations of modern hydrodynamics. In (1.1) and below $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian, the kinematic coefficient of viscosity and fluid density are set equal to unity. Repeated indices denote summation, whereby we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

The problem of finding exact solutions of non-linear equations (1.1) is an important but rather complicated one. There are some ways to solve it. Considerable progress in this field can be achieved by means of making use of a symmetry approach. Equations (1.1) have non-trivial symmetry properties. It was known long ago [37, 2] that they are invariant under the eleven-parametric extended Galilei group. Let us denote it by $G_1(1,3)$. This group includes the Galilei group and scale transformations. The Lie algebra $AG_1(1,3)$ of $G_1(1,3)$ is generated by the operators

$$P_0, \quad J_{ab}, \quad D, \quad P_a, \quad G_a,$$

where

$$\begin{aligned} &P_0 = \partial_t, \quad D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - 2p\partial_p, \\ &J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}, \quad a \neq b, \\ &G_a = t\partial_a + \partial_{u^a}, \quad P_a = \partial_a. \end{aligned}$$

Relatively recently it was found by means of the Lie method [8, 5, 26] that the maximal Lie invariance algebra (MIA) of the NSEs (1.1) is the infinite-dimensional algebra $A(\mathrm{NS})$ with the basis elements

$$\partial_t, \quad D, \quad J_{ab}, \quad R(\vec{m}), \quad Z(\chi), \tag{1.2}$$

Reprinted with permission from J. Nonlinear Math. Phys., 1994, **1**, № 1, P. 75–113; № 2, P. 156–188

where

$$R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p, \tag{1.3}$$

$$Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p, \tag{1.4}$$

$m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (degree of their smoothness is discussed in Note A.1.).

The algebra $AG_1(1,3)$ is a subalgebra of $A(\mathrm{NS})$. Indeed, setting $m^a = \delta_{ab}$, where b is fixed, we obtain $R(\vec{m}) = \partial_b$, and if $m^a = \delta_{ab}t$ then $R(\vec{m}) = G_b$. Here δ_{ab} is the Kronecker symbol ($\delta_{ab} = 1$ if $a = b$, $\delta_{ab} = 0$ if $a \neq b$).

Operators (1.2) generate the following invariance transformations of system (1.1):

$$\begin{aligned}
&\partial_t: && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t+\varepsilon,\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t+\varepsilon,\vec{x}) \\
& && \text{(translations with respect to } t\text{)}, \\
&J_{ab}: && \tilde{\vec{u}}(t,\vec{x}) = B\vec{u}(t,B^T\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t,B^T\vec{x}) \\
& && \text{(space rotations)}, \\
&D: && \tilde{\vec{u}}(t,\vec{x}) = e^{\varepsilon}\vec{u}(e^{2\varepsilon}t,e^{\varepsilon}\vec{x}), \quad \tilde{p}(t,\vec{x}) = e^{2\varepsilon}p(e^{2\varepsilon}t,e^{\varepsilon}\vec{x}) \\
& && \text{(scale transformations)}, \\
&R(\vec{m}): && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t,\vec{x}-\vec{m}(t)) + \vec{m}_t(t), \\
& && \tilde{p}(t,\vec{x}) = p(t,\vec{x}-\vec{m}(t)) - \vec{m}_{tt}\cdot\vec{x} - \tfrac{1}{2}\vec{m}\cdot\vec{m}_{tt} \\
& && \text{(these transformations include the space translations} \\
& && \text{and the Galilei transformations)}, \\
&Z(\chi): && \tilde{\vec{u}}(t,\vec{x}) = \vec{u}(t,\vec{x}), \quad \tilde{p}(t,\vec{x}) = p(t,\vec{x}) + \chi(t).
\end{aligned} \tag{1.5}$$

Here $\varepsilon \in \mathbb{R}$, $B = \{\beta_{ab}\} \in O(3)$, i.e. $BB^T = \{\delta_{ab}\}$, B^T is the transposed matrix.

Besides continuous transformations (1.5) the NSEs admit discrete transformations of the form

$$\begin{aligned}
&\tilde{t} = t, \quad \tilde{x}_a = x_a,\ a \neq b, \quad \tilde{x}_b = -x_b, \\
&\tilde{p} = p, \quad \tilde{u}^a = u^a,\ a \neq b, \quad \tilde{u}^b = -u^b,
\end{aligned} \tag{1.6}$$

where b is fixed. Invariance under transformations (1.5) and (1.6) means that $(\tilde{\vec{u}},\tilde{p})$ is a solution of (1.1) if $(\vec{u},p)$ is a solution of (1.1).

A complete review of exact solutions found for the NSEs before 1963 is contained in [1]. We should like also to mark more modern reviews [16, 7, 36] despite their subjects slightly differ from subjects of our investigations. To find exact solutions of (1.1), symmetry approach in explicit form was used in [2, 31, 32, 6, 20, 21, 4, 17, 15, 12, 10, 11, 30]. This article is a continuation and a extention of our works [15, 12, 10, 11, 30]. In it we make symmetry reduction of the NSEs to systems of PDEs in three and two independent variables and to systems of ODEs, using subalgebraic structure of $A(\mathrm{NS})$. We investigate symmetry properties of the reduced systems of PDEs and construct exact solutions of the reduced systems of ODEs when it is possible. As a result, large classes of exact solutions of the NSEs are obtained.

The reduction problem for the NSEs is to describe ansatzes of the form [9]:

$$u^a = f^{ab}(t,\vec{x})v^b(\omega) + g^a(t,\vec{x}), \quad p = f^0(t,\vec{x})q(\omega) + g^0(t,\vec{x}) \tag{1.7}$$

that reduce system (1.1) in four independent variables to systems of differential equations in the functions v^a and q depending on the variables $\omega = \{\omega_n\}$ $(n = \overline{1,N})$, where N takes on a fixed value from the set $\{1,2,3\}$. In formulas (1.7) f^{ab}, g^a, f^0, g^0, and ω_n are smooth functions to be described. In such a general formulation the reduction problem is too complex to solve. But using Lie symmetry, some ansatzes (1.7) reducing the NSEs can be obtained. According to the Lie method, first a complete set of A(NS)-inequivalent subalgebras of dimension $M = 4 - N$ is to be constructed. For $N = 3$, $N = 2$, and $N = 1$ such sets are given in Subsections A.2, A.3, and A.4, correspondingly. Knowing subalgebraic structure of A(NS), one can find explicit forms for the functions f^{ab}, g^a, f^0, g^0, and ω_n and obtain reduced systems in the functions v^k and q. This is made in Section 2 ($N = 3$), Section 3 ($N = 2$) and Section 4 ($N = 1$). Moreover, in Subsection 2.3 symmetry properties of all reduced systems of PDEs in three independent variables are investigated, and in Subsection 4.3 exact solutions of the reduced systems of ODEs are constructed. Symmetry properties and exact solutions of some reduced systems of PDEs in two independent variables are discussed in Sections 4 and 6. In Section 7 we make symmetry reduction of a some reduced system of PDEs in three independent variables.

In conclusion of the section, for convenience, we give some abbreviations, notations, and default rules used in this article.

Abbreviations:

NSEs: the Navier–Stokes equations

MIA: the maximal Lie invariance algebra (of either a some equation or a some system of equations)

ODE: ordinary differential equation

PDE: partial differential equation

Notations:

$C^\infty((t_0,t_1),\mathbb{R})$**:** the set of infinite-differentiable functions from (t_0,t_1) into $\mathbb{R}$, where $-\infty \le t_0 < t_1 \le +\infty$

$C^\infty((t_0,t_1),\mathbb{R}^3)$**:** the set of infinite-differentiable vector-functions from (t_0,t_1) into $\mathbb{R}^3$, where $-\infty \le t_0 < t_1 \le +\infty$

$\partial_t = \partial/\partial_t$, $\partial_a = \partial/\partial_{x_a}$, $\partial_{u^a} = \partial/\partial_{u^a}$, ...

Default rules:

Repeated indices denote summation whereby we consider the indices a, b to take on values in $\{1,2,3\}$ and the indices i, j to take on values in $\{1,2\}$.

All theorems on the MIAs of PDEs are proved by means of the standard Lie algorithm.

Subscripts of functions denote differentiation.

2. Reduction of the Navier–Stokes equations to systems of PDEs in three independent variables

2.1. Ansatzes of codimension one

In this subsection we give ansatzes that reduce the NSEs to systems of PDEs in three independent variables. The ansatzes are constructed with the subalgebraic analysis of $A(\mathrm{NS})$ (see Subsection A.2) by means of the method described in Section B.

1. $$\begin{aligned} u^1 &= |t|^{-1/2}(v^1\cos\tau - v^2\sin\tau) + \tfrac{1}{2}x_1t^{-1} - \varkappa x_2t^{-1}, \\ u^2 &= |t|^{-1/2}(v^1\sin\tau + v^2\cos\tau) + \tfrac{1}{2}x_2t^{-1} + \varkappa x_1t^{-1}, \\ u^3 &= |t|^{-1/2}v^3 + \tfrac{1}{2}x_3t^{-1}, \\ p &= |t|^{-1}q + \tfrac{1}{2}\varkappa^2t^{-2}r^2 + \tfrac{1}{8}t^{-2}x_ax_a, \end{aligned} \tag{2.1}$$

where

$$y_1 = |t|^{-1/2}(x_1\cos\tau + x_2\sin\tau), \quad y_2 = |t|^{-1/2}(-x_1\sin\tau + x_2\cos\tau),$$
$$y_3 = |t|^{-1/2}x_3, \quad \varkappa \geq 0, \quad \tau = \varkappa\ln|t|.$$

Here and below $v^a = v^a(y_1,y_2,y_3)$, $q = q(y_1,y_2,y_3)$, $r = (x_1^2+x_2^2)^{1/2}$.

2. $$\begin{aligned} u^1 &= v^1\cos\varkappa t - v^2\sin\varkappa t - \varkappa x_2, \\ u^2 &= v^1\sin\varkappa t + v^2\cos\varkappa t + \varkappa x_1, \\ u^3 &= v^3, \\ p &= q + \tfrac{1}{2}\varkappa^2r^2, \end{aligned} \tag{2.2}$$

where

$$y_1 = x_1\cos\varkappa t + x_2\sin\varkappa t, \quad y_2 = -x_1\sin\varkappa t + x_2\cos\varkappa t,$$
$$y_3 = x_3, \quad \varkappa \in \{0;1\}.$$

3. $$\begin{aligned} u^1 &= x_1r^{-1}v^1 - x_2r^{-1}v^2 + x_1r^{-2}, \\ u^2 &= x_2r^{-1}v^1 + x_1r^{-1}v^2 + x_2r^{-2}, \\ u^3 &= v^3 + \eta(t)r^{-1}v^2 + \eta_t(t)\arctan x_2/x_1, \\ p &= q - \tfrac{1}{2}\eta_{tt}(t)(\eta(t))^{-1}x_3^2 - \tfrac{1}{2}r^{-2} + \chi(t)\arctan x_2/x_1, \end{aligned} \tag{2.3}$$

where

$$y_1 = t, \quad y_2 = r, \quad y_3 = x_3 - \eta(t)\arctan x_2/x_1, \quad \eta,\chi \in C^\infty((t_0,t_1),\mathbb{R}).$$

Note 2.1. The expression for the pressure p from ansatz (2.3) is indeterminate in the points $t \in (t_0,t_1)$ where $\eta(t) = 0$. If there are such points t, we will consider ansatz (2.3) on the intervals (t_0^n, t_1^n) that are contained in the interval (t_0,t_1) and that satisfy one of the conditions:

a) $\eta(t) \neq 0 \quad \forall\, t \in (t_0^n, t_1^n)$;

b) $\eta(t) = 0 \quad \forall\, t \in (t_0^n, t_1^n)$.

In the last case we consider $\eta_{tt}/\eta := 0$.

$$4.\quad \vec u = v^i\vec n^i + (\vec m\cdot\vec m)^{-1}v^3\vec m + (\vec m\cdot\vec m)^{-1}(\vec m\cdot\vec x)\vec m_t - y_i\vec n^i_t,$$
$$p = q - \tfrac{3}{2}(\vec m\cdot\vec m)^{-1}((\vec m_t\cdot\vec n^i)y_i)^2 - (\vec m\cdot\vec m)^{-1}(\vec m_{tt}\cdot\vec x)(\vec m\cdot\vec x) + \qquad (2.4)$$
$$+\tfrac{1}{2}(\vec m_{tt}\cdot\vec m)(\vec m\cdot\vec m)^{-2}(\vec m\cdot\vec x)^2,$$

where

$$y_i = \vec n^i\cdot\vec x,\quad y_3 = t,\quad \vec m, \vec n^i\in C^\infty((t_0,t_1),\mathbb{R}^3).$$
$$\vec n^i\cdot\vec m = \vec n^1\cdot\vec n^2 = \vec n^1_t\cdot\vec n^2 = 0,\quad |\vec n^i| = 1. \qquad (2.5)$$

Note 2.2. There exist vector-functions $\vec n^i$ which satisfy conditions (2.5). They can be constructed in the following way: let us fix the vector-functions $\vec k^i = \vec k^i(t)$ such that $\vec k^i\cdot\vec m = \vec k^1\cdot\vec k^2 = 0$, $|\vec k^i| = 1$, and set

$$\begin{aligned}\vec n^1 &= \vec k^1\cos\psi(t) - \vec k^2\sin\psi(t),\\ \vec n^2 &= \vec k^1\sin\psi(t) + \vec k^2\cos\psi(t).\end{aligned} \qquad (2.6)$$

Then $\vec n^1_t\cdot\vec n^2 = \vec k^1_t\cdot\vec k^2 - \psi_t = 0$ if $\psi = \int(\vec k^1_t\cdot\vec k^2)dt$.

2.2. Reduced systems

1–2. Substituting ansatzes (2.1) and (2.2) into the NSEs (1.1), we obtain reduced systems of PDEs with the same general form

$$\begin{aligned}&v^a v^1_a - v^1_{aa} + q_1 + \gamma_1 v^2 = 0,\\ &v^a v^2_a - v^2_{aa} + q_2 - \gamma_1 v^1 = 0,\\ &v^a v^3_a - v^3_{aa} + q_3 = 0,\\ &v^a_a = \gamma_2.\end{aligned} \qquad (2.7)$$

Hereafter subscripts 1, 2, and 3 of functions denote differentiation with respect to y_1, y_2, and y_3, accordingly. The constants γ_i take the values

1. $\gamma_1 = -2\varkappa,\quad \gamma_2 = -\frac{3}{2}$ if $t>0$, $\quad\gamma_1 = 2\varkappa,\quad \gamma_2 = \frac{3}{2}$ if $t<0$.
2. $\gamma_1 = -2\varkappa,\quad \gamma_2 = 0$.

For ansatzes (2.3) and (2.4) the reduced equations have the form

$$3.\quad\begin{aligned}&v^1_1 + v^1v^1_2 + v^3v^1_3 - y_2^{-1}v^2v^2 - \left(v^1_{22} + (1+\eta^2y_2^{-2})v^1_{33}\right) - 2\eta y_2^{-2}v^2_3 + q_2 = 0,\\ &v^2_1 + v^1v^2_2 + v^3v^2_3 + y_2^{-1}v^1v^2 - \left(v^2_{22} + (1+\eta^2y_2^{-2})v^2_{33}\right) +\\ &\quad + 2\eta y_2^{-2}v^1_3 + 2y_2^{-2}v^2 - \eta y_2^{-1}q_3 + \chi y_2^{-1} = 0,\\ &v^3_1 + v^1v^3_2 + v^3v^3_3 - \left(v^3_{22} + (1+\eta^2y_2^{-2})v^3_{33}\right) - 2\eta^2y_2^{-3}v^1_3 + 2\eta_1y_2^{-1}v^2 +\\ &\quad + 2\eta y_2^{-1}(y_2^{-1}v^2)_2 + (1+\eta^2y_2^{-2})q_3 - \eta_{11}\eta^{-1}y_3 - \chi\eta y_2^{-2} = 0,\\ &y_2^{-1}v^1 + v^1_2 + v^3_3 = 0.\end{aligned} \qquad (2.8)$$

$$4.\quad\begin{aligned}&v^i_3 + v^jv^i_j - v^i_{jj} + q_i + \rho^i(y_3)v^3 = 0,\\ &v^3_3 + v^jv^3_j - v^3_{jj} = 0,\\ &v^i_i + \rho^3(y_3) = 0,\end{aligned} \qquad (2.9)$$

where

$$\rho^i = \rho^i(y_3) = 2(\vec{m}\cdot\vec{m})^{-1}(\vec{m}_t\cdot\vec{n}^i),$$
$$\rho^3 = \rho^3(y_3) = (\vec{m}\cdot\vec{m})^{-1}(\vec{m}_t\cdot\vec{m}). \tag{2.10}$$

2.3. Symmetry of reduced systems

Let us study symmetry properties of systems (2.7), (2.8), and (2.9). All results of this subsection are obtained by means of the standard Lie algorithm [28, 27]. First, let us consider system (2.7).

Theorem 2.1. *The MIA of system* (2.7) *is the algebra*

a) $\langle\partial_a, \partial_q, J^1_{12}\rangle$ *if* $\gamma_1 \neq 0$;
b) $\langle\partial_a, \partial_q, J^1_{ab}\rangle$ *if* $\gamma_1 = 0,\ \gamma_2 \neq 0$;
c) $\langle\partial_a, \partial_q, J^1_{ab}, D^1_1\rangle$ *if* $\gamma_1 = \gamma_2 = 0$.

Here $J^1_{ab} = y_a\partial_b - y_b\partial_a + v^a\partial_{v^b} - v^b\partial_{v^a}$, $D^1_1 = y_a\partial_a - v^a\partial_{v^a} - 2q\partial_q$.

Note 2.3. All Lie symmetry operators of (2.7) are induced by operators from $A(\mathrm{NS})$: The operators J^1_{ab} and D^1_1 are induced by J_{ab} and D. The operators $c_a\partial_a$ ($c_a = \mathrm{const}$) and ∂_q are induced by either

$$R(|t|^{1/2}(c_1\cos\tau - c_2\sin\tau, c_1\sin\tau + c_2\cos\tau, c_3)), \quad Z(|t|^{-1}),$$

where $\tau = \varkappa\ln|t|$, for ansats (2.1) or

$$R(c_1\cos\varkappa t - c_2\sin\varkappa t, c_1\sin\varkappa t + c_2\cos\varkappa t, c_3), \quad Z(1)$$

for ansatz (2.2), respectively. Therefore, Lie reductions of system (2.7) give only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of $A(\mathrm{NS})$.

Let us continue to system (2.8). We denote $A^{\max}$ as the MIA of (2.8). Studying symmetry properties of (2.8), one has to consider the following cases:

A. $\eta, \chi \equiv 0$. Then

$$A^{\max} = \langle\partial^1, D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle,$$

where

$$D^1_2 = 2y_1\partial_1 + y_2\partial_2 + y_3\partial_3 - v^a\partial_{v^a} - 2q\partial_q,$$
$$R_1(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3} - \psi_{11}y_3\partial_q, \quad Z^1(\lambda(y_1)) = \lambda(y_1)\partial_q.$$

Here and below $\psi = \psi(y_1)$ and $\lambda = \lambda(y_1)$ are arbitrary smooth functions of $y_1 = t$.

B. $\eta \equiv 0$, $\chi \not\equiv 0$. In this case an extension of $A^{\max}$ exists for $\chi = (C_1y_1 + C_2)^{-1}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$. We can make C_2 vanish by means of equivalence transformation (A.6), i.e., $\chi = Cy_1^{-1}$, where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D^1_2, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

If $C_1 = 0$, $\chi = C = \mathrm{const}$ and

$$A^{\max} = \langle\partial_1, R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

For other values of χ, i.e., when $\chi_{11}\chi \neq \chi_1\chi_1$,

$$A^{\max} = \langle R_1(\psi(y_1)), Z^1(\lambda(y_1))\rangle.$$

C. $\eta \neq 0$. By means of equivalence transformation (A.6) we make $\chi = 0$. In this case an extension of $A^{\max}$ exists for $\eta = \pm|C_1y_1 + C_2|^{1/2}$, where $C_1, C_2 = \text{const}$. Let $C_1 \neq 0$. We can make C_2 vanish by means of equivalence transformation (A.6), i.e., $\eta = C|y_1|^{1/2}$, where $C = \text{const}$. Then

$$A^{\max} = \langle D^1_2, R_2(|y_1|^{1/2}), R_2(|y_1|^{1/2}\ln|y_1|), Z^1(\lambda(y_1))\rangle,$$

where $R_2(\psi(y_1)) = \psi\partial_3 + \psi_1\partial_{v^3}$. If $C_1 = 0$, i.e., $\eta = C = \text{const}$,

$$A^{\max} = \langle \partial^1, \partial_3, y_1\partial_3 + \partial_{v^3} Z^1(\lambda(y_1))\rangle.$$

For other values of η, i.e., when $(\eta^2)_{11} \neq 0$,

$$A^{\max} = \langle R_2(\eta(y_1)), R_2(\eta(y_1)\textstyle\int(\eta(y_1))^{-2}dy_1), Z^1(\lambda(y_1))\rangle.$$

Note 2.4. In all cases considered above the Lie symmetry operators of (2.8) are induced by operators from $A(\mathrm{NS})$: The operators ∂_1, D^1_2, and $Z^1(\lambda(y_1))$ are induced by ∂_t, D, and $Z(\lambda(t))$, respectively. The operator $R(0,0,\psi(t))$ induces the operator $R_1(\psi(y_1))$ for $\eta \equiv 0$ and the operator $R_2(\psi(y_1))$ (if $\psi_{11}\eta - \psi\eta_{11} = 0$) for $\eta \neq 0$. Therefore, the Lie reduction of system (2.8) gives only solutions that can be obtained by reducing the NSEs with two- and three-dimensional subalgebras of $A(\mathrm{NS})$.

When $\eta = \chi = 0$, system (2.8) describes axially symmetric motion of a fluid and can be transformed into a system of two equations for a stream function Ψ^1 and a function Ψ^2 that are determined by

$$\Psi^1_3 = y_2v^1, \quad \Psi^1_2 = -y_2v^3, \quad \Psi^2 = y_2v^2.$$

The transformed system was studied by L.V. Kapitanskiy [20, 21].

Consider system (2.9). Let us introduce the notations

$$\begin{aligned}
&t = y_3, \quad \rho = \textstyle\int \rho^3(t)dt, \quad \bar\psi = (\psi^1,\psi^2),\\
&R_3(\bar\psi) = \psi^i\partial_{y_i} + \psi^i_t\partial_{v^i} - \psi^i_{tt}y_i\partial_q, \quad Z^1(\lambda) = \lambda\partial_q, \quad S = \partial_{v^3} - \rho^i(t)y_i\partial_q,\\
&E(\chi) = 2\chi\partial_t + \chi_t y_i\partial_{y_i} + (\chi_{tt}y_i - \chi_t v^i)\partial_{v^i} - (2\chi_t q + \tfrac12\chi_{ttt}y_jy_j)\partial_q,\\
&J^1_{12} = y_1\partial_2 - y_2\partial_1 + v^1\partial_{v^2} - v^2\partial_{v^1},
\end{aligned}$$

where ψ^i, λ and χ are smooth functions of t.

Theorem 2.2. *For any values of parameter-functions ρ^a system* (2.9) *is invariant with respect to the algebra*

$$A^{\ker} = \langle R_3(\bar\psi),\ Z^1(\lambda),\ S\rangle,$$

where ψ^i and λ run the set of smooth functions of $t = y_3$. Extensions of MIA for system (2.9) *are only in the following cases (for each case we adduce also basis elements from the $A^{\ker}$ compliment in the corresponding MIA):*

1) $\rho^i = 0$: $\quad E(\chi^1)$, $E(\chi^2)$, $v^3\partial_{v^3}$, J^1_{12}, *where $\chi^1 = e^{-\rho}\int e^\rho dt$ and $\chi^2 = e^{-\rho}$;*

2) $\rho^i = \mathrm{const} \neq 0$: $E(\chi^1), E(\chi^2) - 3v^3\partial_{v^3}$, *where* $\chi^1 = e^{-\rho}\int e^{\rho}dt$ *and* $\chi^2 = e^{-\rho}$;

3) $\rho^1 = Ce^{\frac{3}{2}\rho}\hat\rho^{-\frac{3}{2}-a_1}\cos(a_2\ln\hat\rho + \delta)$, $\rho^2 = Ce^{\frac{3}{2}\rho}\hat\rho^{-\frac{3}{2}-a_1}\sin(a_2\ln\hat\rho + \delta)$ *with* $\hat\rho = |\int e^{\rho}dt + a_3|$, $C, \delta = \mathrm{const}$, $C \neq 0$:

$$E(\chi) + 2a_1v^3\partial_{v^3} + 2a_2J^1_{12},$$

where a_1, a_2, *and* a_3 *are fixed constants,* $\chi = e^{-\rho}(\int e^{\rho}dt + a_3)$;

4) $\rho^1 = Ce^{\frac{3}{2}\rho - a_1\hat\rho}\cos(a_2\hat\rho + \delta)$, $\rho^2 = Ce^{\frac{3}{2}\rho - a_1\hat\rho}\sin(a_2\hat\rho + \delta)$ *with* $\hat\rho = \int e^{\rho(t)}dt$, $C, \delta = \mathrm{const}$, $C \neq 0$:

$$E(\chi) + 2a_1v^3\partial_{v^3} + 2a_2J^1_{12},$$

where a_1 *and* a_2 *are fixed constants,* $\chi = e^{-\rho}$.

Note 2.5. If functions ρ^b are determined by (2.10), then $e^{\rho(t)} = C|\vec m(t)|$, where $C = \mathrm{const}$, and the condition $\rho^i = 0$ implies that $\vec m = |\vec m(t)|\vec e$, where $\vec e = \mathrm{const}$ and $|\vec e| = 1$.

Note 2.6. The vector-functions $\vec n^i$ from Note 2.2 are determined up to the transformation

$$\vec n^1 = \vec n^1\cos\delta - \vec n^2\sin\delta, \quad \vec n^2 = \vec n^1\sin\delta + \vec n^2\cos\delta,$$

where $\delta = \mathrm{const}$. Therefore, δ can be chosen such that $C_2 = 0$ (then $C_1 \neq 0$).

Note 2.7. The operators $R_3(\psi^1,\psi^2) + \alpha S$ and $Z^1(\lambda)$ are induced by $R(\vec l) + Z(\chi)$ and $Z(\lambda)$, respectively. Here $\vec l = \psi^i\vec n^i + \psi^3\vec m$, $\psi^3_t(\vec m\cdot\vec m) + 2\psi^i(\vec n^i_t\cdot\vec m) = \alpha$,

$$\chi - \tfrac{3}{2}(\vec m\cdot\vec m)^{-1}((\vec m_t\cdot\vec n^i)\psi^i)^2 - \tfrac{1}{2}(\vec m_{tt}\cdot\vec n^i)\psi^3\psi^i + \tfrac{1}{2}(\vec l_{tt}\cdot\vec n^i)\psi^i = 0.$$

If $\vec m = |\vec m|\vec e$, where $\vec e = \mathrm{const}$ and $|\vec e| = 1$, the operator J^1_{12} is induced by $e^1J_{23} + e^2J_{31} + e^3J_{12}$.

For

$$\vec m = \beta_3e^{\sigma t}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T$$

with $\tau = \varkappa t + \delta$ and $\beta_a = \mathrm{const}$, where $\beta_1^2 + \beta_2^2 = 1$, the operator $\partial_t + \varkappa J_{12}$ induces the operator $\partial_{y_3} - \beta_1\varkappa J^1_{12} + \sigma v^3\partial_{v^3}$ if the following vector-functions $\vec n^i$ are chosen:

$$\vec n^1 = \vec k^1\cos\beta_1\tau + \vec k^2\sin\beta_1\tau, \quad \vec n^2 = -\vec k^1\sin\beta_1\tau + \vec k^2\cos\beta_1\tau, \tag{2.11}$$

where $\vec k^1 = (-\sin\tau, \cos\tau, 0)^T$ and $\vec k^2 = (\beta_1\cos\tau, \beta_1\sin\tau, -\beta_2)^T$.

For

$$\vec m = \beta_3|t + \beta_4|^{\sigma+1/2}(\beta_2\cos\tau, \beta_2\sin\tau, \beta_1)^T$$

with $\tau = \varkappa\ln|t + \beta_4| + \delta$ and $\beta_a, \beta_4 = \mathrm{const}$, where $\beta_1^2 + \beta_2^2 = 1$, the operator $D + 2\beta_4\partial_t + 2\varkappa J_{12}$ induces the operator

$$D^1_3 + 2\beta_4\partial_{y_3} - 2\beta_1\varkappa J^1_{12} + 2\sigma v^3\partial_{v^3},$$

where $D^1_3 = y_i\partial_{y_i} + 2y_3\partial_{y_3} - v^i\partial_{v^i} - 2q\partial_q$, if the vector-functions $\vec n^i$ are chosen in form (2.11). In all other cases the basis elements of the MIA of (2.9) are not induced by operators from $A(\mathrm{NS})$.

Note 2.8. The invariance algebras of systems of form (2.9) with different parameter-functions $\rho^3 = \rho^3(t)$ and $\tilde\rho^3 = \tilde\rho^3(t)$ are similar. It suggests that there exists a local transformation of variables which make ρ^3 vanish. So, let us transform variables in the following way:

$$\begin{aligned}&\tilde y_i = y_i e^{\frac12\rho(t)}, \quad \tilde y_3 = \int e^{\rho(t)}dt,\\ &\tilde v^i = \big(v^i + \tfrac12 y_i\rho^3(t)\big)e^{-\frac12\rho(t)}, \quad \tilde v^3 = v^3,\\ &\tilde q = qe^{-\rho(t)} + \tfrac18 y_iy_i\big((\rho^3(t)^2) - 2\rho^3_t(t)\big)e^{-\rho(t)}.\end{aligned} \tag{2.12}$$

As a result, we obtain the system

$$\begin{aligned}&\tilde v^i_3 + \tilde v^j\tilde v^i_j - \tilde v^i_{jj} + \tilde q_i + \tilde\rho^i(\tilde y_3)\tilde v^3 = 0,\\ &\tilde v^3_3 + \tilde v^j v^3_j - \tilde v^3_{jj} = 0,\\ &\tilde v^i_i = 0\end{aligned}$$

for the functions $\tilde v^a = \tilde v^a(\tilde y_1, \tilde y_2, \tilde y_3)$ and $\tilde q = \tilde q(\tilde y_1, \tilde y_2, \tilde y_3)$. Here subscripts 1, 2, and 3 denote differentiation with respect to $\tilde y_1$, $\tilde y_2$, and $\tilde y_3$, accordingly. Also $\tilde{\rho^i}(\tilde y_3) = \rho^i(t)e^{-\frac32\rho(t)}$.

3. Reduction of the Navier–Stokes equations to systems of PDEs in two independent variables

3.1. Ansatzes of codimension two

In this subsection we give ansatzes that reduce the NSEs to systems of PDEs in two independent variables. The ansatzes are constructed with the subalgebrical analysis of A(NS) (see Subsection A.3) by means of the method described in Section B.

$$\text{1.}\quad\begin{aligned}&u^1 = (rR)^{-1}((x_1 - \varkappa x_2)w^1 - x_2w^2 + x_1x_3r^{-1}w^3),\\ &u^2 = (rR)^{-1}((x_2 + \varkappa x_1)w^1 + x_1w^2 + x_2x_3r^{-1}w^3),\\ &u^3 = x_3(rR)^{-1}w^1 - R^{-1}w^3,\\ &p = R^{-2}s,\end{aligned} \tag{3.1}$$

where $z_1 = \arctan x_2/x_1 - \varkappa\ln R$, $z_2 = \arctan r/x_3$, $\varkappa \ge 0$.

Here and below $w^a = w^a(z_1, z_2)$, $s = s(z_1, z_2)$, $r = (x_1^2 + x_2^2)^{1/2}$, $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $\varkappa$, ε, σ, μ, and ν are real constants.

$$\text{2.}\quad\begin{aligned}&u^1 = |t|^{-1/2}r^{-1}(x_1w^1 - x_2w^2) + \tfrac12 t^{-1}x_1 + x_1r^{-2},\\ &u^2 = |t|^{-1/2}r^{-1}(x_2w^1 + x_1w^2) + \tfrac12 t^{-1}x_2 + x_2r^{-2},\\ &u^3 = |t|^{-1/2}w^3 + \varkappa r^{-1}w^2 + \tfrac12 t^{-1}x_3,\\ &p = |t|^{-1}s - \tfrac12 r^{-2} + \tfrac18 t^{-2}R^2 + \varepsilon|t|^{-1}\arctan x_2/x_1,\end{aligned} \tag{3.2}$$

where $z_1 = |t|^{-1/2}r$, $z_2 = |t|^{-1/2}x_3 - \varkappa \arctan x_2/x_1$, $\varkappa \geq 0$, $\varepsilon \geq 0$.

$$\text{3.}\quad \begin{aligned} u^1 &= r^{-1}(x_1w^1 - x_2w^2) + x_1r^{-2}, \\ u^2 &= r^{-1}(x_2w^1 + x_1w^2) + x_2r^{-2}, \\ u^3 &= w^3 + \varkappa r^{-1}w^2, \\ p &= s - \tfrac{1}{2}r^{-2} + \varepsilon \arctan x_2/x_1, \end{aligned} \tag{3.3}$$

where $z_1 = r$, $z_2 = x_3 - \varkappa \arctan x_2/x_1$, $\varkappa \in \{0;1\}$, $\varepsilon \geq 0$ if $\varkappa = 1$ and $\varepsilon \in \{0;1\}$ if $\varkappa = 0$.

$$\text{4.}\quad \begin{aligned} u^1 &= |t|^{-1/2}(\mu w^1 + \nu w^3)\cos\tau - |t|^{-1/2}w^2 \sin\tau + \\ &\quad + \nu\xi t^{-1}\cos\tau + \tfrac{1}{2}t^{-1}x_1 - \varkappa t^{-1}x_2, \\ u^2 &= |t|^{-1/2}(\mu w^1 + \nu w^3)\sin\tau + |t|^{-1/2}w^2 \cos\tau + \\ &\quad + \nu\xi t^{-1}\sin\tau + \tfrac{1}{2}t^{-1}x_2 + \varkappa t^{-1}x_1, \\ u^3 &= |t|^{-1/2}(-\nu w^1 + \mu w^3) + \mu\xi t^{-1} + \tfrac{1}{2}t^{-1}x_3, \\ p &= |t|^{-1}s - \tfrac{1}{2}t^{-2}\xi^2 + \tfrac{1}{8}t^{-2}R^2 + \tfrac{1}{2}\varkappa^2 t^{-2}r^2 + \\ &\quad + \varepsilon|t|^{-3/2}(\nu x_1\cos\tau + \nu x_2 \sin\tau + \mu x_3), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} &z_1 = |t|^{-1/2}(\mu x_1 \cos\tau + \mu x_2 \sin\tau - \nu x_3), \\ &z_2 = |t|^{-1/2}(x_2\cos\tau - x_1\sin\tau), \\ &\xi = \sigma(\nu x_1\cos\tau + \nu x_2\sin\tau + \mu x_3) + 2\varkappa\nu(x_2\cos\tau - x_1\sin\tau), \\ &\tau = \varkappa \ln|t|, \quad \varkappa > 0, \quad \mu \geq 0, \quad \nu \geq 0, \quad \mu^2 + \nu^2 = 1, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0. \end{aligned}$$

$$\text{5.}\quad \begin{aligned} u^1 &= |t|^{-1/2}w^1 + \tfrac{1}{2}t^{-1}x_1, \\ u^2 &= |t|^{-1/2}w^2 + \tfrac{1}{2}t^{-1}x_2, \\ u^3 &= |t|^{-1/2}w^3 + (\sigma + \tfrac{1}{2})t^{-1}x_3, \\ p &= |t|^{-1}s - \tfrac{1}{2}\sigma^2 t^{-2}x_3^2 + \tfrac{1}{8}t^{-2}R^2 + \varepsilon|t|^{-3/2}x_3, \end{aligned} \tag{3.5}$$

where

$$z_1 = |t|^{-1/2}x_1, \quad z_2 = |t|^{-1/2}x_2, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0.$$

$$\text{6.}\quad \begin{aligned} u^1 &= (\mu w^1 + \nu w^3)\cos t - w^2 \sin t + \nu\xi\cos t - x_2, \\ u^2 &= (\mu w^1 + \nu w^3)\sin t + w^2\cos t + \nu\xi\sin t + x_1, \\ u^3 &= (-\nu w^1 + \mu w^3) + \mu\xi, \\ p &= s - \tfrac{1}{2}\xi^2 + \tfrac{1}{2}r^2 + \varepsilon(\nu x_1\cos t + \nu x_2 \sin t + \mu x_3), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} &z_1 = (\mu x_1\cos t + \mu x_2 \sin t - \nu x_3), \\ &z_2 = (x_2\cos t - x_1 \sin t), \\ &\xi = \sigma(\nu x_1 \cos t + \nu x_2\sin t + \mu x_3) + 2\nu(x_2\cos t - x_1\sin t), \\ &\mu \geq 0, \quad \nu \geq 0, \quad \mu^2 + \nu^2 = 1, \quad \sigma\varepsilon = 0, \quad \varepsilon \geq 0. \end{aligned}$$

$$\begin{aligned} 7.\quad & u^1 = w^1, \quad u^2 = w^2, \quad u^3 = w^3 + \sigma x_3, \\ & p = s - \tfrac{1}{2}\sigma^2 x_3^2 + \varepsilon x_3, \end{aligned} \tag{3.7}$$

where

$$z_1 = x_1, \quad z_2 = x_2, \quad \sigma\varepsilon = 0, \quad \varepsilon \in \{0; 1\}.$$

$$\begin{aligned} 8.\quad & u^1 = x_1 w^1 - x_2 r^{-2}(w^2 - \chi(t)), \\ & u^2 = x_2 w^1 + x_1 r^{-2}(w^2 - \chi(t)), \\ & u^3 = (\rho(t))^{-1}(w^3 + \rho_t(t) x_3 + \varepsilon \arctan x_2/x_1), \\ & p = s - \tfrac{1}{2}\rho_{tt}(t)(\rho(t))^{-1} x_3^2 + \chi_t(t) \arctan x_2/x_1, \end{aligned} \tag{3.8}$$

where

$$z_1 = t, \quad z_2 = r, \quad \varepsilon \in \{0; 1\}, \quad \chi, \rho \in C^\infty((t_0, t_1), \mathbb{R}).$$

$$\begin{aligned} 9.\quad & \vec{u} = \vec{w} + \lambda^{-1}(\vec{n}^i \cdot \vec{x})\vec{m}^i_t - \lambda^{-1}(\vec{k} \cdot \vec{x})\vec{k}_t, \\ & p = s - \tfrac{1}{2}\lambda^{-1}(\vec{m}^i_{tt} \cdot \vec{x})(\vec{n}^i \cdot \vec{x}) - \tfrac{1}{2}\lambda^{-2}(m^i_{tt} \cdot \vec{k})(\vec{n}^i \cdot \vec{x})(\vec{k} \cdot \vec{x}), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} & z_1 = t, \quad z_2 = (\vec{k} \cdot \vec{x}), \quad \vec{m}^i \in C^\infty((t_0, t_1), \mathbb{R}^3), \\ & \vec{m}^1_{tt} \cdot \vec{m}^2 - \vec{m}^1 \cdot \vec{m}^2_{tt} = 0, \quad \vec{k} = \vec{m}^1 \times \vec{m}^2, \quad \vec{n}^1 = \vec{m}^2 \times \vec{k}, \\ & \vec{n}^2 = \vec{k} \times \vec{m}^1, \quad \lambda = \lambda(t) = \vec{k} \cdot \vec{k} \neq 0 \quad \forall\, t \in (t_0, t_1). \end{aligned}$$

3.2. Reduced systems

Substituting ansatzes (3.1)–(3.9) into the NSEs (1.1), we obtain the following systems of reduced equations:

$$\begin{aligned} 1.\quad & w^2 w^1_1 + w^3 w^1_2 - w^1 w^3 \cot z_2 - (w^1)^2 - (w^2 + \varkappa w^1)^2 \sin^2 z_2 - \\ & - (w^3)^2 - \big((\varkappa^2 + \sin^{-2} z_2) w^1_{11} + w^1_{22} - \varkappa w^1_1 - 2w^3_2 - 2w^2_1 - \\ & - 2w^1\big) \sin z_2 + w^1_2 \cos z_2 - w^1 \sin^{-1} z_2 - (2s + \varkappa s_1) \sin^2 z_2 = 0, \\ & w^2 w^2_1 + w^3 w^2_2 + w^3(w^2 + 2\varkappa w^1) \cot z_2 - \\ & - \varkappa\big((w^1)^2 + (w^3)^2 + (w^2 + \varkappa w^1)^2 \sin^2 z_2\big) - \\ & - \big((\varkappa^2 + \sin^{-2} z_2) w^2_{11} + w^2_{22} + 3\varkappa w^2_1 + 2\varkappa(w^3_2 + \varkappa w^1_1 + w^1)\big) \sin z_2 + \\ & + (2w^1_1 + 2w^3_1 \cot z_2 - w^2 - 2\varkappa w^1) \sin^{-1} z_2 - \\ & - (w^2_2 + 2\varkappa w^1_2) \cos z_2 + 2\varkappa s \sin^2 z_2 + (1 + \varkappa^2 \sin^2 z_2) s_1 = 0, \\ & w^2 w^3_1 + w^3 w^3_2 - (w^3)^2 \cot z_2 - (w^2 + \varkappa w^1)^2 \sin z_2 \cos z_2 - \\ & - \big((\varkappa^2 + \sin^{-2} z_2) w^3_{11} + w^3_{22} + \varkappa w^3_1 + 2w^1_2\big) \sin z_2 + \\ & + (2w^1 + w^3_2 + w^2_1 + \varkappa w^1_1) \cos z_2 + s_2 \sin^2 z_2 = 0, \\ & w^1 + w^2_1 + w^3_2 = 0. \end{aligned} \tag{3.10}$$

Hereafter numeration of the reduced systems corresponds to that of the ansatzes in Subsection 3.1. Subscripts 1 and 2 denote differentiation with respect to the variables z_1 and z_2, accordingly.

$$
\begin{aligned}
\text{2–3.}\quad & w^1w^1_1 + w^3w^1_2 - z_1^{-1}w^2w^2 - \big(w^1_{11} + (1+\varkappa^2z_1^{-2})w^1_{22}\big) - \\
& - 2\varkappa z_1^{-2}w^2_2 + s_1 = 0, \\
& w^1w^2_1 + w^3w^2_2 + z_1^{-1}w^1w^2 - \big(w^2_{11} + (1+\varkappa^2z_1^{-2})w^2_{22}\big) + \\
& + 2\varkappa z_1^{-2}w^1_2 + 2z_1^{-2}w^2 - \varkappa z_1^{-1}s_2 + \varepsilon z_1^{-1} = 0, \\
& w^1w^3_1 + w^3w^3_2 - 2\varkappa z_1^{-2}w^1w^2 - \big(w^3_{11} + (1+\varkappa^2z_1^{-2})w^3_{22}\big) + \\
& + 2\varkappa(z_1^{-2}w^2)_1 - 2\varkappa^2z_1^{-3}w^1_2 + (1+\varkappa^2z_1^{-2})s_2 - \varepsilon\varkappa z_1^{-2} = 0, \\
& w^1_1 + w^3_2 + z_1^{-1}w^1 + \gamma = 0,
\end{aligned}
\tag{3.11}
$$

where $\gamma = \pm 3/2$ for ansatz (3.2) and $\gamma = 0$ for ansatz (3.3). Here and below the upper and lower sign in the symbols "$\pm$" and "$\mp$" are associated with $t>0$ and $t<0$, respectively.

4–7. For ansatzes (3.4)–(3.7) the reduced equations can be written in the form

$$
\begin{aligned}
& w^iw^1_i - w^1_{ii} + s_1 + \alpha_2w^2 = 0, \\
& w^iw^2_i - w^2_{ii} + s_2 - \alpha_2w^1 + \alpha_1w^3 = 0, \\
& w^iw^3_i - w^3_{ii} + \alpha_4w^3 + \alpha_5 = 0, \\
& w^i_i = \alpha_3
\end{aligned}
\tag{3.12}
$$

where the constants α_n $(n=\overline{1,5})$, take on the values

4.	$\alpha_1 = \pm 2\varkappa\nu$,	$\alpha_2 = \mp 2\varkappa\mu$,	$\alpha_3 = \mp(\sigma + 3/2)$,	$\alpha_4 = \pm\sigma$,	$\alpha_5 = \varepsilon$.
5.	$\alpha_1 = 0$,	$\alpha_2 = 0$,	$\alpha_3 = \mp(\sigma + 3/2)$,	$\alpha_4 = \pm\sigma$,	$\alpha_5 = \varepsilon$.
6.	$\alpha_1 = 2\nu$,	$\alpha_2 = -2\mu$,	$\alpha_3 = -\sigma$,	$\alpha_4 = \sigma$,	$\alpha_5 = \varepsilon$.
7.	$\alpha_1 = 0$,	$\alpha_2 = 0$,	$\alpha_3 = -\sigma$,	$\alpha_4 = \sigma$,	$\alpha_5 = \varepsilon$.

$$
\begin{aligned}
\text{8.}\quad & w^1_1 + (w^1)^2 - z_2^{-4}(w^2-\chi)^2 + z_2w^1w^1_2 - w^1_{22} - \\
& - 3z_2w^1_2 + z_2^{-1}s_2 = 0,
\end{aligned}
\tag{3.13}
$$

$$
w^2_1 + z_2w^1w^2_2 - w^2_{22} + z_2^{-1}w^2_2 = 0,
\tag{3.14}
$$

$$
w^3_1 + z_2w^1w^3_2 - w^3_{22} - z_2^{-1}w^3_2 + z_2^{-2}(w^2-\chi) = 0,
\tag{3.15}
$$

$$
2w^1 + z_2w^1_2 + \rho_1/\rho = 0.
\tag{3.16}
$$

$$
\text{9.}\quad \vec{w}_1 - \lambda\vec{w}_{22} + s_2\vec{k} + \lambda^{-1}(\vec{n}^i\cdot\vec{w})\vec{m}^i_t + z_2\vec{e} = \vec{0},
\tag{3.17}
$$

$$
\vec{k}\cdot\vec{w}_2 = 0,
\tag{3.18}
$$

where $y_1 = t$ and

$$
\vec{e} = \vec{e}(t) = 2\lambda^{-2}(\vec{m}^1_t\cdot\vec{m}^2 - \vec{m}^1\cdot\vec{m}^2_t)\vec{k}_t\times\vec{k} + \lambda^{-2}(2\vec{k}_t\cdot\vec{k}_t - \vec{k}_{tt}\cdot\vec{k}).
$$

Let us study symmetry properties of reduced systems (3.10) and (3.11).

Theorem 3.1. *The MIA of* (3.10) *is given by the algebra* $\langle\partial_1\rangle$.

Theorem 3.2. *The MIA of* (3.11) *is given by the following algebras:*

$$\begin{array}{ll} a) & \langle\partial_2, \partial_s, D_1^2 = z_i\partial_i - w^a\partial_{w^a} - 2s\partial_s\rangle \quad \textit{if} \quad \gamma = \varkappa = \varepsilon = 0; \\ b) & \langle\partial_2, \partial_s\rangle \quad \textit{if} \quad (\gamma, \varkappa, \varepsilon) \neq (0, 0, 0). \end{array}$$

All the Lie symmetry operators of systems (3.10) and (3.11) are induced by elements of $A(\mathrm{NS})$. So, for system (3.10) the operator ∂_1 is induced by J_{12}. For system (3.11), when $\gamma = 0$ ($\gamma = \pm 3/2$), the operators D_1^2, ∂_2, and ∂_s (∂_2 and ∂_s) are induced by D, $R(0,0,1)$, and $Z(1)$ ($R(0,0,|t|^{-1/2})$ and $Z(|t|^{-1})$), accordingly. Therefore, the Lie reductions of systems (3.10) and (3.11) give only solutions that can be obtained by reducing the NSEs with three-dimensional subalgebras of $A(\mathrm{NS})$ immediately to ODEs.

Investigation of reduced systems (3.13)–(3.16), (3.17)–(3.18), and (3.12) is given in Sections 5 and 6.

4. Reduction of the Navier–Stokes equations to ordinary differential equations

4.1. Ansatzes of codimension three

By means of subalgebraic analysis of $A(\mathrm{NS})$ (see Subsection A.3) and the method described in Section B one can obtain the following ansatzes that reduce the NSEs to ODEs:

$$\begin{array}{ll} 1. & u^1 = x_1R^{-2}\varphi^1 - x_2(Rr)^{-1}\varphi^2 + x_1x_3r^{-1}R^{-2}\varphi^3, \\ & u^2 = x_2R^{-2}\varphi^1 + x_1(Rr)^{-1}\varphi^2 + x_2x_3r^{-1}R^{-2}\varphi^3, \\ & u^3 = x_3R^{-2}\varphi^1 - rR^{-2}\varphi^3, \\ & p = R^{-2}h, \end{array} \tag{4.1}$$

where $\omega = \arctan r/x_3$. Here and below $\varphi^a = \varphi^a(\omega)$, $h = h(\omega)$, $r = (x_1^2 + x_2^2)^{1/2}$, $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$.

$$\begin{array}{ll} 2. & u^1 = r^{-2}(x_1\varphi^1 - x_2\varphi^2), \quad u^2 = r^{-2}(x_2\varphi^1 + x_1\varphi^2), \\ & u^3 = r^{-1}\varphi^3, \quad p = r^{-2}h, \end{array} \tag{4.2}$$

where $\omega = \arctan x_2/x_1 - \varkappa\ln r$, $\varkappa \geq 0$.

$$\begin{array}{ll} 3. & u^1 = x_1|t|^{-1}\varphi^1 - x_2r^{-2}\varphi^2 + \frac{1}{2}x_1t^{-1}, \\ & u^2 = x_2|t|^{-1}\varphi^1 + x_1r^{-2}\varphi^2 + \frac{1}{2}x_2t^{-1}, \\ & u^3 = |t|^{-1/2}\varphi^3 + (\sigma + \frac{1}{2})x_3t^{-1} + \nu|t|^{1/2}t^{-1}\arctan x_2/x_1, \\ & p = |t|^{-1}h + \frac{1}{8}t^{-2}R^2 - \frac{1}{2}\sigma^2x_3^2t^{-2} + \\ & \qquad + \varepsilon_1|t|^{-1}\arctan x_2/x_1 + \varepsilon_2x_3|t|^{-3/2}, \end{array} \tag{4.3}$$

where $\omega = |t|^{-1/2} r, \quad \nu\sigma = 0,\ \varepsilon_2\sigma = 0,\ \varepsilon_1 \geq 0,\ \nu \geq 0.$

$$
\begin{aligned}
4.\quad & u^1 = x_1\varphi^1 - x_2 r^{-2}\varphi^2,\\
& u^2 = x_2\varphi^1 + x_1 r^{-2}\varphi^2,\\
& u^3 = \varphi^3 + \sigma x_3 + \nu \arctan x_2/x_1,\\
& p = h - \tfrac{1}{2}\sigma^2 x_3^2 + \varepsilon_1 \arctan x_2/x_1 + \varepsilon_2 x_3,
\end{aligned}
\tag{4.4}
$$

where $\omega = r$, $\nu\sigma = 0$, $\varepsilon_2\sigma = 0$, and for $\sigma = 0$ one of the conditions

$$
\nu = 1,\ \varepsilon_1 \geq 0; \quad \nu = 0,\ \varepsilon_1 = 1,\ \varepsilon_2 \geq 0; \quad \nu = \varepsilon_1 = 0,\ \varepsilon_2 \in \{0; 1\}
$$

is satisfied.

Two ansatzes are described better in the following way:

5. The expressions for u^a and p are determined by (2.1), where

$$
\begin{aligned}
& v^1 = a_1\varphi^1 + a_2\varphi^3 + b_{1i}\omega_i,\\
& v^2 = \varphi^2 + b_{2i}\omega_i,\\
& v^3 = a_2\varphi^1 - a_1\varphi^3 + b_{3i}\omega_i,\\
& p = h + c_{1i}\omega_i + c_{2i}\omega\omega_i + \tfrac{1}{2}d_{ij}\omega_i\omega_j.
\end{aligned}
\tag{4.5}
$$

In formulas (4.5) we use the following definitions:

$$
\begin{aligned}
& \omega_1 = a_1 y_1 + a_2 y_3, \quad \omega_2 = y_2, \quad \omega = \omega_3 = a_2 y_1 - a_1 y_3;\\
& a_i = \text{const}, \quad a_1^2 + a_2^2 = 1; \quad a_2 = 0 \text{ if } \gamma_1 = 0;\\
& \gamma_1 = -2\varkappa,\ \gamma_2 = -\tfrac{3}{2} \quad \text{if} \quad t > 0 \quad \text{and} \quad \gamma_1 = 2\varkappa,\ \gamma_2 = \tfrac{3}{2} \quad \text{if} \quad t < 0.
\end{aligned}
$$

b_{ai}, B_i, c_{ij}, and d_{ij} are real constants that satisfy the equations

$$
\begin{aligned}
& b_{1i} = a_1 B_i, \quad b_{3i} = a_2 B_i, \quad c_{2i} + a_2\gamma_1 b_{2i} = 0,\\
& b_{21}B_i + b_{22}b_{2i} - \gamma_1 a_1 B_i + d_{2i} = 0,\\
& B_1 B_i + B_2 b_{2i} + \gamma_1 a_1 B_i + d_{1i} = 0,\\
& (B_1 + b_{22})(B_2 + a_1\gamma_1 - b_{21}) = 0.
\end{aligned}
\tag{4.6}
$$

6. The expressions for u^a and p have form (2.2), where v^a and q are determined by (4.5), (4.6), and $\gamma_1 = -2\varkappa$, $\gamma_2 = 0$.

Note 4.1. Formulas (4.5) and (4.6) determine an ansatz for system (2.7), where equations (4.6) are the necessary and sufficient condition to reduce system (2.7) by means of an ansatz of form (4.5).

$$
\begin{aligned}
7.\quad & u^1 = \varphi^1 \cos x_3/\eta^3 - \varphi^2 \sin x_3/\eta^3 + x_1\theta^1(t) + x_2\theta^2(t),\\
& u^2 = \varphi^1 \sin x_3/\eta^3 + \varphi^2 \cos x_3/\eta^3 - x_1\theta^2(t) + x_2\theta^1(t),\\
& u^3 = \varphi^3 + \eta^3_t(\eta^3)^{-1}x_3,\\
& p = h - \tfrac{1}{2}\eta^3_{tt}(\eta^3)^{-1}x_3^2 - \tfrac{1}{2}\eta^j_{tt}\eta^j(\eta^i\eta^i)^{-1}r^2,
\end{aligned}
\tag{4.7}
$$

where $\omega = t$,

$$
\begin{aligned}
& \eta^a \in C^\infty((t_0, t_1), \mathbb{R}), \quad \eta^3 \neq 0, \quad \eta^i\eta^i \neq 0, \quad \eta^1_t\eta^2 - \eta^1\eta^2_t \in \{0; \tfrac{1}{2}\},\\
& \theta^1 = \eta^i_t\eta^i(\eta^j\eta^j)^{-1}, \quad \theta^2 = (\eta^1_t\eta^2 - \eta^1\eta^2_t)(\eta^j\eta^j)^{-1}.
\end{aligned}
$$

$$\begin{aligned} 8.\quad & \vec u = \vec\varphi + \lambda^{-1}(\vec n^a\cdot\vec x)\vec m^a_t, \\ & p = h - \lambda^{-1}(\vec m^a_{tt}\cdot\vec x)(\vec n^a\cdot\vec x) + \tfrac12\lambda^{-2}(\vec m^b_{tt}\cdot\vec m^a)(\vec n^a\cdot\vec x)(\vec n^b\cdot\vec x), \end{aligned} \tag{4.8}$$

where $\omega = t$, $\vec m^a \in C^\infty((t_0,t_1),\mathbb{R})$, $\vec m^a_{tt}\cdot\vec m^b - \vec m^a\cdot\vec m^b_{tt} = 0$,

$$\begin{aligned} & \lambda = \lambda(t) = (\vec m^1\times\vec m^2)\cdot\vec m^3 \neq 0 \quad \forall t\in(t_0,t_1), \\ & \vec n^1 = \vec m^2\times\vec m^3, \quad \vec n^2 = \vec m^3\times\vec m^1, \quad \vec n^3 = \vec m^1\times\vec m^2. \end{aligned}$$

4.2. Reduced systems

Substituting the ansatzes 1–8 into the NSEs (1.1), we obtain the following systems of ODE in the functions φ^a and h:

$$\begin{aligned} 1.\quad & \varphi^3\varphi^1_\omega - \varphi^a\varphi^a - \varphi^1_{\omega\omega} - \varphi^1_\omega\cot\omega - 2h = 0, \\ & \varphi^3\varphi^2_\omega + \varphi^2\varphi^3\cot\omega - \varphi^2_{\omega\omega} - \varphi^2_\omega\cot\omega + \varphi^2\sin^{-2}\omega = 0, \\ & \varphi^3\varphi^3_\omega - \varphi^2\varphi^2\cot\omega - \varphi^3_{\omega\omega} - \varphi^3_\omega\cot\omega + \varphi^3\sin^{-2}\omega - 2\varphi^1_\omega + h_\omega = 0, \\ & \varphi^1 + \varphi^3_\omega + \varphi^3\cot\omega = 0. \end{aligned} \tag{4.9}$$

$$\begin{aligned} 2.\quad & (\varphi^2 - \varkappa\varphi^1)\varphi^1_\omega - (1+\varkappa^2)\varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - \varkappa h_\omega - 2h = 0, \\ & (\varphi^2 - \varkappa\varphi^1)\varphi^2_\omega - (1+\varkappa^2)\varphi^2_{\omega\omega} - 2(\varkappa\varphi^2_\omega + \varphi^1_\omega) + h_\omega = 0, \\ & (\varphi^2 - \varkappa\varphi^1)\varphi^3_\omega - (1+\varkappa^2)\varphi^3_{\omega\omega} - \varphi^1\varphi^3 - \varphi^3 - 2\varkappa\varphi^3_\omega = 0, \\ & \varphi^2_\omega - \varkappa\varphi^1_\omega = 0. \end{aligned} \tag{4.10}$$

$$\begin{aligned} 3\text{–}4.\quad & \varphi^1\varphi^1 - \omega^{-4}\varphi^2\varphi^2 + \omega\varphi^1\varphi^1_\omega - \varphi^1_{\omega\omega} - 3\omega^{-1}\varphi^1_\omega + \omega^{-1}h_\omega = 0, \\ & \omega\varphi^1\varphi^2_\omega - \varphi^2_{\omega\omega} + \omega^{-1}\varphi^2_\omega + \varepsilon_1 = 0, \\ & \omega\varphi^1\varphi^3_\omega + \sigma_1\varphi^3 + \nu\omega^{-2}\varphi^2 - \varphi^3_{\omega\omega} - \omega^{-1}\varphi^3_\omega + \varepsilon_2 = 0, \\ & 2\varphi^1 + \omega\varphi^1_\omega + \sigma_2 = 0, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} 3.\quad & \sigma_1 = \sigma, \qquad \sigma_2 = (\sigma + \tfrac32) \qquad \text{if} \quad t > 0, \\ & \sigma_1 = -\sigma, \qquad \sigma_2 = -(\sigma + \tfrac32) \qquad \text{if} \quad t < 0. \\ 4.\quad & \sigma_1 = \sigma_2 = \sigma. \end{aligned}$$

$$\begin{aligned} 5\text{–}6.\quad & \varphi^3\varphi^1_\omega - \varphi^1_{\omega\omega} - \mu_{1i}\varphi^i + c_{11} + c_{21}\omega = 0, \\ & \varphi^3\varphi^2_\omega - \varphi^2_{\omega\omega} - \mu_{2i}\varphi^i + c_{12} + c_{22}\omega + \gamma_2 a_2\varphi^3 = 0, \\ & \varphi^3\varphi^3_\omega - \varphi^3_{\omega\omega} + \gamma_1 a_2\varphi^2 + h_\omega = 0, \\ & \varphi^3_\omega = \sigma, \end{aligned} \tag{4.12}$$

where $\mu_{11} = -B_1$, $\mu_{12} = -B_2 - \gamma_1 a_1$, $\mu_{21} = -b_{21} + \gamma_1 a_1$, $\mu_{22} = -b_{22}$, $\sigma = \gamma_1 - B_1 - b_{22}$.

$$\begin{aligned} 7.\quad & \varphi^1_\omega + \theta^1\varphi^1 + \theta^2\varphi^2 - (\eta^3)^{-1}\varphi^3\varphi^2 + (\eta^3)^{-2}\varphi^1 = 0, \\ & \varphi^2_\omega - \theta^2\varphi^1 + \theta^1\varphi^2 + (\eta^3)^{-1}\varphi^3\varphi^1 + (\eta^3)^{-2}\varphi^2 = 0, \\ & \varphi^3_\omega + \eta^3_t(\eta^3)^{-1}\varphi^3 = 0, \\ & 2\theta^1 + \eta^3_t(\eta^3)^{-1} = 0. \end{aligned} \tag{4.13}$$

$$\begin{aligned} 8.\quad & \vec\varphi_\omega + \lambda^{-1}(\vec n^b\cdot\vec\varphi)\vec m^b_t = 0, \\ & \vec n^a\cdot\vec m^a_t = 0. \end{aligned} \tag{4.14}$$

4.3. Exact solutions of the reduced systems

1. Ansatz (4.1) and system (4.9) determine the class of solutions of the NSEs (1.1) that are called the steady axially symmetric conically similar flows of a viscous fluid in hydrodynamics. This class of solutions was studied in a number of works (for example, see references in [16]). For $\varphi^2 = 0$ it was shown, by N.A. Slezkin [34], that system (4.9) is reduced to a Riccati equation. The general solution of this equation was expressed in terms of hypergeometric functions. Later similar calculations were made by V.I. Yatseev [38] and H.B. Squire [35]. The particular case in the class of solutions with $\varphi^2 = 0$ is formed by the Landau jets [24]. For swirling flows, where $\varphi^2 \neq 0$, the order of system (4.9) can be reduced too. For example [33], an arbitrary solution of (4.9) satisfies the equation

$$\varphi^2\varphi^2\sin^2\omega - \sin\omega(\Phi_\omega\sin^{-1}\omega)_\omega + 2\Phi_\omega\cot\omega + 2\Phi = \mathrm{const},$$

where $\Phi = (\varphi^3_\omega - \frac{1}{2}\varphi^3\varphi^3)\sin^2\omega - \varphi^3\cos\omega\sin\omega$, and the Yatseev results [38] are completely extended to the case $\varphi^2\sin\omega = \mathrm{const}$.

2. System (4.10) implies that

$$\begin{aligned}
&\varphi^2 = \varkappa\varphi^1 + C_1,\\
&h = \varkappa(1+\varkappa^2)\varphi^1_\omega + (2\varkappa^2 + 2 - \varkappa C_1)\varphi^1 + C_2,\\
&(1+\varkappa^2)\varphi^1_{\omega\omega} + (4\varkappa - C_1)\varphi^1_\omega + \varphi^1\varphi^1 + 4\varphi^1 +\\
&\quad + (1+\varkappa^2)^{-1}(C_1^2 + 2C_2) = 0,\\
&(1+\varkappa^2)\varphi^3_{\omega\omega} - (C_1 - 2\varkappa)\varphi^3_\omega + (1+\varphi^1)\varphi^3 = 0.
\end{aligned} \tag{4.15}$$

If $\varphi^3 = 0$, the solution determined by ansatz (4.10) and formulas (4.15) coincides with the Hamel solution [18, 23]. In Section 6 we consider system (6.14) which is more general than system (4.10).

3–4. Let us integrate the last equation of system (4.11), i.e.,

$$\varphi^1 = C_1\omega^{-2} - \tfrac{1}{2}\sigma_2. \tag{4.16}$$

Taking into account the integration result, the other equations of system (4.11) can be written in the form

$$\begin{aligned}
&h_\omega = \omega^{-3}\varphi^2\varphi^2 + C_1^2\omega^{-3} - \tfrac{1}{4}\sigma_2^2\omega,\\
&\varphi^2_{\omega\omega} - ((C_1+1)\omega^{-1} - \tfrac{1}{2}\sigma_2\omega)\varphi^2_\omega = \varepsilon_1,\\
&\varphi^3_{\omega\omega} - ((C_1-1)\omega^{-1} - \tfrac{1}{2}\sigma_2\omega)\varphi^3_\omega - \sigma_1\varphi^3 = \nu\omega^{-2}\varphi^2 + \varepsilon_2.
\end{aligned} \tag{4.17}$$

Therefore,

$$h = \int\omega^{-3}\varphi^2\varphi^2 d\omega - \tfrac{1}{2}C_1^2\omega^{-2} - \tfrac{1}{8}\sigma_2^2\omega^2, \tag{4.18}$$

$$\begin{aligned}
\varphi^2 = {} & C_2 + C_3\int|\omega|^{C_1+1}e^{-\frac{1}{4}\sigma_2\omega^2}d\omega +\\
& + \varepsilon_1\int|\omega|^{C_1+1}e^{-\frac{1}{4}\sigma_2\omega^2}\left(\int|\omega|^{-C_1-1}e^{\frac{1}{4}\sigma_2\omega^2}d\omega\right)d\omega.
\end{aligned} \tag{4.19}$$

If $\sigma_1 = 0$, it follows that

$$\begin{aligned}
\varphi^3 = {} & C_4 + C_5\int|\omega|^{C_1-1}e^{-\frac{1}{4}\sigma_2\omega^2}d\omega +\\
& + \int|\omega|^{C_1-1}e^{-\frac{1}{4}\sigma_2\omega^2}\left(\int|\omega|^{-C_1+1}e^{\frac{1}{4}\sigma_2\omega^2}(\varepsilon_2 + \nu\omega^{-2}\varphi^2)d\omega\right)d\omega.
\end{aligned} \tag{4.20}$$

Let $\sigma_1 \neq 0$ (and, therefore, $\nu = 0$). Then, if $\sigma_2 \neq 0$, the general solution of equation (4.17) is expressed in terms of Whittaker functions:

$$\varphi^3 = |\omega|^{\frac{1}{2}C_1 - 1} e^{-\frac{1}{8}\sigma_2\omega^2} W(-\sigma_1\sigma_2^{-1} + \tfrac{1}{4}C_1 - \tfrac{1}{2}, \tfrac{1}{4}C_1, \tfrac{1}{4}\sigma_2\omega^2),$$

where $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation

$$4\tau^2 W_{\tau\tau} = (\tau^2 - 4\varkappa\tau + 4\mu^2 - 1)W. \tag{4.21}$$

If $\sigma_2 = 0$, the general solution of equation (4.16) is expressed in terms of Bessel functions:

$$\varphi^3 = |\omega|^{\frac{1}{2}C_1} Z_{\frac{1}{2}C_1}\big((-\sigma_1)^{1/2}\omega\big),$$

where $Z_\nu(\tau)$ is the general solution of the Bessel equation

$$\tau^2 Z_{\tau\tau} + \tau Z_\tau + (\tau^2 - \nu^2)Z = 0. \tag{4.22}$$

Note 4.2. If $\sigma_2 = 0$, all quadratures in formulas (4.18)–(4.20) are easily integrated. For example,

$$\varphi^2 = \begin{cases} C_2 + C_3 \ln|\omega| + \frac{1}{4}\varepsilon_1\omega^2 & \text{if} \quad C_1 = -2, \\ C_2 + C_3\frac{1}{2}\omega^2 + \frac{1}{2}\varepsilon_1\omega^2(\ln\omega - \frac{1}{2}) & \text{if} \quad C_1 = 0, \\ C_2 + C_3(C_1+2)^{-1}|\omega|^{C_1+2} - \frac{1}{2}\varepsilon_1 C_1^{-1}\omega^2 & \text{if} \quad C_1 \neq -2, 0. \end{cases}$$

5–6. Let $\sigma = 0$. Then the last equation of system (4.12) implies that $\varphi^3 = C_0 =$ const. The other equations of system (4.12) can be written in the form

$$\begin{aligned} &h = -\gamma_1 a_2 \textstyle\int \varphi^2(\omega)d\omega, \\ &\varphi^i_{\omega\omega} - C_0\varphi^i_\omega + \mu_{ij}\varphi^j = \nu_{1i} + \nu_{2i}\omega, \end{aligned} \tag{4.23}$$

where $\nu_{11} = c_{11}$, $\nu_{21} = c_{21}$, $\nu_{12} = c_{12} + \gamma_2 a_2 C_0$, $\nu_{22} = c_{22}$. System (4.23) is a linear nonhomogeneous system of ODEs with constant coefficients. The form of its general solution depends on the Jordan form of the matrix $\mathrm{M} = \{\mu_{ij}\}$. Now let us transform the dependent variables

$$\varphi^i = e_{ij}\psi^j,$$

where the constants e_{ij} are determined by means of the system of linear algebraic equations

$$e_{ij}\tilde{\mu}_{jk} = \mu_{ij}e_{jk} \quad (i, j, k = 1, 2)$$

with the condition $\det\{e_{ij}\} \neq 0$. Here $\tilde{\mathrm{M}} = \{\tilde{\mu}_{ij}\}$ is the real Jordan form of the matrix M. The new unknown functions ψ^i have to satisfy the following system

$$\psi^i_{\omega\omega} - C_0\psi^i_\omega + \tilde{\mu}_{ij}\psi^j = \tilde{\nu}_{1i} + \tilde{\nu}_{2i}\omega, \tag{4.24}$$

where $\nu_{1i} = e_{ij}\tilde{\nu}_{1j}$, $\nu_{2i} = e_{ij}\tilde{\nu}_{2j}$. Depending on the form of $\tilde{\mathrm{M}}$, we consider the following cases:

A. $\det\tilde{\mathrm{M}} = 0$ (this is equivalent to the condition $\det \mathrm{M} = 0$).

i. $\tilde{M} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, where $\varepsilon \in \{0; 1\}$. Then

$$\begin{aligned} \psi^2 &= C_1 + C_2 e^{C_0\omega} - \tfrac{1}{2}\tilde{\nu}_{22}C_0^{-1}\omega^2 - (\tilde{\nu}_{12} - \tilde{\nu}_{22}C_0^{-1})C_0^{-1}\omega, \\ \psi^1 &= C_3 + C_4 e^{C_0\omega} - \tfrac{1}{2}\tilde{\nu}_{21}C_0^{-1}\omega^2 - (\tilde{\nu}_{11} - \tilde{\nu}_{21}C_0^{-1})C_0^{-1}\omega + \\ &\quad + \varepsilon\Big(-\tfrac{1}{6}\tilde{\nu}_{22}C_0^{-2}\omega^3 - \tfrac{1}{2}(\tilde{\nu}_{12} - 2\tilde{\nu}_{22}C_0^{-1})C_0^{-2}\omega^2 + \\ &\quad + \big(C_1 + (\tilde{\nu}_{21} - 2\tilde{\nu}_{22}C_0^{-1})C_0^{-2}\big)C_0^{-1}\omega - C_2C_0^{-1}\omega e^{C_0\omega}\Big) \end{aligned} \tag{4.25}$$

for $C_0 \neq 0$, and

$$\begin{aligned} \psi^2 &= C_1 + C_2\omega + \tfrac{1}{6}\tilde{\nu}_{22}\omega^3 + \tfrac{1}{2}\tilde{\nu}_{12}\omega^2, \\ \psi^1 &= C_3 + C_4\omega + \tfrac{1}{6}(\tilde{\nu}_{21} - C_2)\omega^3 + \tfrac{1}{2}(\tilde{\nu}_{11} - C_1)\omega^2 - \tfrac{1}{120}\tilde{\nu}_{22}\omega^5 - \tfrac{1}{24}\tilde{\nu}_{12}\omega^4 \end{aligned} \tag{4.26}$$

for $C_0 = 0$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0\}$. Then the form of ψ^2 is given either by formula (4.25) for $C_0 \neq 0$ or by formula (4.26) for $C_0 = 0$. The form of ψ^1 is given by formula (4.28) (see below).

B. $\det\tilde{\mathrm{M}} \neq 0$ (this is equivalent to the condition $\det\mathrm{M} \neq 0$).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}\backslash\{0\}$. Then

$$\psi^2 = \tilde{\nu}_{22}\varkappa_2^{-1}\omega + (\tilde{\nu}_{12} - C_0\tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1} + C_1\theta^{21}(\omega) + C_2\theta^{22}(\omega), \tag{4.27}$$

$$\psi^1 = \tilde{\nu}_{21}\varkappa_1^{-1}\omega + (\tilde{\nu}_{11} - C_0\tilde{\nu}_{21}\varkappa_1^{-1})\varkappa_1^{-1} + C_3\theta^{11}(\omega) + C_4\theta^{12}(\omega), \tag{4.28}$$

where

$$\theta^{i1}(\omega) = \exp\big(\tfrac{1}{2}(C_0 - \sqrt{D_i})\omega\big), \quad \theta^{i2}(\omega) = \exp\big(\tfrac{1}{2}(C_0 + \sqrt{D_i})\omega\big)$$

if $\quad D_i = C_0^2 - 4\varkappa_i > 0$,

$$\theta^{i1}(\omega) = e^{\frac{1}{2}C_0\omega}\cos\big(\tfrac{1}{2}\sqrt{-D_i}\omega\big), \quad \theta^{i2}(\omega) = e^{\frac{1}{2}C_0\omega}\sin\big(\tfrac{1}{2}\sqrt{-D_i}\omega\big)$$

if $\quad D_i < 0$,

$$\theta^{i1}(\omega) = e^{\frac{1}{2}C_0\omega}, \qquad \theta^{i2}(\omega) = \omega e^{\frac{1}{2}C_0\omega}$$

if $\quad D_i = 0$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_2 & 1 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_2 \in \mathbb{R}\backslash\{0\}$. Then the form of ψ^2 is given by formula (4.27), and

$$\begin{aligned} \psi^1 &= \big(\tilde{\nu}_{11} - (\tilde{\nu}_{12} - C_0\tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1} - C_0(\tilde{\nu}_{21} - \tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1}\big)\varkappa_2^{-1} + \\ &\quad + (\tilde{\nu}_{21} - \tilde{\nu}_{22}\varkappa_2^{-1})\varkappa_2^{-1}\omega + C_3\theta^{21}(\omega) + C_4\theta^{22}(\omega) - C_i\eta^i(\omega), \end{aligned}$$

where

$$\begin{aligned} &\eta^j(\omega) = D_2^{-1}\omega\big(2\theta^{2j}_\omega(\omega) - C_0\theta^{2j}(\omega)\big) && \text{if} \quad D_2 \neq 0, \\ &\eta^1(\omega) = \tfrac{1}{2}\omega^2 e^{\frac{1}{2}C_0\omega}, \quad \eta^2(\omega) = \tfrac{1}{6}\omega^3 e^{\frac{1}{2}C_0\omega} && \text{if} \quad D_2 = 0. \end{aligned}$$

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & -\varkappa_2 \\ \varkappa_2 & \varkappa_1 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}$, $\varkappa_2 \neq 0$. Then

$$\begin{aligned}\psi^1 &= (\varkappa_i\varkappa_i)^{-1}(\tilde\nu_{21}\varkappa_1 + \tilde\nu_{22}\varkappa_2)\omega + (\varkappa_i\varkappa_i)^{-1}(\tilde\nu_{11}\varkappa_1 + \tilde\nu_{12}\varkappa_2) - \\ &\quad - C_0(\varkappa_i\varkappa_i)^{-2}\big(\tilde\nu_{21}(\varkappa_2^2 - \varkappa_1^2) - \tilde\nu_{22}2\varkappa_1\varkappa_2\big) + C_n\theta^{1n}(\omega), \\ \psi^2 &= (\varkappa_i\varkappa_i)^{-1}(-\tilde\nu_{21}\varkappa_2 + \tilde\nu_{22}\varkappa_1)\omega + (\varkappa_i\varkappa_i)^{-1}(-\tilde\nu_{11}\varkappa_2 + \tilde\nu_{12}\varkappa_1) - \\ &\quad - C_0(\varkappa_i\varkappa_i)^{-2}\big(\tilde\nu_{21}2\varkappa_1\varkappa_2 + \tilde\nu_{22}(\varkappa_2^2 - \varkappa_1^2)\big) + C_n\theta^{2n}(\omega),\end{aligned}$$

where $n = \overline{1,4}$,

$$\begin{aligned}&\gamma = \sqrt{(C_0^2 - 4\varkappa_1)^2 + (4\varkappa_2)^2}, \\ &\beta_1 = \tfrac{1}{4}\sqrt{2(\gamma + C_0^2 - 4\varkappa_1)}, \quad \beta_2 = \tfrac{1}{4}\tfrac{|\varkappa_2|}{\varkappa_2}\sqrt{2(\gamma - C_0^2 + 4\varkappa_1)}, \\ &\theta^{11}(\omega) = \theta^{22}(\omega) = \exp\big((\tfrac{1}{2}C_0 - \beta_1)\omega\big)\cos\beta_2\omega, \\ &-\theta^{21}(\omega) = \theta^{12}(\omega) = \exp\big((\tfrac{1}{2}C_0 - \beta_1)\omega\big)\sin\beta_2\omega, \\ &\theta^{13}(\omega) = \theta^{24}(\omega) = \exp\big((\tfrac{1}{2}C_0 + \beta_1)\omega\big)\cos\beta_2\omega, \\ &\theta^{23}(\omega) = -\theta^{14}(\omega) = \exp\big((\tfrac{1}{2}C_0 + \beta_1)\omega\big)\sin\beta_2\omega.\end{aligned}$$

If $\sigma \neq 0$, the last equation of system (4.12) implies that $\psi^3 = \sigma\omega$ (translating ω, the integration constant can be made to vanish). The other equations of system (4.12) can be written in the form

$$\begin{aligned}&h = -\gamma_1 a_2 \textstyle\int \varphi^2(\omega)d\omega - \tfrac{1}{2}\sigma^2\omega^2, \\ &\varphi^i_{\omega\omega} - \sigma\omega\varphi^i_\omega + \mu_{ij}\varphi^j = \nu_{1i} + \nu_{2i}\omega,\end{aligned} \tag{4.29}$$

where $\nu_{11} = c_{11}$, $\nu_{21} = c_{21}$, $\nu_{12} = c_{12}$, $\nu_{22} = c_{22} + \gamma_2 a_2\sigma$. The form of the general solution of system (4.29) depends on the Jordan form of the matrix $\mathrm{M} = \{\mu_{ij}\}$. Now, let us transform the dependent variables

$$\varphi^i = e_{ij}\psi^j,$$

where the constants e_{ij} are determined by means of the system of linear algebraic equations

$$e_{ij}\tilde\mu_{jk} = \mu_{ij}e_{jk} \quad (i,j,k = 1,2)$$

with the condition $\det\{e_{ij}\} \neq 0$. Here $\tilde{\mathrm{M}} = \{\tilde\mu_{ij}\}$ is the real Jordan form of the matrix M. The new unknown functions ψ^i have to satisfy the following system

$$\psi^i_{\omega\omega} - \sigma\omega\psi^i_\omega + \tilde\mu_{ij}\psi^j = \tilde\nu_{1i} + \tilde\nu_{2i}\omega, \tag{4.30}$$

where $\nu_{1i} = e_{ij}\tilde\nu_{1j}$, $\nu_{2i} = e_{ij}\tilde\nu_{2j}$. Depending on the form of $\tilde{\mathrm{M}}$, we consider the following cases:

A. $\det\tilde{\mathrm{M}} = 0$ (this is equivalent to the condition $\det\mathrm{M} = 0$).

i. $\tilde{M} = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$, where $\varepsilon \in \{0; 1\}$. Then

$$\psi^2 = C_1 + C_2\textstyle\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}\tilde\nu_{22}\omega + \tilde\nu_{12}\int e^{\frac{1}{2}\sigma\omega^2}\big(\int e^{-\frac{1}{2}\sigma\omega^2}d\omega\big)d\omega, \tag{4.31}$$

$$\psi^1 = C_3 + C_4\textstyle\int e^{\frac{1}{2}\sigma\omega^2}d\omega - \sigma^{-1}\tilde\nu_{21}\omega + \int e^{\frac{1}{2}\sigma\omega^2}\big(\int e^{-\frac{1}{2}\sigma\omega^2}(\tilde\nu_{11} - \varepsilon\psi^2)d\omega\big)d\omega.$$

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}$. Then the form of ψ^2 is given by formula (4.31), and

$$\begin{aligned}\psi^1 &= C_3\omega + C_4\big(\omega \textstyle\int e^{\frac12\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac12\sigma\omega^2}\big) + \sigma^{-1}\tilde\nu_{11} + \\ &\quad + \sigma^{-1}\tilde\nu_{21}\big(\sigma\omega \textstyle\int e^{\frac12\sigma\omega^2}\lambda^1(\omega)d\omega - e^{\frac12\sigma\omega^2}\lambda^1(\omega)\big),\end{aligned}$$

where $\lambda^1(\omega) = \int e^{-\frac12\sigma\omega^2}d\omega$.

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0;\sigma\}$. Then ψ^2 is determined by (4.31), and the form of ψ^1 is given by (4.33) (see below).

B. $\det\tilde{\mathrm{M}} \neq 0$, $\det\{\tilde\mu_{ij} - \sigma\delta_{ij}\} = 0$ (this is equivalent to the conditions $\det\mathrm{M} \neq 0$, $\det\{\mu_{ij} - \sigma\delta_{ij}\} = 0$; here δ_{ij} is the Kronecker symbol).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \sigma & \varepsilon \\ 0 & \sigma \end{pmatrix}$, where $\varepsilon \in \{0;1\}$. Then

$$\begin{aligned}\psi^2 &= C_1\omega + C_2\big(\omega \textstyle\int e^{\frac12\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac12\sigma\omega^2}\big) + \sigma^{-1}\tilde\nu_{12} + \\ &\quad + \sigma^{-1}\tilde\nu_{22}\big(\sigma\omega \textstyle\int e^{\frac12\sigma\omega^2}\lambda^1(\omega)d\omega - e^{\frac12\sigma\omega^2}\lambda^1(\omega)\big),\end{aligned} \tag{4.32}$$

$$\begin{aligned}\psi^1 &= C_3\omega + C_4\big(\omega \textstyle\int e^{\frac12\sigma\omega^2}d\omega - \sigma^{-1}e^{\frac12\sigma\omega^2}\big) + \sigma^{-1}\tilde\nu_{11} + \\ &\quad + \sigma\omega \textstyle\int e^{\frac12\sigma\omega^2}\lambda^2(\omega)d\omega - e^{\frac12\sigma\omega^2}\lambda^2(\omega) + \sigma^{-1}(\tilde\nu_{21}\omega - \varepsilon\psi^2),\end{aligned}$$

where $\lambda^1(\omega) = \int e^{-\frac12\sigma\omega^2}d\omega$, $\lambda^2(\omega) = \sigma^{-1}\int e^{-\frac12\sigma\omega^2}(\tilde\nu_{21} - \varepsilon\psi^2_\omega)d\omega$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \sigma \end{pmatrix}$, where $\varkappa_1 \in \mathbb{R}\backslash\{0;\sigma\}$. In this case ψ^2 is determined by (4.32), and the form of ψ^1 is given by (4.33) (see below).

C. $\det\tilde{\mathrm{M}} \neq 0$, $\det\{\tilde\mu_{ij} - \sigma\delta_{ij}\} \neq 0$ (this is equivalent to the condition $\det\mathrm{M} \neq 0$, $\det\{\mu_{ij} - \sigma\delta_{ij}\} \neq 0$: here δ_{ij} is the Kronecker symbol).

i. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & 0 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}\backslash\{0;\sigma\}$. Then

$$\begin{aligned}\psi^1 &= \varkappa_1^{-1}\tilde\nu_{11} + (\varkappa_1 - \sigma)^{-1}\tilde\nu_{21}\omega + |\omega|^{-1/2}e^{\frac14\sigma\omega^2} \times \\ &\quad \times \Big(C_3 M\big(\tfrac12\varkappa_1\sigma^{-1} + \tfrac14, \tfrac14, \tfrac12\sigma\omega^2\big) + C_4 M\big(\tfrac12\varkappa_1\sigma^{-1} + \tfrac14, -\tfrac14, \tfrac12\sigma\omega^2\big)\Big),\end{aligned} \tag{4.33}$$

$$\begin{aligned}\psi^2 &= \varkappa_2^{-1}\tilde\nu_{12} + (\varkappa_2 - \sigma)^{-1}\tilde\nu_{22}\omega + |\omega|^{-1/2}e^{\frac14\sigma\omega^2} \times \\ &\quad \times \Big(C_1 M\big(\tfrac12\varkappa_2\sigma^{-1} + \tfrac14, \tfrac14, \tfrac12\sigma\omega^2\big) + C_2 M\big(\tfrac12\varkappa_2\sigma^{-1} + \tfrac14, -\tfrac14, \tfrac12\sigma\omega^2\big)\Big),\end{aligned} \tag{4.34}$$

where $M(\varkappa, \mu, \tau)$ is the Whittaker function:

$$M(\varkappa, \mu, \tau) = \tau^{\frac12+\mu}e^{-\frac12\tau}\,{}_1F_1\big(\tfrac12 + \mu - \varkappa, 2\mu + 1, \tau\big), \tag{4.35}$$

and ${}_1F_1(a, b, \tau)$ is the degenerate hypergeometric function defined by means of the series:

$${}_1F_1(a, b, \tau) = 1 + \sum_{n=1}^{\infty} \frac{a(a+1)\cdots(a+n-1)}{b(b+1)\cdots(b+n-1)}\frac{\tau^n}{n!},$$

$b \neq 0, -1, -2, \ldots$.

ii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_1 & -\varkappa_2 \\ \varkappa_2 & \varkappa_1 \end{pmatrix}$, where $\varkappa_i \in \mathbb{R}$, $\varkappa_2 \neq 0$. Then

$$\psi^1 = (\varkappa_j\varkappa_j)^{-1}(\varkappa_1\tilde{\nu}_{11} + \varkappa_2\tilde{\nu}_{12}) + ((\varkappa_1-\sigma)^2+\varkappa_2^2)^{-1}((\varkappa_1-\sigma)\tilde{\nu}_{21} + \varkappa_2\tilde{\nu}_{22})\omega + \\ + C_1\mathrm{Re}\,\eta^1(\omega) - C_2\mathrm{Im}\,\eta^1(\omega) + C_3\mathrm{Re}\,\eta^2(\omega) - C_4\mathrm{Im}\,\eta^2(\omega),$$

$$\psi^2 = (\varkappa_j\varkappa_j)^{-1}(-\varkappa_2\tilde{\nu}_{11} + \varkappa_1\tilde{\nu}_{12}) + \\ + ((\varkappa_1-\sigma)^2+\varkappa_2^2)^{-1}(-\varkappa_2\tilde{\nu}_{21} + (\varkappa_1-\sigma)\tilde{\nu}_{22})\omega + \\ + C_1\mathrm{Im}\,\eta^1(\omega) + C_2\mathrm{Re}\,\eta^1(\omega) + C_3\mathrm{Im}\,\eta^2(\omega) + C_4\mathrm{Re}\,\eta^2(\omega),$$

where

$$\eta^1(\omega) = M\big(\tfrac{1}{2}(\varkappa_1+\varkappa_2\mathrm{i})\sigma^{-1} + \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2\big),$$
$$\eta^2(\omega) = M\big(\tfrac{1}{2}(\varkappa_1+\varkappa_2\mathrm{i})\sigma^{-1} + \tfrac{1}{4}, -\tfrac{1}{4}, \tfrac{1}{2}\sigma\omega^2\big), \quad \mathrm{i}^2 = -1.$$

iii. $\tilde{\mathrm{M}} = \begin{pmatrix} \varkappa_2 & 1 \\ 0 & \varkappa_2 \end{pmatrix}$, where $\varkappa_2 \in \mathbb{R}\backslash\{0;\sigma\}$. Here the form of ψ^2 is given by (4.34), and

$$\psi^1 = (\tilde{\nu}_{11} - \tilde{\nu}_{12}\varkappa_2^{-1})\varkappa_2^{-1} + \big(\tilde{\nu}_{21} - \tilde{\nu}_{22}(\varkappa_2-\sigma)^{-1}\big)(\varkappa_2-\sigma)^{-1}\omega + \\ + |\omega|^{-1/2}e^{\frac{1}{4}\sigma\omega^2}\Big(C_3\theta^1(\tau) + C_4\theta^2(\tau) - \sigma^{-1}\theta^1(\tau)\int\tau^{-1}\theta^2(\tau)C_i\theta^i(\tau)d\tau + \\ + \sigma^{-1}\theta^2(\tau)\int\tau^{-1}\theta^1(\tau)C_i\theta^i(\tau)d\tau\Big),$$

where $\tau = \frac{1}{2}\sigma\omega^2$,

$$\theta^1(\tau) = M\big(\tfrac{1}{2}\varkappa_2\sigma^{-1} + \tfrac{1}{4}, \tfrac{1}{4}, \tau\big), \quad \theta^2(\tau) = M\big(\tfrac{1}{2}\varkappa_2\sigma^{-1} + \tfrac{1}{4}, -\tfrac{1}{4}, \tau\big).$$

Note 4.3. The general solution of the equation

$$\psi_{\omega\omega} - \sigma\omega\psi_\omega - (n+1)\sigma\psi = 0,$$

where n is an integer and $n \geq 0$, is determined by the formula

$$\psi = \left(\frac{d^n}{d\omega^n}e^{\frac{1}{2}\sigma\omega^2}\right)\left(C_1 + C_2\int e^{\frac{1}{2}\sigma\omega^2}\left(\frac{d^n}{d\omega^n}e^{\frac{1}{2}\sigma\omega^2}\right)^{-2}d\omega\right).$$

Note 4.4. If function ψ satisfies the equation

$$\psi_{\omega\omega} - \sigma\omega\psi_\omega + \varkappa\psi = 0 \quad (\varkappa \neq -\sigma),$$

then $\int\psi(\omega)d\omega = (\varkappa+\sigma)^{-1}(\sigma\omega\psi - \psi_\omega) + C_1$.

7. The last equation of system (4.13) is the compatibility condition of the NSEs (1.1) and ansatz (4.7). Integrating this equation, we obtain that

$$\eta^3 = C_0(\eta^i\eta^i)^{-1}, \quad C_0 \neq 0.$$

As $\varphi^3_\omega = -\eta^3_\omega(\eta^3)^{-1}\varphi^3 = 2\theta^1\varphi^3$, $\varphi^3 = C_3\eta^i\eta^i$. Then system (4.13) is reduced to the equations

$$\begin{aligned}\varphi^1_\omega &= \chi^1(\omega)\varphi^1 - \chi^2(\omega)\varphi^2,\\ \varphi^2_\omega &= \chi^2(\omega)\varphi^1 + \chi^1(\omega)\varphi^2,\end{aligned} \tag{4.36}$$

where $\chi^1 = -C_0^{-2}(\eta^i\eta^i)^2 - \theta^1$ and $\chi^2 = \theta^2 - C_3C_0^{-1}(\eta^i\eta^i)^2$. System (4.36) implies that

$$\begin{aligned}\varphi^1 &= \exp\left(\textstyle\int \chi^1(\omega)d\omega\right)\Big(C_1\cos\left(\textstyle\int\chi^2(\omega)d\omega\right) - C_2\sin\left(\textstyle\int\chi^2(\omega)d\omega\right)\Big),\\ \varphi^2 &= \exp\left(\textstyle\int \chi^1(\omega)d\omega\right)\Big(C_1\sin\left(\textstyle\int\chi^2(\omega)d\omega\right) + C_2\cos\left(\textstyle\int\chi^2(\omega)d\omega\right)\Big).\end{aligned}$$

8. Let us apply the transformation generated by the operator $R(\vec{k}(t))$, where

$$\vec{k}_t = \lambda^{-1}(\vec{n}^b\cdot\vec{k})\vec{m}^b_t - \vec{\varphi},$$

to ansatz (4.8). As a result we obtain an ansatz of the same form, where the functions $\vec{\varphi}$ and h are replaced by the new functions $\tilde{\vec{\varphi}}$ and $\tilde{h}$:

$$\begin{aligned}\tilde{\vec{\varphi}} &= \vec{\varphi} - \lambda^{-1}(\vec{n}^a\cdot\vec{k})\vec{m}^a_t + \vec{k}_t = 0,\\ \tilde{h} &= h - \lambda^{-1}(\vec{m}^a_{tt}\cdot\vec{k})(\vec{n}^a\cdot\vec{k}) + \tfrac{1}{2}\lambda^{-2}(\vec{m}^b_{tt}\cdot\vec{m}^a)(\vec{n}^a\cdot\vec{k})(\vec{n}^b\cdot\vec{k}).\end{aligned}$$

Let us make $\tilde{h}$ vanish by means of the transformation generated by the operator $Z(-\tilde{h}(t))$. Therefore, the functions φ^a and h can be considered to vanish. The equation $(\vec{n}^a\cdot\vec{m}^a_t) = 0$ is the compatibility condition of ansatz (4.8) and the NSEs (1.1).

Note 4.5. The solutions of the NSEs obtained by means of ansatzes 5–8 are equivalent to either solutions (5.1) or solutions (5.5).

5. Reduction of the Navier–Stokes equations to linear systems of PDEs

Let us show that non-linear systems 8 and 9, from Subsection 3.2, are reduced to linear systems of PDEs.

5.1. Investigation of system (3.17)–(3.18)

Consider system 9 from Subsection 3.2, i.e., equations (3.17) and (3.18). Equation (3.18) integrates with respect to z_2 to the following expression:

$$\vec{k}\cdot\vec{w} = \psi(t).$$

Here $\psi = \psi(t)$ is an arbitrary smooth function of $z_1 = t$. Let us make the transformation from the symmetry group of the NSEs:

$$\begin{aligned}\tilde{\vec{u}}(t,\vec{x}) &= \vec{u}(t,\vec{x}-\vec{l}) + \vec{l}_t(t),\\ \tilde{p}(t,\vec{x}) &= p(t,\vec{x}-\vec{l}) - \vec{l}_{tt}(t)\cdot\vec{x},\end{aligned}$$

where $\vec{l}_{tt}\cdot\vec{m}^i-\vec{l}\cdot\vec{m}^i_{tt}=0$ and

$$\vec{k}\cdot\big(\vec{l}_t-\lambda^{-1}(\vec{n}^i\cdot\vec{l})m^i_t+\lambda^{-1}(\vec{k}\cdot\vec{l})\vec{k}_t\big)+\psi=0.$$

This transformation does not modify ansatz (3.9), but it makes the function $\psi(t)$ vanish, i.e., $\vec{k}\cdot\vec{\tilde{w}}=0$. Therefore, without loss of generality we may assume, at once, that $\vec{k}\cdot\vec{w}=0$.

Let $f^i=f^i(z_1,z_2)=\vec{m}^i\cdot\vec{w}$. Since $\vec{m}^1_{tt}\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_{tt}=0$, it follows that $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=C=\mathrm{const}$. Let us multiply the scalar equation (3.17) by $\vec{m}^i$ and $\vec{k}$. As a result we obtain the linear system of PDEs with variable coefficients in the functions f^i and s:

$$\begin{aligned}&f^i_1-\lambda f^i_{22}+C\lambda^{-1}\big((\vec{m}^i\cdot\vec{m}^2)f^1-(\vec{m}^i\cdot\vec{m}^1)f^2\big)-2C\lambda^{-2}\big((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i\big)z_2=0,\\&s_2=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)f^i+\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)z_2.\end{aligned}$$

Consider two possible cases.

