Moving Frames and Moving Coframes

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Dedicated to Jiri Patera and Pavel Winternitz on the occasion of their sixtieth birthdays.

First introduced by Gaston Darboux, and then brought to maturity by Élie Cartan, [**4**], [**5**], the theory of moving frames ("repères mobiles") is widely acknowledged to be a powerful tool for studying the geometric properties of submanifolds under the action of a transformation group. While the basic ideas of moving frames for classical group actions are now ubiquitous in differential geometry, the theory and practice of the moving frame method for more general transformation group actions has remained relatively undeveloped. The famous critical assessment by Weyl in his review, [**27**], of Cartan's seminal book, [**5**], retains its perspicuity to this day:

"I did not quite understand how he [Cartan] does this in general, though in the examples he gives the procedure is clear. ... Nevertheless, I must admit I found the book, like most of Cartan's papers, hard reading."

Implementations of the method of moving frames for certain groups having direct geometrical significance — including the Euclidean, affine, and projective groups — can be found in both Cartan's original treatise, [**5**], as well as many standard texts in differential geometry, e.g., [**13**], [**24**], [**28**]. The method continues to attract the attention of modern day researchers and has been successfully extended to a few additional examples, including, for instance, holomorphic curves in projective spaces and Grassmannians. The papers of Griffiths, [**12**], Green, [**11**], Chern, [**7**], and the lecture notes of Jensen, [**14**], are particularly noteworthy attempts to place Cartan's intuitive constructions on a firm theoretical and differential geometric foundation.

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Recently, [**9**], [**10**], the authors introduced two new methods that enable one to algorithmically implement both the practical and theoretical construction of moving frames for general transformation groups. The first algorithm, which we call the method of "moving coframes", not only reproduces all of the classical moving frame constructions, often in a simpler and more direct fashion, but can be readily applied to a wide variety of new situations, including infinite-dimensional pseudo-groups, intransitive group actions, restricted reparametrization problems, joint group actions, to name a few. The second "regularized" method is applicable to general finite-dimensional transformation group actions, and provides a completely rigorous justification of the general theory. The regularized method bypasses many of the complications inherent in traditional approaches by completely avoiding the usual process of normalization during the general computation. Once a moving frame and coframe, along with the complete system of invariants, are constructed in the regularized framework, one can easily restrict these invariants to particular classes of submanifolds, producing (in nonsingular cases) the standard moving frame. Perhaps Griffiths is the closest in spirit to our guiding philosophy; we fully agree with his statement, [**12**; p. 777], that "The effective use of frames ... goes far beyond the notion that 'frames are essentially the same as studying connections in the principal bundle of the tangent bundle'." Indeed, by de-emphasizing the group theoretical basis for normalization, which, in the past, has hindered the theoretical foundations from covering all the situations to which the practical algorithm could be applied, our formulation of the framework goes beyond what Griffiths envisioned, and successfully realizes Cartan's original vision, [**4**], [**5**]. Significant applications include a new and more general proof of the fundamental theorem on classification of differential invariants, a general classification theorem for syzygies of differential invariants, as well as new explicit commutation formulae for the associated invariant differential operators. We demonstrate a simple but striking generalization of a "replacement theorem" due to T.Y. Thomas, [**25**]. Refined versions of known general theorems on the equivalence, symmetry and rigidity of submanifolds are further direct consequences of our approach.

In this paper, we shall review the results of our investigations, referring the reader to [**9**], [**10**] for proofs, further details, as well as numerous examples and applications. We begin by presenting the basics of the regularized theory and its applications to differential invariants, which is then illustrated by an example arising in classical invariant theory. The moving coframe method is then briefly discussed, and applied to two examples — first, the classical case of equi-affine geometry of curves in the plane, and second, an infinite-dimensional pseudo-group originally studied by Lie.

Throughout this paper, G will denote an r-dimensional Lie group acting smoothly on an m-dimensional manifold M. Let $G_S = \{g \in G \mid g \cdot S = S\}$ denote the *isotropy subgroup* of a subset $S \subset M$, and $G_S^* = \bigcap_{x \in S} G_x$ its *global isotropy subgroup*, which consists of those group elements which fix all points in S. The group G acts *freely* if $G_z = \{e\}$ for all $z \in M$, *effectively* if $G_M^* = \{e\}$, and *effectively on subsets* if $G_U^* = \{e\}$ for every open $U \subset M$. We further incorporate the adjective "locally" in these concepts by replacing $\{e\}$ by a general discrete subgroup of G. If G does not act effectively, one can, without any loss of generality, replace G by the effectively acting quotient group G/G_M^*, which acts in essentially the same manner as G does, cf. [**21**]. Clearly, if G acts effectively on subsets, then G acts effectively. Analytic continuation demonstrates that the converse is true in the analytic category, although not for general smooth actions. To avoid pathology, we shall always assume that G acts locally effectively on subsets. A group acts *semi-regularly* if all its orbits have the same dimension. The action is *regular* if, in addition, each point $x \in M$ has arbitrarily small neighborhoods whose intersection with each orbit is connected.