A. Let $C=0$. Then there exist functions $g^i=g^i(\tau,\omega)$, where $\tau=\int\lambda(t)dt$ and $\omega=z_2$, such that $f^i=g^i_\omega$ and $g^i_\tau-g^i_{\omega\omega}=0$. Therefore,

$$\begin{aligned}&\vec{u}=\lambda^{-1}(g^i_\omega(\tau,\omega)+\vec{m}^i_t\cdot\vec{x})\vec{n}^i-\lambda^{-1}(\vec{k}_t\cdot\vec{x})k,\\&p=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)g^i(\tau,\omega)+\tfrac12\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)\omega^2-\\&\qquad-\tfrac12\lambda^{-1}(\vec{n}^i\cdot\vec{x})(\vec{m}^i_{tt}\cdot\vec{x})-\tfrac12\lambda^{-2}(\vec{k}\cdot\vec{m}^i_{tt})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}),\end{aligned}\tag{5.1}$$

where $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=0$, $\vec{k}=\vec{m}^1\times\vec{m}^2$, $\vec{n}^1=\vec{m}^2\times\vec{k}$, $\vec{n}^2=\vec{k}\times\vec{m}^1$, $\lambda=|\vec{k}|^2$, $\omega=\vec{k}\cdot\vec{x}$, $\tau=\int\lambda(t)dt$, and $g^i_\tau-g^i_{\omega\omega}=0$.

For example, if $\vec{m}=(\eta^1(t),0,0)$ and $\vec{n}=(0,\eta^2(t),0)$ with $\eta^i(t)\neq0$, it follows that

$$\begin{aligned}&u^1=(\eta^1)^{-1}(f^1+\eta^1_tx_1),\quad u^2=(\eta^2)^{-1}(f^2+\eta^2_tx_2),\quad u^3=-(\eta^1\eta^2)_t(\eta^1\eta^2)^{-1}x_3,\\&p=-\tfrac12\eta^1_{tt}(\eta^1)^{-1}x_1^2-\tfrac12\eta^2_{tt}(\eta^2)^{-1}x_2^2+\\&\qquad+\Big(\tfrac12(\eta^1\eta^2)_{tt}(\eta^1\eta^2)^{-1}-\big((\eta^1\eta^2)_t(\eta^1\eta^2)^{-1}\big)^2\Big)x_3^2,\end{aligned}$$

where $f^i=f^i(\tau,\omega)$, $f^i_\tau-f^i_{\omega\omega}=0$, $\tau=\int(\eta^1\eta^2)^2dt$, and $\omega=\eta^1\eta^2x_3$. If $\vec{m}^1=(\eta^1(t),\eta^2(t),0)$ and $\vec{m}^2=(0,0,\eta^3(t))$ with $\eta^3(t)\neq0$ and $\eta^i(t)\eta^i(t)\neq0$, we obtain that

$$\begin{aligned}&u^1=(\eta^i\eta^i)^{-1}\Big\{\eta^1(g_\omega+\eta^i_tx_i)-\eta^2\big(\eta^3_t(\eta^3)^{-2}\omega+\eta^2_tx_1-\eta^1_tx_2\big)\Big\},\\&u^2=(\eta^i\eta^i)^{-1}\Big\{\eta^2(g_\omega+\eta^i_tx_i)+\eta^1\big(\eta^3_t(\eta^3)^{-2}\omega+\eta^2_tx_1-\eta^1_tx_2\big)\Big\},\\&u^3=(\eta^3)^{-1}(f+\eta^3_tx_3),\\&p=2(\eta^3)^{-1}(\eta^1\eta^2_t-\eta^1_t\eta^2)(\eta^i\eta^i)^{-2}g+\tfrac12\lambda^{-1}\times\\&\qquad\times\Big\{\lambda^{-1}\big((\eta^3_{tt}\eta^3-2\eta^3_t\eta^3_t)\eta^i\eta^i-2\eta^3\eta^3_t\eta^i\eta^i_t-2(\eta^3)^2\eta^i_t\eta^i_t\big)\omega^2+\\&\qquad+(\eta^3)^2\big((\eta^2\eta^2_{tt}-\eta^1\eta^1_{tt})(x_1^2-x_2^2)-2(\eta^1_{tt}\eta^2+\eta^1\eta^2_{tt})x_1x_2\big)-\eta^i\eta^i\eta^3\eta^3_{tt}x_3^2\Big\}.\end{aligned}$$

Here $f=f(\tau,\omega)$, $f_\tau-f_{\omega\omega}=0$, $g=g(\tau,\omega)$, $g_\tau-g_{\omega\omega}=0$, $\tau=\int(\eta^3)^2\eta^i\eta^idt$, $\omega=\eta^3(\eta^2x_1-\eta^1x_2)$, and $\lambda=(\eta^3)^2\eta^i\eta^i$.

Note 5.1. The equation

$$\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=0 \tag{5.2}$$

can easily be solved in the following way: Let us fix arbitrary smooth vector-functions $\vec{m}^1,\vec{l}\in C^\infty((t_0,t_1),\mathbb{R}^3)$ such that $\vec{m}^1(t)\neq\vec{0}$, $\vec{l}(t)\neq\vec{0}$, and $\vec{m}^1(t)\cdot\vec{l}(t)=0$ for all $t\in(t_0,t_1)$. Then the vector-function $\vec{m}^2=\vec{m}^2(t)$ is taken in the form

$$\vec{m}^2(t)=\rho(t)\vec{m}^1+\vec{l}(t). \tag{5.3}$$

Equation (5.2) implies

$$\rho(t)=\int(\vec{m}^1\cdot\vec{m}^1)^{-1}(\vec{m}^1_t\cdot\vec{l}-\vec{m}^1\cdot\vec{l}_t)dt. \tag{5.4}$$

B. Let $C\neq 0$. By means of the transformation $\vec{m}^i\to a_{ij}\vec{m}^j$, where $a_{ij}=\mathrm{const}$ and $\det\{a_{ij}\}=C$, we make $C=1$. Then we obtain the following solution of the NSEs (1.1)

$$\begin{aligned}
&\vec{u}=\lambda^{-1}\Big(\theta^{ij}(t)g^j_\omega(\tau,\omega)+\theta^{i0}(t)\omega+\vec{m}^i_t\cdot\vec{x}-\lambda^{-1}((\vec{k}\times\vec{m}^i)\cdot\vec{x})\Big)\vec{n}^i-\lambda^{-1}(\vec{k}_t\cdot\vec{x})\vec{k},\\
&p=2\lambda^{-2}(\vec{n}^i\cdot\vec{k}_t)(\theta^{ij}(t)g^i(\tau,\omega)+\tfrac{1}{2}\theta^{i0}(t)\omega^2)+\tfrac{1}{2}\lambda^{-2}(\vec{k}_{tt}\cdot\vec{k}-2\vec{k}_t\cdot\vec{k}_t)\omega^2-\\
&\quad-\tfrac{1}{2}\lambda^{-1}(\vec{n}^i\cdot\vec{x})(\vec{m}^i_{tt}\cdot\vec{x})-\tfrac{1}{2}\lambda^{-2}(\vec{k}\cdot\vec{m}^i_{tt})(\vec{n}^i\cdot\vec{x})(\vec{k}\cdot\vec{x}).
\end{aligned} \tag{5.5}$$

Here $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=1$, $\vec{k}=\vec{m}^1\times\vec{m}^2$, $\vec{n}^1=\vec{m}^2\times\vec{k}$, $\vec{n}^2=\vec{k}\times\vec{m}^1$, $\lambda=|\vec{k}|^2$, $\omega=\vec{k}\cdot\vec{x}$, $\tau=\int\lambda(t)dt$, and $g^i_\tau-g^i_{\omega\omega}=0$. $(\theta^{1i}(t),\theta^{2i}(t))$ $(i=1,2)$ are linearly independent solutions of the system

$$\theta^i_t+\lambda^{-1}(\vec{m}^i\cdot\vec{m}^2)\theta^1-\lambda^{-1}(\vec{m}^i\cdot\vec{m}^1)\theta^2=0, \tag{5.6}$$

and $(\theta^{10}(t),\theta^{20}(t))$ is a particular solution of the nonhomogeneous system

$$\theta^i_t+\lambda^{-1}(\vec{m}^i\cdot\vec{m}^2)\theta^1-\lambda^{-1}(\vec{m}^i\cdot\vec{m}^1)\theta^2=2\lambda^{-2}\big((\vec{k}\times\vec{k}_t)\cdot\vec{m}^i\big). \tag{5.7}$$

For example, if $\vec{m}^1=(\eta\cos\psi,\eta\sin\psi,0)$ and $\vec{m}^2=(-\eta\sin\psi,\eta\cos\psi,0)$, where $\eta=\eta(t)\neq 0$ and $\psi=-\frac{1}{2}\int(\eta)^{-2}dt$ (therefore, $\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=1$), we obtain

$$\begin{aligned}
&u^1=\eta^{-1}\big(f^1\cos\psi-f^2\sin\psi+\eta_t x_1-\tfrac{1}{2}\eta^{-1}x_2\big),\\
&u^2=\eta^{-1}\big(f^1\sin\psi+f^2\cos\psi+\eta_t x_2+\tfrac{1}{2}\eta^{-1}x_1\big),\\
&u^3=-2\eta_t\eta^{-1}x_3,\\
&p=(\eta_{tt}\eta-3\eta_t\eta_t)\eta^{-2}x_3^2-\tfrac{1}{2}(\eta_{tt}\eta^{-1}-\tfrac{1}{4}\eta^{-4})x_ix_i.
\end{aligned}$$

Here $f^i=f^i(\tau,\omega)$, $f^i_\tau-f^i_{\omega\omega}=0$, $\tau=\int(\eta)^4dt$, and $\omega=(\eta)^2x_3$.

Note 5.2. As in the case $C=0$, the solutions of the equation

$$\vec{m}^1_t\cdot\vec{m}^2-\vec{m}^1\cdot\vec{m}^2_t=1 \tag{5.8}$$

can be sought in form (5.3). As a result we obtain that

$$\rho(t)=\int|\vec{m}^1|^{-2}(\vec{m}^1_t\cdot\vec{l}-\vec{m}^1\cdot\vec{l}_t-1)dt. \tag{5.9}$$

Note 5.3. System (5.6) can be reduced to a second-order homogeneous differential equation either in θ^1, i.e.,

$$\left(\lambda|\vec{m}^1|^{-2}\theta^1_t\right)_t+\left(\left((\vec{m}^1\cdot\vec{m}^2)|\vec{m}^1|^{-2}\right)_t+|\vec{m}^1|^{-2}\right)\theta^1=0 \tag{5.10}$$

(then $\theta^2=|\vec{m}^1|^{-2}(\lambda\theta^1_t+(\vec{m}^1\cdot\vec{m}^2)\theta^1)$), or in θ^2, i.e.,

$$\left(\lambda|\vec{m}^2|^{-2}\theta^2_t\right)_t+\left(-\left((\vec{m}^1\cdot\vec{m}^2)|\vec{m}^2|^{-2}\right)_t+|\vec{m}^2|^{-2}\right)\theta^2=0 \tag{5.11}$$

(then $\theta^1=|\vec{m}^2|^{-2}(-\lambda\theta^2_t+(\vec{m}^1\cdot\vec{m}^2)\theta^2)$). Under the notation of Note 5.1 equation (5.10) has the form:

$$\left((\vec{l}\cdot\vec{l})\theta^1_t\right)_t+|\vec{m}^1|^{-2}(\vec{m}^1_t\cdot\vec{l}-\vec{m}^1\cdot\vec{l}_t)\theta^1=0. \tag{5.12}$$

The vector-functions $\vec{m}^1$ and $\vec{l}$ are chosen in such a way that one can find a fundamental set of solutions for equation (5.12). For example, let $\vec{m}\times\vec{m}_t\neq 0\ \forall t\in(t_0,t_1)$. Let us introduce the notation $\vec{m}:=\vec{m}^1$ and put $\vec{l}=\eta(t)\vec{m}\times\vec{m}_t$, where $\eta\in C^\infty((t_0,t_1),\mathbb{R})$, $\eta(t)\neq 0\ \forall t\in(t_0,t_1)$. Then

$$\begin{aligned}
&\vec{m}\cdot\vec{l}=0,\quad \vec{m}_t\cdot\vec{l}-\vec{m}\cdot\vec{l}_t=0,\quad \vec{m}^2=-\left(\textstyle\int|\vec{m}|^{-2}dt\right)\vec{m}+\eta\vec{m}\times\vec{m}_t,\\
&\vec{k}=\eta\vec{m}\times(\vec{m}\times\vec{m}_t),\quad \lambda=(\eta)^2|\vec{m}|^2|\vec{m}\times\vec{m}_t|^{-2},\\
&\vec{n}^2=\eta|\vec{m}|^2\vec{m}\times\vec{m}_t,\quad \vec{n}^1=\left(\textstyle\int|\vec{m}|^{-2}dt\right)\vec{n}^2+(\eta)^2|\vec{m}\times\vec{m}_t|^{-2}\vec{m},\\
&\theta^{11}(t)=\textstyle\int(\eta)^{-2}|\vec{m}\times\vec{m}_t|^{-2}dt,\quad \theta^{21}(t)=1-\theta^{11}\int|\vec{m}|^{-2}dt,\\
&\theta^{12}(t)=1,\quad \theta^{22}(t)=-\textstyle\int|\vec{m}|^{-2}dt,\\
&\theta^{10}(t)=2\textstyle\int\left(((\vec{m}\times\vec{m}_t)\cdot\vec{m}_{tt})|\vec{m}\times\vec{m}_t|^{-2}+\int\eta^{-1}|\vec{m}|^{-4}dt\right)\eta^{-2}|\vec{m}\times\vec{m}_t|^{-2}dt,\\
&\theta^{20}(t)=-\theta^{10}(t)\textstyle\int|\vec{m}|^{-2}dt+2\int\eta^{-1}|\vec{m}|^{-4}dt.
\end{aligned}$$

Consider the following cases: $\vec{m}\times\vec{m}_t\equiv\vec{0}$, i.e., $\vec{m}=\chi(t)\vec{a}$, where $\chi(t)\in C^\infty((t_0,t_1),\mathbb{R})$, $\chi(t)\neq 0\ \forall t\in(t_0,t_1)$, $\vec{a}=\text{const}$, and $|\vec{a}|=1$. Let us put

$$\vec{l}(t)=\eta^1(t)\vec{b}+\eta^2(t)\vec{c},$$

where $\eta^1,\eta^2\in C^\infty((t_0,t_1),\mathbb{R})$, $(\eta^1(t),\eta^2(t))\neq(0,0)\ \forall t\in(t_0,t_1)$, $\vec{b}=\text{const}$, $|\vec{b}|=1$, $\vec{a}\cdot\vec{b}=0$, and $\vec{c}=\vec{a}\times\vec{b}$. Then

$$\begin{aligned}
&\vec{m}^2=-\left(\chi\textstyle\int\chi^{-2}dt\right)\vec{a}+\eta^1\vec{b}+\eta^2\vec{c},\quad \vec{k}=\chi\eta^1\vec{c}-\chi\eta^2\vec{b},\\
&\lambda=(\chi)^2\eta^i\eta^i,\quad \vec{n}^2=(\chi)^2(\eta^1\vec{b}+\eta^2\vec{c}),\quad \vec{n}^1=\left(\textstyle\int\chi^{-2}dt\right)\vec{n}^2+\chi\eta^i\eta^i\vec{a},\\
&\theta^{11}=\textstyle\int(\eta^i\eta^i)^{-1}dt,\quad \theta^{21}=1-\theta^{11}\int\chi^{-2}dt,\quad \theta^{12}=1,\quad \theta^{22}=-\int\chi^{-2}dt,\\
&\theta^{10}=2\textstyle\int(\eta^2_t\eta^1-\eta^2\eta^1_t)\chi^{-1}(\eta^i\eta^i)^{-1}dt,\quad \theta^{20}=-\theta^{10}\int\chi^{-2}dt.
\end{aligned}$$

Note 5.4. In formulas (5.1) and (5.5) solutions of the NSEs (1.1) are expressed in terms of solutions of the decomposed system of two linear one-dimensional heat equations (LOHEs) that have the form:

$$g^i_\tau=g^i_{\omega\omega}. \tag{5.13}$$

The Lie symmetry of the LOHE are known. Large sets of its exact solutions were constructed [27, 3]. The Q-conditional symmetries of LOHE were investigated in [14]. Moreover, being decomposed system (5.13) admits transformations of the form

$$\tilde g^1(\tau',\omega') = F^1(\tau,\omega,g^1(\tau,\omega)), \quad \tau' = G^1(\tau,\omega), \quad \omega' = H^1(\tau,\omega),$$
$$\tilde g^2(\tau'',\omega'') = F^2(\tau,\omega,g^2(\tau,\omega)), \quad \tau'' = G^2(\tau,\omega), \quad \omega'' = H^2(\tau,\omega),$$

where $(G^1,H^1)\neq(G^2,H^2)$, i.e. the independent variables can be transformed in the functions g^1 and g^2 in different ways. A similar statement is true for system (5.19)–(5.20) (see below) if $\varepsilon=0$.

Note 5.5. It can be proved that an arbitrary Navier–Stokes field $(\vec u,p)$, where

$$\vec u = \vec w(t,\omega) + (\vec k^i(t)\cdot\vec x)\vec l^i(t)$$

with $\vec k^i,\vec l^i \in C^\infty((t_0,t_1),\mathbb{R}^3)$, $\vec k^1\times\vec k^2\neq 0$, and $\omega=(\vec k^1\times\vec k^2)\cdot\vec x$, is equivalent to either a solution from family (5.1) or a solution from family (5.5). The equivalence transformation is generated by $R(\vec m)$ and $Z(\chi)$.

5.2. Investigation of system (3.13)–(3.16)

Consider system 8 from Subsection 3.2, i.e., equations (3.13)–(3.16). Equation (3.16) immediately gives

$$w^1 = -\tfrac12\rho_t\rho^{-1} + (\eta-1)z_2^{-2}, \tag{5.14}$$

where $\eta=\eta(t)$ is an arbitrary smooth function of $z_1=t$. Substituting (5.14) into remaining equations (5.13)–(5.15), we get

$$q_2 = \tfrac12\big((\rho_t\rho^{-1})_t - \tfrac12(\rho_t\rho^{-1})^2\big)z_2 - \eta_t z_2^{-1} - (\eta-1)^2z_2^{-3} + (w^2-\chi)^2z_2^{-3}, \tag{5.15}$$

$$w^2_1 - w^2_{22} + \big(\eta z_2^{-1} - \tfrac12\rho_t\rho^{-1}z_2\big)w^2_2 = 0, \tag{5.16}$$

$$w^3_1 - w^3_{22} + \big(\eta z_2^{-1} - \tfrac12\rho_t\rho^{-1}z_2\big)w^3_2 + \varepsilon(w^2-\chi)z_2^{-2} = 0. \tag{5.17}$$

Recall that $\rho=\rho(t)$ and $\chi=\chi(t)$ are arbitrary smooth functions of t; $\varepsilon\in\{0;1\}$. After the change of the independent variables

$$\tau = \textstyle\int|\rho(t)|dt, \quad z = |\rho(t)|^{1/2}z_2 \tag{5.18}$$

in equations (5.16) and (5.17), we obtain a linear system of a simpler form:

$$w^2_\tau - w^2_{zz} + \hat\eta(\tau)z^{-1}w^2_z = 0, \tag{5.19}$$

$$w^3_\tau - w^3_{zz} + (\hat\eta(\tau)-2)z^{-1}w^3_z + \varepsilon(w^2-\hat\chi(\tau))z^{-2} = 0, \tag{5.20}$$

where $\hat\eta(\tau)=\eta(t)$ and $\hat\chi(\tau)=\chi(t)$. Equation (5.15) implies

$$\begin{aligned} q = {}& \tfrac14\big((\rho_t\rho^{-1})_t - \tfrac12(\rho_t\rho^{-1})^2\big)z_2^2 - \eta_t\ln|z_2| - \\ & - \tfrac12(\eta-1)^2z_2^{-2} + \textstyle\int(w^2(\tau,z)-\hat\chi(\tau))^2z_2^{-3}dz_2. \end{aligned} \tag{5.21}$$

Formulas (5.14), (5.18)–(5.21), and ansatz (3.8) determine a solution of the NSEs (1.1).

If $\varepsilon = 0$ system (5.19)–(5.20) is decomposed and consists of two translational linear equations of the general form

$$f_\tau + \tilde\eta(\tau) z^{-1} f_z - f_{zz} = 0, \tag{5.22}$$

where $\tilde\eta = \hat\eta$ ($\tilde\eta = \hat\eta - 2$) for equation (5.19) ((5.20)). Tilde over η is omitted below. Let us investigate symmetry properties of equation (5.22) and construct some of its exact solutions.

Theorem 5.1. *The MIA of* (5.22) *is given by the following algebras*

a) $L_1 = \langle f\partial_f,\ g(\tau,z)\partial_f\rangle$ *if* $\eta(\tau) \neq \mathrm{const}$;

b) $L_2 = \langle \partial_\tau,\ \hat D,\ \Pi,\ f\partial_f,\ g(\tau,z)\partial_f\rangle$ *if* $\eta(\tau) = \mathrm{const}$, $\eta \notin \{0; -2\}$;

c) $L_3 = \langle \partial_\tau,\ \hat D,\ \Pi,\ \partial_z + \frac{1}{2}\eta z^{-1} f\partial_f,\ G = 2\tau\partial_\tau - (z - \eta z^{-1}\tau) f\partial_f,\ f\partial_f,$ $g(\tau,z)\partial_f\rangle$ *if* $\eta \in \{0; -2\}$.

Here $\hat D = 2\tau\partial_\tau + z\partial_z$, $\Pi = 4\tau^2\partial_\tau + 4\tau z\partial_z - (z^2 + 2(1-\eta)\tau) f\partial_f$; $g = g(\tau,z)$ *is an arbitrary solution of* (5.22).

When $\eta = 0$, equation (5.22) is the heat equation, and, when $\eta = -2$, it is reduced to the heat equation by means of the change $\tilde f = zf$.

For the case $\eta = \mathrm{const}$ equation (5.22) can be reduced by inequivalent one-dimensional subalgebras of L_2. We construct the following solutions:

For the subalgebra $\langle \partial_\tau + af\partial_f\rangle$, where $a \in \{-1; 0; 1\}$, it follows that

$$f = e^{-\tau} z^\nu (C_1 J_\nu(z) + C_2 Y_\nu(z)) \quad \text{if} \quad a = -1,$$
$$f = e^{\tau} z^\nu (C_1 I_\nu(z) + C_2 K_\nu(z)) \quad \text{if} \quad a = 1,$$
$$f = C_1 z^{\eta+1} + C_2 \quad \text{if} \quad a = 0 \quad \text{and} \quad \eta \neq -1,$$
$$f = C_1 \ln z + C_2 \quad \text{if} \quad a = 0 \quad \text{and} \quad \eta = -1.$$

Here J_ν and Y_ν are the Bessel functions of a real variable, whereas I_ν and K_ν are the Bessel functions of an imaginary variable, and $\nu = \frac{1}{2}(\eta + 1)$.

For the subalgebra $\langle \hat D + 2af\partial_f\rangle$, where $a \in \mathbb{R}$, it follows that

$$f = |\tau|^a e^{-\frac{1}{2}\omega} |\omega|^{\frac{1}{2}(\eta-1)} W\left(\tfrac{1}{4}(\eta-1) - a,\ \tfrac{1}{4}(\eta+1), \omega\right)$$

with $\omega = \frac{1}{4} z^2 \tau^{-1}$. Here $W(\varkappa, \mu, \omega)$ is the general solution of the Whittaker equation

$$4\omega^2 W_{\omega\omega} = (\omega^2 - 4\varkappa\omega + 4\mu^2 - 1) W.$$

For the subalgebra $\langle \partial_\tau + \Pi + af\partial_f\rangle$, where $a \in \mathbb{R}$, it follows that

$$f = (4\tau^2 + 1)^{\frac{1}{4}(\eta-1)} \exp(-\tau\omega + \tfrac{1}{2} a \arctan 2\tau) \varphi(\omega)$$

with $\omega = z^2 (4\tau^2 + 1)^{-1}$. The function φ is a solution of the equation

$$4\omega\varphi_{\omega\omega} + 2(1-\eta)\varphi_\omega + (\omega - a)\varphi = 0.$$

For example if $a = 0$, then $\varphi(\omega) = \omega^\mu \Big(C_1 J_\mu(\frac{1}{2}\omega) + C_2 Y_\mu(\frac{1}{2}\omega)\Big)$, where $\mu = \frac{1}{4}(\eta + 1)$.

Consider equation (5.22), where η is an arbitrary smooth function of τ.

Theorem 5.2. *Equation* (5.22) *is Q-conditionally invariant under the operators*

$$Q^1 = \partial_\tau + g^1(\tau,z)\partial_z + \big(g^2(\tau,z)f + g^3(\tau,z)\big)\partial_f \tag{5.23}$$

if and only if

$$\begin{aligned} &g^1_\tau - \eta z^{-1}g^1_z + \eta z^{-2}g^1 - g^1_{zz} + 2g^1_z g^1 - \eta_\tau z^{-1} + 2g^2_z = 0,\\ &g^k_\tau + \eta z^{-1}g^k_z - g^k_{zz} + 2g^1_z g^k = 0, \quad k = 2,3, \end{aligned} \tag{5.24}$$

and

$$Q^2 = \partial_z + B(\tau,z,f)\partial_f \tag{5.25}$$

if and only if

$$B_\tau - \eta z^{-2}B + \eta z^{-1}B_z - B_{zz} - 2BB_{zf} - B^2 B_{ff} = 0. \tag{5.26}$$

An arbitrary operator of Q-conditional symmetry of equation (5.22) *is equivalent to either an operator of form* (5.23) *or an operator of form* (5.25).

Theorem 5.2 is proved by means of the method described in [13].

Note 5.6. It can be shown (in a way analogous to one in [13]) that system (5.24) is reduced to the decomposed linear system

$$f^a_\tau + \eta z^{-1}f^a_z - f^a_{zz} = 0 \tag{5.27}$$

by means of the following non-local transformation

$$\begin{aligned} g^1 &= -\frac{f^1_{zz}f^2 - f^1 f^2_{zz}}{f^1_z f^2 - f^1 f^2_z} + \eta z^{-1},\\ g^2 &= -\frac{f^1_{zz}f^2_z - f^1_z f^2_{zz}}{f^1_z f^2 - f^1 f^2_z},\\ g^3 &= f^3_{zz} - \eta z^{-1}f^3_z + g^1 f^3_z - g^2 f^3. \end{aligned} \tag{5.28}$$

Equation (5.26) is reduced, by means of the change

$$B = -\Phi_\tau/\Phi_f, \quad \Phi = \Phi(\tau,z,f)$$

and the hodograph transformation

$$y_0 = \tau, \quad y_1 = z, \quad y_2 = \Phi, \quad \Psi = f,$$

to the following equation in the function $\Psi = \Psi(y_0,y_1,y_2)$:

$$\Psi_{y_0} + \eta(y_0)y_1^{-1}\Psi_{y_1} - \Psi_{y_1y_1} = 0.$$

Therefore, unlike Lie symmetries Q-conditional symmetries of (5.22) are more extended for an arbitrary smooth function $\eta = \eta(\tau)$. Thus, Theorem 5.2 implies that equation (5.22) is Q-conditionally invariant under the operators

$$\partial_z, \quad X = \partial_\tau + (\eta - 1)z^{-1}\partial_z, \quad G = (2\tau + C)\partial_z - zf\partial_f$$

with $C = \text{const}$. Reducing equation (5.22) by means of the operator G, we obtain the following solution:

$$f = C_2\bigl(z^2 - 2\textstyle\int(\eta(\tau) - 1)d\tau\bigr) + C_1. \tag{5.29}$$

In generalizing this we can construct solutions of the form

$$f = \sum_{k=0}^{N} T^k(\tau)z^{2k}, \tag{5.30}$$

where the coefficients $T^k = T^k(\tau)$ $(k = \overline{0, N})$ satisfy the system of ODEs:

$$T^k_\tau + (2k + 2)(\eta(\tau) - 2k - 1)T^{k+1} = 0, \quad k = \overline{0, N-1}, \quad T^N_\tau = 0. \tag{5.31}$$

Equation (5.31) is easily integrated for arbitrary $N \in \mathbb{N}$. For example if $N = 2$, it follows that

$$f = C_3\Bigl\{z^4 - 4z^2\textstyle\int(\eta(\tau) - 3)d\tau + 8\int\Bigl((\eta(\tau) - 1)\int(\eta(\tau) - 3)d\tau\Bigr)d\tau\Bigr\} +$$
$$+ C_2\Bigl\{z^2 - 2\textstyle\int(\eta(\tau) - 1)d\tau\Bigr\} + C_1.$$

An explicit form for solution (5.30) with $N = 1$ is given by (5.29).

Generalizing the solution

$$f = C_0 \exp\{-z^2(4\tau + 2C)^{-1} + \textstyle\int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\} \tag{5.32}$$

obtained by means of reduction of (5.22) by the operator G, we can construct solutions of the general form

$$f = \sum_{k=0}^{N} S^k(\tau)\bigl(z(2\tau + C)^{-1}\bigr)^{2k} \times \\ \times \exp\Bigl\{-z^2(4\tau + 2C)^{-1} + \textstyle\int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\Bigr\}, \tag{5.33}$$

where the coefficients $S^k = S^k(\tau)$ $(k = \overline{0, N})$ satisfy the system of ODEs:

$$S^k_\tau + (2k + 2)(\eta(\tau) - 2k - 1)(2\tau + C)^{-2}S^{k+1} = 0, \\ k = \overline{0, N-1}, \quad S^N_\tau = 0. \tag{5.34}$$

For example if $N = 1$, then

$$f = \Bigl\{C_1\Bigl(z^2(2\tau + C)^{-2} - 2\textstyle\int(\eta(\tau) - 1)(2\tau + C)^{-2}d\tau\Bigr) + C_0\Bigr\} \times$$
$$\times \exp\Bigl\{-z^2(4\tau + 2C)^{-1} + \textstyle\int(\eta(\tau) - 1)(2\tau + C)^{-1}d\tau\Bigr\}.$$

Here we do not present results for arbitrary N as they are very cumbersome.

Putting $g^2 = g^3 = 0$ in system (5.24), we obtain one equation in the function g^1:

$$g^1_\tau - \eta z^{-1}g^1_z + \eta z^{-2}g^1 - g^1_{zz} + 2g^1_z g^1 - \eta_\tau z^{-1} = 0.$$

It follows that $g^1 = -g_z/g + (\eta - 1)/z$, where $g = g(\tau, z)$ is a solution of the equation

$$g_\tau + (\eta - 2)z^{-1}g_z - g_{zz} = 0. \tag{5.35}$$

Q-conditional symmetry of (5.22) under the operator

$$Q = \partial_\tau + \bigl(-g_z/g + (\eta - 1)/z\bigr)\partial_z \tag{5.36}$$

gives rise to the following

Theorem 5.3. *If g is a solution of equation* (5.35) *and*

$$f(\tau,z)=\int_{z_0}^{z} z'g(\tau,z')dz' + \int_{\tau_0}^{\tau}\Big(z_0g_z(\tau',z_0)-(\eta(\tau')-1)g(\tau',z_0)\Big)d\tau', \tag{5.37}$$

where (τ_0,z_0) is a fixed point, then f is a solution of equation (5.22)*.*

Proof. Equation (5.35) implies

$$(zg)_\tau=(zg_z-(\eta-1)g)_z$$

Therefore, $f_z=zg,\quad f_\tau=zg_z-(\eta-1)g$ and

$$f_\tau+\eta z^{-1}f_z-f_{zz}=zg_z-(\eta-1)g+\eta g-(zg)_z=0. \qquad \text{QED.}$$

The converse of Theorem 5.3 is the following obvious

Theorem 5.4. *If f is a solution of* (5.22)*, the function*

$$g=z^{-1}f_z \tag{5.38}$$

satisfies (5.35)*.*

Theorems 5.3 and 5.4 imply that, when $\eta=2n$ ($n\in\mathbb{Z}$), solutions of (5.22) can be constructed from known solutions of the heat equation by means of applying either formula (5.37) (for $n>0$) or formula (5.38) (for $n<0$) $|n|$ times.

Let us investigate symmetry properties and construct some exact solutions of system (5.19)–(5.20) for $\varepsilon=1$, i.e., the system

$$w^1_\tau-w^1_{zz}+\hat\eta(\tau)z^{-1}w^1_z=0, \tag{5.39}$$

$$w^2_\tau-w^2_{zz}+(\hat\eta(\tau)-2)z^{-1}w^2_z+(w^1-\hat\chi(\tau))z^{-2}=0. \tag{5.40}$$

If (w^1,w^2) is a solution of system (5.39)–(5.40), then (w^1,w^2+g) (where $g=g(\tau,z)$) is also a solution of (5.39)–(5.40) if and only if the function g satisfies the following equation

$$g_\tau-g_{zz}+(\hat\eta(\tau)-2)z^{-1}g_z=0 \tag{5.41}$$

System (5.39)–(5.40), for some $\hat\chi=\hat\chi(\tau)$, has particular solutions of the form

$$w^1=\sum_{k=0}^{N}T^k(\tau)z^{2k},\quad w^2=\sum_{k=0}^{N-1}S^k(\tau)z^{2k},$$

where $T^0(\tau)=\hat\chi(\tau)$. For example, if $\hat\chi(\tau)=-2C_1\int(\hat\eta(\tau)-1)d\tau+C_2$ and $N=1$, then

$$w^1=C_1\big(z^2-2\textstyle\int(\hat\eta(\tau)-1)d\tau\big)+C_2,\quad w^2=-C_1\tau.$$

Let $\hat\chi(\tau)=0$.

Theorem 5.5. *The MIA of system* (5.39)–(5.40) *with* $\hat{\chi}(\tau) = 0$ *is given by the following algebras*

$$
\begin{aligned}
&a)\quad \langle w^i\partial_{w^i},\ \tilde{w}^i(\tau,z)\partial_{w^i}\rangle \quad \textit{if}\ \ \hat{\eta}(\tau)\neq \mathrm{const};\\
&b)\quad \langle 2\tau\partial_\tau + z\partial_z,\ \partial_\tau,\ w^i\partial_{w^i},\ \tilde{w}^i(\tau,z)\partial_{w^i}\rangle \quad \textit{if}\ \ \hat{\eta}(\tau) = \mathrm{const},\ \hat{\eta}\neq 0;\\
&c)\quad \langle 2\tau\partial_\tau + z\partial_z,\ \partial_\tau,\ w^1 z^{-1}\partial_{w^2},\ w^i\partial_{w^i},\ \tilde{w}^i(\tau,z)\partial_{w^i}\rangle \quad \textit{if}\ \ \hat{\eta}\equiv 0.
\end{aligned}
$$

Here $(\tilde{w}^1, \tilde{w}^2)$ *is an arbitrary solution of* (5.39)–(5.40) *with* $\hat{\chi}(\tau) = 0$.

For the case $\hat{\chi}(\tau) = 0$ and $\hat{\eta}(\tau) = \mathrm{const}$ system (5.39)–(5.40) can be reduced by inequivalent one-dimensional subalgebras of its MIA. We obtain the following solutions:

For the subalgebra $\langle\partial_\tau\rangle$ it follows that

$$
\begin{aligned}
w^1 &= C_1 \ln z + C_2,\\
w^2 &= \tfrac14 C_1(\ln^2 z - \ln z) + \tfrac12 C_2 \ln z + C_3 z^{-2} + C_4
\end{aligned}
$$

if $\hat{\eta} = -1$;

$$
\begin{aligned}
w^1 &= C_1 z^2 + C_2,\\
w^2 &= \tfrac14 C_1 z^2 + \tfrac12 C_2 \ln^2 z + C_3 \ln z + C_4
\end{aligned}
$$

if $\hat{\eta} = 1$;

$$
\begin{aligned}
w^1 &= C_1 z^{\hat{\eta}+1} + C_2,\\
w^2 &= \tfrac12 C_1(\hat{\eta}+1)^{-1} z^{\hat{\eta}+1} + C_2(\hat{\eta}-1)^{-1}\ln z + C_3 z^{\hat{\eta}-1} + C_4
\end{aligned}
$$

if $\hat{\eta} \notin \{-1; 1\}$.

For the subalgebra $\langle\partial_\tau - w^i\partial_{w^i}\rangle$ it follows that

$$
w^1 = e^{-\tau} z^{\frac12(\hat{\eta}+1)}\psi^1(z), \quad w^2 = e^{-\tau} z^{\frac12(\hat{\eta}-1)}\psi^2(z),
$$

where the functions ψ^1 and ψ^2 satisfy the system

$$
z^2\psi^1_{zz} + z\psi^1_z + \left(z^2 - \tfrac14(\hat{\eta}+1)^2\right)\psi^1 = 0, \tag{5.42}
$$

$$
z^2\psi^2_{zz} + z\psi^2_z + \left(z^2 - \tfrac14(\hat{\eta}-1)^2\right)\psi^2 = z\psi^1. \tag{5.43}
$$

The general solution of system (5.42)–(5.43) can be expressed by quadratures in terms of the Bessel functions of a real variable $J_\nu(z)$ and $Y_\nu(z)$:

$$
\begin{aligned}
\psi^1 &= C_1 J_{\nu+1}(z) + C_2 Y_{\nu+1}(z),\\
\psi^2 &= C_3 J_\nu(z) + C_4 Y_\nu(z) + \tfrac{\pi}{2} Y_\nu(z)\int J_\nu(z)\psi^1(z)dz - \tfrac{\pi}{2} J_\nu(z)\int Y_\nu(z)\psi^1(z)dz
\end{aligned}
$$

with $\nu = \frac12(\hat{\eta}-1)$;

For the subalgebra $\langle\partial_\tau + w^i\partial_{w^i}\rangle$ it follows that

$$
w^1 = e^{\tau} z^{\frac12(\hat{\eta}+1)}\psi^1(z), \quad w^2 = e^{\tau} z^{\frac12(\hat{\eta}-1)}\psi^2(z),
$$

where the functions ψ^1 and ψ^2 satisfy the system

$$
z^2\psi^1_{zz} + z\psi^1_z - \left(z^2 + \tfrac14(\hat{\eta}+1)^2\right)\psi^1 = 0, \tag{5.44}
$$

$$z^2\psi^2_{zz} + z\psi^2_z - \left(z^2 + \tfrac{1}{4}(\hat{\eta}-1)^2\right)\psi^2 = z\psi^1. \tag{5.45}$$

The general solution of system (5.44)–(5.45) can be expressed by quadratures in terms of the Bessel functions of an imaginary variable $I_\nu(z)$ and $K_\nu(z)$:

$$\psi^1 = C_1 I_{\nu+1}(z) + C_2 K_{\nu+1}(z),$$
$$\psi^2 = C_3 I_\nu(z) + C_4 K_\nu(z) + K_\nu(z)\int I_\nu(z)\psi^1(z)dz - I_\nu(z)\int K_\nu(z)\psi^1(z)dz$$

with $\nu = \frac{1}{2}(\hat{\eta}-1)$.

For the subalgebra $\langle 2\tau\partial_\tau + z\partial_z + aw^i\partial_{w^i}\rangle$ it follows that

$$w^1 = |\tau|^a e^{-\frac{1}{2}\omega}|\omega|^{\frac{1}{4}(\hat{\eta}-1)}\psi^1(\omega), \quad w^2 = |\tau|^a e^{-\frac{1}{2}\omega}|\omega|^{\frac{1}{4}(\hat{\eta}-3)}\psi^2(\omega)$$

with $\omega = \frac{1}{4}z^2\tau^{-1}$, where the functions ψ^1 and ψ^2 satisfy the system

$$4\omega^2\psi^1_{\omega\omega} = \Big(\omega^2 + \big(a - \tfrac{1}{4}(\hat{\eta}-1)\big)\omega + \tfrac{1}{4}(\hat{\eta}+1)^2 - 1\Big)\psi^1, \tag{5.46}$$

$$4\omega^2\psi^2_{\omega\omega} = \Big(\omega^2 + \big(a - \tfrac{1}{4}(\hat{\eta}-3)\big)\omega + \tfrac{1}{4}(\hat{\eta}-1)^2 - 1\Big)\psi^2 + 2|\omega|^{1/2}\psi^1. \tag{5.47}$$

The general solution of system (5.46)–(5.47) can be expressed by quadratures in terms of the Whittaker functions.

6. Symmetry properties and exact solutions of system (3.12)

As was mentioned in Section 3, ansatzes (3.4)–(3.7) reduce the NSEs (1.1) to the systems of PDEs of a similar structure that have the general form (see (3.12)):

$$\begin{aligned} &w^i w^1_i - w^1_{ii} + s_1 + \alpha_2 w^2 = 0, \\ &w^i w^2_i - w^2_{ii} + s_2 - \alpha_2 w^1 + \alpha_1 w^3 = 0, \\ &w^i w^3_i - w^3_{ii} + \alpha_4 w^3 + \alpha_5 = 0, \\ &w^i_i = \alpha_3, \end{aligned} \tag{6.1}$$

where α_n $(n = \overline{1,5})$ are real parameters.

Setting $\alpha_k = 0$ $(k = \overline{2,5})$ in (6.1), we obtain equations describing a plane convective flow that is brought about by nonhomogeneous heating of boundaries [25]. In this case w^i are the coordinates of the flow velocity vector, w^3 is the flow temperature, s is the pressure, the Grasshoff number λ is equal to $-\alpha_1$, and the Prandtl number σ is equal to 1. Some similarity solutions of these equations were constructed in [22]. The particular case of system (6.1) for $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 0$ and $\alpha_3 = 1$ was considered in [31].

In this section we study symmetry properties of system (6.1) and construct large sets of its exact solutions.

Theorem 6.1. *The MIA of* (6.1) *is the algebra*

1. $E_1 = \langle \partial_1,\ \partial_2,\ \partial_s \rangle$ *if* $\alpha_1 \neq 0,\ \alpha_4 \neq 0.$
2. $E_2 = \langle \partial_1,\ \partial_2,\ \partial_s,\ \partial_{w^3} - \alpha_1 z_2 \partial_s \rangle$ *if* $\alpha_1 \neq 0,\ \alpha_4 = 0,\ (\alpha_1, \alpha_2, \alpha_5) \neq (0,0,0).$
3. $E_3 = \langle \partial_1,\ \partial_2,\ \partial_s,\ \partial_{w^3} - \alpha_1 z_2 \partial_s,\ \tilde{D} - 3w^3 \partial_{w^3} \rangle$ *if* $\alpha_1 \neq 0,\ \alpha_k = 0,\ k = \overline{2,5}.$
4. $E_4 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ (w^3 + \alpha_5/\alpha_4)\partial_{w^3} \rangle$ *if* $\alpha_1 = 0,\ \alpha_4 \neq 0.$
5. $E_5 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3} \rangle$ *if* $\alpha_1 = \alpha_4 = 0,\ (\alpha_2, \alpha_3) \neq (0,0),\ \alpha_5 \neq 0.$
6. $E_6 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ w^3 \partial_{w^3} \rangle$ *if* $\alpha_1 = \alpha_4 = \alpha_5 = 0,\ (\alpha_2, \alpha_3) \neq (0,0).$
7. $E_7 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ \tilde{D} + 2w^3 \partial_{w^3} \rangle$ *if* $\alpha_5 \neq 0,\ \alpha_l = 0,\ l = \overline{1,4}.$
8. $E_8 = \langle \partial_1,\ \partial_2,\ \partial_s,\ J,\ \partial_{w^3},\ \tilde{D}, w^3 \partial_{w^3} \rangle$ *if* $\alpha_n = 0,\ n = \overline{1,5}.$

Here $\tilde{D} = z_i \partial_i - w^i \partial_{w^i} - 2s\partial_s$, $J = z_1\partial_2 - z_2\partial_1 + w^1\partial_{w^2} - w^2\partial_{w^1}$, $\partial_i = \partial_{z_i}$.

Note 6.1. The bases of the algebras E_6 and E_8 contain the operator $w^3\partial_{w^3}$ that is not induced by elements of $A(\mathrm{NS})$.

Note 6.2. If $\alpha_4 \neq 0$, the constant α_5 can be made to vanish by means of local transformation

$$\tilde{w}^3 = w^3 + \alpha_5/\alpha_4, \quad \tilde{s} = s - \alpha_1\alpha_5\alpha_4^{-1} z_2, \tag{6.2}$$

where the independent variables and the functions w^i are not transformed. Therefore, we consider below that $\alpha_5 = 0$ if $\alpha_4 \neq 0$.

Note 6.3. Making the non-local transformation

$$\tilde{s} = s + \alpha_2 \Psi, \tag{6.3}$$

where $\Psi_1 = w^2$, $\Psi_2 = -w^1$ (such a function Ψ exists in view of the last equation of (6.1)), in system (6.1) with $\alpha_3 = 0$, we obtain a system of form (6.1) with $\tilde{\alpha}_3 = \tilde{\alpha}_2 = 0$. In some cases ($\alpha_1 \neq 0$, $\alpha_3 = \alpha_4 = \alpha_5 = 0$, $\alpha_2 \neq 0$; $\alpha_1 = \alpha_3 = \alpha_4 = 0$, $\alpha_2 \neq 0$) transformation (6.3) allows the symmetry of (6.1) to be extended and non-Lie solutions to be constructed. Moreover, it means that in the cases listed above system (6.1) is invariant under the non-local transformation

$$\hat{z}_i = e^{\varepsilon} z_i, \quad \hat{w}^i = e^{-\varepsilon} w^i, \quad \hat{w}^3 = e^{\delta\varepsilon} w^3, \quad \hat{s} = e^{-2\varepsilon} s + \alpha_2 (e^{-2\varepsilon} - 1)\Psi,$$

where

$$\begin{aligned}
&\delta = -3 \quad \text{if} \quad \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \alpha_1, \alpha_2 \neq 0;\\
&\delta = 2 \quad \text{if} \quad \alpha_1 = \alpha_3 = \alpha_4 = 0, \quad \alpha_2, \alpha_5 \neq 0;\\
&\delta = 0 \quad \text{if} \quad \alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 0, \quad \alpha_2 \neq 0.
\end{aligned}$$

Let us consider an ansatz of the form:

$$\begin{aligned}
&w^1 = a_1\varphi^1 - a_2\varphi^3 + b_1\omega_2,\\
&w^2 = a_2\varphi^1 + a_1\varphi^3 + b_2\omega_2,\\
&w^3 = \varphi^2 + b_3\omega_2,\\
&s = h + d_1\omega_2 + d_2\omega_1\omega_2 + \tfrac{1}{2} d_3\omega_2^2,
\end{aligned} \tag{6.4}$$

where $a_1^2 + a_2^2 = 1$, $\omega = \omega_1 = a_1 z_2 - a_2 z_1$, $\omega_2 = a_1 z_1 + a_2 z_2$, $B, b_a, d_a = \mathrm{const}$,

$$\begin{aligned} &b_i = B a_i, \quad b_3(B + \alpha_4) = 0, \\ &d_2 = \alpha_2 B - \alpha_1 b_3 a_1, \quad d_3 = -B^2 - \alpha_1 b_3 a_2, \end{aligned} \tag{6.5}$$

Here and below $\varphi^a = \varphi^a(\omega)$ and $h = h(\omega)$. Indeed, formulas (6.4) and (6.5) determine a whole set of ansatzes for system (6.1). This set contains both Lie ansatzes, constructed by means of subalgebras of the form

$$\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2 \partial_s) + a_4\partial_s\rangle, \tag{6.6}$$

and non-Lie ansatzes. Equation (6.5) is the necessary and sufficient condition to reduce (6.1) by means of an ansatz of form (6.3). As a result of reduction we obtain the following system of ODEs:

$$\begin{aligned} &\varphi^3\varphi^1_\omega - \varphi^1_{\omega\omega} + \mu_{1j}\varphi^j + d_1 + d_2\omega + \alpha_2\varphi^3 = 0, \\ &\varphi^3\varphi^2_\omega - \varphi^2_{\omega\omega} + \mu_{2j}\varphi^j + \alpha_5 = 0, \\ &\varphi^3\varphi^3_\omega - \varphi^3_{\omega\omega} + h_\omega - \alpha_2\varphi^1 + \alpha_1 a_1\varphi^2 = 0, \\ &\varphi^3_\omega = \sigma, \end{aligned} \tag{6.7}$$

where $\mu_{11} = -B$, $\mu_{12} = -\alpha_1 a_2$, $\mu_{21} = -b_3$, $\mu_{22} = -\alpha_4$, $\sigma = \alpha_3 - B$. If $\sigma = 0$, system (6.7) implies that

$$\begin{aligned} &\varphi^3 = C_0 = \mathrm{const}, \\ &h = \alpha_2 \textstyle\int \varphi^1(\omega)d\omega - \alpha_1 a_1 \int \varphi^2(\omega)d\omega, \end{aligned}$$

and the functions φ^i satisfy system (4.23), where $\nu_{11} = d_1 + \alpha_2 C_0$, $\nu_{21} = d_2$, $\nu_{12} = \alpha_5$, $\nu_{22} = 0$. If $\sigma \neq 0$, then $\varphi^3 = \sigma\omega$ (translating ω, the integration constant can be made to vanish),

$$h = -\tfrac{1}{2}\sigma^2\omega^2 + \alpha_2 \textstyle\int \varphi^1(\omega)d\omega - \alpha_1 a_1 \int \varphi^2(\omega)d\omega,$$

and the functions satisfy system (4.29), where $\nu_{11} = d_1$, $\nu_{21} = d_2 + \alpha_2\sigma$, $\nu_{12} = \alpha_5$, $\nu_{22} = 0$.

Note 6.4. Step-by-step reduction of the NSEs (1.1) by means of ansatzes (3.4)–(3.7) and (6.4) is equivalent to a particular case of immediate reduction of the NSEs (1.1) to ODEs by means of ansatzes 5 and 6 from Subsection 4.1.

Now let us choose such algebras, among the algebras from Table 1, that can be used to reduce system (6.1) and do not belong to the set of algebras (6.6). By means of the chosen algebras we construct ansatzes that are tabulated in the form of Table 2.

Substituting the ansatzes from Table 2 into system (6.1), we obtain the reduced systems of ODEs in the functions φ^a and h:

$$\begin{aligned} 1.\quad &\varphi^2\varphi^1_\omega - \varphi^1_{\omega\omega} - \varphi^1\varphi^1 - \varphi^2\varphi^2 - 2h + \alpha_1\varphi^3\sin\omega + 2\varphi^2_\omega = 0, \\ &\varphi^2\varphi^2_\omega - \varphi^2_{\omega\omega} + h_\omega - 2\varphi^1_\omega + \alpha_1\varphi^3\cos\omega = 0, \\ &\varphi^2\varphi^3_\omega - \varphi^3_{\omega\omega} - 3\varphi^1\varphi^3 - 9\varphi^3 = 0, \\ &\varphi^2_\omega = 0. \end{aligned} \tag{6.8}$$

Table 1. Complete sets of inequivalent one-dimensional subalgebras of the algebras $E_1 - E_8$ (a and a_l ($l = \overline{1,4}$) are real constants)

Algebra	Subalgebras	Values of parameters
E_1	$\langle a_1\partial_1 + a_2\partial_2 + a_3\partial_s\rangle$, $\langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1$
E_2	$\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s)\rangle$, $\langle\partial_1 + a_4\partial_s\rangle$, $\langle\partial_{w^3} - \alpha_1 z_2\partial_s\rangle$, $\langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1$, $a_4 \neq 0$
E_3	$\langle a_1\partial_1 + a_2\partial_2 + a_3(\partial_{w^3} - \alpha_1 z_2\partial_s)\rangle$, $\langle\partial_1 + a_4\partial_s\rangle$, $\langle\tilde{D} - 3w^3\partial_{w^3}\rangle$, $\langle\partial_{w^3} - \alpha_1 z_2\partial_s\rangle$, $\langle\partial_s\rangle$	$a_1^2 + a_2^2 = 1$, $a_3 \in \{-1; 0; 1\}$, $a_4 \in \{-1; 1\}$
E_4	$\langle J + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle$, $\langle w^3\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_s\rangle$	
E_5	$\langle J + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_{w^3} + a_1\partial_s >, < \partial_s\rangle$	
E_6	$\langle J + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3}\rangle$, $\langle J + a_1\partial_s + a_3\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_3\partial_{w^3}\rangle$, $\langle w^3\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_s\rangle$	$a_2 \neq 0$, $a_3 \in \{-1; 0; 1\}$
E_7	$\langle\tilde{D} + aJ + 2w^3\partial_{w^3}\rangle$, $\langle J + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_{w^3} + a_2\partial_s\rangle$, $\langle\partial_s\rangle$	$a_2 \in \{-1; 0; 1\}$, $a_1 \in \{-1; 0; 1\}$ if $a_2 = 0$
E_8	$\langle\tilde{D} + aJ + a_3 w^3\partial_{w^3}\rangle$, $\langle\tilde{D} + aJ + a_3\partial_{w^3}\rangle$, $\langle J + a_1\partial_s + a_4 w^3\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_4 w^3\partial_{w^3}\rangle$, $\langle J + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle\partial_2 + a_1\partial_s + a_2\partial_{w^3}\rangle$, $\langle w^3\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_{w^3} + a_1\partial_s\rangle$, $\langle\partial_s\rangle$	$a_i \in \{-1; 0; 1\}$, $a_4 \neq 0$

Table 2. Ansatzes reducing system (6.1) ($r = (z_1^2 + z_2^2)^{1/2}$)

N	Values of α_n	Algebra	Invariant variable	Ansatz
1	$\alpha_1 \neq 0$, $\alpha_k = 0$, $k = \overline{2,5}$	$\langle \tilde{D} - 3w^3\partial_{w^3} \rangle$	$\omega = \arctan\frac{z_2}{z_1}$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^{-3}\varphi^3,\ s = r^{-2}h$
2	$\alpha_1 = 0$, $\alpha_5 = 0$	$\langle \partial_2 + a_1\partial_s + a_2 w^3\partial_{w^3} \rangle$, $a_2 \neq 0$	$\omega = z_1$	$w^1 = \varphi^1, \quad w^2 = \varphi^2$, $w^3 = \varphi^3 e^{a_2 z_2}$, $s = h + a_1 z_2$
3	$\alpha_1 = 0$, $\alpha_4 = 0$	$\langle J + a_1\partial_s + a_2\partial_{w^3} \rangle$	$\omega = r$	$w^1 = z_1\varphi^1 - z_2 r^{-2}\varphi^2$, $w^2 = z_2\varphi^1 + z_1 r^{-2}\varphi^2$, $w^3 = \varphi^3 + a_2 \arctan\frac{z_2}{z_1}$, $s = h + a_1 \arctan\frac{z_2}{z_1}$
4	$\alpha_1 = 0$, $\alpha_5 = 0$	$\langle J + a_1\partial_s + a_2 w_3\partial_{w^3} \rangle$ $a_2 \neq 0 \quad \text{if} \quad \alpha_4 = 0$	$\omega = r$	$w^1 = z_1\varphi^1 - z_2 r^{-2}\varphi^2$, $w^2 = z_2\varphi^1 + z_1 r^{-2}\varphi^2$, $w^3 = \varphi^3 e^{a_2 \arctan\frac{z_2}{z_1}}$, $s = h + a_1 \arctan\frac{z_2}{z_1}$
5	$\alpha_5 \neq 0$, $\alpha_l = 0$, $l = \overline{1,4}$	$\langle \tilde{D} + aJ + 2w^3\partial_{w^3} \rangle$	$\omega = \arctan\frac{z_2}{z_1} -$ $- a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^2\varphi^3,\ s = r^{-2}h$
6	$\alpha_n = 0$, $n = \overline{1,5}$	$\langle \tilde{D} + aJ + a_1\partial_{w^3} \rangle$	$\omega = \arctan\frac{z_2}{z_1} -$ $- a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = \varphi^3 + a_1 \ln r$, $s = r^{-2}h$
7	$\alpha_n = 0$, $n = \overline{1,5}$	$\langle \tilde{D} + aJ + a_1 w^3\partial_{w^3} \rangle$, $a_1 \neq 0$	$\omega = \arctan\frac{z_2}{z_1} -$ $- a\ln r$	$w^1 = r^{-2}(z_1\varphi^1 - z_2\varphi^2)$, $w^2 = r^{-2}(z_2\varphi^1 + z_1\varphi^2)$, $w^3 = r^{a_1}\varphi^3,\ s = r^{-2}h$

2. $$\begin{aligned}&\varphi^1\varphi^1_\omega-\varphi^1_{\omega\omega}+\alpha_2\varphi^2+h_\omega=0,\\&\varphi^1\varphi^2_\omega-\varphi^2_{\omega\omega}-\alpha_2\varphi^1+a_1=0,\\&\varphi^1\varphi^3_\omega-\varphi^3_{\omega\omega}+(a_2\varphi^2+\alpha_4-a_2^2)\varphi^3=0,\\&\varphi^1_\omega=\alpha_3.\end{aligned}\tag{6.9}$$

3. $$\begin{aligned}&\omega\varphi^1\varphi^1_\omega-\varphi^1_{\omega\omega}+\varphi^1\varphi^1-\omega^{-4}\varphi^2\varphi^2-3\omega^{-1}\varphi^1_\omega+\alpha_2\omega^{-2}\varphi^2+\omega^{-1}h_\omega=0,\\&\omega\varphi^1\varphi^2_\omega-\varphi^2_{\omega\omega}+\omega^{-1}\varphi^2_\omega-\alpha_2\omega^2\varphi^1+a_1=0,\\&\omega\varphi^1\varphi^3_\omega-\varphi^3_{\omega\omega}+a_2\omega^{-2}\varphi^2-\omega^{-1}\varphi^3_\omega+\alpha_5=0,\\&2\varphi^1+\omega\varphi^1_\omega=\alpha_3.\end{aligned}\tag{6.10}$$

4. $$\begin{aligned}&\omega\varphi^1\varphi^1_\omega-\varphi^1_{\omega\omega}+\varphi^1\varphi^1-\omega^{-4}\varphi^2\varphi^2-3\omega^{-1}\varphi^1_\omega+\alpha_2\omega^{-2}\varphi^2+\omega^{-1}h_\omega=0,\\&\omega\varphi^1\varphi^2_\omega-\varphi^2_{\omega\omega}+\omega^{-1}\varphi^2_\omega-\alpha_2\omega^2\varphi^1+a_1=0,\\&\omega\varphi^1\varphi^3_\omega-\varphi^3_{\omega\omega}+a_2\omega^{-2}\varphi^2\varphi^3-\omega^{-1}\varphi^3_\omega+(\alpha_4-a_2^2\omega^{-2})\varphi^3=0,\\&2\varphi^1+\omega\varphi^1_\omega=\alpha_3.\end{aligned}\tag{6.11}$$

5. $$\begin{aligned}&(\varphi^2-a\varphi^1)\varphi^1_\omega-(1+a^2)\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-ah_\omega-2h=0,\\&(\varphi^2-a\varphi^1)\varphi^2_\omega-(1+a^2)\varphi^2_{\omega\omega}-2(a\varphi^2_\omega+\varphi^1_\omega)+h_\omega=0,\\&(\varphi^2-a\varphi^1)\varphi^3_\omega-(1+a^2)\varphi^3_{\omega\omega}+2\varphi^1\varphi^3-4\varphi^3+4a\varphi^3_\omega+\alpha_5=0,\\&\varphi^2_\omega-a\varphi^1_\omega=0.\end{aligned}\tag{6.12}$$

6. $$\begin{aligned}&(\varphi^2-a\varphi^1)\varphi^1_\omega-(1+a^2)\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-ah_\omega-2h=0,\\&(\varphi^2-a\varphi^1)\varphi^2_\omega-(1+a^2)\varphi^2_{\omega\omega}-2(a\varphi^2_\omega+\varphi^1_\omega)+h_\omega=0,\\&(\varphi^2-a\varphi^1)\varphi^3_\omega-(1+a^2)\varphi^3_{\omega\omega}+a_1\varphi^1=0,\\&\varphi^2_\omega-a\varphi^1_\omega=0.\end{aligned}\tag{6.13}$$

7. $$\begin{aligned}&(\varphi^2-a\varphi^1)\varphi^1_\omega-(1+a^2)\varphi^1_{\omega\omega}-\varphi^1\varphi^1-\varphi^2\varphi^2-ah_\omega-2h=0,\\&(\varphi^2-a\varphi^1)\varphi^2_\omega-(1+a^2)\varphi^2_{\omega\omega}-2(a\varphi^2_\omega+\varphi^1_\omega)+h_\omega=0,\\&(\varphi^2-a\varphi^1)\varphi^3_\omega-(1+a^2)\varphi^3_{\omega\omega}+a_1\varphi^1\varphi^3-a_1^2\varphi^3+2aa_1\varphi^3_\omega=0,\\&\varphi^2_\omega-a\varphi^1_\omega=0.\end{aligned}\tag{6.14}$$

Numeration of reduced systems (6.8)–(6.14) corresponds to that of the ansatzes in Table 2. Let us integrate systems (6.8)–(6.14) in such cases when it is possible. Below, in this section, $C_k=\mathrm{const}$ $(k=\overline{1,6})$.

1. We failed to integrate system (6.8) in the general case, but we managed to find the following particular solutions:

a) $\varphi^1=-6\wp(\omega+C_3,\frac{1}{3}(4-2C_1),C_2)-2,$
$\varphi^2=\varphi^3=0,\quad h=2\varphi^1+C_1;$

b) $\varphi^1=-6C_1^2e^{2C_1\omega}\wp(e^{C_1\omega}+C_3,0,C_2)+3C_1^2-2,$
$\varphi^2=5C_1,\quad \varphi^3=0,$
$h=-12C_1^2e^{2C_1\omega}\wp(e^{C_1\omega}+C_3,0,C_2)-2-\frac{13}{2}C_1^2-\frac{9}{4}C_1^4;$

c) $\varphi^1=C_1,\quad \varphi^2=C_2,\quad \varphi^3=0,\quad h=-\frac{1}{2}(C_1^2+C_2^2).$

Here $\wp(\tau, \varkappa_1, \varkappa_2)$ is the Weierstrass function that satisfies the equation (see [19]):

$$(\wp_\tau)^2 = 4\wp^3 - \varkappa_1\wp - \varkappa_2. \tag{6.15}$$

2. If $\alpha_3 = 0$, the last equation of (6.9) implies that $\varphi^1 = C_1$. It follows from the other equations of (6.9) that

$$\varphi^2 = C_3 + C_2 e^{C_1\omega} - (a_1 C_1^{-1} - \alpha_2)\omega,$$
$$h = C_6 - \alpha_2 C_3\omega - \alpha_2 C_2 C_1^{-1} e^{C_1\omega} + \tfrac{1}{2}\alpha_2(a_1 C_1^{-1} - \alpha_2)\omega^2$$

if $C_1 \neq 0$, and

$$\varphi^2 = C_3 + C_2\omega + \tfrac{1}{2}a_1\omega^2,$$
$$h = C_6 - \alpha_2 C_3\omega - \tfrac{1}{2}\alpha_2 C_2\omega^2 - \tfrac{1}{6}\alpha_2 a_1\omega^3$$

if $C_1 = 0$. The function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega} - C_1\varphi^3_\omega + (a_2^2 - \alpha_4 - a_2\varphi^2)\varphi^3 = 0. \tag{6.16}$$

We solve equation (6.16) for the following cases:

A. $C_2 = a_1 - \alpha_2 C_1 = 0$:

$$\varphi^3 = \begin{cases} e^{\frac{1}{2}C_1\omega}\big(C_4 e^{\mu^{1/2}\omega} + C_5 e^{-\mu^{1/2}\omega}\big), & \mu > 0, \\ e^{\frac{1}{2}C_1\omega}\big(C_4 + C_5\omega\big), & \mu = 0, \\ e^{\frac{1}{2}C_1\omega}\big(C_4\cos((-\mu)^{1/2}\omega) + C_5\sin((-\mu)^{1/2}\omega)\big), & \mu < 0, \end{cases}$$

where $\mu = \frac{1}{4}C_1^2 - a_2^2 + \alpha_4 + a_2 C_3$.

B. $C_1 = a_1 = 0$, $C_2 \neq 0$ ([19]):

$$\varphi^3 = \xi^{1/2} Z_{1/3}\big(\tfrac{2}{3}(-a_2C_2)^{1/2}\xi^{3/2}\big),$$

where $\xi = \omega + (C_3 a_2 - a_2^2 - \alpha_4)/(a_2 C_2)$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

C. $C_1 = 0$, $a_1 \neq 0$ ([19]):

$$\varphi^3 = (\omega + C_2 a_1^{-1})^{-1/2} W\big(\nu, \tfrac{1}{4}, (\tfrac{1}{2}a_1 a_2)^{-1/2}(\omega + C_2 a_1^{-1})^2\big),$$

where $\nu = \frac{1}{4}(\frac{1}{2}a_1 a_2)^{-1/2}\big(a_2^2 - \alpha_4 - a_2 C_3 + \frac{1}{2}a_2 C_3^2 a_1^{-1}\big)$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

D. $C_1 \neq 0$, $C_2 \neq 0$, $a_1 - \alpha_2 C_1 = 0$ ([19]):

$$\varphi^3 = e^{\frac{1}{2}C_1\omega} Z_\nu\big(2C_1^{-1}(-a_2C_2)^{1/2} e^{\frac{1}{2}C_1\omega}\big),$$

where $\nu = C_1^{-1}\big(C_1^2 + 4(\alpha_4 + a_2 C_3 - a_2^2)\big)^{1/2}$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

E. $C_1 \neq 0$, $a_1 - \alpha_2 C_1 \neq 0$, $C_2 = 0$ ([19]):

$$\varphi^3 = e^{\frac{1}{2}C_1\omega}\xi^{1/2} Z_{1/3}\Big(\tfrac{2}{3}\big(a_2(a_1C_1^{-1} - \alpha_2)\big)^{1/2}\xi^{3/2}\Big),$$

where $\xi=\omega+\left(a_2^2-\frac14C_1^2-C_3a_2-\alpha_4\right)/\left(a_2(a_1C_1^{-1}-\alpha_2)\right)$. Here $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

If $\alpha_3\neq0$, then $\varphi^1=\alpha_3\omega$ (translating ω, the integration constant can be made to vanish),

$$\varphi^2=C_1+C_2\int e^{\frac12\alpha_3\omega^2}d\omega+a_1\int e^{\frac12\alpha_3\omega^2}\left(\int e^{-\frac12\alpha_3\omega^2}d\omega\right)d\omega+\alpha_2\omega,$$
$$h=C_3-\tfrac12(\alpha_2^2+\alpha_3^2)\omega^2-\alpha_2C_1\omega-\alpha_2C_2\left(\omega\int e^{\frac12\alpha_3\omega^2}d\omega-\alpha_3^{-1}e^{\frac12\alpha_3\omega^2}\right)-$$
$$-\alpha_2a_1\left(\omega\int e^{\frac12\alpha_3\omega^2}\left(\int e^{-\frac12\alpha_3\omega^2}d\omega\right)d\omega-\alpha_3^{-1}e^{\frac12\alpha_3\omega^2}\int e^{-\frac12\alpha_3\omega^2}d\omega+\alpha_3^{-1}\omega\right),$$

and the function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega}-\alpha_3\omega\varphi^3_\omega+(a_2^2-\alpha_4-a_2\varphi^2)\varphi^3=0. \tag{6.17}$$

We managed to find a solution of (6.17) only for the case $a_1=C_2=0$, i.e.,

$$\varphi^3=e^{\frac14\alpha_3\omega^2}V\left(\alpha_3^{1/2}(\omega+2a_2\alpha_2\alpha_3^{-2}),\nu\right),$$

where $\nu=4\alpha_3^{-1}\left(\alpha_4+a_2C_1-a_2^2(\alpha_2^2\alpha_3^{-2}+1)\right)$. Here $V(\tau,\nu)$ is the general solution of the Weber equation

$$4V_{\tau\tau}=(\tau^2+\nu)V. \tag{6.18}$$

3. The general solution of system (6.10) has the form:

$$\varphi^1=C_1\omega^{-2}+\tfrac12\alpha_3, \tag{6.19}$$

$$\begin{aligned}\varphi^2={}&C_2+C_3\int\omega^{C_1+1}e^{\frac14\alpha_3\omega^2}d\omega-\tfrac12\alpha_2\omega^2+\\&+a_1\int\omega^{C_1+1}e^{\frac14\alpha_3\omega^2}\left(\int\omega^{-C_1-1}e^{-\frac14\alpha_3\omega^2}d\omega\right)d\omega,\end{aligned} \tag{6.20}$$

$$\begin{aligned}\varphi^3={}&C_4+C_5\int\omega^{C_1-1}e^{\frac14\alpha_3\omega^2}d\omega+\\&+\int\omega^{C_1-1}e^{\frac14\alpha_3\omega^2}\left(\int\omega^{1-C_1}e^{-\frac14\alpha_3\omega^2}(\alpha_5+a_2\omega^{-2}\varphi^2)d\omega\right)d\omega,\end{aligned}$$

$$h=C_6-\tfrac18\alpha_3^2\omega^2-\tfrac12C_1^2\omega^{-2}+\int(\varphi^2(\omega))^2\omega^{-3}d\omega-\alpha_2\int\omega^{-1}\varphi^2(\omega)d\omega. \tag{6.21}$$

4. System (6.11) implies that the functions φ^i and h are determined by (6.19)–(6.21), and the function φ^3 satisfies the equation

$$\varphi^3_{\omega\omega}-\left((C_1-1)\omega^{-1}+\tfrac12\alpha_3\omega\right)\varphi^3_\omega+\left(a_2\omega^{-2}(a_2-\varphi^2)-\alpha_4\right)\varphi^3=0. \tag{6.22}$$

We managed to solve equation (6.22) in following cases:

A. $C_3=a_1=0$, $\alpha_3\neq0$:

$$\varphi^3=\omega^{\frac12C_1-1}e^{\frac18\alpha_3\omega^2}W(\varkappa,\mu,\tfrac14\alpha_3\omega^2),$$

where $\varkappa=\frac14\left(2-C_1-(4\alpha_4+2\alpha_2a_2)\alpha_3^{-1}\right)$, $\mu=\frac14(C_1^2-4a_2^2+4a_2C_2)^{1/2}$. Here $W(\varkappa,\mu,\tau)$ is the general solution of the Whittaker equation (4.21).

Let $\alpha_3 = 0$, then

$$\varphi^2 = \begin{cases} C_2 + C_3 \ln\omega + \frac{1}{4}(a_1 + 2\alpha_2)\omega^2, & C_1 = -2, \\ C_2 + \frac{1}{2}C_3\omega^2 + \frac{1}{2}a_1\omega^2(\ln\omega - \frac{1}{2}), & C_1 = 0, \\ C_2 + C_3(C_1+2)^{-1}\omega^{C_1+2} - \frac{1}{2}C_1^{-1}(a_1 - \alpha_2 C_1)\omega^2, & C_1 \neq 0, -2. \end{cases}$$

B. $C_3 = a_1 - \alpha_2 C_1 = 0$:

$$\varphi^3 = \begin{cases} \omega^{\frac{1}{2}C_1} Z_\nu(\mu^{1/2}\omega), & \mu \neq 0, \\ \omega^{\frac{1}{2}C_1}(C_5\omega^\nu + C_6\omega^{-\nu}), & \mu = 0, \nu \neq 0, \\ \omega^{\frac{1}{2}C_1}(C_5 + C_6 \ln\omega), & \mu = 0, \nu = 0, \end{cases} \tag{6.23}$$

where $\mu = -\alpha_4$, $\nu = \frac{1}{2}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$. Here and below $Z_\nu(\tau)$ is the general solution of the Bessel equation (4.22).

C. $C_3 = 0$, $C_1 \neq 0$: φ^3 is determined by (6.23), where

$$\mu = \tfrac{1}{2}a_2C_1^{-1}(a_1 - \alpha_2 C_1) - \alpha_4, \quad \nu = \tfrac{1}{2}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}.$$

D. $C_1 = a_1 = 0$: φ^3 is determined by (6.23), where

$$\mu = -\tfrac{1}{2}a_2C_3 - \alpha_4, \quad \nu = (-a_2^2 + a_2C_2)^{1/2}.$$

E. $C_3 \neq 0$, $C_1 \notin \{0; -2\}$, $a_2(a_1 - \alpha_2C_1) - 2\alpha_4C_1 = 0$:

$$\varphi^3 = \omega^{\frac{1}{2}C_1} Z_\nu(\mu\omega^{1+\frac{1}{2}C_1}),$$

where $\mu = 2C_3^{1/2}(C_1+2)^{-3/2}$, $\nu = (C_1+2)^{-1}(C_1^2 - 4a_2^2 + 4a_2C_2)^{1/2}$.

F. $C_1 = -2$, $C_3 \neq 0$, $a_2(a_1 + 2\alpha_2) + 4\alpha_4 = 0$ ([19]):

$$\varphi^3 = \omega^{-1}\xi^{1/2} Z_{1/3}(\tfrac{2}{3}C_3^{1/2}\xi^{3/2}),$$

where $\xi = \ln\omega + C_3^{-1}(a_2^2 - a_2C_2 - 1)$.

G. $C_1 = 2$, $C_3 < 0$, $1 - a_2^2 + a_2C_2 \geq 0$:

$$\varphi^3 = W(\varkappa, \mu, \tfrac{1}{2}(-C_3)^{1/2}\omega^2),$$

where $\varkappa = \frac{1}{8}(-C_3)^{-1/2}(-4\alpha_4 + a_2^2 - 2\alpha_2a_2)$, $\mu = \frac{1}{2}(1 - a_2^2 + a_2C_2)^{1/2}$. Here $W(\varkappa, \mu, \tau)$ is the general solution of the Whittaker equation (4.21).

5–7. Identical corollaries of system (6.12), (6.13), and (6.14) are the equations

$$\varphi^2 = a\varphi^1 + C_1, \tag{6.24}$$

$$h = a(1+a^2)\varphi^1_\omega + (2 + 2a^2 - aC_1)\varphi^1 + C_2, \tag{6.25}$$

$$(1+a^2)\varphi^1_{\omega\omega} + (4a - C_1)\varphi^1_\omega + \varphi^1\varphi^1 + 4\varphi^1 + (1+a^2)^{-1}(C_1^2 + 2C_2) = 0. \tag{6.26}$$

We found the following solutions of (6.26):

A. If $(1+a^2)^{-1}(C_1^2 + 2C_2) < 4$:

$$\varphi^1 = \left(4 - (1+a^2)^{-1}(C_1^2 + 2C_2)\right)^{1/2} - 2. \tag{6.27}$$

B. If $C_1 = 4a$:

$$\varphi^1 = -6\wp\left(\frac{\omega}{(1+a^2)^{1/2}} + C_4,\ \frac{4}{3} - \frac{(C_1^2+2C_2)}{3(1+a^2)},\ C_3\right) - 2. \tag{6.28}$$

Here and below $\wp(\tau, \varkappa_1, \varkappa_2)$ is the Weierstrass function satisfying equation (6.15). If $C_2 = 2 - 6a^2$ and $C_3 = 0$, a particular case of (6.28) is the function

$$\varphi^1 = -6(1+a^2)\omega^2 - 2 \tag{6.29}$$

(the constant C_4 is considered to vanish).

C. If ${}_1 \neq 4a$, $(1+a^2)^{-1}(C_1^2 + 2C_2) - 4 = -9\mu^4$:

$$\varphi^1 = -6\mu^2 e^{-2\xi}\wp(e^{-\xi} + C_4, 0, C_3) + 3\mu^2 - 2, \tag{6.30}$$

where $\xi = (1+a^2)^{-1/2}\mu\omega$, $\mu = \frac{1}{5}(4a - C_1)(1+a^2)^{-1/2}$. If $C_3 = 0$, a particular case of (6.30) is the function

$$\varphi^1 = -6\mu^2 e^{-2\xi}(e^{-\xi} + C_4)^{-2} + 3\mu^2 - 2, \tag{6.31}$$

where the constant C_4 is considered not to vanish.

The function φ^3 has to be found for systems (6.12), (6.13), and (6.14) individually.

5. The function φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega} - (C_1 + 4a)\varphi^3_\omega - (2\varphi^1 - 4)\varphi^3 - \alpha_5 = 0.$$

If φ^1 is determined by (6.27), we obtain

$$\begin{aligned}\varphi^3 = {} & \exp\left(\tfrac{1}{2}(1+a^2)^{-1}(C_1+4a)\omega\right)\times \\ & \times\left\{\begin{array}{ll} C_5\exp(\nu^{1/2}\omega) + C_6\exp(-\nu^{1/2}\omega), & \nu > 0 \\ C_5\cos((-\nu)^{1/2}\omega) + C_6\sin((-\nu)^{1/2}\omega), & \nu < 0 \\ C_5 + C_6\omega, & \nu = 0 \end{array}\right\} + \\ & + \left\{\begin{array}{lll} -\alpha_5(2\varphi^1-4)^{-1}, & 2\varphi^1 - 4 \neq 0 & \\ -\alpha_5(4a+C_1)^{-1}\omega, & 2\varphi^1 - 4 = 0, & C_1 + 4a \neq 0 \\ \frac{1}{2}\alpha_5(1+a^2)^{-1}\omega^2, & 2\varphi^1 - 4 = 0, & C_1 + 4a = 0 \end{array}\right\},\end{aligned}$$

where $\nu = \frac{1}{4}(1+a^2)^{-2}(C_1+4a)^2 - (1+a^2)^{-1}(4 - 2\varphi^1)$.

6. In this case φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega} - C_1\varphi^3_\omega = a_1\varphi^1.$$

Therefore,

$$\begin{aligned}\varphi^3 = {} & C_5 + C_6\exp\left((1+a^2)^{-1}C_1\omega\right) + a_1C_1^{-1}\Big(\int\varphi^1(\omega)d\omega + \\ & + \exp\left((1+a^2)^{-1}C_1\omega\right)\int\exp\left(-(1+a^2)^{-1}C_1\omega\right)\varphi^1(\omega)d\omega\Big)\end{aligned}$$

for $C_1 \neq 0$, and

$$\varphi^3 = C_5 + C_6\omega + a_1(1+a^2)^{-1}\left(\omega\int\varphi^1(\omega)d\omega - \int\omega\varphi^1(\omega)d\omega\right)$$

for $C_1 = 0$.

7. The function φ^3 satisfy the equation

$$(1+a^2)\varphi^3_{\omega\omega} - (C_1 + 2a_1 a)\varphi^3_\omega + (a_1^2 - a_1\varphi^1)\varphi^3 = 0. \tag{6.32}$$

A. If φ^1 is determined by (6.27), it follows that

$$\varphi^3 = \exp\left(\tfrac{1}{2}(1+a^2)^{-1}(C_1+2a_1a)\omega\right) \times \\ \times \left\{ \begin{array}{ll} C_5\exp(\nu^{1/2}\omega) + C_6\exp(-\nu^{1/2}\omega), & \nu > 0 \\ C_5\cos((-\nu)^{1/2}\omega) + C_6\sin((-\nu)^{1/2}\omega), & \nu < 0 \\ C_5 + C_6\omega, & \nu = 0 \end{array} \right\},$$

where $\nu = \frac{1}{4}(1+a^2)^{-2}(C_1+2a_1a)^2 - (1+a^2)^{-1}(a_1^2 - a_1\varphi^1)$.

B. If $C_1 = 4a$, that is, φ^1 is determined by (6.27), we obtain

$$\varphi^3 = \exp\left(a(a_1+2)(1+a^2)^{-1}\omega\right)\theta(\tau),$$

where $\tau = (1+a^2)^{-1/2}\omega + C_4$. Here the function $\theta = \theta(\tau)$ is the general solution of of the following Lame equation ([19]):

$$\theta_{\tau\tau} + \left(6a_1\wp(\tau) + a_1^2 + 2a_1 - a^2(2+a_1)^2(1+a^2)^{-1}\right)\theta = 0$$

with the Weierstrass function

$$\wp(\tau) = \wp\left(\tau, \tfrac{1}{3}\left(4 - (1+a^2)^{-1}(C_1^2 + 2C_2)\right), C_3\right).$$

Consider the particular case when $C_2 = 2 - 6a^2$ and $C_3 = 0$ additionally, i.e., φ^1 can be given in form (6.29). Depending on the values of a and a_1, we obtain the following expression for φ^3:

Case 1. $a_1 \neq -2$, $a_1 \neq 2a^2$:

$$\varphi^3 = |\omega|^{1/2} exp\left(\frac{a(2+a_1)}{1+a^2}\omega\right) Z_\nu\left(\frac{\left((2+a_1)(a_1-2a^2)\right)^{1/2}}{1+a^2}\omega\right),$$

where $\nu = (\frac{1}{4} - 6a_1)^{1/2}$.

Case 2. $a_1 = -2$: $\varphi^3 = C_5\omega^4 + C_6\omega^{-3}$.

Case 3. $a_1 = 2a^2$:

Case 3.1. $48a^2 < 1$: $\varphi^3 = |\omega|^{1/2}e^{2a\omega}\left(C_5\omega^\sigma + C_6\omega^{-\sigma}\right)$, where $\sigma = \frac{1}{2}\sqrt{1-48a^2}$.

Case 3.2. $48a^2 = 1$, that is, $a = \pm\frac{1}{12}\sqrt{3}$: $\varphi^3 = |\omega|^{1/2}(C_5 + C_6\ln\omega)$.

Case 3.3. $48a^2 > 1$: $\varphi^3 = |\omega|^{1/2}e^{2a\omega}\left(C_5\cos(\gamma\ln\omega) + C_6\sin(\gamma\ln\omega)\right)$, where $\gamma = \frac{1}{2}\sqrt{48a^2-1}$.

C. Let the conditions

$$C_1 \neq 4a, \quad (1+a^2)^{-1}(C_1^2 + 2C_2) - 4 = -9\mu^4$$

be satisfied, that is, let φ^1 be determined by (6.30). Transforming the variables in equation (6.32) by the formulas:

$$\varphi^3 = \tau^{-1/2}\exp\Bigl(\tfrac{1}{2}(C_1 + 2aa_1)(1+a^2)^{-1}\omega\Bigr)\theta(\tau),$$
$$\tau = \exp\bigl(-\mu(1+a^2)^{-1/2}\omega\bigr),$$

we obtain the following equation in the function $\theta = \theta(\tau)$:

$$\tau^2\theta_{\tau\tau} + \bigl(6a_1\tau^2\wp(\tau + C_4, 0, C_3) + \sigma\bigr)\theta = 0, \tag{6.33}$$

where $\sigma = \mu^{-2}\bigl(a_1^2 + 2a_1 - \frac{1}{4}(1+a^2)^{-1}(C_1^2 + 2aa_1)^2\bigr) - 3a_1 + \frac{1}{4}$. If $\sigma = 0$, equation (6.33) is a Lame equation.

In the particular case when φ^1 is determined by (6.31), equation (6.33) has the form:

$$\tau^2(\tau + C_4)^2\theta_{\tau\tau} + \bigl(6a_1\tau^2 + \sigma(\tau + C_4)^2\bigr)\theta = 0. \tag{6.34}$$

By means of the following transformation of variables:

$$\theta = |\xi|^{\nu_1}|\xi - 1|^{\nu_2}\psi(\xi), \quad \xi = -C_4^{-1}\tau,$$

where $\nu_1(\nu_1 - 1) + \sigma = 0$ and $\nu_2(\nu_2 - 1) + 6a_1 = 0$, equation (6.34) is reduced to a hypergeometric equation of the form (see [19]):

$$\xi(\xi - 1)\psi_{\xi\xi} + \bigl(2(\nu_1 + \nu_2)\xi - 2\nu_1\bigr)\psi_\xi + 2\nu_1\nu_2\psi = 0.$$

If $\sigma = 0$, equation (6.34) implies that

$$(\tau + C_4)^2\theta_{\tau\tau} + 6a_1\theta = 0.$$

Therefore,

$$\theta = C_5|\tau + C_4|^{1/2-\nu} + C_6|\tau + C_4|^{1/2+\nu}$$

if $a_1 < \frac{1}{24}$, where $\nu = (\frac{1}{4} - 6a_1)^{1/2}$,

$$\theta = |\tau + C_4|^{1/2}\bigl(C_5 + C_6\ln|\tau + C_4|\bigr)$$

if $a_1 = \frac{1}{24}$, and

$$\theta = |\tau + C_4|^{1/2}\bigl(C_5\cos(\nu\ln|\tau + C_4|) + C_6\sin(\nu\ln|\tau + C_4|)\bigr)$$

if $a_1 > \frac{1}{24}$, where $\nu = (6a_1 - \frac{1}{4})^{1/2}$.

7. Exact solutions of system (2.9)

Among the reduced systems from Section 2, only particular cases of system (2.9) have Lie symmetry operators that are not induced by elements from $A(\mathrm{NS})$. Therefore, Lie reductions of the other systems from Section 2 give only solutions that can be obtained by means of reducing the NSEs with two- and three-dimensional subalgebras of $A(\mathrm{NS})$.

Here we consider system (2.9) with ρ^i vanishing. As mentioned in Note 2.5, in this case the vector-function $\vec{m}$ has the form $\vec{m} = \eta(t)\vec{e}$, where $\vec{e} = \text{const}$, $|\vec{e}| = 1$, and $\eta = \eta(t) = |\vec{m}(t)| \neq 0$. Without loss of generality we can assume that $\vec{e} = (0,0,1)$, i.e.,

$$\vec{m} = (0, 0, \eta(t)).$$

For such vector $\vec{m}$, conditions (2.5) are satisfied by the following vector $\vec{n}^i$:

$$\vec{n}^1 = (1,0,0), \quad \vec{n}^2 = (0,1,0).$$

Therefore, ansatz (2.4) and system (2.9) can be written, respectively, in the forms:

$$\begin{aligned} &u^1 = v^1, \quad u^2 = v^2, \quad u^3 = \big(\eta(t)\big)^{-1}\big(v^3 + \eta_t(t)x_3\big), \\ &p = q - \tfrac{1}{2}\eta_{tt}(t)\big(\eta(t)\big)^{-1}x_3^2, \end{aligned} \tag{7.1}$$

where $v = v(y_1, y_2, y_3)$, $q = q(y_1, y_2, y_3)$, $y_i = x_i$, $y_3 = t$, and

$$\begin{aligned} &v^i_t + v^j v^i_j - v^i_{jj} + q_i = 0, \\ &v^3_t + v^j v^3_j - v^3_{jj} = 0, \\ &v^i_i + \rho^3 = 0, \end{aligned} \tag{7.2}$$

where $\rho^3 = \rho^3(t) = \eta_t/\eta$.

It was shown in Note 2.8 that there exists a local transformation which make ρ^3 vanish. Therefore, we can consider system (7.2) only with ρ^3 vanishing and extend the obtained results in the case $\rho^3 \neq 0$ by means of transformation (2.12). However it will be sometimes convenient to investigate, at once, system (7.2) with an arbitrary function ρ^3.

The MIA of (7.2) with $\rho^3 = 0$ is given by the algebra

$$B = \langle R_3(\bar{\psi}),\ Z^1(\lambda),\ D^1_3,\ \partial_t,\ J^1_{12},\ \partial_{v^3},\ v^3\partial_{v^3}\rangle$$

(see notations in Subsection 2.1). We construct complete sets of inequivalent one-dimensional subalgebras of B and choose such algebras, among these subalgebras, that can be used to reduce system (7.2) and do not lie in the linear span of the operators $R_3(\bar{\psi})$, $Z^1(\lambda)$, J^1_{12}, i.e., the operators which are induced by operators from $A(\mathrm{NS})$ for arbitrary ρ^3. As a result we obtain the following algebras (more exactly, the following classes of algebras):

The one-dimensional subalgebras:

1. $B^1_1 = \langle D^1_3 + 2\varkappa J^1_{12} + 2\gamma v^3\partial_{v^3} + 2\beta\partial_{v^3}\rangle$, where $\gamma\beta = 0$.
2. $B^1_2 = \langle \partial_t + \varkappa J^1_{12} + \gamma v^3\partial_{v^3} + \beta\partial_{v^3}\rangle$, where $\gamma\beta = 0$, $\varkappa \in \{0; 1\}$.
3. $B^1_3 = \langle J^1_{12} + \gamma v^3\partial_{v^3} + Z^1(\lambda(t))\rangle$, where $\gamma \neq 0$, $\lambda \in C^\infty((t_0, t_1), \mathbb{R})$.
4. $B^1_4 = \langle R_3(\bar{\psi}(t)) + \gamma v^3\partial_{v^3}\rangle$, where $\gamma \neq 0$,
   $\bar{\psi}(t) = (\psi^1(t), \psi^2(t)) \neq (0,0)\ \forall\, t \in (t_0, t_1)$, $\psi^i \in C^\infty((t_0, t_1), \mathbb{R})$.

The two-dimensional subalgebras:

1. $B^2_1 = \langle \partial_t + \beta_2\partial_{v^3},\ D^1_3 + \varkappa J^1_{12} + \gamma v^3\partial_{v^3} + \beta_1\partial_{v^3}\rangle$,
   where $\gamma\beta_1 = 0, (\gamma-2)\beta_2 = 0$.
2. $B^2_2 = \langle D^1_3 + 2\gamma_1 v^3\partial_{v^3} + 2\beta_1\partial_{v^3},\ J^1_{12} + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon|t|^{-1})\rangle$,
   where $\gamma_1\beta_1 = 0,\ \gamma_2\beta_2 = 0,\ \gamma_1\beta_2 - \gamma_2\beta_1 = 0$.
3. $B^2_3 = \langle D^1_3 + 2\varkappa J^1_{12} + 2\gamma_1 v^3\partial_{v^3} + 2\beta_1\partial_{v^3},\ R_3(|t|^{\sigma+1/2}\cos\tau, |t|^{\sigma+1/2}\sin\tau) +$
   $+ \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon|t|^{\sigma-1})\rangle$, where $\tau = \varkappa\ln|t|$,
   $(\gamma_1+\sigma)\beta_1 - \gamma_2\beta_1 = 0,\ \sigma\gamma_2 = 0,\ \varepsilon\sigma = 0$.
4. $B^2_4 = \langle \partial_t + \gamma_1 v^3\partial_{v^3} + \beta_1\partial_{v^3},\ J^1_{12} + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3} + Z^1(\varepsilon)\rangle$,
   where $\gamma_1\beta_1 = 0,\ \gamma_2\beta_2 = 0,\ \gamma_1\beta_2 - \gamma_2\beta_1 = 0$.
5. $B^2_5 = \langle \partial_t + \varkappa J^1_{12} + \gamma_1 v^3\partial_{v^3} + \beta_1\partial_{v^3},\ R_3(e^{\sigma t}\cos\varkappa t, e^{\sigma t}\sin\varkappa t) +$
   $+ Z^1(\varepsilon e^{\sigma t}) + \gamma_2 v^3\partial_{v^3} + \beta_2\partial_{v^3}\rangle$, where $(\gamma_1+\sigma)\beta_1 - \gamma_2\beta_1 = 0$,
   $\sigma\gamma_2 = 0,\ \varepsilon\sigma = 0$.
6. $B^2_6 = \langle R_3(\bar\psi^1) + \gamma v^3\partial_{v^3},\ R_3(\bar\psi^2)\rangle$, where $\bar\psi^i = (\psi^{i1}(t), \psi^{i2}(t)) \neq (0,0)$
   $\forall\, t \in (t_0, t_1),\ \psi^{ij} \in C^\infty((t_0,t_1),\mathbb{R}),\ \bar\psi^1_{tt}\cdot\bar\psi^2 - \bar\psi^1\cdot\bar\psi^2_{tt} = 0,\ \gamma \neq 0$.
   Hereafter $\bar\psi^1\cdot\bar\psi^2 := \psi^{1i}\psi^{2i}$.

Let us reduce system (7.2) to systems of PDEs in two independent variables. With the algebras B^1_1–B^1_4 we can construct the following complete set of Lie ansatzes of codimension 1 for system (7.2) with $\rho^3 = 0$:

1.
$$\begin{aligned} v^1 &= |t|^{-1/2}(w^1\cos\tau - w^2\sin\tau) + \tfrac12 y_1 t^{-1} - \varkappa y_2 t^{-1},\\ v^2 &= |t|^{-1/2}(w^1\sin\tau + w^2\cos\tau) + \tfrac12 y_2 t^{-1} + \varkappa y_1 t^{-1},\\ v^3 &= |t|^\gamma w^3 + \beta\ln|t|,\\ q &= |t|^{-1}s + \tfrac12(\varkappa^2 + \tfrac14)t^{-2}r^2, \end{aligned} \tag{7.3}$$

where $\tau = \varkappa\ln|t|$, $\gamma\beta = 0$,

$$z_1 = |t|^{-1/2}(y_1\cos\tau + y_2\sin\tau), \quad z_2 = |t|^{-1/2}(-y_1\sin\tau + y_2\cos\tau).$$

Here and below $w^a = w^a(z_1,z_2)$, $s = s(z_1,z_2)$, $r = (y_1^2 + y_2^2)^{1/2}$.

2.
$$\begin{aligned} v^1 &= w^1\cos\varkappa t - w^2\sin\varkappa t - \varkappa y_2,\\ v^2 &= w^1\sin\varkappa t + w^2\cos\varkappa t + \varkappa y_1,\\ v^3 &= w^3 e^{\gamma t} + \beta t,\\ q &= s + \tfrac12\varkappa^2 r^2, \end{aligned} \tag{7.4}$$

where $\varkappa \in \{0;1\}$, $\gamma\beta = 0$,

$$z_1 = y_1\cos\varkappa t + y_2\sin\varkappa t, \quad z_2 = -y_1\sin\varkappa t + y_2\cos\varkappa t.$$

3.
$$\begin{aligned} v^1 &= y_1 r^{-1}w^3 - y_2 r^{-2}w^1 - \gamma y_2 r^{-2},\\ v^2 &= y_2 r^{-1}w^3 + y_1 r^{-2}w^1 + \gamma y_1 r^{-2},\\ v^3 &= w^2 e^{\gamma\arctan y_2/y_1},\\ q &= s + \lambda(t)\arctan y_2/y_1, \end{aligned} \tag{7.5}$$

where $_1 = t$, $z_2 = r$, $\gamma \neq 0$, $\lambda \in C^\infty((t_0,t_1),\mathbb{R})$.

$$
\begin{aligned}
4.\quad & \bar v = (\bar\psi\cdot\bar\psi)^{-1}\Big((w^1+\gamma)\bar\psi + w^3\bar\theta + (\bar\psi\cdot\bar y)\bar\psi_t - z_2\bar\theta_t\Big)\\
& v^3 = w^2\exp\big(\gamma(\bar\psi\cdot\bar\psi)^{-1}(\bar\psi\cdot\bar y)\big)\\
& q = s - (\bar\psi\cdot\bar\psi)^{-1}(\bar\psi_{tt}\cdot\bar y)(\bar\psi\cdot\bar y) + \tfrac12(\bar\psi\cdot\bar\psi)^{-2}(\bar\psi_{tt}\cdot\bar\psi)(\bar\psi\cdot\bar y)^2,
\end{aligned}
\tag{7.6}
$$

where $z_1 = t$, $z_2 = (\bar\theta\cdot\bar y)$, $\gamma \neq 0$, $\bar v = (v^1, v^2)$, $\bar y = (y_1, y_2)$, $\psi^i \in C^\infty((t_0,t_1),\mathbb{R})$, $\bar\theta = (-\psi^2, \psi^1)$.

Substituting ansatzes (7.3) and (7.4) into system (7.2) with $\rho^3 = 0$, we obtain a reduced system of the form (6.1), where

$$
\begin{aligned}
&\alpha_1 = 0,\quad \alpha_2 = -1,\quad \alpha_3 = -2\varkappa,\quad \alpha_4 = \gamma,\quad \alpha_5 = \beta \quad \text{if}\quad t > 0 \quad \text{and}\\
&\alpha_1 = 0,\quad \alpha_2 = 1,\quad \alpha_3 = 2\varkappa,\quad \alpha_4 = -\gamma,\quad \alpha_5 = -\beta \quad \text{if}\quad t < 0
\end{aligned}
$$

for ansatz (7.3) and

$$
\alpha_1 = 0,\quad \alpha_2 = 0,\quad \alpha_3 = -2\varkappa,\quad \alpha_4 = \gamma,\quad \alpha_5 = \beta
$$

for ansatz (7.4). System (6.1) is investigated in Section 6 in detail.

Because the form of ansatzes (7.3) is not changed after transformation (2.12), it is convenient to substitute their into a system of form (7.2) with an arbitrary function ρ^3. As a result of substituting, we obtain the following reduced systems:

$$
\begin{aligned}
3.\quad & w^3_1 + w^3w^3_2 - z_2^{-3}(w^1+\gamma)^2 - (w^3_{22} + z_2^{-1}w^3_2 - z_2^{-2}w^3) + s_2 = 0,\\
& w^1_1 + w^3w^1_2 - w^1_{22} + z_2^{-1}w^1_2 + \lambda = 0,\\
& w^2_1 + w^3w^2_2 - w^2_{22} - z_2^{-1}w^2_2 + \gamma z_2^{-2}w^1w^2 = 0,\\
& w^3_2 + z_2^{-1}w^3 = -\eta_1/\eta.
\end{aligned}
\tag{7.7}
$$

$$
\begin{aligned}
4.\quad & w^1_1 + w^3w^1_2 - (\bar\psi\cdot\bar\psi)w^1_{22} = 0,\\
& w^3_1 + w^3w^3_2 - (\bar\psi\cdot\bar\psi)w^3_{22} + (\bar\psi\cdot\bar\psi)s_2 + 2(w^1+\gamma)(\bar\psi\cdot\bar\theta)(\bar\psi\cdot\bar\psi)^{-1} -\\
&\quad - 2(\bar\psi_t\cdot\bar\psi)(\bar\psi\cdot\bar\psi)^{-1}w^3 + (2\bar\psi_t\cdot\bar\psi_t - \bar\psi_{tt}\cdot\bar\psi)(\bar\psi\cdot\bar\psi)^{-1}z_2 = 0,\\
& w^2_1 + w^3w^2_2 - (\bar\psi\cdot\bar\psi)w^2_{22} + \gamma(\bar\psi\cdot\bar\psi)^{-1}\big(w^1 + (\bar\psi_t\cdot\bar\theta)(\bar\psi\cdot\bar\psi)^{-1}z_2\big)w^2 = 0,\\
& w^3_2 + \eta_t/\eta = 0.
\end{aligned}
\tag{7.8}
$$

Unlike systems 8 and 9 from Subsection 3.2, systems (7.7) and (7.8) are not reduced to linear systems of PDEs.

Let us investigate system (7.7). The last equation of (7.7) immediately gives

$$
\begin{aligned}
&(w^3_2 + z_2^{-1}w^3)_2 = w^3_{22} + z_2^{-1}w^3_2 - z_2^{-2}w^3 = 0,\\
&w^3 = (\chi - 1)z_2^{-1} - \tfrac12\eta_t\eta^{-1}z_2,
\end{aligned}
\tag{7.9}
$$

where $\chi = \chi(t)$ is an arbitrary differentiable function of $t = z_2$. Then it follows from the first equation of (7.7) that

$$
s = \int z_2^{-3}(w^1+\gamma)^2dz_2 - \tfrac12(\chi-1)^2z_2^{-2} + \tfrac14 z_2^2\Big((\eta_t/\eta)_t - \tfrac12(\eta_t/\eta)^2\Big) - \chi_t\ln|z_2|.
$$

Substituting (7.9) into the remaining equations of (7.7), we get

$$\begin{aligned} &w^1_1 - w^1_{22} + \big(\chi z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)w^1_2 + \lambda = 0,\\ &w^2_1 - w^2_{22} + \big((\chi-2)z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)w^2_2 + \gamma z_2^{-2}w^1w^2 = 0. \end{aligned} \tag{7.10}$$

By means of changing the independent variables

$$\tau = \textstyle\int |\eta(t)|dt, \quad z = |\eta(t)|^{1/2}z_2, \tag{7.11}$$

system (7.10) can be transformed to a system of a simpler form:

$$\begin{aligned} &w^1_\tau - w^1_{zz} + \hat\chi z^{-1}w^2_z + \hat\lambda|\hat\eta|^{-1} = 0,\\ &w^2_\tau - w^2_{zz} + (\hat\chi-2)z^{-1}w^2_z + \gamma z^{-2}w^1w^2 = 0, \end{aligned} \tag{7.12}$$

where $\hat\eta(\tau) = \eta(t)$, $\hat\chi(\tau) = \chi(t)$, and $\hat\lambda(\tau) = \lambda(t)$.

If $\lambda(t) = -2C\eta(t)(\chi(t)-1)$ for some fixed constant C, particular solutions of (7.10) are functions

$$w^1 = C\eta(t)z_2^2, \quad w^2 = f(z_1,z_2)\exp\big(\gamma C\textstyle\int\eta(t)dt\big),$$

where f is an arbitrary solution of the following equation

$$f_1 - f_{22} + \big((\chi-2)z_2^{-1} - \tfrac{1}{2}\eta_t\eta^{-1}z_2\big)f_2 = 0. \tag{7.13}$$

In the variables from (7.11), equation (7.13) has form (5.22) with $\tilde\eta(\tau) = \chi(t)-2$.

In the case $\lambda(t) = 8C(\chi(t)-1)\eta(t)\int\eta(t)(\chi(t)-3)dt$ ($C = \text{const}$), particular solutions of (7.10) are functions

$$\begin{aligned} &w^1 = C\Big((\eta(t))^2z_2^4 - 4z_2^2\eta(t)\textstyle\int\eta(t)(\chi(t)-3)dt\Big),\\ &w^2 = f(z_1,z_2)\exp\big(\tfrac{1}{2}(\gamma C)^{1/2}\eta(t)z_2^2 + \xi(t)\big), \end{aligned}$$

where $\xi(t) = -(\gamma C)^{1/2}\int\eta(t)(\chi(t)-3)dt + 4\gamma C\int\eta(t)\big(\int\eta(t)(\chi(t)-3)dt\big)dt$ and f is an arbitrary solution of the following equation

$$f_1 - f_{22} + \big((\chi-2)z_2^{-1} - (\tfrac{1}{2}\eta_t\eta^{-1} + 2(\gamma C)^{1/2})z_2\big)f_2 = 0. \tag{7.14}$$

After the change of the independent variables

$$\tau = \textstyle\int|\eta(t)|\exp\big(4(\gamma C)^{1/2}\int\eta(t)dt\big)dt, \quad z = |\eta(t)|^{1/2}\exp\big(2(\gamma C)^{1/2}\int\eta(t)dt\big)z_2$$

in (7.14), we obtain equation (5.22) with $\tilde\eta(\tau) = \chi(t)-2$ again.

Let us continue to system (7.8). The last equation of (7.8) integrates with respect to z_2 to the following expression: $w^3 = -\eta_t\eta^{-1}z_2 + \chi$. Here $\chi = \chi(t)$ is an differentiable function of $z_1 = y_3 = t$. Let us make the transformation from the symmetry group of (7.2):

$$\tilde{\bar v}(t,\bar y) = \bar v(t,\bar y - \bar\xi(t)) + \bar\xi_t(t), \quad \tilde v^3 = v^3, \quad \tilde q(t,\bar y) = q(t,\bar y - \bar\xi(t)) - \bar\xi_{tt}(t)\cdot\bar y,$$

where $\bar\xi_{tt}\cdot\bar\psi - \bar\xi\cdot\bar\psi_{tt} = 0$ and

$$\bar\xi_t\cdot\bar\theta + \chi + \eta_t\eta^{-1}(\bar\xi\cdot\bar\theta) - |\bar\psi|^{-2}(\bar\xi\cdot\bar\psi)(\bar\psi_t\cdot\bar\theta) + |\bar\psi|^{-2}(\bar\xi\cdot\bar\theta)(\bar\theta_t\cdot\bar\theta) = 0.$$

Hereafter $|\bar\psi|^2=\bar\psi\cdot\bar\psi$. This transformation does not modify ansatz (7.6), but it makes the function χ vanish, i.e., $\tilde w^3=-\eta_t\eta^{-1}z_2$. Therefore, without loss of generality we may assume, at once, that $w^3=-\eta_t\eta^{-1}z_2$.

Substituting the expression for w^3 in the other equations of (7.8), we obtain that

$$\begin{aligned}s={}&z_2^2|\bar\psi|^{-2}\Big(\big(\tfrac12\bar\psi_{tt}\cdot\bar\psi-\bar\psi_t\cdot\bar\psi_t-(\bar\psi_t\cdot\bar\psi)\eta_t\eta^{-1}\big)|\bar\psi|^{-2}+\tfrac12\eta_{tt}\eta^{-1}-(\eta_t)^2\eta^{-2}\Big)-\\&-2(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-2}\textstyle\int w^1(z_1,z_2)dz_2,\end{aligned}$$

$$\begin{aligned}&w^1_1-\eta_1\eta^{-1}z_2w^1_2-|\bar\psi|^2w^1_{22}=0,\\&w^2_1-\eta_1\eta^{-1}z_2w^2_2-|\bar\psi|^2w^2_{22}+\gamma|\bar\psi|^{-2}\big(2(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-2}z_2+w^1\big)w^2=0.\end{aligned}\qquad(7.15)$$

The change of the independent variables

$$\tau=\int(\eta(t)|\bar\psi|)^2dt,\quad z=\eta(t)z_2$$

reduces system (7.15) to the following form:

$$\begin{aligned}&w^1_\tau-w^1_{zz}=0,\\&w^2_\tau-w^2_{zz}+\gamma|\hat{\bar\psi}|^{-4}\hat\eta^{-2}\big(2(\hat{\bar\psi}_t\cdot\hat{\bar\theta})\hat\eta z+w^1\big)w^2=0,\end{aligned}\qquad(7.16)$$

where $\hat{\bar\psi}(\tau)=\bar\psi(t)$, $\hat{\bar\theta}(\tau)=\bar\theta(t)$, $\hat\eta(\tau)=\eta(t)$.

Particular solutions of (7.15) are the functions

$$\begin{aligned}&w^1=C_1+C_2\eta(t)z_2+C_3\big(\tfrac12(\eta(t)z_2)^2+\textstyle\int(\eta(t)|\bar\psi|)^2dt\big),\\&w^2=f(t,z_2)\exp\big(\xi^2(t)z_2^2+\xi^1(t)z_2+\xi^0(t)\big),\end{aligned}$$

where $(\xi^2(t),\xi^1(t),\xi^0(t))$ is a particular solution of the system of ODEs:

$$\begin{aligned}&\xi^2_t-2\eta_t\eta^{-1}\xi^2-4|\bar\psi|^2(\xi^2)^2+\tfrac12C_3\gamma\eta^2|\bar\psi|^{-2}=0,\\&\xi^1_t-\eta_t\eta^{-1}\xi^1-4|\bar\psi|^2\xi^2\xi^1+2\gamma(\bar\psi_t\cdot\bar\theta)|\bar\psi|^{-4}+C_2\gamma\eta|\bar\psi|^{-2}=0,\\&\xi^0_t-2|\bar\psi|^2\xi^2-|\bar\psi|^2(\xi^1)^2+\gamma\big(C_1+C_3\textstyle\int(\eta(t)|\bar\psi|)^2dt\big)|\bar\psi|^{-2}=0,\end{aligned}$$

and f is an arbitrary solution of the following equation

$$f_1-|\bar\psi|^2f_{22}+\big((\eta_t\eta^{-1}+4|\bar\psi|^2\xi^2)z_2+2|\bar\psi|^2\xi^1\big)f_2=0.\qquad(7.17)$$

Equation (7.17) is reduced by means of a local transformation of the independent variables to the heat equation.

Consider the Lie reductions of system (7.2) to systems of ODEs. The second basis operator of the each algebra B^2_k, $k=\overline{1,5}$ induces, for the reduced system obtained from system (7.2) by means of the first basis operator, either a Lie symmetry operator from Table 2 or a operator giving a ansatz of form (6.4). Therefore, the Lie reduction of system (7.2) with the algebras $B^2_1-B^2_5$ gives only solutions that can be constructed for system (7.2) by means of reducing with the algebras B^1_1 and B^1_2 to system (6.1).

With the algebra B^2_6 we obtain an ansatz and a reduced system of the following forms:

$$\begin{aligned}&\bar v=\bar\phi+\lambda^{-1}(\bar\theta^i\cdot\bar y)\bar\psi^i_t,\quad v^3=\phi^3\exp\big(\gamma\lambda(\bar\theta^1\cdot\bar y)\big),\\&s=h-\tfrac12\lambda^{-1}(\bar\psi^i_{tt}\cdot\bar y)(\bar\theta^i\cdot\bar y),\end{aligned}\qquad(7.18)$$

where $\phi^a = \phi^a(\omega)$, $h = h(\omega)$, $\omega = t$, $\lambda = \psi^{11}\psi^{22} - \psi^{12}\psi^{21} = \bar\psi^1 \cdot \bar\theta^1 = \bar\psi^2 \cdot \bar\theta^2$, $\bar\theta^1 = (\psi^{22}, -\psi^{21})$, $\bar\theta^2 = (-\psi^{12}, \psi^{11})$, and

$$\begin{aligned} &\bar\phi_t + \lambda^{-1}(\bar\theta^i \cdot \bar\phi)\bar\psi^i_t = 0, \quad \phi^3_t + \big(\gamma\lambda^{-1}(\bar\theta^1 \cdot \bar\phi) - \gamma^2\lambda^{-2}(\bar\theta^1 \cdot \bar\theta^1)\big)\phi^3 = 0, \\ &\lambda^{-1}(\bar\theta^i \cdot \bar\psi^i_t) + \eta_t\eta^{-1} = 0. \end{aligned} \tag{7.19}$$

Let us make the transformation from the symmetry group of system (7.2):

$$\tilde{\bar v}(t,\bar y) = \bar v(t, \bar y - \bar\xi) + \bar\xi_t, \quad \tilde v^3(t,\bar y) = v^3(t, \bar y - \bar\xi), \quad \tilde s(t,\bar y) = s(t, \bar y - \bar\xi) - \bar\xi_{tt} \cdot \bar y,$$

where

$$\bar\xi_t + \lambda^{-1}(\bar\theta^i \cdot \bar\xi)\bar\psi^i_t + \bar\phi = 0. \tag{7.20}$$

It follows from (7.20) that $\bar\xi_{tt} = \lambda^{-1}(\bar\theta^i \cdot \bar\xi)\bar\psi^i_{tt}$, i.e., $\bar\theta^i_{tt} \cdot \bar\xi - \bar\theta^i \cdot \bar\xi_{tt} = 0$. Therefore, this transformation does not modify ansatz (7.18), but it makes the functions ϕ^i vanish. And without loss of generality we may assume, at once, that $\phi^i \equiv 0$. Then

$$\phi^3 = C \exp\Big(\int \big(\gamma\lambda^{-1}|\theta|\big)^2 dt\Big), \quad C = \text{const}.$$

The last equation of system (7.19) is the compatibility condition of system (7.2) and ansatz (7.18).

8. Conclusion

In this article we reduced the NSEs to systems of PDEs in three and two independent variables and systems of ODEs by means of the Lie method. Then, we investigated symmetry properties of the reduced systems of PDEs and made Lie reductions of systems which admitted non-trivial symmetry operators, i.e., operators that are not induced by operators from $A(\text{NS})$. Some of the systems in two independent variables were reduced to linear systems of either two one-dimensional heat equations or two translational equations. We also managed to find exact solutions for most of the reduced systems of ODEs.

Now, let us give some remaining problems. Firstly, we failed, for the present, to describe the non-Lie ansatzes of form (1.6) that reduce the NSEs. (These ansatzes include, for example, the well-known ansatzes for the Karman swirling flows (see bibliography in [16]). One can also consider non-local ansatzes for the Navier–Stokes field, i.e., ansatzes containing derivatives of new unknown functions.

Second problem is to study non-Lie (i.e., non-local, conditional, and Q-conditional) symmetries of the NSEs [13].

And finally, it would be interesting to investigate compatibility and to construct exact solutions of overdetermined systems that are obtained from the NSEs by means of different additional conditions. Usually one uses the condition where the nonlinearity has a simple form, for example, the potential form (see review [36]), i.e., $\text{rot}((\vec u \cdot \vec\nabla)\vec u) = \vec 0$ (the NS fields satisfying this condition is called the generalized Beltrami flows). We managed to describe the general solution of the NSEs with the additional condition where the convective terms vanish [29, 30]. But one can give other conditions, for example,

$$\triangle \vec u = \vec 0, \quad \vec u_t + (\vec u \cdot \vec\nabla)\vec u = \vec 0,$$

and so on.

We will consider the problems above elsewhere.

Appendix

A. Inequivalent one-, two-, and three-dimensional subalgebras of $A(NS)$

To find complete sets of inequivalent subalgebras of A(NS), we use the method given, for example, in [27, 28]. Let us describe it briefly.

1. We find the commutation relations between the basis elements of A(NS).

2. For arbitrary basis elements V, W^0 of A(NS) and each $\varepsilon \in \mathbb{R}$ we calculate the adjoint action

$$W(\varepsilon) = \mathrm{Ad}(\varepsilon V)W^0 = \mathrm{Ad}(\exp(\varepsilon V))W^0$$

of the element $\exp(\varepsilon V)$ from the one-parameter group generated by the operator V on W^0. This calculation can be made in two ways: either by means of summing the Lie series

$$W(\varepsilon) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}\{V^n, W^0\} = W^0 + \frac{\varepsilon}{1!}[V, W^0] + \frac{\varepsilon^2}{2!}[V, [V, W^0]] + \cdots, \tag{A.1}$$

where $\{V^0, W^0\} = W^0$, $\{V^n, W^0\} = [V, \{V^{n-1}, W^0\}]$, or directly by means of solving the initial value problem

$$\frac{dW(\varepsilon)}{d\varepsilon} = [V, W(\varepsilon)], \quad W(0) = W^0. \tag{A.2}$$

3. We take a subalgebra of a general form with a fixed dimension. Taking into account that the subalgebra is closed under the Lie bracket, we try to simplify it by means of adjoint actions as much as possible.

A.1. The commutation relations and the adjoint representation of the algebra $A(NS)$

Basis elements (1.2) of A(NS) satisfy the following commutation relations:

$$\begin{aligned}
&[J_{12}, J_{23}] = -J_{31}, \quad [J_{23}, J_{31}] = -J_{12}, \quad [J_{31}, J_{12}] = -J_{23},\\
&[\partial_t, J_{ab}] = [D, J_{ab}] = 0, \quad [\partial_t, D] = 2\partial_t,\\
&[\partial_t, R(\vec m)] = R(\vec m_t), \quad [D, R(\vec m)] = R(2t\vec m_t - \vec m),\\
&[\partial_t, Z(\chi)] = Z(\chi_t), \quad [D, Z(\chi)] = Z(2t\chi_t + 2\chi),\\
&[R(\vec m), R(\vec n)] = Z(\vec m_{tt}\cdot\vec n - \vec m\cdot\vec n_{tt}), \quad [J_{ab}, R(\vec m)] = R(\vec{\tilde m}),\\
&[J_{ab}, Z(\chi)] = [Z(\chi), R(\vec m)] = [Z(\chi), Z(\eta)] = 0,
\end{aligned} \tag{A.3}$$

where $\tilde m^a = m^b$, $\tilde m^b = -m^a$, $\tilde m^c = 0$, $a \neq b \neq c \neq a$.

Note A.1. Relations (A.3) imply that the set of operators (1.2) generates an algebra when, for example, the parameter-functions m^a and χ belong to $C^\infty((t_0, t_1), \mathbb{R})$ ($C_0^\infty((t_0, t_1), \mathbb{R})$, $A((t_0, t_1), \mathbb{R})$), i.e., the set of infinite-differentiable (infinite-differentiable finite, real analytic) functions from (t_0, t_1) in $\mathbb{R}$, where $-\infty \leq t_0 < t_1 \leq +\infty$.

But the NSEs (1.1) admit operators (1.3) and (1.4) with parameter-functions of a less degree of smoothness. Moreover, the minimal degree of their smoothness depends on the smoothness that is demanded for the solutions of the NSEs (1.1). Thus, if $u^a \in C^2((t_0,t_1)\times\Omega,\mathbb{R})$ and $p\in C^1((t_0,t_1)\times\Omega,\mathbb{R})$, where Ω is a domain in $\mathbb{R}^3$, then it is sufficient that $m^a\in C^3((t_0,t_1),\mathbb{R})$ and $\chi\in C^1((t_0,t_1),\mathbb{R})$. Therefore, one can consider the "pseudoalgebra" generated by operators (1.2). The prefix "pseudo-" means that in this set of operators the commutation operation is not determined for all pairs of its elements, and the algebra axioms are satisfied only by elements, where they are defined. It is better to indicate the functional classes that are sets of values for the parameters m^a and χ in the notation of the algebra $A(\mathrm{NS})$. But below, for simplicity, we fix these classes, taking $m^a,\chi\in C^\infty((t_0,t_1),\mathbb{R})$, and keep the notation of the algebra generated by operators (1.2) in the form $A(\mathrm{NS})$. However, all calculations will be made in such a way that they can be translated for the case of a less degree of smoothness.

Most of the adjoint actions are calculated simply as sums of their Lie series. Thus,

$$\begin{aligned}
&\mathrm{Ad}(\varepsilon\partial_t)D = D+2\varepsilon\partial_t, \quad \mathrm{Ad}(\varepsilon D)\partial_t = e^{-2\varepsilon}\partial_t,\\
&\mathrm{Ad}(\varepsilon Z(\chi))\partial_t = \partial_t-\varepsilon Z(\chi_t), \quad \mathrm{Ad}(\varepsilon Z(\chi))D = D-\varepsilon Z(2t\chi_t+2\chi),\\
&\mathrm{Ad}(\varepsilon R(\vec m))\partial_t = \partial_t-\varepsilon R(\vec m_t)-\tfrac12\varepsilon^2 Z(\vec m_t\cdot\vec m_{tt}-\vec m\cdot\vec m_{ttt}),\\
&\mathrm{Ad}(\varepsilon R(\vec m))D = D-\varepsilon R(2t\vec m_t-\vec m)-\\
&\qquad\qquad -\tfrac12\varepsilon^2 Z(2t\vec m_t\cdot\vec m_{tt}-2t\vec m\cdot\vec m_{ttt}-4\vec m\cdot\vec m_{tt}),\\
&\mathrm{Ad}(\varepsilon R(\vec m))J_{ab} = J_{ab}-\varepsilon R(\vec{\tilde m})+\varepsilon^2 Z(m^a m^b_{tt}-m^a_{tt}m^b),\\
&\mathrm{Ad}(\varepsilon R(\vec m))R(\vec n) = R(\vec n)+\varepsilon Z(\vec m_{tt}\cdot\vec n-\vec m\cdot\vec n_{tt}), \quad \mathrm{Ad}(\varepsilon J_{ab})R(\vec m) = R(\vec{\hat m}),\\
&\mathrm{Ad}(\varepsilon J_{ab})J_{cd} = J_{cd}\cos\varepsilon+[J_{ab},J_{cd}]\sin\varepsilon \quad \big((a,b)\neq(c,d)\neq(b,a)\big),
\end{aligned}\tag{A.4}$$

where

$$\begin{aligned}
&\tilde m^a = m^b, \quad \tilde m^b=-m^a, \quad \tilde m^c = 0, \quad a\neq b\neq c\neq a,\\
&\hat m^d = m^d\cos\varepsilon+\tilde m^d\sin\varepsilon, \quad \hat m^c = m^c, \quad a\neq b\neq c\neq a, \quad d\in\{a;b\}.
\end{aligned}$$

Four adjoint actions are better found by means of integrating a system of form (A.2). As a result we obtain that

$$\begin{aligned}
&\mathrm{Ad}(\varepsilon\partial_t)Z(\chi(t)) = Z(\chi(t+\varepsilon)), \quad \mathrm{Ad}(\varepsilon D)Z(\chi(t)) = Z(e^{2\varepsilon}\chi(te^{2\varepsilon})),\\
&\mathrm{Ad}(\varepsilon\partial_t)R(\vec m(t)) = R(\vec m(t+\varepsilon)), \quad \mathrm{Ad}(\varepsilon D)R(\vec m(t)) = R(e^{-\varepsilon}\vec m(te^{2\varepsilon})).
\end{aligned}\tag{A.5}$$

Cases where adjoint actions coincide with the identical mapping are omitted.

Note A.2. If $Z(\chi(t))\in A(NS)[C^\infty((t_0,t_1),\mathbb{R})]$ with $-\infty<t_0$ or $t_1<+\infty$, the adjoint representation $\mathrm{Ad}(\varepsilon\partial_t)$ ($\mathrm{Ad}(\varepsilon D)$) gives an equivalence relation between the operators $Z(\chi(t))$ and $Z(\chi(t+\varepsilon))$ ($Z(\chi(t))$ and $Z(e^{2\varepsilon}\chi(te^{2\varepsilon}))$) that belong to the different algebras

$$\begin{aligned}
&A(NS)[C^\infty((t_0,t_1),\mathbb{R})] \quad\text{and}\quad A(NS)[C^\infty((t_0-\varepsilon,t_1-\varepsilon),\mathbb{R})]\\
&(A(NS)[C^\infty((t_0,t_1),\mathbb{R})] \quad\text{and}\quad A(NS)[C^\infty((t_0e^{-2\varepsilon},t_1e^{-2\varepsilon}),\mathbb{R})])
\end{aligned}$$

respectively. An analogous statement is true for the operator $R(\vec m)$. Equivalence of subalgebras in Theorems A.1 and A.2 is also meant in this sense.

Note A.3. Besides the adjoint representations of operators (1.2) we make use of discrete transformation (1.6) for classifying the subalgebras of $A(\mathrm{NS})$,

To prove the theorem of this section, the following obvious lemma is used.

Lemma A.1. *Let $N \in \mathbb{N}$.*

A. *If $\chi \in C^N((t_0,t_1),\mathbb{R})$, then $\exists\,\eta \in C^N((t_0,t_1),\mathbb{R})$: $2t\eta_t + 2\eta = \chi$.*

B. *If $\chi \in C^N((t_0,t_1),\mathbb{R})$, then $\exists\,\eta \in C^N((t_0,t_1),\mathbb{R})$: $2t\eta_t - \eta = \chi$.*

C. *If $m^i \in C^N((t_0,t_1),\mathbb{R})$ and $a \in \mathbb{R}$, then $\exists\, l^i \in C^N((t_0,t_1),\mathbb{R})$:*
$2tl^1_t - l^1 + al^2 = m^1, \quad 2tl^2_t - l^2 - al^1 = m^2.$

A.2. One-dimensional subalgebras

Theorem A.1. *A complete set of $A(\mathrm{NS})$-inequivalent one-dimensional subalgebras of $A(\mathrm{NS})$ is exhausted by the following algebras:*

1. $A^1_1(\varkappa) = \langle D + 2\varkappa J_{12}\rangle$, *where $\varkappa \geq 0$.*

2. $A^1_2(\varkappa) = \langle \partial_t + \varkappa J_{12}\rangle$, *where $\varkappa \in \{0;1\}$.*

3. $A^1_3(\eta,\chi) = \langle J_{12}+R(0,0,\eta(t))+Z(\chi(t))\rangle$ *with smooth functions η and χ. Algebras $A^1_3(\eta,\chi)$ and $A^1_3(\tilde\eta,\tilde\chi)$ are equivalent if $\exists\,\varepsilon,\delta \in \mathbb{R}$, $\exists\,\lambda \in C^\infty((t_0,t_1),\mathbb{R})$:*

$$\tilde\eta(\tilde t) = e^{-\varepsilon}\eta(t), \quad \tilde\chi(\tilde t) = e^{2\varepsilon}(\chi(t) + \lambda_{tt}(t)\eta(t) - \lambda(t)\eta_{tt}(t)), \tag{A.6}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

4. $A^1_4(\vec m,\chi) = \langle R(\vec m(t))+Z(\chi(t))\rangle$ *with smooth functions $\vec m$ and χ: $(\vec m,\chi) \not\equiv (\vec 0,0)$. Algebras $A^1_4(\vec m,\chi)$ and $A^1_4(\tilde{\vec m},\tilde\chi)$ are equivalent if $\exists\,\varepsilon,\delta \in \mathbb{R}$, $\exists\, C \neq 0$, $\exists\, B \in O(3)$, $\exists\,\vec l \in C^\infty((t_0,t_1),\mathbb{R}^3)$:*

$$\tilde{\vec m}(\tilde t) = Ce^{-\varepsilon}B\vec m(t), \quad \tilde\chi(\tilde t) = Ce^{2\varepsilon}\big(\chi(t) + \vec l_{tt}(t)\cdot\vec m(t) - \vec m_{tt}(t)\cdot\vec l(t)\big), \tag{A.7}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

Proof. Consider an arbitrary one-dimensional subalgebra generated by

$$V = a_1D + a_2\partial_t + a_3J_{12} + a_4J_{23} + a_5J_{31} + R(\vec m) + Z(\chi).$$

The coefficients a_4 and a_5 are omitted below since they always can be made to vanish by means of the adjoint representations $\mathrm{Ad}(\varepsilon_1J_{12})$ and $\mathrm{Ad}(\varepsilon_2J_{31})$.

If $a_1 \neq 0$ we get $\tilde a_1 = 1$ by means of a change of basis. Next, step-by-step we make a_2, $\vec m$, and χ vanish by means of the adjoint representations $\mathrm{Ad}(-\frac12 a_2a_1^{-1}\partial_t)$, $\mathrm{Ad}(R(\vec l))$, and $\mathrm{Ad}(Z(\chi))$, where

$$\vec l \in C^\infty((t_0 + \tfrac12 a_2a_1^{-1}, t_1 + \tfrac12 a_2a_1^{-1}), \mathbb{R}^3),$$
$$\eta \in C^\infty((t_0 + \tfrac12 a_2a_1^{-1}, t_1 + \tfrac12 a_2a_1^{-1}), \mathbb{R}),$$

and $\vec l$, η are solutions of the equations

$$2t\vec l_t - \vec l + a_3a_1^{-1}(l^2,-l^1,0)^T = \hat{\vec m}, \quad 2t\eta_t + 2\eta = \hat\chi + \frac12(\vec l_{tt}\cdot\hat{\vec m} - \vec l\cdot\hat{\vec m}_{tt})$$

with $\vec{m}(t) = a_1^{-1}\vec{m}(t - \frac{1}{2}a_2a_1^{-1})$ and $\hat{\chi}(t) = a_1^{-1}\chi(t - \frac{1}{2}a_2a_1^{-1})$. Such $\vec{l}$ and η exist in virtue of Lemma A.1. As a result we obtain the algebra $A_1^1(\varkappa)$, where $2\varkappa = a_3a_1^{-1}$. In case $\varkappa < 0$ additionally one has to apply transformation (1.6) with $b = 1$.

If $a_1 = 0$ and $a_2 \neq 0$, we make $\tilde{a}_2 = 1$ by means of a change of basis. Next, step-by-step we make $\vec{m}$ and χ vanish by means of the adjoint representations $\mathrm{Ad}(R(\vec{l}))$ and $\mathrm{Ad}(Z(\chi))$, where $\vec{l} \in C^\infty((t_0, t_1), \mathbb{R}^3)$, $\eta \in C^\infty((t_0, t_1), \mathbb{R})$, and

$$a_2\vec{l}_t + a_3(l^2, -l^1, 0)^T = \vec{m}, \quad a_2\eta_t = \chi + \frac{1}{2}(\vec{l}_{tt}\cdot\vec{m} - \vec{l}\cdot\vec{m}_{tt}).$$

If $a_3 = 0$ we obtain the algebra $A_2^1(0)$ at once. If $a_3 \neq 0$, using the adjoint representation $\mathrm{Ad}(\varepsilon D)$ and transformation (1.6) (in case of need), we obtain the algebra $A_2^1(1)$.

If $a_1 = a_2 = 0$ and $a_3 \neq 0$, after a change of basis and applying the adjoint representation $\mathrm{Ad}(R(-a_3^{-1}m^2, a_3^{-1}m^1, 0))$ we get the algebra $A_3^1(\eta, \tilde{\chi})$, where $\eta = a_3^{-1}m^3$ and $\tilde{\chi} = a_3^{-1}\chi + a_3^{-2}(m^1_{tt}m^2 - m^1m^2_{tt})$. Equivalence relation (A.6) is generated by the adjoint representations $\mathrm{Ad}(\varepsilon D)$, $\mathrm{Ad}(\delta\partial_t)$, and $\mathrm{Ad}(R(0, 0, \lambda))$.

If $a_1 = a_2 = a_3 = 0$, at once we get the algebra $A_4^1(\vec{m}, \chi)$. Equivalence relation (A.7) is generated by the adjoint representations $\mathrm{Ad}(\varepsilon D)$, $\mathrm{Ad}(\delta\partial_t)$, $\mathrm{Ad}(R(\vec{l}))$, and $\mathrm{Ad}(\varepsilon_{ab}J_{ab})$.

A.3. Two-dimensional subalgebras

Theorem A.2. *A complete set of $A(\mathrm{NS})$-inequivalent two-dimensional subalgebras of $A(\mathrm{NS})$ is exhausted by the following algebras:*

1. $A_1^2(\varkappa) = \langle \partial_t,\ D + \varkappa J_{12}\rangle$, *where* $\varkappa \geq 0$.

2. $A_2^2(\varkappa, \varepsilon) = \langle D,\ J_{12} + R(0, 0, \varkappa|t|^{1/2}) + Z(\varepsilon t^{-1})\rangle$, *where* $\varkappa \geq 0$, $\varepsilon \geq 0$.

3. $A_3^2(\varkappa, \varepsilon) = \langle \partial_t,\ J_{12} + R(0, 0, \varkappa) + Z(\varepsilon)\rangle$, *where* $\varkappa \in \{0; 1\}$, $\varepsilon \geq 0$ *if* $\varkappa = 1$ *and* $\varepsilon \in \{0; 1\}$ *if* $\varkappa = 0$.

4. $A_4^2(\sigma, \varkappa, \mu, \nu, \varepsilon) = \langle D + 2\varkappa J_{12},\ R\big(|t|^{\sigma+1/2}(\nu\cos\tau,\ \nu\sin\tau,\ \mu)\big) + Z(\varepsilon|t|^{\sigma-1})\rangle$, *where* $\tau = \varkappa\ln|t|$, $\varkappa > 0$, $\mu \geq 0$, $\nu \geq 0$, $\mu^2 + \nu^2 = 1$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

5. $A_5^2(\sigma, \varepsilon) = \langle D,\ R(0, 0, |t|^{\sigma+1/2}) + Z(\varepsilon|t|^{\sigma-1})\rangle$, *where* $\varepsilon\sigma = 0$ *and* $\varepsilon \geq 0$.

6. $A_6^2(\sigma, \mu, \nu, \varepsilon) = \langle \partial_t + J_{12},\ R(\nu e^{\sigma t}\cos t, \nu e^{\sigma t}\sin t, \mu e^{\sigma t}) + Z(\varepsilon e^{\sigma t})\rangle$, *where* $\mu \geq 0$, $\nu \geq 0$, $\mu^2 + \nu^2 = 1$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

7. $A_7^2(\sigma, \varepsilon) = \langle \partial_t,\ R(0, 0, e^{\sigma t}) + Z(\varepsilon e^{\sigma t})\rangle$, *where* $\sigma \in \{-1; 0; 1\}$, $\varepsilon\sigma = 0$, *and* $\varepsilon \geq 0$.

8. $A_8^2(\lambda, \psi^1, \rho, \psi^2) = \langle J_{12} + R(0, 0, \lambda) + Z(\psi^1),\ R(0, 0, \rho) + Z(\psi^2)\rangle$ *with smooth functions (of t) λ, ρ, and ψ^i: $(\rho, \psi^2) \not\equiv (0, 0)$ and $\lambda_{tt}\rho - \lambda\rho_{tt} \equiv 0$. Algebras $A_8^2(\lambda, \psi^1, \rho, \psi^2)$ and $A_8^2(\tilde{\lambda}, \tilde{\psi}^1, \tilde{\rho}, \tilde{\psi}^2)$ are equivalent if $\exists\, C_1 \neq 0$, $\exists\, \varepsilon, \delta, C_2 \in \mathbb{R}$, $\exists\, \theta \in C^\infty((t_0, t_1), \mathbb{R})$:*

$$\begin{aligned}
&\tilde{\lambda}(\tilde{t}) = e^{\varepsilon}(\lambda(t) + C_2\rho(t)), \quad \tilde{\rho}(\tilde{t}) = C_1e^{-\varepsilon}\rho(t),\\
&\tilde{\psi}^1(\tilde{t}) = e^{2\varepsilon}\big(\psi^1(t) + \theta_{tt}(t)\lambda(t) - \theta(t)\lambda_{tt}(t) + \\
&\qquad + C_2(\psi^2(t) + \theta_{tt}(t)\rho(t) - \theta(t)\rho_{tt}(t))\big),\\
&\tilde{\psi}^2(\tilde{t}) = C_1e^{2\varepsilon}(\psi^2(t) + \theta_{tt}(t)\rho(t) - \theta(t)\rho_{tt}(t)),
\end{aligned} \tag{A.8}$$

where $\tilde{t} = te^{-2\varepsilon} + \delta$.

9. $A^2_9(\vec m^1,\chi^1,\vec m^2,\chi^2)=\langle R(\vec m^1(t))+Z(\chi^1(t)),\ R(\vec m^2(t))+Z(\chi^2(t))\rangle$ *with smooth functions* $\vec m^i$ *and* χ^i*:*

$$\vec m^1_{tt}\cdot\vec m^2-\vec m^1\cdot\vec m^2_{tt}=0,\quad \operatorname{rank}\bigl((\vec m^1,\chi^1),(\vec m^2,\chi^2)\bigr)=2.$$

Algebras $A^2_9(\vec m^1,\chi^1,\vec m^2,\chi^2)$ *and* $A^2_9(\vec{\tilde m}^1,\tilde\chi^1,\vec{\tilde m}^2,\tilde\chi^2)$ *are equivalent if* $\exists\,\varepsilon,\delta\in\mathbb{R}$, $\exists\,\{a_{ij}\}_{i,j=1,2}:\det\{a_{ij}\}\neq 0$, $\exists\,B\in O(3)$, $\exists\,\vec l\in C^\infty((t_0,t_1),\mathbb{R}^3)$*:*

$$\begin{aligned}&\vec{\tilde m}^i(\tilde t)=e^{-\varepsilon}a_{ij}B\vec m^j(t),\\ &\tilde\chi^i(\tilde t)=e^{2\varepsilon}a_{ij}\bigl(\chi^j(t)+\vec l_{tt}(t)\cdot\vec m^j(t)-\vec l(t)\cdot\vec m^j_{tt}(t)\bigr),\end{aligned}\tag{A.9}$$

where $\tilde t=te^{-2\varepsilon}+\delta$*.*

10. $A^2_{10}(\varkappa,\sigma)=\langle D+\varkappa J_{12},\ Z(|t|^\sigma)\rangle$*, where* $\varkappa\geq 0$*,* $\sigma\in\mathbb{R}$*.*

11. $A^2_{11}(\sigma)=\langle\partial_t+J_{12},\ Z(e^{\sigma t})\rangle$*, where* $\sigma\in\mathbb{R}$*.*

12. $A^2_{12}(\sigma)=\langle\partial_t,\ Z(e^{\sigma t})\rangle$*, where* $\sigma\in\{-1;0;1\}$*.*

The proof of Theorem A.2 is analogous to that of Theorem A.1. Let us take an arbitrary two-dimensional subalgebra generated by two linearly independent operators of the form

$$V^i=a^i_1D+a^i_2\partial_t+a^i_3J_{12}+a^i_4J_{23}+a^i_5J_{31}+R(\vec m^i)+Z(\chi^i),$$

where $a^i_n=\mathrm{const}$ $(n=\overline{1,5})$ and $[V^1,V^2]\in\langle V^1,V^2\rangle$. Considering the different possible cases we try to simplify V^i by means of adjoint representation as much as possible. Here we do not present the proof of Theorem A.2 as it is too cumbersome.

A.4. Three-dimensional subalgebras

We also constructed a complete set of $A(\mathrm{NS})$-inequivalent three-dimensional subalgebras. It contains 52 classes of algebras. By means of 22 classes from this set one can obtain ansatzes of codimension three for the Navier–Stokes field. Here we only give 8 superclasses that arise from unification of some of these classes:

1. $A^3_1=\langle D,\ \partial_t,\ J_{12}\rangle$.

2. $A^3_2=\langle D+\varkappa J_{12},\ \partial_t,\ R(0,0,1)\rangle$, where $\varkappa\geq 0$. Here and below $\varkappa$, σ, ε_1, ε_2, μ, ν, and a_{ij} are real constants.

3. $A^3_3(\sigma,\nu,\varepsilon_1,\varepsilon_2)=\langle D,\ J_{12}+\nu\bigl(R(0,0,|t|^{1/2}\ln|t|)+Z(\varepsilon_2|t|^{-1}\ln|t|)\bigr)+Z(\varepsilon_1|t|^{-1}),$ $R(0,0,|t|^{\sigma+1/2})+Z(\varepsilon_2|t|^{\sigma-1})\rangle$, where $\nu\sigma=0$, $\varepsilon_1\geq 0$, $\nu\geq 0$, and $\sigma\varepsilon_2=0$.

4. $A^3_4(\sigma,\nu,\varepsilon_1,\varepsilon_2)=\langle\partial_t,\ J_{12}+Z(\varepsilon_1)+\nu\bigl(R(0,0,t)+Z(\varepsilon_2 t)\bigr),\ R(0,0,e^{\sigma t})+Z(\varepsilon_2e^{\sigma t})\rangle$, where $\nu\sigma=0$, $\sigma\varepsilon_2=0$, and, if $\sigma=0$, the constants ν, ε_1, and ε_2 satisfy one of the following conditions:

$$\nu=1,\ \varepsilon_1\geq 0;\quad \nu=0,\ \varepsilon_1=1,\ \varepsilon_2\geq 0;\quad \nu=\varepsilon_1=0,\ \varepsilon_2\in\{0;1\}.$$

5. $A^3_5(\varkappa,\vec m^1,\vec m^2,\chi^1,\chi^2)=\langle D+2\varkappa J_{12},\ R(\vec m^1)+Z(\chi^1),\ R(\vec m^2)+Z(\chi^2)\rangle$, where $\varkappa\geq 0$, $\operatorname{rank}(\vec m^1,\vec m^2)=2$,

$$\begin{aligned}&t\vec m^i_t-\tfrac12\vec m^i+\varkappa(m^{i2},-m^{i1},0)^T=a_{ij}\vec m^j,\\ &t\chi^i_t+\chi^i=a_{ij}\chi^j,\quad a_{ij}=\mathrm{const},\end{aligned}$$

$$(a_{11}+a_{22})\big(a_{21}\vec m^1\cdot\vec m^1+(a_{22}-a_{11})\vec m^1\cdot\vec m^2-a_{12}\vec m^2\cdot\vec m^2+ \\ +2\varkappa(m^{12}m^{21}-m^{11}m^{22})\big)=0. \tag{A.10}$$

This superclass contains eight inequivalent classes of subalgebras that can be obtained from it by means of a change of basis and the adjoint actions

$$\mathrm{Ad}(\delta_1 D),\quad \mathrm{Ad}(\delta_2 J_{12}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta)),$$
$$\big(\mathrm{Ad}(\delta D),\quad \mathrm{Ad}(\varepsilon_{ab}J_{ab}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta))\big)$$

if $\varkappa>0$ ($\varkappa=0$) respectively. Here the functions $\vec n$ and η satisfy the following equations:

$$t\vec n_t-\tfrac12\vec n+\varkappa(n^2,-n^1,0)^T=b_i\vec m^i,$$
$$t\eta_t+\eta=b_i\chi_i+\tfrac12 t(\vec n_{ttt}\cdot\vec n-\vec n_{tt}\cdot\vec n_t)+\vec n_{tt}\cdot\vec n+\varkappa(n^1n^2_{tt}-n^1_{tt}n^2).$$

6. $A^3_6(\varkappa,\vec m^1,\vec m^2,\chi^1,\chi^2)=\langle\partial_t+\varkappa J_{12},\ R(\vec m^1)+Z(\chi^1),\ R(\vec m^2)+Z(\chi^2)\rangle$, where $\varkappa\in\{0;1\}$, $\mathrm{rank}(\vec m^1,\vec m^2)=2$,

$$\vec m^i_t-\varkappa(m^{i2},-m^{i1},0)^T=a_{ij}\vec m^j,\quad t\chi^i_t=a_{ij}\chi^j,$$

and a_{ij} are constants satisfying (A.10). This superclass contains eight inequivalent classes of subalgebras that can be obtained from it by means of a change of basis and the adjoint actions

$$\mathrm{Ad}(\delta_1\partial_t),\quad \mathrm{Ad}(\delta_2 J_{12}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta)),$$
$$\big(\mathrm{Ad}(\delta_1\partial_t),\quad \mathrm{Ad}(\delta_2 D),\quad \mathrm{Ad}(\varepsilon_{ab}J_{ab}),\quad \mathrm{Ad}(R(\vec n)+Z(\eta))\big)$$

if $\varkappa=1$ ($\varkappa=0$) respectively. Here the functions $\vec n$ and η satisfy the following equations:

$$\vec n_t+\varkappa(n^2,-n^1,0)^T=b_i\vec m^i,$$
$$\eta_t=b_i\chi_i+\tfrac12(\vec n_{ttt}\cdot\vec n-\vec n_{tt}\cdot\vec n_t)+\varkappa(n^1n^2_{tt}-n^1_{tt}n^2).$$

7. $A^3_7(\eta^1,\eta^2,\eta^3,\chi)=\langle J_{12}+R(0,0,\eta^3),\ R(\eta^1,\eta^2,0),\ R(-\eta^2,\eta^1,0)\rangle$, where

$$\eta^a\in C^\infty((t_0,t_1),\mathbb R),\quad \eta^1_{tt}\eta^2-\eta^1\eta^2_{tt}\equiv0,\quad \eta^i\eta^i\not\equiv0,\quad \eta^3\neq0.$$

Algebras $A^3_7(\eta^1,\eta^2,\eta^3)$ and $A^3_7(\tilde\eta^1,\tilde\eta^2,\tilde\eta^3)$ are equivalent if $\exists\,\delta_a\in\mathbb R$, $\exists\,\delta_4\neq0$:

$$\begin{aligned}&\tilde\eta^1(\tilde t)=\delta_4(\eta^1(t)\cos\delta_3-\eta^2(t)\sin\delta_3),\\&\tilde\eta^2(\tilde t)=\delta_4(\eta^1(t)\sin\delta_3+\eta^2(t)\cos\delta_3),\\&\tilde\eta^3(\tilde t)=e^{-\delta_1}\eta^3(t),\end{aligned}\tag{A.11}$$

where $\tilde t=te^{-2\delta_1}+\delta_2$.

8. $A^3_8(\vec m^1,\vec m^2,\vec m^3)=\langle R(\vec m^1),\ R(\vec m^2),\ R(\vec m^3)\rangle$, where

$$\vec m^a\in C^\infty((t_0,t_1),\mathbb R^3),\quad \mathrm{rank}(\vec m^1,\vec m^2,\vec m^3)=3,\quad \vec m^a_{tt}\cdot\vec m^b-\vec m^a\cdot\vec m^b_{tt}=0.$$

Algebras $A^3_8(\vec m^1,\vec m^2,\vec m^3)$ and $A^3_8(\vec{\tilde m}^1,\vec{\tilde m}^2,\vec{\tilde m}^3)$ are equivalent if $\exists\,\delta_i\in\mathbb R^3$, $\exists\,B\in O(3)$, $\exists\,\{d_{ab}\}:\det\{d_{ab}\}\neq0$ such that

$$\vec{\tilde m}^a(\tilde t)=d_{ab}B\vec m^b(t),\tag{A.12}$$

where $\tilde t=te^{-2\delta_1}+\delta_2$.

B. On construction of ansatzes for the Navier–Stokes field by means of the Lie method

The general method for constructing a complete set of inequivalent Lie ansatzes of a system of PDEs are well known and described, for example, in [27, 28]. However, in some cases when the symmetry operators of the system have a special form, this method can be modified [9]. Thus, in the case of the NSEs, coefficients of an arbitrary operator

$$Q = \xi^0\partial_t + \xi^a\partial_a + \eta^a\partial_{u^a} + \eta^0\partial_p$$

from $A(\mathrm{NS})$ satisfy the following conditions:

$$\begin{array}{l} \xi^0 = \xi^0(t,\vec{x}), \quad \xi^a = \xi^a(t,\vec{x}), \quad \eta^a = \eta^{ab}(t,\vec{x})u^b + \eta^{a0}(t,\vec{x}), \\ \eta^0 = \eta^{01}(t,\vec{x})p + \eta^{00}(t,\vec{x}). \end{array} \tag{B.1}$$

(The coefficients ξ^a, ξ^0, η^a, and η^0 also satisfy stronger conditions than (B.1). For example if $Q \in A(NS)$, then $\xi^0 = \xi^0(t)$, $\eta^{ab} = \mathrm{const}$, and so on. But conditions (B.1) are sufficient to simplify the general method.) Therefore, ansatzes for the Navier–Stokes field can be constructing in the following way:

1. We fix a M-dimensional subalgebra of $A(\mathrm{NS})$ with the basis elements

$$Q^m = \xi^{m0}\partial_t + \xi^{ma}\partial_a + (\eta^{mab}u^b + \eta^{ma0})\partial_{u^a} + (\eta^{m01}p + \eta^{m00})\partial_p, \tag{B.2}$$

where $M \in \{1;2;3\}$, $m = \overline{1,M}$, and

$$\mathrm{rank}\{(\xi^{m0}, \xi^{m1}, \xi^{m2}, \xi^{m3}),\ m = \overline{1,M}\} = M. \tag{B.3}$$

To construct a complete set of inequivalent Lie ansatzes of codimension M for the Navier–Stokes field, we have to use the set of M-dimensional subalgebras from Section A. Condition (B.3) is needed for the existence of ansatzes connected with this subalgebra.

2. We find the invariant independent variables $\omega_n = \omega_n(t,\vec{x})$, $n = \overline{1,N}$, where $N = 4 - M$, as a set of functionally independent solutions of the following system:

$$L^m\omega = Q^m\omega = \xi^{m0}\partial_t\omega + \xi^{ma}\partial_a\omega = 0, \quad m = \overline{1,M}, \tag{B.4}$$

where $L^m := \xi^{m0}\partial_t + \xi^{ma}\partial_a$.

3. We present the Navier–Stokes field in the form:

$$u^a = f^{ab}(t,\vec{x})v^b(\bar{\omega}) + g^a(t,\vec{x}), \quad p = f^0(t,\vec{x})q(\bar{\omega}) + g^0(t,\vec{x}), \tag{B.5}$$

where v^a and q are new unknown functions of $\bar{\omega} = \{\omega_n,\ n = \overline{1,N}\}$. Acting on representation (B.5) with the operators Q^m, we obtain the following equations on functions f^{ab}, g^a, f^0, and g^0:

$$\begin{array}{l} L^m f^{ab} = \eta^{mac}f^{cb}, \quad L^m g^a = \eta^{mab}g^b + \eta^{ma0}, \quad c = \overline{1,3}, \\ L^m f^0 = \eta^{m01}f^0, \quad L^m g^0 = \eta^{m01}g^0 + \eta^{m00}. \end{array} \tag{B.6}$$

If the set of functions f^{ab}, f^0, g^a, and g^0 is a particular solution of (B.6) and satisfies the conditions $\mathrm{rank}\{(f^{1b}, f^{2,b}, f^{3b}),\ b = \overline{1,3}\} = 3$ and $f^0 \neq 0$, formulas (B.5) give an ansatz for the Navier–Stokes field.

The ansatz connected with the fixed subalgebra is not determined in an unique manner. Thus, if

$$\tilde\omega_l = \tilde\omega_l(\bar\omega), \quad \det\left\{\frac{\partial\tilde\omega_l}{\partial\omega_n}\right\}_{l,n=\overline{1,N}} \neq 0,$$
$$\tilde f^{ab}(t,\vec x) = f^{ac}(t,\vec x)F^{cb}(\bar\omega), \quad \tilde g^a(t,\vec x) = g^a(t,\vec x) + f^{ac}(t,\vec x)G^c(\bar\omega),$$
$$\tilde f^0(t,\vec x) = f^0(t,\vec x)F^0(\bar\omega), \quad \tilde g^0(t,\vec x) = g^0(t,\vec x) + f^0(t,\vec x)G^0(\bar\omega), \tag{B.7}$$

the formulas

$$u^a = \tilde f^{ab}(t,\vec x)\tilde v^b(\bar{\tilde\omega}) + \tilde g^a(t,\vec x), \quad p = \tilde f^0(t,\vec x)q(\bar{\tilde\omega}) + \tilde g^0(t,\vec x) \tag{B.8}$$

give an ansatz which is equivalent to ansatz (B.5). The reduced system of PDEs on the functions $\tilde v^a$ and $\tilde q$ is obtained from the system on v^a and q by means of a local transformation. Our problem is to find or "to guess", at once, such an ansatz that the corresponding reduced system has a simple and convenient form for our investigation. Otherwise, we can obtain a very complicated reduced system which will be not convenient for investigation and we can not simplify it.

Consider a simple example.

Let $M=1$ and let us give the algebra $\langle\partial_t + \varkappa J_{12}\rangle$, where $\varkappa\in\{0;1\}$. For this algebra, the invariant independent variables $y_a = y_a(t,\vec x)$ are functionally independent solutions of the equation $Ly=0$ (see (B.4)), where

$$L := \partial_t + \varkappa(x_1\partial_{x_2} - x_2\partial_{x_1}). \tag{B.9}$$

There exists an infinite set of choices for the variables y_a. For example, we can give the following expressions for y_a:

$$y_1 = \arctan\frac{x_1}{x_2} - \varkappa t, \quad y_2 = (x_1^2+x_2^2)^{1/2}, \quad y_3 = x_3.$$

However choosing y_a in such a way, for $\varkappa\neq 0$ we obtain a reduced system which strongly differs from the "natural" reduced system for $\varkappa=0$ (the NSEs for steady flows of a viscous fluid in Cartesian coordinates). It is better to choose the following variables y_a:

$$y_1 = x_1\cos\varkappa t + x_2\sin\varkappa t, \quad y_2 = -x_1\sin\varkappa t + x_2\cos\varkappa t, \quad y_3 = x_3.$$

The vector-functions $\vec f^{\,b} = (f^{1b}, f^{2b}, f^{3b})$, $b=\overline{1,3}$, should be linearly independent solutions of the system

$$Lf^1 = -\varkappa f^2, \quad Lf^2 = \varkappa f^1, \quad Lf^3 = 0$$

and the function f^0 should satisfy the equation $Lf^0 = 0$ and the condition $f^0\neq 0$. Here the operator L is defined by (B.9). We give the following values of these functions:

$$\vec f^{\,1} = (\cos\varkappa t, \sin\varkappa t, 0), \quad \vec f^{\,2} = (-\sin\varkappa t, \cos\varkappa t, 0), \quad \vec f^{\,3} = (0,0,1), \quad f^0 = 1.$$

The functions g^a and g^0 are solutions of the equations

$$Lg^1 = -\varkappa g^2, \quad Lg^2 = \varkappa g^1, \quad Lg^3 = 0, \quad Lg^0 = 0.$$

We can make, for example, g^a and g^0 vanish. Then the corresponding ansatz has the form:

$$u^1 = \tilde{v}^1 \cos \varkappa t - \tilde{v}^2 \sin \varkappa t, \quad u^2 = \tilde{v}^1 \sin \varkappa t + \tilde{v}^2 \cos \varkappa t, \quad u^3 = \tilde{v}^3, \quad p = \tilde{q}, \tag{B.10}$$

where $\tilde{v}^a = \tilde{v}^a(y_1, y_2, y_3)$ and $\tilde{q} = \tilde{q}(y_1, y_2, y_3)$ are the new unknown functions. Substituting ansatz (B.10) into the NSEs, we obtain the following reduced system:

$$\begin{aligned}
&\tilde{v}^a \tilde{v}^1_a - \tilde{v}^1_{aa} + \tilde{q}_1 + \varkappa y_2 \tilde{v}^1_1 - \varkappa y_1 \tilde{v}^1_2 - \varkappa \tilde{v}^2 = 0,\\
&\tilde{v}^a \tilde{v}^2_a - \tilde{v}^2_{aa} + \tilde{q}_2 + \varkappa y_2 \tilde{v}^2_1 - \varkappa y_1 \tilde{v}^2_2 + \varkappa \tilde{v}^1 = 0,\\
&\tilde{v}^a \tilde{v}^3_a - \tilde{v}^3_{aa} + \tilde{q}_3 + \varkappa y_2 \tilde{v}^3_1 - \varkappa y_1 \tilde{v}^3_2 = 0,\\
&\tilde{v}^a_a = 0.
\end{aligned} \tag{B.11}$$

Here subscripts 1,2, and 3 of functions in (B.11) denote differentiation with respect to y_1, y_2, and y_3 accordingly. System (B.11), having variable coefficients, can be simplified by means of the local transformation

$$\tilde{v}^1 = v^1 - \varkappa y_2, \quad \tilde{v}^2 = v^2 + \varkappa y_1, \quad \tilde{v}^3 = v^3, \quad \tilde{q} = q + \tfrac{1}{2}(y_1^2 + y_2^2). \tag{B.12}$$

Ansatz (B.10) and system (B.11) are transformed under (B.12) into ansatz (2.2) and system (2.7), where

$$g^1 = -\varkappa x_2, \quad g^2 = \varkappa x_1, \quad g_3 = 0, \quad g^0 = \tfrac{1}{2}\varkappa^2(x_1^2 + x_2^2), \tag{B.13}$$

$\gamma_1 = -2\varkappa$, and $\gamma_2 = 0$. Therefore, we can give the values of g^a and g^0 from (B.13) and obtain ansatz (2.2) and system (2.7) at once.

The above is a good example how a reduced system can be simplified by means of modifying (complicating) an ansatz corresponding to it. Thus, system (2.7) is simpler than system (B.11) and ansatz (2.2) is more complicated than ansatz (B.10).

Finally, let us make several short notes about constructing other ansatzes for the Navier–Stokes field.

Ansatz corresponding to the algebra $A^1_4(\vec{m}, \chi)$ (see Subsection A.2) can be constructed only for such t that $\vec{m}(t) \neq \vec{0}$. For these values of t, the parameter-function χ can be made to vanish by means of equivalence transformations (A.7).

Ansatz corresponding to the algebra $A^2_8(\lambda, \psi^1, \rho, \psi^2)$ (see Subsection A.3) can be constructed only for such t that $\rho(t) \neq 0$. For these values of t, the parameter-function ψ^2 can be made to vanish by means of equivalence transformations (A.8). Moreover, it can be considered that $\lambda_t \rho - \lambda \rho_t \in \{0; 1\}$. The algebra obtained finally is denoted by $A^2_8(\lambda, \chi, \rho, 0)$.

Ansatz corresponding to the algebra $A^2_9(\vec{m}^1, \chi^1, \vec{m}^2, \chi^2)$ (see Subsection A.3) can be constructed only for such t that $\operatorname{rank}(\vec{m}^1, \vec{m}^2) = 2$. For these values of t, the parameter-functions χ^i can be made to vanish by means of equivalence transformations (A.9).

The algebras $A^2_{10}(\varkappa, \sigma)$, $A^2_{11}(\sigma)$, and $A^2_{12}(\sigma)$ can not be used to construct ansatzes by means of the Lie algorithm.

In view of equivalence transformation (A.11), the functions η^i in the algebra $A^3_7(\eta^1, \eta^2, \eta^3)$ (see Subsection A.4) can be considered to satisfy the following condition:

$$\eta^1_t \eta^2 - \eta^1 \eta^2_t \in \{0; \tfrac{1}{2}\}.$$

1. Berker R., Integration des equations du mouvement d'un fluid visqueux incompressible, in Handbuch der Physik, Vol. 8/2, Berlin, Springer, 1963, 1–384.
2. Birkgoff G., Hydrodynamics, Princeton, Princeton, University Press, Princeton, 1960.
3. Bluman G.W., Cole J.D., The general similarity solutions of the heat equation, *J. Math. Mech.*, 1969, **18**, № 11, 1025.
4. Boisvert R.E., Ames W.F., Srivastava U.N., Group properties and new solutions of Navier–Stokes equations, *J. Eng. Math.*, 1983, **17**, 203–221.
5. Bytev V.O., Group properties of the Navier–Stokes equations, *Chislennye Metody Mehaniki Sploshnoy Sredy*, 1972, **3**, 13–17.
6. Bytev V.O., Invariant solutions of the Navier–Stokes equations, *PTMF J. Appl. Mech. Tech. Phys.*, 1972, № 6, 56–64.
7. Craik A.D.D., Criminale W.O., Evolution of wavelike disturbance in shear flows: a class of exact solutions of the Navier–Stokes equations, *Proc. Roy. Soc. London A*, 1986, **406**, 13–29.
8. Danilov Yu.A., Group properties of the Maxwell and Navier–Stokes equations, Preprint, Moscow, Kurchatov In-t of Atomic Energy, Acad. Sci. USSR, 1967.
9. Fushchych W.I., Symmetry in the problems of mathematical physics, in Theoretic-Algebraic Investigations in Mathematical Physics, Kyiv, Institute of Mathematics, 1981, 6–28.
10. Fushchych W.I., Popovych R.O., Symmetry reduction of the Navier–Stokes equations to linear two-dimensional systems of equations, *Dopovidi Acad. Sci. Ukrainy*, 1992, № 8, 29–37.
11. Fushchych W.I., Popovych R.O., Symmetry reduction and exact solution of the Navier–Stokes equations, Preprint 93.7, Kyiv, Institute of Mathematics Acad. Sci. Ukraine, 1993.
12. Fushchych W.I., Shtelen W.M., Popovych R.O., On reduction of the Navier–Stokes equations to linear heat equations, *Dopovidi Acad. Sci. Ukrainy*, 1992, № 2, 23–30.
13. Fushchych W.I., Shtelen W.M., Serov M.I., Symmetry analysis and exact solutions equations of non-linear mathematical physics, Dordrecht, Kluwer Academic Publisher, 1992.
14. Fushchych W.I., Shtelen W.M., Serov M.I., Popovych R.O., Q-conditional symmetry of the linear heat equation, *Dopovidi Acad. Sci. Ukrainy*, 1992, № 12, 29–37.
15. Fushchych W.I., Shtelen W.M., Slavutskiy S.L., Reduction and exact solutions of the Navier–Stokes equations, *J. Phys. A: Math. Gen.*, 1991, **24**, № 4, 971–984.
16. Goldshtik M.A., Shtern V.N., Yavorskiy N.I., Viscous flows with paradoxical properties, Novosibirsk, Nauka, 1989.
17. Grauel A., Steeb W.-H., Similarity of the Euler equation and the Navier–Stokes equation in two space dimensions, *Int. J. Theor. Phys.*, 1985, **24**, № 3, 255–265.
18. Hamel G., Spiralförmige bewegungen zäher flüssigkeiten, *Jahresbericht der deutschen Mathematiker Vereinigung*, 1916, **25**, 34–60.
19. Kamke E., Differentialgleichungen Lösungsmethoden und Lösungen, Chelsea, New York, 1971.
20. Kapitanskiy L.V., Group analysis of the Navier–Stokes and Euler equations in the presence of rotational symmetry and new exact solutions of these equations, *Dokl. Acad. Sci. USSR*, 1978, **243**, № 4, 901–904.
21. Kapitanskiy L.V., Group analysis of the Navier–Stokes equations in the presence of rotational symmetry and somenew exact solutions, *Zapiski Nauchnogo Sem. LOMI*, 1979, **84**, 89–107.
22. Katkov V.L., Exact solutions of some problems of convection, *Prikl. Matem. and Mechan.*, 1968, **32**, № 3, 482–486.
23. Kochin N.E., Kibel I.A., Roze N.B., Theoretical hydrodynamics, V.2, Moscow, Fysmatgis, 1963.
24. Landau L.D., On a new exact solution of the Navier–Stokes equations, *Dokl. Acad. Sci. USSR*, 1944, **28**, № 7, 299–301.
25. Landau L.D., Lifshits E.M., Hydrodynamics, Vol. 6, Moscow, Nauka, 1988.
26. Lloyd S.P., The infinitesimal group of the Navier–Stokes equations, *Acta Mech.*, 1981, **38**, № 1–2, 85–98.
27. Olver P., Applications of Lie groups to differential equations, New-York, Springer-Verlag, 1989.
28. Ovsiannikov L.V., Group analysis of differential equations, New York, Acadamic Press, 1978.

29. Popovych R.O., On the general solution of the Navier–Stokes equations with an additional condition, *Dopovidi Acad. Sci. Ukrainy*, 1992, № 4, 21–24.

30. Popovych R.O., Symmetry reduction and exact solution of the Navier–Stokes equations, PhD thesis, Kyiv, Institute of Mathematics Acad. Sci. Ukraine, 1992.

31. Pukhnachev V.V., Group properties of the equations of Navier–Stokes in the planar case, *PMTF J. Appl. Mech. Tech. Phys.*, 1960, № 1, 83–90.

32. Pukhnachev V.V., Invariant solutions of the Navier–Stokes equations describing motion with a free boundary, *Dokl. Acad. Sci. USSR*, 1972, **202**, № 2, 302–305.

33. Sedov L.I., Methods of similarity and dimensionality in mechanics, Moscow, Nauka, 1987.

34. Slezkin N.A., On a case of integrebility of the complete differential equations of viscous fluid motion, *Uchonye Zapiski MGU*, 1934, **2**, 89–91.

35. Squire H.B., Some viscous fluid flow problem 1. jet emerging from a hole in a plane wall, *Philos. Mag.*, 1952, **43**, № 7, 942–945.

36. Wang C.Y., Exact solutions of the Navier–Stokes equations-the generalized beltrami flows, review and extention, *Acta Mech.*, 1990, **81**, № 1–2, 69–74.

37. Wilzynski E.J., An application of group theory to hydrodynamics, *Trans. Amer. Math. Soc.*, 1900, **1**, 339–352.

38. Yatseev V.I., On a class of exact solutions of equations of viscous fluid motion, *Zurn. Experim. Teor. Phys.*, 1950, **20**, № 11, 1031–1034.

*In the framework of the Ovsiannikov's program "Submodels" (see e.g. [*J. Appl. Math. Mech.*, 1994, V.58, N 4, 601–627]), this paper can be considered as a one containing complete, with the Lie symmetry point of view, investigation of Lie submodels of the Navier–Stokes equations. *Editors' Remark.*

Antireduction and exact solutions of nonlinear heat equations

W.I. FUSHCHYCH, R.Z. ZHDANOV

We construct a number of ansatzes that reduce one-dimensional nonlinear heat equations to systems of ordinary differential equations. Integrating these, we obtain new exact solution of nonlinear heat equations with various nonlinearities.

By the term antireduction for a partial differential equation (PDE) we mean the construction of an ansatz which transforms the PDE to a system of differential equations for several unknown differentiable functions. As a rule, such procedure reduces the PDE under consideration to a system of PDE with fewer numbers of independent variables and greater number of dependent variables [1–4].

Antireduction of the nonlinear acoustics equation

$$u_{x_0x_1} - (u_{x_1}u)_{x_1} - u_{x_2x_2} - u_{x_3x_3} = 0 \tag{1}$$

is carried out in the paper [2] with the use of the ansatz

$$u = \frac{1}{3}x_1\varphi_1(x_0, x_2, x_3) - \frac{1}{6}x_1^2\varphi_2(x_0, x_2, x_3) + \varphi_3(x_0, x_2, x_3). \tag{2}$$

In [3] antireduction of the equation for short waves in gas dynamics

$$2u_{x_0x_1} - 2(2x_1 + u_{x_1})u_{x_1x_1} + u_{x_2x_2} + 2\lambda u_{x_1} = 0 \tag{3}$$

is carried out via the following ansatz:

$$u = x_1\varphi_1 + x_1^2\varphi_2 + x_1^{3/2}\varphi_3 + \varphi_4, \quad \varphi_i = \varphi_i(x_0, x_2). \tag{4}$$

Ansatzes (2), (4) reduce equations (1), (3) to system of PDE for three and four functions, respectively.

In the present paper we adduce some new results on antireduction for the nonlinear heat equations of the form

$$u_t = \big(a(u)u_x\big)_x + F(u). \tag{5}$$

The antireduction of equation (5) is performed by means of the ansatz

$$h\big(t, x, u, \varphi_1(\omega), \varphi_2(\omega), \dots, \varphi_N(\omega)\big) = 0 \tag{6}$$

where $\omega = \omega(t, x, u)$ is a new independent variable. Ansatz (6) reduces equation (5) to a system of ordinary differential equations (ODE) for the unknown functions $\varphi_i(\omega)$, $i = \overline{1, N}$.

Reprinted with permission from J. Nonlinear Math. Phys., 1994, **1**, № 1, P. 60–64

Below we list, without derivation, explicit forms of $a(u)$ and $F(u)$, such that equation (5) admits an antireduction of the form (6). For each case the reduced ODE are given.

1. $a(u)=\ddot\theta(u)\theta(u),\quad F(u)=\big(\lambda_1+\lambda_2\dot\theta(u)\big)\big(\ddot\theta(u)\big)^{-1},$
$\dot\theta(u)=\varphi_1(t)+\varphi_2(t)x,\quad \dot\varphi_1=(\lambda_2+\varphi_2^2)\varphi_1+\lambda_1,\quad \dot\varphi_2=(\lambda_2+\varphi_2^2)\varphi_2;$
2. $a(u)=u\dot\theta(u),\quad F(u)=\big(\lambda_1+\lambda_2\theta(u)\big)\big(\dot\theta(u)\big)^{-1},$
$\theta(u)=\varphi_1(t)+\varphi_2(t)x,\quad \dot\varphi_1=\lambda_2\varphi_1+\varphi_2^2+\lambda_1,\quad \dot\varphi_2=\lambda_2\varphi_2;$
3. $a(u)=\dot\theta(u),\quad F(u)=\big(\lambda_1+\lambda_2\theta(u)\big)\big(\dot\theta(u)\big)^{-1},$
$\theta(u)=\varphi_1(t)+\varphi_2(t)x,\quad \dot\varphi_1=\lambda_2\varphi_1+\lambda_1,\quad \dot\varphi_2=\lambda_2\varphi_2;$
4. $a(u)=\lambda u^k,\quad F(u)=\lambda_1 u+\lambda_2 u^{1-k},\quad u^k=\varphi_1(t)+\varphi_2(t)x+\varphi_3(t)x^2,$
$\dot\varphi_1=2\lambda\varphi_1\varphi_3+\lambda k^{-1}\varphi_2^2+k\lambda_2,\quad \dot\varphi_2=2\lambda(1+2k^{-1})\varphi_2\varphi_3+k\lambda_1\varphi_2,$
$\dot\varphi_3=2\lambda(1+2k^{-1})\varphi_3^2+k\lambda_1\varphi_3;$
5. $a(u)=\lambda e^u,\quad F(u)=\lambda_1+\lambda_2 e^{-u},\quad e^u=\varphi_1(t)+\varphi_2(t)x+\varphi_3(t)x^2,$
$\dot\varphi_1=2\lambda\varphi_1\varphi_3+\lambda_1\varphi_1+\lambda_2,\quad \dot\varphi_2=2\lambda\varphi_2\varphi_3+\lambda_1\varphi_2,\quad \dot\varphi_3=2\lambda\varphi_3^2+\lambda_1\varphi_3;$
6. $a(u)=\lambda u^{-3/2},\quad F(u)=\lambda_1 u+\lambda_2 u^{5/2},$
$u^{-3/2}=\varphi_1(t)+\varphi_2(t)x+\varphi_3(t)x^2+\varphi_4(t)x^3,$
$\dot\varphi_1=2\lambda\varphi_1\varphi_3-\frac{2}{3}\lambda\varphi_2^2-\frac{3}{2}\lambda_1\varphi_1-\frac{3}{2}\lambda_2,$
$\dot\varphi_2=-\frac{2}{3}\lambda\varphi_2\varphi_3+6\lambda\varphi_1\varphi_4-\frac{3}{2}\lambda_1\varphi_2,$
$\dot\varphi_3=-\frac{2}{3}\lambda\varphi_3^2+2\lambda\varphi_2\varphi_4-\frac{3}{2}\lambda_1\varphi_3,\quad \dot\varphi_4=-\frac{3}{2}\lambda_1\varphi_4;$
7. $a(u)=1,\quad F(u)=(\alpha+\beta\ln u)u,\quad \ln u=\varphi_1(t)+\varphi_2(t)x,$
$\dot\varphi_1=\beta\varphi_1+\varphi_2^2+\alpha,\quad \dot\varphi_2=\alpha\varphi_2;$
8. $a(u)=1,\quad F(u)=\big(\alpha+\beta\ln u-\gamma^2(\ln u)^2\big)u,\quad \ln u=\varphi_1(t)+\varphi_2(t)e^{\gamma x},$
$\dot\varphi_1=\alpha+\beta\varphi_1-\gamma^2\varphi_1^2,\quad \dot\varphi_2=(\beta+\gamma^2-2\gamma^2\varphi_1)\varphi_2;$
9. $a(u)=1,\quad F(u)=-u(1+\ln u^2)\big(\alpha+\beta(\ln u)^{-1/2}\big),$
$\int^{\ln u}\Big(2\alpha\tau+4\beta\tau^{1/2}+\varphi_2(t)\Big)^{-1/2}d\tau=x+\varphi_1(t),$
$\dot\varphi_1=0,\quad \dot\varphi_2=4\beta^2-2\alpha\varphi_2;$
10. $a(u)=1,\quad F(u)=-2(u^3+\alpha u^2+\beta u),$
 (a) $\alpha=\beta=0$
 $u=\big(\varphi_1(t)+2\varphi_2(t)x\big)\big(1+\varphi_1(t)x+\varphi_2(t)x^2\big)^{-1},$
 $\dot\varphi_1=-6\varphi_1\varphi_2,\quad \dot\varphi_2=-6\varphi_2^2;$
 (b) $\alpha^2=4\beta\neq 0$
 $u=\left(-\frac{\alpha}{2}\varphi_1(t)+\left(1-\frac{\alpha}{2}x\right)\varphi_2(t)\right)\left(e^{\alpha x/2}+\varphi_1(t)+\varphi_2(t)x\right)^{-1},$
 $\dot\varphi_1=-\frac{\alpha^2}{4}\varphi_1-\alpha\varphi_2,\quad \dot\varphi_2=-\frac{\alpha^2}{4}\varphi_2;$
 (c) $\alpha^2>4\beta$
 $u=\big((A+B)\varphi_1(t)e^{Bx}+(A-B)\varphi_2(t)e^{-Bx}\big)\times$

$$\times\left(e^{-Ax}+\varphi_1(t)e^{Bx}+\varphi_2(t)e^{-Bx}\right)^{-1},$$

$$A=-\frac{\alpha}{2},\quad B=\frac{1}{2}(\alpha^2-4\beta)^{1/2},$$

$$\dot\varphi_1=\left(\frac{\alpha^2}{2}-3\beta-\frac{\alpha}{2}(\alpha^2-4\beta)^{1/2}\right)\varphi_1,$$

$$\dot\varphi_2=\left(\frac{\alpha^2}{2}-3\beta+\frac{\alpha}{2}(\alpha^2-4\beta)^{1/2}\right)\varphi_2;$$

(d) $\alpha^2<4\beta$

$$u=\big(\varphi_1(t)(A\cos Bx-B\sin Bx)+\varphi_2(t)(A\sin Bx+$$

$$+B\cos Bx)\big)\left(e^{-Ax}+\varphi_1(t)\cos Bx+\varphi_2(t)\sin Bx\right)^{-1},$$

$$\dot\varphi_1=\left(\frac{\alpha^2}{2}-3\beta\right)\varphi_1-\frac{\alpha}{2}(4\beta-\alpha^2)^{1/2}\varphi_2,$$

$$\dot\varphi_2=\left(\frac{\alpha^2}{2}-3\beta\right)\varphi_2+\frac{\alpha}{2}(4\beta-\alpha^2)^{1/2}\varphi_1.$$

In the above formulae $\theta=\theta(u)\in C^2(\mathbb{R}^1,\mathbb{R}^1)$ is an arbitrary function; λ, λ_1, λ_2, α, β, γ are arbitrary real constants; overdot means differentiation with respect to the corresponding argument.

Most of above adduced system of ODE can be integrated. As a result, one obtains a number of new exact solutions of the nonlinear heat equation (5). Detailed study of reduced systems of ODE and construction of exact solutions of equation (5) will be a topic of our future paper. Here we present some exact solutions of the nonlinear heat equation

$$u_t=u_{xx}+F(u)$$

obtained with the help of ansatzes 7–10 which are listed above.

1) $F(u)=\big(\alpha+\beta\ln u-\gamma^2(\ln u)^2\big)u$,

(a) $\Delta=\beta^2+4\alpha\gamma^2>0$

$$u=C\left(\cos\frac{\Delta^{1/2}t}{2}\right)^{-2}e^{\gamma x+\gamma^2t}+\frac{1}{2\gamma^2}\left(\beta-\Delta^{1/2}\mathrm{tg}\,\frac{\Delta^{1/2}t}{2}\right);$$

(b) $\Delta=-\beta^2-4\alpha\gamma^2>0$

$$u=C\left(\mathrm{ch}\,\frac{\Delta^{1/2}t}{2}\right)^{-2}e^{\gamma x+\gamma^2t}+\frac{1}{2\gamma^2}\left(\beta+\Delta^{1/2}\mathrm{th}\,\frac{\Delta^{1/2}t}{2}\right);$$

(c) $\Delta=\beta^2+4\alpha\gamma^2=0$

$$u=Ct^{-2}e^{\gamma x+\gamma^2t}+\frac{1}{2\gamma^2t}(\beta t+2);$$

2) $F(u)=-u(1+\ln u^2)\big(\alpha+\beta(\ln u)^{-1/2}\big)$,

(a) $\alpha\neq0$

$$\int^{\ln u}\left(2\alpha\tau+4\beta\tau^{1/2}+Ce^{-2\alpha t}+2\beta^2\alpha^{-1}\right)^{-1/2}d\tau=x;$$

(b) $\alpha=0$

$$\int^{\ln u}\left(4\beta\tau^{1/2}+4\beta^2t\right)^{-1/2}d\tau=x;$$

3) $F(u) = -2u(u^2 + \alpha u + \beta)$,

(a) $\alpha^2 = 4\beta$

$$u = \left(1 - \frac{\alpha}{2}(x - \alpha t)\right)\left(x - \alpha t + C\exp\left(\frac{\alpha}{2}\left(x + \frac{\alpha t}{2}\right)\right)\right)^{-1};$$

(b) $\alpha^2 > 4\beta$

$$u = \Big((A+B)C_1\exp\big((A+B)(x-\alpha t)\big) + (A-B)C_2 \times$$
$$\times \exp\big((A-B)(x-\alpha t)\big)\Big)\Big(\exp(3\beta t) + C_1\exp\big((A+B)(x-\alpha t)\big) +$$
$$+ C_2\exp\big((A-B)(x-\alpha t)\big)\Big)^{-1},$$
$$A = -\frac{\alpha}{2}, \quad B = \frac{1}{2}(\alpha^2 - 4\beta)^{1/2};$$

(c) $\alpha^2 < 4\beta$

$$u = \big((\alpha AC_1 - BC_2)\cos B(x-\alpha t) + (AC_2 + BC_1)\times$$
$$\times \sin B(x-\alpha t)\big)\Big(\exp\big(3\beta t - A(x-\alpha t)\big) +$$
$$+ C_1\cos B(x-\alpha t) + C_2\sin B(x-\alpha t)\Big)^{-1},$$
$$A = -\frac{\alpha}{2}, \quad B = \frac{1}{2}(4\beta - \alpha^2)^{1/2}.$$

In the above formulae C, C_1, C_2 are arbitrary constants.

It is worth noting that the above solutions can not be obtained with the use of the classical Lie symmetry reduction technique [6]. That is why they are essentially new. Another important feature is that solutions 3(a) and 3(c) are soliton-like solutions. Consequently, nonlinear heat equation with cubic nonlinearity admits soliton-like solutions.

1. Fushchych W.I., Conditional symmetry of equations of nonlinear mathematical physics, *Ukr. Math. J.*, 1991, **43**, № 11, 1456–1470.
2. Fushchych W.I., Myronyuk P.I., Conditional symmetry and exact solutions of nonlinear acoustics equation, *Dopovidi of Ukr. Acad. Sci.*, 1991, № 6, 23–29.
3. Fushchych W.I., Repeta V., Exact solutions of equations of gas dynamics and nonlinear acoustics, *Dopovidi of Ukr. Acad. Sci.*, 1991, № 8, 35–38.
4. Fushchych W.I., Conditional symmetry of equations of mathematical physics, in Proceedings of the International Workshop "Modern Group Analysis", Editors N. Ibragimov, M. Torrisi and A. Valenti, Kluwer Academic Publishers, 1993, 231–240.
5. Fushchych W.I., Zhdanov R.Z., Antireduction of the nonlinear wave equations, *Dopovidi of Ukr. Acad. Sci.*, 1993, № 11, 37–41.
6. Fushchych W., Shtelen W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kluwer Academic Publishers, 1993.

Nonlinear representations for Poincaré and Galilei algebras and nonlinear equations for electromagnetic fields

W.I. FUSHCHYCH, I.M. TSYFRA, V.M. BOYKO

We construct nonlinear representations of the Poincaré, Galilei, and conformal algebras on a set of the vector-functions $\Psi = (\vec{E}, \vec{H})$. A nonlinear complex equation of Euler type for the electromagnetic field is proposed. The invariance algebra of this equation is found.

1. Introduction

It is well known that the linear representations of the Poincaré algebra $AP(1,3)$ and conformal algebra $AC(1,3)$, with the basis elements

$$P_\mu = ig^{\mu\nu}\partial_\nu, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + S_{\mu\nu}, \tag{1}$$

$$D = x_\nu P^\nu - 2i, \tag{2}$$

$$K_\mu = 2x_\mu D - (x_\nu x^\nu)P_\mu + 2x^\nu S_{\mu\nu}, \tag{3}$$

is realized on the set of solutions of the Maxwell equations for the electromagnetic field in vacuum (see e.g. [1, 2])

$$\frac{\partial \vec{E}}{\partial t} = \operatorname{rot} \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\operatorname{rot} \vec{E}, \tag{4}$$

$$\operatorname{div} \vec{E} = 0, \quad \operatorname{div} \vec{H} = 0. \tag{5}$$

Here $S_{\mu\nu}$ realize the representation $D(0,1) \oplus D(1,0)$ of the Lorentz group.

Operators (1)–(3) satisfy the following commutation relations:

$$[P_\mu, P_\nu] = 0, \quad [P_\mu, J_{\alpha\beta}] = i(g_{\mu\alpha}P_\beta - g_{\mu\beta}P_\alpha), \tag{6}$$

$$[J_{\alpha\beta}, J_{\mu\nu}] = i(g_{\beta\mu}J_{\alpha\nu} + g_{\alpha\nu}J_{\beta\mu} - g_{\alpha\mu}J_{\beta\nu} - g_{\beta\nu}J_{\alpha\mu}), \tag{7}$$

$$[D, P_\mu] = -iP_\mu, \quad [D, J_{\mu\nu}] = 0, \tag{8}$$

$$[K_\mu, P_\alpha] = i(2J_{\alpha\mu} - 2g_{\mu\alpha}D), \quad [K_\mu, J_{\alpha\beta}] = i(g_{\mu\nu}K_\beta - g_{\mu\beta}K_\alpha), \tag{9}$$

$$[K_\mu, D] = -iK_\mu, \quad [K_\mu, K_\nu] = 0, \quad \mu, \nu, \alpha, \beta = 0, 1, 2, 3. \tag{10}$$

In this paper the nonlinear representations of the Poincaré, Galilei, and conformal algebras for the electromagnetic field $\vec{E}$, $\vec{H}$ are constructed. In particular, we prove that the continuity equation for the electromagnetic field is not invariant under the Lorentz group if the velocity of the electromagnetic field is taken in accordance with the Poynting definition. Conditional symmetry of the continuity equation is studied. The complex Euler equation for the electromagnetic field is introduced. The symmetry of this equation is investigated.

Reprinted with permission from J. Nonlinear Math. Phys., 1994, **1**, № 2, P. 210–221

2. Formulation of the main results

The operators, realizing the nonlinear representations of the Poincaré algebras $AP(1,3) = \langle P_\mu, J_{\mu\nu}\rangle$, $AP_1(1,3) = \langle P_\mu, J_{\mu\nu}, D\rangle$, and conformal algebra $AC(1,3) = \langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$, have the structure

$$P_\mu = \partial_{x_\mu}, \tag{11}$$

$$J_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + S_{kl}, \tag{12}$$

$$J_{0k} = x_0\partial_{x_k} + x_k\partial_{x_0} + S_{0k}, \quad k,l = 1,2,3, \tag{13}$$

$$D = x_\mu\partial_{x_\mu}, \tag{14}$$

$$K_0 = x_0^2\partial_{x_0} + x_0x_k\partial_{x_k} + (x_k - x_0E^k)\partial_{E^k} - x_0H^k\partial_{H^k}, \tag{15}$$

$$\begin{aligned} K_l = {} & x_0x_l\partial_{x_0} + x_lx_k\partial_{x_k} + [x_kE^l - x_0(E^lE^k - H^lH^k)]\partial_{E^k} + \\ & + [x_kH^l - x_0(H^lE^k + E^lH^k)]\partial_{H^k}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} S_{kl} &= E^k\partial_{E^l} - E^l\partial_{E^k} + H^k\partial_{H^l} - H^l\partial_{H^k}, \\ S_{0k} &= \partial_{E^k} - (E^kE^l - H^kH^l)\partial_{E^l} - (E^kH^l + H^kE^l)\partial_{H^l}. \end{aligned}$$

The operators, realizing the nonlinear representations of the Galilei algebras $AG^{(2)}(1,3) = \langle P_\mu, J_{kl}, G_k^{(2)}\rangle$, $AG_1^{(2)}(1,3) = \langle P_\mu, J_{kl}, G_k^{(2)}, D\rangle$ have the form:

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + S_{kl}, \tag{17}$$

$$G_k^2 = x_k\partial_{x_0} - (E^kE^l - H^kH^l)\partial_{E^l} - (E^kH^l + H^kE^l)\partial_{H^l}, \tag{18}$$

$$D = x_0\partial_{x_0} + 2x_k\partial_{x_k} + E^k\partial_{E^k} + H^k\partial_{H^k}. \tag{19}$$

We see by direct verification that all represented operators satisfy the commutation relations of the algebras $AP(1,3)$, $AC(1,3)$, $AG(1,3)$.

3. Construction of nonlinear representations

In order to construct the nonlinear representations of Euclid-, Poincaré-, and Galilei groups and their extensions the following idea was proposed in [2, 3]: to use nonlinear equations invariant under these groups; it is necessary to find (point out, guess) the equations, which admit symmetry operators having a nonlinear structure. Such equation for the scalar field $u(x_0, x_1, x_2, x_3)$ is the eikonal equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} = 0, \quad \mu = 0,1,2,3 \tag{20}$$

which is invariant under the conformal algebra $AC(1,3)$ with the nonlinear operator K_μ [2, 3].

The nonlinear Euler equation for an ideal fluid

$$\frac{\partial v_k}{\partial t} + v_l \frac{\partial v_k}{\partial x_l} = 0, \quad k = 1, 2, 3 \tag{21}$$

which is invariant under nonlinear representation of the $AP(1,3)$ algebra, with basis elements

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + v_k \partial_{v_l} - v_l \partial_{v_k}, \tag{22}$$

$$J_{0k} = x_k \partial_0 + x_0 \partial_{x_k} + \partial_{v_k} - v_k v_l \partial_{v_l}, \tag{23}$$

was proposed in [3] to construct the nonlinear representation for the vector field. Note that equation (21) is also invariant with respect to the Galilei algebra $AG(1,3)$ with the basis elements

$$P_\mu = \partial_{x_\mu}, \quad J_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + v_k \partial_{v_l} - v_l \partial_{v_k}, \quad G_a = x_0 \partial_{x_a} + \partial_{v_a}. \tag{24}$$

As mentioned in [2, 3] both the Lorentz–Poincaré–Einstein and Galilean principles of relativity are valid for system (21). We use the following nonlinear system of equations [4]

$$\frac{\partial E^k}{\partial x_0} + H^l \frac{\partial E^k}{\partial x_l} = 0, \quad \frac{\partial H^k}{\partial x_0} + E^l \frac{\partial H^k}{\partial x_l} = 0, \tag{25}$$

for constructing a nonlinear representation of the $AP(1,3)$ and $AG(1,3)$ algebras for the electromagnetic field. To construct the basis elements of the $AP(1,3)$ and $AG(1,3)$ algebras in explicit form we investigate the symmetry of system (25). We search for the symmetry operators of equations (25) in the form:

$$X = \xi^\mu \partial_{x_\mu} + \eta^l \partial_{E^l} + \beta^l \partial_{H^l}, \tag{26}$$

where $\xi^\mu = \xi^\mu(x, \vec{E}, \vec{H})$, $\eta^l = \eta^l(x, \vec{E}, \vec{H})$, $\beta^l = \beta^l(x, \vec{E}, \vec{H})$.

Theorem 1. *The maximal invariance algebra of system* (25) *in the class of operators* (26) *is the* 20*-dimensional algebra, whose basis elements are given by the formulas*

$$P_\mu = \partial x_\mu, \tag{27}$$

$$J^{(1)}_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + E^k \partial_{E^l} - E^l \partial_{E^k} + H^k \partial_{H^l} - H^l \partial_{H^k}, \tag{28}$$

$$J^{(2)}_{kl} = x_k \partial_{x_l} + x_l \partial_{x_k} + E^k \partial_{E^l} + E^l \partial_{E^k} + H^k \partial_{H^l} + H^l \partial_{H^k}, \tag{29}$$

$$G^{(1)}_a = x_0 \partial_{x_a} + \partial_{E^a} + \partial_{H^a}, \tag{30}$$

$$G^{(2)}_a = x_a \partial_{x_0} - E^a E^k \partial_{E^k} - H^a H^k \partial_{H^k}, \tag{31}$$

$$D_0 = x_0 \partial_{x_0} - E^l \partial_{E^l} - H^l \partial_{H^l}, \tag{32}$$

$$D_1 = x_1 \partial_{x_1} + E^1 \partial_{E^1} + H^1 \partial_{H^1}, \tag{33}$$

$$D_2 = x_2 \partial_{x_2} + E^2 \partial_{E^2} + H^2 \partial_{H^2}, \tag{34}$$

$$D_3 = x_3 \partial_{x_3} + E^3 \partial_{E^3} + H^3 \partial_{H^3}. \tag{35}$$

Proof. To prove theorem 1 we use Lie's algorithm. The condition of invariance of the system $L(\vec{E},\vec{H})$, i.e. (25), with respect to operator X has the form

$$\underset{1}{X} L\Big|_{L=0} = 0, \tag{36}$$

where

$$\underset{1}{X} = X + [D_\alpha(\eta^l) - E^l_j D_\alpha(\xi^j)]\partial_{E^l_\alpha} + [D_\alpha(\beta^l) - H^l_j D_\alpha(\xi^j)]\partial_{H^l_\alpha},$$
$$E^l_\alpha = \frac{\partial E^l}{\partial x_\alpha}, \quad H^l_\alpha = \frac{\partial H^l}{\partial x_\alpha}, \quad l = 1,2,3; \quad \alpha = 0,1,2,3$$

is the prolonged operator. From the invariance condition (36) we obtain the system of equations which determine the coefficient functions ξ^μ, η^l, β^l of the operator (26):

$$\begin{aligned}
&\eta^l_k = 0, \quad \eta^l_0 = 0, \quad \beta^l_k = 0, \quad \beta^l_0 = 0, \quad \xi^\mu_{\alpha\nu} = 0, \quad \xi^\mu_{E^a} = 0, \quad \xi^\mu_{H^a} = 0,\\
&\eta^k = -E^k\xi^0_0 + \xi^k_0 + E^a\xi^k_a - E^aE^k\xi^0_a,\\
&\beta^k = -H^k\xi^0_0 + \xi^k_0 + H^a\xi^k_a - H^aH^k\xi^0_a,
\end{aligned} \tag{37}$$

where

$$\eta^l_k = \frac{\partial\eta^l}{\partial x_k}, \quad \eta^l_0 = \frac{\partial\eta^l}{\partial x_0}, \quad \xi^\mu_{E^a} = \frac{\partial\xi^\mu}{\partial E^a}, \quad \xi^\mu_{\alpha\nu} = \frac{\partial^2\xi^\mu}{\partial x_\alpha\partial x_\nu}.$$

Having found the general solution of system (37), we get the explicit form of all the linear independent symmetry operators of system (25), which have the structure (27)–(35). Operators of Lorentz rotations J_{0k} is given by the linear combination of the Galilean operators $G^{(1)}_k$ and $G^{(2)}_k$:

$$J_{0k} = G^{(1)}_k + G^{(2)}_k. \tag{38}$$

All the following statements, given here without proofs, can be proved in analogy with the above-mentioned scheme.

4. The finite transformations and invariants

We present some finite transformations which are generated by the operators J_{0k}:

$$\begin{aligned}
J_{01}:\quad &x_0 \to x'_0 = x_0\operatorname{ch}\theta_1 + x_1\operatorname{sh}\theta_1,\\
&x_1 \to x'_1 = x_1\operatorname{ch}\theta_1 + x_0\operatorname{sh}\theta_1,\\
&x_2 \to x'_2 = x_2, \quad x_3 \to x'_3 = x_3,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&E^1 \to E^{1'} = \frac{E^1\operatorname{ch}\theta_1 + \operatorname{sh}\theta_1}{E^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, \quad H^1 \to H^{1'} = \frac{H^1\operatorname{ch}\theta_1 + \operatorname{sh}\theta_1}{H^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1},\\
&E^2 \to E^{2'} = \frac{E^2}{E^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, \quad H^2 \to H^{2'} = \frac{H^2}{H^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1},\\
&E^3 \to E^{3'} = \frac{E^3}{E^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}, \quad H^3 \to H^{3'} = \frac{H^3}{H^1\operatorname{sh}\theta_1 + \operatorname{ch}\theta_1}.
\end{aligned} \tag{40}$$

The operators J_{02}, J_{03} generate analogous transformations. θ_1 is the real group parameter of the geometric Lorentz transformation. Operators $G_k^{(2)}$ generate the following transformations:

$$G_1^{(2)}: \quad x_0 \to x_0' = x_0 + \theta_1 x_1, \quad x_k \to x_k' = x_k,$$
$$E^k \to E^{k'} = \frac{E^k}{1+\theta_1 E^1}, \quad H^k \to H^{k'} = \frac{H^k}{1+\theta_1 H^1}.$$

Analogous transformations are generated by the operators $G_2^{(2)}$, $G_3^{(2)}$. Operators $G_k^{(1)}$ generate the following transformations:

$$\begin{aligned} G_1^{(1)}: \quad & x_0 \to x_0' = x_0, \quad x_1 \to x_1' = x_1 + x_0\theta_1, \\ & x_2 \to x_2' = x_2, \quad x_3 \to x_3' = x_3, \\ & E^1 \to E^{1'} = E^1 + \theta_1, \quad H^1 \to H^{1'} = H^1 + \theta_1, \\ & E^2 \to E^{2'} = E^2, \quad E^3 \to E^{3'} = E^3, \\ & H^2 \to H^{2'} = H^2, \quad H^3 \to H^{3'} = H^3. \end{aligned}$$

The operators $G_2^{(1)}$, $G_3^{(1)}$ generate analogous transformations.

It is easy to verify that

$$I_1 = \frac{\left(1 - \vec{E}\vec{H}\right)^2}{\left(1-\vec{E}^2\right)\left(1-\vec{H}^2\right)}, \quad \vec{E}^2 \neq 1, \quad \vec{H}^2 \neq 1 \tag{41}$$

is invariant with respect to the nonlinear transformations of the Poincaré group which are generated by representations (28), (38).

The invariant of the Galilei group which is generated by representations (28), (31) has the form:

$$I_2 = \frac{\vec{E}^2\vec{H}^2}{\left(\vec{E}\vec{H}\right)^2}, \tag{42}$$

whereas the Galilei group which is generated by representations (28), (30) has the invariant

$$I_3 = (\vec{E} - \vec{H})^2. \tag{43}$$

5. Complex Euler equation for the electromagnetic field

Let us consider the system of equations

$$\frac{\partial \Sigma^k}{\partial x_0} + \Sigma^l \frac{\partial \Sigma^k}{\partial x_l} = 0, \quad \Sigma^k = E^k + iH^k. \tag{44}$$

The complex system (44) is equivalent to the real system of equations for $\vec{E}$ and $\vec{H}$

$$\begin{aligned} &\frac{\partial E^k}{\partial x_0} + E^l \frac{\partial E^k}{\partial x_l} - H^l \frac{\partial H^k}{\partial x_l} = 0, \\ &\frac{\partial H^k}{\partial x_0} + H^l \frac{\partial E^k}{\partial x_l} + E^l \frac{\partial H^k}{\partial x_l} = 0. \end{aligned} \tag{45}$$

The following statement has been proved with the help of Lie's algorithm.