Let $\mathrm{J}^n = \mathrm{J}^n(M,p)$ denote the n^{th} order (extended) jet bundle consisting of equivalence classes of p-dimensional submanifolds $S \subset M$ under the equivalence relation of n^{th} order contact, cf. [**20**; Chapter 3]. We let $\mathrm{j}_n S \subset \mathrm{J}^n$ denote the n-jet of the submanifold S. We introduce local coordinates $z = (x,u)$ on M, considering the first p components $x = (x^1, \ldots, x^p)$ as independent variables, and the latter $q = m - p$ components $u = (u^1, \ldots, u^q)$ as dependent variables. The induced local coordinates on J^n are denoted by $z^{(n)} = (x, u^{(n)})$, with components u^α_J, where $J = (j_1, \ldots, j_k)$, $1 \le j_\nu \le p$, representing the partial derivatives of the dependent variables with respect to the independent variables.

Since G preserves the order of contact between submanifolds, there is an induced action of G on the jet bundle J^n known as its n^{th} *prolongation*, and denoted by $G^{(n)}$. We choose a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ for the Lie algebra $\mathfrak{g}$ of infinitesimal generators on M, and let $\{\mathrm{pr}^{(n)}\mathbf{v}_1, \ldots, \mathrm{pr}^{(n)}\mathbf{v}_r\}$ denote the corresponding basis for the Lie algebra $\mathfrak{g}^{(n)}$ of infinitesimal generators of the prolonged group action $G^{(n)}$. The prolonged generators are obtained by truncating, at order n, the infinitely prolonged vector fields

$$\begin{aligned} \mathrm{pr}\,\mathbf{v}_\kappa &= \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{k=\#J\ge 0} \varphi^\alpha_{J,\kappa}(x,u^{(k)})\,\frac{\partial}{\partial u^\alpha_J}\,, \qquad \text{where} \\ \varphi^\alpha_{J,\kappa} &= D_J Q^\alpha_\kappa + \sum_{i=1}^{p} \xi^i_\kappa\, u^\alpha_{J,i}, \qquad Q^\alpha_\kappa(x,u^{(1)}) = \varphi^\alpha_\kappa(x,u) - \sum_{i=1}^{p} \xi^i_\kappa(x,u)\,u^\alpha_i. \end{aligned} \tag{1}$$

Here $Q_\kappa = (Q^1_\kappa, \ldots, Q^q_\kappa)$ is the usual characteristic of $\mathbf{v}_\kappa$, and $D_J = D_{j_1} \cdots D_{j_k}$ denotes total differentiation of order $k = \#J$.

Let $s_n = \max\{\dim \mathfrak{g}^{(n)}|_{z^{(n)}}\}$ denote the maximal orbit dimension of the prolonged action $G^{(n)}$ on J^n. The *stable orbit dimension* is $s = \max s_n$. The *stabilization order* of G is the minimal n such that $s_n = s$. The *regular subset* $\mathcal{V}^n \subset \mathrm{J}^n$ is the open subset consisting of all prolonged group orbits of dimension equal to the stable orbit dimension, while the *singular subset* is $\mathcal{S}^n = \mathrm{J}^n \setminus \mathcal{V}^n$. Note that, by this definition, $\mathcal{V}^n = \varnothing$ and $\mathcal{S}^n = \mathrm{J}^n$ if n is less than the stabilization order of $G^{(n)}$. Ovsiannikov's stabilization theorem, [**23**], [**21**], completely characterizes the stable orbit dimension. A correct version can be stated as follows; see [**22**] for details.

Theorem 1. *A Lie group G acts locally effectively on subsets of M if and only if its stable orbit dimension equals its dimension, $s = r = \dim G$, which means that G acts locally freely on the regular subset $\mathcal{V}^n \subset \mathrm{J}^n$.*

A submanifold $S \subset M$ is called *totally singular* if all its jets never intersect the regular subset. Such submanifolds can be geometrically characterized as follows.

Theorem 2. *A submanifold $S \subset M$ is totally singular, meaning that $\mathrm{j}_n S \subset \mathcal{S}^n$ for all $n = 0, 1, \ldots$, if and only if its isotropy subgroup G_S does not act locally freely on S.*

Our approach to the theory of moving frames is based on the following simple but remarkably powerful device. In general, any complicated transformation group action can be "regularized" by lifting it to a suitable bundle sitting over the original manifold. Let $\mathcal{B}^n = G \times \mathrm{J}^n$ denote the trivial (left) principal G–bundle over the jet space. The n^{th} order *regularization* of the prolonged action $G^{(n)}$ is the action of G on $\mathcal{B}^n$ given by

$$g \cdot (h, z^{(n)}) = (g \cdot h, g^{(n)} \cdot z^{(n)}). \tag{2}$$

The key, elementary result is that regularizing any group action immediately eliminates *all* singularities and irregularities, e.g., lower dimensional orbits, non-embedded orbits, etc.

Theorem 3. *For any $n \geq 0$, the regularized action* (2) *defines a regular, free action of G on the bundle $\mathcal{B}^n = G \times \mathrm{J}^n$.*

Recall that a *differential invariant* is a function $I\colon \mathrm{J}^n \to \mathbb{R}$ which is invariant under the action of $G^{(n)}$. Similarly, a *lifted differential invariant* is defined as a function $L\colon \mathcal{B}^n \to \mathbb{R}$ which is invariant under the regularized action (2). Remarkably, all the lifted differential invariants are trivial to construct; they are the components of the order n evaluation map $w^{(n)}\colon \mathcal{B}^n \to \mathrm{J}^n$ which is given by $w^{(n)}(g, z^{(n)}) = (g^{(n)})^{-1} \cdot z^{(n)}$.

Proposition 4. *Every lifted differential invariant can be locally written as a function of the fundamental lifted differential invariants $w^{(n)}(g, z^{(n)})$.*

In particular, an ordinary differential invariant $I\colon \mathrm{J}^n \to \mathbb{R}$ also defines a lifted differential invariant $L = I \circ \pi_n\colon \mathcal{B}^n \to \mathbb{R}$. Conversely, any lifted invariant $L(g, x, u^{(n)})$ that does not depend on the g coordinates automatically defines an ordinary differential invariant.

Theorem 5. *Let $I(z^{(n)})$ be an ordinary differential invariant. Then we can write $I(z^{(n)}) = I(w^{(n)})$ as the* same *function of the lifted differential invariants.*

In Riemannian geometry, Theorem 5 reduces to the striking Thomas Replacement Theorem, [**25**; p. 109]. See [**2**] for recent applications of Thomas' result.

The introduction of local coordinates $z = (x, u)$ on M also partitions the fundamental zero$^{\mathrm{th}}$ order lifted invariants $w = (w^1, \ldots, w^m) = g^{-1} \cdot z$ into two components, $w = (y, v)$, where $y = (y^1, \ldots, y^p)$ will be considered as "lifted independent variables", and $v = (v^1, \ldots, v^q)$ as "lifted dependent variables". The lifted differential invariants can be found via a process of invariant differentiation, which we now describe.

The identification of independent variables on M induces a splitting of the differential forms on J^n into horizontal and contact components, cf. [**1**], [**21**]. Given a differential function $F(x, u^{(n)})$, let

$$d_H F = \sum_{i=1}^{p} (D_i F)\, dx^i \tag{3}$$

denote the horizontal component of its exterior derivative, known as the *total differential* of F. Formula (3) extends without change to lifted functions $F(g, x, u^{(n)})$. Let

$$\eta^i = d_H y^i = \sum_{j=1}^{p} (D_j y^i)\, dx^j, \qquad i = 1, \ldots, p,$$

denote the horizontal differentials of the lifted independent variables. We then rewrite (3) in invariant form

$$d_H F = \sum_{j=1}^{p} (\mathcal{E}_j F)\, \eta^j, \qquad \text{where} \qquad \mathcal{E}_j F = \frac{\mathbf{D}(y^1,\ldots,y^{j-1},F,y^{j+1},\ldots y^p)}{\mathbf{D}(y^1,\ldots,y^p)}.$$

We can identify the *lifted invariant differential operator* $\mathcal{E}_j = D_{y^j}$ with total differentiation with respect to the lifted invariant y^j. In column vector notation, $\mathcal{E} = (\mathbf{D}y)^{-T}\cdot\mathbf{D}$, where $\mathbf{D}y$ is the total Jacobian matrix of y and $\mathbf{D} = (D_1,\ldots,D_p)^T$ is the "total gradient operator". A very important point is that, unlike the usual invariant differential operators, the lifted invariant differential operators *always* mutually commute, $[\mathcal{E}_j,\mathcal{E}_k] = 0$.