Theorem 2. *The maximal invariance algebra of the system* (45) *is the* 24*-dimensional Lie algebra whose basis elements are given by the formulas*

$$\begin{aligned} &P_\mu = \partial_{x_\mu}, \\ &J^{(1)}_{kl} = x_k \partial_{x_l} - x_l \partial_{x_k} + E^k \partial_{E^l} - E^l \partial_{E^k} + H^k \partial_{H^l} - H^l \partial_{H^k}, \\ &J^{(2)}_{kl} = x_k \partial_{x_l} + x_l \partial_{x_k} + E^k \partial_{E^l} + E^l \partial_{E^k} + H^k \partial_{H^l} + H^l \partial_{H^k}, \\ &G^{(1)}_a = x_0 \partial_{x_a} + \partial_{E^a}, \\ &G^{(2)}_a = x_a \partial_{x_0} - (E^a E^k - H^a H^k)\partial_{E^a} - (E^a H^k + H^a E^k)\partial_{H^k}, \\ &D_0 = x_0 \partial_{x_0} - E^k \partial_{E^k} - H^k \partial_{H^k}, \\ &D_a = x_a \partial_{x_a} + E^a \partial_{E^a} + H^a \partial_{H^a} \quad \text{(no sum over } a), \\ &K_0 = x_0^2 \partial_{x_0} + x_0 x_k \partial_{x_k} + (x_k - x_0 E^k)\partial_{E^k} - x_0 H^k \partial_{H^k}, \\ &K_a = x_0 x_a \partial_{x_0} + x_a x_k \partial_{x_k} + [x_k E^a - x_0(E^a E^k - H^a H^k)]\partial_{E^k} + \\ &\qquad + [x_k H^a - x_0(H^a E^k + E^a H^k)]\partial_{H^k}. \end{aligned} \tag{46}$$

The algebra, engendered by the operators (46), include the Galilei algebras $AG^{(1)}(1,3)$, $AG^{(2)}(1,3)$ and Poincaré algebra $AP(1,3)$, and conformal algebra $AC(1,3)$ as subalgebras. Operators $G^{(2)}_a$ generate the linear geometrical transformations in $\mathbb{R}(1,3)$

$$x_0 \to x_0' = x_0 + \theta_a x_a \quad \text{(no sum over } a), \quad x_l \to x_l', \tag{47}$$

as well as the nonlinear transformations of the fields

$$\begin{aligned} &E^l + iH^l \to E^{l'} + iH^{l'} = \frac{E^l + iH^l}{1 + \theta_a(E^a + iH^a)} \quad \text{(no sum over } a), \\ &E^l - iH^l \to E^{l'} - iH^{l'} = \frac{E^l - iH^l}{1 + \theta_a(E^a - iH^a)}. \end{aligned} \tag{48}$$

The invariant of the group $G^{(2)}(1,3)$ is

$$I_4 = \frac{(\vec{E}^2 - \vec{H}^2) + 4(\vec{E}\vec{H})^2}{(\vec{E}^2 + \vec{H}^2)^2}. \tag{49}$$

Operators J_{0k} generate the linear transformations in $\mathbb{R}(1,3)$

$$\begin{aligned} &x_0 \to x_0' = x_0 \operatorname{ch}\theta_k + x_0 \operatorname{sh}\theta_k, \\ &x_k \to x_k' = x_k \operatorname{ch}\theta_k + x_0 \operatorname{sh}\theta_k \quad \text{(no sum over } k), \\ &x_l \to x_l' = x_l, \quad \text{if } l \neq k, \end{aligned} \tag{50}$$

as well as the nonlinear transformations of the fields

$$E^k + iH^k \to E^{k'} + iH^{k'} = \frac{(E^k + iH^k)\operatorname{ch}\theta_k + \operatorname{sh}\theta_k}{(E^k + iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k},$$
$$E^k - iH^k \to E^{k'} - iH^{k'} = \frac{(E^k - iH^k)\operatorname{ch}\theta_k + \operatorname{sh}\theta_k}{(E^k - iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k}.$$

If $l \neq k$, then

$$\begin{aligned} &E^l + iH^l \to E^{l'} + iH^{l'} = \frac{E^l + iH^l}{(E^k + iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k}, \\ &E^l - iH^l \to E^{l'} - iH^{l'} = \frac{E^l - iH^l}{(E^k - iH^k)\operatorname{sh}\theta_k + \operatorname{ch}\theta_k} \quad \text{(no sum over } k\text{)}. \end{aligned} \tag{51}$$

The invariant of group $P(1,3)$ is

$$I_5 = \frac{1 - 2\left[(\vec{E}^2 - \vec{H}^2) - \frac{1}{2}(\vec{E}^2 - \vec{H}^2)^2 - 2(\vec{E}\vec{H})^2\right]}{\left[1 - (\vec{E}^2 + \vec{H}^2)\right]^2}, \quad \vec{E}^2 + \vec{H}^2 \neq 1. \tag{52}$$

The operator K_0 generates the following nonlinear transformations in $\mathbb{R}(1,3)$ and linear transformations of the fields

$$\begin{aligned} &x_\mu \to x'_\mu = \frac{x_\mu}{1 - \theta_0 x_0}, \\ &E^k \to E^{k'} = E^k + \theta_0(x_k - x_0 E^k), \\ &H^k \to H^{k'} = H^k(1 - \theta_0 x_0). \end{aligned} \tag{53}$$

The operators K_a generate nonlinear transformations in both $\mathbb{R}(1,3)$ and of the fields

$$x_0 \to x'_0 = \frac{x_0}{1 - x_a\theta_a}, \quad x_a \to x'_a = \frac{x_a}{1 - x_a\theta_a}.$$

If $k \neq a$, then

$$x_k \to x'_k = \frac{x_k}{1 - x_a\theta_a},$$
$$E^a + iH^a \to E^{a'} + iH^{a'} = \frac{E^a + iH^a}{1 + \theta_a[x_0(E^a + iH^a) - x_a]},$$
$$E^a - iH^a \to E^{a'} - iH^{a'} = \frac{E^a - iH^a}{1 + \theta_a[x_0(E^a - iH^a) - x_a]}.$$

If $k \neq a$, then

$$\begin{aligned} &E^k + iH^k \to E^{k'} + iH^{k'} = \frac{E^k + iH^k + \theta_a(E^a + iH^a)x_k}{1 + \theta_a[x_0(E^a + iH^a) - x_a]}, \\ &E^k - iH^k \to E^{k'} - iH^{k'} = \frac{E^k - iH^k + \theta_a(E^a - iH^a)x_k}{1 + \theta_a[x_0(E^a - iH^a) - x_a]} \quad \text{(no sum over } a\text{)}. \end{aligned} \tag{54}$$

Note 1. Setting $\vec{\Sigma} = a\vec{E} + ib\vec{H}$, where a, b are arbitrary functions of the invariants $\vec{E}^2$, $\vec{H}^2$, $\vec{E}\vec{H}$, we obtain more general form of the equation (44). The equation

$$\frac{\partial \Sigma^k}{\partial x_0} + \Sigma^l \frac{\partial \Sigma^k}{\partial x_l} = F(\vec{E}\vec{H}, \vec{E}^2, \vec{H}^2)\Sigma^k$$

is invariant only under some subalgebras of algebra (46) depending on the choice of function F.

Note 2. If we analyse the symmetry of the following equations

$$\left(\frac{\partial}{\partial x_0} + E^l\frac{\partial}{\partial x_l} + H^l\frac{\partial}{\partial x_l}\right)E^k = 0,$$
$$\left(\frac{\partial}{\partial x_0} + E^l\frac{\partial}{\partial x_l} + H^l\frac{\partial}{\partial x_l}\right)H^k = 0; \qquad (*)$$

or

$$\frac{\partial E^k}{\partial x_0} = \pm\left(E^l\frac{\partial}{\partial x_l} + H^l\frac{\partial}{\partial x_l}\right)H^k,$$
$$\frac{\partial H^k}{\partial x_0} = \pm\left(E^l\frac{\partial}{\partial x_l} + H^l\frac{\partial}{\partial x_l}\right)E^k, \qquad (**)$$

we obtain concrete examples of nonlinear representations for the Poincaré and Galilei algebras. This problem will be considered in a future paper.

6. Symmetry of the continuity equation and the Poynting vector

Let us consider the continuity equation for the electromagnetic field

$$L(\vec{E},\vec{H}) \equiv \frac{\partial\rho}{\partial x_0} + \operatorname{div}\rho\vec{v} = 0. \qquad (55)$$

According to the Poynting definition ρ and ρv^k have the forms

$$\rho = \frac{1}{2}(\vec{E}^2 + \vec{H}^2), \quad \rho v^k = \varepsilon_{kln}E^lH^n. \qquad (56)$$

Theorem 3. *The nonlinear system* (55), (56) *is not invariant under the Lorentz algebra, with basis elements:*

$$J_{kl} = x_k\partial_{x_l} - x_l\partial_{x_k} + E^k\partial_{E^l} - E^l\partial_{E^k} + H^k\partial_{H^l} - H^l\partial_{H^k},$$
$$J_{0k} = x_k\partial_{x_0} + x_0\partial_{x_k} + \varepsilon_{kln}(E^l\partial_{H^n} - H^l\partial_{E^n}), \quad k,l,n = 1,2,3. \qquad (57)$$

To prove theorem 3 it is necessary to substitute ρ and ρv^k, from formulas (56), to equation (55) and to apply Lie's algorithm, i.e., it is necessary to verify that the invariance condition

$$\underset{1}{J}_{\mu\nu}\left(L(\vec{E},\vec{H})\right)\Big|_{L=0} \equiv 0 \qquad (58)$$

is not satisfied, where $\underset{1}{J}_{\mu\nu}$ is the first prolongation of the operator $J_{\mu\nu}$.

Theorem 4. *The continuity equation* (55), (56) *is conditionally invariant with respect to the operators* $J_{\mu\nu}$*, given in* (57) *if and only if* $\vec{E}$*,* $\vec{H}$ *satisfy the Maxwell equation* (4), (5).

Thus the continuity equation, which is the mathematical expression of the conservation law of the electromagnetic field energy and impulse is not Lorentz-invariant if

$\vec{E}$, $\vec{H}$ does not satisfy the Maxwell equation. A more detailed discussion on conditional symmetries can be found in [1, 2].

The following statement can be proved in the case when

$$\rho = F^0(\vec{E}, \vec{H}) \quad \text{and} \quad \rho v^k = F^k(\vec{E}, \vec{H}), \tag{59}$$

where F^0, F^k are arbitrary smooth functions $F^0 \not\equiv 0$, $F^k \not\equiv 0$.

Theorem 5. *The continuity equation* (55), (59) *is invariant with respect to the classic geometrical Lorentz transformations if and only if*

$$r(B) = 4, \tag{60}$$

where $r(B)$ is the rank of the Jacobi matrix of functions F^μ.

In conclusion we present some statements about the symmetry of the following systems:

$$\begin{aligned} &\frac{\partial \vec{E}}{\partial x_0} = \mathrm{rot}\,\vec{H} + \vec{F}_1(\vec{E}, \vec{H}), \quad \frac{\partial \vec{H}}{\partial x_0} = -\mathrm{rot}\,\vec{E} + \vec{F}_2(\vec{E}, \vec{H}), \\ &\mathrm{div}\,\vec{E} = R_1(\vec{E}, \vec{H}), \quad \mathrm{div}\,\vec{H} = R_2(\vec{E}, \vec{H}), \end{aligned} \tag{61}$$

$$\begin{aligned} &\frac{\partial (R\vec{E})}{\partial x_0} = \mathrm{rot}\,(R\vec{H}), \quad \frac{\partial N\vec{H}}{\partial x_0} = -\mathrm{rot}\,(N\vec{E}), \\ &\mathrm{div}\,(R\vec{E}) = 0, \quad \mathrm{div}\,(N\vec{H}) = 0. \end{aligned} \tag{62}$$

Here

$$R = R(W_1, W_2), \quad N = N(W_1, W_2), \quad W_1 = \vec{E}^2 - \vec{H}^2, \quad W_2 = \vec{E}\vec{H}.$$

$$\mathrm{div}\,(R\vec{E} + N\vec{H}) = 0. \tag{63}$$

Theorem 6. *The system of equations* (61) *is invariant under the Lorentz algebra with the basis elements* (57) *if and only if*

$$\vec{F}_1 \equiv \vec{F}_2 \equiv 0, \quad R_1 \equiv R_2 \equiv 0.$$

Theorem 7. *The system of equations* (62) *is invariant under the Lorentz algebra* (57) *if R and N are arbitrary functions of the invariants $W_1 = \vec{E}^2 - \vec{H}^2$, $W_2 = \vec{E}\vec{H}$.*

Theorem 8. *The equation* (63) *is invariant under the Lorentz algebra with the basis elements* (57) *if and only if $\vec{E}$, $\vec{H}$ satisfy the system of equations*

$$\frac{\partial (R\vec{E} + N\vec{H})}{\partial x_0} = \mathrm{rot}\,(R\vec{H} - N\vec{E}).$$

Thus it is established that, besides the generally recognized linear representation of the Lorentz group discovered by Henry Poincaré in 1905 [5], there exists the nonlinear representation constructed by using the nonlinear equations of hydrodynamical type [4]. It is obvious that for instance the linear superposition principle does not hold for a non-Maxwell electrodynamic theory based on the equation (25) or (45).

The nonlinear representations for the algebras $AG(1,3)$, $A\tilde{P}(1,2)$, $A\tilde{P}(2,2)$, $AC(1,2)$, $AC(2,2)$ for a scalar field have been considered in [6], $AP(1,1)$ in [7], and $AP(1,2)$ in [8].

1. Fushchych W., Nikitin A., Symmetries of Maxwell's equations, Dordrecht, Reidel Publ. Comp., 1987.
2. Fushchych W., Shtelen W., Serov N., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Publishers, 1993.
3. Fushchych W., On symmetry and exact solutions of the multidimensional equations of mathematical physic, in Algebraic-Theoretical Studies in Mathematical Physics Problems, Kyiv, Institute of Mathematics, 1983, 4–23.
4. Fushchych W., New nonlinear equations for electromagnetic field having the velocity different from c, *Dopovidi of Ukr. Acad. Sci.*, 1982, № 4, 24–28.
5. Poincaré H., On the dynamics of the electron, *Comptes Rendus*, 1905, **140**, 1504–1510.
6. Fushchych W., Zhdanov R., Lahno V., On nonlinear representation of the conformal algebra $AC(2,2)$, *Dopovidi of Ukr. Acad. Sci.*, 1993, № 9, 44–47.
7. Rideau G., Winternitz P., Nonlinear equations invariant under the Poincaré, similitude and conformal groups in two-dimensional space-time, *J. Math. Phys.*, 1990, **31**, 1095–1105.
8. Yehorchenko I., Nonlinear representation of the Poincaré algebra and invariant equations, in Symmetry Analysis of Equations of Mathematical Physics, Kyiv, Institute of Mathematics, 1992, 62–66.

*For more precise definitions and detailed reviews of results on realizations of Lie algebras in Lie vector fields see [Zhdanov R.Z., Lahno V.I., Fushchych W.I., On covariant realizations of the Euclid group, *Commun. Math. Phys.*, 2000, **212**, 535–556] and [Popovych R., Boyko V., Nesterenko M., Lutfullin M., Realizations of real low-dimensional Lie algebras, *J. Phys. A: Math. Gen.*, 2003, **36**, N 26, 7337–7360; math-ph/0301029]. *Editors' Remark.*

Ansatz '95

W.I. FUSHCHYCH

In this talk I am going to present a brief review of some key ideas and methods which were given start and were developed in Kyiv, at the Institute of Mathematics of National Academy of Sciences of Ukraine during recent years.

Plan of the talk

The simplest classification of equations.

What is ansatz? The problem of PDE reduction without symmetry.

Conditional symmetry. How can we expand symmetry of PDE?

Conditional symmetry of Maxwell and Schrödinger systems.

Q-conditional symmetry of the nonlinear wave equation, which is not invariant with respect to the Lorentz group.

Conditional symmetry of the Poincaré–invariant d'Alembert equation.

Conditional symmetry of the nonlinear heat equation.

Reduction and Antireduction.

Antireduction of the nonlinear acoustics equation.

Antireduction of the equation for short waves in gas dynamics.

Antireduction of nonlinear heat equation.

Nonlocal symmetry, new relativity principles.

Non–Lie symmetry of the Schrödinger equation.

Time is absolute in relativistic physics.

New equations of motions.

High–order parabolic equation in Quantum Mechanics.

Nonlinear generalization of the Maxwell equations.

Equations for fields with the spin 1/2.

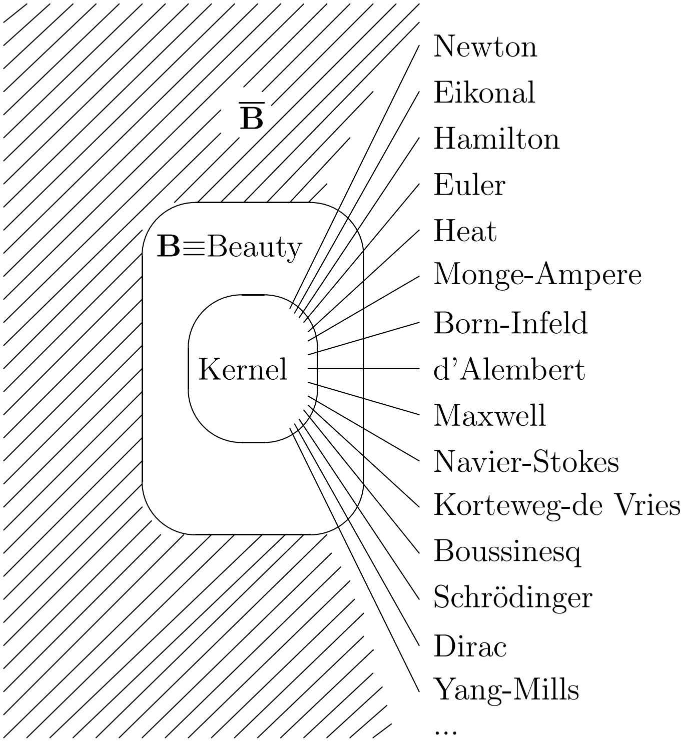
How to extend symmetry of on equation with arbitrary coefficients?

1. Classification of equations

Every field of science must begin from some classification. We have today a lot of classifications of differential equations: parabolic, hyperbolic, elliptic, ultrahyperbolic etc. I believe that it is most appropriate for our Conference to divide all equations of

Reprinted with permission from J. Nonlinear Math. Phys., 1995, **2**, N 3–4, P. 216–235

mathematics into two classes: B and $\bar{B}$



It is seen from the adduced picture that all fundamental equations of mathematical physics are united into one class B. From the point of view of existing now classifications they belong to essentially different classes. Equations from the class B have wide symmetry, and by this feature they are substantially different from other equations of mathematics.

It is important to point out that there are close relations among these different equations, which have not been investigated yet till now. For example, if we know solutions of the heat equation, we can construct solutions for the wave (d'Alembert) equation. By means of solutions of the Dirac equation, solutions of the Maxwell, heat, Yang–Mills, and other equations [18] can be obtained.

2. Ansatz reduction of PDE without using symmetry

Let us consider a PDE

$$
\begin{gathered}
L(x, u, u_{(1)}, u_{(2)}, \dots u_{(n)}) = 0, \\
u = u(x), \quad x = (x_0, x_1, \dots, x_n), \quad u_{(1)} = (u_0, u_1, u_2, \dots, u_n), \ u_\mu = \frac{\partial u}{\partial x_\mu}, \\
u_{(2)} = (u_{00}, u_{01}, \dots, u_{nn}), \ u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}.
\end{gathered}
\tag{2.1}
$$

Depending on the explicit form of L, equation (2.1) can belong to B or $\bar{B}$. In mathematical physics we often come across equations of the following type:

$$Lu \equiv \Box u - F(x, u, u_{(1)}) = 0. \tag{2.2}$$

What can we say today about solutions of equations (2.1), (2.2)? The answer is trivial: Nothing.

If equation (2.2) belongs to the class B and is invariant with respect to the Poincaré group $P(1,n)$, that is, a nonlinear function $F(x, u, u_{(1)})$ has the special form

$$F(x, u, u_{(1)}) = F\left(u, \frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu}\right) \tag{2.3}$$

then for equation (2.2) we can construct some classes of exact solutions, study Painlevé properties, construct approximate solutions, study asymptotic properties, etc.

Definition 1 (W. Fushchych, 1981, 1983 [1, 2, 3]). *We shall call a formula*

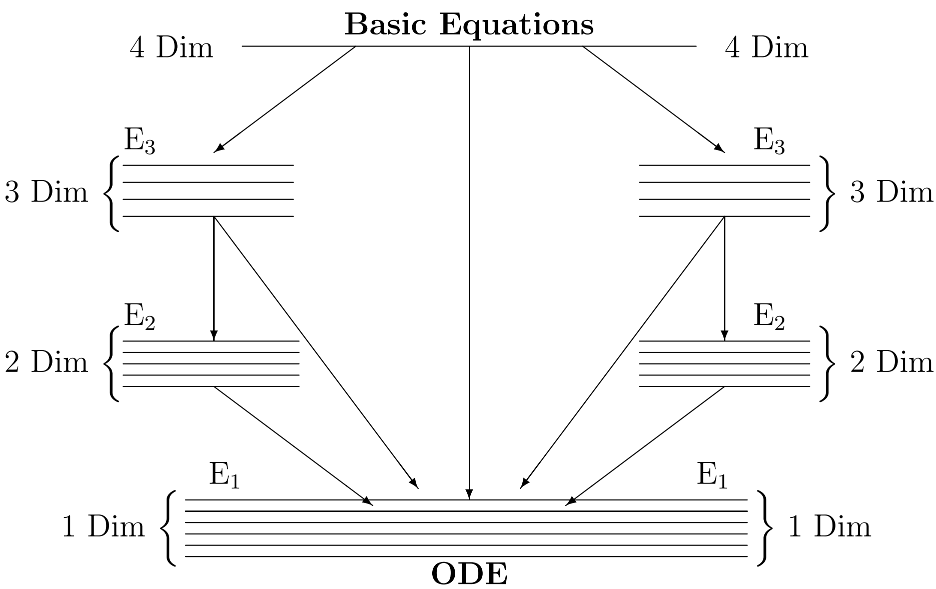
$$u = f(x)\varphi(\omega) + g(x), \tag{2.4}$$

an ansatz for equation (2.2) *if after substitution of* (2.4) *we get an equation for the function* $\varphi(\omega)$ *which depends only on new variables* $\omega = (\omega_1, \omega_2, \dots, \omega_{n-1})$*, where* $f(x)$*,* $g(x)$ *are given functions.*

If (2.4) is an ansatz for (2.2), then the latter is reduced (the number of independent variables decreases by one) to an equation for the function $\varphi(\omega)$.

Thus the problem of reduction of an equation reduces to description of three functions $\langle f(x), g(x), \omega\rangle$ which leads to an equation for $\varphi(\omega)$ with less number of variables.

We can display schematically the process of reduction for an 4-dimensional equation in the following way:



E_3 is a set of three-dimensional equations, E_2 is a set of two-dimensional equations, E_1 is a set of one-dimensional equations with the following inclusion $E_3 \subset E_2 \subset E_1$.

That is, from one principal equation we obtain the whole set of ODE. Having solved the ODE, we find exact solutions of a multidimensional equation.

Description of ansatzes of the form (2.4) for the nonlinear wave equation is an extremely difficult nonlinear problem. In the simplest case, when we put $f(x) = 1$, $g(x) = 0$ for the nonlinear Poincaré–invariant d'Alembert equation

$$\Box u = F(u), \tag{2.5}$$

the problem of reduction of (2.5) to ODE reduces to construction of solutions for the following overdetermined system for ω (Fushchych W., Serov M. 1983 [3])

$$\begin{gathered}\Box \omega = F_2(\omega),\\ \frac{\partial \omega}{\partial x_\mu}\frac{\partial \omega}{\partial x^\mu} = \left(\frac{\partial \omega}{\partial x_0}\right)^2 - \left(\frac{\partial \omega}{\partial x_1}\right)^2 - \left(\frac{\partial \omega}{\partial x_2}\right)^2 - \cdots - \left(\frac{\partial \omega}{\partial x_n}\right)^2 = F_2(\omega).\end{gathered} \tag{2.6}$$

If ω is a solution of the system (2.6), then the multidimensional equation (2.5) reduces to ODE with variable coefficients

$$a_2(\omega)\ddot{\varphi}(\omega) + a_1(\omega)\dot{\varphi}(\omega) + a_0(\omega)\varphi(\omega)\ F(\varphi) = 0 \tag{2.7}$$

A solution of equation (2.5) has the form

$$u(x_0, \ldots x_n) = \varphi(\omega), \quad \omega = \omega(x_0, x_1, \ldots, x_n), \tag{2.8}$$

φ is a solution of equation (2.7).

Compatibility and general solutions of system (2.6) are described in detail in papers of Zhdanov, Revenko, Yehorchenko, Fushchych (1987–1993, [4–6]). As we see, without using explicitly the symmetry of equation (2.5), we can reduce a multidimensional wave equation to ODE. It is obvious that all ansatzes and solutions, which are constructed on the basis of the classical method by Sophus Lie, can be obtained within the framework of our approach. The subgroup analysis of the Poincaré group $P(1, n)$ (Patera J., Winternitz P., Zassenhaus H., 1975–1983, [7, 8] Fedorchuk V., Barannyk A., Barannyk L., Fushchych W., 1985–1991 [9–11]) gives only a part of possible ansatzes.

Note 1. P. Clarkson and M. Kruskal (1989 [12]) implemented the approach suggested by us in 1981–1983 [1, 2, 3] for the one-dimensional Boussinesq equation and constructed in explicit form ansatzes and solutions which cannot be obtained within the framework of the classical S. Lie method. In the literature, this approach is often called the "direct method of reduction". I believe that it would be more consistent and correct to call this method of construction of PDE solutions a method of ansatzes.

3. Conditional symmetry

The Lie symmetry, as known, is a local symmetry of the whole set of solutions. The Lie algorithm enables us to define the invariance algebra for an arbitrary given equation and to construct ansatzes.

The term and the concept "conditional symmetry" was introduced and developed in our papers (1983–1993, [2, 3, 13–18]). This extremely simple concept has appeared to be efficient and enabled us to discover a nature of many ansatzes which could not be obtained within the framework of the Lie method.

Conditional symmetry is the symmetry of subsets of equation's solutions. Knowing conditional symmetry of an equation, we can construct non–Lie ansatzes and solutions. It is more difficult to study conditional symmetry of a given equation than to study its classical Lie symmetry. The difficulty is related to the fact that to find conditional symmetry of an equation, it is necessary to solve nonlinear determining equations.

During recent years, there are intensive studies in this promising direction, and today we can make following general conclusion:

Corollary 1. *Principal nonlinear equations of mathematical physics have conditional symmetry.*

Let us denote by the symbol

$$Q = \langle Q_1, Q_2, \ldots, Q_r \rangle \tag{3.1}$$

some set of operators which does not belong to the invariance algebra (IA) of equation (2.1).

Definition 2 (Fushchych W., Nikitin A., Shtelen W. and Serov M., 1987 [13, 14, 18], Fushchych W. and Tsyfra I. (1987 [15])). *Equation* (2.1) *is said to be conditionally invariant under the operators Q from* (3.1)*, if there exists a supplementary condition on the solutions of* (3.1) *of the form*

$$L_1(x, u, u_{(1)}, \ldots, u_{(n)}) = 0 \tag{3.2}$$

such that (3.1) *together with* (3.2) *is invariant under Q.*

Thus, one has the following criterion of conditional invariance [13, 15, 18]

$$Q_s L = \lambda_0 L + \lambda_1 L_1, \tag{3.3}$$

$$Q_s L_1 = \lambda_2 L + \lambda_3 L_1, \tag{3.4}$$

where λ_0, λ_1, λ_2, λ_3 are some differential expressions, Q_s is the s-th prolongation by Lie.

Definition 3. *We shall say that an equation is Q-conditionally invariant if the additional equation $L_1 = 0$ is a quasilinear equation of the first order*

$$L_1(x, u, u_{(1)}) \equiv Qu = 0, \tag{3.5}$$

$$Q = \xi_\mu(x, u)\frac{\partial}{\partial x^\mu} + \eta(x, u)\frac{\partial}{\partial u}, \tag{3.6}$$

with η being a smooth function.

Thus, the problem of finding the conditional symmetry of a equation reduces to the solution of equations (3.3), (3.4). As a rule, the determining equations for calculating ξ_μ and η are nonlinear equations.

As is known, in the classical approach ξ_μ, η satisfy a linear system of differential equations which, because of being overdetermined, can be solved.

3.1. Conditional symmetry of the Maxwell equations

The first equation where we had noticed conditional symmetry was the Maxwell subsystem [13]

$$\frac{\partial \vec{E}}{\partial t} = \operatorname{rot} \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\operatorname{rot} \vec{E}. \tag{3.7}$$

It is possible to prove by means the standard Lie method that the maximal invariance algebra of system (3.7) is an 8-dimensional extended Euclid algebra $AE_1(4)$ with basis elements:

$$P_\mu = i\frac{\partial}{\partial x_\mu}, \quad J_{ab} = x_a p_b - x_b p_a + S_{ab}, \quad D = x_\mu P^\mu, \tag{3.8}$$

where S_{ab} are 6×6 matrices, which realize a reduced representation of the Lie algebra of the group $SU(2)$.

Thus, system (3.7) is not invariant with respect to the Lorentz transformations, which are generated by operators

$$J_{oa} = x_o P_a - x_a P_0 + S_{0a}, \tag{3.9}$$

$\langle S_{ab}, S_{0a}\rangle$ are matrices which realize a finite-dimensional representation of the Lie algebra of the Lorentz group $S(1,3)$.

Theorem 1 (Fushchych W. and Nikitin A. 1983 [13]). *System* (3.7) *is conditionally invariant under the Lorentz boosts* (3.9) *if and only if the solutions of* (3.7) *satisfy the conditions*

$$\operatorname{div} \vec{E} = 0, \quad \operatorname{div} \vec{H} = 0. \tag{3.10}$$

Thus, system (3.7) *only together with equations* (3.10) *is invariant under the Lorentz group.*

Note 2. 90 years ago H. Lorentz (1904, April 23), H. Poincaré (1905, June 5, July 23), A. Einstein (1905, June 30) discovered the theorem about invariance of the full Maxwell system (3.7), (3.10) with respect to rotations in the four-dimensional pseudo-Euclidean space-time. This theorem is a mathematical formulation of the fundamental Lorentz–Poincaré–Einstein principle of relativity.

3.2. Conditional symmetry of linear Schrödinger systems

Let us consider the multicomponent system of disconnected Schrödinger equations:

$$\begin{aligned} &S\Psi = \left(p_0 - \frac{p_a^2}{2m}\right)\Psi_r = 0, \quad r = 1,2,\ldots,n, \\ &p_0 = i\frac{\partial}{\partial x_0}, \quad p_a = -i\frac{\partial}{\partial x_a}, \quad a = 1,2,3, \\ &\Psi = (\Psi_1, \Psi_2, \ldots, \Psi_n), \quad \Psi = \Psi(x_0 = t, x_1, x_2, x_3). \end{aligned} \tag{3.11}$$

It is evident that every separate Schrödinger equation (3.11) is invariant with respect to a scalar representation of the group $G_2(1,3)$, a full Galilei group.

Let us consider a problem of existence of nontrivial vector, spinor, tensor representations of the full Galilei group, which are realized on the set of solutions of system (3.11).

We demand system (3.11) be invariant with respect to the following linear representations of the algebra $AG_2(1,3)$

$$\begin{aligned} &P_0 = i\frac{\partial}{\partial x_0}, \quad P_a = -i\frac{\partial}{\partial x_a}, \quad M = im, \\ &J_a = x_a p_b - x_b p_a + S_a, \quad S_a = \frac{1}{2}\varepsilon_{abc}S_{bc}, \\ &G_a = x_0 p_a - x_a p_0 + \lambda_a, \quad D = 2x_0 P_0 - x_k P_k + \lambda_0, \\ &A = x_0 D - x_0^2 P_0 + \frac{1}{2}m x_k^2 - \lambda_a x_a, \end{aligned} \tag{3.12}$$

where matrices S_a, λ_0, λ_a satisfy the commutation relations [29]

$$\begin{aligned} &[S_a, S_b] = i\varepsilon_{abc}S_c, \quad [\lambda_a, \lambda_b] = 0, \quad [\lambda_0, S_a] = 0, \\ &[\lambda_a S_b] = i\varepsilon_{abc}S_c, \quad [\lambda_0, \lambda_a] = i\lambda_a. \end{aligned} \tag{3.13}$$

Theorem 2 (Fushchych and Shtelen, 1983, [19]). *System of equations* (3.11) *is conditional invariant under representation* $AG_2(1,3)$ (3.12) *if*

$$\left(\lambda_0 - \frac{3}{2}i - \frac{1}{m}\lambda_k P_k\right)\Psi = 0, \tag{3.14}$$

$$(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\Psi = 0. \tag{3.15}$$

3.3. *Q*-conditional symmetry of Lorentz noninvariant nonlinear wave equation

Let us consider the following wave equation (Fushchych and Tsyfra 1987, [15])

$$Lu \equiv \Box u + F(x, \underset{1}{u}) = 0 \tag{3.16}$$

$$\begin{aligned} F(x, \underset{1}{u}) = &-\left(\frac{\lambda_0}{x_0}\right)^2\left(\frac{\partial u}{\partial x_0}\right)^2 + \left(\frac{\lambda_1}{x_1}\right)^2\left(\frac{\partial u}{\partial x_1}\right)^2 + \\ &+\left(\frac{\lambda_2}{x_2}\right)^2\left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\lambda_3}{x_3}\right)^2\left(\frac{\partial u}{\partial x_3}\right)^2, \quad x_\mu \neq 0, \end{aligned} \tag{3.17}$$

λ_μ are arbitrary parameters.

Equation (3.16) is invariant only with respect to scale transformations and translations:

$$x_\mu \to x'_\mu = e^b x_\mu, \quad u \to u' = e^{2b}u, \quad u \to u' = u + c,$$

b is a real parameter.

Let us consider a Lorentz-invariant ansatz

$$u = \varphi(\omega), \quad \omega = x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{3.18}$$

This ansatz, despite the fact that (3.16) is not invariant with respect to the Lorentz group, reduces equation (3.16) to ODE

$$\omega \frac{d^2\varphi}{d\omega^2} + 2\frac{d\varphi}{d\omega} + \lambda^2 \left(\frac{d\varphi}{d\omega}\right)^2 = 0 \tag{3.19}$$

whose solutions are given by the functions

$$\begin{aligned}
&\lambda = \lambda_0^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2,\\
&\varphi(\omega) = 2(-\lambda^2)^{1/2} \tan^{-1} \omega(-\lambda^2)^{-1/2} \quad \text{for} - \lambda^2 > 0,\\
&\varphi(\omega) = -(\lambda^2)^{-1/2} \ln \left\{ \frac{(\lambda^2)^{1/2} + \omega}{(\lambda^2)^{1/2} - \omega} \right\} \quad \text{for} - \lambda^2 < 0.
\end{aligned}$$

What is the reason of such reduction? From the classical point of view, ansatz (3.18) must not reduce the Lorentz non–invariant equation (3.16) to ODE.

The reason of all this is the fact that equation (3.16) is conditionally invariant with respect to the Lorentz group.

Theorem 3 (Fushchych and Tsyfra, 1987 [15]). *Equation* (3.16), (3.17) *is conditionally invariant with respect to the Lorentz group if the following six conditions are added:*

$$J_{\mu\nu} u = 0, \quad J_{\mu\nu} = x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}, \quad \mu, \nu = 0, 1, 2, 3. \tag{3.20}$$

Thus, equation (3.16) together with the additional condition (3.20) is invariant with respect to the Lorentz group. The condition (3.20) picks out the subset from the whole set of solutions which is invariant with respect to the Lorentz group.

3.4. Conditionally conformal symmetry of the Poincaré-invariant d'Alembert equation

Let us consider the nonlinear d'Alembert equation with an additional condition

$$\Box u + F(u) = 0, \tag{3.21}$$

$$\frac{\partial u}{\partial x_\mu} \frac{\partial u}{\partial x^\mu} = F_1(u). \tag{3.22}$$

Theorem 4 (Fushchych, Zhdanov, Serov 1989 [18]). *Equation* (3.21) *is conditionally invariant under the conformal group if*

$$F = 3\lambda(u + c)^{-1}, \tag{3.23}$$

$$\frac{\partial u}{\partial x_\mu} \frac{\partial u}{\partial x^\mu} = \lambda, \tag{3.24}$$

where λ, c *are arbitrary constants. The operators of conformal symmetry are*

$$K_\mu = 2x_\mu D - (x_\alpha x^\alpha - u^2) \frac{\partial}{\partial x^\mu}, \quad \mu = 0, 1, 2, 3 \tag{3.25}$$

$$D = x^\mu \frac{\partial}{\partial x^\mu} + u\frac{\partial}{\partial u}. \tag{3.26}$$

Remark 3. Formulae (3.25), (3.26) give a nonlinear representation for the conformal algebra $AC(1,3)$.

An ansatz for the system

$$\Box u = u^{-1}, \quad \partial_\mu u \partial^\mu u = 1 \tag{3.27}$$

has the form (Fushchych and Zhdanov, 1989 [4])

$$u^2 = (a_\mu x^\mu + g_1)^2 - (b_\mu x^\mu + g_2)^2, \tag{3.28}$$

where $g_1 = g_1(\theta_\mu x^\mu)$, $g_2 = g_2(\theta_\mu x^\mu)$ are arbitrary smooth functions, a_μ, b_μ, θ_μ are arbitrary complex parameters satisfying the condition

$$a_\mu a^\mu = -b_\mu b^\mu = 1, \quad a_\mu b^\mu = a_\mu \theta^\mu = b_\mu \theta^\mu = \theta_\mu \theta^\mu = 0.$$

Remark 5. The problem of compatibility and construction of solutions of the d'Alembert–Hamilton system are considered in detail in [5, 6].

3.5. Conditional symmetry of the nonlinear heat equation

Let us consider the equation

$$u_0 + \vec{\nabla}[f(u)\vec{\nabla}u] = 0, \quad f(u) \neq \text{const}. \tag{3.29}$$

Ovsyannikov L. (1962, [20]) carried out the complete classification of the one-dimensional equation (3.29). Dorodnitsyn A., Knyaseva Z., Svirshchevskii S. (1983, [21]) carried out group classification of the three-dimensional equation (3.29) From the analysis of these results it follows.

Conclusion 1 (Fushchych 1983 [2]). *Among equations of the class* (3.29)*, there are no nonlinear equations invariant with respect to Galilei transformations which are generated by the operators*

$$G_a = x_0 \partial_a + M(u) x_a \frac{\partial}{\partial u}, \tag{3.30}$$

$M(u)$ is constant.

Theorem 5 (Fushchych, Serov, Chopyk 1988 [16]). *The equation* (3.29) *is conditional invariant under the Galilean operators* (3.30) *if*

$$u_0 + \frac{(\vec{\nabla}u)^2}{2M(u)} = 0, \tag{3.31}$$

$$M(u) = \frac{u}{2f(u)}. \tag{3.32}$$

Conclusion 2. *The nonlinear equation* (3.29) *with the additional condition* (3.31) *is compatible with the Galilei relativity principle.*

Conclusion 3. *If*

$$f(u) = \frac{1}{2m} u^k, \qquad M(u) = \frac{2m}{kn+2} u^{1-k}, \tag{3.33}$$

$$f(u) = e^u, \qquad M(u) = 1, \tag{3.34}$$

where m, k are arbitrary constants, $kn + 2 \neq 0$, then equation (3.29) *is conditionally invariant with respect to Galilei transformations.*

Q-conditional symmetry of the one-dimensional equation

$$u_0 - u_{11} = F(u)$$

was studied in detail (Fushchych and Serov, 1990, [22, 23]). Recently these results were obtained by Clarkson P. and Mansfield E. (1994, [24]).

4. Reduction and antireduction

Under the term "reduction–antireduction", we understand a decreasing of dimension of an equation with respect to independent variables and increasing (antireduction) by the number of dependent variables. That is we have simultaneously the process of reduction (by the number of independent variables) and antireduction (increasing the number of reduced systems with respect to the original equation) [25].

In the classical Lie approach as a rule the number of components of dependent variables for reduced systems does not increase.

Example 1. Let us consider the nonlinear acoustics equation (Khokhlov–Zabolotskaja equation)

$$u_{01} - (u_1 u)_1 - u_{22} - u_{33} = 0, \tag{4.1}$$

$$u = u(x_1, x_2, x_3).$$

The ansatz (Fushchych and Myronyuk, 1991 [26])

$$u = \frac{1}{3} x_1 \varphi^{(1)}(\omega_0, \omega_2, \omega_3) + \frac{1}{6} x_1^2 \varphi^{(2)}(\omega_0, \omega_2, \omega_3) + \varphi^{(3)}(\omega_0, \omega_2, \omega_3), \tag{4.2}$$

$$\omega_0 = x_0, \quad \omega_2 = x_2, \quad \omega = x_3$$

antireduces four-dimensional equation (4.1) to the system of coupled three-dimensional equations for functions $\varphi^{(1)}$, $\varphi^{(2)}$, $\varphi^{(3)}$

$$\begin{aligned}
&\frac{\partial^2 \varphi^{(1)}}{\partial \omega_2^2} + \frac{\partial^2 \varphi^{(2)}}{\partial \omega_3^2} = (\varphi^{(2)})^2, \\
&\frac{\partial^2 \varphi^{(1)}}{\partial \omega_2^2} + \frac{\partial^2 \varphi^{(1)}}{\partial \omega_3^2} + \frac{\partial \varphi^{(1)}}{\partial \omega_0} - \varphi^{(1)} \varphi^{(2)} = 0, \\
&\frac{\partial^2 \varphi^{(3)}}{\partial \omega_2^2} + \frac{\partial^2 \varphi^{(3)}}{\partial \omega_3^2} - \frac{1}{3} \varphi^{(2)} \varphi^{(3)} - \frac{1}{3} \frac{\partial \varphi^{(1)}}{\partial \omega_0} + \frac{1}{9} (\varphi^{(1)})^2 = 0.
\end{aligned} \tag{4.3}$$

The formula (4.2) gives a non-Lie ansatz for equation (4.1).

Example 2. Let us consider the equation for short waves in gas dynamics

$$\begin{aligned} &2u_{01} - 2(2x_1 + u_1)u_{11} + u_{22} + 2\lambda u_1 = 0, \\ &u = u(x_0 = t, x_1, x_2). \end{aligned} \tag{4.4}$$

The ansatz (Fushchych and Repeta 1991, [27])

$$\begin{aligned} &u = x_1\varphi^{(1)}(\omega_0, \omega_2) + x_1^2\varphi^{(2)}(\omega_0, \omega_2) + x_1^{3/2}\varphi^{(3)} + \varphi^{(4)}, \\ &\omega_0 = x_0, \qquad \omega_2 = x_2 \end{aligned} \tag{4.5}$$

antireduces one three-dimensional scalar equation (4.4) to a system of two-dimensional equations for four functions

$$\begin{aligned} &\varphi^{(3)} = 0, \quad \frac{\partial^2\varphi^{(1)}}{\partial\omega_2^2} = 0, \quad \frac{\partial^2\varphi^{(2)}}{\partial\omega_2^2} = 0, \\ &\frac{\partial^2\varphi^{(4)}}{\partial\omega_2^2} = \frac{9}{4}\big(\varphi^{(1)}\big)^2, \quad \frac{\partial\varphi^{(1)}}{\partial\omega_0} = \varphi^{(1)}\left(3\varphi^{(2)} + \frac{1}{2} - \lambda\right). \end{aligned} \tag{4.6}$$

4.1. Antireduction and ansatzes for the nonlinear heat equation

Let us consider the nonlinear one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{a(u)\frac{\partial u}{\partial x}\right\} + F(u), \tag{4.7}$$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(u). \tag{4.8}$$

We consider an implicit ansatz

$$h\big(t, x, u, \varphi^{(1)}(\omega), \varphi^{(2)}(\omega), \dots, \varphi^{(N)}(\omega)\big) = 0, \tag{4.9}$$

which reduces the two–dimensional equation (4.7) to the system of ODE for functions $\varphi^{(1)}, \dots, \varphi^{(N)}$. We have constructed a quite long list of ansatzes which reduce equation (4.7) to the system of ODE (Zhdanov R. and Fushchych W. 1994, [33]).

Example 3. If in (4.7)

$$a(u) = \lambda u^{-3/2}, \quad F(u) = \lambda_1 u + \lambda_2 u^{5/2}, \tag{4.10}$$

then the ansatz [33] is as follows

$$u^{-3/2} = \varphi^{(1)}(t) + \varphi^{(2)}(t)x + \varphi^{(3)}(t)x^2 + \varphi^{(4)}(t)x^3, \tag{4.11}$$

$$\begin{aligned} &\dot\varphi^{(1)} = 2\lambda\varphi^{(1)}\varphi^{(3)} - \frac{2}{3}\lambda(\varphi^{(2)})^2 - \frac{3}{2}\lambda_1\varphi^{(1)} - \frac{3}{2}\lambda_2, \\ &\dot\varphi^{(2)} = -\frac{2}{3}\lambda\varphi^{(2)}\varphi^{(3)} + 6\lambda\varphi^{(1)}\varphi^{(4)} - \frac{3}{2}\lambda_1\varphi^{(2)}, \\ &\dot\varphi^{(3)} = -\frac{2}{3}\lambda(\varphi^{(3)})^2 + 2\lambda\varphi^{(2)}\dot\varphi^{(4)} - \frac{3}{2}\lambda_1\varphi^{(3)}, \\ &\dot\varphi^{(4)} = -\frac{3}{2}\lambda_1\varphi^{(4)}. \end{aligned} \tag{4.12}$$

Having solved the system of ODE (4.12), by formula (4.11) we construct exact solutions of the equation (4.7).

Example 4. If in (4.8)

$$F(u) = \left\{\alpha + \beta \ln u - \gamma^2 (\ln u)^2\right\} u, \tag{4.13}$$

then the ansatz

$$\ln u = \varphi^{(1)}(t) + e^{\gamma x}\varphi^{(2)}(t) \tag{4.14}$$

reduces (4.8) to the system of ODE

$$\begin{aligned} \dot\varphi^{(1)} &= 2 + \beta\varphi^{(1)} - \gamma^2(\varphi^{(1)})^2, \\ \dot\varphi^{(2)} &= \left\{\beta + \gamma^2 - 2\gamma^2\varphi^{(1)}\right\}\varphi^{(2)}. \end{aligned} \tag{4.15}$$

It is possible to construct solutions of system (4.15) in the explicit form. Depending on the sign of the quantity $d = \beta^2 + 4\alpha\gamma^2$ we get the following solutions of the nonlinear equation (4.8), (4.13).

Case 4.1 $d > 0$

$$u = c\left(\cos\frac{d^{1/2}t}{2}\right)^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2}\left(\beta - d^{1/2}\mathrm{tg}\,\frac{d^{1/2}t}{2}\right). \tag{4.16}$$

Case 4.2 $d < 0$

$$u = c\left(\mathrm{ch}\,\frac{|d|^{1/2}t}{2}\right)^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2}\left(\beta + |d|^{1/2}\mathrm{th}\,\frac{|d|^{1/2}t}{2}\right). \tag{4.17}$$

Case 4.3 $d = 0$

$$u = ct^{-2}\exp\left(\gamma x + \gamma^2 t\right) + \frac{1}{2\gamma^2 t}(\beta t + 2). \tag{4.18}$$

Example 5. If in (4.7)

$$a(u) = \lambda u^k, \quad F(u) = \lambda_1 u + \lambda_2 u^{1-k}, \tag{4.19}$$

then the ansatz

$$u^k = \varphi^{(1)}(t) + \varphi^{(2)}(t)x + \varphi^{(3)}(t)x^2 \tag{4.20}$$

antireduces (4.7) to the system of ODE

$$\begin{aligned} \dot\varphi^{(1)} &= 2\lambda\varphi^{(1)}\varphi^{(3)} + \lambda k^{-1}(\varphi^{(2)})^2 + k\lambda_2, \\ \dot\varphi^{(2)} &= 2\lambda(1 + 2k^{-1})\varphi^{(2)}\varphi^{(3)} + k\lambda_1\varphi^{(2)}, \\ \dot\varphi^{(3)} &= 2\lambda(1 + 2k^{-1})(\varphi^{(3)})^2 + k\lambda_1\varphi^{(3)}. \end{aligned} \tag{4.21}$$

5. Non-Lie symmetry, new relativity principles

5.1. Non-Lie symmetry Schrödinger equation

Let us consider the Schrödinger equation

$$\left(i\frac{\partial}{\partial x_0} - \frac{p_a^2}{2n}\right) u(x_0, \vec{x}) = 0. \tag{5.1}$$

It is well known that the maximal (in the Lie sense) invariance algebra (5.1) is the full Galilei algebra $AG_2(1,3) = \langle P_0, P_a, J_{ab}, G_a, D, A\rangle$

$$\begin{aligned} &P_0 = i\frac{\partial}{\partial x_0}, \quad P_a = -i\frac{\partial}{\partial x_0}, \quad a = 1,2,3, \\ &J_{ab} = x_a p_b - x_b p_a, \quad G_a = x_0 p_a - m x_a, \\ &D = 2x_0 P_0 - x_k P_k, \quad A = x_0 D - x_0^2 P_0 + \frac{1}{2} m x_a^2. \end{aligned} \tag{5.2}$$

Operators G_a generate the standard Galilei transformations:

$$t \to t' = \exp\{iG_a v_a\}\, t \exp\{-iG_a v_a\} = t, \tag{5.3}$$

$$x_a \to x_a' = \exp\{iG_b v_b\}\, x_a \exp\{-iG_c v_c\} = x_a + v_a t. \tag{5.4}$$

Let us put the following question: do symmetries which are not reduced for the algebra (5.2) exhaust for equation (5.1)?

Answer: The Schrödinger equation (5.1) has additional symmetries (supersymmetries, non-Lie, nonlocal) which are not reduced to the Galilei algebra $AG_2(1,3)$ [29].

One of results in this direction is the following:

Theorem 6 (Fushchych and Seheda 1977 [28]). *The Schrödinger equation* (5.1) *is invariant with respect to the Lorentz algebra* $AL(1,3)$

$$J_{ab} = x_a p_b - x_b p_a, \tag{5.5}$$

$$J_{0a} = \frac{1}{2m}(pG_a + G_a p), \quad p = (p_1^2 + p_2^2 + p_3^2)^{1/2} = (-\Delta)^{1/2}. \tag{5.6}$$

It is not difficult to check that the operators $\langle J_{ab}, J_{0c}\rangle \equiv AL(1,3)$ satisfy the commutation relations

$$[J_{ab}, J_{0c}] = i(g_{ac} J_{b0} - g_{bc} J_{a0}), \quad [J_{0a}, J_{0b}] = -iJ_{ab}.$$

It is important to point out that J_{0a} are integral-differential symmetry operators and generate nonlocal transformations

$$x_a \to x_a' = \exp\{iJ_{ob} V_b\}\, x_a \exp\{-iJ_{0c} V_c\} \neq \text{ Galilei transform. (5.4)}, \tag{5.7}$$

$$t \to t' = \exp\{iJ_{0a} V_a\}\, t \exp\{-iJ_{0b} V_b\} = t. \tag{5.8}$$

Hence the operators J_{0a} (5.6) generate new transformations which do not coincide with the known Galilei and Lorentz transformation. Thus we have new relativity principle. It is defined by formulae (5.7), (5.8).

5.2. Time is absolute in relativistic physics

The four-component Dirac equation lies in the foundation of the modern quantum mechanics

$$\gamma_\mu p^\mu \Psi = m\Psi(x_0, x_1, x_2, x_3). \tag{5.9}$$

Here γ_μ are 4×4 Dirac matrices.

Since the time of discovery of this equation it is known that (5.9) is invariant with respect to the Poincaré algebra $AP(1,3) = \langle P_\mu, J_{\mu\nu} \rangle$ with the basis elements

$$P_\mu = i\frac{\partial}{\partial x^\mu}, \quad J^{(1)}_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \tag{5.10}$$

Operators $J^{(1)}_{\mu\nu}$ generate the standard Lorentz transformations

$$t \to t' = \exp\left\{iJ^{(1)}_{0a} v_a\right\} t \exp\left\{-iJ_{0b}v_b\right\}, \tag{5.11}$$

$$x_a \to x'_a = \exp\left\{iJ^{(1)}_{0b} v_b\right\} x_a \exp\left\{-iJ_{0c}v_c\right\}. \tag{5.12}$$

Hence, the fundamental statement follows that time $t \in T(1)$ and space $\vec{x} \in R(3)$ are the single pseudo-Euclidean space-time with the metric

$$s^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2. \tag{5.13}$$

Let us put another question: Do there exist symmetries in equation (5.10) which cannot be reduced to the algebra AP(1,3) (5.11)?

Answer: The Dirac equation (5.9) has a wide additional symmetry (supersymmetry, non-Lie symmetry) which cannot be reduced to the algebra $AP(1,3)$ (5.10) [13, 29].

I shall say here briefly about one of such symmetries.

Theorem 7 (Fushchych 1971, 1974 [30, 31]. *The Dirac equation* (5.9) *is invariant with respect to the following representation of the Poincaré algebra*

$$P^{(2)}_0 = H = \gamma_0\gamma_a p_a + \gamma_0 m, \quad P^{(2)}_a = -i\frac{\partial}{\partial x_a}, \quad a = 1, 2, 3, \tag{5.14}$$

$$J^{(2)}_{ab} = x_a p_b - x_b p_a + S_{ab}, \qquad S_{ab} = \frac{i}{4}(\gamma_a\gamma_b - \gamma_b\gamma_a), \tag{5.15}$$

$$J^{(2)}_{0a} = x_0 p_a - \frac{1}{2}(x_a H + H x_a). \tag{5.16}$$

Thus we have two different representations of the Poincaré algebra $AP(1,3)$ (5.10) and (5.14)–(5.16).

The representation (5.15) and (5.16) generates nonlocal transformations

$$x_a \to x'_a = \exp\{iJ^{(2)}_{ab} v_b\} x_a \exp\{iJ^{(2)}_{0c} v_c\} \neq \text{Lorentz transform}, \tag{5.17}$$

$$t \to t' = \exp\{iJ^{(2)}_{0b} v_b\} t \exp\{-iJ^{(2)}_{0c} v_c\} = t. \tag{5.18}$$

Thus, time does not change in relativistic physics. Time is absolute in relativistic physics.

There are two nonequivalent possibilities (duality) for transformations of coordinates and time: Lorentz transformation (5.11), (5.12) and non-Lorentz transformation (5.17), (5.18).

The Maxwell and Klein–Gordon–Fock equations are also invariant under nonlocal transformations (5.17), (5.18) when time does not change. However energy and momentum are transformed by the Lorentz law [31,32]. We have new relativity principle (5.17), (5.18).

What is the reason of such a paradoxical statement? The reason is that the operators $J_{0a}^{(2)}$ are non-Lie symmetry operators and the standard relation (S. Lie's theorems) between Lie groups and Lie algebras is broken.

So, physics is not equivalent to geometry and geometry is not physics. Physics is Nature. Theoretical Physics is only a Model of Nature!

6. On some new motion equations

Some new motion equations are adduced in this section. These equations are generalizations of known classical equations. Symmetry of these equations has not been investigated.

6.1. High order parabolic equation in quantum mechanics

The Schrödinger equation (5.1) is not the only equation compatible with the Galilei relativity principle. A more general equation was suggested in [1, 2]

$$
\begin{aligned}
&(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n) u = \lambda u, \\
&S \equiv p_0 - \frac{p_a^2}{2m}, \quad S^2 = S \cdot S, \quad S^n = S^{n-1} S,
\end{aligned}
\tag{6.1}
$$

$\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n$ are arbitrary parameters. Equation (6.1) as well as the classical equation (5.1) is invariant with respect to the Galilei transformations but it is not invariant with respect to scale and projective transformations.

A new equation for two particles (waves):

$$
\begin{aligned}
&p_0 u_1 = \frac{1}{2m_1} p_a^2 u_1 + V_1\big(t, x_1, x_2, \ldots, x_6, u_1, u_2\big), \\
&p_0 u_2 = \frac{1}{2m_2} p_{a+3}^2 u_2 + V_2\big(t, x_1, x_2, \ldots, x_6, u_1, u_2\big), \\
&u_1 = u_1(t, x_1, x_2, x_3), \quad u_2 = u_2(t, x_4, x_5, x_6), \quad V_1 \text{ and } V_2 \text{ are potentials.}
\end{aligned}
$$

6.2. Nonlinear generalization of Maxwell equations

If we assume that the light velocity is not constant [34], we can suggest some generalizations of the Maxwell equations

$$
\begin{aligned}
&\tfrac{\partial \vec{E}}{\partial t} = \mathrm{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}\}, \qquad \frac{\partial \vec{H}}{\partial t} = -\mathrm{rot}\,\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\}, \\
&\mathrm{div}\,\{a(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\} = 0, \qquad \mathrm{div}\,\{b(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}\} = 0,
\end{aligned}
\tag{6.2}
$$

where a, b and c are some functions of electromagnetic field;

$$\begin{aligned} &\frac{\partial \vec{E}}{\partial t} = \operatorname{rot}\{c(\vec{B}^2, \vec{D}^2, \vec{B}\vec{D})\vec{B}\} + \vec{j}, & & \frac{\partial \vec{D}}{\partial t} = \operatorname{rot}\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\} + \vec{j}, \\ & & \text{or} & \\ &\frac{\partial \vec{H}}{\partial t} = -\operatorname{rot}\{c(\vec{B}^2, \vec{D}^2, \vec{B}\vec{D})\vec{D}\}, & & \frac{\partial \vec{B}}{\partial t} = -\operatorname{rot}\{c(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E}\}, \end{aligned} \tag{6.3}$$

$$\begin{aligned} \lambda_1 \vec{D} + \lambda_2 \Box \vec{D} &= F_1(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E} + F_2(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}, \\ \lambda_3 \vec{B} + \lambda_4 \Box \vec{B} &= R_1(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{E} + R_2(\vec{E}^2, \vec{H}^2, \vec{E}\vec{H})\vec{H}, \end{aligned} \tag{6.4}$$

$$\operatorname{div} \vec{D} = \rho, \quad \operatorname{div} \vec{B} = 0, \tag{6.5}$$

where F_1, F_2, R_1, R_2 are functions of fields $\vec{E}$ and $\vec{H}$, c in equations (6.2), (6.3) can be a function of $(t, \vec{x})$, $c = c(t, \vec{x})$, or depend on the gravity potential V, $c = C(V)$. Nonlinear wave equations for $\vec{E}$ and $\vec{H}$ have form

$$\frac{\partial^2 \vec{E}}{\partial t^2} - c^2 \Delta \vec{E} = 0, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - c^2 \Delta \vec{H} = 0, \tag{6.6}$$

or

$$\frac{\partial^2 \vec{E}}{\partial t^2} - \Delta(c^2 \vec{E}) = 0, \quad \frac{\partial^2 \vec{H}}{\partial t^2} - \Delta(c^2 \vec{H}) = 0; \tag{6.7}$$

or

$$\frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}\vec{E}\right) - \Delta \vec{E} = 0, \quad \frac{\partial^2}{\partial t^2}\left(\frac{1}{c^2}\vec{H}\right) - \Delta H = 0, \tag{6.8}$$

with one of the conditions

$$c^2 = \frac{1}{2} \frac{\left(\frac{\partial \vec{E}}{\partial t}\right) + \left(\frac{\partial \vec{H}}{\partial t}\right)}{(\operatorname{rot} \vec{H})^2 + (\operatorname{rot} \vec{E})^2} \tag{6.9}$$

or

$$\frac{\partial c^2}{\partial x^\mu} \frac{\partial c^2}{\partial x^\mu} = 0. \tag{6.10}$$

or

$$c_\mu \frac{\partial c_2}{\partial x_\mu} = \lambda(E^2 \vec{H}^2, \vec{E}\vec{H}) F_{\alpha\beta} c^\beta, \tag{6.11}$$

c_α is the four-velocity of the light (electromagnetic field), $c^2 = c_\alpha c^\alpha$.

Equations of hydrodynamical type for electromagnetic field have form

$$\begin{aligned} \frac{\partial \vec{E}}{\partial t} &= a_1 \left\{\vec{\nabla} \times (\vec{c} \times \vec{H})\right\} + a_2 \left\{\vec{\nabla} \times (\vec{c} \times \vec{E})\right\}, \\ \frac{\partial \vec{H}}{\partial t} &= b_1 \left\{\vec{\nabla} \times (\vec{c} \times \vec{E})\right\} + b_2 \left\{\vec{\nabla} \times (\vec{c} \times \vec{H})\right\}, \\ \frac{\partial \vec{c}}{\partial t} &+ (\vec{c}\vec{\nabla})\vec{c} = R_1 \vec{E} + R_2 \vec{H}, \end{aligned} \tag{6.12}$$

$\vec{c}$ is the three-velocity of the light, where a_1, a_2, b_1, b_2, R_1, R_2 are functions of $\vec{E}^2$, $\vec{H}^2$, $\vec{E}\vec{H}$.

Maxwell's equations in a moving frame with the velocity can be generalized in such forms

$$\frac{\partial \vec{E}}{\partial t} + \lambda_1 v_k \frac{\partial \vec{E}}{\partial x_k} + \lambda_2 \operatorname{rot} \vec{H} = 0, \quad \frac{\partial \vec{H}}{\partial t} + \lambda_3 v_k \frac{\partial \vec{H}}{\partial x_k} + \lambda_4 \operatorname{rot} \vec{E} = 0,$$

or

$$\frac{\partial \vec{E}}{\partial t} + \lambda_1 v_k \frac{\partial \vec{H}}{\partial x_k} + \lambda_2 \operatorname{rot} \vec{H} = 0, \quad \frac{\partial \vec{H}}{\partial t} + \lambda_3 v_k \frac{\partial \vec{E}}{\partial x_k} + \lambda_4 \operatorname{rot} \vec{E} = 0,$$

with the conditions $\frac{\partial v_k}{\partial t} + v_l \frac{\partial v_k}{\partial x_l} = 0$.

6.3. Equations for fields with the spin 1/2

Fields with the spin 1/2 are described, as a rule, by first-order equations, by the Dirac equation. However, such fields can be also described by second-order equations. Some of such equations are adduced below:

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad \bar{\psi}\gamma_\mu p^\mu \Psi = F_2(\bar{\psi}\Psi); \tag{6.13}$$

$$p_\mu p^\mu \Psi = R_1(\bar{\psi}\psi)\Psi, \quad (\bar{\psi}\gamma_\mu \Psi) p^\mu \Psi = F_2(\bar{\psi}\psi)\Psi; \tag{6.14}$$

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad (\bar{\psi}\gamma_\mu \Psi)(\bar{\psi} p^\mu \Psi) = F_3(\bar{\psi}\psi); \tag{6.15}$$

$$p_\mu p^\mu \Psi + \lambda \gamma_\mu p^\mu \Psi = F(\bar{\psi}\psi)\Psi; \tag{6.16}$$

$$p_\mu p^\mu \Psi = F_1(\bar{\psi}\psi)\Psi, \quad p_0 \Psi = \{(\bar{\psi}\gamma_0 \Psi)(\bar{\psi}\gamma_k \psi) p_k + m \bar{\Psi}\gamma_0 \Psi\}\Psi.$$

6.4. How to extend symmetry of an equation with arbitrary coefficients?

Let us consider the a second-order equation

$$a_{\mu\nu}(x) \frac{\partial^2 u}{\partial x^\mu \partial x^\nu} + b_\mu(x) \frac{\partial u}{\partial x^\mu} + F(u) = 0. \tag{6.17}$$

Equation (6.17) with arbitrary fixed coefficients has only a trivial symmetry ($x \to x' = x$, $u \to u' = u$). However, if we do not fix coefficient functions $a_{\mu\nu}(x), b_\mu(x)$, such an equation can have wide symmetry. E.g., if $a_{\mu\nu}$, b_μ satisfy the equations

$$\Box a_{\mu\nu} = \frac{\partial u}{\partial x_\mu} \frac{\partial u}{\partial x_\nu} F_1(u) \tag{6.18}$$

or

$$\Box b_\mu = F_2(u) \frac{\partial u}{\partial x_\mu}, \quad \Box a_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu} F_3(u), \tag{6.19}$$

then the nonlinear system (6.17), (6.18), (6.19) is invariant with respect to the Poincaré group P(1,3). Let us emphasize that here even if we put $F_1 = 0$, $F_2 = 0$, equations

(6.17), (6.18), (6.19) are a nonlinear system of equations. With some particular functions F_1 and F_2, it is possible to construct ansatzes which reduce system (6.17), (6.18), (6.19) to the system of ordinary differential equations.

So, considering (6.17) as a nonlinear equation with additional conditions for $a_{\mu\nu}$, b_ν, we can construct the exact solution for equation (6.17). The adduced idea about extension of the symmetry of (6.17) can be used for construction of exact solutions for motion equations in gravity theory.

The second example of equations which have wide symmetry is

$$v_\mu v_\nu \frac{\partial^2 F_{\alpha\beta}}{\partial x^\mu \partial x^\nu} = 0, \qquad (6.20)$$

$$v_\mu \frac{\partial v_\nu}{\partial x^\mu} = 0. \qquad (6.21)$$

If in (6.20) v_μ are fixed functions the equation, as a rule, has trivial symmetry.

1. Fushchych W., Symmetry in problems of mathematical physics, in Algebraic-Theoretic Studies in Mathematical Physics, Kyiv, Inst. of Math., 1981, 6–44.
2. Fushchych W., On symmetry and exact solutions of some multidimensional equations of mathematical physics problems, in Algebraic-Theoretical Methods in Mathematical Physics, Kyiv, Inst. of Math., 1983, 4–23.
3. Fushchych W., Serov M., The symmetry and exact solutions of the nonlinear multi-dimensional Liouville, d'Alembert and eikonal equations, *J. Phys. A: Math. Gen.*, 1983, **16**, № 15, 3645–3658.
4. Fushchych W., Zhdanov R., On some new exact solutions of the nonlinear d'Alembert–Hamilton system, *Phys. Lett. A*, 1989, **141**, № 3–4, 113–115.
5. Fushchych W., Zhdanov R., Revenko I., General solutions of the nonlinear wave and eikonal equations, *Ukr. Math. J.*, 1991, **43**, № 11, 1471–1486.
6. Fushchych W., Zhdanov R., Yehorchenko I., On the reduction of the nonlinear multi-dimensional wave eqautions and compatibility of the d'Alembert–Hamilton system, *J. Math. Anal. Appl.*, 1991, **161**, № 2, 352–360.
7. Patera J., Winternitz P., Zassenhaus H., Continuous subgroups of the fundamental groups of physics, *J. Math. Phys.*, 1975, **16**, № 8, 1597–1624.
8. Patera J., Winternitz P., Zassenhaus H., Maximal Abelian subalgebras of real and complex symplectic Lie algebras, *J. Math. Phys.*, 1983, **24**, № 8, 1973–1985.
9. Fushchych W., Barannyk A., Barannyk L., Fedorchuk V., Continuous subgroups of the Poincaré group $P(1,4)$, *J. Phys. A: Math. Gen.*, 1985, **18**, № 5, 2893–2899.
10. Barannyk L., Fushchych W., On continuous subgroups of the generalized Schrödinger groups, *J. Math. Phys.*, 1989, **30**, № 2, 280–290.
11. Fushchych W., Barannyk L., Barannyk A., Subgroup analysis of the Galilei and Poincaré groups and reductions of nonlinear equations, Kyiv, Naukova Dumka,1991, 300 p.
12. Clarkson P., Kruskal M., New similarity reductions of the Boussinesq equation, *J. Math. Phys.*, 1989, **30**, № 10, 2201–2213.
13. Fushchych W., Nikitin A., Symmetries of Maxwell's equation, Reidel Dordrecht, 1987, 230 p. (Russian version 1983, Naukova Dumka 1983).
14. Fushchych W., How to extend symmetry of differential equations?, in Symmetry and Solutions of Nonlinear Mathematical Physics, Kyiv, Inst. of Math., 1987, 4–16.
15. Fushchych W., Tsyfra I., On reduction and solutions of nonlinear wave equations with broken symmetry, *J.Phys. A: Math. Gen.*, 1987, **20**, № 2, 45–47.
16. Fushchych W., Serov M., Chopyk V., Conditional invariance and nonlinear heat equations, *Dopovidi Ukr. Acad. Sci.*, 1988, № 9, 17–20.

17. Fushchych W., Serov M., Conditional invariance and exact solutions of nonlinear acoustic equation, *Dopovidi Ukr. Acad. Sci.*, 1988, № 10, 27–31.
18. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kluwer Academic Publishers, 1993, 430 p. (Russian version, 1989, Naukova Dumka).
19. Fushchych W., Shtelen W., On linear and nonlinear systems of differential equations invariant under the Schrödinger group, *Theor. and Math. Fisika*, 1983, **56**, № 3, 387–394.
20. Ovsyannikov L., Group analysis of differential equations, Moscow, Nauka, 1978.
21. Dorodnyzyn W., Knyazeva I., Svirshchevski S., Group properties of the heat equation with a source in two and three dimensions, *Defferentsial'nye Uravneniya*, 1983, **19**, № 7, 1215–1224.
22. Fushchych W., Serov M., Conditional symmetry and reduction of nonlinear heat equation, *Dopovidi Ukrainian Acad. Sci.*, 1990, № 7, 24–28.
23. Serov M., Conditional invariance and exact solutions of nonlinear heat equation, *Ukr. Math. J.,* 1990, **42**, № 10, 1370–1376.
24. Clarkson P., Mansfield E., Algorithms for the nonclassical method of symmetry reductions, *SIAM J. Appl. Math.*, 1994, **54**, № 6, 1693–1719.
25. Fushchych W., Conditional symmetry of the equations of mathematical physics, *Ukr. Math. J.*, 1991, **43**, № 1, 1456–1470.
26. Fushchych W., Myronyuk P., Conditional symmetry and exact solutions of nonlinear acoustics equation, *Dopovidi Ukr. Acad. Sci.*, 1991, № 8, 23–26.
27. Fushchych W., Repeta V., Exact solutions of equations of gas dynamics and nonlinear acoustics, *Dopovidi Ukr. Acad. Sci.*, 1991, № 8, 35–38.
28. Fushchych W., Seheda Yu., On a new invariance algebra for the free Schrödinger equation, *Doklady Acad. Sci. USSR*, 1977, **232**, № 4, 800–801.
29. Fushchych W., Nikitin A., Symmetries of equations of quantum mechanics, Allerton Press, 1994, 500 p. (in Russian 1990, Moscow, Nauka).
30. Fushchych W., On additional invariance of relativistic equations of motion, *Teor. Mat. Fizika*, 1971, **7**, № 1, 3–12.
31. Fushchych W., On additional invariance of the Dirac and Maxwell equations, *Lettere Nuovo Cimento*, 1974, **11**, № 10, 508–512.
32. Fushchych W., On additional invariance of the Klein–Gordon–Fock equation, *Dokl. Acad. Sci. USSR*, 1976, **230**, № 3, 570–573.
33. Fushchych W., Zhdanov R., Antireduction and exact solutions of nonlinear heat equation, *J. Nonlinear Math. Phys.*, 1994, **1**, № 1, 60–64.
34. Fushchych W., New nonlinear equations for electromagnetic field having the velocity different from c, *Dopovidi Ukr. Acad. Sci.*, 1992, № 4, 24–28.

Symmetry reduction and exact solutions of the Yang–Mills equations

V.I. LAHNO, R.Z. ZHDANOV, W.I. FUSHCHYCH

We present a detailed account of symmetry properties of $SU(2)$ Yang–Mills equations. Using a subgroup structure of the Poincaré group $P(1,3)$ we have constructed all $P(1,3)$-inequivalent ansatzes for the Yang–Mills field which are invariant under the three-dimensional subgroups of the Poincaré group. With the aid of these ansatzes reduction of Yang–Mills equations to systems of ordinary differential equations is carried out and wide families of their exact solutions are constructed.

1. Introduction

Since Newton's and Euler's works, exact solutions of differential equations describing physical processes were highly estimated. Green, Lame, Liouville, Cayley, Donkin, Stokes, Kirchhoff, Poincaré, Stieltjes, Forsyth, Volterra, Appel, MacDonald, Weber, Bateman, Whittaker, Sommerfeld and many other famous researchers constructed different classes of exact solutions of linear Laplace, d'Alembert, heat, and Maxwell equations.

Nowadays, this constructive branch of mathematical physics is not so popular as earlier. But if one wants to have some nontrivial information on solutions of basic motion equations in quantum mechanics, field theory, gravitation theory, acoustics, and hydrodynamics, then the more intensive research work should be carried out in order to develop analytical methods of solution of partial differential equations (PDE). And what is more, unlike the mathematical physics of the 19th century, modern mathematical physics is essentially nonlinear. It means that all principal equations of modern physics, biology and chemistry are nonlinear. This fact complicates very much the problem of constructing their exact solutions (see, e.g. [1] and references therein).

Up to now, we have comparatively few papers devoted to construction of exact solutions of nonlinear multi-dimensional d'Alembert, Maxwell, Schrödinger, Dirac, Maxwell–Dirac, Yang–Mills equations. Whereas, a huge amount of papers and monographs are devoted to construction of exact solutions of equations for gravitational field. It is difficult even to estimate the number of papers and monographs, where the soliton solutions of the one-dimensional nonlinear KdV, Schrödinger and Sine-Gordon equations are studied. We are sure that the above mentioned equations should deserve much more attention of researchers in mathematical physics.

With the present paper we start a series of papers devoted to construction of new classes of exact solutions of the classical Yang–Mills equations (YME) with the use of their Lie and non-Lie symmetry. Here we study in detail symmetry reduction of YME by Poincaré-invariant ansatzes and obtain wide families of its exact Poincaré-invariant solutions.

Reprinted with permission from J. Nonlinear Math. Phys., 1995, **2**, № 1, P. 51–72

By the classical YME, we mean the following nonlinear system of twelve second-order PDE:

$$\partial_\nu\partial^\nu \vec{A}_\mu - \partial^\mu\partial_\nu \vec{A}_\nu + e[(\partial_\nu \vec{A}_\nu)\times\vec{A}_\mu - 2(\partial_\nu\vec{A}_\mu)\times\vec{A}_\nu + (\partial^\mu\vec{A}_\nu)\times\vec{A}^\nu] + \\ + e^2\vec{A}_\nu\times(\vec{A}^\nu\times\vec{A}_\mu) = \vec{0}. \tag{1.1}$$

Here $\partial_\nu = \frac{\partial}{\partial x_\nu}$, $\mu,\nu = \overline{0,3}$, $e = \text{const}$, $\vec{A}_\mu = \vec{A}_\mu(x_0,x_1,x_2,x_3)$ is the three-component vector-potential of the Yang–Mills field (called, for brevity, the Yang–Mills field). Hereafter, the summation over the repeated indices μ,ν from 0 to 3 is understood. Raising and lowering the vector indices is performed with the aid of the metric tensor

$$g_{\mu\nu} = \begin{cases} 1, & \mu=\nu=0,\\ -1, & \mu=\nu=1,2,3,\\ 0, & \mu\neq\nu \end{cases}$$

(i.e. $\partial^\mu = g_{\mu\nu}\partial_\nu$).

It should be said that there were several reviews devoted to classical solutions of YME (see [2] and the literature cited there). But, in fact, symmetry properties of YME were not used. The solutions were obtained with the help of ad hoc substitutions suggested by Wu and Yang, Rosen, 't Hooft, Corrigan and Fairlie, Wilczek, Witten (for more detail, see [2]).

The structure of our paper is as follows. In the second Section we give all necessary information about symmetry properties of YME and about a solution generation procedure by virtue of the finite transformations of the symmetry group admitted by YME. In Section 3 we construct $P(1,3)$-inequivalent ansatzes for the Yang–Mills field invariant under the three-parameter subgroups of the Poincaré group. Section 4 is devoted to reduction of YME to systems of ordinary differential equations (ODE). Integrating these in Section 5 we construct multi-parameter families of exact solutions of YME. In Section 6 we consider some generalizations of the solutions obtained and, in particular, construct the generalization of Coleman's solution.

2. Symmetry and solution generation for the Yang–Mills equations

It was known long ago that YME are invariant with respect to the group $C(1,3)\otimes SU(2)$, where $C(1,3)$ is the 15-parameter conformal group having the following generators:

$$\begin{aligned} &P_\mu = \partial_\mu,\\ &J_{\alpha\beta} = x^\alpha\partial_\beta - x^\beta\partial_\alpha + A^{a\alpha}\partial_{A^a_\beta} - A^{a\beta}\partial_{A^a_\alpha},\\ &D = x_\mu\partial_\mu - A^a_\mu\partial_{A^a_\mu},\\ &K_\mu = 2x^\mu D - (x_\nu x^\nu)\partial_\mu + 2A^{a\mu}x_\nu\partial_{A^a_\nu} - 2A^a_\nu x^\nu\partial_{A^a_\mu}, \end{aligned} \tag{2.1}$$

and $SU(2)$ is the infinite-parameter special unitary group with the following basis generator:

$$Q = (\varepsilon_{abc}A^b_\mu w^c(x) + e^{-1}\partial_\mu w^a(x))\partial_{A^a_\mu}. \tag{2.2}$$

In (2.1), (2.2) $\partial_{A_\mu^a} = \frac{\partial}{\partial A_\mu^a}$, $w^c(x)$ are arbitrary smooth functions, ε_{abc} is the third-order anti-symmetrical tensor with $\varepsilon_{123} = 1$. Hereafter, summation over the repeated indices a, b, c from 1 to 3 is understood.

But the fact that the group with generators (2.1), (2.2) is a maximal (in Lie's sense) invariance group admitted by YME was established only recently [3] with the use of a symbolic computation technique. The only explanation for this situation is a very cumbersome structure of the system of PDE (1.1). As a consequence, realization of the Lie algorithm of finding the maximal invariance group admitted by YME demands a huge amount of computations. This difficulty had been overcome with the aid of computer facilities.

One of the remarkable possibilities provided by the fact that the considered equation admits a nontrivial symmetry group gives the possibility of getting new solutions from the known ones by the solution generation technique [1, 4]. This technique is based on the following assertion.

Lemma. *Let*

$$x'_\mu = f_\mu(x, u, \tau), \quad \mu = \overline{0, n-1},$$
$$u'_a = g_a(x, u, \tau), \quad a = \overline{1, N},$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_r)$ *be the* r*-parameter invariance group of some system of PDE and* $U_a(x)$, $a = \overline{1, N}$ *be its particular solution. Then the* N*-component function* $u_a(x)$ *determined by implicit formulae*

$$U_a(f(x, u, \tau)) = g_a(x, u, \tau), \quad a = \overline{1, N} \tag{2.3}$$

is also a solution of the same system of PDE.

To make use of the above assertion we need formulae for finite transformations generated by infinitesimal operators (2.1), (2.2). We adduce these formulae following [1, 2].

1. The group of translations (generator $X = \tau_\mu P_\mu$)

$$x'_\mu = x_\mu + \tau_\mu, A_\mu^{d\prime} = A_\mu^d.$$

2. The Lorentz group $O(1,3)$

a) the group of rotations (generator $X = \tau J_{ab}$)

$$x'_0 = 0, \quad x'_c = x_c, \quad c \neq a, \quad c \neq b,$$
$$x'_a = x_a \cos\tau + x_b \sin\tau,$$
$$x'_b = x_b \cos\tau - x_a \sin\tau,$$
$$A_0^{d\prime} = A_0^d, \quad A_c^{d\prime} = A_c^d, \quad c \neq a, \quad c \neq b,$$
$$A_a^{d\prime} = A_a^d \cos\tau + A_b^d \sin\tau,$$
$$A_b^{d\prime} = A_b^d \cos\tau - A_a^d \sin\tau;$$

b) the group of Lorentz transformations (generator $X = \tau J_{0a}$)

$$x'_0 = x_0 \cosh\tau + x_a \sinh\tau,$$
$$x'_a = x_a \cosh\tau + x_0 \sinh\tau, \quad x'_b = x_b, \quad b \neq a,$$
$$A_0^{d\prime} = A_0^d \cosh\tau + A_a^d \sinh\tau,$$
$$A_a^{d\prime} = A_a^d \cosh\tau + A_0^d \sinh\tau, \quad A_b^{d\prime} = A_b^d, \quad b \neq a.$$

3. The group of scale transformations (generator $X = \tau D$)

$$x'_\mu = x_\mu e^\tau, \quad A^{d\,\prime}_\mu = A^d_\mu e^{-\tau}.$$

4. The group of conformal transformations (generator $X = \tau_\mu K_\mu$)

$$x'_\mu = (x_\mu - \tau_\mu x_\nu x^\nu)\sigma^{-1}(x),$$
$$A^{d\,\prime}_\mu = [g_{\mu\nu}\sigma(x) + 2(x_\mu\tau_\nu - x_\nu\tau_\mu + 2\tau_\alpha x^\alpha \tau_\mu x_\nu - x_\alpha x^\alpha \tau_\mu\tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu]A^{d\nu}.$$

5. The group of gauge transformations (generator $X = Q$)

$$x'_\mu = x_\mu,$$
$$A^{d\,\prime}_\mu = A^d_\mu \cos w + \varepsilon_{dbc}A^b_\mu n^c \sin w + 2n^d n^b A^b_\mu \sin^2\frac{w}{2} + $$
$$+ e^{-1}\left[\frac{1}{2}n^d\partial_\mu w + \frac{1}{2}(\partial_\mu n^d)\sin w + \varepsilon_{dbc}(\partial_\mu n^b)n^c\right].$$

In the above formulae $\sigma(x) = 1 - \tau_\alpha x^\alpha + (\tau_\alpha\tau^\alpha)(x_\beta x^\beta)$, $n^a = n^a(x)$ is a unit vector determined by the equality $w^a(x) = w(x)n^a(x)$, $a = \overline{1,3}$.

Using the Lemma it is not difficult to obtain formulae for generating solutions of YME by the above transformation groups. We adduce them omitting derivation (see also [3]).

1. The group of translations

$$A^a_\mu(x) = u^a_\mu(x + \tau).$$

2. The Lorentz group

$$A^d_\mu(x) = a_\mu u^d_0(ax, bx, cx, dx) + b_\mu u^d_1(ax, bx, cx, dx) +$$
$$+ c_\mu u^d_2(ax, bx, cx, dx) + d_\mu u^d_3(ax, bx, cx, dx).$$

3. The group of scale transformations

$$A^d_\mu(x) = e^\tau u^d_\mu(xe^\tau).$$

4. The group of conformal transformations

$$A^d_\mu(x) = [g_{\mu\nu}\sigma^{-1}(x) + 2\sigma^{-2}(x)(x_\mu\tau_\nu - x_\nu\tau_\mu + 2\tau_\alpha x^\alpha\tau_\mu x_\nu -$$
$$- x_\alpha x^\alpha \tau_\mu\tau_\nu - \tau_\alpha\tau^\alpha x_\mu x_\nu)]u^{d\nu}((x - \tau(x_\alpha x^\alpha))\sigma^{-1}(x)).$$

5. The group of gauge transformations

$$A^d_\mu(x) = u^d_\mu \cos w + \varepsilon_{dbc}u^b_\mu n^c \sin w + 2n^d n^b u^b_\mu \sin^2\frac{w}{2} +$$
$$+ e^{-1}\left[\frac{1}{2}n^d\partial_\mu w + \frac{1}{2}(\partial_\mu n^d)\sin w + \varepsilon_{dbc}(\partial_\mu n^b)n^c\right].$$

Here $u^d_\mu(x)$ is an arbitrary given solution of YME; $A^d_\mu(x)$ is a new solution of YME; τ, τ_μ are arbitrary parameters; a_μ, b_μ, c_μ, d_μ are arbitrary parameters satisfying the equalities

$$a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,$$
$$a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.$$

Besides that, we use the following designations: $x + \tau = \{x_\mu + \tau_\mu, \mu = \overline{0,3}\}$, $ax = a_\mu x^\mu$.

Thus, each particular solution of YME gives rise to a multi-parameter family of exact solutions by virtue of the above solution generation formulae.

3. Ansatzes for the Yang–Mills field

A key idea of the symmetry approach to the problem of reduction of PDE is a special choice of the form of a solution. This choice is dictated by a structure of the symmetry group admitted by the equation under study.

In the case involved, to reduce YME by N variables one has to construct ansatzes for the Yang–Mills field $A^a_\mu(x)$ invariant under N-dimensional subalgebras of the algebra with the basis elements (2.1), (2.2) [1, 5]. Since we are looking for Poincaré-invariant ansatzes reducing YME to systems of ODE, N is equal to 3. Due to invariance of YME under the Poincaré group $P(1,3)$, it is enough to consider only subalgebras which can not be transformed one into another by group transformation, i.e. $P(1,3)$-inequivalent subalgebras. Complete description of $P(1,3)$-inequivalent subalgebras of the Poincaré algebra was obtained in [6] (see also [7]).

According to the classical symmetry approach, to construct the ansatz invariant under the invariance algebra having the basis elements

$$X_a = \xi_{a\mu}(x,A)\partial_\mu + \eta^b_{a\mu}(x,A)\partial_{A^b_\mu}, \quad a=\overline{1,3}, \tag{3.1}$$

where $A=\{A^a_\mu, a=\overline{1,3}, \mu=\overline{0,3}\}$, one has

1) to construct a complete system of functionally-independent invariants of the operators (3.1) $\Omega=\{w_i(x,A), i=\overline{1,13}\}$;

2) to resolve relations

$$F_j(w_1(x,A),\ldots,w_{13}(x,A))=0, \quad j=\overline{1,13} \tag{3.2}$$

with respect to the function A^a_μ.

As a result, one gets the ansatz for the field $A^a_\mu(x)$ which reduces YME to the system of twelve nonlinear ODE.

Note. Equalities (3.2) can be resolved with respect to A^a_μ, $a=\overline{1,3}$, $\mu=\overline{0,3}$ if the condition

$$\operatorname{rank}\|\xi_{a\mu}(x,A)\|^{3}_{a=1}{}^{3}_{\mu=0}=3 \tag{3.3}$$

holds. If (3.3) does not hold, the above procedure leads to partially-invariant solutions [5], which are not considered in the present paper.

In [1, 4] we established that the procedure of construction of invariant ansatzes could be essentially simplified if coefficients of operators X_a have the following structure:

$$\xi_{a\mu}=\xi_{a\mu}(x), \quad \eta^b_{a\mu}=\rho^{bc}_{a\mu\nu}(x)A^c_\nu \tag{3.4}$$

(i.e. basis elements of the invariance algebra realize the linear representation). In this case, the invariant ansatz for the field $A^a_\mu(x)$ is searched for in the form

$$A^a_\mu(x)=Q^{ab}_{\mu\nu}(x)B^b_\nu(w(x)). \tag{3.5}$$

Here $B^b_\nu(w)$ are arbitrary smooth functions and $w(x)$, $Q^{ab}_{\mu\nu}(x)$ are particular solutions of the system of PDE

$$\begin{gathered}\xi_{a\mu}w_{x_\mu}=0, \quad a=\overline{1,3},\\ (\xi_{a\nu}\partial_\nu-\rho^{bc}_{a\mu\alpha})Q^{cd}_{\alpha\beta}=0, \quad \mu=\overline{0,3}, \quad a,b,d=\overline{1,3}.\end{gathered} \tag{3.6}$$

Basis elements of the Poincaré algebra P_μ, $J_{\alpha\beta}$ from (2.1) evidently satisfy the conditions (3.4) and besides the equalities

$$\eta^b_{a\mu} = \rho_{a\mu\nu}(x)A^b_\nu, \quad a,b=\overline{1,3}, \quad \mu=\overline{0,3} \tag{3.7}$$

hold.

This fact permits further simplification of formulae (3.5), (3.6). Namely, the ansatz for the Yang–Mills field invariant under the 3-dimensional subalgebra of the Poincaré algebra with basis elements of the form (3.1), (3.7) should be looked for in the form

$$A^a_\mu = Q_{\mu\nu}(x)B^a_\nu(w(x)), \tag{3.8}$$

where $B^a_\nu(w)$ are arbitrary smooth functions and $w(x)$, $Q_{\mu\nu}(x)$ are particular solutions of the following system of PDE:

$$\xi_{a\mu}w_{x_\mu} = 0, \quad a=\overline{1,3}, \tag{3.9}$$

$$\xi_{a\alpha}\partial_\alpha Q_{\mu\nu} - \rho_{a\mu\alpha}Q_{\alpha\nu} = 0, \quad a=\overline{1,3}, \quad \mu,\nu=\overline{0,3}. \tag{3.10}$$

Thus, to obtain the complete description of $P(1,3)$-inequivalent ansatzes for the field $A^a_\mu(x)$ invariant under 3-dimensional subalgebras of the Poincaré algebra, one has to integrate the over-determined system of PDE (3.9), (3.10) for each $P(1,3)$-inequivalent subalgebra. Let us note that compatibility of (3.9), (3.10) is guaranteed by the fact that operators X_1, X_2, X_3 form a Lie algebra.

Consider, as an example, the procedure of constructing ansatz (3.8) invariant under the subalgebra $\langle P_1, P_2, J_{03}\rangle$. In this case system (3.9) reads

$$w_{x_1} = 0, \quad w_{x_2} = 0, \quad x_0 w_{x_3} + x_3 w_{x_0} = 0,$$

whence $w = x_0^2 - x_3^2$.

Next, we note that coefficients $\rho_{1\mu\nu}$, $\rho_{2\mu\nu}$ of the operators P_1, P_2 are equal to zero, while coefficients $\rho_{3\mu\nu}$ form the following (4×4) matrix

$$\|\rho_{3\mu\nu}\|^3_{\mu,\nu=0} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$$

(we designate this constant matrix by the symbol S).

With account of the above fact, equations (3.10) take the form

$$Q_{x_1} = 0, \quad Q_{x_2} = 0, \quad x_0 Q_{x_3} + x_3 Q_{x_0} - SQ = 0, \tag{3.11}$$

where $Q = \|Q_{\mu\nu}(x)\|^3_{\mu,\nu=0}$ is a (4×4)-matrix.

From the first two equations of system (3.11) it follows that $Q = Q(x_0, x_3)$. Since S is a constant matrix, a solution of the third equation can be looked for in the form (see, for example, [4])

$$Q = \exp\{f(x_0,x_3)S\}.$$

Substituting this expression into (3.11) we get

$$(x_0 f_{x_3} + x_3 f_{x_0} - 1)\exp\{fS\} = 0$$

or, equivalently,

$$x_0 f_{x_3} + x_3 f_{x_0} = 1,$$

whence $f = \ln(x_0 + x_3)$.

Consequently, a particular solution of equations (3.11) reads

$$Q = \exp\{\ln(x_0 + x_3)S\}.$$

Using an evident identity $S = S^3$ we get the following equalities:

$$\begin{aligned} Q &= \sum_{n=0}^{\infty}(n!)^{-1}(\ln(x_0+x_3))^n S^n = \\ &= I + S[\ln(x_0+x_3) + (3!)^{-1}(\ln(x_0+x_3))^3 + \cdots] + \\ &\quad + S^2[(2!)^{-1}(\ln(x_0+x_3))^2 + (4!)^{-1}(\ln(x_0+x_3))^4 + \cdots] = \\ &= I + S\sinh(\ln(x_0+x_3)) + S^2(\cosh(\ln(x_0+x_3)) - 1), \end{aligned}$$

where I is a unit (4×4)-matrix.

Substitution of the obtained expressions for functions $w(x)$, $Q_{\mu\nu}(x)$ into (3.8) yields the ansatz for the Yang–Mills field $A^a_\mu(x)$ invariant under the algebra $\langle P_1, P_2, J_{03}\rangle$

$$\begin{aligned} A^a_0 &= B^a_0(x_0^2 - x_3^2)\cosh\ln(x_0+x_3) + B^a_3(x_0^2 - x_3^2)\sinh\ln(x_0+x_3), \\ A^a_1 &= B^a_1(x_0^2 - x_3^2), \quad A^a_2 = B^a_2(x_0^2 - x_3^2), \\ A^a_3 &= B^a_3(x_0^2 - x_3^2)\cosh\ln(x_0+x_3) + B^a_0(x_0^2 - x_3^2)\sinh\ln(x_0+x_3). \end{aligned} \tag{3.12}$$

Substituting (3.12) into YME we get a system of ODE for functions B^a_μ. If we will succeed in constructing its general or particular solutions, then substituting it into formulae (3.12) we get an exact solution of YME. But such a solution will have an unpleasant feature: independent variables x_μ will be included into it in asymmetrical way. At the same time, in the initial equation (1.1) all independent variables are on equal rights. To remove this defect one has to apply solution generation procedure by transformations from the Lorentz group. As a result, we will obtain an ansatz for the Yang–Mills field in the manifestly-covariant form with symmetrical dependence on x_μ.

In the same way, we construct the rest of ansatzes invariant under three-dimensional subalgebras of the Poincaré algebra. They are represented in the unified form

$$\begin{aligned} A^a_\mu(x) = \{&(a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 + \\ &+ 2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu + \\ &+ (\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] + (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 - \\ &- (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu)\}B^{a\nu}(w). \end{aligned} \tag{3.13}$$

Here θ_μ, $\mu = \overline{0,3}$, w are some functions whose explicit form is determined by the choice of a subalgebra of the Poincaré algebra $AP(1,3)$.

Below, we adduce a complete list of 3-dimensional $P(1,3)$-inequivalent subalgebras of the Poincaré algebra following [7]

$L_1 = \langle P_0, P_1, P_2\rangle$; $L_2 = \langle P_1, P_2, P_3\rangle$;

$L_3 = \langle P_0 + P_3, P_1, P_2\rangle$; $L_4 = \langle J_{03} + \alpha J_{12}, P_1, P_2\rangle$;

$$
\begin{array}{ll}
L_5 = \langle J_{03}, P_0 + P_3, P_1\rangle; & L_6 = \langle J_{03} + P_1, P_0, P_3\rangle;\\
L_7 = \langle J_{03} + P_1, P_0 + P_3, P_2\rangle; & L_8 = \langle J_{12} + \alpha J_{03}, P_0, P_3\rangle;\\
L_9 = \langle J_{12} + P_0, P_1, P_2\rangle; & L_{10} = \langle J_{12} + P_3, P_1, P_2\rangle;\\
L_{11} = \langle J_{12} + P_0 - P_3, P_1, P_2\rangle; & L_{12} = \langle G_1, P_0 + P_3, P_2 + \alpha P_1\rangle;\\
L_{13} = \langle G_1 + P_2, P_0 + P_3, P_1\rangle; & L_{14} = \langle G_1 + P_0 - P_3, P_0 + P_3, P_2\rangle;\\
L_{15} = \langle G_1 + P_0 - P_3, P_0 + P_3, P_1 + \alpha P_2\rangle; & L_{16} = \langle J_{12}, J_{03}, P_0 + P_3\rangle;\\
L_{17} = \langle G_1 + P_2, G_2 - P_1 + \alpha P_2, P_0 + P_3\rangle; & L_{18} = \langle J_{03}, G_1, P_2\rangle;\\
L_{19} = \langle G_1, J_{03}, P_0 + P_3\rangle; & L_{20} = \langle G_1, J_{03} + P_2, P_0 + P_3\rangle;\\
L_{21} = \langle G_1, J_{03} + P_1 + \alpha P_2, P_0 + P_3\rangle; & L_{22} = \langle G_1, G_2, J_{03} + \alpha J_{12}\rangle;\\
L_{23} = \langle G_1, P_0 + P_3, P_1\rangle; & L_{24} = \langle J_{12}, P_1, P_2\rangle;\\
L_{25} = \langle J_{03}, P_0, P_3\rangle; & L_{26} = \langle J_{12}, J_{13}, J_{23}\rangle;\\
L_{27} = \langle J_{01}, J_{02}, J_{12}\rangle. &
\end{array}
\tag{3.14}
$$

Here $G_i = J_{0i} - J_{i3}$ $(i = 1, 2)$, $\alpha \in \mathbb{R}$.