Proposition 6. *The components $w^{(n)} = (y, v^{(n)})$ of the fundamental lifted invariants are found by successively applying the invariant differential operators $\mathcal{E}_j = D_{y^j}$ associated with the first p lifted invariants $y = (y^1,\ldots,y^p)$ to the remaining zero*$^{\text{th}}$ *order invariants $v = (v^1,\ldots,v^q)$, so that $v^\alpha_J = \mathcal{E}_J v^\alpha$, where $\mathcal{E}_J = \mathcal{E}_{j_1}\cdot\ldots\cdot\mathcal{E}_{j_k}$.*

The primary use of a moving frame is that it enables one to pass from lifted invariant objects, which are trivial, to their ordinary invariant counterparts back on the original manifold and its jet spaces. This allows us to systematically analyze the invariants via the particularities of the moving frame. We first discuss the theory of completely determined moving frames, meaning ones that do not depend on any group parameters.

Definition 7. An n^{th} order (left) *moving frame* is a map $\rho^{(n)}\colon \mathrm{J}^n \to G$ which is (locally) G-equivariant with respect to the prolonged action $G^{(n)}$ on J^n, and the left action $h \mapsto g\cdot h$ of G on itself.

We can identify a moving frame with an equivariant section $\sigma^{(n)}\colon \mathrm{J}^n \to \mathcal{B}^n = G\times \mathrm{J}^n$ given by $\sigma^{(n)}(x,u^{(n)}) = (\rho^{(n)}(x,u^{(n)}), x, u^{(n)})$. Note that any n^{th} order moving frame also defines a moving frame on all higher order jet bundles by composition with the standard projections $\pi^k_n\colon \mathrm{J}^k \to \mathrm{J}^n$, $k > n$.

Theorem 8. *If G acts effectively on subsets, then an n^{th} order moving frame exists in a neighborhood of a point $z^{(n)} \in \mathrm{J}^n$ if and only if $z^{(n)} \in \mathcal{V}^n$ is a regular jet.*

In particular, the minimal order at which any moving frame exists is the stabilization order of the group. In practical implementations, Cartan's normalization procedure for constructing moving frames amounts to choosing a (local) cross-section $\mathcal{K}^n \subset \mathcal{V}^n$ to the regular prolonged group orbits. Let $\mathcal{O}^n$ denote the $G^{(n)}$ orbit passing through the regular jet $z^{(n)} \in \mathcal{O}^n \subset \mathcal{V}^n$, and suppose that $\mathcal{O}^n$ intersects the cross-section at the unique point $k^{(n)} \in \mathcal{O}^n \cap \mathcal{K}^n$; we can view $k^{(n)}$ as the "canonical form" of the jet $z^{(n)}$. Finally, let $g = \rho^{(n)}(z^{(n)})$ denote the group element which maps $k^{(n)}$ to $z^{(n)} = g^{(n)}\cdot k^{(n)}$. The resulting map $\rho^{(n)}\colon \mathrm{J}^n \to G$ from the jet space to the group is the moving frame defined by the chosen cross-section.

Assuming $G^{(n)}$ acts locally freely, the simplest local cross-sections are obtained by setting $r = \dim G$ of the jet coordinates $z^{(n)} = (x, u^{(n)})$ to be constant. Let $z_1,\ldots,z_r$ denote the chosen coordinates, so that each z_ν is either one of the x^i's or one of the u^α_J's.

Let $w_1, \ldots, w_r$ be the corresponding lifted invariants, so that w_ν is the corresponding y^i or v^α_J. The normalization constants $c_1, \ldots, c_r$ are chosen so that the *normalization equations*

$$w_1(g, x, u^{(n)}) = c_1, \qquad \ldots \qquad w_r(g, x, u^{(n)}) = c_r, \tag{4}$$

can be (locally) uniquely solved for $g = \rho^{(n)}(x, u^{(n)})$ in terms of the jet coordinates; the resulting map defines the moving frame associated with the chosen cross-section.