Ansatzes for the Yang–Mills field $A^a_\mu(x)$ are of the form (3.13), functions $\theta_\mu(x)$, $\mu = \overline{0,3}$, $w(x)$ being determined by one of the following formulae:

$$
\begin{array}{ll}
L_1: & \theta_\mu = 0, \quad w = dx; \quad L_2: \quad \theta_\mu = 0, \quad w = ax; \quad L_3: \quad \theta_\mu = 0, w = kx;\\
L_4: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = \alpha\ln|kx|, \quad w = (ax)^2 - (dx)^2;\\
L_5: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = cx;\\
L_6: & \theta_0 = -bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = cx;\\
L_7: & \theta_0 = -bx, \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad w = bx - \ln|kx|;\\
L_8: & \theta_0 = \alpha\arctan(bx(cx)^{-1}), \quad \theta_1 = \theta_2 = 0,\\
 & \theta_3 = -\arctan(bx(cx)^{-1}), \quad w = (bx)^2 + (cx)^2;\\
L_9: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -ax, \quad w = dx;\\
L_{10}: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = dx, \quad w = ax;\\
L_{11}: & \theta_0 = \theta_1 = \theta_2 = 0, \quad \theta_3 = -\dfrac{1}{2}kx, \quad w = ax - dx;\\
L_{12}: & \theta_0 = 0, \quad \theta_1 = \dfrac{1}{2}(bx - \alpha cx)(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad w = kx;\\
L_{13}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = \tfrac{1}{2}cx, \quad w = kx;\\
L_{14}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\dfrac{1}{4}kx, \quad w = 4bx + (kx)^2;\\
L_{15}: & \theta_0 = \theta_2 = \theta_3 = 0, \quad \theta_1 = -\dfrac{1}{4}kx, \quad w = 4(\alpha bx - cx) + \alpha(kx)^2;\\
L_{16}: & \theta_0 = -\ln|kx|, \quad \theta_1 = \theta_2 = 0, \quad \theta_3 = -\arctan(bx(cx)^{-1}),\\
 & w = (bx)^2 + (cx)^2;\\
L_{17}: & \theta_0 = \theta_3 = 0, \quad \theta_1 = \dfrac{1}{2}(cx + (\alpha + kx)bx)(1 + kx(\alpha + kx))^{-1},\\
 & \theta_2 = -\dfrac{1}{2}(bx - cxkx)(1 + kx(\alpha + kx))^{-1}, \quad w = kx;\\
L_{18}: & \theta_0 = -\ln|kx|, \quad \theta_1 = \dfrac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0,\\
 & w = (ax)^2 - (bx)^2 - (dx)^2;
\end{array}
\tag{3.15}
$$

$$\begin{aligned}
L_{19}: &\quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, w = cx;\\
L_{20}: &\quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \theta_3 = 0, \quad w = \ln|kx| - cx;\\
L_{21}: &\quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}(bx - \ln|kx|)(kx)^{-1}, \quad \theta_2 = \theta_3 = 0,\\
&\quad w = \alpha\ln|kx| - cx;\\
L_{22}: &\quad \theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \frac{1}{2}cx(kx)^{-1},\\
&\quad \theta_3 = \alpha\ln|kx|, \quad w = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.
\end{aligned}$$

Here $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = c_\mu x^\mu$, $dx = d_\mu x^\mu$, $\mu = \overline{0,3}$, $kx = ax + dx$.

Note. Basis elements of subalgebras L_{23}, L_{24}, L_{25}, L_{26}, L_{27} do not satisfy (3.3). That is why, ansatzes invariant under these subalgebras are partially-invariant solutions and are not considered here.

4. Reduction of the Yang–Mills equations

In order to reduce YME to ODE it is necessary to substitute ansatz (3.13) into (1.1) and convolute the expression obtained with $Q^\mu_\alpha(x)$. As a result, we get a system of twelve nonlinear ODE for functions $B^a_\nu(w)$ of the form

$$\begin{aligned}
&k_{\mu\gamma}\ddot{\vec{B}}^\gamma + l_{\mu\gamma}\dot{\vec{B}}^\gamma + m_{\mu\gamma}\vec{B}^\gamma + eg_{\mu\nu\gamma}\dot{\vec{B}}^\nu \times \vec{B}^\gamma + eh_{\mu\nu\gamma}\vec{B}^\nu \times \vec{B}^\gamma +\\
&\qquad + e^2\vec{B}_\gamma \times (\vec{B}^\gamma \times \vec{B}_\mu) = \vec{0}.
\end{aligned} \tag{4.1}$$

Coefficients of the reduced ODE are given by the following formulae:

$$\begin{aligned}
&k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu G_\gamma, \quad l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2S_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma,\\
&m_{\mu\gamma} = R_{\mu\gamma} - G_\mu \dot{H}_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma,\\
&h_{\mu\nu\gamma} = (1/2)(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma},
\end{aligned} \tag{4.2}$$

where $g_{\mu\nu}$ is a metric tensor of the Minkowski space $\mathbb{R}(1,3)$ and $F_1, F_2, G_\mu, \ldots, T_{\mu\nu\gamma}$ are functions on w determined by the relations

$$\begin{aligned}
&F_1 = w_{x_\mu}w_{x^\mu}, \quad F_2 = \Box w, \quad G_\mu = Q_{\alpha\mu}w_{x_\alpha}, \quad H_\mu = Q_{\alpha\mu x_\alpha},\\
&S_{\mu\nu} = Q^\alpha_\mu Q_{\alpha\nu x_\beta}w_{x^\beta}, \quad R_{\mu\nu} = Q^\alpha_\mu \Box Q_{\alpha\nu},\\
&T_{\mu\nu\gamma} = Q^\alpha_\mu Q_{\alpha\nu x_\beta}Q_{\beta\gamma} + Q^\alpha_\nu Q_{\alpha\gamma x_\beta}Q_{\beta\mu} + Q^\alpha_\gamma Q_{\alpha\mu x_\beta}Q_{\beta\nu}.
\end{aligned} \tag{4.3}$$

Substituting functions $Q_{\mu\nu}(x)$ from (3.13), where $\theta_\mu(x)$, $w(x)$ are determined by one of the formulae (3.15) into (4.2), (4.3) we obtain coefficients of the corresponding systems of ODE (4.1)

$$\begin{aligned}
L_1: &\quad k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0,\\
&\quad g_{\mu\nu\gamma} = g_{\mu\gamma}d_\nu + g_{\nu\gamma}d_\mu - 2g_{\mu\nu}d_\gamma, \quad h_{\mu\nu\gamma} = 0;\\
L_2: &\quad k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0,\\
&\quad g_{\mu\nu\gamma} = g_{\mu\gamma}a_\nu + g_{\nu\gamma}a_\mu - 2g_{\mu\nu}a_\gamma, \quad h_{\mu\nu\gamma} = 0;\\
L_3: &\quad k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = m_{\mu\gamma} = 0, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma,\\
&\quad h_{\mu\nu\gamma} = 0;
\end{aligned}$$

$$
\begin{aligned}
L_4: \quad & k_{\mu\gamma} = 4g_{\mu\gamma}w - a_\mu a_\gamma(w+1)^2 - d_\mu d_\gamma(w-1)^2 - (a_\mu d_\gamma + a_\gamma d_\mu)(w^2-1),\\
& l_{\mu\gamma} = 4(g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma)) - 2k_\mu(a_\gamma - d_\gamma + k_\gamma w), \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu w) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu w) -\\
& \qquad - 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma w)),\\
& h_{\mu\nu\gamma} = \frac{\epsilon}{2}[g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma] + \alpha\epsilon[(b_\mu c_\nu - c_\mu b_\nu)k_\gamma + (b_\nu c_\gamma - c_\nu b_\gamma)k_\mu +\\
& \qquad + (b_\gamma c_\mu - c_\gamma b_\mu)k_\nu];\\
L_5: \quad & k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = -\epsilon c_\mu k_\gamma, \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma, \quad h_{\mu\nu\gamma} = \frac{\epsilon}{2}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma);\\
L_6: \quad & k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma), \quad g_{\mu\nu\gamma} = g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma,\\
& h_{\mu\nu\gamma} = -[(a_\mu d_\nu - a_\nu d_\mu)b_\gamma + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu];\\
L_7: \quad & k_{\mu\gamma} = -g_{\mu\gamma} - (b_\mu - \epsilon k_\mu)(b_\gamma - \epsilon k_\gamma), \quad l_{\mu\gamma} = -2(a_\mu d_\gamma - a_\gamma d_\mu),\\
& m_{\mu\gamma} = -(a_\mu a_\gamma - d_\mu d_\gamma),\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}(b_\nu - \epsilon k_\nu) + g_{\nu\gamma}(b_\mu - \epsilon k_\mu) - 2g_{\mu\nu}(b_\gamma - \epsilon k_\gamma),\\
& h_{\mu\nu\gamma} = -[(a_\mu d_\nu - a_\nu d_\mu)b_\gamma + (a_\nu d_\gamma - a_\gamma d_\nu)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu];\\
L_8: \quad & k_{\mu\gamma} = -4w(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma),\\
& m_{\mu\gamma} = -\frac{1}{w}(\alpha^2(a_\mu a_\gamma - d_\mu d_\gamma) + b_\mu b_\gamma),\\
& g_{\mu\nu\gamma} = 2\sqrt{w}(g_{\mu\gamma}c_\nu + g_{\nu\gamma}c_\mu - 2g_{\mu\nu}c_\gamma),\\
& h_{\mu\nu\gamma} = \frac{1}{2\sqrt{w}}(g_{\mu\gamma}c_\nu - g_{\mu\nu}c_\gamma) + \frac{\alpha}{\sqrt{w}}((a_\mu d_\nu - a_\nu d_\mu)b_\gamma +\\
& \qquad + (a_\nu d_\gamma - d_\nu a_\gamma)b_\mu + (a_\gamma d_\mu - a_\mu d_\gamma)b_\nu);\\
L_9: \quad & k_{\mu\gamma} = -g_{\mu\gamma} - d_\mu d_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = b_\mu b_\gamma + c_\mu c_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}d_\nu + g_{\nu\gamma}d_\mu - 2g_{\mu\nu}d_\gamma,\\
& h_{\mu\nu\gamma} = a_\gamma(b_\mu c_\nu - c_\mu b_\nu) + a_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + a_\nu(b_\gamma c_\mu - c_\gamma b_\mu);\\
L_{10}: \quad & k_{\mu\gamma} = g_{\mu\gamma} - a_\mu a_\gamma, \quad l_{\mu\gamma} = 0,\\
& m_{\mu\gamma} = -(b_\mu b_\gamma + c_\mu c_\gamma), \quad g_{\mu\nu\gamma} = g_{\mu\gamma}a_\nu + g_{\nu\gamma}a_\mu - 2g_{\mu\nu}a_\gamma,\\
& h_{\mu\nu\gamma} = -[d_\gamma(b_\mu c_\nu - c_\mu b_\nu) + d_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + d_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];\\
L_{11}: \quad & k_{\mu\gamma} = -(a_\mu - d_\mu)(a_\gamma - d_\gamma), \quad l_{\mu\gamma} = -2(b_\mu c_\gamma - c_\mu b_\gamma), \quad m_{\mu\gamma} = 0,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}(a_\nu - d_\nu) + g_{\nu\gamma}(a_\mu - d_\mu) - 2g_{\mu\nu}(a_\gamma - d_\gamma),\\
& h_{\mu\nu\gamma} = \tfrac{1}{2}[k_\gamma(b_\mu c_\nu - c_\mu b_\nu) + k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];\\
L_{12}: \quad & k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\frac{1}{w}k_\mu k_\gamma, \quad m_{\mu\gamma} = -\frac{\alpha^2}{w^2}k_\mu k_\gamma,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma,\\
& h_{\mu\nu\gamma} = \frac{1}{2w}(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma) +\\
& \qquad + \frac{\alpha}{w}((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);\\
L_{13}: \quad & k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = 0, \quad m_{\mu\gamma} = -k_\mu k_\gamma,\\
& g_{\mu\nu\gamma} = g_{\mu\gamma}k_\nu + g_{\nu\gamma}k_\mu - 2g_{\mu\nu}k_\gamma, \quad h_{\mu\nu\gamma} = -((k_\mu b_\nu - k_\nu b_\mu)c_\gamma +
\end{aligned}
\tag{4.4}
$$

$$+ (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu + (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);$$

$$L_{14}: \quad k_{\mu\gamma} = -16(g_{\mu\gamma} + b_\mu b_\gamma), \quad l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0,$$
$$g_{\mu\nu\gamma} = 4(g_{\mu\gamma} b_\nu + g_{\nu\gamma} b_\mu - 2g_{\mu\nu} b_\gamma);$$

$$L_{15}: \quad k_{\mu\gamma} = -16[(1+\alpha^2)g_{\mu\gamma} + (c_\mu - \alpha b_\mu)(c_\gamma - \alpha b_\gamma)],$$
$$l_{\mu\gamma} = m_{\mu\gamma} = h_{\mu\nu\gamma} = 0,$$
$$g_{\mu\nu\gamma} = -4[g_{\mu\gamma}(c_\nu - \alpha b_\nu) + g_{\nu\gamma}(c_\mu - \alpha b_\mu) - 2g_{\mu\nu}(c_\gamma - \alpha b_\gamma)];$$

$$L_{16}: \quad k_{\mu\gamma} = -4w(g_{\mu\gamma} + c_\mu c_\gamma), \quad l_{\mu\gamma} = -4(g_{\mu\gamma} + c_\mu c_\gamma) - 2\epsilon k_\gamma c_\mu \sqrt{w},$$
$$m_{\mu\gamma} = -\frac{1}{w} b_\mu b_\gamma, \quad g_{\mu\nu\gamma} = 2\sqrt{w}(g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma),$$
$$h_{\mu\nu\gamma} = \frac{1}{2}[\epsilon(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma) + \frac{1}{\sqrt{w}}(g_{\mu\gamma} c_\nu - g_{\mu\nu} c_\gamma)];$$

$$L_{17}: \quad k_{\mu\gamma} = -k_\mu k_\gamma, \quad l_{\mu\gamma} = -\frac{2w+\alpha}{w(w+\alpha)+1} k_\mu k_\gamma,$$
$$m_{\mu\gamma} = -4k_\mu k_\gamma (1 + w(\alpha+w))^{-2}, \quad g_{\mu\nu\gamma} = g_{\mu\gamma} k_\nu + g_{\nu\gamma} k_\mu - 2g_{\mu\nu} k_\gamma,$$
$$h_{\mu\nu\gamma} = \tfrac{1}{2}(\alpha + 2w)(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma)(1 + w(\alpha+w))^{-1} -$$
$$2(1 + w(w+\alpha))^{-1}((k_\mu b_\nu - k_\nu b_\mu)c_\gamma + (k_\nu b_\gamma - k_\gamma b_\nu)c_\mu +$$
$$+ (k_\gamma b_\mu - k_\mu b_\gamma)c_\nu);$$

$$L_{18}: \quad k_{\mu\gamma} = 4w g_{\mu\gamma} - (k_\mu w + a_\mu - d_\mu)(k_\gamma w + a_\gamma - d_\gamma),$$
$$l_{\mu\gamma} = 6g_{\mu\gamma} + 4(a_\mu d_\gamma - a_\gamma d_\mu) - 3k_\gamma(k_\mu w + a_\mu - d_\mu), \quad m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(k_\nu w + a_\nu - d_\nu) + g_{\nu\gamma}(k_\mu w + a_\mu - d_\mu) -$$
$$- 2g_{\mu\nu}(k_\gamma w + a_\gamma - d_\gamma)),$$
$$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma);$$

$$L_{19}: \quad k_{\mu\gamma} = -g_{\mu\gamma} - c_\mu c_\gamma, \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu, \quad m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = g_{\mu\gamma} c_\nu + g_{\nu\gamma} c_\mu - 2g_{\mu\nu} c_\gamma, \quad h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma);$$

$$L_{20}: \quad k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \epsilon k_\mu)(c_\gamma - \epsilon k_\gamma), \quad l_{\mu\gamma} = 2\epsilon k_\gamma c_\mu - 2k_\mu k_\gamma,$$
$$m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = g_{\mu\gamma}(\epsilon k_\nu - c_\nu) + g_{\nu\gamma}(\epsilon k_\mu - c_\mu) - 2g_{\mu\nu}(\epsilon k_\gamma - c_\gamma),$$
$$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma);$$

$$L_{21}: \quad k_{\mu\gamma} = -g_{\mu\gamma} - (c_\mu - \alpha\epsilon k_\mu)(c_\gamma - \alpha\epsilon k_\gamma), \quad l_{\mu\gamma} = 2(\epsilon k_\gamma c_\mu - \alpha k_\mu k_\gamma),$$
$$m_{\mu\gamma} = -k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = -g_{\mu\gamma}(c_\nu - \alpha\epsilon k_\nu) - g_{\nu\gamma}(c_\mu - \alpha\epsilon k_\mu) + 2g_{\mu\nu}(c_\gamma - \alpha\epsilon k_\gamma),$$
$$h_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma);$$

$$L_{22}: \quad k_{\mu\gamma} = 4w g_{\mu\gamma} - (a_\mu - d_\mu + k_\mu w)(a_\gamma - d_\gamma + k_\gamma w),$$
$$l_{\mu\gamma} = 4[2g_{\mu\gamma} + \alpha(b_\mu c_\gamma - c_\mu b_\gamma) - a_\mu a_\gamma + d_\mu d_\gamma - w k_\mu k_\gamma],$$
$$m_{\mu\gamma} = -2k_\mu k_\gamma,$$
$$g_{\mu\nu\gamma} = \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu w) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu w) -$$
$$- 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma w),$$
$$h_{\mu\nu\gamma} = \frac{3\epsilon}{2}(g_{\mu\gamma} k_\nu - g_{\mu\nu} k_\gamma) - \epsilon\alpha[k_\gamma(b_\mu c_\nu - c_\mu b_\nu) +$$
$$+ k_\mu(b_\nu c_\gamma - c_\nu b_\gamma) + k_\nu(b_\gamma c_\mu - c_\gamma b_\mu)];$$

where $k_\mu = a_\mu + d_\mu$, $\epsilon = 1$ for $ax + dx > 0$ and $\epsilon = -1$ for $ax + dx < 0$.

5. Exact solutions of the Yang–Mills equations

When applying the symmetry reduction procedure to the nonlinear Dirac equation, we succeeded in constructing general solutions of a large part of reduced systems of ODE. In the case involved we are not so lucky. Nevertheless, we obtain some particular solutions of equations (4.2), (4.4).

The principal idea of our approach to integration of systems of ODE (4.2), (4.4) is rather simple and quite natural. It is a reduction of these systems by the number of components with the aid of ad hoc substitutions. Using this trick we construct particular solutions of equations 1, 2, 5, 8, 14, 15, 16, 18, 19, 20, 21, 22 ($\alpha = 0$). Below we adduce substitutions for $\vec{B}_\mu(w)$ and corresponding equations.

$$
\begin{array}{ll}
1. & \vec{B}_\mu = a_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w) + c_\mu \vec{e}_3 h(w), \\
& \ddot{f} - e^2(g^2+h^2)f = 0, \quad \ddot{g} + e^2(f^2-h^2)g = 0, \quad \ddot{h} + e^2(f^2-g^2)h = 0. \\
2. & \vec{B}_\mu = b_\mu \vec{e}_1 f(w) + c_\mu \vec{e}_2 g(w) + d_\mu \vec{e}_3 h(w), \\
& \ddot{f} + e^2(g^2+h^2)f = 0, \quad \ddot{g} + e^2(f^2+h^2)g = 0, \quad \ddot{h} + e^2(f^2+g^2)h = 0. \\
5. & \vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w), \quad \ddot{f} - e^2 g^2 f = 0, \ddot{g} = 0. \\
8.1. & (\alpha = 0)\ \vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w), \\
& 4w\ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4w\ddot{g} + 4\dot{g} - w^{-1} g = 0. \\
8.2. & \vec{B}_\mu = a_\mu \vec{e}_1 f(w) + d_\mu \vec{e}_2 g(w) + b_\mu \vec{e}_3 h(w), \\
& 4w\ddot{f} + 4\dot{f} - \dfrac{\alpha^2}{w} f - \dfrac{2\alpha e}{\sqrt{w}} gh - e^2(h^2+g^2)f = 0, \\
& 4w\ddot{g} + 4\dot{g} + \dfrac{\alpha^2}{w} g + \dfrac{2\alpha e}{\sqrt{w}} fh + e^2(f^2-h^2)g = 0, \\
& 4w\ddot{h} + 4\dot{h} - w^{-1} h + \dfrac{2\alpha e}{\sqrt{w}} fg + e^2(f^2-g^2)h = 0. \\
14.1. & \vec{B}_\mu = a_\mu \vec{e}_1 f(w) + d_\mu \vec{e}_2 g(w) + c_\mu \vec{e}_3 h(w), \\
& 16\ddot{f} - e^2(h^2+g^2)f = 0, \quad 16\ddot{g} + e^2(f^2-h^2)g = 0, \\
& 16\ddot{h} + e^2(f^2-g^2)h = 0. \\
14.2. & \vec{B}_\mu = k_\mu \vec{e}_1 f(w) + c_\mu \vec{e}_2 g(w), \quad 16\ddot{f} - e^2 g^2 f = 0, \ddot{g} = 0. \\
15.1. & \vec{B}_\mu = a_\mu \vec{e}_1 f(w) + d_\mu \vec{e}_2 g(w) + (1+\alpha^2)^{-\frac{1}{2}}(\alpha c_\mu + b_\mu)\vec{e}_3 h(w), \\
& 16(1+\alpha^2)\ddot{f} - e^2(h^2+g^2)f = 0, \quad 16(1+\alpha^2)\ddot{g} + e^2(f^2-h^2)g = 0, \\
& 16(1+\alpha^2)\ddot{h} + e^2(f^2-g^2)h = 0. \\
15.2. & \vec{B}_\mu = k_\mu \vec{e}_1 f(w) + (1+\alpha^2)^{-\frac{1}{2}}(\alpha c_\mu + b_\mu)\vec{e}_2 g(w), \\
& 16(1+\alpha^2)\ddot{f} - e^2 f g^2 = 0, \quad \ddot{g} = 0. \\
16. & \vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w), \\
& 4w\ddot{f} + 4\dot{f} - e^2 g^2 f = 0, \quad 4w\ddot{g} + 4\dot{g} - w^{-1} g = 0. \\
18. & \vec{B}_\mu = b_\mu \vec{e}_1 f(w) + c_\mu \vec{e}_2 g(w), \\
& 4w\ddot{f} + 6\dot{f} + e^2 g^2 f = 0, \quad 4w\ddot{g} + 6\dot{g} + e^2 f^2 g = 0. \\
19. & \vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w), \quad \ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.
\end{array}
\tag{5.1}
$$

20. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w), \quad \ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$
21. $\vec{B}_\mu = k_\mu \vec{e}_1 f(w) + b_\mu \vec{e}_2 g(w), \quad \ddot{f} - e^2 g^2 f = 0, \quad \ddot{g} = 0.$
22. $(\alpha = 0)\ \vec{B}_\mu = b_\mu \vec{e}_1 f(w) + c_\mu \vec{e}_2 g(w),$
$4w\ddot{f} + 8\dot{f} + e^2 g^2 f = 0, \quad 4w\ddot{g} + 8\dot{g} + e^2 f^2 g = 0.$

In the above formulae we use designations $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$.

Thus, combining symmetry reduction by the number of independent variables and reduction by the number of dependent variables we reduce YME to rather simple ODE. It is worth reminding that effectiveness of the widely used ansatz for the Yang–Mills field suggested by t'Hooft et al [2] is closely connected with the fact that it reduces the system of twelve PDE to one nonlinear wave equation.

Next, we will briefly consider a procedure of integration of equations (5.1).

Substitution $f = 0$, $g = h = u(w)$ reduces the system of ODE 1 from (5.1) to the equation

$$\ddot{u} = e^2 u^3, \tag{5.2}$$

which is integrated in elliptic functions [8]. Besides that, ODE (5.2) has a solution which is expressed in terms of elementary functions $u = \sqrt{2}(ew - C)^{-1}$, $C \in \mathbb{R}^1$.

ODE 2 with $f = g = h = u(w)$ reduces to the form $\ddot{u} + 2e^2 u^3 = 0$.

This equation is also integrated in elliptic functions [8].

Integrating the second equation of system of ODE 5 we get $g = C_1 w + C_2$, $C_i \in \mathbb{R}^1$. If $C_1 \neq 0$, then the constant C_2 can be neglected, and we may put $C_2 = 0$. Provided $C_1 \neq 0$, the first equation from system 5 reads

$$\ddot{f} - e^2 C_1^2 w^2 f = 0. \tag{5.3}$$

A general solution of ODE (5.3) is given by formula $f = w^{1/2} Z_{\frac{1}{4}}(\frac{ie}{2} C_1 w^2)$.

Hereafter, we use the designation $Z_\nu(w) = C_3 J_\nu(w) + C_4 Y_\nu(w)$, where J_ν, Y_ν are Bessel functions, C_3, C_4 are arbitrary constants.

In the case $C_1 = 0$, $C_2 \neq 0$ a general solution of the first equation from system 5 reads $f = C_3 \cosh C_2 ew + C_4 \sinh C_2 ew$, where C_3, C_4 are arbitrary constants.

At last, provided $C_1 = C_2 = 0$, a general solution of the first equation from system 5 has the form $f = C_3 w + C_4$, $C_3, C_4 \in \mathbb{R}^1$.

A general solution of the second ODE from system 8.1 is of the form $g = C_1\sqrt{w} + C_2(\sqrt{w})^{-1}$, where C_1, C_2 are arbitrary constants.

Substituting the expression obtained into the first equation we get

$$4w^2 \ddot{f} + 4w\dot{f} - e^2 (C_1 w + C_2)^2 f = 0. \tag{5.4}$$

Under $C_1, C_2 \neq 0$ a solution of ODE (5.4) is not known. In the remaining cases its general solution reads

$$\begin{array}{llll} a) & C_1 \neq 0, & C_2 = 0 & f = Z_0\left[\dfrac{ie}{2} C_1 w\right], \\ b) & C_1 = 0, & C_2 \neq 0 & f = C_3 w^{\frac{eC_2}{2}} + C_4 w^{-\frac{eC_2}{2}}, \\ c) & C_1 = 0, & C_2 = 0 & f = C_3 \ln w + C_4. \end{array}$$

Here C_3, C_4 are arbitrary constants.

We do not succeed in obtaining particular solutions of system 8.2. Equations 14.1 coincide with equations 1, if one changes e by $\frac{e}{4}$. Similarly, equations 14.2 coincide with equations 5, if one changes e by $\frac{e}{4}$. Next, equations 15.1 coincide with equations 1 and equations 15.2 – with equations 5, if one replaces e by $\frac{e}{4}(1+\alpha^2)^{-\frac{1}{2}}$.

System of ODE 16 coincides with system 8.1 and systems 19, 20, 21 – with system 5. We did not succeed in integrating equations 18.

At last, system 22 ($\alpha = 0$) with the substitution $f = g = u(w)$ reduces to the form

$$w\ddot{u} + 2\dot{u} + \frac{e^2}{4}u^3 = 0. \tag{5.5}$$

ODE (5.5) is Emden–Fowler equation and the function $u = e^{-1}w^{-\frac{1}{2}}$, is its particular solution.

Substituting the results obtained into corresponding formulae from (5.1) and then into the ansatz (3.13), we get exact solutions of the nonlinear YME (1.1). Let us note that solutions of systems of ODE 5, 8.1, 14.2, 15.2, 16, 19, 20, 21 satisfying the condition $g = 0$ give rise to Abelian solutions of YME. We do not adduce them and present only non-Abelian solutions of YME.

1. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\sqrt{2}(edx - \lambda)^{-1};$

2. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\left[\lambda\,\mathrm{sn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\mathrm{dn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\right]\left[\mathrm{cn}\left(\frac{\sqrt{2}}{2}e\lambda dx\right)\right]^{-1};$

3. $\vec{A}_\mu = (\vec{e}_2 b_\mu + \vec{e}_3 c_\mu)\lambda[\mathrm{cn}\,(e\lambda dx)]^{-1};$

4. $\vec{A}_\mu = (\vec{e}_1 b_\mu + \vec{e}_2 c_\mu + \vec{e}_3 d_\mu)\lambda\,\mathrm{cn}\,(e\lambda ax);$

5. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{cx}Z_{\frac{1}{4}}\left[\frac{i}{2}e\lambda(cx)^2\right] + \vec{e}_2 b_\mu \lambda cx;$

6. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(e\lambda cx) + \lambda_2\sinh(e\lambda cx)] + \vec{e}_2 b_\mu\lambda;$

7. $\vec{A}_\mu = \vec{e}_1 k_\mu Z_0\left[\frac{i}{2}e\lambda((bx)^2 + (cx)^2)\right] + \vec{e}_2(b_\mu cx - c_\mu bx)\lambda;$

8. $\vec{A}_\mu = \vec{e}_1 k_\mu[\lambda_1((bx)^2 + (cx)^2)^{\frac{e\lambda}{2}} + \lambda_2((bx)^2 + (cx)^2)^{-\frac{e\lambda}{2}}] + {}$
   $+ \vec{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2 + (cx)^2)^{-1};$

9. $\vec{A}_\mu = \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\lambda\,\mathrm{sn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx + (kx)^2)\right)\times$
   $\times\,\mathrm{dn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx + (kx)^2)\right)\left(\mathrm{cn}\left(\frac{e\sqrt{2}}{8}\lambda(4bx + (kx)^2)\right)\right)^{-1};$

10. $\vec{A}_\mu = \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\times$
    $\times\,\lambda\left[\mathrm{cn}\left(\frac{e\sqrt{2}\lambda}{8}(4bx + (kx)^2)\right)\right]^{-1};$

11. $\vec{A}_\mu = \left[\vec{e}_2\left(\frac{1}{8}(d_\mu - k_\mu(kx)^2) + \frac{1}{2}b_\mu kx\right) + \vec{e}_3 c_\mu\right]\times$
    $\times\,4\sqrt{2}(e(4bx + (kx)^2) - \lambda)^{-1};$

$$
\begin{aligned}
12.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu \sqrt{4bx+(kx)^2}\, Z_{\frac{1}{4}}\left(\frac{ie\lambda}{8}(4bx+(kx)^2)^2\right) + \vec{e}_2 c_\mu \lambda (4bx+(kx)^2);\\
13.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu \left(\lambda_1 \cosh\left(\frac{e\lambda}{4}(4bx+(kx)^2)\right) + \right.\\
&\quad \left. + \lambda_2 \sinh\left(\frac{e\lambda}{4}(4bx+(kx)^2)\right)\right) + \vec{e}_2 c_\mu \lambda;\\
14.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu (kx)^2 - \frac{1}{2} b_\mu kx\right) + \right.\\
&\quad \left. + \vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\} \times\\
&\quad \times \lambda\, \mathrm{sn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx) + \alpha (kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right] \times\\
&\quad \times \mathrm{dn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx) + \alpha (kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right] \times\\
&\quad \times \left\{\mathrm{cn}\left[\frac{e\lambda\sqrt{2}}{8}(4(\alpha bx - cx) + \alpha (kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\right\}^{-1};\\
15.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu (kx)^2 - \frac{1}{2} b_\mu kx\right) + \right.\\
&\quad \left. + \vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\} \times\\
&\quad \times \left\{\mathrm{cn}\left[\frac{e\lambda}{4}(4(\alpha bx - cx) + \alpha (kx)^2)(1+\alpha^2)^{-\frac{1}{2}}\right]\right\}^{-1};\\
16.\quad \vec{A}_\mu &= \left\{\vec{e}_2\left(d_\mu - \frac{1}{8}k_\mu (kx)^2 - \frac{1}{2} b_\mu kx\right) + \right.\\
&\quad \left. + \vec{e}_3\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)(1+\alpha^2)^{-\frac{1}{2}}\right\} \times\\
&\quad \times 4\sqrt{2}(1+\alpha^2)^{\frac{1}{2}}[e(4(\alpha bx - cx) + \alpha (kx)^2)]^{-1};\\
17.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu \left\{\sqrt{4(\alpha bx - cx) + \alpha (kx)^2} \times \right.\\
&\quad \left. \times Z_{\frac{1}{4}}\left(\frac{ie\lambda}{8}(4(\alpha bx - cx) + \alpha (kx)^2)^2 (1+\alpha^2)^{-\frac{1}{2}}\right)\right\} +\\
&\quad + \vec{e}_2\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)\lambda(4(\alpha bx - cx) + \alpha (kx)^2)(1+\alpha^2)^{-\frac{1}{2}};\\
18.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu \left\{\lambda_1 \cosh\left[\frac{e\lambda}{4}(1+\alpha^2)^{-\frac{1}{2}}(4(\alpha bx - cx) + \alpha (kx)^2)\right] + \right.\\
&\quad \left. + \lambda_2 \sinh\left[\frac{e\lambda}{4}(1+\alpha^2)^{-\frac{1}{2}}(4(\alpha bx - cx) + \alpha (kx)^2\right]\right\} +\\
&\quad + \vec{e}_2\left(\alpha c_\mu + b_\mu + \frac{1}{2}k_\mu kx\right)\lambda(1+\alpha^2)^{-\frac{1}{2}};\\
19.\quad \vec{A}_\mu &= \vec{e}_1 k_\mu |kx|^{-1} Z_0\left[\frac{ie\lambda}{2}((bx)^2 + (cx)^2)\right] + \vec{e}_2 (b_\mu cx - c_\mu bx)\lambda;
\end{aligned}
\tag{5.6}
$$

20. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1((bx)^2+(cx)^2)^{\frac{e\lambda}{2}} + \lambda_2((bx)^2+(cx)^2)^{-\frac{e\lambda}{2}}] +$
$+ \vec{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2+(cx)^2)^{-1};$

21. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{cx}Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(cx)^2\right) + \vec{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda cx;$

22. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(\lambda ecx) + \lambda_2\sinh(\lambda ecx)] + \vec{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda;$

23. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{\ln|kx| - cx}Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(\ln|kx| - cx)^2\right) +$
$+ \vec{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda(\ln|kx| - cx);$

24. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(\lambda e(\ln|kx| - cx)) + \lambda_2\sinh(\lambda e(\ln|kx| - cx))] +$
$+ \vec{e}_2(b_\mu - k_\mu bx(kx)^{-1})\lambda;$

25. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}\sqrt{\alpha\ln|kx| - cx}Z_{\frac{1}{4}}\left(\frac{ie\lambda}{2}(\alpha\ln|kx| - cx)^2\right) +$
$+ \vec{e}_2(b_\mu - k_\mu(bx - \ln|kx|)(kx)^{-1})\lambda(\alpha\ln|kx| - cx);$

26. $\vec{A}_\mu = \vec{e}_1 k_\mu |kx|^{-1}[\lambda_1\cosh(\lambda e(\alpha\ln|kx| - cx)) +$
$+ \lambda_2\sinh(\lambda e(\alpha\ln|kx| - cx))] + \vec{e}_2(b_\mu - k_\mu(bx - \ln|kx|)(kx)^{-1})\lambda;$

27. $\vec{A}_\mu = \{\vec{e}_1(b_\mu - k_\mu bx(kx)^{-1}) + \vec{e}_2(c_\mu - k_\mu cx(kx)^{-1})\}e^{-1}(x_\mu x^\mu)^{-\frac{1}{2}};$

28. $\vec{A}_\mu = \{\vec{e}_1(b_\mu - k_\mu bx(kx)^{-1}) + \vec{e}_2(c_\mu - k_\mu cx(kx)^{-1})\}f(x_\mu x^\mu),$
$w\ddot{f} + 2\dot{f} + (e^2 f^3/4) = 0, \quad w = x_\mu x^\mu = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.$

In the above formulae $Z_\alpha(w)$ is the Bessel function; sn, dn, cn are Jacobi elliptic functions having the modulus $\frac{\sqrt{2}}{2}$; $\lambda, \lambda_1, \lambda_2 = \text{const}$.

In the present paper we do not analyze in detail the obtained solution. We only note that the solutions numbered by 27 is nothing more but the meron solution of YME [2]. In the Euclidean space meron and instanton solutions were obtained by Alfaro, Fubini, Furlan [9] and Belavin, Polyakov, Schwartz, Tyupkin [10] with the use of the ansatz suggested by 't Hooft [11], Corrigan and Fairlie [12] and Wilczek [13].

Another important point is that we can obtain new exact solutions of YME by applying to solutions (5.6) the solution generation technique. We do not adduce corresponding formulae because of their cumbersomity.

6. Some generalizations

It was noticed in [14] that group-invariant solutions of nonlinear PDE could provide us with rather general information about the structure of solutions of the equation under study. Using this fact, we constructed in [4, 14] a number of new exact solutions of the nonlinear Dirac equation which could not be obtained by symmetry reduction procedure. We will demonstrate that the same idea will be effective for constructing new solutions of YME.

Solutions of YME numbered by 7, 8, 19, 20 can be presented in the following unified form:

$$\vec{A}_\mu = k_\mu \vec{B}(kx, cx) + b_\mu \vec{C}(kx, cx), \tag{6.1}$$

where $kx = k_\mu x^\mu$, $cx = c_\mu x^\mu$, $k_\mu = a_\mu + d_\mu$.

Substituting the ansatz (6.1) into YME and splitting the equality obtained with respect to linearly-independent four-vectors with components k_μ, b_μ, c_μ, we get

$$\begin{aligned} &1. \quad \vec{C}_{w_1 w_1} = \vec{0}, \\ &2. \quad \vec{C} \times \vec{C}_{w_1} = \vec{0}, \\ &3. \quad \vec{B}_{w_1 w_1} + e\vec{C}_{w_0} \times \vec{C} + e^2 \vec{C} \times (\vec{C} \times \vec{B}) = \vec{0}. \end{aligned} \tag{6.2}$$

Here we use designations $w_0 = kx$, $w_1 = cx$.

A general solution of the first two equations from (6.2) is given by one of the formulae

$$\begin{aligned} &\text{I.} \quad \vec{C} = \vec{f}(w_0), \\ &\text{II.} \quad \vec{C} = (w_1 + v_0(w_0))\vec{f}(w_0), \end{aligned}$$

where v_0, $\vec{f}$ are arbitrary smooth functions.

Consider the case $\vec{C} = \vec{f}(w_0)$. Substituting this expression into the third equation from (6.2) we have

$$\vec{B}_{w_1 w_1} + e\vec{f}_{w_0} \times \vec{f} + e^2 \vec{f}(\vec{f}\vec{B}) - e^2 \vec{f}^2 \vec{B} = \vec{0}. \tag{6.3}$$

Since equations (6.3) do not contain derivatives of $\vec{B}$ with respect to w_0, they can be considered as a system of ODE with respect to the variable w_1. Multiplying (6.3) by $\vec{f}$ we arrive at the relation $(\vec{B}\vec{f})_{w_1 w_1} = 0$, whence

$$\vec{B}\vec{f} = v_1(w_0)w_1 + v_2(w_0). \tag{6.4}$$

In (6.4) v_1, v_2 are arbitrary smooth enough functions.

With account of (6.4) system (6.3) reads

$$\vec{B}_{w_1 w_1} - e^2 \vec{f}^2 \vec{B} = e\vec{f} \times \vec{f}_{w_0} - e^2 (v_1 w_1 + v_2)\vec{f}.$$

The above linear system of ODE is easily integrated. Its general solution is given by the formula

$$\begin{aligned} \vec{B} = {} & \vec{g}(w_0) \cosh e|\vec{f}|w_1 + \vec{h}(w_0) \sinh e|\vec{f}|w_1 + \\ & + e^{-1}|\vec{f}|^{-2} \vec{f}_{w_0} \times \vec{f} + |\vec{f}|^{-2}(v_1 w_1 + v_2)\vec{f}, \end{aligned} \tag{6.5}$$

where $\vec{g}$, $\vec{h}$ are arbitrary smooth functions.

Substituting (6.5) into (6.4) we get the following restrictions on the choice of the functions $\vec{g}$, $\vec{h}$:

$$\vec{f}\vec{g} = 0, \quad \vec{f}\vec{h} = 0. \tag{6.6}$$

Thus, provided $\vec{C}_{w_1} = 0$, a general solution of the system of ODE (6.3) is given by the formulae (6.5), (6.6). Substituting (6.5) into the initial ansatz (6.1) we obtain the following family of exact solutions of YME:

$$\begin{aligned} \vec{A}_\mu = {} & k_\mu \{\vec{g}(kx) \cosh e|\vec{f}|cx + \vec{h}(kx) \sinh e|\vec{f}|cx + \\ & + e^{-1}|\vec{f}|^{-2} \dot{\vec{f}} \times \vec{f} + (v_1(kx)cx + v_2(kx))\vec{f}\} + b_\mu \vec{f} \end{aligned}$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions satisfying (6.6), $\dot{\vec{f}} = \frac{d\vec{f}}{d\omega_0}$.

The case $\vec{C} = (w_1 + v_0(w_0))\vec{f}(w_0)$ is treated in analogous way. As a result, we obtain the following family of exact solutions of YME:

$$\vec{A}_\mu = k_\mu \Bigg\{ (cx + v_0(kx))^{\frac{1}{2}} \left[\vec{g}(kx) J_{\frac{1}{4}} \left(\frac{ie}{2} |\vec{f}| (\vec{c}\vec{x} + v_0(kx))^2 \right) + \right.$$
$$\left. + \vec{h}(kx) Y_{\frac{1}{4}} \left(\frac{ie}{2} |\vec{f}| (cx + v_0(kx))^2 \right) \right] +$$
$$+ (v_1(kx)cx + v_2(kx))\vec{f} + e^{-1}|\vec{f}|^{-2}\dot{\vec{f}} \times \vec{f} \Bigg\} + b_\mu (cx + v_0(kx))\vec{f},$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_0(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions satisfying (6.6), $J_{\frac{1}{4}}(w)$, $Y_{\frac{1}{4}}(w)$ are the Bessel functions.

Another effective ansatz for the Yang–Mills field is obtained if one replaces in (6.1) cx by bx

$$\vec{A}_\mu = k_\mu \vec{B}(kx, bx) + b_\mu \vec{C}(kx, bx). \tag{6.7}$$

Substitution of (6.7) into YME yields the following system of PDE for $\vec{B}$, $\vec{C}$:

$$\vec{B}_{w_1 w_1} - \vec{C}_{w_0 w_1} - e(\vec{B} \times \vec{C}_{w_1} + 2\vec{B}_{w_1} \times \vec{C} + \vec{C} \times \vec{C}_{w_0}) + e^2 \vec{C} \times (\vec{C} \times \vec{B}) = \vec{0}. \tag{6.8}$$

We succeeded in integrating system (6.8), provided $\vec{C} = \vec{f}(w_0)$. Substituting the result obtained into (6.7), we come to the following family of exact solutions of YME:

$$\vec{A}_\mu = k_\mu \{ (\vec{g} + |\vec{f}|^{-1} \vec{g} \times \vec{f} bx) \cos(e|\vec{f}|bx) + (\vec{h} + |\vec{f}|^{-1} \vec{h} \times \vec{f} bx) \sin(e|\vec{f}|bx) +$$
$$+ e^{-1}|\vec{f}|^{-2} \dot{\vec{f}} \times \vec{f} + (v_1(kx)bx + v_2(kx))\vec{f} \} + b_\mu \vec{f},$$

where $\vec{f}(kx)$, $\vec{g}(kx)$, $\vec{h}(kx)$, $v_1(kx)$, $v_2(kx)$ are arbitrary smooth functions.

Besides that, we obtained the following class of exact solutions of YME:

$$\vec{A}_\mu = k_\mu \vec{e}_1 v_0(kx) u^2(bx) + b_\mu \vec{e}_2 u(bx),$$

where $\vec{e}_1 = (1, 0, 0)$, $\vec{e}_2 = (0, 1, 0)$; $v_0(kx)$ is an arbitrary smooth function; $u(bx)$ is a solution of the nonlinear ODE $\ddot{u} = e^2 u^5$, which is integrated in elliptic functions.

In conclusion of this Section we will obtain a generalization of the plane-wave Coleman solution [15]]

$$\vec{A}_\mu = k_\mu (\vec{f}(kx)bx + \vec{g}(kx)cx). \tag{6.9}$$

It is not difficult to verify that (6.9) satisfy YME with arbitrary $\vec{f}$, $\vec{g}$.

Evidently, solution (6.9) is a particular case of the ansatz

$$\vec{A}_\mu = k_\mu \vec{B}(kx, bx, cx). \tag{6.10}$$

Substituting (6.10) into YME we get

$$\vec{B}_{w_1 w_1} + \vec{B}_{w_2 w_2} = \vec{0}, \tag{6.11}$$

where $w_1 = bx$, $w_2 = cx$.

Integrating the Laplace equations (6.11) and substituting the result obtained into (6.10) we have

$$\vec{A}_\mu = k_\mu(\vec{U}(kx, bx + icx) + \vec{U}(kx, bx - icx)).$$

Here $\vec{U}(kx, z)$ is an arbitrary analytical with respect to z function. Choosing $\vec{U} = \frac{1}{2}(\vec{f}(kx) - i\vec{g}(kx))z$ we get Coleman solution (6.9).

7. Conclusion

Thus, starting from the invariance of YME under the Poincaré group we have obtained wide families of its exact solutions including arbitrary functions. In our future papers we intend to describe exact solutions of YME invariant under the extended Poincaré group and conformal group.

Besides that, we will study exact solutions which correspond to the conditional and non-local symmetries of the Yang–Mills equations (1.1)

Acknowledgments. One of the authors (Wilhelm Fushchych) is indebted to DKNT of Ukraina for financial support (project № 11. 3/42). R.Z. Zhdanov was supported by the Alexander von Humboldt Foundation.

1. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989.
2. Actor A., Classical solutions of $SU(2)$ Yang–Mills theories, *Rev. Mod. Phys.*, 1979, **51**, № 3, 461–525.
3. Schwarz F., Symmetry of $SU(2)$-invariant Yang–Mills theories, *Lett. Math. Phys.*, 1982, **6**, № 5, 355–359.
4. Fushchych W.I., Zhdanov R.Z. Symmetry and exact solutions of nonlinear spinor equations, *Phys. Repts.*, 1989, **46**, № 2, 325–365.
5. Ovsiannikov L.V., Group analysis of differential equations, Moscow, Nauka, 1978.
6. Patera J., Winternitz P., Zassenhaus H., Continuous subgroups of the fundamental groups of physics. 1. General method and the Poincaré group, *J. Math. Phys.*, 1975, **16**, 1597–1624.
7. Fushchych W.I., Barannik L.F., Barannik A.F., Subgroup analysis of the Galilei and Poincaré groups and reduction of nonlinear equations, Kyiv, Naukova Dumka, 1991.
8. Kamke E., Handbook on ordinary differential equations, Moscow, Nauka, 1976.
9. De Alfaro V., Fubini S., Furlan G., A new classical solutions of the Yang–Mills field equations, *Phys. Lett. B*, 1977, **65**, № 2, 163–166.
10. Belavin A.A., Polyakov A.M., Tyupkin Yu.S., Schwartz A.S., Pseudoparticle solutions of the Yang–Mills equations, *Phys. Lett. B*, 1975, **59**, № 1, 85–87.
11. 't Hooft G., Computation of the quantum effects due to a four-dimensional pseudoparticle, *Phys. Rev. D*, 1976, **14**, № 12, 3432–3450.
12. Corrigan E., Fairlie D.B., Scalar field theory and exact solutions to a classical $SU(2)$ gange theory, *Phys. Lett. B*, 1977, **67**, 69–74.
13. Wilczek F., Geometry and interaction of instantons, in Quark Confinement and Field Theory, New York, Wiley, 1977, 211–219.
14. Fushchych W.I., Zhdanov R.Z., Nonlinear spinor equations: symmetry and exact solutions, Kyiv, Naukova Dumka, 1992.
15. Coleman S., Classical lumps and their quantum descendants, in New Phenomena in Subnuclear Physics (Erice, 1975), Editor A. Zichichi, New York, 1977.

On the new approach to variable separation in the time-dependent Schrödinger equation with two space dimensions

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

We suggest an effective approach to separation of variables in the Schrödinger equation with two space variables. Using it we classify inequivalent potentials $V(x_1, x_2)$ such that the corresponding Schrödinger equations admit separation of variables. Besides that, we carry out separation of variables in the Schrödinger equation with the anisotropic harmonic oscillator potential $V = k_1x_1^2 + k_2x_2^2$ and obtain a complete list of coordinate systems providing its separability. Mojority of these coordinate systems depend essentially on the form of the potential and do not provide separation of variables in the free Schrödinger equation ($V = 0$).

1. Introduction

The problem of separation of variables (SV) in the two-dimensional Schrödinger equation

$$iu_t + u_{x_1x_1} + u_{x_2x_2} = V(x_1, x_2)u \tag{1}$$

as well as the most of classical problems of mathematical physics can be formulated in a very simple way (but this simplicity does not, of course, imply an existence of easy way to its solution). To separate variables in Eq. (1) one has to construct such functions $R(t, \boldsymbol{x})$, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$ that the Schrödinger equation (1) after being rewritten in the new variables

$$\begin{aligned} &z_0 = t, \quad z_1 = \omega_1(t, \boldsymbol{x}), \quad z_2 = \omega_2(t, \boldsymbol{x}), \\ &v(z_0, \boldsymbol{z}) = R(t, \boldsymbol{x})u(t, \boldsymbol{x}) \end{aligned} \tag{2}$$

separates into three ordinary differential equations (ODEs). From this point of view the problem of SV in Eq. (1) is studied in [1–4].

But no less of an important problem is the one of description of potentials $V(x_1, x_2)$ such that the Schrödinger equation admits variable separation. That is why saying about SV in Eq. (1) we imply two mutually connected problems. The first one is to describe all such functions $V(x_1, x_2)$ that the corresponding Schrödinger equation (1) can be separated into three ODEs in some coordinate system of the form (2) (classification problem). The second problem is to construct for each function $V(x_1, x_2)$ obtained in this way all coordinate systems (2) enabling us to carry out SV in Eq. (1).

Up to our knowledge, the second problem has been solved provided $V = 0$ [2, 3] and $V = \alpha x_1^{-2} + \beta x_2^{-2}$ [1]. The first one was considered in a restricted sense in [4]. Authors using symmetry approach to classification problem obtained some potentials providing

Reprinted with permission from J. Math. Phys., 1995, **36**, № 10, P. 5506–5521

Original article is available at http://scitation.aip.org/jmp/

separability of Eq. (1) and carried out SV in the corresponding Schrödinger equation. But their results are far from being complete and systematic. The necessary and sufficient conditions imposed on the potential $V(x_1, x_2)$ by the requirement that the Schrödinger equation admits symmetry operators of an arbitrary order are obtained in [5]. But so far there is no systematic and exhaustive description of potentials $V(x_1, x_2)$ providing SV in Eq. (1).

To be able to discuss the description of *all* potentials and *all* coordinate systems making it possible to separate the Schrödinger equation one has to give a definition of SV. One of the possible definitions of SV in partial differential equations (PDEs) is proposed in our article [6]. It is based on the concept of Ansatz suggested by Fushchych [7] and on ideas contained in the article by Koornwinder [8]. The said definition is quite algorithmic in the sense that it contains a regular algorithm of variable separation in partial differential equations which can be easily adapted to handle both linear [6, 9] and nonlinear [10] PDEs. In the present article we apply the said algorithm to solve the problem of SV in Eq. (1).

Consider the following system of ODEs:

$$\begin{aligned} &i\frac{d\varphi_0}{dt} = U_0(t, \varphi_0; \lambda_1, \lambda_2),\\ &\frac{d^2\varphi_1}{d\omega_1^2} = U_1\left(\omega_1, \varphi_1, \frac{d\varphi_1}{d\omega_1}; \lambda_1, \lambda_2\right), \quad \frac{d^2\varphi_2}{d\omega_2^2} = U_2\left(\omega_2, \varphi_2, \frac{d\varphi_2}{d\omega_2}; \lambda_1, \lambda_2\right), \end{aligned} \tag{3}$$

where U_0, U_1, U_2 are some smooth functions of the corresponding arguments, $\lambda_1, \lambda_2 \subset \mathbb{R}^1$ are arbitrary parameters (separation constants) and what is more

$$\operatorname{rank} \left\| \frac{\partial U_\mu}{\partial \lambda_a} \right\|_{\mu=0}^{2} {}_{a=1}^{2} = 2 \tag{4}$$

(the last condition ensures essential dependence of the corresponding solution with separated variables on λ_1, λ_2, see [8]).

Definition 1. *We say that Eq.* (1) *admits SV in the system of coordinates* t, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$ *if substitution of the Ansatz*

$$u = Q(t, \boldsymbol{x})\varphi_0(t)\varphi_1\big(\omega_1(t, \boldsymbol{x})\big)\varphi_2\big(\omega_2(t, \boldsymbol{x})\big) \tag{5}$$

into Eq. (1) *with subsequent exclusion of the derivatives* $d\varphi_0/dt$, $d^2\varphi_1/d\omega_1^2$, $d^2\varphi_2/d\omega_2^2$ *according to Eqs.* (3) *yields an identity with respect to* φ_0, φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2.

Thus, according to the above definition to separate variables in Eq. (1) one has

(i) to substitute the expression (5) into (1),

(ii) to exclude derivatives $d\varphi_0/dt$, $\frac{d^2\varphi_1}{d\omega_1^2}$, $d^2\varphi_2/d\omega_2^2$ with the help of Eqs. (3),

(iii) to split the obtained equality with respect to the variables φ_0, φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2 considered as independent.

As a result one gets some over-determined system of PDEs for the functions $Q(t, \boldsymbol{x})$, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$. On solving it one obtains a complete description of all coordinate systems and potentials providing SV in the Schrödinger equation. Naturally, an expression *complete description* makes sense only within the framework of our definition. So if one uses a more general definition it may be possible to construct new coordinate systems and potentials providing separability of Eq. (1). But all solutions of the Schrödinger equation with separated variables known to us fit into the scheme suggested by us and can be obtained in the above described way.

2. Classification of potentials $V(x_1, x_2)$

We do not adduce in full detail computations needed because they are very cumbersome. We shall restrict ourselves to pointing out main steps of the realization of the above suggested algorithm.

First of all we make a remark, which makes life a little bit easier. It is readily seen that a substitution of the form

$$\begin{gathered} Q \to Q' = Q\Psi_1(\omega_1)\Psi_2(\omega_2), \\ \omega_a \to \omega_a' = \Omega_a(\omega_a), \quad a = 1,2, \quad \lambda_a \to \lambda_a' = \Lambda_a(\lambda_1, \lambda_2), \quad a = 1,2, \end{gathered} \tag{6}$$

does not alter the structure of relations (3), (4), and (5). That is why, we can introduce the following equivalence relation:

$$(\omega_1, \omega_2, Q) \ \sim \ (\omega_1', \omega_2', Q')$$

provided Eq. (6) holds with some Ψ_a, Ω_a, Λ_a.

Substituting Eq. (5) into Eq. (1) and excluding the derivatives $d\varphi_0/dt$, $d^2\varphi_1/d\omega_1^2$, $d^2\varphi_2/d\omega_2^2$ with the use of equations (3) we get

$$\begin{aligned} & i(Q_t\varphi_0\varphi_1\varphi_2 + QU_0\varphi_1\varphi_2 + Q\omega_{1t}\varphi_0\dot\varphi_1\varphi_2 + Q\omega_{2t}\varphi_0\varphi_1\dot\varphi_2) + (\triangle Q)\varphi_0\varphi_1\varphi_2 + \\ & \quad + 2Q_{x_a}\omega_{1x_a}\varphi_0\dot\varphi_1\varphi_2 + 2Q_{x_a}\omega_{2x_a}\varphi_0\varphi_1\dot\varphi_2 + Q\big((\triangle\omega_1)\varphi_0\dot\varphi_1\varphi_2 + \\ & \quad + (\triangle\omega_2)\varphi_0\varphi_1\dot\varphi_2 + \omega_{1x_a}\omega_{1x_a}\varphi_0U_1\varphi_2 + \omega_{2x_a}\omega_{2x_a}\varphi_0\varphi_1U_2 + \\ & \quad + 2\omega_{1x_a}\omega_{2x_a}\varphi_0\dot\varphi_1\dot\varphi_2\big) = VQ\varphi_0\varphi_1\varphi_2, \end{aligned}$$

where the summation over the repeated index a from 1 to 2 is understood. Hereafter an overdot means differentiation with respect to a corresponding argument and $\triangle = \partial_{x_1}^2 + \partial_{x_2}^2$.

Splitting the equality obtained with respect to independent variables φ_1, φ_2, $d\varphi_1/d\omega_1$, $d\varphi_2/d\omega_2$, λ_1, λ_2 we conclude that ODEs (3) are linear and up to the equivalence relation (6) can be written in the form

$$\begin{aligned} & i\frac{d\varphi_0}{dt} = \big(\lambda_1R_1(t) + \lambda_2R_2(t) + R_0(t)\big)\varphi_0, \\ & \frac{d^2\varphi_1}{d\omega_1^2} = \big(\lambda_1B_{11}(\omega_1) + \lambda_2B_{12}(\omega_1) + B_{01}(\omega_1)\big)\varphi_1, \\ & \frac{d^2\varphi_2}{d\omega_2^2} = \big(\lambda_1B_{21}(\omega_2) + \lambda_2B_{22}(\omega_2) + B_{02}(\omega_2)\big)\varphi_2 \end{aligned}$$

and what is more, functions ω_1, ω_2, Q satisfy an over-determined system of nonlinear PDEs

$$\begin{aligned} & (1) \quad \omega_{1x_b}\omega_{2x_b} = 0, \\ & (2) \quad B_{1a}(\omega_1)\omega_{1x_b}\omega_{1x_b} + B_{2a}(\omega_2)\omega_{2x_b}\omega_{2x_b} + R_a(t) = 0, \quad a = 1,2, \\ & (3) \quad 2\omega_{ax_b}Q_{x_b} + Q(i\omega_{at} + \triangle\omega_a), \quad a = 1,2, \end{aligned} \tag{7}$$

$$(4)\quad \big(B_{01}(\omega_1)\omega_{1x_b}\omega_{1x_b} + B_{02}(\omega_1)\omega_{2x_b}\omega_{2x_b}\big)Q + iQ_t + \triangle Q + R_0(t)Q - \\ - V(x_1, x_2)Q = 0.$$

Thus, to solve the problem of SV for the linear Schrödinger equation it is necessary to construct general solution of system of nonlinear PDEs (7). Roughly speaking, to solve a linear equation one has to solve a system of *nonlinear equations*! This is the reason why so far there is no complete description of all coordinate systems providing separability of the four-dimensional wave equation [3].

But in the case involved we have succeeded in integrating system of nonlinear PDEs (7). Our approach to integration of it is based on the following change of variables (hodograph transformation)

$$z_0 = t, \quad z_1 = Z_1(t, \omega_1, \omega_2), \quad z_2 = Z_2(t, \omega_1, \omega_2), \quad v_1 = x_1, \quad v_2 = x_2,$$

where z_0, z_1, z_2 are new independent and v_1, v_2 are new dependent variables correspondingly.

Using the hodograph transformation determined above we have constructed the general solution of Eqs. (1)–(3) from Eq. (7). It is given up to the equivalence relation (6) by one of the following formulas:

$$\begin{aligned}
&(1)\quad \omega_1 = A(t)x_1 + W_1(t), \quad \omega_2 = B(t)x_2 + W_2(t),\\
&\qquad Q(t,\boldsymbol{x}) = \exp\left\{-\frac{i}{4}\left(\frac{\dot A}{A}x_1^2 + \frac{\dot B}{B}x_2^2\right) - \frac{i}{2}\left(\frac{\dot W_1}{A}x_1 + \frac{\dot W_2}{B}x_2\right)\right\};\\
&(2)\quad \omega_1 = \frac{1}{2}\ln(x_1^2 + x_2^2) + W(t), \quad \omega_2 = \arctan\frac{x_1}{x_2},\\
&\qquad Q(t,\boldsymbol{x}) = \exp\left\{-\frac{i\dot W}{4}(x_1^2 + x_2^2)\right\};\\
&(3)\quad x_1 = \frac{1}{2}W(t)(\omega_1^2 - \omega_2^2) + W_1(t), \quad x_2 = W(t)\omega_1\omega_2 + W_2(t),\\
&\qquad Q(t,\boldsymbol{x}) = \exp\left\{\frac{i\dot W}{4W}\big((x_1 - W_1)^2 + (x_2 - W_2)^2\big) + \frac{i}{2}(\dot W_1 x_1 + \dot W_2 x_2)\right\};\\
&(4)\quad x_1 = W(t)\cosh\omega_1\cos\omega_2 + W_1(t), \quad x_2 = W(t)\sinh\omega_1\sin\omega_2 + W_2(t),\\
&\qquad Q(t,\boldsymbol{x}) = \exp\left\{\frac{i\dot W}{4W}\big((x_1 - W_1)^2 + (x_2 - W_2)^2\big) + \frac{i}{2}(\dot W_1 x_1 + \dot W_2 x_2)\right\};
\end{aligned} \tag{8}$$

Here A, B, W, W_1, W_2 are arbitrary smooth functions on t.

Substituting the obtained expressions for the functions Q, ω_1, ω_2 into the last equation from the system (7) and splitting with respect to variables x_1, x_2 we get explicit forms of potentials $V(x_1, x_2)$ and systems of nonlinear ODEs for unknown functions $A(t)$, $B(t)$, $W(t)$, $W_1(t)$, $W_2(t)$. We have succeeded in integrating these and in constructing all coordinate systems providing SV in the initial equation (1).

Here we consider in detail integration of the fourth equation of system (7) for the case 2 from Eq. (8), since computations needed are not so lengthy as for other cases.

First, we make several important remarks which introduce an equivalence relation on the set of potentials $V(x_1, x_2)$.

Remark 1. The Schrödinger equation with the potential

$$V(x_1, x_2) = k_1 x_1 + k_2 x_2 + k_3 + V_1(k_2 x_1 - k_1 x_2), \tag{9}$$

where k_1, k_2, k_3 are constants, is transformed to the Schrödinger equation with the potential

$$V'(x_1', x_2') = V_1(k_2 x_1' - k_1 x_2') \tag{10}$$

by the following change of variables:

$$\begin{aligned} &t' = t, \quad \boldsymbol{x}' = \boldsymbol{x} + t^2\boldsymbol{k}, \\ &u' = u \exp\left\{\frac{i}{3}(k_1^2 + k_2^2)t^3 + it(k_1 x_1 + k_2 x_2) + ik_3 t\right\}. \end{aligned} \tag{11}$$

It is readily seen that the class of Ansätze (5) is transformed into itself by the above change of variables. That is why, potentials (9) and (10) are considered as equivalent.

Remark 2. The Schrödinger equation with the potential

$$V(x_1, x_2) = k(x_1^2 + x_2^2) + V_1\left(\frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1} \tag{12}$$

with $k = \mathrm{const}$ is reduced to the Schrödinger equation with the potential

$$V'(x_1, x_2) = V_1\left(\frac{x_1'}{x_2'}\right)(x_1'^2 + x_1'^2)^{-1} \tag{13}$$

by the change of variables

$$t' = \alpha(t), \quad \boldsymbol{x}' = \beta(t)\boldsymbol{x}, \quad u' = u\exp\{i\gamma(t)(x_1^2 + x_2^2) + \delta(t)\},$$

where $\big(\alpha(t), \beta(t), \gamma(t), \delta(t)\big)$ is an arbitrary solution of the system of ODEs

$$\dot\gamma - 4\gamma^2 = k, \quad \dot\beta - 4\gamma\beta = 0, \quad \dot\alpha - \beta^2 = 0, \quad \dot\delta + 4\gamma = 0$$

such that $\beta \neq 0$.

Since the above change of variables does not alter the structure of the Ansatz (5), when classifying potentials $V(x_1, x_2)$ providing separability of the corresponding Schrödinger equation, we consider potentials (12), (13) as equivalent.

Remark 3. It is well-known (see e.g. [11, 12]) that the general form of the invariance group admitted by Eq. (1) is as follows

$$t' = F(t, \boldsymbol{\theta}), \quad x_a' = g_a(t, \boldsymbol{x}, \boldsymbol{\theta}), \quad a = 1, 2, \quad u' = h(t, \boldsymbol{x}, \boldsymbol{\theta})u + U(t, \boldsymbol{x}),$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$ are group parameters and $U(t, \boldsymbol{x})$ is an arbitrary solution of Eq. (1).

The above transformations also do not alter the structure of the Ansatz (5). That is why, systems of coordinates t', x_1', x_2' and t, x_1, x_2 are considered as equivalent.

Now we turn to the integration of the fourth equation of system (7). Substituting into it the expressions for the functions ω_1, ω_2, Q given by formulas (2) from Eq. (8) we get

$$\begin{aligned} V(x_1, x_2) = {} & \big(B_{01}(\omega_1) + B_{02}(\omega_2)\big)\exp\{-2(\omega_1 - W)\} + \frac{1}{4}(\ddot W - \dot W^2) \times \\ & \times \exp\{2(\omega_1 - W)\} + R_0(t) - i\dot W. \end{aligned} \tag{14}$$

In the above equality B_{01}, B_{02}, $R_0(t)$, $W(t)$ are unknown functions to be determined from the requirement that the right-hand side of (14) does not depend on t.

Differentiating Eq. (14) with respect to t and taking into account the equalities

$$\omega_{1t} = \dot{W}, \quad \omega_{2t} = 0$$

we have

$$\dot{W} \exp\{-2(\omega_1 - W)\} \dot{B}_{01} + \dot{\alpha}(t) \exp\{2(\omega_1 - W)\} + \dot{\beta}(t) = 0, \tag{15}$$

where $\alpha(t) = \frac{1}{4}(\ddot{W} - \dot{W}^2)$, $\beta(t) = R_0 - i\dot{W}$.

Cases $\dot{W} = 0$ and $\dot{W} \neq 0$ have to be considered separately.

Case 1. $\dot{W} = 0$. In this case $W = C = \text{const}$, $R_0 = 0$. Since coordinate systems ω_1, ω_2 and $\omega_1 + C_1$, $\omega_2 + C_2$ are equivalent with arbitrary constants C_1, C_2, choosing $C_1 = -C$, $C_2 = 0$ we can put $C = 0$. Hence it immediately follows that

$$V(x_1, x_2) = \left[B_{01} \left(\frac{1}{2} \ln(x_1^2 + x_2^2) \right) + B_{02} \left(\arctan \frac{x_1}{x_2} \right) \right] (x_1^2 + x_2^2)^{-1},$$

where B_{01}, B_{02} are arbitrary functions. And what is more, the Schrödinger equation (1) with such potential separates only in one coordinate system

$$\omega_1 = \frac{1}{2} \ln(x_1^2 + x_2^2), \quad \omega_2 = \arctan \frac{x_1}{x_2}. \tag{16}$$

Case 2. $\dot{W} \neq 0$. Dividing Eq. (14) into $\dot{W} \exp\{-2(\omega_1 - W)\}$ and differentiating the equality obtained with respect to t we get

$$\exp\{4\omega_1\} \frac{d}{dt} \left(\dot{\alpha} (\dot{W})^{-1} \exp\{-4W\} \right) + \exp\{2\omega_1\} \frac{d}{dt} \left(\dot{\beta} (\dot{W})^{-1} \exp\{-2W\} \right) = 0,$$

whence

$$\frac{d}{dt} \left(\dot{\alpha} (\dot{W})^{-1} \exp\{-4W\} \right) = 0, \quad \frac{d}{dt} \left(\dot{\beta} (\dot{W})^{-1} \exp\{-2W\} \right) = 0.$$

Integration of the above ODEs yields the following result:

$$\alpha = C_1 \exp\{4W\} + C_2, \quad \beta = C_3 \exp\{2W\} + C_4,$$

where $C_\jmath$, $\jmath = \overline{1, 4}$ are arbitrary real constants.

Inserting the result obtained into Eq. (15) we get an equation for B_{01}

$$\dot{B}_{01} = -4C_1 \exp\{4\omega_1\} - 2C_3 \exp\{2\omega_1\},$$

which general solution reads

$$B_{01} = -C_1 \exp\{4\omega_1\} - C_3 \exp\{2\omega_1\} + C_5.$$

In the above equality C_5 is an arbitrary real constant.

Substituting the expressions for α, β, B_{01} into Eq. (14) we have the explicit form of the potential $V(x_1, x_2)$

$$V(x_1, x_2) = \left[B_{02} \left(\arctan \frac{x_1}{x_2} \right) + C_5 \right] (x_1^2 + x_2^2)^{-1} + C_2 (x_1^2 + x_2^2) + C_4,$$

where B_{02} is an arbitrary function.

By force of the Remarks 1, 2 we can choose $C_2 = C_4 = 0$. Furthermore, due to arbitrariness of the function B_{02} we can put $C_5 = 0$.

Thus, the case $\dot{W} \neq 0$ leads to the following potential:

$$V(x_1, x_2) = B_{02}\left(\arctan\frac{x_1}{x_2}\right)(x_1^2 + x_2^2)^{-1}. \tag{17}$$

Substitution of the above expression into Eq. (14) yields second-order nonlinear ODE for the function $W = W(t)$

$$\ddot{W} - \dot{W}^2 = 4C_1 \exp\{4W\}, \tag{18}$$

while the function R_0 is given by the formula

$$R_0 = i\dot{W} + C_3 \exp\{2W\}.$$

Integration of ODE (18) is considered in detail in the Appendix A. Its general solution has the form

under $C_1 \neq 0$

$$W = -\frac{1}{2}\ln\big((at - b)^2 - 4C_1\big) + \frac{1}{2}\ln a,$$

under $C_1 = 0$

$$W = a - \ln(t + b).$$

Substituting obtained expressions for W into formulas (2) from (8) and taking into account the Remark 3 we arrive at the conclusion that the Schrödinger equation (1) with the potential (17) admits SV in two coordinate systems. One of them is the polar coordinate system (16) and another one is the following:

$$\omega_1 = \frac{1}{2}\ln(x_1^2 + x_2^2) - \frac{1}{2}\ln(t^2 \pm 1), \quad \omega_2 = \arctan\frac{x_1}{x_2}. \tag{19}$$

Consequently, the case 2 from Eq. (8) gives rise to two classes of the separable Schrödinger equations (1).

Cases 1, 3, 4 from Eq. (8) are considered in an analogous way but computations involved are much more cumbersome. As a result, we obtain the following list of inequivalent potentials $V(x_1, x_2)$ providing separability of the Schrödinger equation.

(1) $V(x_1, x_2) = V_1(x_1) + V_2(x_2)$;

(a) $V(x_1, x_2) = k_1x_1^2 + k_2x_1^{-2} + V_2(x_2)$, $k_2 \neq 0$;

(i) $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_1^{-2} + k_4x_2^{-2}$, $k_3k_4 \neq 0$, $k_1^2 + k_2^2 \neq 0$, $k_1 \neq k_2$;

(ii) $V(x_1, x_2) = k_1x_1^2 + k_2x_1^{-2}$, $k_1k_2 \neq 0$;

(iii) $V(x_1, x_2) = k_1x_1^{-2} + k_2x_2^{-2}$;

(b) $V(x_1, x_2) = k_1x_1^2 + V_2(x_2)$;

(i) $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_2^{-2}$, $k_1k_3 \neq 0$, $k_1 \neq k_2$;

(ii) $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2$, $k_1k_2 \neq 0$, $k_1 \neq k_2$;

(iii) $V(x_1, x_2) = k_1x_1^2 + k_2x_2^{-2}$, $k_1 \neq 0$;

(2) $V(x_1,x_2) = V_1(x_1^2+x_2^2) + V_2(x_1/x_2)(x_1^2+x_2^2)^{-1}$;
 (a) $V(x_1,x_2) = V_2(x_1/x_2)(x_1^2+x_2^2)^{-1}$;
 (b) $V(x_1,x_2) = k_1(x_1^2+x_2^2)^{-1/2}$, $k_1 \neq 0$;
(3) $V(x_1,x_2) = (V_1(\omega_1)+V_2(\omega_2))(\omega_1^2+\omega_2^2)^{-1}$, where $\omega_1^2-\omega_2^2 = 2x_1$, $\omega_1\omega_2 = x_2$;
(4) $V(x_1,x_2) = (V_1(\omega_1)+V_2(\omega_2))(\sinh^2\omega_1+\sin^2\omega_2)^{-1}$, where $\cosh\omega_1\cos\omega_2 = x_1$, $\sinh\omega_1\sin\omega_2 = x_2$;
(5) $V(x_1,x_2) = 0$.

In the above formulas V_1, V_2 are arbitrary smooth functions, k_1, k_2, k_3, k_4 are arbitrary constants.

It should be emphasized that the above potentials are not inequivalent in a usual sense. These potentials differ from each other by the fact that the coordinate systems providing separability of the corresponding Schrödinger equations are different. As an illustration, we give the Fig. 1, where $r = (x_1^2+x_2^2)^{1/2}$ and by the symbol $V^{(\jmath)}$, $\jmath = 1,4$ we denote the potential given in the above list under the number $\jmath$. Down arrows in the Fig. 1 indicate specifications of the potential $V(x_1,x_2)$ providing new possibilities to separate the corresponding Schrödinger equation (1).

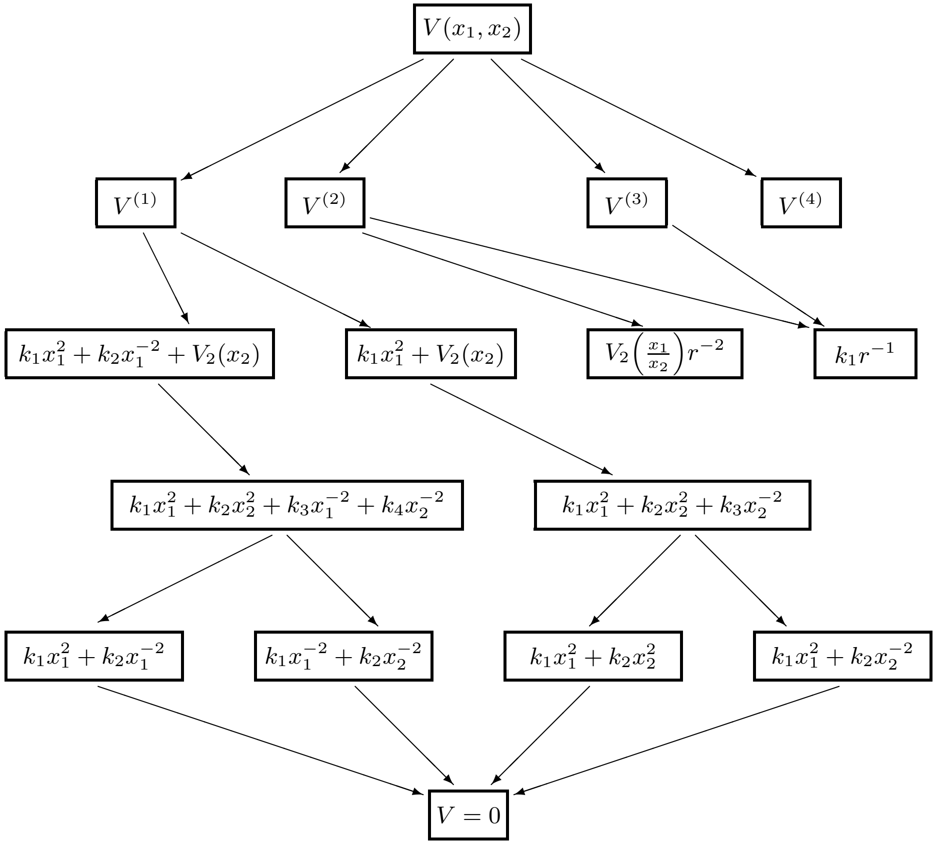


Figure 1.

The Schrödinger equation (1) with arbitrary function $V(x_1,x_2)$ (level 1 of the Fig. 1) admits no separation of variables. Next, Eq. (1) with the "root" potentials $V^{(\jmath)}$

(level 2), V_1, V_2 being arbitrary smooth functions, separates in the Cartesian ($\jmath = 1$), polar ($\jmath = 2$), parabolic ($\jmath = 3$) and elliptic ($\jmath = 4$) coordinate systems, correspondingly. Specifying the functions V_1, V_2 (i.e. going down to the lower levels) new possibilities to separate variables in the Schrödinger equation (1) arise. For example, Eq. (1) with the potential $V_2(x_1/x_2)r^{-2}$, which is a particular case of the potential $V^{(2)}$, separates not only in the polar coordinate system (16) but also in the coordinate systems (19). The Schrödinger equation with the Coulomb potential $k_1 r^{-1}$, which is a particular case of the potentials $V^{(2)}$, $V^{(3)}$, separates in two coordinate systems (namely, in the polar and parabolic coordinate systems, see below the Theorem 4). An another characteristic example is a transition from the potential $V^{(1)}$ to the potential $k_1 x_1^2 + V_2(x_2)$. The Schrödinger equation with the potential $V^{(1)}$ admits SV in the Cartesian coordinate system $\omega_0 = t$, $\omega_1 = x_1$, $\omega_2 = x_2$ only, while the one with the potential $k_1 x_1^2 + V_2(x_2)$ separates in seven ($k_1 < 0$) or in three ($k_1 > 0$) coordinate systems.

A complete list of coordinate systems providing SV in the Schrödinger equations with the above given potentials takes two dozen pages. Therefore, we restrict ourself to considering the Schrödinger equation with anisotropic harmonic oscillator potential $V(x_1, x_2) = k_1 x_1^2 + k_2 x_2^2$, $k_1 \neq k_2$ and Coulomb potential $V(x_1, x_2) = k_1(x_1^2 + x_2^2)^{-1/2}$.

3. Separation of variables in the Schrödinger equation with the anisotropic harmonic oscillator and the Coulomb potentials

Here we will obtain all coordinate systems providing separability of the Schrödinger equation with the potential $V(x_1, x_2) = k_1 x_1^2 + k_2 x_2^2$

$$iu_t + u_{x_1 x_1} + u_{x_2 x_2} = (k_1 x_1^2 + k_2 x_2^2)u. \qquad (20)$$

In the following, we consider the case $k_1 \neq k_2$, because otherwise Eq. (1) is reduced to the free Schrödinger equation (see the Remark 2) which has been studied in detail in [1–3].

Explicit forms of the coordinate systems to be found depend essentially on the signs of the parameters k_1, k_2. We consider in detail the case, when $k_1 < 0$, $k_2 > 0$ (the cases $k_1 > 0$, $k_2 > 0$ and $k_1 < 0$, $k_2 < 0$ are handled in an analogous way). It means that Eq. (20) can be written in the form

$$iu_t + u_{x_1 x_1} + u_{x_2 x_2} + \frac{1}{4}(a^2 x_1^2 - b^2 x_2^2)u = 0, \qquad (21)$$

where a, b are arbitrary non-null real constants (the factor $\frac{1}{4}$ is introduced for further convenience).

As stated above to describe all coordinate systems t, $\omega_1(t, \boldsymbol{x})$, $\omega_2(t, \boldsymbol{x})$ providing separability of Eq. (20) one has to construct the general solution of system (8) with $V(x_1, x_2) = -\frac{1}{4}(a^2 x_1^2 - b^2 x_2^2)$. The general solution of Eqs. (1)–(3) from Eq. (7) splits into four inequivalent classes listed in Eq. (8). Analysis shows that only solutions belonging to the first class can satisfy the fourth equation of (7).

Substituting the expressions for ω_1, ω_2, Q given by the formulas (1) from (8) into the equation 4 from (7) with $V(x_1,x_2) = -\frac{1}{4}(a^2x_1^2 - b^2x_2^2)$ and splitting with respect to x_1, x_2 one gets

$$B_{01}(\omega_1) = \alpha_1\omega_1^2 + \alpha_2\omega_1, \quad B_{02}(\omega_2) = \beta_1\omega_2^2 + \beta_2\omega_2,$$

$$\left(\frac{\dot{A}}{A}\right)^{\cdot} - \left(\frac{\dot{A}}{A}\right)^2 - 4\alpha_1 A^4 + a^2 = 0, \tag{22}$$

$$\left(\frac{\dot{B}}{B}\right)^{\cdot} - \left(\frac{\dot{B}}{B}\right)^2 - 4\beta_1 B^4 - b^2 = 0, \tag{23}$$

$$\ddot{\theta}_1 - 2\dot{\theta}_1\frac{\dot{A}}{A} - 2(2\alpha_1\theta_1 + \alpha_2)A^4 = 0, \tag{24}$$

$$\ddot{\theta}_2 - 2\dot{\theta}_2\frac{\dot{B}}{B} - 2(2\beta_1\theta_2 + \beta_2)B^4 = 0. \tag{25}$$

Here α_1, α_2, β_1, β_2 are arbitrary real constants.

Integration of the system of nonlinear ODEs (22)–(25) is carried out in the Appendix A. Substitution of the formulas (A.2), (A.4)–(A.6), (A.8)–(A.11) into the corresponding expressions 1 from (8) yields a complete list of coordinate systems providing separability of the Schrödinger equation (21). These systems can be transformed to canonical form if we use the Remark 3.

The invariance group of Eq. (21) is generated by the following basis operators [11]:

$$\begin{aligned}
&P_0 = \partial_t, \quad I = u\partial_u, \quad M = iu\partial_u, \quad Q_\infty = U(t,\boldsymbol{x})\partial_u,\\
&P_1 = \cosh at\partial_{x_1} + \frac{ia}{2}(x_1\sinh at)u\partial_u,\\
&P_2 = \cos bt\partial_{x_2} - \frac{ib}{2}(x_2\sin bt)u\partial_u,\\
&G_1 = \sinh at\partial_{x_1} + \frac{ia}{2}(x_1\cosh at)u\partial_u,\\
&G_2 = \sin bt\partial_{x_2} + \frac{ib}{2}(x_2\cos bt)u\partial_u,
\end{aligned} \tag{26}$$

where $U(t,\boldsymbol{x})$ is an arbitrary solution of Eq. (21).

Using the finite transformations generated by the infinitesimal operators (26) and the Remark 3 we can choose in the formulas (A.4)–(A.6), (A.8), (A.10), (A.11) $C_3 = C_4 = D_1 = 0$, $D_3 = D_4 = 0$, $C_2 = D_2 = 1$. As a result, we come to the following assertion.

Theorem 1. *The Schrödinger equation* (21) *admits SV in* 21 *inequivalent coordinate systems of the form*

$$\omega_0 = t, \quad \omega_1 = \omega_1(t,\boldsymbol{x}), \quad \omega_2 = \omega_2(t,\boldsymbol{x}), \tag{27}$$

where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 1.

Table 1. Coordinate systems proving SV in Eq. (21).

$\omega_1(t,\boldsymbol{x})$	$\omega_2(t,\boldsymbol{x})$
$x_1\big(\sinh a(t+C)\big)^{-1}+\alpha\big(\sinh a(t+C)\big)^{-2}$	$x_2(\sin bt)^{-1}+\beta(\sin bt)^{-2}$
$x_1\big(\cosh a(t+C)\big)^{-1}+\alpha\big(\cosh a(t+C)\big)^{-2}$	$x_2(\beta+\sin 2bt)^{-1/2}$
$x_1\exp(\pm at)+\alpha\exp(\pm 4at)$	x_2
$x_1\big(\alpha+\sinh 2a(t+C)\big)^{-1/2}$	
$x_1\big(\alpha+\cosh 2a(t+C)\big)^{-1/2}$	
$x_1\big(\alpha+\exp(\pm 2at)\big)^{-1/2}$	
x_1	

Here C, α, β are arbitrary real constants.

There is no necessity to consider specially the case when in Eq. (20) $k_1>0$, $k_2<0$, since such an equation by the change of independent variables $u(t,x_1,x_2)\to u(t,x_2,x_1)$ is reduced to Eq. (21).

Below we adduce without proof the assertions describing coordinate systems providing SV in Eq. (20) with $k_1<0$, $k_2<0$ and $k_1>0$, $k_2>0$.

Theorem 2. *The Schrödinger equation*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}+\frac{1}{4}(a^2x_1^2+b^2x_2^2)u=0 \tag{28}$$

with $a^2\neq 4b^2$ admits SV in 49 inequivalent coordinate systems of the form (27), where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 2. Provided $a^2=4b^2$ one more coordinate system should be included into the above list, namely

$$\omega_0=t,\quad \omega_1^2-\omega_2^2=2x_1,\quad \omega_1\omega_2=x_2. \tag{29}$$

Table 2. Coordinate systems proving SV in Eq. (28).

$\omega_1(t,\boldsymbol{x})$	$\omega_2(t,\boldsymbol{x})$
$x_1\big(\sinh a(t+C)\big)^{-1}+\alpha\big(\sinh a(t+C)\big)^{-2}$	$x_2(\sinh bt)^{-1}+\beta(\sinh bt)^{-2}$
$x_1\big(\cosh a(t+C)\big)^{-1}+\alpha\big(\cosh a(t+C)\big)^{-2}$	$x_2(\cosh bt)^{-1}+\beta(\cosh bt)^{-2}$
$x_1\exp(\pm at)+\alpha\exp(\pm 4at)$	$x_2\exp(\pm bt)+\beta\exp(\pm 4bt)$
$x_1\big(\alpha+\sinh 2a(t+C)\big)^{-1/2}$	$x_2(\beta+\sinh 2bt)^{-1/2}$
$x_1\big(\alpha+\cosh 2a(t+C)\big)^{-1/2}$	$x_2(\beta+\cosh 2bt)^{-1/2}$
$x_1\big(\alpha+\exp(\pm 2at)\big)^{-1/2}$	$x_2\big(\beta+\exp(\pm 2bt)\big)^{-1/2}$
x_1	x_2

Here C, α, β are arbitrary constants.

Theorem 3. *The Schrödinger equation*

$$iu_t+u_{x_1x_1}+u_{x_2x_2}-\frac{1}{4}(a^2x_1^2+b^2x_2^2)u=0 \tag{30}$$

with $a^2 \neq 4b^2$ admits SV in 9 inequivalent coordinate systems of the form (27)*, where ω_1 is given by one of the formulas from the first and ω_2 by one of the formulas from the second column of the Table 3. Provided $a^2 = 4b^2$, the above list should be supplemented by the coordinate system* (29)*.*

Table 3. Coordinate systems proving SV in Eq. (30).

$\omega_1(t, \boldsymbol{x})$	$\omega_2(t, \boldsymbol{x})$
$x_1\big(\sin a(t+C)\big)^{-1} + \alpha\big(\sin a(t+C)\big)^{-2}$	$x_2(\sin bt)^{-1} + \beta(\sin bt)^{-2}$
$x_1\big(\beta + \sin 2a(t+C)\big)^{-1/2}$	$x_2(\beta + \sin 2bt)^{-1/2}$
x_1	x_2

Here C, α, β are arbitrary constants.

Remark 4. If we consider Eq. (1) as an equation for a complex-valued function u of three complex variables t, x_1, x_2, then the cases considered in the Theorems 1–3 are equivalent. Really, replacing, when necessary, a with ia and b by ib we can always reduce Eqs. (21), (28) to the form (30). It means that coordinate systems presented in the Tables 1, 2 are complex equivalent to those listed in the Table 3. But if u is a complex-valued function of real variables t, x_1, x_2 it is not the case.

Theorem 4. *The Schrödinger equation with the Coulomb potential*

$$iu_t + u_{x_1x_1} + u_{x_2x_2} - k_1(x_1^2 + x_2^2)^{-1/2}u = 0$$

admits SV in two coordinate systems (16)*,* (29)*.*

It is important to note that explicit forms of coordinate systems providing separability of Eqs. (21), (28), (30) depend essentially on the parameters a, b contained in the potential $V(x_1, x_2)$. It means that the free Schrödinger equation ($V = 0$) does not admit SV in such coordinate systems. Consequently, they are essentially new.

4. Conclusion

In the present paper we have studied the case when the Schrödinger equation (1) separates into one first-order and two second-order ODEs. It is not difficult to prove that there are no functions $Q(t, \boldsymbol{x})$, $\omega_\mu(t, \boldsymbol{x})$, $\mu = 0, 1, 2$ such that the Ansatz

$$u = Q(t, \boldsymbol{x})\varphi_0(\omega_0(t, \boldsymbol{x}))\varphi_1\big(\omega_1(t, \boldsymbol{x})\big)\varphi_2\big(\omega_2(t, \boldsymbol{x})\big)$$

separates Eq. (1) into three second-order *ODEs* (see Appendix B). Nevertheless, there exists a possibility for Eq. (1) to be separated into two first-order and one second-order ODEs or into three first-order ODEs. This is a probable source of new potentials and new coordinate systems providing separability of the Schrödinger equation. It should be said that separation of the two-dimensional wave equation

$$u_{tt} - u_{xx} = V(x)u$$

into one first-order and one second-order ODEs gives no new potentials as compared with separation of it into two second-order ODEs. But for some already known potentials new coordinate system providing separability of the above equation are obtained [9].

Let us briefly analyze a connection between separability of Eq. (1) and its symmetry properties. It is well-known that each solution of the free Schrödinger equation with separated variables is a common eigenfunction of two mutually commuting second-order symmetry operators of the said equation [2, 3]. And what is more, separation constants λ_1, λ_2 are eigenvalues of these symmetry operators.

We will establish that the same assertion holds for the Schrödinger equation (1). Let us make in Eq. (1) the following change of variables:

$$u = Q(t, \boldsymbol{x}) U\big(t, \omega_1(t, \boldsymbol{x}), \omega_2(t, \boldsymbol{x})\big), \tag{31}$$

where (Q, ω_1, ω_2) is an arbitrary solution of the system of PDEs (7).

Substituting the expression (31) into (1) and taking into account equations (7) we get

$$Q\big(iU_t + \big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big)\omega_{1x_a}\omega_{1x_a} + \big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big)\omega_{2x_a}\omega_{2x_a}\big) = 0. \tag{32}$$

Resolving Eqs. (2) from the system (7) with respect to $\omega_{1x_a}\omega_{1x_a}$ and $\omega_{2x_a}\omega_{2x_a}$ we have

$$\omega_{1x_a}\omega_{1x_a} = \frac{1}{\delta}\big(R_2(t)B_{21}(\omega_2) - R_1(t)B_{22}(\omega_2)\big),$$
$$\omega_{2x_a}\omega_{2x_a} = \frac{1}{\delta}\big(R_1(t)B_{12}(\omega_1) - R_2(t)B_{11}(\omega_1)\big),$$

where $\delta = B_{11}(\omega_1)B_{22}(\omega_2) - B_{12}(\omega_1)B_{21}(\omega_2)$ ($\delta \neq 0$ by force of the condition (4)).

Substitution of the above equalities into Eq. (32) with subsequent division by $Q \neq 0$ yields the following PDE:

$$\begin{gathered} iU_t + \frac{R_1(t)}{\delta}\big(B_{12}(\omega_1)\big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big) - B_{22}(\omega_2)\big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big)\big) + \\ + \frac{R_2(t)}{\delta}\big(B_{21}(\omega_2)\big(U_{\omega_1\omega_1} - B_{01}(\omega_1)U\big) - B_{11}(\omega_1)\big(U_{\omega_2\omega_2} - B_{02}(\omega_2)U\big)\big) = 0. \end{gathered} \tag{33}$$

Thus, in the new coordinates t, ω_1, ω_2, $U(t, \omega_1, \omega_2)$ Eq. (1) takes the form (33).

By direct (and very cumbersome) computation one can check that the following second-order differential operators:

$$X_1 = \frac{B_{22}(\omega_2)}{\delta}\big(\partial^2_{\omega_1} - B_{01}(\omega_1)\big) - \frac{B_{12}(\omega_1)}{\delta}\big(\partial^2_{\omega_2} - B_{02}(\omega_2)\big),$$
$$X_2 = -\frac{B_{21}(\omega_2)}{\delta}\big(\partial^2_{\omega_1} - B_{01}(\omega_1)\big) + \frac{B_{11}(\omega_1)}{\delta}\big(\partial^2_{\omega_2} - B_{02}(\omega_2)\big),$$

commute under arbitrary B_{0a}, B_{ab}, $a, b = 1, 2$, i.e.

$$[X_1, X_2] \equiv X_1X_2 - X_2X_1 = 0. \tag{34}$$

After being rewritten in terms of the operators X_1, X_2 Eq. (33) reads

$$\big(i\partial_t - R_1(t)X_1 - R_2(t)X_2\big)U = 0.$$

Since the relations

$$\big[i\partial_t - R_1(t)X_1 - R_2(t)X_2, X_a\big] = 0, \quad a = 1, 2 \tag{35}$$

hold, operators X_1, X_2 are mutually commuting symmetry operators of Eq. (33). Furthermore, solution of Eq. (33) with separated variables $U = \varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)$ satisfies the identities

$$X_a U = \lambda_a U, \quad a = 1, 2. \tag{36}$$

Consequently, if we designate by X_1', X_2' the operators X_1, X_2 written in the initial variables t, $\boldsymbol{x}$, u, then we get from (34)–(36) the following equalities:

$$\begin{aligned}&\left[i\partial_t + \triangle - V(x_1, x_2), X_a'\right] = 0, \quad a = 1, 2,\\ &\left[X_1', X_2'\right] = 0, \quad X_a' u = \lambda_a u, \quad a = 1, 2.\end{aligned}$$

where $u = Q(t, \boldsymbol{x})\varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2)$.

It means that each solution with separated variables is a common eigenfunction of two mutually commuting symmetry operators X_1', X_2' of the Schrödinger equation (1), separation constants λ_1, λ_2 being their eigenvalues.

Detailed study of the said operators as well as analysis of separated ODEs for functions φ_μ, $\mu = \overline{0, 2}$ (in the way as it is done for the free Schrödinger equation in [2, 3]) is in progress and will be a topic of our future publications.

Acknowledgments. When the paper was in the last stage of preparation, one of the authors (R. Zhdanov) was supported by the Alexander von Humboldt Foundation. Taking an opportunity he wants to express his gratitude to Director of Arnold Sommerfeld Institute for Mathematical Physics Professor H.-D. Doebner for hospitality.

Appendix A. Integration of nonlinear ODEs (22)–(25)

Evidently, equations (22)–(25) can be rewritten in the following unified form:

$$\left(\frac{\dot{y}}{y}\right)^{\cdot} - \left(\frac{\dot{y}}{y}\right)^2 - 4\alpha y^4 = k, \quad \ddot{z} - 2\dot{z}\frac{\dot{y}}{y} - 2(2\alpha z + \beta)y^4 = 0. \tag{A.1}$$

Provided $k = -a^2 < 0$, system (A.1) coincides with Eqs. (22), (24) and under $k = b^2 > 0$ – with Eqs. (23), (25).

First of all, we note that the function $z = z(t)$ is determined up to addition of an arbitrary constant. Really, the coordinate functions ω_a have the following structure:

$$\omega_a = yx_a + z, \quad a = 1, 2.$$

But the coordinate system t, ω_1, ω_2 is equivalent to the coordinate system t, $\omega_1 + C_1$, $\omega_2 + C_2$, $C_a \in \mathbb{R}^1$. Hence it follows that the function $z(t)$ is equivalent to the function $z(t) + C$ with arbitrary real constant C. Consequently, provided $\alpha \neq 0$, we can choose in (A.1) $\beta = 0$.

The case 1. $\alpha = 0$. On making in (A.1) the change of variables

$$w = \dot{y}/y, \quad v = z/y \tag{A.2}$$

we get

$$\dot{w} = w^2 + k, \quad \ddot{v} + kv = 2\beta y^3. \tag{A.3}$$

First, we consider the case $k = -a^2 < 0$. Then the general solution of the first equation from (A.3) is given by one of the formulas

$$w = -a\coth a(t + C_1), \quad w = -a\tanh a(t + C_1), \quad w = \pm a, \quad C_1 \in \mathbb{R}^1,$$

whence

$$\begin{aligned} &y = C_2 \sinh^{-1} a(t + C_1), \quad y = C_2 \cosh^{-1} a(t + C_1), \\ &y = C_2 \exp(\pm at), \quad C_2 \in \mathbb{R}^1. \end{aligned} \tag{A.4}$$

The second equation of system (A.3) is a linear inhomogeneous ODE. Its general solution after being substituted into (A.2) yields the following expression for $z(t)$:

$$\begin{aligned} &(C_3 \cosh at + C_4 \sinh at) \sinh^{-1} a(t + C_1) + \frac{\beta C_2^4}{a^2} \sinh^{-2} a(t + C_1), \\ &(C_3 \cosh at + C_4 \sinh at) \cosh^{-1} a(t + C_1) + \frac{\beta C_2^4}{a^2} \cosh^{-2} a(t + C_1), \\ &(C_3 \cosh at + C_4 \sinh at) \exp(\pm at) + \frac{\beta C_2^4}{4a^2} \exp(\pm 4at), \quad C_3, C_4 \subset \mathbb{R}^1. \end{aligned} \tag{A.5}$$

The case $k = b^2 > 0$ is treated in an analogous way, the general solution of (A.1) being given by the formulas

$$\begin{aligned} &y = D_2 \sin^{-1} b(t + D_1), \\ &z = (D_3 \cos bt + D_4 \sin bt) \sin^{-1} b(t + D_1) + \frac{\beta D_2^4}{b^2} \sin^{-2} b(t + D_1), \end{aligned} \tag{A.6}$$

where D_1, D_2, D_3, D_4 are arbitrary real constants.

The case 2. $\alpha \neq 0$, $\beta = 0$. On making in Eq. (A.1) the change of variables

$$y = \exp w, \quad v = z/y$$

we have

$$\ddot{w} - \dot{w}^2 = k + \alpha \exp 4w, \quad \ddot{v} + kv = 0. \tag{A.7}$$

The first ODE from Eq. (A.7) is reduced to the first-order linear ODE

$$\frac{1}{2}\frac{dp(w)}{dw} - p(w) = k + \alpha \exp 4w$$

by the substitution $\dot{w} = (p(w))^{1/2}$, whence

$$p(w) = \alpha \exp 4w + \gamma \exp 2w - k, \quad \gamma \in \mathbb{R}^1.$$

Equation $\dot{w} = (p(w))^{1/2}$ has a singular solution $w = C = \text{const}$ such that $p(C) = 0$. If $\dot{w} \neq 0$, then integrating the equation $\dot{w} = p(w)$ and returning to the initial variable y we get

$$\int^{y(t)} \frac{d\tau}{\tau(\alpha\tau^4 + \gamma\tau^2 - k)^{1/2}} = t + C_1.$$

Taking the integral in the left-hand side of the above equality we obtain the general solution of the first ODE from Eq. (A.1). It is given by the following formulas:

under $k = -a^2 < 0$

$$y = C_2\big(\alpha + \sinh 2a(t + C_1)\big)^{-1/2},$$
$$y = C_2\big(\alpha + \cosh 2a(t + C_1)\big)^{-1/2}, \tag{A.8}$$
$$y = C_2\big(\alpha + \exp(\pm 2at)\big)^{-1/2},$$

under $k = b^2 > 0$

$$y = D_2\big(\alpha + \sin 2b(t + D_1)\big)^{-1/2}. \tag{A.9}$$

Here C_1, C_2, D_1, D_2 are arbitrary real constants.

Integrating the second ODE from Eq. (A.7) and returning to the initial variable z we have
under $k = -a^2 < 0$

$$z = y(t)(C_3 \cosh at + C_4 \sinh at) \tag{A.10}$$

under $k = b^2 > 0$

$$z = y(t)(D_3 \cos bt + D_4 \sin bt), \tag{A.11}$$

where C_3, C_4, D_3, D_4 are arbitrary real constants.

Thus, we have constructed the general solution of the system of nonlinear ODEs (A.1) which is given by the formulas (A.5)–(A.11).

Appendix B. Separation of Eq. (1) into three second-order ODEs

Suppose that there exists an Ansatz

$$u = Q(t, \boldsymbol{x})\varphi_0(\omega_0(t, \boldsymbol{x}))\varphi_1\big(\omega_1(t, \boldsymbol{x})\big)\varphi_2\big(\omega_2(t, \boldsymbol{x})\big) \tag{A.12}$$

which separates the Schrödinger equation into three second-order ODEs

$$\begin{gathered}\frac{d^2\varphi_0}{d\omega_0^2} = U_0\left(\omega_0, \varphi_0, \frac{d\varphi_0}{d\omega_0}; \lambda_1, \lambda_2\right), \quad \frac{d^2\varphi_1}{d\omega_1^2} = U_1\left(\omega_1, \varphi_1, \frac{d\varphi_1}{d\omega_1}; \lambda_1, \lambda_2\right), \\ \frac{d^2\varphi_2}{d\omega_2^2} = U_2\left(\omega_2, \varphi_2, \frac{d\varphi_2}{d\omega_2}; \lambda_1, \lambda_2\right)\end{gathered} \tag{A.13}$$

according to the Definition 1.

Substituting the Ansatz (A.12) into Eq. (1) and excluding the second derivatives $d^2\varphi_\mu/d\omega_\mu^2$, $\mu = \overline{0,2}$ according to Eqs. (A.13) we get

$$\begin{aligned} & i(Q_t\varphi_0\varphi_1\varphi_2 + Q\omega_{0t}\dot\varphi_0\varphi_1\varphi_2 + Q\omega_{1t}\varphi_0\dot\varphi_1\varphi_2 + Q\omega_{2t}\varphi_0\varphi_1\dot\varphi_2) + (\triangle Q)\varphi_0\varphi_1\varphi_2 + \\ & \quad + 2Q_{x_a}\omega_{0x_a}\dot\varphi_0\varphi_1\varphi_2 + 2Q_{x_a}\omega_{1x_a}\varphi_0\dot\varphi_1\varphi_2 + 2Q_{x_a}\omega_{2x_a}\varphi_0\varphi_1\dot\varphi_2 + \\ & \quad + Q\big((\triangle\omega_0)\dot\varphi_0\varphi_1\varphi_2 + (\triangle\omega_1)\varphi_0\dot\varphi_1\varphi_2 + (\triangle\omega_2)\varphi_0\varphi_1\dot\varphi_2 + \omega_{0x_a}\omega_{0x_a}U_0\varphi_1\varphi_2 + \\ & \quad + \omega_{1x_a}\omega_{1x_a}\varphi_0 U_1\varphi_2 + \omega_{2x_a}\omega_{2x_a}\varphi_0\varphi_1 U_2 + 2\omega_{0x_a}\omega_{1x_a}\dot\varphi_0\dot\varphi_1\varphi_2 + \\ & \quad + 2\omega_{0x_a}\omega_{2x_a}\dot\varphi_0\varphi_1\dot\varphi_2 + 2\omega_{1x_a}\omega_{2x_a}\varphi_0\dot\varphi_1\dot\varphi_2\big) = VQ\varphi_0\varphi_1\varphi_2. \end{aligned}$$

Splitting the above equality with respect to $\dot\varphi_0\dot\varphi_1$, $\dot\varphi_0\dot\varphi_2$, $\dot\varphi_1\dot\varphi_2$ we obtain the equalities:

$$\omega_{0x_a}\omega_{1x_a} = 0, \quad \omega_{0x_a}\omega_{2x_a} = 0, \quad \omega_{1x_a}\omega_{2x_a} = 0. \tag{A.14}$$

Since the functions ω_μ, $\mu = \overline{0,2}$ are real-valued, equalities (A.14) mean that there are three real two-component vectors which are mutually orthogonal. This is possible only if one of them is a null-vector. Without loss of generality we may suppose that $(\omega_{0x_1}, \omega_{0x_2}) = (0,0)$, whence $\omega_0 = f(t) \sim t$.

Consequently, Ansatz (A.12) necessarily takes the form (5). But Ansatz (5) can not separate Eq. (1) into three second-order ODEs, since it contains no second-order derivative with respect to t.

Thus, we have proved that the Schrödinger equation (1) is not separable into three second-order ODEs.

1. Boyer C., *SIAM J. Math. Anal.*, 1976, **7**, 230.
2. Boyer C., Kalnins E., Miller W., *J. Math. Phys.*, 1975, **16**, 499.
3. Miller W., Symmetry and separation of variables, Massachusetts, Addison-Wesley, 1977.
4. Shapovalov V.N., Sukhomlin N.B., *Izvestiya Vuzov, Fizika*, 1974, № 12, 268.
5. Nikitin A.G., Onufriychuk S.P., Fushchych W.I., *Teoret. Mat. Fiz.*, 1992, **12**, 268.
6. Fushchych W.I., Zhdanov R.Z., Revenko I.V., *Proc. Ukrain. Acad. Sci.*, 1993, № 1, 27.
7. Fushchych W.I., in Symmetry in Mathematical Physics Problems, Kiev, Inst. of Math., 1981, 6.
8. Koornwinder T.H., *Lect. Notes in Math.*, 1980, **810**, 240.
9. Zhdanov R.Z., Revenko I.V., Fushchych W.I., *J. Phys. A: Math. Gen.*, 1993, **26**, 5959.
10. Zhdanov R.Z., *J. Phys. A: Math. Gen.*, 1994, **27**, L291.
11. Boyer C., *Helv. Phys. Acta*, 1974, **47**, 589.
12. Niederer U., *Helv. Phys. Acta*, 1972, **45**, 802.

On the general solution of the d'Alembert equation with a nonlinear eikonal constraint and its applications

R.Z. ZHDANOV, I.V. REVENKO, W.I. FUSHCHYCH

We construct the general solutions of the system of nonlinear differential equations $\Box_n u = 0$, $u_\mu u^\mu = 0$ in the four- and five-dimensional complex pseudo-Euclidean spaces. The results obtained are used to reduce the multi-dimensional nonlinear d'Alembert equation $\Box_4 u = F(u)$ to ordinary differential equations and to construct its new exact solutions.

1. Introduction

Kaluza [1] was the first who put forward an idea of extension of the four-dimensional Minkowski space in order to use it as a geometric basis for unification of the electromagnetic and gravitational fields. Nowadays, Kaluza's idea is well-known and there are a lot of papers where further development and various generalizations of this idea are obtained [2].

In [3–5] it was proposed to apply five-dimensional wave equations to describe particles (fields) having variable spins and masses. Such physical interpretation of the five-dimensional equations is based on the fact that the generalized Poincaré group $P(1,4)$ acting in the five-dimensional de Sitter space contains the Poincaré group $P(1,3)$ as a subgroup. It means that the mass and spin Casimir operators have continuous and discrete spectrum, respectively, in the space of irreducible representations of the group $P(1,4)$ [3–6].

The simplest $P(1,4)$-invariant scalar linear equation has the form

$$\Box_5 u + \chi^2 u = 0, \quad \chi = \text{const}, \tag{1}$$

where $\Box_5$ is the d'Alembert operator in the five-dimensional Minkowski space with the signature $(+,-,-,-,-)$.

The problem of construction of exact solutions of the above equation is, in fact, completely open. One can obtain some its particular solutions applying the symmetry reduction procedure or the method of separation of variables (both approaches use essentially symmetry properties of the whole set of solutions of Eq. (1)). In the present paper we suggest a method for construction of solutions of partial differential equation (1) which utilizes implicitly the symmetry of a *subset* of the set of its solutions. Namely, a special subset of its exact solutions obtained by imposing an additional constraint

$$u_{x_0}^2 - u_{x_1}^2 - u_{x_2}^2 - u_{x_3}^2 - u_{x_4}^2 = 0,$$

which is the eikonal equation in the five-dimensional space, will be investigated. As shown in [7, 8], the system obtained is compatible if and only if $\chi = 0$. We will

Reprinted with permission from J. Math. Phys., 1995, **36**, № 12, P. 7109–7127

Original article is available at http://scitation.aip.org/jmp/

construct general solutions of multi-dimensional systems of partial differential equations (PDE)

$$\Box_n u = 0, \quad u_\mu u^\mu = 0 \tag{2}$$

in the four- and five-dimensional complex pseudo-Euclidean spaces.

In (2) $u = u(x_0, x_1, \ldots, x_{n-1}) \in C^2(\mathbb{C}^n, \mathbb{C}^1)$. Hereafter, the summation over the repeated indices in the pseudo-Euclidean space $M(1, n-1)$ with the metric tensor $g_{\mu\nu} = \mathrm{diag}\,(1, \underbrace{-1, \ldots, -1}_{n-1})$ is understood, e.g. $\Box_n \equiv \partial_\mu \partial^\mu = \partial_0^2 - \partial_1^2 - \cdots - \partial_{n-1}^2$, $\partial_\mu = \partial/\partial x_\mu$.

It occurs that solutions of system of PDE (2), being very interesting by itself, can be used to reduce the *nonlinear* d'Alembert equation

$$\Box_4 u = F(u), \quad F(u) \in C(\mathbb{R}^1, \mathbb{R}^1), \tag{3}$$

to ordinary differential equations, thus giving rise to families of principally new exact solutions of (3). More precisely, we will establish that there exists a nonlinear map from the set solutions of the system of PDE (2) into the set of solutions of the nonlinear d'Alembert equation, such that each solution of (2) corresponds to a family of exact solutions of Eq. (3) containing two arbitrary functions of one argument. It will be shown that solutions of the nonlinear d'Alembert equation obtained in this way can be related to its *conditional* symmetry.

The paper is organized as follows. In Section 2 we give assertions describing the general solution of system of PDE (2) in the n-dimensional real and in the four- and five-dimensional complex pseudo-Euclidean spaces. In Section 3 we prove these assertions. Section 4 is devoted to discussion of the connection between exact solutions of system (2) and the problem of reduction of the nonlinear d'Alembert equation (3). In Section 5 we construct principally new exact solutions of Eq. (3).

2. Integration of the system (2): the list of principal results

Below we adduce assertions giving general solutions of the system of PDE (2) with arbitrary $n \in \mathbb{N}$ provided $u(x) \in C^2(\mathbb{R}^n, \mathbb{R}^1)$, and with $n = 4, 5$, provided $u(x) \in C^2(\mathbb{C}^n, \mathbb{C}^1)$.

Theorem 1. *Let $u(x)$ be a sufficiently smooth real function of n real variables $x_0, \ldots, x_{n-1}$. Then, the general solution of the system of nonlinear PDE* (2) *is given by the following formula:*

$$A_\mu(u) x^\mu + B(u) = 0, \tag{4}$$

where $A_\mu(u)$, $B(u)$ are arbitrary real functions which satisfy the condition

$$A_\mu(u) A^\mu(u) = 0. \tag{5}$$

Note 1. As far as we know, Jacobi, Smirnov and Sobolev were the first who obtained the formulas (4), (5) with $n = 3$ [9, 10]. That is why, it is natural to call (4), (5) the Jacoby–Smirnov–Sobolev formulas (JSSF). Later on, in 1944 Yerugin generalized

JSSF for the case $n = 4$ [11]. Recently, Collins [12] has proved that JSSF give the general solution of system (2) for an arbitrary $n \in \mathbb{N}$. He applied rather complicated differential geometry technique. Below we show that to integrate Eqs. (2) it is quite enough to make use of the classical methods of mathematical physics only.

Theorem 2. *The general solution of the system of nonlinear PDE* (2) *in the class of functions* $u = u(x_0, x_1, x_2, x_3, x_4) \in C^2(\mathbb{C}^5, \mathbb{C}^1)$ *is given by one of the following formulas:*

$$(1)\ A_\mu(\tau, u)x^\mu + C_1(\tau, u) = 0, \tag{6}$$

where $\tau = \tau(u, x)$ *is a complex function determined by the equation*

$$B_\mu(\tau, u)x^\mu + C_2(\tau, u) = 0, \tag{7}$$

and $A_\mu, B_\mu, C_1, C_2 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ *are arbitrary functions satisfying the conditions*

$$A_\mu A^\mu = A_\mu B^\mu = B_\mu B^\mu = 0, \quad B^\mu \frac{\partial A_\mu}{\partial \tau} = 0, \tag{8}$$

and what is more,

$$\Delta = \det \left\| \begin{matrix} x^\mu \dfrac{\partial A_\mu}{\partial \tau} + \dfrac{\partial C_1}{\partial \tau} & x^\mu \dfrac{\partial A_\mu}{\partial u} + \dfrac{\partial C_1}{\partial u} \\ x^\mu \dfrac{\partial B_\mu}{\partial \tau} + \dfrac{\partial C_2}{\partial \tau} & x^\mu \dfrac{\partial B_\mu}{\partial u} + \dfrac{\partial C_2}{\partial u} \end{matrix} \right\| \neq 0; \tag{9}$$

$$(2)\ A_\mu(u)x^\mu + C_1(u) = 0, \tag{10}$$

where $A_\mu(u)$, $C_1(u)$ *are arbitrary smooth functions satisfying the relations*

$$A_\mu A^\mu = 0 \tag{11}$$

(in the formulas (6)–(11) *the index* μ *takes the values* $0, 1, 2, 3, 4$*).*

Theorem 3. *The general solution of the system of nonlinear PDE* (2) *in the class of functions* $u = u(x_0, x_1, x_2, x_3) \in C^2(\mathbb{C}^4, \mathbb{C}^1)$ *is given by the formulas* (6)–(11), *where the index* μ *is supposed to take the values* $0, 1, 2, 3$.

Note 2. Investigating particular solutions of the Maxwell equations, Bateman [13] arrived at the problem of integrating the d'Alembert equation $\Box_4 u = 0$ with an additional nonlinear condition (the eikonal equation) $u_{x_\mu} u_{x^\mu} = 0$. He has obtained the following class of exact solutions of the said system of PDE:

$$u(x) = c_\mu(\tau)x^\mu + c_4(\tau), \tag{12}$$

where $\tau = \tau(x)$ is a complex-valued function determined in implicit way

$$\dot{c}_\mu(\tau)x^\mu + \dot{c}_4(\tau) = 0, \tag{13}$$

and $c_\mu(\tau)$, $c_4(\tau)$ are arbitrary smooth functions satisfying conditions

$$c_\mu c^\mu = \dot{c}_\mu \dot{c}^\mu = 0. \tag{14}$$

(hereafter, a dot over a symbol means differentiation with respect to a corresponding argument).

It is not difficult to check that solutions (12)–(14) are complex (see the Lemma 1 below). An another class of complex solutions of the system (2) with $n = 4$ was constructed by Yerugin [11]. But neither the Bateman's formulas (12)–(14) nor the Yerugin's results give the general solution of the system (2) with $n = 4$.

3. Proofs of Theorems 1–3

It is well-known that the maximal symmetry group admitted by equation (1) is finite-dimensional (we neglect a trivial invariance with respect to an infinite-parameter group $u(x) \to u(x) + U(x)$, where $U(x)$ is an arbitrary solution of Eq. (1), which is due to its linearity). But being restricted to a set of solutions of the eikonal equation the set solutions of PDE (1) admits an infinite-dimensional symmetry group [14]! It is this very fact that enables us to construct the general solution of (2).

Proof of the Theorem 1. Let us make in (2) the hodograph transformation

$$z_0 = u(x), \quad z_a = x_a, \quad a = \overline{1, n-1}, \quad w(z) = x_0. \tag{15}$$

Evidently, the transformation (15) is defined for all functions $u(x)$, such that $u_{x_0} \not\equiv 0$. But the system (2) with $u_{x_0} = 0$ takes the form

$$\sum_{a=1}^{n-1} u_{x_a x_a} = 0, \quad \sum_{a=1}^{n-1} u_{x_a}^2 = 0,$$

whence $u_{x_a} \equiv 0$, $a = \overline{1, n-1}$ or $u(x) = \text{const}$.

Consequently, the change of variables (9) is defined on the whole set of solutions of the system (2) with the only exception $u(x) = \text{const}$.

Being rewritten in the new variables z, $w(z)$ the system (2) takes the form

$$\sum_{a=1}^{n-1} w_{z_a z_a} = 0, \quad \sum_{a=1}^{n-1} w_{z_a}^2 = 1. \tag{16}$$

Differentiating the second equation with respect to z_b, z_c we get

$$\sum_{a=1}^{n-1} (w_{z_a z_b z_c} w_{z_a} + w_{z_a z_b} w_{z_a z_c}) = 0.$$

Choosing in the above equality $c = b$ and summing up we have

$$\sum_{a,b=1}^{n-1} (w_{z_a z_b z_b} w_{z_a} + w_{z_a z_b} w_{z_a z_b}) = 0,$$

whence, by force of (16),

$$\sum_{a,b=1}^{n-1} w_{z_a z_b}^2 = 0. \tag{17}$$

Since $u(z)$ is a real-valued function, it follows from (17) that an equality $w_{z_a z_b} = 0$ holds for all $a, b = \overline{1, n-1}$, whence

$$w(z) = \sum_{a=1}^{n-1} \alpha_a(z_0) z_a + \alpha(z_0). \tag{18}$$

In (18) $\alpha_a, \alpha \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting (18) into the second equation of system (16), we have

$$\sum_{a,b=1}^{n-1} \alpha_a^2(z_0) = 1. \tag{19}$$

Thus, the formulas (18), (19) give the general solution of the system of nonlinear PDE (16). Rewriting (18), (19) in the initial variables x, $u(x)$, we get

$$x_0 = \sum_{a=1}^{n-1} \alpha_a(u)x_a + \alpha(u), \quad \sum_{a=1}^{n-1} \alpha_a^2(u) = 1. \tag{20}$$

To represent the formulas (20) in a manifestly covariant form (4), (5) we redefine the functions $\alpha_a(u)$ in the following way:

$$\alpha_a(u) = \frac{A_a(u)}{A_0(u)}, \quad \alpha(u) = -\frac{B(u)}{A_0(u)}, \quad a = \overline{1, n-1}.$$

Substituting the above expressions into (20) we arrive at the formulas (4), (5).

Next, as $u = \text{const}$ is contained in the class of functions $u(x)$ determined by the formulas (4), (5) under $A_\mu \equiv 0$, $\mu = \overline{0, n-1}$, $B(u) = u + \text{const}$, JSSF (4), (5) give the general solution of the system of the PDE (2) with an arbitrary $n \in \mathbb{N}$. The theorem is proved.

Let us emphasize that the reasonings used above can be applied to the case of a real-valued function $u(x)$ only. If a solution of the system (2) is looked for in a class of complex-valued functions $u(x)$, then JSSF (4), (5) do not give its general solution with $n > 3$. Each case $n = 4, 5 \ldots$ requires a special consideration.

Proof of the Theorem 2. *Case 1:* $u_{x_0} \neq 0$. In this case the hodograph transformation (15) reducing the system (2) with $n = 5$ to the form

$$\sum_{a=1}^{4} w_{z_a z_a} = 0, \quad \sum_{a=1}^{4} w_{z_a}^2 = 1, \quad w_{z_0} \not\equiv 0 \tag{21}$$

is defined.

The general solution of nonlinear complex Eqs. (21) was constructed in [15]. It is given by one of the following formulas:

$$(1)\ w(z) = \sum_{a=1}^{4} \alpha_a(\tau, z_0) z_a + \gamma_1(\tau, z_0), \tag{22}$$

where $\tau = \tau(z_0, \ldots, z_4)$ is a function determined in implicit way

$$\sum_{a=1}^{4} \beta_a(\tau, z_0) z_a + \gamma_2(\tau, z_0) = 0 \tag{23}$$

and $\alpha_a, \beta_a, \gamma_1, \gamma_2 \in C^2(\mathbb{C}^2, \mathbb{C}^1)$ are arbitrary smooth functions satisfying the relations

$$\sum_{a=1}^{4} \alpha_a^2 = 1, \quad \sum_{a=1}^{4} \alpha_a \beta_a = \sum_{a=1}^{4} \beta_a^2 = 0, \quad \sum_{a=1}^{4} \alpha_a \frac{\partial \beta_a}{\partial \tau} = 0; \tag{24}$$

$$(2)\ w(z) = \sum_{a=1}^{4} \alpha_a(z_0) z_a + \gamma_1(z_0), \tag{25}$$

where $\alpha_a, \gamma_1 \in C^2(\mathbb{C}^1, \mathbb{C}^1)$ are arbitrary functions satisfying the relation

$$\sum_{a=1}^{4} \alpha_a^2 = 1. \tag{26}$$

Rewriting the formulas (23), (24) in the initial variables x, $u(x)$, we have

$$x_0 = \sum_{a=1}^{4} \alpha_a(\tau, u) x_a + \gamma_1(\tau, u), \tag{27}$$

where $\tau = \tau(u, x)$ is a function determined in implicit way

$$\sum_{a=1}^{4} \beta_a(\tau, u) x_a + \gamma_2(\tau, u) = 0, \tag{28}$$

and the relations (24) hold.

Evidently, the formulas (27), (28) are obtained from (6)–(8) with a particular choice of functions A_μ, B_μ, C_1, C_2

$$\begin{aligned} &A_0 = 1, \quad A_a = \alpha_a, \quad C_1 = -\gamma_1, \\ &B_0 = 0, \quad B_a = \beta_a, \quad C_2 = -\gamma_2, \end{aligned} \tag{29}$$

where $a = \overline{1,4}$.

Next, by force of inequality $w_{z_0} \not\equiv 0$ we get from (22)

$$\sum_{a=1}^{4} (\alpha_{az_0} + \alpha_{a\tau}\tau_{z_0}) x_a + \gamma_{1z_0} + \gamma_{1\tau}\tau_{z_0} \neq 0. \tag{30}$$

Differentiation of (23) with respect to z_0 yields the following expression for τ_{z_0}:

$$\tau_{z_0} = -\left(\sum_{a=1}^{4} \beta_{az_0} x_a + \gamma_{2z_0}\right)\left(\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau}\right)^{-1}$$

Substitution of the above result into (30) yields the relation

$$\left(\sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau}\right)^{-1} \begin{vmatrix} \sum_{a=1}^{4} \alpha_{az_0} x_a + \gamma_{1z_0} & \sum_{a=1}^{4} \alpha_{a\tau} x_a + \gamma_{1\tau} \\ \sum_{a=1}^{4} \beta_{az_0} x_a + \gamma_{2z_0} & \sum_{a=1}^{4} \beta_{a\tau} x_a + \gamma_{2\tau} \end{vmatrix} \neq 0.$$

As the direct check shows, the above inequality is equivalent to (9) provided the conditions (29) hold.

Now we turn to solutions of the system (21) of the form (25). Rewriting the formulas (25), (26) in the initial variables x, $u(x)$ we get

$$x_0 = \sum_{a=1}^{4} \alpha_a(u)x_a + \gamma_1(u), \quad \sum_{a=1}^{4} \alpha_a^2(u) = 1.$$

Making in the equalities obtained the change $\alpha_a = A_a A_0^{-1}$, $a = \overline{1,4}$, $\gamma_1 = -C_1 A_0^{-1}$, we arrive at the formulas (10), (11).

Thus, under $u_{x_0} \not\equiv 0$ the general solution of the system (2) is contained in the class of functions $u(x)$ given by the formulas (6)–(9) or (10), (11).

Case 2: $u_{x_0} \equiv 0$, $u \neq$ const. It is a common knowledge that the system of PDE (2) is invariant under the generalized Poincaré group $P(1, n-1)$ (see, e.g. [16])

$$x'_\mu = \Lambda_{\mu\nu} x^\nu + \Lambda_\mu, \quad u'(x') = u(x),$$

where $\Lambda_{\mu\nu}$, Λ_μ are arbitrary complex parameters satisfying the relations $\Lambda_{\alpha\mu}\Lambda^\alpha{}_\nu = g_{\mu\nu}$, $\mu, \nu = \overline{0, n-1}$. Hence, it follows that the transformation

$$u(x) \to u(x') = u(\Lambda_{\mu\nu} x^\nu) \tag{31}$$

leaves the set of solutions of the system (2) invariant. Consequently, provided $u(x) \neq$ const we can always transform u to such a form that $u_{x_0} \neq 0$. Thus, in the case 2 the general solution is also given by the formulas (6)–(11) within the transformation (31).

Case 3: $u =$ const. Choosing in (10), (11) $A_\mu = 0$, $\mu = \overline{0,4}$, $C_1 = u =$ const we come to the conclusion that this solution is described by the formulas (6)–(11).

Thus, we have proved that, within a transformation from the group $P(1,4)$ (31), the general solution of the system of PDE (2) with $n = 5$ is given by the formulas (6)–(11). But these formulas are represented in a manifestly covariant form and are not altered with the transformation (31). Consequently, to complete the proof of the theorem it is enough to demonstrate that each function $u = u(x)$ determined by the equalities (6)–(11) is a solution of the system of equations (2).

Differentiating the relations (6), (7) with respect to x_μ, we have

$$\begin{aligned} &A^\mu + \tau_{x_\mu}(A_{\nu\tau}x^\nu + C_{1\tau}) + u_{x_\mu}(A_{\nu u}x^\nu + C_{1u}) = 0, \\ &B^\mu + \tau_{x_\mu}(B_{\nu\tau}x^\nu + C_{2\tau}) + u_{x_\mu}(B_{\nu u}x^\nu + C_{2u}) = 0. \end{aligned}$$

Resolving the above system of linear algebraic equations with respect to u_{x_μ}, τ_{x_μ}, we get

$$\begin{aligned} u_{x_\mu} &= \frac{1}{\Delta}\big(B_\mu(A_{\nu\tau}x^\nu + C_{1\tau}) - A_\mu(B_{\nu\tau}x^\nu + C_{2\tau})\big), \\ \tau_{x_\mu} &= \frac{1}{\Delta}\big(A_\mu(B_{\nu u}x^\nu + C_{1u}) - B_\mu(A_{\nu u}x^\nu + C_{2u})\big), \end{aligned} \tag{32}$$

where $\Delta \neq 0$ by force of (9). Consequently,

$$\begin{aligned} u_{x_\mu}u_{x^\mu} = \Delta^{-2}\big(&B_\mu B^\mu(A_{\nu\tau}x^\nu + C_{1\tau})^2 - 2A_\mu B^\mu(A_{\nu\tau}x^\nu + C_{1\tau})(B_{\nu\tau}x^\nu + C_{2\tau}) + \\ &+ A_\mu A^\mu(B_{\nu\tau}x^\nu + C_{2\tau})^2\big) = 0. \end{aligned}$$

Analogously, differentiating (32) with respect to x_ν and convoluting the expression obtained with the metric tensor $g_{\mu\nu}$, we get $g^{\mu\nu}u_{x_\mu x_\nu} \equiv \Box_5 u = 0$.

Next, differentiating (10) with respect to x_μ we have

$$u_{x_\mu} = -A_\mu(\dot{A}_\nu x^\nu + \dot{C}_1)^{-1}, \quad \mu = \overline{0,4},$$

whence

$$u_{x_\mu x_\nu} = -(\dot{A}^\mu A^\nu + \dot{A}^\nu A^\mu)(\dot{A}_\alpha x^\alpha + \dot{C}_1)^{-2} + A^\mu A^\nu(\ddot{A}_\alpha x^\alpha + \ddot{C}_1)(\dot{A}_\alpha x^\alpha + \dot{C}_1)^{-2}.$$

Consequently,

$$\begin{aligned} &u_{x_\mu} u_{x^\mu} = A_\mu A^\mu(\dot{A}_\nu x^\mu + \dot{C}_1)^{-2} = 0, \\ &\Box_5 u \equiv u_{x_\mu x^\mu} = -2(A_\mu \dot{A}^\mu)(\dot{A}_\nu x^\nu + \dot{C}_1)^{-2} + \\ &\qquad + A_\mu A^\mu(\ddot{A}_\nu x^\nu + \ddot{C}_1)(\dot{A}_\nu x^\nu + \dot{C}_1)^{-2} = 0. \end{aligned}$$

The Theorem 2 is proved.

The Theorem 3 is a direct consequence of the Theorem 2. Really, solutions of the system of PDE (2) with $n = 4$ are obtained from solutions of the system of PDE (2) with $n = 5$ provided $u_{x_4} \equiv 0$. Imposing on functions $u(x)$ determined by the formulas (6)–(11) a condition $u_{x_4} \equiv 0$ we arrive at the following restrictions on the functions A_μ, B_μ, C_1, C_2:

$$A_4 = 0, \quad B_4 = 0$$

the same as what was to be proved.

4. Applications: reduction of the nonlinear d'Alembert equation

Following [8, 15, 16], we look for a solution of the nonlinear d'Alembert equation

$$\Box_4 w = F(w), \quad F \in C^1(\mathbb{R}^1, \mathbb{R}^1) \tag{33}$$

in the form

$$w = \varphi(\omega_1, \omega_2), \tag{34}$$

where $\omega_i = \omega_i(x) \in C^2(\mathbb{R}^4, \mathbb{R}^1)$ are supposed to be functionally-independent. The functions $\omega_1(x)$, $\omega_2(x)$ are determined by the requirement that the substitution of (34) into (33) yields two-dimensional PDE for a function $\varphi = \varphi(\omega_1, \omega_2)$. As a result, we obtain an over-determined system of PDE [16]

$$\begin{aligned} &\Box_4 \omega_1 = f_1(\omega_1, \omega_2), \quad \Box_4 \omega_2 = f_2(\omega_1, \omega_2), \\ &\omega_{1x_\mu}\omega_{1x^\mu} = g_1(\omega_1, \omega_2), \quad \omega_{2x_\mu}\omega_{2x^\mu} = g_2(\omega_1, \omega_2), \\ &\omega_{1x_\mu}\omega_{2x^\mu} = g_3(\omega_1, \omega_2), \quad \operatorname{rank} \left\| \frac{\partial \omega_i}{\partial x_\mu} \right\|_{i=1}^{2}\,{}_{\mu=0}^{3} = 2, \end{aligned} \tag{35}$$

and besides, the function $\varphi(\omega_1, \omega_2)$ satisfies a two-dimensional PDE,

$$g_1\varphi_{\omega_1\omega_1} + g_2\varphi_{\omega_2\omega_2} + 2g_3\varphi_{\omega_1\omega_2} + f_1\varphi_{\omega_1} + f_2\varphi_{\omega_2} = F(\varphi). \tag{36}$$

Consider the following problem: to describe all smooth real functions $\omega_1(x)$, $\omega_2(x)$ such that the Ansatz (34) reduces Eq. (33) to an ordinary differential equation (ODE) with respect to the variable ω_1. It means that one has to put coefficients g_2, g_3, f_2 in (36) equal to zero. In other words, it is necessary to construct a general solution of the system of nonlinear PDE

$$\begin{aligned} &\Box_4\omega_1 = f_1(\omega_1,\omega_2), \quad \omega_{1x_\mu}\omega_{1x^\mu} = g_1(\omega_1,\omega_2), \\ &\omega_{1x_\mu}\omega_{2x^\mu} = 0, \quad \omega_{2x^\mu}\omega_{2x_\mu} = 0, \quad \Box_4\omega_2 = 0. \end{aligned} \tag{37}$$

The above system includes Eqs. (2) as a subsystem. So, the d'Alembert-eikonal system (2) arises in a natural way when solving the problem of reduction of Eq. (33) to PDE having a smaller dimension (see, also [15, 17]).

With an appropriate choice of a function $G(\omega_1,\omega_2)$ the change of variables

$$v = G(\omega_1,\omega_2), \quad u = \omega_2$$

reduces the system (37) to the form

$$\Box_4 v = f(u,v), \quad v_{x_\mu}v_{x^\mu} = \lambda, \tag{38}$$

$$u_{x_\mu}v_{x^\mu} = 0, \quad u_{x_\mu}u_{x^\mu} = 0, \quad \Box_4 u = 0, \tag{39}$$

$$\operatorname{rank}\left\|\begin{matrix} v_{x_0} v_{x_1} v_{x_2} v_{x_3} \\ u_{x_0} u_{x_1} u_{x_2} u_{x_3} \end{matrix}\right\| = 2, \tag{40}$$

where λ is a real parameter taking the values -1, 0, 1.

Before formulating the principal assertion, we will prove an auxiliary lemma.

Lemma 1. *Let $a = (a_0,a_1,a_2,a_3)$, $b = (b_0,b_1,b_2,b_3)$ be four-vectors defined in the real Minkowski space $M(1,3)$. Suppose they satisfy the relations*

$$a_\mu b^\mu = b_\mu b^\mu = 0, \quad \sum_{\mu=0}^{3} b_\mu^2 \neq 0. \tag{41}$$

Then, an inequality $a_\mu a^\mu \leq 0$ holds.

Proof. It is known that any isotropic non-null vector b in the space $M(1,3)$ can be reduced to the form $b' = (\alpha,\alpha,0,0)$, $\alpha \neq 0$ by means of a transformation from the group $P(1,3)$. Substituting $b' = (\alpha,\alpha,0,0)$ into the first equality from (41), we get

$$\alpha(a'_0 - a'_2) = 0 \;\Leftrightarrow\; a'_0 = a'_3.$$

Consequently, the vector a' has the following components: a'_0, a'_1, a'_2, a'_0. That is why, $a'_\mu a'^\mu = a'^2_0 - a'^2_1 - a'^2_2 - a'^2_0 = -(a'^2_1 + a'^2_2) \leq 0$. As the quadratic form $a_\mu a^\mu$ is invariant with respect to the group $P(1,3)$, hence it follows that $a_\mu a^\mu \leq 0$.

Let us note that $a_\mu a^\mu = 0$ if and only if $a_2 = a_3$, i.e. $a_\mu a^\mu = 0$ if and only if the vectors a and b are parallel.

Theorem 4. *Eqs.* (38)–(40) *are compatible if and only if*

$$\lambda = -1, \quad f = -N\big(v + h(u)\big)^{-1}, \tag{42}$$

where $h \in C^1(\mathbb{R}^1,\mathbb{R}^1)$ is an arbitrary function, $N = 0,1,2,3$.

Theorem 5. *The general solution of the system of Eqs.* (38)–(40) *being determined within a transformation from the group* $P(1,3)$ *is given by the following formulas:*

a) under $f = -3\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\begin{gathered}\big(v + h(u)\big)^2 = (-\dot A_\nu \dot A^\nu)^{-1}(\dot A_\mu x^\mu + \dot B)^2 + \\ + (-\dot A_\nu \dot A^\nu)^{-3}(\varepsilon^{\mu\nu\alpha\beta} A_\mu \dot A_\nu \ddot A_\alpha x_\beta + C)^2, \\ A_\mu x^\mu + B = 0;\end{gathered} \tag{43}$$

b) under $f = -2\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\big(v + h(u)\big)^2 = (-\dot A_\nu \dot A^\nu)^{-1}(\dot A_\mu x^\mu + \dot B)^2, \quad A_\mu x^\mu + B = 0, \tag{44}$$

where $A_\mu = A_\mu(u)$, $B = B(u)$, $C = C(u)$ *are arbitrary smooth functions satisfying the relations*

$$A_\mu A^\mu = 0, \quad \dot A_\mu \dot A^\mu \neq 0, \tag{45}$$

c) under $f = -\big(v + h(u)\big)^{-1}$, $\lambda = -1$

$$\begin{gathered}\big(v + h(x_0 - x_3)\big)^2 = \big(x_1 + C_1(x_0 - x_3)\big)^2 + \big(x_2 + C_2(x_0 - x_3)\big)^2, \\ u = C_0(x_0 - x_3),\end{gathered} \tag{46}$$

where C_0, C_1, C_2 *are arbitrary smooth functions;*
d) under $f = 0$, $\lambda = -1$

$$(1) \quad v = (-\dot A_\nu \dot A^\nu)^{-3/2} \varepsilon^{\mu\nu\alpha\beta} A_\mu \dot A_\nu \ddot A_\alpha x_\beta + C, \quad A_\mu x^\mu + B = 0, \tag{47}$$

where $A_\mu = A_\mu(u)$, $B = B(u)$, $C = C(u)$ *are arbitrary smooth functions satisfying the relations* (45)*;*

$$\begin{aligned}(2) \quad & v = x_1 \cos\big(C_1(x_0 - x_3)\big) + x_2 \sin\big(C_1(x_0 - x_3)\big) + C_2(x_0 - x_3), \\ & u = C_0(x_0 - x_3),\end{aligned} \tag{48}$$

where C_0, C_1, C_2 *are arbitrary smooth functions.*

In the above formulas (43), (47) *we denote by* $\varepsilon_{\mu\nu\alpha\beta}$ *the completely anti-symmetric fourth-order tensor (the Levi-Civita tensor), i.e.*

$$\varepsilon_{\mu\nu\alpha\beta} = \begin{cases} 1, & (\mu,\nu,\alpha,\beta) = \text{cycle}\,(0,1,2,3), \\ -1, & (\mu,\nu,\alpha,\beta) = \text{cycle}\,(1,0,2,3), \\ 0, & \text{in the remaining cases.}\end{cases}$$

Proof of the Theorems 4, 5. By force of (40) $u \neq \text{const}$. Consequently, within a transformation from the group $P(1,3)$ $u_{x_0} \neq 0$. That is why, one can apply to Eqs. (38)–(40) the hodograph transformation

$$z_0 = u(x), \quad z_a = x_a, \quad a = \overline{1,3}, \quad w(z) = x_0, \quad v = v(z_0, z_a).$$

As a result, the system (38), (39) reads

$$\sum_{a=1}^{3} w_{z_a}^2 = 1, \quad \sum_{a=1}^{3} w_{z_a z_a} = 0, \tag{49}$$

$$\sum_{a=1}^{3} v_{z_a} w_{z_a} = 0, \tag{50}$$

$$\sum_{a=1}^{3} v_{z_a}^2 = -\lambda, \quad \sum_{a=1}^{3} (v_{z_a z_a} + 2w_{z_0}^{-1} v_{z_a} w_{z_a z_0}) = -f(v, z_0). \tag{51}$$

As $v(z)$ is a real-valued function, $\lambda \leq 0$. Scaling, if necessary, the function v we can put $\lambda = -1$ or $\lambda = 0$.

Case 1: $\lambda = -1$. As it is shown in the Section 2, the general solution of the system (49) in the class of real-valued functions $w(z)$ is given by the formulas (18), (19) with $n = 4$. Substituting (18) into (50), we obtain a first-order linear PDE

$$\sum_{a=1}^{3} \alpha_a(z_0) v_{z_a} = 0, \tag{52}$$

whose general solution is represented in the form

$$v = v(z_0, \rho_1, \rho_2). \tag{53}$$

In (53),

$$z_0, \quad \rho_1 = \left(\sum_{a=1}^{3} \dot{\alpha}_a^2\right)^{-1/2} \left(\sum_{a=1}^{3} \dot{\alpha}_a z_a + \dot{\alpha}\right),$$
$$\rho_2 = \left(\sum_{a=1}^{3} \dot{\alpha}_a^2\right)^{-3/2} \sum_{a,b,c=1}^{3} \varepsilon_{abc} z_a \alpha_b \dot{\alpha}_c$$

are the first integrals of Eq. (52) and what is more, $\sum\limits_{a=1}^{3} \dot{\alpha}_a^2 \neq 0$ (the case $\alpha_a = \text{const}$, $a = \overline{1,3}$ will be treated separately), ε_{abc} is the third-order anti-symmetric tensor with $\varepsilon_{123} = 1$.

Substitution the expression (53) into (51) yields the system of two PDE for a function $v = v(z_0, \rho_1, \rho_2)$

$$v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1} = -f(v, z_0), \tag{54}$$

$$v_{\rho_1}^2 + v_{\rho_2}^2 = 1. \tag{55}$$

To get rid of an arbitrary element (function) $f(v, z_0)$ from (54) we consider instead of system (54), (55) its differential consequence

$$v_{\rho_2}(v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1})_{\rho_1} - v_{\rho_1}(v_{\rho_1\rho_1} + v_{\rho_2\rho_2} + 2\rho_1^{-1} v_{\rho_1})_{\rho_1} = 0, \tag{56}$$

$$v_{\rho_1}^2 + v_{\rho_2}^2 = 1, \tag{57}$$

that is obtained by differentiating the first equation with respect to ρ_1, ρ_2, multiplying the expressions obtained by v_{ρ_2} and $-v_{\rho_1}$, respectively, and summing.

Further, we will consider the subcases $v_{\rho_2\rho_2} = 0$ and $v_{\rho_2\rho_2} \neq 0$ separately.

Subcase 1.A: $v_{\rho_2\rho_2} = 0$. Then,

$$v = g_1(z_0, \rho_1)\rho_2 + g_2(z_0, \rho_1), \tag{58}$$

where $g_1, g_2 \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting (58) into (57) and splitting an equality obtained by the powers of ρ_2, we have

$$g_{1\rho_1} = 0, \quad g_1^2 + (g_{2\rho_2})^2 = 1,$$

whence

$$v = \alpha\rho_1 \pm \sqrt{1-\alpha^2}\rho_2 - h(z_0). \tag{59}$$

Here $\alpha \in \mathbb{R}^1$, h is an arbitrary smooth function.

Inserting (59) into (56) we get an algebraic equation $\alpha\sqrt{1-\alpha^2} = 0$, whence $\alpha = 0, \pm 1$.

Finally, substitution of (59) into (54) yields the equation for $f(v, z_0)$

$$2\alpha\rho_1^{-1} = -f\left(\alpha\rho_1 \pm \sqrt{1-\alpha^2}\rho_2 - h(z_0), z_0\right). \tag{60}$$

From Eq. (60) it follows that, under $\alpha = 0$,

$$f = 0, \quad v = \pm\rho_2 - h(z_0) \tag{61}$$

and under $\alpha = \pm 1$,

$$f = -2\big(v + h(z_0)\big)^{-1}, \quad v = \pm\rho_1 - h(z_0). \tag{62}$$

Subcase 1.B: $v_{\rho_2\rho_2} \neq 0$. In this case one can apply to Eqs. (56), (57) the Euler–Ampére transformation

$$\begin{aligned} &z_0 = y_0, \quad \rho_1 = y_1, \quad \rho_2 = G_{y_2}, \quad v + G = \rho_2 y_2, \quad v_{\rho_1} = -G_{y_1}, \quad v_{\rho_2} = y_2, \\ &v_{\rho_2\rho_2} = (G_{y_2y_2})^{-1}, \quad v_{\rho_1\rho_2} = -G_{y_1y_2}(G_{y_2y_2})^{-1}, \\ &v_{\rho_1\rho_1} = (G_{y_1y_2}^2 - G_{y_1y_1}G_{y_2y_2})(G_{y_2y_2})^{-1}. \end{aligned} \tag{63}$$

Here y_0, y_1, y_2 are new independent variables, $G = G(y_0, y_1, y_2)$ is a new function.

Being rewritten in the new variables y, $G(y)$ the Eq. (57), becomes linear

$$G_{y_1} = \pm\sqrt{1 - y_2^2},$$

whence

$$G = \pm y_1\sqrt{1-y_2^2} + H(y_0, y_2), \quad H \in C^2(\mathbb{R}^2, \mathbb{R}^1). \tag{64}$$

Making in the Eq. (56) the change of variables (63) and inserting the expression (64), we transform it as follows

$$\big(y_2 - (1-y_2^2)^{3/2}H_{y_2y_2}\big)^{-2}\big(3y_2H_{y_2y_2} + (y_2^2-1)H_{y_2y_2y_2}\big) + 2y_1^{-2}y_2H_{y_2y_2} = 0. \tag{65}$$

Splitting (65) by the powers of y_1 and integrating the equations obtained, we get

$$H = h_1(y_0)y_2 + h_2(y_0).$$

Substituting the above result into (64) and returning to the initial variables z_0, ρ_1, ρ_2, $v(z_0,\rho_1,\rho_2)$ we obtain the general solution of the system of PDE (56), (57)

$$v + h_2(z_0) = \pm\big([\rho_2 - h_1(z_0)]^2 + \rho_1^2\big)^{1/2}. \tag{66}$$

At last, inserting (66) into the equation (54), we arrive at the conclusion that the function f is determined by the formula (42) with $N = 3$.

If $\alpha_a = \text{const}$, $a = \overline{1,3}$, then the equality $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ holds. Applying, if necessary, a transformation from the group $P(1,3)$ one can put $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 1$, i.e. $u = C_0(x_0 - x_3)$, $C_0 \in C^2(\mathbb{R}^1,\mathbb{R}^2)$.

As a consequence of Eqs. (39) we get $v = v(\xi, x_1, x_2)$, where $\xi = x_0 - x_3$, and what is more, Eqs. (38) take the form

$$v_{x_1}^2 + v_{x_2}^2 = 1, \quad v_{x_1x_1} + v_{x_2x_2} = -f\big(v, C_0(\xi)\big). \tag{67}$$

It is known [15, 18] that Eqs. (67) are compatible if and only if $f = 0$ or $f = -(v + h(u))^{-1}$, $h \in C^1(\mathbb{R}^1,\mathbb{R}^1)$. And besides, the general solution of (67) is given by the formulas (48) and (46), respectively.

Thus, we have completely investigated the case $\lambda = -1$.

Case 2: $\lambda = 0$. By force of the fact that the function v is a real one, it follows from (51) that $v = v(z_0)$. Consequently, an equality $v = v(u)$ holds that breaks the condition (40) which means that under $\lambda = 0$ the system (38)–(40) is incompatible.

Thus, we have proved that the system of nonlinear PDE (38)–(40) is compatible if and only if the relations (42) hold and that its general solution is given by one of the formulas (46), (48), (61), (62), and (66). To complete the proof, one has to rewrite the expressions (61), (62), (66) in the manifestly covariant from (43), (44), (47).

Consider, as an example, the formula (62)

$$v = \pm\rho_1 - h(z_0) \equiv \pm\left(\sum_{a=1}^{3}\dot\alpha_a^2(u)\right)^{-1/2}\left(\sum_{a=1}^{3}x_a\dot\alpha_a(u) + \dot\alpha(u)\right) - h(u), \tag{68}$$

the function $u(x)$ being determined by the formula (20),

$$\sum_{a=1}^{3}\alpha_a(u)x_a + \alpha(u) = x_0, \quad \sum_{a=1}^{3}\alpha_a^2(u) = 1. \tag{69}$$

Let us make in (68), (69) a substitution $\alpha_a = A_aA_0^{-1}$, $\alpha = -BA_0^{-1}$, whence

$$\begin{aligned}
&A_\mu(u)x^\mu + B(u) = 0, \quad A_\mu A^\mu = 0,\\
&v = \pm\left(\sum_{a=1}^{3}(\dot A_aA_0^{-1} - A_a\dot A_0A_0^{-2})^2\right)^{-1/2} \times\\
&\quad\times\left(\sum_{a=1}^{3}x_a(\dot A_aA_0^{-1} - A_a\dot A_0A_0^{-2}) + B\dot A_0A_0^{-2} - \dot BA_0^{-1}\right) - h(u) =
\end{aligned}$$

$$= \pm\left(\sum_{a=1}^{3}(\dot A_a^2 A_0^{-2} + A_a^2\dot A_0^2 A_0^{-4} - 2\dot A_a A_a \dot A_0 A_0^{-3})^{-1/2}\right)\times$$
$$\times\left(\sum_{a=1}^{3} x_a(\dot A_a A_0^{-1} - A_a\dot A_0 A_0^{-2}) + B\dot A_0 A_0^{-2} - \dot B A_0\right) - h(u) =$$
$$= \pm\left(-\dot A_\mu \dot A^\mu A_0^{-2} - A_\mu A^\mu \dot A_0^2 A_0^{-4} + 2\dot A_\mu A^\mu \dot A_0 A_0^{-3}\right)^{-1/2}\times$$
$$\times\left(-A_0^{-1}(x_\mu\dot A^\mu + \dot B) + A_0^{-2}\dot A_0(x_\mu A^\mu + B)\right) - h(u) =$$
$$= \mp(-\dot A_\mu\dot A^\mu)^{-1/2}(x_\mu\dot A^\mu + \dot B) - h(u).$$

The only thing left is to prove that $\dot A_\mu\dot A^\mu < 0$. Since $A_\mu A^\mu = 0$, the equality $\dot A_\mu A^\mu = 0$ holds. Consequently, by force of the Lemma $-\dot A_\mu\dot A^\mu \geq 0$, and what is more, the equality $\dot A_\mu\dot A^\mu = 0$ holds if and only if $\dot A_\mu = k(u)A_\mu$. General solution of the above system of ordinary differential equations reads $A_\mu = l(u)\theta_\mu$, where $l(u)$ is an arbitrary function, θ_μ are arbitrary real parameters obeying the equality $\theta_\mu\theta^\mu = 0$. Hence it follows that $\alpha_a = A_a A_0^{-1} = \theta_a\theta_0^{-1} = \text{const}$, and the condition $\sum\limits_{a=1}^{3}\dot\alpha_a^2 \neq 0$ does not hold. We come to the contradiction, whence it follows that $\dot A_\mu\dot A^\mu < 0$.

Thus, we have obtained the formula (44). Derivation of the remaining formulas from (43), (47) is carried out in the same way. The theorems are proved.

Substitution of the results obtained above into the formula (34) yields the following collection of Ansätze for the nonlinear d'Alembert equation (33):

$$\begin{aligned}
&(1)\ w(x) = \varphi\Big(\big[\big(-\dot A_\nu(u)\dot A^\nu(u)\big)^{-1}\big(\dot A_\mu(u)x^\mu + \dot B(u)\big)^2 + \\
&\qquad + \big(-\dot A_\nu(u)\dot A^\nu(u)\big)^{-3}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot A_\nu(u)\ddot A_\alpha(u)x_\beta + C(u)\big)^2\big]^{1/2}, u\Big);\\
&(2)\ w(x) = \varphi\left(\big(-\dot A_\nu(u)\dot A^\nu(u)\big)^{1/2}\big(\dot A_\mu(u)x^\mu + \dot B(u)\big), u\right);\\
&(3)\ w(x) = \varphi\left(\big[\big(x_1 + C_1(x_0 - x_3)\big)^2 + \big(x_2 + C_2(x_0 - x_3)\big)^2\big]^{1/2}, x_0 - x_3\right);\\
&(4)\ w(x) = \varphi\left(\big(-\dot A_\nu(u)\dot A^\nu(u)\big)^{-3/2}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot A_\nu(u)\ddot A_\alpha(u)x_\beta + C(u)\big), u\right);\\
&(5)\ w(x) = \varphi\big(x_1\cos C_1(x_0 - x_3) + x_2\sin C_1(x_0 - x_3) + C_2(x_0 - x_3), x_0 - x_3\big).
\end{aligned}\tag{70}$$

Here B, C, C_1, C_2 are arbitrary smooth functions of the corresponding arguments, $A_\mu(u)$ are arbitrary smooth functions satisfying the condition $A_\mu A^\mu = 0$ and the function $u = u(x)$ is determined by JSSF (10) with $C_1(u) = B(u)$, $n = 4$. Note that arbitrary functions h contained in the functions $v(x)$ (see above the formulas (43), (44), (46)) are absorbed by the function $\varphi(v, u)$ at the expense of the second argument.

Substitution of the expressions (70) into (33) gives the following equations for $\varphi = \varphi(u, v)$:

$$(1)\ \varphi_{vv} + \frac{3}{v}\varphi_v = -F(\varphi), \tag{71}$$

$$(2)\ \varphi_{vv} + \frac{2}{v}\varphi_v = -F(\varphi), \tag{72}$$

$$(3)\ \varphi_{vv} + \frac{1}{v}\varphi_v = -F(\varphi), \tag{73}$$

$$(4)\ \varphi_{vv} = -F(\varphi), \tag{74}$$

$$(5)\ \varphi_{vv} = -F(\varphi), \tag{75}$$

Equations (4), (5) from (71)–(75) are known to be integrable in quadratures. Therefore, any solution of the d'Alembert-eikonal system (2) corresponds to some class of exact solutions of the nonlinear wave equation (33) that contains arbitrary functions. Saying it in another way, the formulas (70) make it possible to construct wide families of exact solutions of the nonlinear PDE (33) using exact solutions of the linear d'Alembert equation $\Box_4 u = 0$ satisfying an additional constraint $u_{x_\mu} u_{x^\mu} = 0$.

It is interesting to compare our approach to the problem of reduction of Eq. (33) with the classical Lie approach. Within the framework of the Lie approach functions $\omega_1(x)$, $\omega_2(x)$ from (34) are looked for as invariants of the symmetry group of the equation under study (in the case involved it is the Poincaré group $P(1,3)$). Since the group $P(1,3)$ is a finite-parameter group, its invariants cannot contain an arbitrary function (a complete description of invariants of the group $P(1,3)$ had been carried out in [19]). Therefore, the Ansätze (70) cannot, in principle, be obtained by means of the Lie symmetry of the PDE (33).

All Ansätze listed in (70) correspond to a *conditional invariance* of the nonlinear d'Alembert equation (33). It means that for each Ansatz from (70) there exist two differential operators $Q_a = \xi_{a\mu}(x)\partial_{x_\mu}$, $a = 1, 2$ such that

$$Q_a w(x) \equiv Q_a \varphi(\omega_1, \omega_2) = 0, \quad a = \overline{1,2}$$

and besides, the system of PDE

$$\Box_4 w - F(w) = 0, \quad Q_a w = 0, \quad a = 1, 2$$

is invariant in Lie's sense under the one-parameter groups with the generators Q_1, Q_2. For example, the fourth Ansatz from (16) is invariant with respect to the operators: $Q_1 = A_\mu(u)\partial_\mu$, $Q_2 = \dot{A}_\mu(u)\partial_\mu$. A direct computation shows that the following relations hold:

$$\underset{2}{Q_i}(\Box_4 \omega) = -(\dot{A}^\alpha x_\alpha + \dot{B})^{-1} A^\mu \partial_\mu Q_i w, \quad i = 1, 2,$$
$$[Q_1, Q_2] = 0,$$

where $\underset{2}{Q_i}$ stands for the second prolongation of the operator Q_i. Hence it follows that the nonlinear d'Alembert equation (33) is conditionally-invariant under the two-dimensional commutative Lie algebra having the basis elements Q_1, Q_2 (for more details about conditional symmetry of PDE see [20, 21]). It should be said that the notion of conditional symmetry of PDE is closely connected with the "non-classical reduction" [22–24] and "direct reduction" [25] methods.

5. On the new exact solutions of the nonlinear d'Alembert equation

According to [26], general solutions of Eqs. (74), (75) are given by the following quadrature:

$$v + D(u) = \int_0^{\varphi(u,v)} \left(-2 \int_0^{\tau} F(z)dz + C(u) \right)^{-1/2} d\tau, \tag{76}$$

where $D(u), C(u) \in C^2(\mathbb{R}^1, \mathbb{R}^1)$ are arbitrary functions.

Substituting the expressions for $u(x)$, $v(x)$ given by the formulas (4), (5) from (70) into (76) we obtain two classes of exact solutions of the nonlinear d'Alembert equation (33) that contain several arbitrary functions of one variable.

Equations (71) and (72) with $F(\varphi) = \lambda\varphi^k$ are Emden–Fowler type equations. They were investigated by many authors (see, e.g. [26]). In particular, it is known that the equations

$$\varphi_{vv} + 2v^{-1}\varphi_v = -\lambda\varphi^5, \tag{77}$$

$$\varphi_{vv} + 3v^{-1}\varphi_v = -\lambda\varphi^3 \tag{78}$$

are integrated in quadratures. In the paper [27] it has been established that Eqs. (77), (78) possess a Painlevé property. This fact makes it possible to integrate these by applying rather complicated technique. In [28] we have integrated Eqs. (77), (78) using a standard technique based on their Lie symmetry. Substituting the results obtained into the corresponding Ansätze from (70) we get exact solutions of the nonlinear PDE (33) with $F(w) = \lambda w^3$, λw^5, which include an arbitrary solution of the system (2) with $n = 4$. Consequently, we have constructed principally new exact solutions of the nonlinear d'Alembert equation (33) depending on several arbitrary functions. Let us stress that following the classical Lie symmetry reduction procedure one can not in principle obtain solutions with arbitrary functions since the maximal symmetry group of Eq. (33) is finite-dimensional (see, e.g. [16]).

Below we give new exact solutions of the nonlinear d'Alembert equation (33) obtained with the use of the technique described above. We adduce only those ones that can be written down explicitly

1. $F(w) = \lambda w^3$

(1) $$w(x) = \frac{1}{a\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 - x_0^2)^{-1/2} \times \tan\left\{ -\frac{\sqrt{2}}{4} \ln\big(C(u)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\big) \right\},$$

where $\lambda = -2a^2 < 0$,

(2) $$w(x) = \frac{2\sqrt{2}}{a} C(u)\big(1 \pm C^2(u)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\big)^{-1},$$

where $\lambda = \pm a^2$;

2. $F(w) = \lambda w^5$

(1) $$w(x) = a^{-1}(x_1^2 + x_2^2 - x_0^2)^{-1/4}\left\{\sin\ln\bigl(C(u)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\bigr) + 1\right\}^{1/2} \times$$
$$\times \left\{2\sin\ln\bigl(C(u)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\bigr) - 4\right\}^{-1/2},$$

where $\lambda = a^4 > 0$,

(2) $$w(x) = \frac{3^{1/4}}{\sqrt{a}}C(u)\bigl(1 \pm C^4(u)(x_1^2 + x_2^2 - x_0^2)\bigr)^{-1/2},$$

where $\lambda = \pm a^2$.

In the above formulas $C(u)$ is an arbitrary twice continuously differentiable function on

$$u(x) = \frac{x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2},$$

$a \neq 0$ is an arbitrary real parameter.

6. Conclusion

The present paper demonstrates once more that possibilities to construct in explicit form new exact solutions of the nonlinear d'Alembert equation (33) (as compared with those obtainable by the standard symmetry reduction technique [16, 19, 27]) are far from being exhausted. A source of new (non-Lie) reductions is the conditional symmetry of Eq. (33).

Roughly speaking, a principal idea of the method of conditional symmetries is the following: to be able to reduce given PDE it is enough to require an invariance of a *subset* of its solutions with respect to some Lie transformation group. And what is more, this subset is not obliged to coincide with the whole set. This specific subsets can be chosen in different ways: one can fix in some way an Ansatz for a solution to be found (the method of Ansätze [16, 17] or the direct reduction method [25]) or one can impose an additional differential constraint (the method of non-classical [22–24] or conditional symmetries [20, 21]). But all the above approaches have a common feature: to find new (non-Lie) reduction of a given PDE one has to solve some nonlinear over-determined system of differential equations. For example, to describe Ansätze of the form (34) reducing Eq. (33) to ODE one has to integrate system of five nonlinear PDE (37). This is a "price" to be paid for the new possibilities to reduce a given nonlinear PDE to equations with less number of independent variables and to construct its explicit solutions.

As mentioned in the Introduction, the Ansatz (34) can also be interpreted as a map (more exactly, a family of maps) from the set of solutions of the linear d'Alembert equation,

$$\Box_4 u = 0 \tag{79}$$

into the set of solutions of the nonlinear d'Alembert equation (33).

Really, we started with a subset of solutions of Eq. (79) which was chosen by an additional eikonal constraint $u_{x_\mu}u_{x^\mu} = 0$. Then, we constructed the functions

$v(x)$ and $\varphi(v,u)$ in such a way that the function $w(x)$ determined by the equality $w = \varphi(v(x), u(x))$ satisfied the nonlinear d'Alembert equation (33) (see below the Fig. 1).

There is an analogy between the map described above and Bäcklund transformations of partial differential equations. System of PDE (38)–(40) and the Ansatz (34) (level 2 of the Fig. 1) can be interpreted as a Bäcklund transformation of a set of solutions of linear PDE (level 1 of the Fig. 1) into a set of solutions of nonlinear PDE (level 3). A principal difference is that a classical Bäcklund transformation acts on the whole spaces of solutions of equations under study and the above map acts on subsets of solutions of the linear and nonlinear d'Alembert equations. It is known that technique of linearization of PDE with the use of Bäcklund transformations can be effectively applied to two-dimensional equations only. The results obtained in the present paper imply the following way of extension of applicability of Bäcklund transformations: one should consider Bäcklund transformations connecting subsets of solutions of linear and nonlinear equations. And these subsets may not coincide with the whole sets of solutions.

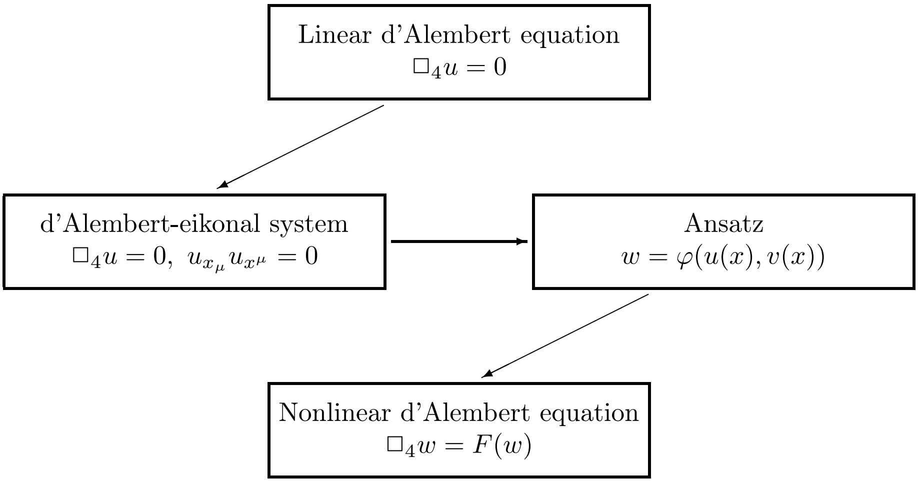


Figure 1.

As an illustration we consider the case when in (33) $F(w) = 0$, i.e. the case when the map constructed above transforms a subset of solutions of the linear d'Alembert equation into another subset of solutions of the same equation. Integrating ODE (71)–(75) we obtain explicit forms of functions $\varphi(v,u)$

(1) $\varphi(v,u) = f_1(u)v^{-2} + f_2(u)$,

(2) $\varphi(v,u) = f_1(u)v^{-1} + f_2(u)$,

(3) $\varphi(v,u) = f_1(u)\ln v + f_2(u)$,

(4) $\varphi(v,u) = f_1(u)v + f_2(u)$,

(5) $\varphi(v,u) = f_1(u)v + f_2(u)$,

where f_1, f_2 are arbitrary smooth enough functions. Consequently, we have the following maps transforming subsets of solutions of the linear d'Alembert equation (79) into another subsets of its solutions:

$$
\begin{aligned}
&(1)\quad u \to f_1(u)\big[\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-1}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big)^2 + \\
&\qquad + \big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big)^2\big]^{-1} + f_2(u),\\
&(2)\quad u \to f_1(u)\big[\big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{1/2}\big(\dot{A}_\mu(u)x^\mu + \dot{B}(u)\big)\big]^{-1} + f_2(u),\\
&(3)\quad x_0 - x_3 u \to f_1(x_0 - x_3)\ln\big[\big(x_1 + C_1(x_0 - x_3)\big)^2 + \\
&\qquad + \big(x_2 + C_2(x_0 - x_3)\big)^2\big]^{-1/2} + f_2(x_0 - x_3),\\
&(4)\quad u \to \big(-\dot{A}_\nu(u)\dot{A}^\nu(u)\big)^{-3/2}\big(\varepsilon^{\mu\nu\alpha\beta}A_\mu(u)\dot{A}_\nu(u)\ddot{A}_\alpha(u)x_\beta + C(u)\big),\\
&(5)\quad x_0 - x_3 \to f_1(x_0 - x_3)(x_1\cos C_1(x_0 - x_3) + \\
&\qquad + x_2\sin C_1(x_0 - x_3) + C_2(x_0 - x_3).
\end{aligned}
$$

Note that in the cases 4, 5 function f_2 is absorbed by arbitrary functions C, C_2.

And one more remark seems to be noteworthy. If one takes as a particular solution of the system (2) the function $u(x) = (x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2})/(x_1^2 + x_2^2)$ and substitutes it into the first, second and fourth Ansätze from (70), then the following Ansätze are obtained:

$$
\begin{aligned}
&(1)\quad w(x) = \varphi\left(x_1^2 + x_2^2 + x_3^2 - x_0^2, \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right),\\
&(2)\quad w(x) = \varphi\left(x_1^2 + x_2^2 - x_0^2, \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right),\\
&(4)\quad w(x) = \varphi\left(x_3, \frac{x_0x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2}\right).
\end{aligned}
$$

Provided the above Ansätze do not depend on the second argument, the usual Lie Ansätze are obtained which are invariant under some subgroups of the Poincaré group $P(1,3)$ [19]. Consequently, if we imagine invariant solutions as dots in a solution space of the nonlinear d'Alembert equation, then through some of them one can draw curves which are conditionally-invariant solutions. In this respect a number of interesting questions arise, let us mention two of these:

(1) Is any invariant solution of the nonlinear d'Alembert equation (33) a particular case of some more general conditionally-invariant solution?

(2) Does there exist such conditionally-invariant solution of Eq. (33) that all invariant solutions of Eq. (33) are its particular cases? (saying about invariant solutions we mean solutions invariant under some subgroup of the symmetry group of Eq. (33)).

An answer to the first question seems to be positive. A positive answer to the second one would provide us with a concept of a "general invariant solution". But so far this problem is completely open and needs further investigation.

Acknowledgments. One of the authors (R. Zhdanov) is supported by the Alexander von Humboldt Foundation.

1. Kaluza Th., *Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl.*, 1921, **996**.
2. Khodos A., *Uspekhi Fiz. Nauk*, 1985, **146**, 647.
3. Fushchych W., Krivsky I., *Nuclear Physics*, 1968, **17**, 79.
4. Fushchych W., Krivsky I., *Nuclear Physics*, 1969, **18**, 573.
5. Fushchych W., *Theor. Math. Phys.*, 1970, **4**, 360.
6. Fushchych W., Nikitin A., *J. Phys. A: Math. Gen.*, 1980, **13**, 2319.
7. Cieciura G., Grundland A., *J. Math. Phys.*, 1984, **25**, 3460.
8. Fushchych W., Zhdanov R., Yegorchenko I., *J. Math. Anal. Appl.*, 1991, **161**, 352.
9. Smirnov V., Sobolev S., *Trudy Seismol. Instituta AN SSSR*, 1932, **20**, 37.
10. Smirnov V., Sobolev S., *Trudy Seismol. Instituta AN SSSR*, 1933, **29**, 43.
11. Yerugin N.P., *Doklady AN SSSR*, 1944, **20**, 385.
12. Collins C.B., *J. Math. Phys.*, 1983, **24**, 22.
13. Bateman H., The mathematical analysis of electrical and optical wave motion of the basis of Maxwell's equations, Cambridge, Univ. Press, 1915.
14. Shulga M., in Group-Theoretical Studies of the Mathematical Physics Equations, Kyiv, Inst. of Math., 1985, 36.
15. Fushchych W., Zhdanov R., Revenko I., *Ukr. Math. J.*, 1991, **43**, 1364.
16. Fushchych W., Shtelen W., Serov M., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Dordrecht, Kluwer Academic Publisher, 1992.
17. Fushchych W., Zhdanov R., *Phys. Reports*, 1989, **172** 123.
18. Collins C.B., *Math. Proc. Cambr. Phyl. Soc.*, 1976, **80**, 165.
19. Patera J., Sharp R.T., Winternitz P., Zassenhaus H., *J. Math. Phys.*, 1976, **17**, 977.
20. Fushchych W., *Ukr. Math. J.*, 1991, **43**, 1456.
21. Fushchych W., Zhdanov R., *Ukr. Math. J.*, 1992, **44**, 970.
22. Bluman G., Cole J., *J. Math. Mech.*, 1969, **18**, 1025.
23. Olver P., Rosenau Ph., *Phys. Lett. A*, 1987, **112**, 107.
24. Levi D., Winternitz P., *J. Phys. A: Math. Gen.*, 1989, **22**, 2922.
25. Clarkson P., Kruskal M., *J. Math. Phys.*, 1989, **30**, 2201.
26. Kamke E., Differentialgleichungen Lösungmethoden und Lösungen, Leipzig, Akademische Verlagsgesellschaft, 1976.
27. Winternitz P., Grundland A., Tuszyński J., *J. Math. Phys.*, 1987, **28**, 2194.
28. Fushchych W., Zhdanov R., Revenko I., in Symmetry Analysis of Equations of Mathematical Physics, Kyiv, Inst. of Math., 1992, 68.

*This paper is last in the series of works of Professor Fushchych and collaborators, which are devoted to investigation of compatibility, integration and symmetries of overdetermined d'Alembert–Hamilton systems (see [165, 187, 211, 236, 237, 238, 258, 315] in *List of Scientific Works of W.I. Fushchych*). *Editors' Remark.*

Симетрія рівнянь лінійної та нелінійної квантової механіки

В.І. ФУЩИЧ

We describe local and nonlocal symmetries of linear and nonlinear wave equations and present a classification of nonlinear multi-dimensional equations compatible with the Galilei relativity principle. We propose new systems of nonlinear equations for the description of physical phenomena in classical and quantum mechanics.

Описані локальні і нелокальні симетрії лінійних та нелінійних хвильових рівнянь, класифікації нелінійних багатовимірних рівнянь, сумісних з принципом відносності Галілея. Запропоновано нові системи нелінійних рівнянь для опису фізичних процесів у класичній та квантовій механіці.

Проблема побудови нелінійних математичних моделей для опису процесів у механіці, фізиці, біології була і є однією з головних задач математичної фізики [1–4]. Сьогодні ми не можемо вважати, що класичні рівняння Ньютона–Лоренца, Даламбера, Нав'є–Стокса, Максвелла, Шрьодінгера, Дірака та інші рівняння руху послідовно і повно описують реальні фізичні процеси. У зв'язку з цим досить сказати, що нині ми не знаємо жодного рівняння руху в квантовій механіці для двох частинок, яке було б сумісне з принципом відносності Лоренца–Пуанкаре–Ейнштейна. Широкий спектр рівнянь, які запропоновані багатьма дослідниками, як правило мають принципові недоліки і часто приводять до абсурдних фізичних наслідків.

Характерна особливість сучасного математичного опису реальних процесів полягає в тому, що рівняння руху для частинок, хвиль, полів є складними нелінійними системами диференціальних і інтегро-диференціальних рівнянь. Як будувати такі рівняння? Як розв'язувати і досліджувати такі системи? Очевидно, що підхід Лагранжа–Ойлера (механічний у своїй основі) до побудови рівняння руху у багатьох випадках є обмеженим. Досить нагадати, що в рамках класичного методу Лагранжа–Ойлера неможливо одержати без переходу до потенціалів рівняння Максвелла для електромагнітних хвиль.

В наших роботах [3–12] запропоновано нелагранжевий підхід для побудови і класифікації рівнянь руху. В основі цього підходу лежать принципи відносності Галілея та Лоренца–Пуанкаре–Ейнштейна. Короткий огляд деяких результатів у цьому напрямку подається далі.

1. Короткий коментар про відкриття Шрьодінгера. Перш за все нагадуємо, що 70 років тому Ервін Шрьодінгер відкрив рівняння руху і цим самим заклав математичну основу квантової механіки. 21 червня 1926 р. Шрьодінгер представив до друку роботу [2], в якій запропонував рівняння

$$
\begin{gathered}
S\Psi = 0, \quad S = p_0 - \frac{p_a^2}{2m} - V(t, x), \\
p_0 = i\hbar\frac{\partial}{\partial t}, \quad p_a = -i\hbar\frac{\partial}{\partial x_a}, \quad a = 1, 2, 3,
\end{gathered}
\tag{1}
$$

де $\Psi = \Psi(x_0 = t, \vec{x})$ — комплекснозначна хвильова функція, V — потенціал.

Передруковано з Укр. мат. журн., 1997, **49**, № 1, С. 164–177 за дозволом

Ця робота була останньою з серії чотирьох статей під однією назвою, в яких розв'язана проблема квантування в атомній фізиці.

Чи можна сказати, що Ервін Шрьодінгер вивів своє рівняння?

Знайомство з оригінальною роботою [2] дає нам однозначну відповідь на це питання. Шрьодінгер не вивів рівняння. Рівняння (1) було написано без строгого обґрунтування, більше того, Шрьодінгер вважав, що правильним рівнянням руху в квантовій механіці повинно бути рівняння четвертого порядку для дійсної функції, а не рівняння (1) для комплексної функції. Шрьодінгер розглядував рівняння (1) як деяке допоміжне, проміжне рівняння, яке дає змогу спрощувати обчислення.

В основі попередніх його робіт були рівняння

$$\Delta\Psi - \frac{2(E-V)}{E^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{2}$$

$$\Delta\Psi + \frac{8\pi^2}{\hbar^2}(E-V)\Psi = 0, \tag{3}$$

де Ψ — дійсна функція.

Коли потенціал V не залежить від часу, Шрьодінгер виводить з (2), (3) хвильове рівняння четвертого порядку

$$\left(\Delta - \frac{8\pi^2}{\hbar^2}V\right)^2\Psi + \frac{16\pi^2}{\hbar^2}\frac{\partial^2\Psi}{\partial t^2} = 0, \tag{4}$$

де Ψ — дійсна функція.

Про рівняння (4) Шрьодінгер пише: "… рівняння (4) є єдиним і загальним хвильовим рівнянням для польового скаляра Ψ … хвильове рівняння (4) заключає в собі закон дисперсії і може служити основою розвинутої мною теорії консервативних систем. Його узагальнення на випадок потенціалу вимагає деяку обережність … спроба перенести рівняння (4) на неконсервативні системи зустрічається зі складністю, яка виникає через член $\frac{\partial V}{\partial t}$. Тому далі я піду по іншому шляху, більш простому з обчислювальної точки зору. Цей шлях я вважаю принципово самим правильним. (4) є рівняння коливання пластинки."

У листі до Лоренца (6 червня 1926 р., Цюріх) Шрьодінгер пише: "… з рівнянь (2) і (3) ми одержуємо загальне хвильове рівняння (4), яке не залежить від константи інтегрування E. Воно точно співпадає з рівнянням коливання пластинки, яке містить квадрат оператора Лапласа. Відкриття цього простого факту забрало у мене багато часу."

У листі до Планка (14 червня 1926 р., Цюріх) Шрьодінгер пише: "… отже, справжнім хвильовим рівнянням є рівняння четвертого порядку відносно координат …".

І далі Шрьодінгер виписує рівняння (1) для комплексної функції Ψ. Якраз у цьому місці статті [2] Шрьодінгер робить геніальний (і алогічний) крок, записуючи рівняння (1) для комплексної функції.

Відносно рівняння (1) Шрьодінгер пише: "Деяка трудність, без сумніву, виникає в застосуванні комплексних хвильових функцій. Якщо вони принципово необхідні, а не є тільки способом полегшення (спрощення) обчислень, то це буде означати, що існують принципово дві функції, які тільки разом дають опис стану системи ... Справжнє хвильове рівняння, найбільш вірогідно, має бути рівняння четвертого порядку. Для неконсервативної системи $\left(\frac{\partial V}{\partial t} \neq 0\right)$ мені, одначе, не вдалось знайти таке рівняння".

З наведеного ми можемо зробити такі висновки.

Висновок 1. *В* 1926 *році Шрьодінгер вважав, що правильним рівнянням руху в квантовій механіці має бути рівняння четвертого порядку. Для випадку, коли потенціал не залежить від часу, це рівняння має вигляд* (4).

Висновок 2. *В червні* 1926 *року Шрьодінгер вважав, що рівняння* (1)*, першого порядку за часом і другого порядку за просторовими змінними, для комплексної функції є проміжним (не основним), яке треба використати тільки для спрощення обчислень.*

Висновок 3. *Шрьодінгер вважав, що в тому випадку, коли потенціал V залежить від часу, рівняння руху має бути також четвертого порядку для дійсної функції (йому його не вдалось одержати).*

Висновок 4. *Шрьодінгер ніколи пізніше не обговорював рівняння четвертого порядку.*

Сьогодні можна однозначно сказати, що Шрьодінгер помилявся відносно важливості (фундаментальності) рівнянь (1), (4). Дійсно, рівняння (1) є основним рівнянням руху квантової механіки, а рівняння (4) не може бути рівнянням руху, оскільки воно не сумісне з принципом відносності Галілея.

Це твердження є наслідком симетрійного аналізу рівнянь (1) і (4) [3]: рівняння (1) інваріантне відносно групи Галілея. У зв'язку з наведеним у наступному параграфі дано відповідь на такі питання:

1. Які лінійні рівняння другого, четвертого, n-го порядку сумісні з принципом відносності Галілея?
2. Чи існують лінійні рівняння першого порядку за часовою змінною і четвертого порядку за просторовими змінними, які сумісні з принципом відносності Галілея?

Під принципом відносності Галілея ми розуміємо інваріантність (у сенсі Лі) рівняння відносно перетворень

$$t \to t' = t, \quad \vec{x} \to \vec{x}' = \vec{x} + \vec{v}t, \tag{5}$$

коли хвильова функція перетворюється за лінійним зображенням групи (5) [4]:

$$\Psi \to \Psi' = T_g \Psi. \tag{6}$$

Перш ніж дати відповідь на сформульовані питання наведемо добре відомі факти про локальну симетрію лінійного вільного ($V = 0$) рівняння Шрьодінгера (1).

Теорема 1 [3]. *Максимальною (у сенсі Лі) алгеброю інваріантності* (1) *є* 13*-вимірна алгебра Лі*

$$AG_2(1,3) = \langle P_0, P_a, J_{ab}, G_a, D, \Pi, Q \rangle,$$

з базисними елементами

$$P_0 = i\frac{\partial}{\partial x_0} = p_0, \quad P_a = -i\frac{\partial}{\partial x_a} = p_a, \quad J_{ab} = x_a p_b - x_b p_a,$$
$$G_a = tp_a - mx_a, \quad a = 1,2,3, \quad D = 2x_0 p_0 - x_a p_a, \tag{7}$$
$$\Pi = x_0^2 p_0 - x_0 x_a p_a + \frac{in}{2}x_0 - \frac{m}{2}x_a^2, \quad Q = i\left(\Psi\frac{\partial}{\partial\Psi} - \Psi^*\frac{\partial}{\partial\Psi^*}\right).$$

Оператори G_a породжують (генерують) перетворення Галілея (5) і таке перетворення для хвильової функції:

$$\Psi \to \Psi' = \exp\left\{i\left(\vec{v}\vec{x} + \frac{\vec{v}^{\,2}t}{2}\right)\right\}\left\{\Psi(t,x)\Big|_{\vec{x}\to\vec{x}+\vec{v}t}\right\} \tag{8}$$

(деталі доведення див. у [4] і цитованій там літературі).

Ми вживаємо наступні позначення:

$AG(1,3) = \langle P_0, P_a, J_{ab}, G_a\rangle$ — 10-вимірна алгебра Галілея;

$AG_1(1,3) = \langle P_0, P_a, J_{ab}, G_a, D\rangle$ — розширена алгебра Галілея;

$AG_2(1,3) = \langle P_0, P_a, J_{ab}, G_a, D, \Pi\rangle$ — повна алгебра Галілея;

$AE(1,3) = \langle P_0, P_a, J_{ab}\rangle$ — алгебра Евкліда;

$AE_1(1,3) = \langle P_0, P_a, J_{ab}, D\rangle$ — розширена алгебра Евкліда.

Теорема 2 [5]. *Максимальною алгеброю інваріантності рівняння* (4) ($V = 0$) *є розширена алгебра Евкліда* $AE_1(1,3)$.

З наведених теорем маємо такі наслідки.

Наслідок 1. *Рівняння* (4) *несумісне з принципом відносності Галілея* (5). *Це означає, що* (4) *не може розглядатись, як рівняння руху частинки (поля) в квантовій механіці. Вся множина розв'язків рівняння* (4) *не інваріантна відносно перетворень Галілея* (5), (6).

Зауважимо, що будь-який гладкий розв'язок рівняння (1) є розв'язком рівняння (4) (при $V = 0$), тобто множина розв'язків (4) містить у собі розв'язки (2).

2. Виведення рівняння Шрьодінгера і рівняння високого порядку. Виведемо рівняння Шрьодінгера з вимоги інваріантності рівняння відносно перетворень Галілея (5), (8) і групи часових і просторових трансляцій.

Розглянемо довільне лінійне рівняння першого порядку за часом і другого порядку за просторовими змінними

$$i\frac{\partial\Psi}{\partial t} = a_{lk}(t,\vec{x})\frac{\partial^2\Psi}{\partial x_l\partial x_k} + b_l(t,\vec{x})\frac{\partial\Psi}{\partial x_l} + c(t,\vec{x})\Psi, \tag{9}$$

де $a_{lk}(t,\vec{x})$, $b_l(t,\vec{x})$, $c(t,\vec{x})$ — довільні гладкі функції.

Теорема 3 [5, 6]. *Серед множини рівнянь* (9), *інваріантних відносно групи* (5) *і групи трансляцій, для комплексної функції* Ψ *є тільки одне рівняння, яке локально еквівалентне рівнянню Шрьодінгера* (1).

Отже, клас лінійних рівнянь, які сумісні з класичним принципом відносності Галілея, зводиться до одного рівняння (1).

Зауваження 1. Якщо в (9) Ψ — дійсна функція, то єдиним рівнянням, сумісним з принципом Галілея, є рівняння теплопровідності

$$\frac{\partial u}{\partial t} = \lambda \Delta u, \tag{10}$$

де λ — довільний параметр.

В [7] запропоноване таке узагальнення рівняння ($V=0$) Шрьодінгера (1):

$$\begin{gathered}\left(\lambda_1 S + \lambda_2 S^2 + \cdots + \lambda_n S^n\right)\Psi = \lambda\Psi,\\ S^2 = \left(p_0 - \frac{p_a^2}{2m}\right)^2, \quad \ldots, \quad S^n = \left(p_0 - \frac{p_a^2}{2m}\right)^n,\end{gathered} \tag{11}$$

$\lambda, \lambda_1, \lambda_2, \ldots, \lambda_n$ — довільні параметри.

Рівняння (11) сумісне з принципом відносності Галілея і інваріантне відносно алгебри Галілея $AG(1,3)$, але не інваріантне відносно масштабного D і проективного Π операторів ($\lambda_1 \neq 0$, $\lambda_2 \neq 0$).

Повну інформацію про симетрію рівняння (11) дає наступна теорема.

Теорема 4 [13]. *Серед лінійних рівнянь довільного порядку є тільки рівняння* (11)*, яке інваріантне відносно алгебри* $AG(1,3)$*. У випадку, коли* $\lambda = \lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 0$*, рівняння* (11) *інваріантне відносно алгебри* $AG_2(1,3)$*.*

Таким чином, клас лінійних галілей-інваріантних рівнянь довільного порядку досить вузький і зводиться до рівняння (11). Всі інші галілей-інваріантні рівняння локально еквівалентні рівнянню (11).

3. Алгебра Лоренца для рівняння Шрьодінгера. Лінійне рівняння Шрьодінгера (коли $V=0$ і при деяких специфічних видах потенціалів $V(t,x)$) має крім локальної (теорема 1) і нелокальну симетрії (див. [4] і цитовану там літературу). Наведемо одну з таких незвичних (нелокальних) симетрій.

Теорема 5 [8]. *Рівняння Шрьодінгера* (1) *(коли* $V=0$*) інваріантне відносно алгебри Лоренца* $AL(1,3) = \langle J_{ab}, J_{0a}\rangle$*, базисні елементи якої задаються операторами*

$$\begin{gathered}J_{ab} = x_a p_b - x_b p_a, \quad J_{0a} = \frac{1}{2m}(pG_a + G_a p),\\ p \equiv (p_1^2 + p_2^2 + p_3^2)^{1/2} = (-\Delta)^{1/2},\\ G_a \equiv x_0 p_a - m x_a, \quad [J_{0a}, J_{0b}] = -iJ_{ab}.\end{gathered} \tag{12}$$

Важливо підкреслити, що псевдодиференціальні оператори $\langle J_{0a}\rangle$ не генерують ні перетворення Лоренца, ні перетворення Галілея:

$$x_a \to x_a' = \exp\{iJ_{0a}v_a\}x_a \exp\{-J_{0b}v_b\} \neq \text{ лоренц-перетворення}, \tag{13}$$

$$x_0 \to x_0' = \exp\{iJ_{0a}v_a\}x_0 \exp\{-J_{0b}v_b\} = x_0. \tag{14}$$

Час при таких нелокальних перетвореннях не міняється.

4. Нелокальна галілей-симетрія еволюційного рівняння четвертого порядку. Розглянемо рівняння першого порядку за часовою змінною і четвертого порядку за просторовими змінними

$$p_0\Psi = \mathcal{H}(p^2)\Psi, \quad \mathcal{H}(p^2) = a_0 m_0 + a_2 p^2 + a_4 \frac{p^4}{8}, \tag{15}$$

$p^2 = p_a^2 = p_1^2 + p_2^2 + p_3^2 = -\Delta$, $a_2 = \frac{1}{2m_0}$, a_0, a_2, a_4, m_0 — довільні дійсні константи.

Гамільтоніан (15), коли $a_0 = 1$, $a_4 = -m_0^{-3}$, являє собою перші три члени розкладу в ряд Тейлора релятивістського гамільтоніана

$$\mathcal{H}(p^2) = (p^2 + m_0^2)^{1/2} = m_0 + \frac{p^2}{2m_0} - \frac{p^4}{8m_0^3}.$$

У тому випадку, коли $a_0 = a_4 = 0$, рівняння (15) співпадає з рівнянням Шрьодінгера (1).

Зі стандартної (загально прийнятої) фізичної точки зору рівняння (15) не можна розглядувати як рівняння руху в квантовій механіці, оскільки воно не інваріантне ні відносно групи Галілея, ні відносно групи Лоренца. Тобто ні один з відомих принципів відносності (Галілея або Лоренца–Пуанкаре–Ейнштейна) не виконується для рівняння (15).

Застосовуючи метод Лі, можна довести, що максимальною алгеброю інваріантності рівняння (15) є алгебра Евкліда $AE(1,3) = \langle P_0, P_a, J_{ab}, I\rangle$, I — одиничний оператор. Виявляється, що крім локальної симетрії рівняння (15) має широку нелокальну симетрію. Зокрема, рівняння (15) інваріантне відносно алгебри Галілея $AG(1,3)$, базисні елементи (оператори G_a) якої задаються операторами 3-го порядку. Більш точно, справедливе наступне твердження.

Теорема 6 [9, 10]. *Рівняння* (15) *інваріантне відносно* 20-*вимірної алгебри Лі, базисні елементи якої задаються операторами*

$$P_0 = i\frac{\partial}{\partial t}, \quad P_a = -i\frac{\partial}{\partial x_a}, \quad J_{ab} = x_a p_b - x_b p_a, \tag{16}$$

$$G_a = (tV_a - x_a)m_0, \tag{17}$$

$$V_a = \frac{1}{m_0}\left(1 + a_4\frac{p^2}{2m_0^2}\right)p_a, \tag{18}$$

$$R_{ab} = a_4\left(P_aP_b + \frac{1}{2}\delta_{ab}P^2\right). \tag{19}$$

Оператори (16)–(19) задовольняють комутаційні співвідношення

$$[J_{ab}, G_c] = i(\delta_{ac}G_b - \delta_{bc}G_a), \quad [P_a, G_b] = i\delta_{ab}I, \quad [G_a, G_b] = 0,$$
$$[P_0, G_a] = iV_a, \quad [V_a, G_b] = i(R_{ab} - a_2\delta_{ab}I),$$
$$[J_{ab}, R_{cd}] = i(\delta_{ac}R_{bd} + \delta_{bd}R_{ac} - \delta_{bc}R_{ad} - \delta_{ad}R_{bc}),$$
$$[J_{ab}, V_c] = i(\delta_{ac}V_b - \delta_{bc}V_a), \quad [G_a, R_{bc}] = ia_4(\delta_{ab}P_c + \delta_{bc}P_a + \delta_{ac}P_b).$$

Підкреслимо, що оператори (17)–(19) є операторами третього і другого порядку, а це означає, що вони породжують нелокальні перетворення. Так, оператори Галілея G_a (17) генерують стандартні локальні перетворення для часу і координат

$$t \to t' = \exp(iu_aG_a)t\exp(-iu_bG_b) = t,$$
$$x_a \to x_a' = \exp(iv_bG_b)x_a\exp(-v_lG_l) = x_a + v_at,$$

і нелокальні перетворення для хвильової функції [3]

$$\Psi(x) \to \Psi'(x) = \exp\left\{ im_0 \left(x_a u_a + \frac{\vec{u}^{\,2}}{2} t - \frac{1}{2} a_4 t u_a P_a P^2 \right) \right\} \Psi. \tag{20}$$

Як добре відомо, швидкість частинки в релятивістській механіці визначається за формулою

$$v_a = \frac{p_a}{m}, \quad m = m(\vec{v}^{\,2}), \quad m = m_0 (1 - \vec{v}^{\,2})^{-1/2}. \tag{21}$$

У механіці, побудованій на базі рівняння (15), відповідна формула має вигляд

$$v_a = \frac{p_a}{m_0} + \frac{a_4 p^2}{2} p_a. \tag{22}$$

Якщо швидкість частинки задати формулою (21) і використати (22), то одержимо формулу залежності маси (в новій механіці) від швидкості

$$\frac{m}{m_0} + \frac{a_4}{2} m^3 v^2 - 1 = 0. \tag{23}$$

Розв'язавши кубічне рівняння (23), одержимо (в залежності від знака коефіцієнта a_4) такі формули:

$$m = m_0 \frac{3}{\omega} \sin\left\{ \frac{1}{3} \operatorname{arctg} \frac{\omega}{\sqrt{1 - \omega^2}} \right\}, \quad a_4 < 0, \quad \omega \neq 1, \tag{24}$$

$$m = m_0 \frac{3}{\omega} \operatorname{sh}\left\{ \frac{1}{3} \ln(\omega + \sqrt{1 + \omega^2}) \right\}, \quad a_4 > 0, \tag{25}$$

$$\omega = \left(\frac{3}{2} \right)^{3/2} (\vec{v})^2 \sqrt{m_0^3 |a_4|}.$$

Отже, у квантовій механіці, побудованій на рівнянні (15), виконується нестандартний принцип відносності Галілея (формула (20)) і маса частинки (поля) залежить від швидкості згідно з формулами (24), (25).

5. Принцип відносності Галілея і нелінійні рівняння типу Шрьодінгера. За останні роки багато авторів, виходячи з різних мотивів і міркувань, запропонували широкий спектр нелінійних узагальнень рівнянь Шрьодінгера. Багато з нелінійних рівнянь, запропонованих для опису нелінійних ефектів у плазмі, оптиці, квантовій механіці, не задовольняють принцип відносності Галілея. У зв'язку з цим в серії наших робіт [3, 4, 6, 7, 11] проведено симетрійну класифікацію нелінійних рівнянь типу Шрьодінгера, які інваріантні відносно групи Галілея та різних її розширень.

У цьому пункті наведемо деякі результати про класифікацію нелінійних рівнянь типу Шрьодінгера, які мають таку ж симетрію (або ширшу), як і лінійне рівняння Шрьодінгера (1).

Розглянемо нелінійне рівняння другого порядку

$$i \frac{\partial \Psi}{\partial t} + \frac{1}{2} \Delta \Psi + i \frac{\Delta \varphi(\Psi^* \Psi)}{2 \Psi^* \Psi} \Psi = F\left(\Psi^* \Psi, (\vec{\nabla}(\Psi^* \Psi))^2, \Delta(\Psi^* \Psi) \right) \Psi, \tag{26}$$

де φ, F — довільні гладкі функції.

Теорема 7 [7, 11]. *Рівняння* (26) *у випадку, коли* $\varphi = 0$, *а функція* $F(\Psi^*\Psi)$ *не залежить від похідних, інваріантне відносно повної алгебри* $AG_2(1,n)$ *з базисними елементами* (7) *тоді і тільки тоді, коли*

$$F(\Psi^*\Psi) = \lambda|\Psi|^{4/n}, \tag{27}$$

де n *— число просторових змінних.*

Теорема 8 [12]. *Рівняння* (26) *інваріантне відносно алгебри* $AG_2(1,n)$ *і оператора* I *тоді і тільки тоді, коли*

$$F\left(\Psi^*\Psi, (\vec{\nabla}(\Psi^*\Psi))^2, \Delta(\Psi^*\Psi)\right) = \frac{\Delta|\Psi|}{|\Psi|} N\left(\frac{|\Psi|\Delta|\Psi|}{(\vec{\nabla}|\Psi|)^2}\right), \quad \varphi(\Psi^*\Psi) = |\Psi|^2, \tag{28}$$

де N *— довільна гладка функція.*

В тому випадку, коли $N = 1/2$, $\varphi = 0$, рівняння (26) набуває вигляду

$$i\frac{\partial\Psi}{\partial t} + \frac{1}{2}\Delta\Psi = \frac{1}{2}\frac{\Delta|\Psi|}{|\Psi|}\Psi. \tag{29}$$

Рівняння (29) запропоновано у роботах [14–18]. Воно має унікальну симетрію.

Теорема 9 [19]. *Рівняння* (29) *інваріантне відносно алгебри Лі з базисними операторами*

$$\begin{gathered}
P_0 = i\frac{\partial}{\partial t}, \ P_a = -i\frac{\partial}{\partial x_a}, \ I = \Psi\frac{\partial}{\partial\Psi} + \Psi^*\frac{\partial}{\partial\Psi^*}, \\
J_{ab} = x_aP_b - x_bP_a, \quad a,b = 1,2,\dots,n, \\
G_a = tP_a + \frac{x_a}{2}Q, \quad Q = i\left(\Psi\frac{\partial}{\partial\Psi} - \Psi^*\frac{\partial}{\partial\Psi^*}\right), \\
D = 2tP_0 + x_aP_a - \frac{n}{2}I, \quad \Pi = t^2P_0 + tx_aP_a + \frac{|\vec{x}|}{4}Q - \frac{nt}{2}I,
\end{gathered} \tag{30}$$

$$\begin{gathered}
G_a^{(1)} = -i\ln\frac{\Psi}{\Psi^*}P_a + x_aP_0, \quad D^{(1)} = -i\frac{\Psi}{\Psi^*}Q + x_aP_a, \\
\Pi^{(1)} = -\left(\ln\frac{\Psi}{\Psi^*}\right)Q - 2i\left(\ln\frac{\Psi}{\Psi^*}\right)x_aP_a + |\vec{x}|^2P_0 + in\left(\ln\frac{\Psi}{\Psi^*}\right)I, \\
K_a = tx_aP_0 - \left(\frac{|\vec{x}|^2}{2} + it\ln\frac{\Psi}{\Psi^*}\right)P_a + x_ax_bP_b - \frac{n}{2}x_aI - i\frac{x_a}{2}\left(\ln\frac{\Psi}{\Psi^*}\right)Q.
\end{gathered} \tag{31}$$

Виписана алгебра еквівалентна конформній алгебрі $AC(2,n)$ в $(2+n)$-вимірному просторі Мінковського. Якщо від комплексної функції Ψ перейти до амплітуди-фази

$$\Psi = A(t,x)\exp\{i\Theta(t,x)\},$$

то наведені формули значно спрощуються. Алгебра симетрії рівняння (29) еквівалентна алгебрі симетрії класичного рівняння Гамільтона [3]

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_k}.$$

Отже, нелінійне рівняння (29) має значно ширшу симетрію, ніж лінійне рівняння Шрьодінгера (1). Аналогічний ефект має місце і для пуанкаре-інваріантного нелінійного хвильового рівняння [16, 17]

$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\Psi. \tag{32}$$

6. Нелокальна симетрія лінійного пуанкаре-інваріантного хвильового рівняння. Сімдесят років тому, у 1926 р. майже одночасно сім учених: Шрьодінгер, де Броль, Дондер ван Дунген, Клейн, Фок, Гордон і Кудар відкрили рівняння

$$(p_0^2 - p_a^2)u(x_0, \vec{x}) = m^2 u \tag{33}$$

для скалярної комплексної функції u. У випадку, коли $m = 0$ (33) співпадає з хвильовим рівнянням Даламбера.

Відомо, що рівняння (33) інваріантне відносно алгебри Пуанкаре $AP(1,3)$ з базисними елементами

$$\begin{aligned} &P_0 = p_0, \quad P_k = p_k, \quad k = 1,2,3, \\ &J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad \mu,\nu = 0,1,2,3, \end{aligned} \tag{34}$$

тобто виконуються умови:

$$[p_0^2 - p_a^2 - m^2, J_{\mu\nu}] = 0, \quad [p_0^2 - p_a^2 - m^2, P_\mu] = 0. \tag{35}$$

Алгебра $AP(1,3) = \langle P_\mu, J_{\mu\nu}\rangle$ є максимальною (у сенсі Лі) алгеброю інваріантності рівняння (33).

Оператори $\langle J_{0a}\rangle$ генерують стандартні перетворення Лоренца

$$x_\mu \to x'_\mu = \exp(iJ_{0a}v_a)x_\mu \exp(-iJ_{0b}v_b) = \text{ перетворення Лоренца.}$$

В [20] поставлено і дано позитивну відповідь на таке питання: чи має рівняння (33) додаткову симетрію, відмінну від (34)?

Щоб виявити додаткову (нелокальну) симетрію (33), перепишемо його у вигляді системи двох рівнянь першого порядку за часовою змінною і другого порядку за просторовими змінними

$$\begin{aligned} &i\frac{\partial\Phi}{\partial t} = H\Phi, \\ &H = \frac{1}{2\kappa}\Big\{(E^2+\kappa^2)\sigma_1 + i(E^2-\kappa^2)\sigma_2\Big\}, \\ &E^2 = -\Delta + m^2, \quad \kappa \neq 0, \quad \Phi = \begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}, \quad \kappa\Phi_1 = i\frac{\partial u}{\partial t}, \quad \Phi_2 = u, \end{aligned} \tag{36}$$

κ — довільна константа, σ_1 і σ_2 — (2×2)-матриці Паулі.

Теорема 10 [20]. *Рівняння* (36) *інваріантне відносно алгебри Пуанкаре, базисні оператори якої мають вигляд*

$$P_0^{(1)} = H, \quad P_k^{(1)} = p_k, \quad J_{ab}^{(1)} = x_a p_b - x_b p_a = J_{ab}, \tag{37}$$

$$J_{0a}^{(1)} = x_0 P_a - \frac{1}{2}(Hx_a + x_a H) \neq J_{0a}. \tag{38}$$

Прямою перевіркою можна переконатись, що оператори (37), (38) задовольняють умови

$$\left[i\frac{\partial}{\partial t} - H, J_{0a}\right] = 0, \quad \left[i\frac{\partial}{\partial t} - H, J_{ab}\right] = 0. \tag{39}$$

Істотна різниця між операторами $J_{0a}^{(1)}$ і J_{0a} полягає в тому, що $J_{0a}^{(1)}$ — оператори другого порядку і генерують нелокальні перетворення; J_{0a} — оператори першого порядку і генерують стандартні локальні перетворення Лоренца.

Підкреслимо, що оператори $J_{0a}^{(1)}$ генерують тотожне перетворення для часу, тобто час інваріантний відносно операторів $J_{0a}^{(1)}$:

$$t \to t' = \exp(iJ_{0a}^{(1)} v_a) t \exp(-iJ_{0b}^{(1)} v_b) = t. \tag{40}$$

Просторові перетворення змінних x_a, які генеруються операторами $J_{0a}^{(1)}$, не співпадають з перетвореннями Лоренца:

$$x_k \to x'_k = \exp(iJ_{0a}^{(1)} v_a) x_k \exp(-iJ_{0b}^{(1)} v_b) \neq \text{ перетворення Лоренца}. \tag{41}$$

Таким чином, ми встановили, що множина розв'язків рівняння (33) має дуальну симетрію:

1) лоренцову (локальну) симетрію; час змінюється при переході від однієї інерційної системи до іншої за формулами Лоренца.
2) нелоренцову (нелокальну) симетрію (40), (41); час не змінюється при переході від однієї інерційної системи до іншої.

7. Нелокальна галілей-симетрія релятивістського псевдодиференціального хвильового рівняння. Розглянемо псевдодиференціальне рівняння

$$p_0 u = Eu, \quad E \equiv (p_a^2 + m^2)^{1/2}, \quad u = u(x_0, \vec{x}). \tag{42}$$

Рівняння (42) можна розглядувати як "корінь квадратний з хвильового оператора (33)" для скалярної комплексної функції u. Прямим обчисленням можна переконатись, що рівняння (42) інваріантне відносно стандартного зображення алгебри Пуанкаре (34) і не інваріантне відносно стандартного зображення алгебри Галілея (7).

Теорема 11 [9]. *Рівняння* (42) *інваріантне відносно* 11*-вимірної алгебри Галілея з такими базисними операторами:*

$$\begin{gathered}
P_0^{(2)} = \frac{p^2}{2m} = -\frac{\Delta}{2m}, \quad P_a^{(2)} = p_a = -\frac{\partial}{\partial x_a}, \quad J_{ab}^{(2)} = x_a p_b - x_b p_a \equiv J_{ab}, \\
G_a^{(2)} = t\widetilde{p}_a - m x_a, \qquad \widetilde{p}_a \equiv \frac{m}{E} p_a, \quad E = (p_a^2 + m^2)^{1/2}.
\end{gathered} \tag{43}$$

Доведення теореми зводиться до перевірки умови інварінтності

$$[p_0 - E, Q_l] u = 0, \tag{44}$$

де Q_l — будь-який оператор з набору (43).

Оператори (43) задовольняють комутаційні співвідношення алгебри Галілея; $G_a^{(2)}$ — псевдодиференціальні оператори, які генерують на відміну від стандартних операторів G_a нелокальні перетворення.

Отже, множина розв'язків рівняння руху (42) для скалярної частинки (поля) з позитивною енергією має нелокальну галілеєву симетрію, алгебра Лі якої задається операторами (43).

8. Нелокальна галілей-симетрія рівняння Дірака. Відомо, що рівняння Дірака

$$p_0\Psi = (\gamma_0\gamma_a p_a + \gamma_0\gamma_4 m)\Psi = H(p)\Psi \tag{45}$$

інваріантне відносно алгебри Пуанкаре з базисними операторами (див., наприклад, [3, 4])

$$\begin{aligned} &P_0 = i\frac{\partial}{\partial x_0}, \quad P_k = -i\frac{\partial}{\partial x_k}, \\ &J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \quad S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]. \end{aligned} \tag{46}$$

Рівняння Дірака, як це встановлено в наших роботах (див., наприклад, літературу в [3]), має широку нелокальну симетрію.

У цьому пункті встановимо нелокальну галілей-симетрію рівняння Дірака. Для цієї мети, наслідуючи метод [4], за допомогою інтегрального оператора

$$W = \frac{1}{\sqrt{2}}\left(1 + \gamma_0\frac{H}{E}\right), \quad E = (p_a^2 + m^2)^{1/2}, \quad H = \gamma_0\gamma_a p_a + \gamma_0\gamma_4 m \tag{47}$$

перетворимо систему чотирьох зв'язаних диференціальних рівнянь першого порядку на систему незв'язаних псевдодиференціальних рівнянь

$$i\frac{\partial\Phi}{\partial t} = \gamma_0 E\Phi, \quad \gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{48}$$

$$\Phi = W\Psi, \quad \gamma_0 E = WHW^{-1}. \tag{49}$$

Встановлюючи додаткову симетрію рівняння (48), одночасно встановлюємо симетрію рівняння Дірака 45).

Теорема 12 [9]. *Рівняння* (48) *інваріантне відносно* 11*-вимірної алгебри Галілея з базисними операторами*

$$\begin{aligned} &P_0^{(3)} = \frac{\vec{p}^{\,2}}{2m}, \quad P_a^{(3)} = p_a = -\frac{\partial}{\partial x_a}, \quad I, \\ &J_{ab}^{(3)} = x_a p_b - x_b p_a + S_{ab}, \quad G_a^{(3)} = t\widetilde{p_a} - m x_a, \qquad \widetilde{p_a} \equiv \gamma_0\frac{m}{E}p_a. \end{aligned} \tag{50}$$

Оператори (50) задовольняють комутаційні співвідношення алгебри Галілея $AG(1,3)$.

Для доведення теореми треба переконатися, що умова інваріантності

$$[p_0 - \gamma_0 E, Q_l]\Psi = 0 \tag{51}$$

виконується для довільного оператора Q_l з набору (50); $G_a^{(3)}$ — інтегральний оператор, що генерує нелокальні перетворення, які не співпадають з класичними перетвореннями Галілея.

Отже, рівняння (48), а тому і рівняння Дірака (45), має нелокальну симетрію, яка задається операторами (50). Явний вигляд операторів (50) для рівняння (45) обчислюється за формулою

$$\widetilde{Q}_l = W^{-1} Q_l W. \tag{52}$$

9. Деякі нові рівняння нелінійної математичної фізики. У цьому пункті наведено серію нових нелінійних рівнянь, які можна розглядати як математичні моделі для опису нелінійних процесів у класичній та квантовій механіці, електродинаміці, гідродинаміці.

1. Рівняння Ньютона–Лоренца для зарядженої частинки природно узагальнити так:

$$\begin{aligned} \frac{d}{dt}(m\vec{v}) &= \lambda_1 \vec{D} + \lambda_2 \vec{B} + \lambda_3 (\vec{v} \times \vec{D}) + \lambda_4 (\vec{v} \times \vec{B}) + \\ &\quad + a_1 (\vec{E} \times \vec{D}) + a_2 (\vec{E} \times \vec{B}) + a_3 (\vec{H} \times \vec{D}) + a_4 (\vec{H} \times \vec{B}), \end{aligned} \tag{53}$$

де $m = m(\vec{v}^{\,2}, \vec{E}^2, \vec{H}^2, \vec{E}\vec{H}, \vec{v}\vec{E}, \vec{v}\vec{H})$ — маса частинки, яка залежить від швидкості $\vec{v}^{\,2}$ і $(\vec{E}, \vec{H})$ — електромагнітного поля, яке створює сама заряджена частинка; $(\vec{D}, \vec{B})$ — зовнішнє електромагнітне поле; $\lambda_1, \lambda_2, \ldots, a_1, a_2, \ldots$ — деякі параметри.

У випадку, коли маса m є константою і $a_1 = a_2 = a_3 = a_4 = 0$, $\lambda_2 = \lambda_3 = 0$, рівняння (53) співпадає з класичним рівнянням Ньютона з силою Лоренца.

Явна залежність маси від $\vec{v}^{\,2}$ і власного електромагнітного поля $(\vec{E}, \vec{H})$ може бути встановлена з вимоги інваріантності (53) відносно групи Галілея або групи Пуанкаре.

Гідроелектродинамічні узагальнення рівняння Ойлера для зарядженої частинки мають вигляд

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + v_l \frac{\partial}{\partial x_l}\right) m(\vec{v}^{\,2}, \vec{E}^2, \ldots)\vec{v} = \lambda_1 \vec{D} + \lambda_2 \vec{B} + a_1 (\vec{E} \times \vec{D}) + \\ &\qquad + a_2 (\vec{E} \times \vec{B}) + \cdots, \quad l = 1, 2, 3. \end{aligned} \tag{54}$$

Пуанкаре-інваріантне рівняння для зарядженої частинки має вигляд

$$\left(v_\alpha \frac{\partial}{\partial x^\alpha}\right) m(v_\nu v^\nu, \vec{E}^2 - \vec{H}^2, \vec{E}\vec{H}) v_\mu = \lambda R_{\mu\nu} v^\nu,$$

де $R_{\mu\nu}$ — антисиметричний тензор зовнішнього електромагнітного поля $(\vec{D}, \vec{B})$.

Нелокальне (псевдодиференціальне) узагальнення рівняння Ньютона для частинки можна подати у вигляді

$$\left(m^2 \frac{d^4}{dx^4} + \lambda\right)^{1/2} \vec{x}(t) = F(t, \vec{x}, \dot{\vec{x}}, \ddot{\vec{x}}). \tag{55}$$

У випадку, коли параметр $\lambda = 0$, рівняння (55) співпадає з класичним рівнянням руху Ньютона.

2. Рівняння для скалярного комплексного поля u зі змінною швидкістю v можна задати так:

$$\left(-\hbar^2 \frac{\partial^2}{\partial t^2} + \hbar^2 v^2 \Delta - m^2 v^4\right) u = F(|u|) u, \tag{56}$$

$$\frac{\partial v_k}{\partial t} + v_l \frac{\partial v_k}{\partial x_l} = g(|u|) \frac{\partial |u|}{\partial x_k}, \quad v^2 \equiv v_1^2 + v_2^2 + v_3^2, \tag{57}$$

де $g(|u|)$ — довільна гладка функція.

Швидкість поширення поля u задається рівнянням (57). Отже, хвильове рівняння (56) (і при $F(|u|) = 0$) з умовою (57) є нелінійним рівнянням. При стандартному підході $v^2 = c^2$, де c — постійна швидкість поширення світла у вакуумі; у цьому випадку рівняння (56) лінійне. Явно пуанкаре-інваріантне рівняння для поля u має вигляд

$$\left(v_\mu v_\nu \frac{\partial^2 u}{\partial x^\mu \partial x^\nu} - m^2 v^4 \right) u = 0, \tag{58}$$

$$v_\alpha \frac{\partial v_\mu}{\partial x^\alpha} = g(|u|) \frac{\partial |u|}{\partial x_\mu}, \quad v_\mu v^\mu \equiv v_0^2 - v_1^2 - v_2^2 - v_3^2 > 0. \tag{59}$$

Важливою властивістю цієї системи є те, що вона лоренц-інваріантна, швидкість поля v_μ не є сталою величиною і залежить від амплітуди і швидкості зміни амплітуди поля.

3. Стандартна класична і квантова електродинаміка побудована в термінах потенціалів A_μ. Однак до цього часу не використані інші можливості (моделі) формулювання електродинаміки. Не вводячи потенціалів, можна запропонувати таку пуанкаре-інваріантну систему рівнянь для тензора електромагнітного поля $F_{\mu\nu}$ і спінорного поля Ψ:

$$\begin{gathered} \frac{\partial F_{\mu\nu}}{\partial x^\nu} = j_\mu, \qquad j_\mu = g_1 \bar{\Psi} \gamma_\mu \Psi + g_2 \bar{\Psi} p_\mu \Psi, \\ \frac{\partial F_{\mu\nu}}{\partial x_\alpha} + \frac{\partial F_{\nu\alpha}}{\partial x_\mu} + \frac{\partial F_{\alpha\mu}}{\partial x_\nu} = g \left(\frac{\partial \bar{\Psi} S_{\mu\nu} \Psi}{\partial x_\alpha} + \frac{\partial \bar{\Psi} S_{\nu\alpha} \Psi}{\partial x_\mu} + \frac{\partial \bar{\Psi} S_{\alpha\mu} \Psi}{\partial x_\nu} \right), \end{gathered} \tag{60}$$

$$\begin{gathered} \gamma^\mu \left(p_\mu - \gamma^\alpha F_{\mu\alpha} \right) \Psi = m \Psi, \quad p_\mu = i g_{\mu\nu} \frac{\partial}{\partial x^\nu}, \\ S_{\mu\nu} = \frac{i}{4} [\gamma_\mu, \gamma_\nu] \equiv \frac{i}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu). \end{gathered} \tag{61}$$

Другу модель електродинаміки, без потенціалів, можна будувати на основі нелінійних рівнянь другого порядку

$$\Box F_{\mu\nu} = g \bar{\Psi} S_{\mu\nu} \Psi, \tag{62}$$

$$(p_\mu - \lambda \gamma_\nu F_{\mu\nu})(p^m - \lambda F^{\mu\alpha} \gamma_\alpha) \Psi = m^2 \Psi. \tag{63}$$

4. Одне з можливих нелінійних узагальнень рівнянь Максвелла для електромагнітного поля, яке поширюється зі змінною швидкістю v, має вигляд [21]

$$\begin{gathered} \frac{d\vec{E}}{dt} = v \operatorname{rot} \vec{H} + \vec{j}, \quad \operatorname{div} \vec{E} = \rho, \\ \frac{d\vec{H}}{dt} = -v \operatorname{rot} \vec{E}, \qquad \operatorname{div} \vec{H} = 0, \quad v = (v_1^2 + v_2^2 + v_3^2)^{1/2}, \end{gathered} \tag{64}$$

$$\lambda_1\left(\frac{\partial}{\partial t}+v_l\frac{\partial}{\partial x_l}\right)v_k+\lambda_2\left(\frac{\partial}{\partial t}+v_l\frac{\partial}{\partial x_l}\right)^2v_k+\lambda_3 v_k= \\ =a_1E_k+a_2H_k+a_3\varepsilon_{kln}E_lH_n, \quad k,l,n=1,2,3, \tag{65}$$

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+b_1E_l\frac{\partial}{\partial x_l}+b_2H_l\frac{\partial}{\partial x_l}+b_3v_l\frac{\partial}{\partial x_l},$$

де λ_1, λ_2, λ_3, a_1, a_2, a_3, b_1, b_2, b_3 — функції, які залежать від інваріантів $\vec{E}^2-\vec{H}^2$, $\vec{E}\vec{H}$, $\vec{v}^2$.

Виписана система співпадає з класичним рівнянням Максвелла при умові, що v є сталою величиною і всі λ_1, λ_2, $\dot{b}_3$ дорівнюють нулеві.

Рівняння другого порядку для електромагнітного поля $(\vec{E},\vec{H})$ із змінною швидкістю має вигляд

$$\left(\frac{\partial^2}{\partial t^2}-v^2\Delta\right)\vec{E}=c_1\vec{E}+c_2\vec{H}+c_3(\vec{E}\times\vec{H})+c_4(\vec{v}\times\vec{E})+c_5(\vec{v}\times\vec{H}),$$
$$\left(\frac{\partial^2}{\partial t^2}-v^2\Delta\right)\vec{H}=d_1\vec{E}+d_2\vec{H}+d_3(\vec{E}\times\vec{H})+d_4(\vec{v}\times\vec{E})+d_5(\vec{v}\times\vec{H}).$$

Швидкість $\vec{v}$ електромагнітного поля $(\vec{E},\vec{H})$ визначається з рівняння (65).

5. Пуанкаре-інваріантне узагальнення класичного рівняння Ойлера має вигляд

$$(\lambda_1L+\lambda_2L^2)v_\mu=r_1v_\mu+r_2\frac{\partial P}{\partial x_\mu}+r_3\left(v_\alpha\frac{\partial v_\nu}{\partial x_\alpha}\right)^2v_\mu, \\ L\equiv v_\alpha\frac{\partial}{\partial x^\alpha}, \quad L^2\equiv\left(v_\alpha\frac{\partial}{\partial x^\alpha}\right)\left(v_\alpha\frac{\partial}{\partial x^\alpha}\right), \tag{66}$$

де r_1, r_2, r_3 — гладкі функції від інваріантів $v_\alpha v^\mu$, P.

Застосування виписаних нелінійних рівнянь до опису конкретних фізичних процесів дає можливість уточнити довільні функції, які входять у рівняння. Вимога інваріантності до запропонованих рівнянь відносно групп Галілея, групи Пуанкаре та їх різних розширень дозволяє істотно звузити класи допустимих моделей.

1. *Боголюбов Н.Н., Митропольский Ю.А.* Асимптотические методы в теории нелинейных колебаний. — М.: Наука, 1974. — 408 с.

2. *Schrödinger E.* Quantisierung als Eigenwertprobleme // Annalen der Physik. — 1926. — **81**. — S. 109–139.

3. *Fushchych W., Shtelen W., Serov N.* Symmetry analysis and exact solutions of equations of nonlinear mathematical physics. — Dordrecht: Kluwer Academic Publishers, 1993. — 436 p.

4. *Fushchych W., Nikitin A.* Symmetries of equations of quantum mechanics. — New York: Allerton Press, 1994. — 480 p.

5. *Фущич В.И.* Как расширить симметрию дифференциальных уравнений? // Симметрия и решения нелинейных уравнений математической физики. — Киев: Ин-т математики АН УССР, 1987. — С. 4–16.

6. *Fushchych W.I., Cherniha R.M.* The Galilei relativistic principle and nonlinear partial differential equations // J. Phys. A: Math. Gen. — 1985. — **18**. — P. 3491–3503.

7. *Фущич В.И.* Симметрия в задачах математической физики // Теоретико-алгебраические исследования в математической физике. — Киев: Ин-т математики АН УССР, 1981. — С. 6–28.
8. *Фущич В.И., Сегеда Ю.Н.* О новой алгебре инвариантности свободного уравнения Шредингера // Докл. АН СССР. — 1977. — **232**, № 4. — С. 800–801.
9. *Фущич В.И.* Групповые свойства дифференциальных уравнений квантовой механики // Проблемы асимптотической теории нелинейных колебаний. — Киев: Наук. думка, 1977. — С. 238–246.
10. *Фущич В.И., Никитин А.Г.* О группе инвариантности квази-релятивистского уравнения движения // Докл. АН СССР. — 1978. — **238**, № 1. — С. 46–49.
11. *Fushchych W.I., Serov N.I.* On some exact solutions of the three-dimensional non-linear Schrödinger equation // J. Phys. A: Math. Gen. — 1987. — **20**. — L929–L933.
12. *Fushchych W.I., Boyko V.M.* Continuity equation in nonlinear quantum mechanics and the Galilei relativity principle // J. Nonlinear Math. Phys. — 1997. — **4**, N 1–2. — P. 124–128.
13. *Fushchych W.I., Symenoh Z.I.* High-order equations of motion in quantum mechanics and Galilean relativity // J. Phys. A: Math. Gen. — 1997. — **30**, N 6. — L131–L135.
14. *Schiller R.* Quasi-classical transformation theory // Phys. Rev. — 1962. — **125**, N 3. — P. 1109.
15. *Rosen N.* The relation between classical and quantum mechanics // American J. Phys. — 1965. — **32**. — P. 597–600.
16. *Guerra F., Pusterla M.* Nonlinear Klein–Gordon equation carrying a nondispersive solitonlike singularity // Lett. Nuovo Cimento. — 1982. — **35**. — P. 256–259.
17. *Guéret Ph., Vigier J.-P.* Relativistic wave equation with quantum potential nonlinearity // Lett. Nuovo Cimento. — 1983. — **38**. — P. 125–128.
18. *Doebner H.-D., Goldin G.A.* Properties of nonlinear Schrödinger equations associated with diffeomorphism group representations // J. Phys. A: Math. Gen. — 1994. — **27**. — P. 1771–1780.
19. *Fushchych W.I., Cherniha R.M., Chopyk V.I.* On unique symmetry of two nonlinear generalizations of the Schrödinger equation // J. Nonlinear Math. Phys. — 1996. — **3**, N 3–4. — P. 296–301.
20. *Фущич В.И.* О дополнительной инвариантности уравнения Клейна–Гордона–Фока // Докл. АН СССР. — 1976. — **230**, № 3. — С. 570–572.
21. *Fushchych W.I.* New nonlinear equations for electromagnetic field having velocity different from c // Доп. АН України. — 1992. — № 4. — С. 24–27.

Higher symmetries and exact solutions of linear and nonlinear Schrödinger equation

W.I. FUSHCHYCH, A.G. NIKITIN

A new approach for the analysis of partial differential equations is developed which is characterized by a simultaneous use of higher and conditional symmetries. Higher symmetries of the Schrödinger equation with an arbitrary potential are investigated. Nonlinear determining equations for potentials are solved using reductions to Weierstrass, Painlevé, and Riccati forms. Algebraic properties of higher order symmetry operators are analyzed. Combinations of higher and conditional symmetries are used to generate families of exact solutions of linear and nonlinear Schrödinger equations.

1. Introduction

Higher order symmetry operators (SOs) have many important applications in modern mathematical physics. These operators correspond to hidden symmetries of partial differential equations, including Lie–Bäcklund symmetries [1, 2], as well as super- and parasupersymmetries [3–7]. Higher order SOs can be used to construct new conservation laws which cannot be found in the classical Lie approach [3, 8]. These operators are applied to separate variables [9]. Moreover, one should use SOs whose order is higher than the order of the equation whose variables are separated [10].

In the present paper we investigate higher order SOs of the Schrödinger equation, which are "non-Lie symmetries" [8, 11]. The simplest non-Lie symmetries are considered in detail and all related SOs are explicitly calculated. The potentials admitting these symmetries are found as solutions of the corresponding nonlinear compatibility conditions. It is shown that the higher order SOs extend the class of potentials which were previously obtained in the Lie symmetry analysis.

Algebraic properties of higher order SOs are investigated and used to construct exact solutions of the linear and related nonlinear Schrödinger equations. We propose a new method to generate extended families of exact solutions by using both the conditional symmetries [8, 12–14] and higher order SOs.

The Schrödinger equation with a time-independent potential $V = V(x)$ is studied mainly. Time-dependent potentials $V = V(t, x)$ are discussed briefly in Section 6. By this, we recover the old result [15] connected with the Lax representation for the Boussinesq equation, and generate some other nonlinear equations admitting this representation.

The distinguishing feature of our approach is that coefficients of symmetry operators and the corresponding potentials are defined as solutions of differential equations which can be easily generalized to the case of multidimensional Schrödinger equation contrary to the method of inverse scattering problem. This paper continues (and in some sense completes) our works [16–18] where non-Lie symmetries of the Schrödinger equation were considered. A detailed analysis of higher symmetries of multidimensional Schrödinger equations will be a subject of our subsequent paper.

Reprinted with permission from J. Math. Phys., 1997, **38**, № 11, P. 5944–5959

2. Symmetry operators of the Schrödinger equation

Let us formulate the concept of higher order SO for the Schrödinger equation

$$L\Psi(t,x) = 0, \quad L = i\partial_t - H,$$
$$H = \frac{1}{2}\left(-\partial_x^2 + U(x)\right), \quad \partial_t \equiv \frac{\partial}{\partial t}, \quad \partial_x \equiv \frac{\partial}{\partial x}. \tag{2.1}$$

In every sense of the word, a SO of equation (2.1) is any (linear, nonlinear, differential, integro-differential, etc.) operator Q transforming solutions into solutions. Restricting ourselves to linear differential operators of finite order n we represent Q in the form

$$Q = \sum_{i=0}^{n}(h_i \cdot p)_i, \quad (h_i \cdot p)_i = \{(h_i \cdot p)_{i-1}, p\}, \quad (h_i \cdot p)_0 = h_i, \tag{2.2}$$

where h_i are unknown functions of (t,x), $\{A,B\} = AB + BA$, $p = -i\partial_x$.

Operator (2.2) includes no derivatives w.r.t. t which can be expressed as $\frac{1}{2}\left(p^2 + U\right)$ on the set of solutions of equation (2.1).

Definition [8]. *Operator* (2.2) *is a SO of order n of equation* (2.1) *if*

$$[Q, L] = 0. \tag{2.3}$$

Remark. The more general invariance condition [3] $[Q,L] = \alpha_Q L$, where α_Q is a linear operator, reduces to relation (2.3) if L and Q are operators defined in (2.1), (2.2). Terms proportional to $i\frac{\partial}{\partial t}$ cannot appear as a result of commutation of Q and L; hence, without loss of generality, $\alpha_Q = 0$.

For $n = 1, 2$ SOs (2.2) reduce to differential operators of the first order and can be interpreted as generators of the invariance group of the equation in question. For $n > 2$ these operators (which we call higher order SO) correspond to non-Lie [8, 11] symmetries.

Lie symmetries of equation (2.1) were described in Refs. [19–21]. The general form of potentials admitting nontrivial (i.e., distinct from time displacements) symmetries is as follows

$$U = a_0 + a_1 x + a_2 x^2 + \frac{a_3}{(x + a_4)^2}, \tag{2.4}$$

where $a_0, \ldots, a_4$ are arbitrary constants. No other potentials admitting local invariance groups exist.

Group properties of equation (2.1) with potentials (2.4) were used to solve the equation exactly, to establish connections between equations with different potentials, to separate variables, etc. [9]. Unfortunately, all these applications are valid for a very restricted class of potentials given by formula (2.4).

The class of admissible potentials can be essentially extended if we require that equation (2.1) admits higher order SOs [17]. The problem of describing such potentials (and the corresponding SOs) reduces to solving operator equations (2.2), (2.3). Evaluating the commutators and equating the coefficients for linearly independent differentials we arrive at the following system of determining equations (which is valid for *arbitrary* n) [5]:

$$\partial_x h_n = 0, \quad \partial_x h_{n-1} + 2\partial_t h_n = 0,$$

$$\begin{aligned}
&\partial_x h_{n-m} + 2\partial_t h_{n-m+1} - \\
&\qquad - \sum_{k=0}^{\left[\frac{m-2}{2}\right]} (-1)^k \frac{2(n-m+2+2k)!}{(2k+1)!(n-m+1)!} h_{n-m+2k+2} \partial_x^{2k+1} U = 0, \\
&\partial_t h_0 + \sum_{p=0}^{\left[\frac{n-1}{2}\right]} (-1)^{p+1} h_{2p+1} \partial_x^{2p+1} U = 0,
\end{aligned} \tag{2.5}$$

where $m = 2, 3, \ldots, n$, and $[y]$ is the entire part of y.

Formulae (2.5) define a system of nonlinear equations in h_i and U. For $n = 2$ the general solution for U is given by formula (2.4).

Let us consider the case $n = 3$, which corresponds to the simplest non-Lie symmetry, in more detail. The corresponding system (2.5) reduces to

$$h_3' = 0, \quad h_2' + 2\dot{h}_3 = 0, \quad 2\dot{h}_2 + h_1' - 6h_3 U' = 0, \tag{2.6a}$$

$$2\dot{h}_1 + h_0' - 4h_2 U' = 0, \quad \dot{h}_0 - h_1 U' + h_3 U''' = 0, \tag{2.6b}$$

where the dots and primes denote derivatives w.r.t. t and x respectively.

Excluding h_0 from (2.6b) and using (2.6a) we arrive at the following equation:

$$\begin{aligned}
&F(a, b, c; U, x) \equiv aU'''' - (2\ddot{a}x^2 + 6aU + c - 2\dot{b}x)U'' - \\
&\qquad - 6(2\ddot{a}x + aU' - \dot{b})U' - 12\ddot{a}U - 2(2\partial_t^4 a x^2 - 2\dddot{b}\, x + \ddot{c}) = 0,
\end{aligned} \tag{2.7}$$

where a, b, c are arbitrary functions of t.

Equation (2.7) is nothing but the compatibility condition for system (2.6). If potential U satisfies (2.7) then the corresponding coefficients of the SO have the form

$$\begin{aligned}
&h_3 = a, \quad h_2 = -2\dot{a}x + b, \quad h_1 = g_1 + 6aU, \\
&h_0 = -\frac{4}{3}\dddot{a}\, x^3 + 2\ddot{b}x^2 - 2\dot{c}x - 4\dot{a}\varphi + 4(b - 2\dot{a}x)U + d,
\end{aligned} \tag{2.8}$$

where

$$g_1 = 2\ddot{a}x^2 - 2\dot{b}x + c, \quad \varphi = \int U dx, \quad u = \varphi', \quad d = d(t). \tag{2.9}$$

3. Equations for potential

Equation (2.7) was obtained earlier [17] (see Ref. [22]) and, moreover, particular solutions for U were found [17]. Here we analyze this equation in detail.

First of all, let us reduce the order of equation (2.7). Integrating it twice w.r.t. x and choosing the new dependent variable φ defined in (2.9) we obtain

$$a[\varphi''' - 3(\varphi')^2] - (g_1\varphi)' = \frac{1}{3}\partial_t^4 a x^4 - \frac{2}{3}\dddot{b}\, x^3 + \ddot{c}x^2 + dx + e. \tag{3.1}$$

Using the fact that φ depends on x only while a, b, c, d, e are functions of t, it is possible to separate variables in (3.1). Indeed, dividing any term of (3.1) by $a \neq 0$, differentiating w.r.t. t and integrating over x we obtain the following consequence

$$\frac{\dot{g}_1 a - g_1\dot{a}}{a^2}\varphi = \partial_t \frac{1}{a}\left(\frac{1}{15}\partial_t^4 a x^5 - \frac{1}{6}\dddot{b}\, x^4 + \frac{1}{3}\ddot{c}x^2 + \frac{1}{2}dx^2 + ex + f\right). \tag{3.2}$$

Consider equation (3.2) separately in two following cases:

$$\dot{g}_1 a - g_1 \dot{a} \neq 0, \tag{3.3a}$$

$$\dot{g}_1 a - g_1 \dot{a} = 0. \tag{3.3b}$$

Let condition (3.3a) be valid. Then dividing the l.h.s. and r.h.s. of (3.2) by $\partial_t(g_1/a)$ we come to the following general expression for φ

$$\varphi = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0 + \frac{\alpha_4}{x+\alpha_5} + \frac{\beta_1 x + \beta_2}{x^2 + \beta_3 x + \beta_4}, \tag{3.4}$$

where $\alpha_0, \ldots, \alpha_5$, $\beta_1, \ldots, \beta_4$ are constants.

It is possible to verify by a straightforward but cumbersome calculation that relation (3.4) is compatible with (3.1) only for $\beta_1 = \beta_2 = 0$. We will not analyze solutions (3.4) inasmuch as they correspond to potentials (2.4) and to SOs which are products of the usual Lie symmetries [19–21].

If condition (3.3a) is valid, we obtain from equation (3.2)

$$\ddot{a} = a k_1, \quad \dot{b} = k_2 a, \quad c = k_3 a, \tag{3.5}$$

where k_1, k_2, k_3 are arbitrary constants. The corresponding equation (3.1) reduces to

$$\varphi''' - 3(\varphi')^2 - (G''\varphi)' = 2k_1 G + k_4 x + k_5, \tag{3.6}$$

where

$$G = \frac{1}{6}k_1 x^4 - \frac{1}{3}k_2 x^3 + \frac{1}{2}k_3 x^2, \quad G'' = g_1 = 2k_1 x^2 - 2k_2 x + k_3, \tag{3.7}$$

k_4 and k_5 are constants.

Let us prove that, up to equivalence, equation (3.6) can be reduced to one of the following forms:

$$U'' - 3U^2 + 3\omega_1 = 0, \tag{3.8a}$$

$$U'' - 3U^2 - 8\omega_2 x = 0, \tag{3.8b}$$

$$(U'' - 3U^2)' - 2\omega_3(xU' + 2U) = 0, \tag{3.8c}$$

$$\varphi''' - 3(\varphi')^2 - 2\omega_4(x^2\varphi)' = \frac{1}{3}\omega_4^2 x^4 + \omega_5, \quad U = \varphi', \tag{3.8d}$$

where $\omega_1, \ldots \omega_5$ are arbitrary constants. Indeed, by using invertible transformations

$$\varphi \to \varphi + C_1 x + C_2, \quad x \to x + C_3, \tag{3.9}$$

where C_k $(k = 1, 2, 3)$ are constants, it is possible to simplify the r.h.s. of (3.6). These transformations cannot change the order of polynomial G, and so there exist four nonequivalent possibilities:

$$k_1 = 0, \quad k_2 = 0, \quad k_4 = 0, \tag{3.10a}$$

$$k_1 = 0, \quad k_2 = 0, \quad k_4 \neq 0, \tag{3.10b}$$

$$k_1 = 0, \quad k_2 \neq 0, \tag{3.10c}$$

$$k_1 \neq 0. \tag{3.10d}$$

Setting in (3.9)

$$C_1 = -\frac{1}{6}k_3, \quad C_2 = C_3 = 0, \quad k_5 - \frac{1}{12}k_3^2 = \omega_1, \tag{3.11a}$$

$$C_1 = -\frac{1}{6}k_3, \quad C_2 = 0, \quad C_3 = -\frac{k_5}{k_4} + \frac{k_3^2}{12k_4}, \quad k_4 = 8\omega_2, \tag{3.11b}$$

$$C_1 = \frac{k_4}{4k_2}, \quad C_2 = \frac{k_5}{2k_2} + \frac{3k_4^2}{32k_2^3} + \frac{k_3k_4}{8k_2^2}, \quad C_3 = \frac{k_3}{2k_2} + \frac{3k_4}{4k_2^2}, \quad k_2 = -\omega_3, \tag{3.11c}$$

$$\begin{aligned} &C_1 = -\frac{1}{6}k_3 + \frac{k_2^2}{12k_1}, \quad C_2 = -\frac{k_4}{4k_1} - \frac{k_2k_3}{6k_1} + \frac{k_2^3}{24k_1^2}, \\ &C_3 = \frac{k_2}{2k_1}, \quad k_1 = \omega_4, \quad k_5 - \frac{k_3^2}{12} + \frac{k_2k_4}{2k_1} + \frac{k_2^2k_3}{3k_1} - \frac{k_2^4}{16k_1^2} = \omega_5 \end{aligned} \tag{3.11d}$$

for cases (3.10a)–(3.10d) correspondingly, we reduce (3.6) to one of the forms (3.8a)–(3.8d) respectively.

From (2.2), (2.8), (3.4), (3.9)–(3.11) we find the corresponding symmetry operators

$$Q = p^3 + \frac{3}{4}\{U, p\} \equiv 2pH + \frac{1}{2}Up + \frac{i}{4}U', \tag{3.12a}$$

$$Q = p^3 + \frac{3}{4}\{U, p\} - \omega_2 t, \tag{3.12b}$$

$$Q = p^3 + \frac{3}{4}\{U, p\} + \omega_3\left(tH - \frac{1}{4}\{x, p\}\right), \tag{3.12c}$$

$$\begin{aligned} Q_\pm = \frac{1}{\sqrt{24}}\Big[&p^3 \pm \frac{i}{4}\omega\{\{x, p\}, p\} + \frac{1}{4}\{3\varphi' - \omega^2x^2, p\}\pm \\ &\pm \frac{i}{2}\omega\left(\varphi + 2x\varphi' - \frac{\omega^2}{3}x^3\right)\Big]\exp(\pm i\omega t), \quad \omega = \sqrt{-\omega_4}, \end{aligned} \tag{3.12d}$$

where U and φ are solutions of (3.2) and H is the related Hamiltonian (2.1).

Thus, *the Schrödinger equation* (2.1) *admits a third-order SO if potential U satisfies one of the equations* (3.8). The explicit form of the corresponding SOs is present in (3.12).

4. Algebraic properties of SOs

Let us investigate algebraic properties of SOs defined by relations (3.12). We shall see that these properties are predetermined by the type of equations (3.8) satisfied by U. By direct calculations, using (2.3), (2.1) and (3.12), we find the following relations

$$[Q, H] = 0, \tag{4.1a}$$

$$Q^2 = 8H^3 - \frac{3}{2}\omega_1 H - \frac{C}{8} \tag{4.1b}$$

if the potential satisfies equation (3.8a) (C is the first integral of equation (3.8a), refer to (5.1));

$$[Q, H] = i\omega_2 I, \quad [Q, I] = [H, I] = 0 \tag{4.2}$$

if the potential satisfies equation (3.8b);

$$[Q, H] = -i\omega_3 H \tag{4.3}$$

if the potential satisfies equation (3.8c), and

$$[H, Q_\pm] = \pm\omega Q_\pm, \tag{4.4a}$$

$$[Q_+, Q_-] = \omega \left(H^2 + \frac{1}{48}(2\omega^2 + \omega_5) \right) \tag{4.4b}$$

if the potential satisfies (3.8d).

It follows from (4.1)–(4.3) that non-Lie SOs Q and Hamiltonians H form consistent Lie algebras which can have rather nontrivial applications.

Formula (4.1b) presents an example of the general theorem [23, 24] stating that commuting ordinary differential operators are connected by a polynomial algebraic relation with constant coefficients. In Section 7 we use relations (4.1) to integrate the related equations (2.1).

Relations (4.2) define the Heisenberg algebra. The linear combinations $a_\pm = \frac{1}{\sqrt{2}}(H \pm iQ)$ realize the unusual representation of creation and annihilation operators in terms of third-order differential operators.

In accordance with (4.3), Q plays a role of dilatation operator which continuously changes eigenvalues of H. Indeed, let

$$H\Psi_E = E\Psi_E, \tag{4.5}$$

then the function $\Psi' = \exp(i\lambda Q)\Psi_E$ (where λ is a real parameter) is also an eigenvector of the Hamiltonian H with the eigenvalue λE.

It follows from (4.4) that for $\omega_4 < 0$ the operators Q_+ and Q_- are raising and lowering operators for the corresponding Hamiltonian. In other words, if Ψ_E satisfies (4.5) then $Q_\pm\Psi_E$ are also eigenfunctions of the Hamiltonian which, however, correspond to the eigenvalues $E \pm \omega$:

$$H(Q_\pm\Psi_E) = (E \pm \omega)(Q_\pm\Psi_E). \tag{4.6}$$

Relations (4.6) are typical for creation and annihilation operators of the quantum oscillator. This observation shows a way for constructing exact solutions of the Schrödinger equation whose potential satisfies relation (3.8d). Moreover, relations (4.4a) allow Q to be interpreted as a conditional symmetry [8, 12]; such symmetries are of particular interest in the analysis of partial differential equations [14, 25, 26]. Thus, third-order SOs of equation (2.1) generate algebras of certain interest. Moreover, algebraic properties of these SOs are the same for wide classes of potentials described by one of equations (3.8).

5. Reduction of equations for potentials

Let us consider equations (3.8) in detail and describe the corresponding classes of potentials. A solution of some of these nonlinear equations is a complicated problem which, however, can be simplified by using reductions to other well–studied equations.

5.a. The Weierstrass equation. Formula (3.8a) defines the Weierstrass equation whose solutions are expressed via either elementary functions or via the Weierstrass function, depending on values of the parameter ω_1 and the integration constant. Here we represent these well-known solutions (refer, e.g. to the classic monograph of E.T. Whittaker and G.N. Watson [28]) in the form convenient for our purposes.

Multiplying the l.h.s. of (3.8a) by U' and integrating we obtain

$$\frac{1}{2}(U')^2 - U^3 + 3\omega_1 U = C, \tag{5.1}$$

where C is an integration constant which appeared above in (4.1b). Then by changing roles of dependent and independent variables it becomes possible to integrate (5.1) and to find U as an implicit function of x. We will distinguish five qualitatively different cases:

$$C^2 - 4\omega_1^3 = 0, \quad C > 0, \tag{5.2a}$$

$$C^2 - 4\omega_1^3 = 0, \quad C < 0, \tag{5.2b}$$

$$C = \omega_1 = 0, \tag{5.2c}$$

$$C^2 - 4\omega_1^3 < 0. \tag{5.3a}$$

$$C^2 - 4\omega_1^3 > 0. \tag{5.3b}$$

For (5.2a)–(5.2c), solutions of (5.1) can be expressed via elementary functions, while (5.3a,b) generate solutions in elliptic functions.

For our purposes, it is convenient to transform (5.1) to another equivalent form. Using the substitution

$$U = V - \frac{\mu}{2}, \tag{5.4}$$

where μ is a real root of the cubic equation

$$\mu^3 - 3\omega_1\mu + C = 0, \tag{5.5}$$

we obtain

$$\frac{1}{2}(V')^2 - V^3 - \bar{\omega}_0 V^2 + 4\bar{\omega}_1 V + 8\bar{\omega}_0\bar{\omega}_1 = 0, \tag{5.6}$$

where $\bar{\omega}_0 = \frac{3}{2}\mu$ and $\bar{\omega}_1 = \frac{3}{4}(\omega_1 - \mu^2)$ are arbitrary real numbers.

The substitution (5.4), (5.5) transforms conditions (5.2), (5.3) to the following form:

$$\bar{\omega}_1\left(\bar{\omega}_1 - \bar{\omega}_0^2\right)^2 = 0, \quad \bar{\omega}_0 < 0, \tag{5.7a}$$

$$\bar{\omega}_1\left(\bar{\omega}_1 - \bar{\omega}_0^2\right)^2 = 0, \quad \bar{\omega}_0 > 0, \tag{5.7b}$$

$$\bar{\omega}_1\left(\bar{\omega}_1-\bar{\omega}_0^2\right)^2=0, \quad \bar{\omega}_0=0, \tag{5.7c}$$

$$\bar{\omega}_1\left(\bar{\omega}_1-\bar{\omega}_0^2\right)\neq 0, \quad \bar{\omega}_1>0, \tag{5.8a}$$

$$\bar{\omega}_1\left(\bar{\omega}_1-\bar{\omega}_0^2\right)\neq 0, \quad \bar{\omega}_1<0. \tag{5.8b}$$

If relations (5.7a) are satisfied, then $\bar{\omega}_1=\bar{\omega}_0^2$ or $\bar{\omega}_1=0$. Moreover, the corresponding solutions for V differ by a constant shift: $V\to V+2\bar{\omega}_0$, $\bar{\omega}_0\to\bar{\omega}_0/2$. Without loss of generality we restrict ourselves to the former case, then solutions of equation (5.6) corresponding to conditions (5.7a-c) have the following forms:

$$V=\nu^2\left[2\tanh^2\left(\nu(x-k)\right)-1\right], \quad \bar{\omega}_0=-\frac{1}{2}\nu^2, \quad \bar{\omega}_1=\frac{1}{4}\nu^4, \tag{5.9a}$$

$$V=\nu^2\left[2\coth^2\left(\nu(x-k)\right)-1\right], \quad \bar{\omega}_0=-\frac{1}{2}\nu^2, \quad \bar{\omega}_1=\frac{1}{4}\nu^4, \tag{5.9a$'$}$$

$$V=\nu^2\left[2\tan^2\left(\nu(x-k)-1\right)\right], \quad \bar{\omega}_0=\frac{1}{2}\nu^2, \quad \bar{\omega}_1=\frac{1}{4}\nu^4, \tag{5.9b}$$

$$V=\frac{2}{(x-k)^2}. \tag{5.9c}$$

Here, k and ν are arbitrary real numbers.

For the cases (5.8) the general solution of (5.1) has the form

$$V=2\wp(x-k)+\frac{1}{2}\mu, \tag{5.10}$$

where $\wp$ is a two-periodic Weierstrass function, which is meromorphic on all the complex plane. The invariants of this function are $g_2=-\frac{4}{3}\left(\bar{\omega}_0^2+3\bar{\omega}_1\right)$ and $g_3=-\frac{4}{27}\bar{\omega}_0\left(\bar{\omega}_0^2-9\bar{\omega}_1\right)$. Moreover, if condition (5.8a) holds, the corresponding solutions are bounded and can be expressed via the elliptic Jacobi functions

$$V=B\mathrm{cn}^2(Dx+k)+F, \tag{5.11a}$$

where

$$B=(e_3-e_2), \quad D=\sqrt{(e_1-e_3)/2}, \quad F=e_2, \tag{5.11b}$$

$e_1>e_2>e_3$ are real solutions of the cubic equation from the r.h.s. of (5.6).

We note that formulae (5.9) present the set of well-known potentials which correspond to the exactly solvable Schrödinger equations [27]. In accordance with the above, these equations admit extended Lie symmetries.

5.b. Painlevé and Riccati equations. Relation (3.8b) defines the first Painlevé transcendent. Its solutions are meromorphic on all the complex plane but cannot be expressed via elementary or special functions.

Equation (3.8c) is more complicated. However, by using the special change of variables and applying the Miura [29] ansatz, we shall reduce it to the Painlevé form also. Indeed, making the following change of variables

$$U=-\sqrt[3]{\frac{\omega_3^2}{6}}V, \quad x=-\sqrt[3]{\frac{1}{6\omega_3}}y, \tag{5.12}$$

we obtain

$$V''' + VV' - \frac{1}{3}xV' - \frac{2}{3}V = 0, \quad V' = \partial V/\partial y. \tag{5.13}$$

The ansatz

$$V = W' - \frac{1}{6}W^2 \tag{5.14}$$

reduces (5.13) to

$$\left(\partial_y - \frac{1}{3}W\right)\left(W''' - \frac{1}{6}W^2W' - \frac{1}{3}yW' - \frac{1}{3}W\right) = 0.$$

Equating the expression in the second brackets to zero and integrating it we come to the second Painlevé transcendent

$$W'' = \frac{1}{18}W^3 + \frac{1}{3}yW + K, \tag{5.15}$$

where K is an arbitrary constant.

To make one more reduction of equation (3.8c) we take $U = \varphi'$. Then, integrating the resultant equation, we obtain

$$\varphi''' - 3\left(\varphi'\right)^2 - 2\omega_3\left(x\varphi\right)' = C. \tag{5.16}$$

Then, defining

$$\begin{aligned} &\varphi = 2\sqrt[3]{2\omega_3}\xi + \frac{1}{4}y^2 + \frac{C}{2\omega_3}, \quad y = \sqrt[3]{2\omega_3}x, \\ &\hat{W} = \xi' - \xi^2 - \frac{1}{2}y, \quad \xi' = \frac{\partial\xi}{\partial y} \end{aligned} \tag{5.17}$$

we represent (5.16) as

$$\hat{W}'' - 4\xi'\hat{W} + 2\xi\hat{W}' - y\hat{W} = 0. \tag{5.18}$$

The trivial solutions of (5.18) correspond to the following Riccati equation for ξ:

$$\xi' - \xi^2 - \frac{1}{2}y = 0. \tag{5.19}$$

It follows from the above that any solution of equations (5.15) or (5.19) generates a potential U defined by relations (5.12), (5.14) or (5.17). The corresponding Schrödinger equation admits a third-order SO.

The last of the equations considered, i.e., equation (3.8d), is the most complicated. The change

$$\varphi = 2f - \frac{1}{3}\omega_4x^3 \tag{5.20}$$

reduces it to the following form:

$$f''' - 6(f')^2 + 4\omega_4(f'x^2 - xf) = \omega_4 + \frac{1}{2}\omega_5. \tag{5.21}$$

Multiplying (5.21) by f'' and integrating we obtain the first integral

$$\frac{1}{2}(f'')^2 - 2(f')^3 + 2\omega_4(f - xf')^2 - \left(\omega_4 + \frac{1}{2}\omega_5\right) f' = C \tag{5.22}$$

which is still a very complicated nonlinear equation.

Let us demonstrate that (5.21) can be reduced to the Riccati equation. To realize this we rewrite (5.21) as follows

$$F'' + 2fF' - 4f'F = \frac{1}{2}\omega_5 - \omega_4, \tag{5.23}$$

where

$$F = f' - f^2 - \omega_4 x^2.$$

Choosing $\omega_5 = 2\omega_4$ we conclude that any solution of the Riccati equation

$$f' = f^2 + \omega_4 x^2 \tag{5.24}$$

generates a solution of equation (3.8d), given by relation (5.20).

One more possibility in solving of equation (3.8d) consists in its reduction to the Painlevé form. Making the change of variables $\varphi = \sqrt{-w_4}\chi, x = \frac{1}{\sqrt{-\omega_4}}y$ and differentiating equation (3.8d) w.r.t. y, we obtain

$$\left(\tilde{U}'' - 3\tilde{U}^2\right)'' + \left(6\tilde{U} + 6x\tilde{U}' + 2\tilde{U}''\right) = 4x^2, \tag{5.25}$$

where $\tilde{U} = \frac{\partial\chi}{\partial y} = -\frac{1}{\omega_4}U$.

Using the following generalized Miura ansatz

$$\tilde{U} = -V' + V^2 + 2Vy + y^2 - 1, \tag{5.26}$$

we reduce equation (5.25) to the form

$$\partial_y\left(\partial_y - 2V - 2y - 2\right) \times \\ \times \left(V''' - 6V^2V' - 4V_2 - 12yVV' - 4yV - 4V'y^2 - 2V'\right) = 0.$$

Equating the expression in the right brackets to zero, integrating and dividing it by $2V$, we come to the fourth Painlevé transcendent

$$V'' = \frac{V'^2}{2V} + \frac{3}{2}V^3 + 8yV^2 + \left(2y^2 - 1\right)V + \frac{b}{V}. \tag{5.27}$$

We note that the double differentiation and consequent change of variables

$$\varphi' = -\sqrt{\frac{\omega_4}{3}}\left(\Phi + \frac{1}{6}y^2\right), \quad x = \frac{1}{\sqrt[4]{4\omega_4}}y$$

transform equation (3.8d) to the form

$$\partial^4\Phi + \Phi''\Phi + \Phi'\Phi' - \frac{1}{3}\left(8\Phi + x^2\Phi'' + 7x\Phi'\right) = 0$$

which coincides with the reduced Boussinesq equation [3, 12]. The procedures outlined above reduces the equation either to the fourth Painlevé transcendent (5.27) or to the Riccati equation (5.24).

Thus, the third-order SO are admitted by a very extended class of potentials described above. We should like to emphasize that in general the corresponding Schrödinger equation does not possess any nontrivial (distinct from time displacements) Lie symmetry.

6. Equations for time-dependent potentials

Consider briefly the case of time-dependent potentials $U = U(x,t)$. The determining equations (2.6) are valid in this case also. Moreover, the compatibility condition for system (2.6) takes the form

$$F(a,b,c;x,U) + 12a\ddot{U} - 4(b - 2\dot{a}x)\dot{U}' = 0, \tag{6.1}$$

where $F(a,b,c;x,U)$ is defined in (2.7).

Equation (6.1) is much more complicated than (2.7) due to the time dependence of U, which makes it impossible to separate variables. For any fixed set of functions $a(t)$, $b(t)$, and $c(t)$, formula (6.1) defines a nonlinear equation for potential. Moreover, any of these equations admits the Lax representation

$$[H,Q] = i\frac{\partial Q}{\partial t}, \tag{6.2}$$

cf. (2.3). See Refs. [30, 31] for the general results connected with arbitrary ordinary differential operators satisfying (6.2).

We will not analyze equations (6.1) here, but present a few simple examples concerning particular choices of arbitrary functions a, b, and c.

$a = \text{const}$, $b = c = 0$:

$$-12\ddot{U} + U'''' - 6(UU')' = 0; \tag{6.3}$$

a, b are constants, $c = 0$:

$$12\ddot{U} - (4b\dot{U} - U''' + 6UU')' = 0; \tag{6.4}$$

$\dot{a} = c = 0$, $\dot{b} = \omega_3 a$:

$$12\ddot{U} - 4(\omega_3 t - 2x)\dot{U}' + (U'' - 3U^2)'' + 2\omega_3(xU' + 2U)' = 0; \tag{6.5}$$

$a = \exp(t)$, $b = c = 0$:

$$12\ddot{U} + 8x\dot{U}' + (U'' - U^2)'' - 12(Ux)' - 2x^2U'' - 4x^2 = 0. \tag{6.6}$$

Formula (6.3) defines the Boussinesq equation. The Lax representation (6.2) for this equation is well known [15]. Formulae (6.4)–(6.6) present other examples of non-linear equations admitting this representation and arise naturally under the analysis of third-order SOs of the Schrödinger equation.

7. Exact solutions

Let us regard the case of potentials satisfying (3.8a) or (5.4), (5.6). Taking into account commutativity of the corresponding SO (3.12a) with Hamiltonian (2.1) it is convenient to search for solutions of the Schrödinger equation in the form

$$\Psi(t,x) = \exp(-iEt)\psi(x), \tag{7.1}$$

where $\psi(x)$ are eigenfunctions of the commuting operators H and Q

$$H\psi(x) = E\psi(x), \tag{7.2a}$$

$$Q\psi(x) = \lambda\psi(x). \tag{7.2b}$$

Using (7.2a), (3.12a), and (5.4) we reduce (7.2b) to the first-order equation

$$\left(2E + \frac{V}{2} + \bar{\omega}_0\right)\psi' = \left(\frac{1}{4}V' + i\lambda\right)\psi \tag{7.3}$$

whose general solution has the form

$$\psi = A\sqrt{V + 4E + 2\bar{\omega}_0}\exp\left(2i\lambda\int\frac{dx}{V + 4E + 2\bar{\omega}_0}\right), \tag{7.4}$$

where A is an arbitrary constant. Then, expressing ψ' via ψ in accordance with (7.3) and using (5.6), we reduce (7.2a) to the following *algebraic* relation for E and λ (compare with (4.1b)):

$$\lambda^2 = 8E^2(E + \bar{\omega}_0). \tag{7.5}$$

Thus there exists a remarkably simple way to integrate the Schrödinger equation which admits a third order SO. The integration reduces to the problem of solving the first-order ordinary differential equation (7.3) and algebraic equation (7.5).

Let us show that the existence of a third-order SO for the linear Schrödinger equation enables one to find exact solutions for the following *nonlinear* equation:

$$i\partial_t\tilde{\Psi} = \frac{1}{2}p^2\tilde{\Psi} + \frac{1}{2A^2}(\tilde{\Psi}^*\tilde{\Psi})\tilde{\Psi}. \tag{7.6}$$

Indeed, if $\lambda^2 > 0$, solutions (7.1), (7.4) satisfy the following relations

$$\Psi^*\Psi = A^2(V + 4E + 2\bar{\omega}_0). \tag{7.7}$$

Using (7.2a) and (7.7) we make sure that the functions

$$\tilde{\Psi} = \exp(i\varepsilon t)\psi(x), \quad \varepsilon = -3E - \bar{\omega}_0 \tag{7.8}$$

(where $\psi(x)$ are functions defined in (7.4)) are exact solutions of (7.6).

Thus, we obtain a wide class of exact solutions of the nonlinear Schrödinger equation, which depend on arbitrary parameters ε, $\bar{\omega}_0$, $\bar{\omega}_1$, k (see (7.8), (7.4), (5.6), (5.8)). Properties of these (and some more general) solutions are discussed in the following section.

8. Lie symmetries and generation of solutions

It is well known that equation (7.6) is invariant under the Galilei transformations (refer, e.g., to Refs. [2, 3])

$$\begin{aligned} &x \to x' = x + vt, \\ &\Psi(t,x) \to \Psi'(t,x') = \exp\left[i\left(vx + \frac{v^2t}{2} + \varphi_0\right)\right]\Psi(t,x), \end{aligned} \tag{8.1}$$

where v and φ_0 are real parameters. Using (8.1) and starting with (7.8) it is possible to generate a more extended family of solutions

$$\bar{\Psi} = A\sqrt{V(x-k+vt)+4E+2\bar{\omega}_0}\times \\ \times\exp\left\{i\left[(2\varepsilon+v^2)\frac{t}{2}+vx+\varphi_0+2\lambda\int_0^{x-k-vt}\frac{dy}{V(y)+4E+2\bar{\omega}_0}\right]\right\}. \tag{8.2}$$

Here, V is an arbitrary solution of equation (5.6), v, $\bar{\omega}_0$, $\bar{\omega}_1$, k, φ_0 and E are real parameters, λ and ε are defined in (7.5), (7.8).

In order for λ to be real we require $\varepsilon \geq 0$, other parameters are arbitrary.

Solutions (8.2) are qualitatively different for different values of free parameters enumerated in (5.7). If $\bar{\omega}_0$ and $\bar{\omega}_1$ satisfy (5.7a) or (5.7c), possible V are given by formulae (5.9a), (5.9a$'$) or (5.9c). Solutions (8.2), (5.9a) are bounded for any x and t, whereas solutions (8.2), (5.9a$'$) and (8.2), (5.9c) are singular at $x-k-vt=0$. For $\bar{\omega}_0$ and $\bar{\omega}_1$ satisfying (5.7b) the modulus of the complex function (8.2), (5.9b) is periodic and singular at $x-k-vt=(2n+1)\pi/2\nu$. All the above mentioned singularities are simple poles. If $\bar{\omega}_0$ and $\bar{\omega}_1$ satisfy relations (5.8a), the solutions (8.2) are expressed via the two-periodic Weierstrass function $\wp$ (refer to (5.10)) and are, generally speaking, unbounded. But if we restrict ourselves to solutions (5.11) for potential, the corresponding solutions (8.2) are periodic and bounded.

To inquire into a physical content of the obtained solutions let us consider in more detail the cases (8.2), (5.9a) and (8.2), (5.11).

For potentials (5.9a) the corresponding relation (7.5) reduces to

$$\lambda^2 = 4E^2\epsilon, \quad \epsilon = 2E-\nu^2, \tag{8.3}$$

and the integral in (8.2) can be easily calculated. This enables us to represent solutions (8.2), (5.9a) as follows

$$\tilde{\Psi} = \frac{A\nu}{\cosh[\nu(x-k+vt)]}\exp\left\{i\left[\left(\frac{\nu^2+v^2}{2}\right)t+vx+\varphi_0\right]\right\}, \quad E=0; \tag{8.4}$$

$$\tilde{\Psi} = A\left\{\nu\tanh[\nu(x-k+vt)]\pm i\sqrt{\epsilon}\right\}\times \\ \times\exp\left\{i\left[\left(\frac{\nu^2+v^2}{2}-3E\right)t+(v\mp\sqrt{\epsilon})x+\varphi_0\right]\right\}, \quad E\neq 0, \quad \epsilon\geq 0. \tag{8.5}$$

For potentials (5.11) we obtain from (8.2)

$$\tilde{\Psi} = \tilde{\Psi}_1 = A\sqrt{B}\,\mathrm{cn}\,[D(x+vt)+k]\exp[if_1(t,x)], \quad E=0; \tag{8.6}$$

$$\tilde{\Psi} = \tilde{\Psi}_2 = A\sqrt{B\,\mathrm{cn}^2[D(x+vt)+k]+F}\exp(if_2(t,x)], \quad E+\bar{\omega}_0=0, \tag{8.7}$$

where

$$f_1(t,x) = f_2(t,x)+\frac{3}{2}Ft = \left(F+\frac{v^2}{2}\right)t+vx+\varphi_0,$$

B, D and F are parameters defined in (5.11b).

For other values of E solutions (8.2), (5.11) are also reduced to the form (8.7) where the phase $f_2(t,x)$ is expressed via elliptic integrals.

Formula (8.4) presents a fast decreasing one-soliton solution [31]. Relation (8.5) defines a soliton solution whose behavior at $x \to \infty$ is typical of solitons with a finite density. Formulae (8.6), (8.7) describe "cnoidal" solutions for the nonlinear Schrödinger equation.

9. Conditional symmetry and generation of solutions

Let us return to the linear Schrödinger equation (2.1) with the potential U satisfying (3.8a). Generally speaking it possesses no non-trivial (distinct from time displacements) Lie symmetry. Nevertheless, its solutions can be generated within the framework of the concept of conditional symmetry [2, 3, 12, 14, 32]. Indeed, these solutions satisfy (7.7), and equation (2.1) with the additional condition (7.7) *is* invariant under the Galilei transformations (8.1) (i.e., condition (7.7) extends the symmetry of equation (2.1)).

This conditional symmetry enables us to generate new solutions. Starting with (7.1), (7.4) and using (8.1) we obtain

$$\Psi = A\sqrt{V(x-k+vt)+4E+2\bar{\omega}_0}\times \\ \times \exp\left\{ i\left[(-2E+v^2)\frac{t}{2} + vx + \varphi_0 + 2\lambda \int\limits_0^{x-k-vt} \frac{dy}{V(y)+4E+2\bar{\omega}_0} \right]\right\}. \tag{9.1}$$

Functions (9.1) satisfy the Schrödinger equation with a potential $V(x-k+vt)$ where $V(x)$ is a solution of equation (5.6). In the particular case $E = -\frac{\bar{\omega}_0}{2}$ these functions are reduced to solutions (8.2) of the nonlinear equation (7.6).

One more generation of solutions can be made using a third-order SO. Inasmuch as $V(x)$ satisfies (5.6), then $V(x+vt)$ satisfies the Boussinesq equation (6.3). It means that the corresponding linear Schrödinger equation admits a third-order SO. In accordance with (2.2), (2.6) this SO can be represented in the form

$$Q = p^3 + \frac{1}{4}\{3V + 2\bar{\omega}_0 + 6v^2, p\} + \frac{3}{2}vV \equiv \\ \equiv 2pH + \frac{1}{2}(V + 2\bar{\omega}_0 + 6v^2)p + \frac{3}{2}vV + \frac{i}{4}V'. \tag{9.2}$$

Formula (9.2) generalizes (3.12a) to the case of time-dependent potential.

Acting by operator (9.2) on Ψ in (9.1) we obtain a new family of solutions

$$\Psi' = Q\Psi = a\psi + iv^2\Psi_1, \tag{9.3}$$

where $a = \lambda + 4Ev + \bar{\omega}_0 v - 4v^3$, Ψ is the initial solution (9.1),

$$\Psi_1 = \frac{V' + 4i\lambda}{2(4E + V + 2\bar{\omega}_0)}\Psi. \tag{9.4}$$

We note that if Ψ is a soliton solution

$$\Psi = \frac{\nu A}{\cosh[\nu[x+vt)]}\exp\left[i\left(\frac{v^2}{2}t + vx + \varphi_0\right)\right] \tag{9.5}$$

(the corresponding potential is present in (5.9a)), then (9.4) is a soliton solution too:

$$\Psi_1 = \frac{\nu^2 A \sinh[\nu(x+vt)]}{\cosh^2[\nu(x+vt)]} \exp\left[i\left(\frac{v^2}{2}t + vx + \varphi_0\right)\right]. \tag{9.6}$$

Starting with the potential (5.11) we obtain from (9.1) a particular solution

$$\Psi = A\sqrt{B\,\mathrm{cn}^2 z + F}\exp\left[i\left(\frac{v^2}{2}t + vx + \varphi_0\right)\right], \quad z = D(x - vt). \tag{9.7}$$

The corresponding generated solution (9.4) reads

$$\Psi_1 = -\frac{ABD\,\mathrm{cn}\,z\,\mathrm{sn}\,z\,\mathrm{dn}\,z}{B\,\mathrm{cn}^2 z + 2F}\exp\left[i\left(\frac{v^2}{2}t + vx + \varphi_0\right)\right] \tag{9.8}$$

and is also bounded.

Acting by SO (9.2) on solutions (9.3), (9.8) we again obtain new solutions. Moreover, this procedure can be repeated. In particular, in this way it is possible to construct multisoliton solutions of the linear Schrödinger equation.

We see that higher order SOs present efficient possibilities for solving equations of motion and generating new solutions starting with known ones.

10. Conclusion

Higher order SOs present a powerful tool for analyzing and solving the Schrödinger equation. The concept of higher symmetries enables us to extend the class of privileged potentials (2.4) and to investigate invariance algebras of the equations whose potentials satisfy one of relations (3.8).

We note that potentials (5.9) can be represented in the form $V = W^2 + W'$ where $W = \nu\tanh[\nu(x-k)]$ for solution (5.9a) (superpotentials W for solutions (5.9a)–(5.9c) can be also easily calculated). Moreover, the corresponding superpartners $\tilde{V} = W^2 - W'$ reduce to constants, therefore it is possible to integrate easily the Schrödinger equation with potentials (5.9) using the Darboux transformation [33].

It is worth to note that invariance condition (2.3) for operators (2.1), (3.12) can be treated as a zero curvature condition for equations associated with the eigenvalue problem for operator Q, or as the Lax condition where a role of the Lax operator L is played by a SO, refer to (6.2). The reasons stimulating our research of such a well-studied subject and distinguishing features of our approach are the following:

(1) The main goal of our paper is to present a constructive description of potentials for the Schrödinger equation which admit higher symmetries. In this way we extend the fundamental results [19–21] connected with the search for potentials admitting usual Lie symmetries.

To solve the deduced determining equations for potentials we use direct reductions to the Painlevé or Riccati forms. The obtained results can be used for analysis and solution of the Schrödinger equation as well as for construction of exact solutions of the Boussinesq equation, see item 5 in the following.

In the method of inverse problem, description of pairs of operators (2.1), (2.8) satisfying the Lax condition (6.2) is reduced to the Gelfand–Marchenko–Levitan equations [34] or to the Riemann problem [15, 31] which can be solved explicitly for a restricted class of potentials.

(2) We use non-Lie symmetries of the Schrödinger equation for construction and generation of exact solutions. Moreover, we are interested not so much in finding *new solutions* as in developing a *new method* of their derivation, which consists in simultaneous using of higher order and conditional symmetries. Nevertheless, the cnoidal solutions (9.7), (9.8) and (8.6), (8.7) for the linear and nonlinear Schrödinger equations can be of interest for physicists as well as infinite series of soliton and cnoidal solutions generated by a repeated application of the procedure described in Section 9.

We believe that the combination “higher order symmetries + conditional symmetries” may be used effectively in the investigations and analysis of other equations of mathematical physics.

(3) Our approach admits a direct generalization to multidimensional Schrödinger equations. Note that higher symmetries of the three-dimension Schrödinger equation were investigated in [18, 35] for particular potentials.

(4) Algebraic relations (4.1)–(4.4) are valid for extended classes of potentials. They open additional possibilities in the application of algebraic methods to investigate the Schrödinger equation, in particular, the use of raising and lowering operators for this equation with potentials satisfying (3.8d). We note that relations (3.8d) are valid also for time-independent operators $\tilde{Q}_\pm = \exp(\mp i\omega t)Q_\pm$ where $Q_\pm$ are given by relations (3.12d).

(5) Equations (3.8) which describe potentials that admit third-order symmetries are equivalent to the reduced versions of the Boussinesq equation, which appear under the similarity reduction [36] (this is the case for (3.8a,d)) and the reduction with using symmetries [14, 25, 26] (the last is valid for (3.8b,c)). Thus, the results obtained in Section V can be used to construct exact solutions of the Boussinesq equation.

A systematic study of higher symmetries of multidimensional Schrödinger equations is planned to be carried out elsewhere.

Acknowledgements. We are indebted to our anonymous referee for the rigorous criticism and helpful suggestions. This work is partly supported by the Ukrainian DFFD foundation (Project 1.4/356).

1. Anderson R.L., Ibragimov N.H., Lie–Bäcklund Transformations in Applications, Philadelphia, SIAM, 1979.
2. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of equations of nonlinear mathematical physics, Kyiv, Naukova Dumka, 1990; D. Reidel, Dordrecht, 1993.
3. Fushchych W.I., Nikitin A.G., Symmetries of equations of quantum mechanics, Moscow, Nauka, 1990; New York, Allerton Press Inc., 1994.
4. Fushchych W.I., Nikitin A.G., *J. Phys. A*, 1987, **20**, 537.
5. Beckers J., Debergh N., Nikitin A.G., *Mod. Phys. Lett. A*, 1992, **7**, 1609.
6. Beckers J., Debergh N., Nikitin A.G., *Mod. Phys. Lett. A*, 1993, **8** , 435.
7. Beckers J., Debergh N., Nikitin A.G., *Phys. Lett. B*, 1992, **279**, 333.
8. Fushchych W.I., Nikitin A.G., Symmetries of Maxwell’s equations, Kyiv, Naukova Dumka, 1983; Dordrecht, D. Reidel, 1987.
9. Miller W., Symmetry and separation of variables, Massachusets, Addison-Wesley, 1977.
10. Fels M., Kamran N., *Proc. Roy. Soc. Lond. A*, 1990, **428**, 229.
11. Fushchych W.I., *Teor. Math. Phys.*, 1971, **7**, 3.
12. Fushchych W.I., Symmetry in problems of mathematical physics, in Group-Theoretical Studies in Mathematical Physics, Kyiv, Inst. of Math., 1981, 6–28 (in Russian).

13. Fushchych W.I., Shtelen V.M., *DAN USSR*, 1983, **269**, 88.

14. Fushchych W.I., Serov N.I., Conditional invariance and exact solutions of the Boussinesq equation, in Symmetry and Solutions of Nonlinear Equations of Mathematical Physics, Kyiv, Inst. of Math., 1989, 96–103 (in Russian).

15. Krichever I.M., *Funk. Analiz*, 1978, **12**, 20;
Krichever I.M., Novikov S.P., *Dokl. AN SSSR*, 1979, **247**, 33.

16. Fushchych W.I., Segeda Yu.N., *Dokl. AN SSSR*, 1977, **232**, 800.

17. Beckers J., Debergh N., Nikitin A.G., *J. Phys. A*, 1992, **24**, L1269.

18. Nikitin A.G., Onufriichuk S.P., Fushchych W.I., *Teor. Math. Phys.*, 1992, **91**, 268 (in Russian); English translation in *Theor. Math. Phys.*, 1992, **91**, 514.

19. Anderson R.L., Kumei S., Wulfman C., *Rev. Mexicana Fis.*, 1972, **21**, 35.

20. Boyer C., *Helv. Phys. Acta*, 1974, **47**, 589.

21. Niederer U., *Helv. Phys. Acta*, 1972, **45**, 802.

22. This formula is present in [17] with a misprint: the coefficient 2 for ∂_t^4 is missing there.

23. Burchnall J.L., Chaundy T.W., *Proc. London Math. Soc.*, 1922, **21**, 420; *Proc. Royal Soc. London*, 1928, **118**, 557.

24. Ince E.L., Ordinary differential equations, London, Longmans Green, 1927.

25. Olver P.J., Rosenau Ph., *Phys. Lett. A*, 1986, **114**, 107.

26. Clarkson P.A., Kruskal M.D., *J. Math. Phys.*, 1989, **30**, 2201.

27. Bagrov V.G., Gitman D.M., Exact solutions of relativistic wave equations, Dordrecht, Kluwer Academic, 1990.

28. Whittaker E.T., Watson G.N., A Course of modern analysis, Cambridge University Press, 1927.

29. Miura R.M., *J. Math. Phys.*, 1968, **9**, 1202.

30. Gelfand I.M., Dikyi L.A., *Uspekhi Mat. Nauk*, 1975, **30**, 67–100; *Funk. Analiz*, 1976, **10**, 13.

31. Takhtadjian L.A., Faddeev L.D., Hamiltonian Approach in soliton theory, Moscow, Nauka, 1986 (in Russian).

32. Fushchych W.I., Tsyfra I.M., *J. Phys. A*, 1987, **20**, 45.

33. Darboux G., *Compt. Rend.*, 1882, **94**, 1456.

34. Gelfand I.M., Levitan B.M., *Izv. AN SSSR, Mat. Ser.*, 1951, **15**, 309;
Marchenko V.A., *Proc. Moskow Math. Soc.*, 1952, **1**, 357.

35. Nikitin A.G., *Ukrain. Math. J.*, 1991, **43**, 1521 (in Russian); English translation in *Ukr. Math. J.*, 1991, **43**, 1413.

36. Boiti M., Pempinelli F., *Nuovo Cim. B*, 1980, **56**, 148.

On the classification of subalgebras of the conformal algebra with respect to inner automorphisms

L.F. BARANNYK, P. BASARAB-HORWATH, W.I. FUSHCHYCH

We give a complete justification of the classification of inequivalent subalgebras of the conformal algebra with respect to the inner automorphisms of the conformal group, and we perform the classification of the subalgebras of the conformal algebra $AC(1,3)$.

1. Introduction

The necessity of classifying the subalgebras of the conformal algebra is motivated by many problems in mathematics and mathematical physics [1, 2]. The conformal algebra $AC(1,n)$ of Minkowski space $\mathbb{R}_{1,n}$ contains the extended Poincaré algebra $A\tilde{P}(1,n)$ and the full Galilei algebra $AG_4(n-1)$ (also known as the optical algebra). The classification of the subalgebras of the conformal algebra $AC(1,n)$ is almost reducible to the classification of the subalgebras of the algebras $A\tilde{P}(1,n)$ and $AG_4(n-1)$.

Patera, Winternitz and Zassenhaus [1] have given a general method for the classification of the subalgebras of inhomogeneous transformations. Using this method, the classification of the subalgebras $AP(1,n)$, $A\tilde{P}(1,n)$, and $AG_4(n-1)$ was carried out in Refs. [1–9] for $n=2,3,4$. In Refs. [7–11], this general method was supplemented by many structural results which made possible the algorithmization of the classification of the subalgebras of the Euclidean, Galilean, and Poincaré algebras for spaces of arbitrary dimensions. Indeed, this was done in Refs. [9] and [10], where the subalgebras of $AC(1,n)$ were classified up to conjugation under the conformal group $C(1,n)$ for $n=2,3,4$.

In order to perform the symmetry reduction of differential equations, it is necessary to identify the subalgebras of the symmetry algebra (of the equation) which give the same systems of basic invariants. This observation has led to the introduction in Ref. [12] of the concept of I-maximal subalgebras: a subalgebra F is said to be I-maximal if it contains every subalgebra of the symmetry algebra with the same invariants as F. In Ref. [13], all I-maximal subalgebras of $AC(1,4)$, classified up to $C(1,4)$-conjugation, were found in the representation defined on the solutions of the eikonal equation. Using these subalgebras, reductions of the eikonal and Hamilton–Jacobi equations to differential equations of lower order were obtained in Refs. [9] and [12]. We note that the list of I-maximal subalgebras for a given algebra can differ according to the equation being investigated.

In the above works, the question of the connection between conjugation of the subalgebras of the algebra $A\tilde{P}(1,n)$ under the group $\tilde{P}(1,n)$ (or the group $\mathrm{Ad}\,A\tilde{P}(1,n)$ of inner automorphisms of the algebra $A\tilde{P}(1,n)$) and the conjugacy of these subalgebras under the group $C(1,n)$ was not dealt with. This, and the same problem for

Reprinted with permission from J. Math. Phys., 1998, **39**, № 9, P. 4899–4922

subalgebras of the Galilei algebra $AG_4(n-1)$, is the problem we address in the present article.

Since the group analysis of differential equations is of a local nature, we concentrate on conjugacy of the subalgebras under the group of inner automorphisms of the algebra $AC(1,n)$. Going over to conjugacy under $C(1,n)$ is not complicated, and requires only a further identification of the subalgebras under the action of at most three discrete symmetries. The results of this paper allow us to obtain a full classification of the subalgebras of $AC(1,n)$ for low values of n. On the basis of these results, we give at the end of this paper a classification of the algebra $AC(1,3)$ with respect to its group of inner automorphisms. The list of subalgebras obtained in this way can be used for the symmetry reduction of any system of differential equations which are invariant under $AC(1,3)$.

2. Maximal subalgebras of the conformal algebra

We denote by $\mathrm{Ad}\,L$ the group of inner automorphisms of the Lie algebra L. Unless otherwise stated, conjugacy of subalgebras of L means conjugacy with respect to the group $\mathrm{Ad}\,L$. We consider $\mathrm{Ad}\,L_1$ as a subgroup of $\mathrm{Ad}\,L_2$ whenever L_1 is a subalgebra of L_2. The connected identity component of a Lie group H is denoted by H_1.

Let $\mathbb{R}_{1,n}$ $(n \geq 2)$, be Minkowski space with metric $g_{\alpha\beta}$, where $(g_{\alpha\beta}) = \mathrm{diag}\,[1, -1, \ldots, -1]$ and $\alpha, \beta = 0, 1, \ldots, n$. The transformation defined by the equations

$$x_\alpha = x_\alpha(y_0, y_1, \ldots, y_n), \quad \alpha = 0, 1, \ldots, n$$

of a domain $U \subset \mathbb{R}_{1,n}$ into $\mathbb{R}_{1,n}$, is said to be conformal if

$$\frac{\partial x_\mu}{\partial y^\alpha}\frac{\partial x_\nu}{\partial y^\beta} g^{\mu\nu} = \lambda(x) g_{\alpha\beta},$$

where $\lambda(x) \neq 0$ and $x = (x_0, x_1, \ldots, x_n)$. The conformal transformations of $\mathbb{R}_{1,n}$ form a Lie group, the conformal group $C(1,n)$. The Lie algebra $AC(1,n)$ of the group $C(1,n)$ has as its basis the generators of pseudorotations $J_{\alpha\beta}$, the translations P_α, the nonlinear conformal translations K_α, and the dilatations D, where $\alpha, \beta = 0, 1, \ldots, n$. These generators satisfy the following commutation relations:

$$\begin{aligned}
&[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta} J_{\beta\gamma} + g_{\beta\gamma} J_{\alpha\delta} - g_{\alpha\gamma} J_{\beta\delta} - g_{\beta\delta} J_{\alpha\gamma},\\
&[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta, \quad [P_\alpha, P_\beta] = 0, \quad [K_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} K_\gamma - g_{\alpha\gamma} K_\beta,\\
&[K_\alpha, K_\beta] = 0, \quad [D, P_\alpha] = P_\alpha, \quad [D, K_\alpha] = -K_\alpha, \quad [D, J_{\alpha\beta}] = 0,\\
&[K_\alpha, P_\beta] = 2(g_{\alpha\beta} D - J_{\alpha\beta}).
\end{aligned} \tag{1}$$

The pseudo-orthogonal group $O(2, n+1)$ is the multiplicative group of all $(n+3) \times (n+3)$ real matrices C satisfying $C^t E_{2,n+1} C = E_{2,n+1}$, where $E_{2,n+1} = \mathrm{diag}\,[1, 1, -1, \ldots, -1]$. We denote by I_{ab} the $(n+3) \times (n+3)$ matrix whose entries are zero except for 1 in the (a, b) position, with $a, b = 1, 2, \ldots, n+3$. The Lie algebra $AO(2, n+1)$ of $O(2, n+1)$ has as its basis the following operators:

$$\begin{aligned}
&\Omega_{12} = I_{12} - I_{21}, \quad \Omega_{ab} = -I_{ab} + I_{ba} \ \ (a < b;\ a, b = 3, \ldots, n+3),\\
&\Omega_{ia} = -I_{ia} - I_{ai} \ \ (i = 1, 2;\ a = 3, \ldots, n+3),
\end{aligned}$$

which satisfy the commutation relations

$$[\Omega_{ab}, \Omega_{cd}] = \rho_{ad}\Omega_{bc} + \rho_{bc}\Omega_{ad} - \rho_{ac}\Omega_{bd} - \rho_{bd}\Omega_{ac} \quad (a, b, c, d = 1, 2, \ldots, n+3),$$

where $(\rho_{ab}) = E_{2,n+1}$. Let us denote by $\mathbb{R}_{2,n+1}$ the pseudo-Euclidean space of $n+3$ dimensions with metric ρ_{ab}. The matrices of the group $O(2, n+1)$ and the algebra $AO(2, n+1)$ will be identified with operators acting on the left on $\mathbb{R}_{2,n+1}$. Then, with this convention, $O(2, n+1)$ is the group of isometries of $\mathbb{R}_{2,n+1}$.

It is known (see for instance Ref. [9]) that there is a homomorphism $\Psi : O(2, n+1) \to C(1, n)$ with kernel $\{\pm E_{n+3}\}$, where $\{E_{n+3}\}$ is the unit $(n+3)\times(n+3)$ matrix. Thus we are able to identify $O(2, n+1)$ with $C(1, n)$. This homomorphism of groups induces an isomorphism f of the corresponding Lie algebras, $f : AO(2, n+1) \to AC(1, n)$, which is given by

$$\begin{aligned}&f(\Omega_{\alpha+2,\beta+2}) = J_{\alpha\beta}, \quad f(\Omega_{1,\alpha+2} - \Omega_{\alpha+2,n+3}) = P_\alpha,\\ &f(\Omega_{1,\alpha+2} + \Omega_{\alpha+2,n+3}) = K_\alpha, \quad f(\Omega_{1,n+3}) = -D \quad (\alpha, \beta = 0, 1, \ldots, n).\end{aligned}$$

We shall in this article identify the two algebras, using this isomorphism, so that we can write the previous equations as

$$\begin{aligned}&\Omega_{\alpha+2,\beta+2} = J_{\alpha\beta}, \quad \Omega_{1,\alpha+2} - \Omega_{\alpha+2,n+3} = P_\alpha,\\ &\Omega_{1,\alpha+2} + \Omega_{\alpha+2,n+3} = K_\alpha, \quad \Omega_{1,n+3} = -D \quad (\alpha < \beta;\ \alpha, \beta = 0, 1, \ldots, n).\end{aligned}$$

We shall use the matrix realization of the conformal algebra.

Each matrix C which belongs to the identity component $O_1(2, n+1)$ of the group $O(2, n+1)$ is a product of matrices which are rotations in the x_1x_2 and x_ax_b planes ($a < b$; $a, b = 3, \ldots, n+3$) and hyperbolic rotations in the x_ix_a planes ($i = 1, 2$; $a = 3, \ldots, n+3$). Thus each such matrix C can be given as a finite product of matrices of the form $\exp X$, where $X \in AO(2, n+1)$. From this, it follows that each inner automorphism of the algebra $AO(2, n+1)$ is a mapping

$$\varphi_C : Y \to CYC^{-1}, \tag{2}$$

where $Y \in AO(2, n+1)$ and $C \in O_1(2, n+1)$, and conversely each mapping of this type is an inner automorphism of the algebra $AO(2, n+1)$.

In the process of our investigation mappings of the above type (2) will occur for certain matrices $C \in O(2, n+1)$, so we call these types of mappings $O(2, n+1)$-automorphisms of the algebra $AO(2, n+1)$ corresponding to the matrix C.

If G is the group of $O(2, n+1)$-automorphisms of the algebra $AO(2, n+1)$, and H is the subgroup of G consisting of its inner automorphisms, then H is normal in G and $[G : H] \leq 4$. Representatives of the cosets of G/H different from the identity will be

$$\begin{aligned}&C_1 = \mathrm{diag}\,[-1, 1, \ldots, 1, -1], \quad C_2 = \mathrm{diag}\,[1, 1, -1, 1 \ldots, 1],\\ &C_3 = \mathrm{diag}\,[-1, 1, -1, 1, \ldots, 1, -1],\end{aligned} \tag{3}$$

or

$$\begin{aligned}&C_1 = \mathrm{diag}\,[1, -1, 1, \ldots, 1, -1, 1], \quad C_2 = \mathrm{diag}\,[1, 1, -1, 1 \ldots, 1],\\ &C_3 = \mathrm{diag}\,[1, -1, -1, 1, \ldots, 1, -1, 1].\end{aligned} \tag{4}$$

Given a subspace V of $\mathbb{R}_{2,n+1}$, there is a maximal subalgebra of $AO(2,n+1)$ which leaves V invariant. We call this algebra the normalizer in $AO(2,n+1)$ of the subspace V.

Let $Q_1,\ldots,Q_{n+3}$ be a system of unit vectors in $\mathbb{R}_{2,n+1}$. Then the normalizer in $AO(2,n+1)$ of the isotropic subspace $\langle Q_1+Q_{n+3}\rangle$ is the extended Poincaré algebra

$$A\tilde{P}(1,n)=\langle P_0,P_1,\ldots,P_n\rangle \uplus (AO(1,n)\oplus\langle D\rangle),$$

where $\uplus$ denotes semidirect sum, and $\oplus$ denotes direct sum of algebras; $AO(1,n)=\langle J_{\alpha,\beta}:\alpha,\beta=0,1,\ldots,n\rangle$. The normalizer in $AO(2,n+1)$ of the completely isotropic subspace $\langle Q_1+Q_{n+3},Q_2+Q_{n+2}\rangle$ is the ful1 Galilei algebra

$$AG_4(n-1)=\langle M,P_1,\ldots,P_{n-1},G_1,\ldots,G_{n-1}\rangle \uplus (AO(n-1)\oplus\langle R,S,T\rangle\oplus\langle Z\rangle),$$

where

$$M=P_0+P_n,\quad G_a=J_{0a}-J_{an}\quad (a=1,\ldots,n-1),\quad R=-(J_{0n}+D),$$
$$S=\frac{1}{2}(K_0+K_n),\quad T=\frac{1}{2}(P_0-P_n),\quad Z=J_{0n}-D.$$

The generators of the algebra $AG_4(n-1)$ satisfy the following commutation relations:

$$[J_{ab},J_{cd}]=g_{ad}J_{bc}+g_{bc}J_{ad}-g_{ac}J_{bd}-g_{bd}J_{ac},\quad [G_a,J_{bc}]=g_{ab}G_c-g_{ac}G_b,$$
$$[P_a,J_{bc}]=g_{ab}P_c-g_{ac}P_b,\quad [G_a,G_b]=0,\quad [P_a,G_b]=\delta_{ab}M,\quad [G_a,M]=0,$$
$$[P_a,M]=0,\quad [J_{ab},M]=0,\quad [R,S]=2S,\quad [R,T]=-2T,\quad [T,S]=R,$$
$$[Z,R]=[Z,S]=[Z,T]=[Z,J_{ab}]=0,\quad [R,G_a]=G_a,\quad [R,P_a]=-P_a,$$
$$[R,M]=0,\quad [R,J_{ab}]=0,\quad [S,G_a]=0,\quad [S,P_a]=-G_a,\quad [S,M]=0,$$
$$[S,J_{ab}]=0,\quad [T,G_a]=P_a,\quad [T,P_a]=0,\quad [T,M]=0,\quad [T,J_{ab}]=0,$$
$$[Z,G_a]=-G_a,\quad [Z,P_a]=-P_a,\quad [Z,M]=-2M,$$

with $a,b,c,d=1,\ldots,n-1$.

From these commutation relations we find that

$$\langle R,S,T\rangle=ASL(2,\mathbb{R}),\quad \langle R,S,T\rangle\oplus\langle Z\rangle=AGL(2,\mathbb{R}),$$

where $\mathbb{R}$ denotes the field of real numbers.

Let F be a reducible subalgebra of $AO(2,n+1)$. That is, there exists in $\mathbb{R}_{2,n+1}$ a nontrivial subspace W which is invariant under F. If W is isotropic, then there exists a totally isotropic subspace $W_0\subset W$ which is invariant under F. Since $\dim W_0$ is 1 or 2, then, by Witt's theorem [14] there exists an isometry $C\in O(2,n+1)$ such that CW_0 is either $\langle Q_1+Q_{n+3}\rangle$ or $\langle Q_1+Q_{n+3},Q_2+Q_{n+2}\rangle$. Taking into account that the matrices (3) do not change these subspaces and represent all the components of the group $O(2,n+1)$ different from the identity component $O_1(2,n+1)$, then we may assume that the above C lies in $O_1(2,n+1)$, the identity component. Thus there exists an inner automorphism φ of the algebra $AO(2,n+1)$ such that either $\varphi(F)\subset A\tilde{P}(1,n)$ or $\varphi(F)\subset AG_4(n-1)$.

If W is a nondegenerate subspace, then, by Witt's theorem, it is isometric with one of the following subspaces: $\mathbb{R}_{1,k}$ $(k\geq 2)$, $\mathbb{R}_{2,k}$ $(k\geq 1)$, $\mathbb{R}_k$ $(k\geq 1)$. Each of the isometrics (3) leaves invariant each of these subspaces, so that we may assume that the isometry which maps W onto one of these subspaces belongs to $O_1(2, n+1)$. From this, it follows that a subalgebra F is conjugate under the group of inner automorphisms of the algebra $AO(2, n+1)$ to a subalgebra of one of the following algebras:

(1) $AO'(1, k) \oplus AO''(1, n-k+1)$,
where $AO'(1, k) = \langle \Omega_{ab} : a, b = 1, 3, \ldots, k+2\rangle$ and
$AO''(1, n-k+1) = \langle \Omega_{ab} : a, b = 2, k+3, \ldots, n+3\rangle$ with $n \geq 3$
and $k = 2, \ldots, [(n+1)/2]$;
(2) $AO(2, k) \oplus AO(n-k+1)$, where
$AO(n-k+1) = \langle \Omega_{ab} : a, b = k+3, \ldots, n+3\rangle$ with $k = 0, 1, \ldots, n$.

In order to classify the subalgebras of these direct sums it is necessary to know the irreducible subalgebras of algebras of the type $AO(1, m)$ ($m \geq 2$) and $AO(2, m)$ ($m \geq 3$). It has been shown in Ref. [15] that $AO(1, m)$ has no irreducible subalgebras different from $AO(1, m)$. In Refs. [16] and [17] it has been shown that every semisimple irreducible subalgebra of $AO(2, m)$ ($m \geq 3$) can be mapped by an automorphism of this algebra onto one of the following algebras:

(1) $AO(2, m)$;
(2) $ASU[1, (m/2)]$ when m is even;
(3) $\langle \Omega_{14} + \sqrt{3}\Omega_{13} + \Omega_{25}, -\Omega_{15} + \Omega_{24} - \sqrt{3}\Omega_{23}, \Omega_{12} - 2\Omega_{45}\rangle$ when $m = 3$.

It follows then that when $m > 3$ is odd, the algebra $AO(2, m)$ has no irreducible subalgebras other than $AO(2, m)$. If $m = 2k$ and $k \geq 2$, then, up to inner automorphisms, $AO(2, m)$ has two nontrivial maximal irreducible subalgebras: $ASU(1, k) \oplus \langle Y\rangle$, and $ASU(1, k)' \oplus \langle Y'\rangle$, where

$$Y = \mathrm{diag}\,[J, \ldots, J], \quad Y' = \mathrm{diag}\,[J, -J, J \ldots, J]$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We note that a subalgebra L of $AG_4(n-1)$ is conjugate under $\mathrm{Ad}\, AO(2, n+1)$ with a subalgebra the algebra $A\tilde{P}(1, n)$ if and only if the projection of L onto $AGL(2, \mathbb{R}) = \langle R, S, T\rangle \oplus \langle Z\rangle$ is conjugate under $\mathrm{Ad}\, AGL(2, \mathbb{R})$ with a subalgebra of the algebra $\langle R, T, Z\rangle$.

3. Conjugacy under Ad $AP(1, n)$ of subalgebras of the Poincaré algebra $AP(1, n)$

The Poincaré group $P(1, n)$ is the multiplicative group of matrices

$$\begin{pmatrix} \Delta & Y \\ 0 & 1 \end{pmatrix},$$

where $\Delta \in O(1,n)$ and $Y \in \mathbb{R}_{n+1}$. Let I'_{ab}, $a,b = 0,1,\ldots,n+1$ be the $(n+2)\times(n+2)$ matrix whose entries are all zero except for the ab-entry, which is unity. Then a basis for $AP(1,n)$ is given by the matrices

$$J_{0a} = -I'_{0a} - I'_{a0}, \quad J_{ab} = -I'_{ab} + I'_{ba}, \quad P_0 = I'_{0,n+1}, \quad P_a = I'_{a,n+1},$$

with $a < b$; $a,b = 1,\ldots,n$. These basis elements obey the commutation relations (1). It is sometimes useful in calculations to identify elements of $AO(1,n)$ with matrices of the form

$$X = \begin{pmatrix} 0 & \beta_{01} & \beta_{02} & \cdots & \beta_{0n} \\ \beta_{01} & 0 & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{02} & -\beta_{12} & 0 & \cdots & \beta_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \beta_{0n} & -\beta_{1n} & -\beta_{2n} & \cdots & 0 \end{pmatrix}$$

and elements of the space $U = \langle P_0,\ldots,P_n\rangle$ are represented by $n+1$-dimensional columns Y. In this case, we take

$$P_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, P_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

and with this notation it is easy to see that $[X,Y] = XY$. We endow the space U with the metric of the pseudo-Euclidean space $\mathbb{R}_{1,n}$, so that the inner product of two vectors

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}$$

is $x_0y_0 - x_1y_1 - \cdots - x_ny_n$. The projection of $AP(1,n)$ onto $AO(1,n)$ is denoted by $\hat{\varepsilon}$. We also note that $AO(n)$, contained in $AO(1,n)$, is generated by J_{ab} ($a<b$; $a,b = 1,\ldots,n$).

Let B be a Lie subalgebra of the algebra $AO(1,n)$ which has no invariant isotropic subspaces in $\mathbb{R}_{1,n}$. Then B is conjugate under $\operatorname{Ad} AO(1,n)$ to a subalgebra of $AO(n)$ or to $AO(1,k)\oplus C$, where $k \ge 2$ and C is a subalgebra of the orthogonal algebra $AO'(n-k)$ generated by the matrices J_{ab} ($a,b = k+1,\ldots,n$). In the first case, B is not conjugate to any subalgebra of $AO(n-1)$.

Proposition 1. *Let B be a subalgebra of $AO(n)$ which is not conjugate to a subalgebra of $AO(n-1)$. If L is a subalgebra of $AP(1,n)$ and $\hat{\varepsilon}(L) = B$, then L is conjugate to an algebra $W \uplus C$, where W is a subalgebra of $\langle P_1,\ldots,P_n\rangle$, and C is a subalgebra of $B\oplus\langle P_0\rangle$. Two subalgebras $W_1 \uplus C_1$ and $W_2 \uplus C_2$ of this type are conjugate to each other under* $\operatorname{Ad} AP(1,n)$ *if and only if they are conjugate under* $\operatorname{Ad} AO(n)$.

Proof. The algebra B is a completely reducible algebra of linear transformations of the space U and annuls only the subspace $\langle P_0\rangle$ (other than the null subspace itself). Thus, by Theorem 1.5.3 [9], the algebra L is conjugate to an algebra of the form

$W \uplus C$ where $W \subset \langle P_1, \ldots P_n \rangle$ and $C \subset B \oplus \langle P_0 \rangle$. Now let $W_1 \uplus C_1$, and $W_2 \uplus C_2$ be of this form, conjugate under $\operatorname{Ad} AP(1,n)$. Then there exists a matrix $\Gamma \in P_1(1,n)$ such that $\varphi_\Gamma(W_1 \uplus C_1) = W_2 \uplus C_2$, and from this it follows that $\varphi_\Lambda(B_1) = B_2$ for some $\Lambda \in O_1(1,n)$. Let $V = \langle P_1, \ldots, P_n \rangle$. Since $[B_1, V] = V$, then $[B_2, \varphi_\Lambda(V)] = \varphi_\Lambda(V)$ and $\varphi_\Lambda(V) = V$. Thus we can assume that $\Lambda = \operatorname{diag}[1, \Lambda_1]$ where $\Lambda_1 \in SO(n)$, so that the given algebras are conjugate under $\operatorname{Ad} AO(n)$. The converse is obvious.

Proposition 2. *Let $B = AO(1,k) \oplus C$, where $k \geq 2$ and $C \subset AO'(n-k)$. If L is a subalgebra of $AP(1,n)$ and $\hat{\varepsilon}(L) = B$ then L is conjugate to $L_1 \oplus L_2$ where $L_1 = AO(1,k)$ or $L_1 = AP(1,k)$, and L_2 is a subalgebra of the Euclidean algebra $AE'(n-k)$ with basis P_a, J_{ab} $(a, b = k+1, \ldots, n)$. Two subalgebras of this form, $L_1 \oplus L_2$ and $L_1' \oplus L_2'$ are conjugate under* $\operatorname{Ad} AP(1,n)$ *if and only if $L_1 = L_1'$ and L_2 is conjugate to L_2' under the group of $E'(n-k)$-automorphisms.*

Proof. The proof is as in the proof of Proposition 1.

Lemma 1. *If $C \in O(1,n)$ and $C(P_0 + P_n) = \lambda(P_0 + P_n)$ then $\lambda \neq 0$ and*

$$C = \begin{pmatrix} \dfrac{1 + \lambda^2(1 + \boldsymbol{v}^2)}{2\lambda} & \lambda \boldsymbol{v}^t B & \dfrac{-1 + \lambda^2(1 - \boldsymbol{v}^2)}{2\lambda} \\ \boldsymbol{v} & B & -\boldsymbol{v} \\ \dfrac{-1 + \lambda^2(1 + \boldsymbol{v}^2)}{2\lambda} & \lambda \boldsymbol{v}^t B & \dfrac{1 + \lambda^2(1 - \boldsymbol{v}^2)}{2\lambda} \end{pmatrix}, \tag{5}$$

where $B \in B(n-1)$, $\boldsymbol{v}$ is an $(n-1)$-dimensional column vector, $\boldsymbol{v}^2$ is the scalar square of $\boldsymbol{v}$ and $\boldsymbol{v}^t$ is the transpose of $\boldsymbol{v}$. Conversely, every matrix C of this form satisfies $C(P_0 + P_n) = \lambda(P_0 + P_n)$.

Proof. Proof is by direct calculation.

Lemma 2. *Let $C \in O(1,n)$ have the form* (5)*, with $\lambda > 0$. Then*

$$C = \operatorname{diag}[1, B, 1] \exp[(-\ln \lambda) J_{0n}] \exp(-\beta_1 G_1 - \cdots - \beta_{n-1} G_{n-1}),$$

where $G_a = J_{0a} - J_{an}$ and

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{n-1} \end{pmatrix} = B^{-1} \boldsymbol{v}.$$

Proof. Direct calculation gives us

$$\exp(-\theta J_{0n}) = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & E_{n-1} & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}$$

and

$$\exp(-\beta_1 G_1 - \cdots - \beta_{n-1} G_{n-1}) = \begin{pmatrix} 1 + \dfrac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & \dfrac{\boldsymbol{b}^2}{2} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \dfrac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & 1 - \dfrac{\boldsymbol{b}^2}{2} \end{pmatrix},$$

where $\boldsymbol{b} = (\beta_1, \ldots, \beta_{n-1})^t$. On putting $\lambda \exp\theta$ we have

$$\cosh\theta = \frac{\lambda^2+1}{2\lambda}, \quad \sinh\theta = \frac{\lambda^2-1}{2\lambda}.$$

Since we have

$$\begin{pmatrix} \frac{\lambda^2+1}{2\lambda} & 0 & \frac{\lambda^2-1}{2\lambda} \\ 0 & E_{n-1} & 0 \\ \frac{\lambda^2-1}{2\lambda} & 0 & \frac{\lambda^2+1}{2\lambda} \end{pmatrix} \begin{pmatrix} 1+\frac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & \frac{\boldsymbol{b}^2}{2} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \frac{\boldsymbol{b}^2}{2} & \boldsymbol{b}^t & 1-\frac{\boldsymbol{b}^2}{2} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{1+\lambda^2(1+\boldsymbol{b}^2)}{2\lambda} & \lambda\boldsymbol{b}^t & \frac{-1+\lambda^2(1-\boldsymbol{b}^2)}{2\lambda} \\ \boldsymbol{b} & E_{n-1} & -\boldsymbol{b} \\ \frac{-1+\lambda^2(1+\boldsymbol{b}^2)}{2\lambda} & \lambda\boldsymbol{b}^t & \frac{1+\lambda^2(1-\boldsymbol{b}^2)}{2\lambda} \end{pmatrix},$$

then

$$\exp(-\theta J_{0n})\exp(-\beta_1 G_1 - \cdots - \beta_{n-1}G_{n-1}) = \operatorname{diag}[1, \beta^{-1}, 1]C$$

from which it follows directly that

$$C = \operatorname{diag}[1, B, 1]\exp[(-\ln\lambda)J_{0n}]\exp(-\beta_1 G_1 - \cdots - \beta_{n-1}G_{n-1})$$

and the lemma is proved.

The set of F of matrices of the form (5) with $\lambda > 0$ is a group under multiplication. The mapping

$$C \to \begin{pmatrix} \lambda B & \lambda\boldsymbol{v} \\ 0 & 1 \end{pmatrix}$$

is an isomorphism of the group F onto the extended Euclidean group $\tilde{E}(n-1)$. Thus we shall mean the group F when talking of the extended Euclidean group, and the connected identity component $\tilde{E}_1(n-1)$ will be identified with the group of matrices of the form (5) with $\lambda > 0$ and $B \in SO(n-1)$. From Lemma 2 it follows that the Lie algebra AF of the group F is generated by the basis elements J_{ab}, G_a, J_{0n} ($a < b$; $a, b = 1, \ldots, n-1$).

Lemma 3. *If $C \in O_1(1,n)$ and $C(P_0 + P_n) = \lambda(P_0 + P_n)$ then $\lambda > 0$ and $B \in SO(n-1)$ in (5).*

Proof. Since

$$\frac{1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} > 0,$$

then we have $\lambda > 0$. From Lemma 2, $\operatorname{diag}[1, B, 1] \in O_1(1,n)$, so that $\det B > 0$. Thus $B \in SO(n-1)$ and the lemma is proved.

Lemma 4. *If $C \in O(1,n)$ and $\pm C \notin \tilde{E}(n-1)$ then $C = \pm A_1 C' A_2$ where $A_1, A_2 \in \tilde{E}(n-1)$ and $C' = \operatorname{diag}[1, \ldots, 1, -1]$.*

Proof. We can choose a matrix $\Lambda \in O(n-1)$ so that $\Lambda C(P_0+P_n) = \alpha P_0+\beta P_1+\gamma P_n$ where $\alpha^2-\beta^2-\gamma^2=0$. If $\beta \neq 0$ then $\alpha-\gamma \neq 0$. Let $\theta = \beta/(\alpha-\gamma)$. Then,

$$\exp(\theta G_1)(\alpha P_0+\beta P_1+\gamma P_n) = \frac{\alpha-\gamma}{2}(P_0-P_n)$$

and so there exists a matrix $\Gamma \in \tilde{E}(n-1)$ such that $\Gamma C(P_0+P_n) = \lambda(P_0+P_n)$ or $\Gamma C(P_0+P_n) = \lambda(P_0-P_n)$. In the first case, $\pm\Gamma C \in \tilde{E}(n-1)$, so that then we have $\pm C \in \tilde{E}(n-1)$, which is impossible. In the second case, $C'\Gamma C(P_0+P_n) = \lambda(P_0+P_n)$. For $\lambda > 0$ we find $C'\Gamma C \in \tilde{E}(n-1)$. Put $C'\Gamma C = A_2$, $\Gamma = A_1^{-1}$. Then $C = A_1 C' A_2$. If $\lambda < 0$ then we put $-C'\Gamma C = A_2$, in which case $C = -A_1 C' A_2$, and the lemma is proved.

Lemma 5. *If $C \in O_1(1,n)$ and $C \notin \tilde{E}_1(n-1)$, then $C = D_1 Q D_2$, where $D_1, D_2 \in \tilde{E}_1(n-1)$, and $Q = \mathrm{diag}\,[1,-1,1,\ldots,1,-1]$.*

Proof. If $\pm C \in \tilde{E}(n-1)$, then $C(P_0+P_n) = \gamma(P_0+P_n)$. By Lemma 3, $\gamma > 0$ and $C \in \tilde{E}_1(n-1)$, which contradicts the assumption. Thus, $\pm C \notin \tilde{E}(n-1)$. By Lemma 4, $C = \pm A_1 C' A_2$. From this it follows that $C = D_1 \Gamma D_2$, where $D_1, D_2 \in \tilde{E}_1(n-1)$, and Γ is one of the matrices $\pm C'$, $\pm Q$. However, $\Gamma \in O_1(1,n)$, since $\Gamma = D_1^{-1} C D_2^{-1}$, find from this it follows that $\Gamma = Q$. The Lemma is proved.

Direct calculation shows that the normalizer of the space $\langle P_0+P_n\rangle$ in $AO(1,n)$ is generated by the matrices G_a, J_{ab}, J_{0n} $(a,b=1,\ldots,n-1)$, which satisfy the commutation relations

$$[G_a, J_{bc}] = g_{ab}G_c - g_{ac}G_b, \quad [G_a, G_b] = 0, \quad [G_a, J_{0n}] = G_a.$$

This means that the normalizer of the space $\langle P_0+P_n\rangle$ in the algebra $AO(1,n)$ is the extended Euclidean algebra

$$A\tilde{E}(n-1) = \langle G_1,\ldots,G_{n-1}\rangle \uplus (AO(n-1) \oplus \langle J_{0n}\rangle)$$

in an $(n-1)$-dimensional space, where the generators of translations are $G_1,\ldots,G_{n-1}$ and the generator of dilatations is the matrix J_{0n}.

Let K be a subalgebra of $AP(1,n)$ such that its projection onto $AO(1,n)$ has an invariant isotropic subspace in Minkowski space $\mathbb{R}_{1,n}$. The subalgebra K is conjugate under $\mathrm{Ad}\,AP(1,n)$ with a subalgebra of the algebra $\mathcal{A} = AG_1(n-1) \uplus \langle J_{0n}\rangle$ where $AG_1(n-1)$ is the usual Galilei algebra with basis M, T, P_a, G_a, J_{ab} $(a,b=1,\ldots,n-1)$, and $M = P_0+P_n$, $T = \frac{1}{2}(P_0-P_n)$.

Proposition 3. *Let L_1 and L_2 be subalgebras of $\mathcal{A}$, with L_1 not conjugate under $\mathrm{Ad}\,\mathcal{A}$ to any subalgebra having zero projection onto $\langle G_1,\ldots,G_{n-1}\rangle$. If $\varphi(L_1) = L_2$ for some $\varphi \in \mathrm{Ad}\,AP(1,n)$, then there exists an inner automorphism ψ of the algebra $\mathcal{A}$ with $\psi(L_1) = L_2$.*

Proof. Since $\mathrm{Ad}\,\mathcal{A}$ contains automorphisms which correspond to matrices of the form

$$\exp\left(\sum_{\gamma=}^{n} a_\gamma P_\gamma\right) \tag{6}$$

and since $P(1,n)$ is a semidirect product of the group of matrices of the form (6) and the group $O(1,n)$ of matrices of the form $\mathrm{diag}\,[\Delta, 1]$, then we may assume that

$\varphi = \varphi_C$ with $C \in O_1(1,n)$. If $C \notin \tilde{E}_1(n-1)$, then by Lemma 5, $C = D_1 Q D_2$. In that case we find that

$$(D_1 Q D_2)\hat{\varepsilon}(L_1)(D_2^{-1} Q D_1^{-1}) = \hat{\varepsilon}(L_2),$$

whence

$$Q(D_2 \hat{\varepsilon}(L_1) D_2^{-1})Q = D_1^{-1}\hat{\varepsilon}(L_2) D_1. \tag{7}$$

However,

$$QG_aQ = Q(J_{0a} - J_{an})Q = \begin{cases} J_{0a} + J_{an}, & \text{when } a \neq 1, \\ -(J_{01} + J_{1n}), & \text{when } a = 1. \end{cases}$$

This means that $QG_aQ \notin \mathcal{A}$. Because of this, the left-hand side of (7) does not belong to $\mathcal{A}$, whereas the right-hand side of (7) is a subalgebra of $\mathcal{A}$. This then implies that we must have $C \in \tilde{E}_1(n-1)$ and thus we have $\psi(L_1) = L_2$ for some $\psi \in \operatorname{Ad}\mathcal{A}$.

Proposition 4. *Let $\tilde{\mathcal{A}}$ be a Lie algebra with basis P_0, P_a, P_n, J_{ab}, J_{0n} $(a,b = 1,\ldots,n-1)$ and let L_1, L_2 be subalgebras of $\tilde{\mathcal{A}}$ such that at least one of them has a nonzero projection onto $\langle J_{0n}\rangle$. If $\varphi(L_1) = L_2$ for some $\varphi \in \operatorname{Ad} AP(1,n)$, then there exists an inner automorphism $\psi \in \tilde{\mathcal{A}}$ so that either $\psi(L_1) = L_2$ or $\psi(L_1) = \varphi_Q(L_2)$ where $Q = \operatorname{diag}[1,-1,1,\ldots,1,-1]$.*

Proof. As in the proof of Proposition 3, we may assume that $\varphi = \varphi_C$ where $C \in O_1(1,n)$. We shall also assume that the projection of L_1 onto $\langle J_{0n}\rangle$ is nonzero. If $C \in \tilde{E}_1(n-1)$ and $C \notin \tilde{O}_1(n-1)$ then the projection of the algebra $\varphi(L_1)$ onto $\langle G_1,\ldots,G_{n-1}\rangle$ is nonzero, and hence the projection of L_2 onto $\langle G_1,\ldots,G_{n-1}\rangle$ is nonzero, which contradicts the assumptions of the proposition. Thus, if $C \in \tilde{E}_1(n-1)$ then $\varphi \in \operatorname{Ad}\tilde{\mathcal{A}}$.

Let $C \notin \tilde{E}_1(n-1)$. By Lemma 5, $C = D_1 Q D_2$ where $D_1, D_2 \in \tilde{E}_1(n-1)$. Then $\varphi(L_1) = L_2$ can be written as

$$\varphi_Q(\sigma_{D_2}(L_1)] = \varphi_{D_1^{-1}}(L_2).$$

If $D_2 \notin \tilde{O}_1(n-1)$ then the projection of $\varphi_{D_2}(L_1)$ onto $\langle G_1,\ldots,G_{n-1}\rangle$ is nonzero and hence $\varphi_Q[\varphi_{D_2}(L_1)]$ does not belong to $\mathcal{A}$. But then $\varphi_{D_1^{-1}}(L_2)$ is also not in $\mathcal{A}$. This is a contradiction. Thus $D_1, D_2 \in \tilde{O}_1(n-1)$. From this it follows that $\varphi_Q(\psi(L_1)) = L_2$ where $\psi = \varphi_D$ is an inner automorphism of the algebra $\tilde{\mathcal{A}}$. This proves the proposition.

Proposition 5. *Suppose $2 \leq m \leq n-1$. Let F be a subalgebra of the algebra $AO(m)$ which is not conjugate under $\operatorname{Ad} AO(m)$ to a subalgebra of $AO(m-1)$, and let L be a subalgebra of $\langle P_0, P_1,\ldots,P_n\rangle \uplus F$ such that $\hat{\varepsilon}(L) = F$. Then L is conjugate to an algebra $W \uplus K$, where W is a subalgebra of $\langle P_1,\ldots,P_m\rangle$ and K is a subalgebra of $F \oplus \langle P_0, P_{m+1},\ldots,P_n\rangle$. Two subalgebras $W_1 \uplus K_1$ and $W_2 \uplus K_2$ of this type are conjugate under $\operatorname{Ad} AP(1,n)$ if and only if there exists an automorphism $\psi \in \operatorname{Ad} AO(m) \times \operatorname{Ad} AO(1,n-m)$ such that $\psi(W_1 \uplus K_1) = W_2 \uplus K_2$ or $\psi(W_1 \uplus K_1) = Q(W_2 \uplus K_2)Q$ where*

$$AO(1,n-m) = \langle J_{\alpha\beta} : \alpha,\beta = 0, m+1,\ldots,n\rangle$$

and $Q = \operatorname{diag}[1,-1,1,\ldots,1,-1]$.

4. Conjugacy of subalgebras of the extended Poincaré algebra $A\tilde{P}(1,n)$ under $\mathrm{Ad}\, AC(1,n)$

Lemma 6. *If $C \in O(2,n+1)$ and $C(Q_1+Q_{n+3}) = \lambda(Q_1+Q_{n+3})$ then $\lambda \neq 0$ and*

$$C = \begin{pmatrix} \dfrac{1+\lambda^2(1-\boldsymbol{v}^2)}{2\lambda} & -\lambda \boldsymbol{v}^t E_{1,n} B & \dfrac{-1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} \\ \boldsymbol{v} & B & -\boldsymbol{v} \\ \dfrac{-1+\lambda^2(1-\boldsymbol{v}^2)}{2\lambda} & -\lambda \boldsymbol{v}^t E_{1,n} B & \dfrac{1+\lambda^2(1+\boldsymbol{v}^2)}{2\lambda} \end{pmatrix}, \tag{8}$$

where $B \in O(1,n)$, $E_{1,n} = \mathrm{diag}\,[1,-1,\ldots,-1]$, $\boldsymbol{v}$ is an $(n+1)\times 1$ matrix and $\boldsymbol{v}^2$ is its scalar square in $\mathbb{R}_{1,n}$. Conversely, every matrix C of the form (8) satisfies the condition $C(Q_1+Q_{n+3}) = \lambda(Q_1+Q_{n+3})$.

Proof. Direct calculation.

Lemma 7. *Let $C \in O(2,n+1)$ have the form (8), with $\lambda > 0$. Then*

$$C = \mathrm{diag}\,[1,B,1] \exp[(\ln\lambda)D] \exp(-\beta_0 P_0 - \beta_1 P_1 - \cdots - \beta_n P_n),$$

where

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = B^{-1}\boldsymbol{v}.$$

Proof. The proof of Lemma 7 is similar to that of Lemma 2.

The mapping

$$f : C \to \begin{pmatrix} \lambda B & \lambda \boldsymbol{v} \\ 0 & 1 \end{pmatrix}$$

is a homomorphism of the group of matrices (8) onto the extended Poincaré group $\tilde{P}(1,n)$. The kernel of this homomorphism is the group of order two, $\{-E_{n+3}, E_{n+3}\}$. Let us denote by H the set of matrices of the form (8) with $\lambda > 0$. Then f is an isomorphism of H onto $\tilde{P}(1,n)$. For this reason we shall, in the remainder of this article, mean the group H when referring to $\tilde{P}(1,n)$. Its Lie algebra is the extended Poincaré algebra $A\tilde{P}(1,n)$ given in Section 2.

Lemma 8. *Let $C \in O_1(2,n+1)$ and let it be of the form (8) with $\lambda > 0$. Then $B \in B_1(1,n)$.*

Remark 1. Note that when $\lambda < 0$ it is possible that B does not belong to $O_1(2,n+1)$.

Lemma 9. *If $C \in O(2,n+1)$ and $\pm C \notin \tilde{P}(1,n)$ then either $C = \pm A_1 Q A_2$ or $C = A_1 F(\theta) A_2$, where $A_1, A_2 \in \tilde{P}(1,n)$, $Q = \mathrm{diag}\,[1,\ldots,1-1]$ and $F(\theta) = \exp[(\theta/2)(K_0 + P_0 + K_n - P_n)]$.*

Proof. There exists a matrix $\Lambda\tilde{P}(1,n)$ such that

$$\Lambda C(Q_1+Q_{n+3}) = \alpha_1 Q_1 + \alpha_2 Q_2 + \alpha_3 Q_{n+2} + \alpha_4 Q_{n+3},$$

where $\alpha_1^2 + \alpha_2^2 - \alpha_3^2 - \alpha_4^2 = 0$ and $\alpha_2\alpha_3 \geq 0$. If $\alpha_1 \neq \alpha_4$ then, as in the proof of Lemma 4, we obtain that

$$\exp(\beta_0 P_0 + \beta_n P_n)\Lambda C(Q_1 + Q_{n+3}) = \gamma(Q_1 \pm Q_{n+3})$$

for some real numbers β_0, β_n, γ. From this it follows that

$$\Gamma \exp(\beta_0 P_0 + \beta_n P_n)\Lambda C(Q_1 + Q_{n+3}) = \lambda(Q_1 + Q_{n+3}),$$

where $\lambda > 0$ and $\Gamma = \pm E_{n+3}$ or $\Gamma = \pm Q$. By Lemma 6 and Lemma 7, we obtain

$$\Gamma \exp(\beta_0 P_0 + \beta_n P_n)\Lambda C = \tilde{\Lambda}, \quad \tilde{\Lambda} \in \tilde{P}(1,n).$$

Since $\pm C \notin \tilde{P}(1,n)$, then $\Gamma = \pm Q$, and so $C = \pm A_1 Q A_2$, where $A_1 = \Lambda^{-1}\exp(-\beta_0 P_0 -\beta_n P_n)$, $A_2 = \tilde{\Lambda}$.

If $\alpha_1 = \alpha_4$, then also $\alpha_2 = \alpha_3$. It is easy to verify that

$$F(\theta)\Lambda C(Q_1 + Q_{n+3}) = (\alpha_1 \cos\theta + \alpha_2 \sin\theta)(Q_1 + Q_{n+3}) + \\ + (\alpha_2 \cos\theta - \alpha_1 \sin\theta)(Q_2 + Q_{n+2}).$$

If $\alpha_1 = 0$ then we put $\theta = (\pi/2)$, when $\alpha_2 > 0$ and $\theta = -(\pi/2)$, when $\alpha_2 < 0$. If $\alpha_1 \neq 0$ then we let $\alpha_2 \cos\theta - \alpha_1 \sin\theta = 0$. In that case,

$$\tan\theta = \frac{\alpha_2}{\alpha_1}, \quad \alpha_1 \cos\theta + \alpha_2 \sin\theta = \alpha_1 \cos\theta(1 + \tan^2\theta).$$

We choose the value of θ so that $\alpha_1 \cos\theta > 0$. With this choice of θ we have

$$F(\theta)\Lambda C(Q_1 + Q_{n+3}) = \lambda(Q_1 + Q_{n+3}),$$

where $\lambda > 0$. But then, as a result of Lemma 6 and Lemma 7, $F(\theta)\Lambda C = \tilde{\Lambda}$, $\tilde{\Lambda} \in \tilde{P}(1,n)$, and so $C = A_1 F(-\theta) A_2$, where $A_1 = \Lambda^{-1}$, $A_2 = \tilde{\Lambda}$. The result is proved.

Lemma 10. *Let L_1 and L_2 be subalgebras of $A\tilde{P}(1,n)$ which are not conjugate under $A\tilde{P}(1,n)$ to subalgebras of $A\tilde{O}(1,n) = AO(1,n) \oplus \langle D\rangle$. Then L_1, L_2 are conjugate under* $\mathrm{Ad}\, AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\, A\tilde{P}(1,n)$ *or if one of the following conditions holds:*

(1) *n is an odd number and there exists an automorphism $\psi \in \mathrm{Ad}\, A\tilde{P}(1,n)$ with $\psi(L_1) = C_2 L_2 C_2^{-1}$ (see Eq.* (3) *for notation);*

(2) *there exist automorphisms $\psi_1, \psi_2 \in A\tilde{P}(1,n)$ with*

$$\psi_1(L_1) = F(\theta)[\psi_2(L_2)]F(-\theta).$$

Proof. Let $CL_1C^{-1} = L_2$ for some $C \in O_1(2, n+1)$. By Lemma 9, we may assume that $\pm C \in \tilde{P}(1,n)$ or that C is one of the matrices $\pm A_1 Q A_2$, $A_1 F(\theta) A_2$ (we use the notation of Lemma 9). If $C \in \tilde{P}(1,n)$ then, by Lemma 8, C belongs to the identity component of the group $\tilde{P}(1,n)$ and thus φ_C is an inner automorphism of the algebra $A\tilde{P}(1,n)$. Now suppose $-C \in \tilde{P}(1,n)$. Then by Lemma 7, $C = -\mathrm{diag}\,[1, B, 1]$, where $B \in O(1,n)$ and $\Delta \in \tilde{P}_1(1,n)$. Thus we may assume that $C = -\mathrm{diag}\,[1, B, 1]$. From this it follows that $B \in O_1(1,n)$ for odd n and we have

$$\mathrm{diag}\,[1, 1, -1, 1, \ldots, 1, 1]B \in O_1(1,n)$$

For even n this means that the algebras L_1, L_2 are conjugate to each other under $\mathrm{Ad}\,A\tilde{P}(1,n)$ or that there exists an automorphism $\psi \in \mathrm{Ad}\,A\tilde{P}(1,n)$ such that $\psi(L_1) = C_2L_2C_2^{-1}$.

Let $C = \pm A_1QA_2$. Then $C = \Gamma_1\Delta\Gamma_2$ with $\Gamma_1, \Gamma_2 \in \tilde{P}(1,n)$ and $\Delta = \pm\mathrm{diag}\,[1, \varepsilon_1, 1, \ldots, 1, \varepsilon_2, -1]$ with $\varepsilon_1, \varepsilon_2 \in \{-1, 1\}$. Clearly, $\Delta \in O_1(2, n+1)$. When $C = A_1QA_2$ we have $\varepsilon_1 = 1$, $\varepsilon_2 = -1$ and when $C = -A_1QA_2$, $\varepsilon_1 = 1$, $\varepsilon_2 = (-1)^n$. Since

$$\Delta P_n \Delta^{-1} = \pm K_n, \quad \Delta P_\alpha \Delta^{-1} = \pm K_\alpha$$

with $\alpha < n$, then from $\Gamma_1^{-1}L_2\Gamma_1 = \Delta(\Gamma_2L_1\Gamma_2^{-1})\Delta^{-1}$ it follows that the algebra $\Gamma_1^{-1}L_2\Gamma_1$ has a nonzero projection onto $\langle K_0, K_1, \ldots, K_n\rangle$, which is impossible. Thus the matrix C is different from $\pm A_1QA_2$.

Now let $C = A_1F(\theta)A_2$. If Γ is one of the matrices (4), then $\Gamma F(\theta)\Gamma^{-1} = F(\pm\theta)$, so that

$$C = A_1'F(\theta)A_2'\Delta,$$

where $A_1', A_2' \in \tilde{P}(1,n)$ and $\Delta = E$ or Δ is one of the matrices (4). Since Δ can be represented as a product of matrices in $O_1(2,n)$, then the last case is impossible, and we have proved the Lemma.

Theorem 1. *Let L_1 and L_2 be subalgebras of $A\tilde{P}(1,n)$ which are not conjugate under $A\tilde{P}(1,n)$ to subalgebras of $A\tilde{O}(1,n)$ and such that their projections onto $AO(1,n)$ have no invariant isotropic subspace in $\mathbb{R}_{1,n}$. The subalgebras L_1 and L_2 are conjugate under* $\mathrm{Ad}AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,A\tilde{P}(1,n)$ *or when there exists an automorphism $\psi \in \mathrm{Ad}\,A\tilde{P}(1,n)$ such that $\psi(L_1) = C_2L_2C_2^{-1}$, where $C_2 =$* $\mathrm{diag}\,[1, 1, -1, 1, \ldots, 1]$.

Proof. By Lemma 10 we may assume that $\psi_1(L_1) = F(\theta)[\psi_2(L_2)]F(-\theta)$ for some $\psi_1, \psi_2 \in A\tilde{P}(1,n)$. Under the given assumptions, the projection of $\psi_2(L_2)$ onto $AO(1,n)$ contains an element of the form

$$X = \sum_{b=1}^{n-1}(\alpha_b J_{0b} + \gamma_b J_{bn}) + \sum_{b,c=1}^{n-1} \sigma_{bc}J_{bc},$$

where $\alpha_q \neq -\gamma_q$ for some q $(1 \leq q \leq n-1)$. Since

$$F(\theta)J_{0q}F(-\theta) = J_{0q}\cos\theta + \frac{1}{2}(K_q + P_q)\sin\theta$$

and

$$F(\theta)J_{qn}F(-\theta) = J_{qn}\cos\theta + \frac{1}{2}(K_q - P_q)\sin\theta$$

we have that $F(\theta)XF(-\theta)$ contains the term

$$F(\theta)[\alpha_q J_{0q} + \gamma_q J_{qn}]F(-\theta) = (\alpha_q J_{0q} + \gamma_q J_{qn})\cos\theta + \\ + \frac{1}{2}[\alpha_q(K_q + P_q) + \gamma_q(K_q - P_q)]\sin\theta$$

and from this it follows that $(\alpha_q + \gamma_q)\sin\theta = 0$ so that $\sin\theta = 0$. But then $\theta = m\pi$. When $m = 2d$ we have $F(\theta) = E_{n+3}$. When $m = 2d + 1$ then $F(\theta) = \mathrm{diag}\,[-1, -1, E_{n-1}, -1, -1]$. However,

$$F(\theta)[\psi_2(L_2)]F(-\theta) = (-F(\theta))[\psi_2(L_2)](-F(-\theta))$$

from which it follows that we may assume that $\psi_1(L_1) = C[\psi_2(L_2)]C^{-1}$ where $C = \operatorname{diag}[1, 1, -E_{n-1}, 1, 1]$. If n is odd, then φ_C is an inner automorphism of $A\tilde{P}(1,n)$. If n is even, then $\varphi_{C_2}\varphi_C$ is an inner automorphism of the algebra $A\tilde{P}(1,n)$. In the first case, $\psi_3(L_1) = L_2$ where $\psi_3 = \psi_2^{-1}\varphi_C^{-1}\psi_1$ is an inner automorphism of the algebra $A\tilde{P}(1,n)$. In the second case, $\psi(L_1) = \varphi_{C_2}(L_2)$ for some $\psi \in \operatorname{Ad} A\tilde{P}(1,n)$. The theorem is proved.

Theorem 2. *Let L_1 and L_2 be subalgebras of $A\tilde{O}(1,n)$ having no invariant isotropic subspaces in $\mathbb{R}_{1,n}$. The subalgebras L_1, L_2 are conjugate under $\operatorname{Ad} AC(1,n)$ if and only if they are conjugate under $\operatorname{Ad} A\tilde{O}(1,n)$ or when there exists an automorphism $\psi \in \operatorname{Ad} A\tilde{O}(1,n)$ such that $\psi(L_1) = CL_2C^{-1}$ where C is one of the $(n+3)\times(n+3)$ matrices*

$$\operatorname{diag}[1, 1, -1, 1, \ldots, 1], \quad \operatorname{diag}[1, \ldots, 1, -1], \quad \operatorname{diag}[1, \ldots, 1, -1, -1].$$

We note that $A\tilde{O}(1,n) \subset AO(2, n+1)$ and that the matrix C is $(n+3)\times(n+3)$.

5. Subalgebras of the full Galilei algebra

Lemma 11. *Let $C \in O(2, n+1)$ and $W = \langle Q_1 + Q_{n+3}, Q_2 + Q_{n+2}\rangle$. If $CW = W$, then*

$$C = \exp[\theta(S+T)] \operatorname{diag}[1, \varepsilon, K, \varepsilon, 1] \exp(\alpha R + \beta Z) \times \\ \times \exp\left(\sum_{i=1}^{n-1} \gamma_i G_i\right) \exp\left(\delta M + \lambda T + \sum_{i=1}^{n-1} \mu_i P_i\right), \tag{9}$$

where $\varepsilon = \pm 1$, $K \in O(n-1)$.

Proof. We have

$$C(Q_1 + Q_{n+3}) = \alpha_1(Q_1 + Q_{n+3}) + \alpha_2(Q_2 + Q_{n+2})$$

and so

$$F(-\theta)C(Q_1 + Q_{n+3}) = (\alpha_1 \cos\theta - \alpha_2 \sin\theta)(Q_1 + Q_{n+3}) + \\ + (\alpha_2 \cos\theta + \alpha_1 \sin\theta)(Q_2 + Q_{n+2}).$$

If $\alpha_1 = 0$ then we put $\theta = (3\pi/2)$ when $\alpha_2 > 0$ and $\theta = (\pi/2)$ when $\alpha_2 < 0$. If $\alpha_1 \neq 0$ then we put $\alpha_1 \sin\theta + \alpha_2 \cos\theta = 0$ and then $\tan\theta = -\alpha_2/\alpha_1$ and $\alpha_1 \cos\theta - \alpha_2 \sin\theta = \alpha_1 \cos\theta(1 + \tan^2\theta)$. We choose θ so that $\alpha_1 \cos\theta > 0$. For this choice of θ we have $F(-\theta)C(Q_1 + Q_{n+3}) = \xi(Q_1 + Q_{n+3})$, where $\xi > 0$. Using Lemma 7, we obtain

$$F(-\theta)C = A = \operatorname{diag}[1, B, 1] \exp([\ln\xi]D) \exp\left(-\sum_{i=0}^{n} \beta_i P_i\right) \in \tilde{P}(1,n),$$

where $B \in O(1,n)$. Then $C = F(\theta)A$. The matrix A has the form (8). Direct calculation gives

$$A(Q_2 + Q_{n+2}) = \alpha(Q_1 + Q_{n+3}) + \beta Q_2 + \gamma Q_{n+2} + \sum_{i=3}^{n+1} \delta_i Q_i.$$

From this it follows that

$$F(\theta)A(Q_2+Q_{n+2}) = (\alpha\cos\theta+\beta\sin\theta)Q_1 + (-\alpha\sin\theta+\beta\cos\theta)Q_2 + \\ + (\gamma\cos\theta-\alpha\sin\theta)Q_{n+2} + (\gamma\sin\theta+\alpha\cos\theta)Q_{n+3} + \sum_{i=3}^{n+1}\delta_i Q_i.$$

Now we have $F(\theta)A(Q_2+Q_{n+2}) \in W$, from which we have

$$\alpha\cos\theta+\beta\sin\theta = \gamma\sin\theta+\alpha\cos\theta, \quad -\alpha\sin\theta+\beta\cos\theta = \gamma\cos\theta-\alpha\sin\theta$$

and so we conclude that $\beta=\gamma$ and $\delta_j=0$, $j=3,\ldots,n+1$. But in that case we have

$$\mathrm{diag}\,[1,B,1](Q_2+Q_{n+2}) = \beta(Q_2+Q_{n+2}).$$

By Lemma 2, we have

$$\pm B = \mathrm{diag}\,[1,K,1]\exp[(-\ln|\beta|)J_{0n}]\exp\left(\sum_{i=1}^{n-1}\gamma_i G_i\right),$$

where $K\in O(n-1)$. We note that

$$K_0+P_0-K_n-P_n = 2(S+T), \quad J_{0n} = \frac{1}{2}(Z-R), \quad D = -\frac{1}{2}(Z+R),$$
$$P_0 = \frac{1}{2}(M+2T), \quad P_n = \frac{1}{2}(M-2T), \quad [D,G_a]=0, \quad [D,J_{0n}]=0.$$

The lemma is proved.

Lemma 12. *Let $C\in O_1(2,n+1)$ and $W=\langle Q_1+Q_{n+3}, Q_2+Q_{n+2}\rangle$. If $CW=W$ then the matrix C has the form* (9) *with $\varepsilon=1$ and $K\in SO(n-1)$.*

Proof. From the conditions of Lemma 1 1 and the fact that we ask for $C\in O_1(2,n+1)$, it follows that $\mathrm{diag}\,[1,\varepsilon,K,\varepsilon,1]\in O_1(2,n+1)$. It follows now that $\varepsilon>0$ and that

$$\begin{vmatrix} K & 0 \\ 0 & \varepsilon \end{vmatrix} > 0$$

and thus we have $\varepsilon=1$ and $|K|>0$, whence $K\in SO(n-1)$. This proves the lemma.

The matrices of the form (9) with $\varepsilon=1$ and $K\in SO(n-1)$ form a group under multiplication, which we denote by $G_4(n-1)$ since its Lie algebra is the full Galilei algebra $AG_4(n-1)$. It is easy to see that $G_4(n-1)\subset O_1(2,n+1)$.

Lemma 13. *If $C\in O_1(2,n+1)$ but $C\notin G_4(n-1)$, then $C=A_1\Gamma A_2$, where $A_1,A_2\in G_4(n-1)$ and Γ is one of the matrices*

$$\Gamma_1 = \mathrm{diag}\,[1,\ldots,1,-1], \quad \Gamma_2 = \mathrm{diag}\,[1,1,-1,1,\ldots,1,-1,1]. \tag{10}$$

Proof. Let

$$C(Q_1+Q_{n+3}) = \sum_{i=1}^{n+3}\alpha_i Q_i, \quad \alpha_1^2+\alpha_2^2-\alpha_3^3-\cdots-\alpha_{n+3}^2=0.$$

There exists a matrix $\Lambda=\mathrm{diag}\,[1,1,\Delta,1,1]$ with $\Delta\in SO(n-1)$ such that $\Lambda C(Q_1+Q_{n+3})$ does not contain $Q_4,\ldots,Q_{n+1}$. Hence we may assume $\alpha_1^2+\alpha_2^2-\alpha_3^2-\alpha_{n+2}^2-\alpha_{n+3}^2=0$.

Since

$$S + T = \frac{1}{2}(K_0 + P_0 + K_n - P_n) = \Omega_{12} + \Omega_{n+2,n+3},$$

then, up to a factor $\exp[\theta(S+T)]$, we may suppose that $\alpha_1 \neq 0$, $\alpha_2 = 0$. If $\alpha_1^2 = \alpha_{n+3}^2$ then $\alpha_3 = 0$, $\alpha_{n+2} = 0$. Assume $\alpha_1 \neq \alpha_{n+3}$. As in the proof of Lemma 4, we find that

$$\begin{aligned}&\exp(\beta_1 P_1 + \beta_2 P_2)(\alpha_1 Q_1 + \alpha_3 Q_3 + \alpha_{n+2} Q_{n+2} + \alpha_{n+3} Q_{n+3}) = \\ &\quad = \alpha_1' Q_1 + \alpha_{n+3}' Q_{n+3},\end{aligned}$$

where $\alpha_1'^2 - \alpha_{n+3}'^2 = 0$. Thus there exists a matrix $A_1 \in G_4(n-1)$ such that

$$\begin{aligned}&A_1^{-1} C(Q_1 + Q_{n+3}) = \gamma(Q_1 \pm Q_{n+3}),\\ &A_1^{-1} C(Q_2 + Q_{n+2}) = \delta_1 Q_1 + \delta_2 Q_2 + \delta_3 Q_3 + \delta_4 Q_{n+2} + \delta_5 Q_{n+3}.\end{aligned} \tag{11}$$

Since the pseudo-orthogonal transformations preserve the scalar product, it follows that the right-hand sides in (11) are also orthogonal, which implies that $\gamma(\delta_1 \mp \delta_5) = 0$ so that $\delta_5 = \pm\delta_1$. If $\delta_2 \neq \delta_4$ then multiplying the left- and right-hand sides in (11) by $\exp(\theta G_1)$ does not change the right-hand side of the first equality, and allows us to eliminate δ_3 by transforming it into 0. If $\delta_2 = \delta_4$, then one easily deduces that $\delta_3 = 0$. Thus we may assume that $\delta_3 = 0$. But then we have $\delta_4 = \pm\delta_2$ because $\delta_5 = \pm\delta_1$ and $\delta_1^2 + \delta_2^2 - \delta_4^2 - \delta_5^2 = 0$.

Let $W = \langle Q_1 + Q_{n+3}, Q_2 + Q_{n+2}\rangle$. The above reasoning implies that for some matrix $A_1 \in G_4(n-1)$ we have $\Gamma A_1^{-1} C W = W$ where Γ is one of the matrices (10). The fact that $\Gamma A_1^{-1} C \in O_1(2, n+1)$ implies, using Lemma 12, $\Gamma A_1^{-1} C = A_2 \in G_4(n-1)$. Thus $C = A_1 \Gamma A_2$ and the lemma is proved.

Lemma 14. *The subalgebras L_1 and L_2 of $AG_4(n-1)$ are conjugate under* $\mathrm{Ad}AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,AG(n-1)$ *or if there exist automorphisms ψ_1, ψ_2 in* $\mathrm{Ad}\,AG_4(n-1)$ *with $\psi_1(L_1) = \Gamma[\psi_2(L_2)]\Gamma^{-1}$, where Γ is one of the matrices* (10).

Proof. The result follows immediately from Lemma 13.

In the following table we give the action on the full Galilei algebra $AG_4(n-1)$ of the automorphisms φ_{Γ_1}, φ_{Γ_2}, φ_{C_1}, φ_{C_4}, φ_{C_5}, where

$$C_4 = \exp\left(\frac{\pi}{2}(S+T)\right), \quad C_5 = \exp(\pi(S+T))$$

(see (3) and (10) for the notation).

Theorem 3. *Let L_1 and L_2 be subalgebras of $AG_4(n-1)$ which are not conjugate under* $\mathrm{Ad}\,AG_4(n-1)$ *with subalgebras of*

$$\langle M, T, P_1, \ldots, P_{n-1}\rangle \uplus (AO(n-1) \oplus \langle D, J_{0n}\rangle$$

and

$$AO(n-1) \oplus \langle S+T, Z\rangle.$$

Then the subalgebras L_1 and L_2 are conjugate under $\mathrm{Ad}\,AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\,AG_4(n-1)$.

Table 1. Action of automorphisms on elements of $AG_4(n-1)$ for $n \geq 2$.

Element of $AG_4(n-1)$	φ_{Γ_1}	φ_{Γ_2}	φ_{C_1}	φ_{C_4}	φ_{C_5}	Restrictions
P_1	K_1	$-P_1$	$-P_1$	$-G_1$	$-P_1$	
P_a	K_a	P_a	$-P_a$	$-G_a$	$-P_a$	$a = 2, \ldots, n-1$
M	$K_0 - K_n$	$2T$	$-M$	M	M	
G_1	$J_{01} + J_{1n}$	$-(J_{01} + J_{1n})$	G_1	P_1	$-G_1$	
G_a	$J_{0a} + J_{an}$	$J_{0a} + J_{an}$	G_a	P_a	$-G_a$	$a = 2, \ldots, n-1$
J_{1a}	J_{1a}	$-J_{1a}$	J_{1a}	J_{1a}	J_{1a}	$a = 2, \ldots, n-1$
J_{ab}	J_{ab}	J_{ab}	J_{ab}	J_{ab}	J_{ab}	$a, b = 2, \ldots, n-1$
R	$-R$	Z	R	$-R$	R	
S	T	$\frac{1}{2}(K_0 - K_n)$	$-S$	T	S	
T	S	$\frac{1}{2}M$	$-T$	S	T	
Z	$-Z$	R	Z	Z	Z	

Proof. If the subalgebras L_1 and L_2 are conjugate under $\mathrm{Ad}\, AG_4(n-1)$ then they are conjugate under $\mathrm{Ad}AC(1,n)$. Now suppose that they are conjugate under $\mathrm{Ad}AC(1,n)$. In order to prove their conjugacy under $\mathrm{Ad}\, AG_4(n-1)$ it is sufficient (by Lemma 14) to show that for an arbitrary $\psi \in \mathrm{Ad}\, AG_4(n-1)$ and for each matrix Γ of the form (10), the subalgebra $\Gamma\psi(L_1)\Gamma^{-1}$ either equals $\psi(L_1)$ or is not contained in $AG_4(n-1)$, for then the only possibility is that they are conjugate under $\mathrm{Ad}\, AG_4(n-1)$.

If the projection of $\psi(L_1)$ onto $\langle G_1, \ldots, G_{n-1}\rangle$ is nonzero, then, using Table 1, the subalgebra $\Gamma\psi(L_1)\Gamma^{-1}$ contains an element Y whose projection for some a, $1 \leq a \leq n-1$ onto $\langle J_{0a}, J_{an}\rangle$ is of the form $\lambda(J_{0a}+J_{an})$ with $\lambda \neq 0$. If $\Gamma\psi(L_1)\Gamma^{-1} \subset AG_4(n-1)$, then the projection of Y onto $\langle J_{0a}, J_{an}\rangle$ would have the form $\mu(J_{0a}-J_{an})$ which would imply $\lambda = \mu = -\mu = 0$, an obvious contradiction.

Now let the projection of $\psi(L_1)$ onto $\langle G_1, \ldots, G_{n-1}\rangle$ be zero. Denote by $\tau\psi(L_1)$ the projection of $\psi(L_1)$ onto $\langle R, S, T\rangle$. If $\tau\psi(L_1) = \langle R, S, T\rangle$, then $\langle R, S, T\rangle \subset \psi(L_1)$. From this it follows that $\Gamma_2\psi(L_1)\Gamma_2^{-1}$ is not a subset of $AG_4(n-1)$. If we assume that $\Gamma_1\psi(L_1)\Gamma_1^{-1} \subset AG_4(n-1)$, we obtain, from Table 1, that the projection of $\psi(L_1)$ onto $\langle P_1, \ldots, P_n, M\rangle$ is zero, and consequently we have either $\psi(L_1) = \langle R, S, T\rangle$ or $\psi(L_1) = \langle R, S, T\rangle \oplus \langle Z\rangle$. In this case, $\Gamma_1\psi(L_1)\Gamma_1^{-1} = \psi(L_1)$. If $\tau\psi(L_1) = \langle R + \alpha S, T + \beta S\rangle$, with $\alpha \neq 0$, then $\Gamma_2\psi(L_1)\Gamma_2^{-1}$ is not contained in $AG_4(n-1)$. If we had $\Gamma_1\psi(L_1)\Gamma_1^{-1} \subset AG_4(n-1)$, then the projection of $\psi(L_1)$ onto $\langle P_1, \ldots, P_n, M\rangle$ would be zero. But then $\psi(L_1)$ would be conjugate under $\mathrm{Ad}\, AG_4(n-1)$ with a subalgebra of $AO(n-1) \oplus \langle R, T, Z\rangle$, which contradicts the assumptions of the theorem. The theorem is proved.

Theorem 4. *Let L_1 and L_2 be subalgebras of the algebra*

$$L = \langle M, T, P_1, \ldots, P_{n-1}\rangle \uplus (AO(n-1) \oplus \langle D, J_{0n}\rangle)$$

having nonzero projection on $\langle J_{0n}\rangle$ and $\langle D\rangle$ and are not conjugate under $\mathrm{Ad}\, L$ *with subalgebras of the algebra $\langle M, T\rangle \uplus (AO(n-1) \oplus \langle D, J_{0n}\rangle)$. Then L_1 and L_2 are conjugate under* $\mathrm{Ad}\, AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\, L$ *or if there exists an automorphism $\psi \in \mathrm{Ad}\, L$ such that $\psi(L_1) = \Lambda L_2 \Lambda^{-1}$ where Λ is one of the matrices Γ_2, C_5, $\Gamma_2 C_5$ (see Table 1).*

Proof. If $\psi \in \mathrm{Ad}\, AG_4(n-1)$, then $\psi = \varphi_C$ where C is a matrix of the form (9). By theorem IV.3.4 of Ref. [9], the subalgebra L_1 is, up to an automorphism of $\mathrm{Ad}\, AG_4(n-$

1), one of the following algebras:

(1) $(U_1 + U_2 + U_3) \uplus F$, where $U_1 \subset \langle M \rangle$, $U_2 \subset \langle T \rangle$, $U_3 \subset \langle P_1, \ldots, P_{n-1} \rangle$ and $F \subset AO(n-1) \oplus \langle D, J_{0n} \rangle$;

(2) $(U_1 + U_2) \uplus F$, where $U_1 \subset \langle T \rangle$, $U_2 \subset \langle P_1, \ldots, P_{n-1} \rangle$ and F is a subalgebra of $AO(n-1) \oplus \langle R, M \rangle$;

(3) $(U_1 + U_2) \uplus F$, where $U_1 \subset \langle M \rangle$, $U_2 \subset \langle P_1, \ldots, P_{n-1} \rangle$ and F is a subalgebra of $AO(n-1) \oplus \langle Z, T \rangle$.

By assumption, the projection of L_1 onto $\langle P_1, \ldots, P_{n-1} \rangle$ is nonzero.

If $\psi(L_1) = L_2$, then in formula (9) $\theta = 0$ or $\theta = \pi$ because for other values of θ the projection of $\psi(L_1)$ onto $\langle G_1, \ldots, G_{n-1} \rangle$ is nonzero. For this reason, $\gamma_1 = \cdots = \gamma_{n-1} = 0$ and so $\psi \in \mathrm{Ad}\, L$ or $\varphi_{C_5}\psi \in \mathrm{Ad}\, L$. Let there be automorphisms $\psi_1, \psi_2 \in \mathrm{Ad}\, AG_4(n-1)$ with $\Gamma\psi_1(L_1)\Gamma = \psi_2(L_2)$ where Γ is one of the matrices (10). If $\mathrm{Ad}\, L$ did not contain ψ_1 and $\varphi_{C_5}\psi_1$, then the projection of $\psi_1(L_1)$ on $\langle G_1, \ldots, G_{n-1} \rangle$ would be nonzero, and so, by Table 1, $\psi_2(L_2)$ would not be in $AG_4(n-1)$. Thus ψ_j or $\varphi_{C_5}\psi_j$ belongs to $\mathrm{Ad}\, L$ for each $j = 1, 2$. For $\Gamma = \Gamma_1$ the projection of $\Gamma\psi_1(L_1)\Gamma$ onto $\langle K_1, \ldots, K_{n-1} \rangle$ is nonzero, so we have $\Gamma = \Gamma_2$. In this case $\Gamma\psi_2(L_2)\Gamma = \psi_2'(\Gamma L_2 \Gamma)$. Using Lemma 14, the theorem is proved.

In a similar way, one proves the following results.

Theorem 5. *Let B be a subalgebra of the algebra*

$$N = \langle M, P_1, \ldots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle D, T \rangle)$$

and let B have nonzero projection onto $\langle D \rangle$. Then B is conjugate under $\mathrm{Ad}\, AC(1,n)$ *to the algebra*

$$F = (W_1 \oplus W_2) \uplus E, \tag{12}$$

where E is a subalgebra of the algebra $AO(n-1) \oplus \langle D \rangle$, $W_1 \subset \langle P_1, \ldots, P_{n-1} \rangle$ and W_2 is one of the algebras 0, $\langle P_0 \rangle$, $\langle P_n \rangle$, $\langle M \rangle$, $\langle P_0, P_n \rangle$. If $W_2 = \langle P_n \rangle$, or $W_2 = \langle P_0, P_n \rangle$ then the subalgebra $W_1 \uplus E$ is not conjugate under $\mathrm{Ad}\, AO(n-1)$ *with any subalgebra of $\langle P_1, \ldots, P_{n-2} \rangle \uplus (AO(n-2) \oplus \langle D \rangle)$. Subalgebras F_1, F_2 of the type* (12) *of the algebra N with nonzero projection onto $\langle D \rangle$, which are not conjugate under* $\mathrm{Ad}\, N$ *to subalgebras of $\langle M, T \rangle \uplus (AO(n-1) \oplus \langle D \rangle)$, will be conjugate under $AC(1,n)$ if and only if they are conjugate under* $\mathrm{Ad}\, L$ *or when there exists an automorphism* $\psi \in \mathrm{Ad}\, L$ *with $\psi(F_1) = \Gamma_2 F_2 \Gamma_2^{-1}$ (see (10)), where $L = AO(n-1)$ (we consider* $\mathrm{Ad}\, AO(n-1)$ *to be a subgroup of* $\mathrm{Ad}\, AC(1,n)$*).*

Theorem 6. *Let B be a subalgebra of the algebra*

$$N = \langle M, P_1, \ldots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle J_{0n}, T \rangle)$$

and let B have nonzero projection onto $\langle J_{0n} \rangle$. Then B is conjugate under $\mathrm{Ad}\, AC(1,n)$ *with the algebra*

$$F = W \uplus E, \tag{13}$$

where E is a subalgebra of the algebra $\langle P_1, \ldots, P_{n-1} \rangle \uplus (AO(n-1) \oplus \langle J_{0n} \rangle)$ and W is one of the algebras 0, $\langle M \rangle$, $\langle P_0, P_n \rangle$. Let $L = N \uplus \langle D \rangle$. Subalgebras F_1, F_2 of the *type* (13) *of the algebra* N *which are not conjugate under* $\mathrm{Ad}\, N$ *with subalgebras of the algebra* $\langle M\rangle \uplus (AO(n-1) \oplus \langle J_{0n}, T\rangle)$, *will be conjugate under* $\mathrm{Ad}\, AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\, L$ *or if there exists an automorphism* $\psi \in \mathrm{Ad}\, L$ *with* $\psi(F_1) = \Lambda F_2 \Lambda^{-1}$ *where* Λ *is one of the matrices* Γ_2, C_5, $\Gamma_2 C_5$ *(see Table 1).*

Theorem 7. *Let* L_1, L_2 *be subalgebras of the algebra* $L = \langle M, S+T, Z\rangle \oplus AO(n-1)$ *which have nonzero projection onto* $\langle S+T\rangle$. *The algebras* L_1 *and* L_2 *are conjugate under* $\mathrm{Ad}\, AC(1,n)$ *if and only if they are conjugate under* $\mathrm{Ad}\, L$ *or if there exists an automorphism* $\psi \in \mathrm{Ad}\, L$ *such that* $\psi(L_1) = \Gamma_1 L_2 \Gamma_1^{-1}$ *(see Table 1).*

6. Subalgebras of $AC(1,3)$

We recall that in this article the conformal algebra $AC(1,3)$ is realized as the pseudo-orthogonal algebra $AO(2,4)$. It turns out that it is convenient to divide the subalgebras of $AO(2,4)$ into seven classes:

(1) subalgebras not having invariant isotropic subspaces in $\mathbb{R}_{2,4}$;

(2) subalgebras conjugate to subalgebras of $AG_1(2)$;

(3) subalgebras conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}\rangle$ and having nonzero projection onto $\langle J_{03}\rangle$;

(4) subalgebras conjugate to subalgebras of $AP(1,3)$ but not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}\rangle$;

(5) subalgebras conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}, D\rangle$ but not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}\rangle$;

(6) subalgebras conjugate to subalgebras of $A\tilde{P}(1,3)$ but not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03}, D\rangle$;

(7) subalgebras conjugate to subalgebras of $AG_4(2)$ but not conjugate to subalgebras of $A\tilde{P}(1,3)$.

Since subalgebras conjugate under $\mathrm{Ad}\, AC(1,3)$ are identified, we omit mentioning conjugacy when referring to classes. So, for instance, we shall consider the second class as consisting of subalgebras of $AG_1(2)$. In order to have a better survey of subalgebras it is convenient to split the classes into subclasses corresponding to certain properties of the projections of the subalgebras of a class onto the homogeneous part of the algebra.

The division of the set of subalgebras of $AC(1,3)$ into the classes (1)–(7) allows us easily to construct the set of subalgebras of each of the algebras $AG_1(2)$, $AP(1,3)$, $A\tilde{P}(1,3)$, $AG_4(2)$. Up to conjugacy under $\mathrm{Ad}\, AC(1,3)$ we have

(a) the set of subalgebras of $AG_1(2)$ coincides with class (2);

(b) the set of subalgebras of $AP(1,3)$ is the union of classes (2), (3) and (4);

(c) the set of subalgebras of $A\tilde{P}(1,3)$ coincides with the union of classes (2)–(6);

(d) the set of subalgebras of $AG_4(2)$ is the union of classes (2), (3), (5), and (7).

We use the notation $F : U_1, \ldots, U_m$ for $U_1 \uplus F, \ldots, U_m \uplus F$.

A. Subalgebras not possessing invariant isotropic subspaces in $\mathbb{R}_{2,4}$

This class is divided into subclasses by the existence for the subalgebras of invariant irreducible subspaces of a particular kind in the space $\mathbb{R}_{2,4}$.

1. Irreducible subalgebras of $AO(2,4)$

$$AC(1,3);$$

$$ASU(1,2) = \langle P_0 + K_0 + 2J_{12}, P_0 + K_0 + K_3 - P_3, P_1 + K_1 + 2J_{02}, P_3 + K_3 + K_0 - P_0, K_2 - P_2 + 2J_{13}, P_2 + K_2 - 2J_{01}, D + J_{03}, K_1 - P_1 - 2J_{23}\rangle;$$

$$ASU'(1,2) = \langle P_0 + K_0 - 2J_{12}, P_0 + K_0 + K_3 - P_3, P_1 + K_1 - 2J_{02}, P_3 + K_3 + K_0 - P_0, K_2 - P_2 - 2J_{13}, P_2 + K_2 + 2J_{01}, D + J_{03}, K_1 - P_1 + 2J_{23}\rangle;$$

$$ASU(1,2) \oplus \langle P_0 + K_0 - 2J_{12} - K_3 + P_3\rangle;$$

$$ASU'(1,2) \oplus \langle P_0 + K_0 + 2J_{12} - K_3 + P_3\rangle;$$

$$\langle P_0 + K_0 - 2J_{12} - K_3 + P_3\rangle \oplus \langle P_1 + K_1 + 2J_{02}, P_3 + K_3 + K_0 - P_0, K_2 - P_2 + 2J_{13}\rangle;$$

$$\langle P_0 + K_0 + 2J_{12} - K_3 + P_3\rangle \oplus \langle P_1 + K_1 - 2J_{02}, P_3 + K_3 + K_0 - P_0, K_2 - P_2 - 2J_{13}\rangle.$$

2. Irreducible subalgebras $AO(1,4)$

$$AC(3).$$

3. Irreducible subalgebras of $AO(2,3)$

$$AC(1,2);$$

$$\langle P_2 + K_2 + \sqrt{3}(P_1 + K_1) + K_0 - P_0, D + J_{02} - \sqrt{3}J_{01}, P_0 + K_0 - 2(K_2 - P_2)\rangle;$$

$$\langle P_2 + K_2 - \sqrt{3}(P_1 + K_1) + K_0 - P_0, D + J_{02} + \sqrt{3}J_{01}, P_0 + K_0 - 2(K_2 - P_2)\rangle.$$

4. Subalgebras of $AO(2,2) \oplus AO(2)$ with irreducible projection onto $AO(2,2)$

$$\langle J_{01} - D, K_0 - P_0 - P_1 - K_1, P_0 + K_0 - K_1 + P_1\rangle \oplus \oplus \langle P_0 + K_0 + K_1 - P_1\rangle \oplus F, \quad \text{where} \quad F = 0 \quad \text{or} \quad F = \langle J_{23}\rangle;$$

$$\langle J_{01} + D, K_0 - P_0 + P_1 + K_1, P_0 + K_0 + K_1 - P_1\rangle \oplus \oplus \langle P_0 + K_0 - K_1 + P_1\rangle \oplus F, \quad \text{where} \quad F = 0 \quad \text{or} \quad F = \langle J_{23}\rangle;$$

$$AC(1,1), \quad AC(1,1) \oplus \langle J_{23}\rangle, \quad \text{where} \quad AC(1,1) = \langle P_0, P_1, K_0, K_1, J_{01}, D\rangle;$$

$$\langle J_{01} - D, K_0 - P_0 - P_1 - K_1, P_0 + K_0 - K_1 + P_1\rangle \oplus \oplus \langle P_0 + K_0 + K_1 - P_1 + \alpha J_{23}\rangle \quad (\alpha \neq 0);$$

$$\langle J_{01} + D, K_0 - P_0 + P_1 + K_1, P_0 + K_0 + K_1 - P_1\rangle \oplus \oplus \langle P_0 + K_0 - K_1 + P_1 + \alpha J_{23}\rangle \quad (\alpha \neq 0).$$

5. Subalgebras of the type $AO(2,1) \oplus F$ with $F \subset AO(3)$

$AC(1) \oplus L$, where $AC(1) = \langle D, P_0, K_0\rangle$, and L is one of the algebras: 0, $\langle J_{12}\rangle$, $\langle J_{12}, J_{13}, J_{23}\rangle$.

6. Subalgebras of $AO(2) \oplus AO(4)$ having an irreducible projection

$\langle P_0 + K_0 \rangle$; $\langle P_0 + K_0 \rangle \oplus \langle 2J_{12} + \alpha(K_3 - P_3) \rangle$ $(|\alpha| \le 1)$;
$\langle P_0 + K_0 \rangle \oplus \langle J_{12}, K_3 - P_3 \rangle$; $\langle P_0 + K_0 \rangle \oplus \langle J_{12}, J_{13}, J_{23} \rangle$;
$\langle P_0 + K_0 \rangle \oplus \langle 2J_{12} + \varepsilon(K_3 - P_3), 2J_{13} - \varepsilon(K_2 - P_2),$
$2J_{23} + \varepsilon(K_1 - P_1) \rangle$ $(\varepsilon = \pm 1)$;
$\langle P_0 + K_0 \rangle \oplus \langle 2J_{12} + \varepsilon(K_3 - P_3), 2J_{13} - \varepsilon(K_2 - P_2), 2J_{23} + \varepsilon(K_1 - P_1) \rangle \oplus$
$\oplus \langle 2J_{12} - \varepsilon(K_3 - P_3) \rangle$ $(\varepsilon = \pm 1)$;
$\langle P_0 + K_0 \rangle \oplus \langle K_1 - P_1, K_2 - P_2, K_3 - P_3, J_{12}, J_{13}, J_{23} \rangle$;
$\langle P_0 + K_0 + 2\alpha J_{12} \rangle$ $(\alpha \ne 0,\ |\alpha| \ne 1)$;
$\langle P_0 + K_0 + 2\alpha J_{12} + \beta(K_3 - P_3) \rangle$ $(\alpha \ne 0,\ |\alpha| \ne 1,\ \beta \ge \alpha,\ \beta \ne 1)$;
$\langle 2J_{12} + \alpha(P_0 + K_0), K_3 - P_3 + \beta(P_0 + K_0) \rangle$
$(\alpha \ne 0,\ \beta \ge 0,$ with $|\alpha| \ne 1$ when $\beta = 0)$;
$\langle \alpha(P_0 + K_0) + 2\varepsilon J_{12} - K_3 + P_3 \rangle \oplus \langle 2\varepsilon J_{12} + K_3 - P_3, 2\varepsilon J_{13} - K_2 + P_2,$
$2\varepsilon J_{23} + K_1 - P_1 \rangle$ $(\alpha \ge 0)$;
$\langle 2\varepsilon J_{12} + K_3 - P_3, 2\varepsilon J_{13} - K_2 + P_2, 2\varepsilon J_{23} + K_1 - P_1 \rangle$ $(\varepsilon = \pm 1)$;
$\langle 2\varepsilon J_{12} + K_3 - P_3, 2\varepsilon J_{13} - K_2 + P_2, 2\varepsilon J_{23} + K_1 - P_1 \rangle \oplus$
$\oplus \langle 2\varepsilon J_{12} - K_3 + P_3 \rangle$ $(\varepsilon = \pm 1)$;
$\langle K_1 - P_1, K_2 - P_2, K_3 - P_3, J_{12}, J_{13}, J_{23} \rangle$.

7. Subalgebras of $AO(1,2) \oplus AO(1,2)$

$\langle P_1 + K_1, P_2 + K_2, J_{12} \rangle \oplus \langle K_0 - P_0, K_3 - P_3, J_{03} \rangle$;
$\langle P_1 + K_1 + 2\varepsilon J_{03}, P_2 + K_2 + K_0 - P_0, 2\varepsilon J_{12} + K_3 - P_3 \rangle$ $(\varepsilon = \pm 1)$;
$\langle P_1 + K_1, P_2 + K_2, J_{12} \rangle \oplus \langle K_3 - P_3 \rangle$.

B. Subalgebras of $AG_1(2)$

The classical Galilei algebra $AG_1(2)$ is the semidirect sum of a solvable ideal, generated by $\langle P_1, P_2, M, T \rangle$, and the Euclidean algebra $AE(2) = \langle G_1, G_2, J_{12} \rangle$. The projection of $AG_1(2)$ onto $AO(1,3)$ coincides with $AE(2)$, which has, up to inner automorphisms, the subalgebras 0, $\langle J_{12} \rangle$, $\langle G_1 \rangle$, $\langle G_1, G_2 \rangle$, $\langle G_1, G_2, J_{12} \rangle$. The first two subalgebras are completely reducible algebras of linear transformations of Minkowski space $\mathbb{R}_{1,3}$, whereas the others are not of this type. Thus we divide this class into two subclasses A and B.

1. Subalgebras with completely reducible projection onto $AO(1,3)$

0, $\langle P_0 \rangle$, $\langle P_1 \rangle$, $\langle M \rangle$, $\langle P_0, P_3 \rangle$, $\langle M, P_1 \rangle$, $\langle P_1, P_2 \rangle$, $\langle M, P_1, P_2 \rangle$, $\langle P_0, P_1, P_2 \rangle$,
$\langle P_1, P_2, P_3 \rangle$, $\langle P_0, P_1, P_2, P_3 \rangle$;
$\langle J_{12} \rangle$: 0, $\langle P_0 \rangle$, $\langle P_3 \rangle$, $\langle M \rangle$, $\langle P_0, P_3 \rangle$, $\langle P_1, P_2 \rangle$, $\langle P_0, P_1, P_2 \rangle$, $\langle M, P_1, P_2 \rangle$,
$\langle P_1, P_2, P_3 \rangle$, $\langle P_0, P_1, P_2, P_3 \rangle$;
$\langle J_{12} + P_0 \rangle$: 0, $\langle P_3 \rangle$, $\langle P_1, P_2 \rangle$, $\langle P_1, P_2, P_3 \rangle$;
$\langle J_{12} \pm P_3 \rangle$: 0, $\langle P_0 \rangle$, $\langle P_1, P_2 \rangle$, $\langle P_0, P_1, P_2 \rangle$;
$\langle J_{12} \pm 2T \rangle$: 0, $\langle M \rangle$, $\langle P_1, P_2 \rangle$, $\langle M, P_1, P_2 \rangle$.

2. Subalgebras whose projection onto $AO(1,3)$ is not completely reducible

$\langle G_1\rangle:\ \langle P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P_2\rangle\ (\alpha\neq 0),\ \langle M,P_1,P_2\rangle,$
$\langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$\langle G_1\pm P_2\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle P_0,P_1,P_3\rangle;$

$\langle G_1+2T\rangle:\ 0,\ \langle P_2\rangle,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P_2\rangle\ (\alpha\neq 0),$
$\langle M,P_1,P_2\rangle;$

$\langle G_1,G_2\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$\langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,M\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1+\alpha P_2,M\rangle\ (\varepsilon=\pm1,\ \alpha\neq 0);$

$\langle G_1+\alpha P_2,G_2+2T,M,P_1\rangle\ (\alpha\in\mathbb{R});$

$\langle G_1\pm P_2,G_2,M,P_1\rangle,\ \langle G_1,G_2+2T,M,P_1,P_2\rangle;$

$\langle G_1,G_2,J_{12}\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$\langle G_1,G_2,J_{12}\pm 2T,M,P_1,P_2\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,J_{12},M\rangle\ (\varepsilon=\pm1).$

C. Subalgebras of $AG_1(2)\uplus\langle J_{03}\rangle$ with nonzero projection onto $\langle J_{03}\rangle$

We divide also this class into two subclasses which are distinguished by whether or not they have a completely reducible projection onto $AO(1,3)$.

1. Subalgebras with completely reducible projection onto $AO(1,3)$

$\langle J_{03}\rangle:\ 0,\ \langle P_1\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle M,P_1\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_1,P_3\rangle,\ \langle M,P_1,P_2\rangle,$
$\langle P_0,P_1,P_2,P_3\rangle;$

$\langle J_{03}+P_1\rangle:\ 0,\ \langle P_2\rangle,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle M,P_2\rangle,\ \langle P_1,P_2,P_3\rangle;$

$\langle J_{12}+\alpha J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,$
$\langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$

$\langle J_{12},J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle.$

2. Subalgebras with projections onto $AO(1,3)$ which are not completely reducible

$\langle G_1,J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P_2\rangle\ (\alpha\neq 0),$
$\langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$\langle G_1,J_{03}+P_2\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_1+\alpha P_2\rangle\ (\alpha\neq 0),\ \langle P_0,P_1,P_3\rangle;$

$\langle G_1,J_{03}+P_1\rangle:\ \langle M\rangle,\ \langle M,P_2\rangle;$

$\langle G_1,J_{03}+P_1+\alpha P_2,M\rangle\ (\alpha\neq 0);$

$\langle G_1,G_2,J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$

$\langle G_1,G_2,J_{03}+P_1,M\rangle,\quad \langle G_1,G_2,J_{03}+P_2,M,P_1\rangle;$

$\langle G_1,G_2,J_{12}+\alpha J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$

$\langle G_1,G_2,J_{12},J_{03}\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle.$

D. Subalgebras of $AP(1,3)$ which are not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03} \rangle$

This class consists of those subalgebras of the Poincaré algebra $AP(1,3)$ whose projection onto $AO(1,3)$ do not possess isotropic invariant subspaces in $\mathbb{R}_{1,3}$. Since the projections are simple algebras, then each subalgebra of the fourth class splits. The full list of such algebras is

$AO(1,2): \ 0, \ \langle P_3 \rangle, \ \langle P_0, P_1, P_2 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$

$AO(3): \ 0, \ \langle P_0 \rangle, \ \langle P_1, P_2, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$

$AO(1,3): \ 0, \ \langle P_0, P_1, P_2, P_3 \rangle.$

E. Subalgebras of $AG_1(2) \uplus \langle J_{03}, D \rangle$ which are not conjugate to subalgebras of $AG_1(2) \uplus \langle J_{03} \rangle$

Let K be a subalgebra of $AG_1(2) \uplus \langle J_{03}, D \rangle$ with nonzero projection onto $\langle D \rangle$, and let $\hat{\theta}$ be the projection of K onto $\langle J_{03}, D \rangle$. By Propositions IV.2.3 and IV.2.5 in Ref. [9], the algebra K, as a subalgebra of $A\tilde{P}(1,3)$, is split whenever $\hat{\theta}(K)$ is one of the subalgebras 1) $\langle D \rangle$; 2) $\langle \gamma D - J_{03} \rangle$ ($\gamma \neq \pm 1, 0, 2$); 3) $\langle D, J_{03} \rangle$. This leads us to dividing this class of subalgebras into two subclasses of nonsplittable subalgebras K of $A\tilde{P}(1,3)$, denoted by D and E, for which the projection onto $\langle G_1, G_2 \rangle$ is non-zero, and for which $\hat{\theta}(K)$ is $\langle J_{03} \mp D \rangle$ and $\langle J_{03} - 2D \rangle$ respectively. It is also useful to distinguish the subclass A of subalgebras having zero projection onto $\langle G_1, G_2 \rangle$. The subalgebras in this subclass differ from the other subalgebras in that their projections onto $AO(1,3)$ are completely reducible algebras of linear transformations of Minkowski space $\mathbb{R}_{1,3}$. All the other subalgebras are split, and we divide them formally into subclasses B and C, depending on the dimension of their projection onto $\langle D, J_{03} \rangle$.

1. Subalgebras with zero projection on $\langle G_1, G_2 \rangle$

$\langle D \rangle: \ \langle P_0 \rangle, \ \langle P_0, P_3 \rangle, \ \langle P_0, P_1, P_2 \rangle, \ \langle P_1, P_2, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$

$\langle J_{12} + \alpha D \rangle: \ \langle P_0 \rangle, \ \langle P_3 \rangle: \ \langle P_0, P_3 \rangle, \ \langle P_0, P_1, P_2 \rangle, \ \langle P_1, P_2, P_3 \rangle,$
$\quad \langle P_0, P_1, P_2, P_3 \rangle \ (\alpha > 0);$

$\langle J_{12}, D \rangle: \ \langle P_0 \rangle, \ \langle P_3 \rangle: \ \langle P_0, P_3 \rangle, \ \langle P_0, P_1, P_2 \rangle, \ \langle P_1, P_2, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$

$\langle J_{03} + \alpha D \rangle \ (0 < \alpha \leq 1);$

$\langle J_{03} + \alpha D, M \rangle \ (0 < |\alpha| \leq 1);$

$\langle J_{03} + \alpha D \rangle: \ \langle P_1 \rangle, \ \langle P_0, P_3 \rangle, \ \langle P_1, P_2 \rangle, \ \langle P_0, P_1, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle \ (\alpha > 0);$

$\langle J_{03} + \alpha D \rangle: \ \langle M, P_1 \rangle, \ \langle M, P_1, P_2 \rangle \ (\alpha \neq 0);$

$\langle J_{03} - D \pm 2T \rangle: \ 0, \ \langle P_1 \rangle, \ \langle M \rangle, \ \langle P_1, P_2 \rangle, \ \langle M, P_1 \rangle, \ \langle M, P_1, P_2 \rangle;$

$\langle J_{03}, D \rangle: \ 0, \ \langle P_1 \rangle, \ \langle M \rangle, \ \langle P_0, P_3 \rangle, \ \langle P_1, P_2 \rangle, \ \langle M, P_1 \rangle, \ \langle M, P_1, P_2 \rangle,$
$\quad \langle P_0, P_1, P_3 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle;$

$\langle \varepsilon J_{12} + \alpha J_{03} + \beta D \rangle \ (0 < \alpha \leq \beta, \ \varepsilon = \pm 1);$

$\langle J_{12} + \alpha J_{03} + \beta D, M \rangle \ (0 < |\alpha| \leq |\beta|);$

$\langle \varepsilon J_{12} + \alpha J_{03} + \beta D \rangle: \ \langle P_0, P_3 \rangle, \ \langle P_1, P_2 \rangle, \ \langle P_0, P_1, P_2, P_3 \rangle \ (\varepsilon = \pm 1, \ \alpha, \beta > 0);$

$\langle J_{12} + \alpha J_{03} + \beta D, M, P_1, P_2 \rangle \ (\alpha \neq 0, \ \beta \neq 0);$

$\langle J_{12} + \alpha (J_{03} - D \pm 2T) \rangle: \ 0, \ \langle M \rangle, \langle P_1, P_2 \rangle, \ \langle M, P_1, P_2 \rangle \ (\alpha \neq 0);$

$\langle J_{12}+\alpha J_{03}, D\rangle:\ 0,\ \langle M\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_3\rangle,\ \langle M,P_1,P_2\rangle,$
$\langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle J_{03}+\alpha D, J_{12}+\beta D\rangle:\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha^2+\beta^2\neq 0);$
$\langle J_{03}+\alpha D, J_{12}+\beta D\rangle:\ (|\alpha|\leq 1,\ \beta\geq 0,\ |\alpha|+\beta\neq 0);$
$\langle J_{03}+\alpha D, J_{12}+\beta D, M\rangle:\ (|\alpha|\leq 1,\ \beta\geq 0,\ |\alpha|+\beta\neq 0);$
$\langle J_{03}+\alpha D, J_{12}+\beta D, M, P_1, P_2\rangle:\ (\alpha,\beta\in\mathbb{R},\ \alpha^2+\beta^2\neq 0);$
$\langle J_{03}-D\pm 2T, J_{12}+2\alpha T\rangle:\ 0,\ \langle M\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle;$
$\langle J_{03}-D, J_{12}\pm T\rangle:\ 0,\ \langle M\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle;$
$\langle J_{03}, J_{12}, D\rangle:\ 0,\ \langle M\rangle,\ \langle P_0,P_3\rangle,\ \langle P_1,P_2\rangle,\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle.$

2. Subalgebras with two-dimensional projection onto $\langle J_{03}, D\rangle$ and nonzero projection onto $\langle G_1, G_2\rangle$

$\langle G_1, J_{03}, D\rangle:\ \langle P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\alpha P_2\rangle\ (\alpha\neq 0),\ \langle M,P_1,P_2\rangle,$
$\langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1, G_2, J_{03}, D\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1, G_2, J_{12}+\alpha J_{03}, D\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle G_1, G_2, J_{03}+\alpha D, J_{12}+\beta D, P_1, P_2\rangle\ (|\alpha|\leq 1,\ \beta\geq 0,\ |\alpha|+\beta\neq 0);$
$\langle G_1, G_2, J_{03}+\alpha D, J_{12}+\beta D, P_0, P_1, P_2, P_3\rangle\ (\alpha^2+\beta^2\neq 0);$
$\langle G_1, G_2, J_{03}, J_{12}, D\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_2,P_3\rangle.$

3. Split subalgebras with one-dimensional projection onto $\langle J_{03}, D\rangle$ and nonzero projection onto $\langle G_1, G_2\rangle$

$\langle G_1+D\rangle:\ \langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1, D\rangle:\ \langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle;$
$\langle G_1+D, G_2, P_0, P_1, P_2, P_3\rangle,\ \langle G_1, G_2, D, P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, J_{03}+\alpha D\rangle:\ \langle P_2\rangle,\ \langle M,P_1\rangle,\ \langle M,P_2\rangle,\ \langle M,P_1+\beta P_2\rangle$
$(|\alpha|\leq 1,\ \alpha\neq 0,\ \beta\neq 0);$
$\langle G_1, J_{03}+\alpha D\rangle:\ \langle M,P_1,P_2\rangle,\ \langle P_0,P_1,P_3\rangle,\ \langle P_0,P_1,P_2,P_3\rangle\ (\alpha\neq 0);$
$\langle G_1, G_2, J_{03}+\alpha D, M, P_1, P_2\rangle\ (0<|\alpha|\leq 1);$
$\langle G_1, G_2, J_{03}+\alpha D, P_0, P_1, P_2, P_3\rangle\ (\alpha\neq 0);$
$\langle G_1, G_2, J_{12}+\alpha D, P_0, P_1, P_2, P_3\rangle\ (\alpha\neq 0);$
$\langle G_1, G_2, J_{12}, D, P_0, P_1, P_2, P_3\rangle;$
$\langle G_1, G_2, J_{12}+\alpha J_{03}+\beta D, M, P_1, P_2\rangle\ (0<|\alpha|\leq|\beta|);$
$\langle G_1, G_2, J_{12}+\alpha J_{03}+\beta D, P_0, P_1, P_2, P_3\rangle\ (\beta\neq 0).$

4. Nonsplit subalgebras of $AG_1(2)\uplus\langle J_{03}\mp D\rangle$ with nonzero projection onto $\langle G_1, G_2\rangle$ and $\langle J_{03}\mp D\rangle$

$\langle J_{03}-D, G_1\pm P_2\rangle:\ 0,\ \langle M\rangle,\ \langle M,P_1\rangle,\ \langle P_0,P_1,P_3\rangle;$
$\langle J_{03}-D\pm 2T, G_1+\alpha P_2, M, P_1\rangle;$
$\langle J_{03}-D\pm 2T, G_1, M, P_1, P_2\rangle,\ \langle J_{03}-D+M, G_1, P_2\rangle;$

$\langle J_{03} - D, G_1 + \varepsilon P_2, G_2 - \varepsilon P_1 + \alpha P_2, M\rangle$ ($\varepsilon = \pm 1$, $\alpha \in \mathbb{R}$);
$\langle J_{03} - D, G_1 \pm P_2, G_2, M, P_1\rangle$, $\langle J_{03} - D \pm 2T, G_1, G_2, P_1, P_2, M\rangle$;
$\langle J_{12} + \alpha(J_{03} - D), G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$ ($\varepsilon = \pm 1$, $\alpha \neq 0$);
$\langle J_{12} + \alpha(J_{03} - D \pm 2T), G_1, G_2, M, P_1, P_2\rangle$ ($\alpha \neq 0$);
$\langle J_{12} \pm 2T, J_{03} - D, G_1, G_2, M, P_1, P_2\rangle$;
$\langle J_{12} + 2\alpha T, J_{03} - D \pm 2T, G_1, G_2, M, P_1, P_2\rangle$ ($\alpha \in \mathbb{R}$);
$\langle J_{12}, J_{03} - D, G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$ ($\varepsilon = \pm 1$).

5. Nonsplit subalgebras of $AG_1(2) \uplus \langle J_{03} - 2D\rangle$ with nonzero projection onto $\langle G_1, G_2\rangle$ and $\langle J_{03} - 2D\rangle$

$\langle J_{03} - 2D, G_1 + 2T\rangle$: 0, $\langle M\rangle$, $\langle P_2\rangle$, $\langle M, P_1\rangle$, $\langle M, P_2\rangle$, $\langle M, P_1 + \alpha P_2\rangle$ ($\alpha \neq 0$), $\langle M, P_1, P_2\rangle$;
$\langle J_{03} - 2D, G_1, G_2 + 2T\rangle$: $\langle M, P_1\rangle$, $\langle M, P_1, P_2\rangle$.

F. Subalgebras of $A\tilde{P}(1,3)$ not conjugate to subalgebras of $AP(1,3)$ and of $AG_1(2) \uplus \langle J_{03}, D\rangle$

This class consists of those subalgebras of $AP(1,3)$ whose projection onto $AO(1,3)$ do not have invariant isotropic subspaces in $\mathbb{R}_{1,3}$ and with a nonzero projection onto $\langle D\rangle$. We have

$AO(1,2) \oplus \langle D\rangle$: 0, $\langle P_3\rangle$, $\langle P_0, P_1, P_2\rangle$, $\langle P_0, P_1, P_2, P_3\rangle$;
$AO(3) \oplus \langle D\rangle$: 0, $\langle P_0\rangle$, $\langle P_1, P_2, P_3\rangle$, $\langle P_0, P_1, P_2, P_3\rangle$;
$AO(1,3) \oplus \langle D\rangle$: 0, $\langle P_0, P_1, P_2, P_3\rangle$.

G. Subalgebras of $AG_4(2)$ which are not conjugate to subalgebras of $A\tilde{P}(1,3)$

Let K be a subalgebra of $AG_4(2)$ and $\tau(K)$ its projection onto $AGL(2,\mathbb{R})$. By Propositions V.2.1 and V.2.2 of Ref. [9], the algebra K belongs to this class if and only if $\tau(K)$ is conjugate to one of the following algebras: $\langle S + T\rangle$, $\langle S + T\rangle \dot{+} \langle Z\rangle$ (subdirect sum), $ASL(2,\mathbb{R}) = \langle R, S, T\rangle$, $AGL(2,\mathbb{R}) = \langle R, S, T, Z\rangle$. Because of this, we divide this seventh class into three subclasses, each of which consists of subalgebras having a corresponding projection onto $AGL(2,\mathbb{R})$; those subalgebras whose projections are either $ASL(2,\mathbb{R})$ or $AGL(2,\mathbb{R})$ are put into the same subclass.

1. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ is $\langle S + T\rangle$

$\langle S + T\rangle$: 0, $\langle M\rangle$, $\langle G_1, P_1, M\rangle$, $\langle G_1 - \alpha^{-1}P_2, G_2 + \alpha P_1, M\rangle$ ($0 < |\alpha| \leq 1$), $\langle G_1, G_2, P_1, P_2, M\rangle$;
$\langle S + T \pm M\rangle$, $\langle S + T + \alpha J_{12} \pm M\rangle$ ($\alpha \neq 0$);
$\langle S + T + \alpha J_{12}\rangle$: 0, $\langle M\rangle$, $\langle G_1 + \varepsilon P_2, G_2 - \varepsilon P_1, M\rangle$, $\langle G_1, G_2, P_1, P_2, M\rangle$ ($\varepsilon = \pm 1, \alpha \neq 0$);
$\langle S + T + \varepsilon J_{12}\rangle$: $\langle G_1 + \varepsilon P_2\rangle$, $\langle G_1 + \varepsilon P_2, M\rangle$, $\langle G_1 + \varepsilon P_2, G_1 - \varepsilon P_2, G_2 + \varepsilon P_1, M\rangle$ ($\varepsilon = \pm 1$);
$\langle S + T + \varepsilon J_{12} \pm M, G_1 + \varepsilon P_2\rangle$ ($\varepsilon = \pm 1$);

$\langle S+T+\varepsilon J_{12}+\varepsilon G_1+P_2\rangle:\ 0,\ \langle M\rangle,\ \langle G_2-\varepsilon P_1,M\rangle,$
$\langle G_1-\varepsilon P_2,G_2+\varepsilon P_1,M\rangle,\ \langle G_2-\varepsilon P_1,G_1-\varepsilon P_2,G_2+\varepsilon P_1,M\rangle\ (\varepsilon=\pm1);$
$\langle J_{12},S+T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,M\rangle,$
$\langle G_1,G_2,P_1,P_2,M\rangle\ (\varepsilon=\pm1);$
$\langle J_{12}\pm M,S+T+\alpha M\rangle\ (\alpha\in\mathbb{R});$
$\langle J_{12},S+T\pm M\rangle.$

2. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ is the subdirect sum $\langle S+T\rangle+\langle Z\rangle$

$\langle S+T+\alpha Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1,P_1,M\rangle,\ \langle G_1-\beta^{-1}P_2,G_2+\beta P_1,M\rangle,$
$\langle G_1,G_2,P_1,P_2,M\rangle\ (0<|\beta|\leq1,\ \alpha\neq0);$
$\langle S+T,Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1,P_1,M\rangle,\ \langle G_1-\alpha^{-1}P_2,G_2+\alpha P_1,M\rangle,$
$\langle G_1,G_2,P_1,P_2,M\rangle\ (0<|\alpha|\leq1);$
$\langle S+T+\alpha J_{12}+\beta Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,M\rangle,$
$\langle G_1,G_2,P_1,P_2,M\rangle\ (\varepsilon=\pm1,\alpha\neq0,\ \beta>0);$
$\langle S+T+\alpha J_{12},Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,M\rangle,$
$\langle G_1,G_2,P_1,P_2,M\rangle\ (\varepsilon=\pm1,\ \alpha\neq0);$
$\langle S+T+\varepsilon J_{12}+\alpha Z\rangle:\ \langle G_1+\varepsilon P_2\rangle,\ \langle G_1+\varepsilon P_2,M\rangle,$
$\langle G_1+\varepsilon P_2,G_1-\varepsilon P_2,G_2+\varepsilon P_1,M\rangle\ (\varepsilon=\pm1,\ \alpha\neq0);$
$\langle S+T+\varepsilon J_{12},Z\rangle:\ \langle G_1+\varepsilon P_2\rangle,\ \langle G_1+\varepsilon P_2,M\rangle,$
$\langle G_1+\varepsilon P_2,G_1-\varepsilon P_2,G_2+\varepsilon P_1,M\rangle\ (\varepsilon=\pm1);$
$\langle J_{12}+\alpha Z,S+T+\beta Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,M\rangle,$
$\langle G_1,G_2,P_1,P_2,M\rangle\ (\varepsilon=\pm1,\ |\alpha|+|\beta|\neq0);$
$\langle J_{12},S+T,Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1+\varepsilon P_2,G_2-\varepsilon P_1,M\rangle,$
$\langle G_1,G_2,P_1,P_2,M\rangle\ (\varepsilon=\pm1).$

3. Subalgebras whose projection onto $AGL(2,\mathbb{R})$ contains $ASL(2,\mathbb{R})$

$\langle R,S,T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1,P_1,M\rangle,\ \langle G_1,G_2,P_1,P_2,M\rangle;$
$\langle J_{12}\rangle\oplus\langle R,S,T\rangle:\ 0,\ \langle M\rangle,\ \langle G_1,G_2,P_1,P_2,M\rangle;$
$\langle J_{12}\pm M\rangle\oplus\langle R,S,T\rangle;$
$\langle R,S,T,Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1,P_1,M\rangle,\ \langle G_1,G_2,P_1,P_2,M\rangle;$
$\langle R,S,T\rangle\oplus\langle J_{12}+\alpha Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1,G_2,P_1,P_2,M\rangle\ (\alpha\neq0);$
$\langle R,S,T\rangle\oplus\langle J_{12},Z\rangle:\ 0,\ \langle M\rangle,\ \langle G_1,G_2,P_1,P_2,M\rangle.$

Acknowledgments

This work was supported in part by INTAS and DKNT of Ukraine. P. Basarab-Horwath thanks the Wallenberg fund of Linköping University and the Magnusson Fund of the Swedish Academy of Sciences for travel grants, and the Mathematics Institute of the Ukrainian Academy of Sciences in Kyiv for its hospitality.

Professor Wilhelm Fuschych died on April 7, 1997, after a short illness. This is a great loss for his family, his many students, and for the scientific community. His

many and deep contributions to the field of symmetry analysis of differential equations have made the Kyiv school of symmetries known throughout the world. We take this opportunity to express our deep sense of loss as well as our gratitude for all the encouragement in research that Wilhelm Fushchych gave during the years we knew him.

1. Patera J., Winternitz P., Zassenhaus H., Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group, *J. Math. Phys.*, 1975, **16**, 1597–1614.

2. Burdet G., Patera J., Perrin M., Winternitz P., The optical group and its subgroups, *J. Math. Phys.*, 1978, **19**, 1758–1780.

3. Patera J., Winternitz P., Zassenhaus H., Continuous subgroups of the fundamental groups of physics. II. The similitude group, *J. Math. Phys.*, 1975, **16**, 1615–1624.

4. Patera J., Winternitz P., Sharp R.T., Zassenhaus H., Subgroups of the similitude group of three-dimensional Minkowski space, *Can. J. Phys.*, 1976, **54**, 950–961.

5. Burdet G., Patera J., Perrin M., Winternitz P., Sous-algèbres de Lie de l'algèbre de Schrödinger, *Ann. Sc. Math Quebec*, 1978, **2**, 81–108.

6. Fushchych W.I., Barannik A.F., Barannik L.F., Fedorchuk V.M., Continuous subgroups of the Poincaré group $P(1,4)$, *J. Phys. A*, 1985, **18**, 2893–2899.

7. Barannik L.F., Fushchych W.I., On subalgebras of the Lie algebra of the extended Poincaré group $P(1,n)$, *J. Math. Phys.*, 1987, **28**, 1445–1458.

8. Barannyk L.F., Fushchych W.I., On continuous subgroups of the generalized Schrödinger groups, *J. Math. Phys.*, 1989, **30**, 280–290.

9. Fushchych W.I., Barannyk L.F., Barannyk A.F., Subgroup analysis of the Galilei and Poincaré groups and reduction of nonlinear equations, Kyiv, Naukova Dumka, 1991 (in Russian).

10. Barannik L.F., Fushchych W.I., On continuous subgroups of the conformal group of Minkowski space $\mathbb{R}_{1,n}$, Preprint 88.34, Kyiv, Inst. of Math., 1988 (in Russian).

11. Fushchych W.I., Barannyk L.F., Barannyk A.F., Continuous subgroups of the generalized Euclidean group, *Ukr. Math. J.*, 1986, **38**, 67–72 (in Russian).

12. Barannyk A.F., Barannyk L.F., Fushchych W.I., Reduction and exact solutions of the eikonal equation, *Ukr. Math. J.*, 1991, **43**, 461–474 (in Russian).

13. Barannyk A.F., Barannyk L.F., Fushchych W.I., Connected subgroups of the conformal group $C(1,4)$, *Ukr. Math. J.*, 1991, **43**, 870–884 (in Russian).

14. Lang S., Algebra, Addison-Wesley, Reading, 1965.

15. Chen Su-Shing, On subgroups of the noncompact real exceptional Lie group F_4^*, *Math. Ann.*, 1973, **204**, 271–284.

16. Taufik M.S., On semisimple subalgebras of some simple real Lie algebras, *Usp. Mat. Nauk*, 1975, **30**, 261–262 (in Russian).

17. Taufik M.S., On maximal subalgebras in classical real Lie algebras, in Voprosy Teorii Grupp i Gomologicheskoj Algebry, Jaroslavl Univ., 1979, 148–168 (in Russian).

Зміст

Наукове видання

НАЦІОНАЛЬНА АКАДЕМІЯ НАУК УКРАЇНИ
ІНСТИТУТ МАТЕМАТИКИ

ФУЩИЧ
ВІЛЬГЕЛЬМ ІЛЛІЧ
ВИБРАНІ ПРАЦІ

Київ, видавництво “Наукова думка”

Художник *М.О. Коваленко*
Художній редактор *Є.І. Муштенко*
Комп'ютерна верстка *В.М. Бойко*

Підп. до друку 26.04.2005. Формат 70 × 100/16. Папір офс. № 1. Гарн. Таймс.
Друк. офс. Фіз. друк. арк. 28,0 + 1 вкл. на крейд. пап. Ум. друк. арк. 36,56.
Ум. фарбо-відб. 36,56. Обл.-вид. арк. 32,45. Тираж 300 прим. Зам. № 5-166

Видавництво “Наукова думка”
Р.с. № 05417561 від 16.03.95.
01601 Київ 4, вул. Терещенківська, 3

ВАТ “Книжкова друкарня наукової книги”
04107, Київ 107
вул. Багговутівська, 17-21