Definition 9. The *fundamental* n^{th} *order normalized differential invariants* associated with a moving frame $\rho^{(n)}$ of order n (or less) are given by

$$I^{(n)}(z^{(n)}) = w^{(n)} \circ \sigma^{(n)}(z^{(n)}) = \rho^{(n)}(z^{(n)})^{-1} \cdot z^{(n)}.$$

Note that $I^{(n)}(z^{(n)}) = k^{(n)} \in \mathcal{K}^n$ can be identified with the canonical form of the jet $z^{(n)}$. In terms of the invariant local coordinates $w^{(n)} = (y, v^{(n)})$ on $\mathcal{B}^n$, the fundamental normalized differential invariants are

$$\begin{aligned} J^i(x, u^{(n)}) &= y^i(\rho^{(n)}(x, u^{(n)}), x, u), && i = 1, \ldots, p, \\ I^\alpha_K(x, u^{(k)}) &= v^\alpha_K(\rho^{(n)}(x, u^{(n)}), x, u^{(k)}), && \alpha = 1, \ldots, q, \quad k = \#K \geq 0. \end{aligned} \tag{5}$$

In particular, the normalized differential invariants corresponding to the components w_i being normalized via (4) will be constant. We shall call these the *phantom differential invariants*. The other components of $w^{(n)}$ will define a complete system of functionally independent differential invariants defined on the domain of definition of the moving frame map.

Theorem 10. *Let n be greater than or equal to the order of the moving frame. Every n^{th} order differential invariant can be locally written as a function of the normalized n^{th} order differential invariants $I^{(n)}$. The function is unique provided it does not depend on the phantom invariants.*

Given an arbitrary differential function $F\colon \mathrm{J}^n \to \mathbb{R}$, then $L = F \circ w^{(n)}\colon \mathcal{B}^n \to \mathbb{R}$ defines a lifted differential invariant, and hence $J = L \circ \sigma^{(n)} = F \circ I^{(n)}$ defines a differential invariant, called the *invariantization* of F with respect to the given moving frame. Thus, a moving frame provides a natural way to construct a differential invariant from any differential function! Theorem 5 just says that if F itself is a differential invariant, then $F \circ w^{(n)}$ is independent of the group parameters, and hence $J = F$. In other words, invariantization defines a projection, depending on the moving frame, from the space of differential functions to the space of differential invariants.

The higher order differential invariants can also be obtained by invariant differentiation. The *normalized contact-invariant coframe* is the pull-back of the lifted contact-invariant coframe: $\boldsymbol{\omega} = (\sigma^{(n)})^* \boldsymbol{\eta} = (\mathbf{D}y \circ \sigma^{(n)})\, d\mathbf{x}$. The associated invariant differential operators $\mathcal{D} = (\mathbf{D}y \circ \sigma^{(n)})^{-T} \cdot \mathbf{D}$ are obtained by normalizing the lifted invariant differential operators $\mathcal{E}$. The invariant differential operators $\mathcal{D}_j$ do not necessarily commute; explicit commutation formulae are presented below.

The invariant differential operators will map differential invariants to higher order differential invariants. However, unlike their lifted counterparts, they do not directly produce the normalized differential invariants; in other words, $\mathcal{D}_K I^\alpha$ is *not*, in general, equal to I^α_K. The moving frame method will effectively resolve the computational difficulties in the usual (unlifted) theory. The fundamental *recurrence formulae* for the differential invariants (5) are

$$\mathcal{D}_j J^i = \delta^i_j + M^i_j, \qquad \mathcal{D}_j I^\alpha_K = I^\alpha_{K,j} + M^\alpha_{K,j}. \tag{6}$$

The "correction terms" M^i_j, $M^\alpha_{K,j}$ can be effectively computed using the following algorithm. For any n greater than or equal to the order of the moving frame, let $q^{(n)} = p + q\binom{p+n}{n} = \dim \mathrm{J}^n$. Let $V = V^{(n)}$ denote the $r \times q^{(n)}$ matrix whose entries are the coefficients ξ^i, φ^α_J of the n^{th} order prolonged infinitesimal generators (1). Let $W = V \circ I^{(n)}$ be its invariantized version, obtained by replacing the jet coordinates $z^{(n)}$ by the associated differential invariants $I^{(n)}$. We perform a Gauss–Jordan row reduction on the matrix W so as to reduce the $r \times r$ minor whose columns correspond to the chosen normalization variables $z_1, \ldots, z_r$ to be the identity matrix; let P be the resulting $r \times q^{(n)}$ matrix of invariants. Let $S = (S^\kappa_i)$ denote the $p \times r$ matrix whose entries are the total derivatives $S^\kappa_i = D_i z_\kappa$ of the normalization coordinates. Let $T = S \circ I^{(n)}$ be its invariantization. Then the correction terms in (6) are the entries of the $p \times q^{(n)}$ matrix product $M = -T \cdot P$.

A *syzygy* is a functional dependency $H(\ldots \mathcal{D}_J I_\nu \ldots) \equiv 0$ among the fundamental differentiated invariants. The normalization procedure not only gives us a generating system of fundamental differential invariants, but also classifies all syzygies among the normalized differential invariants.

Theorem 11. *A generating system of differential invariants consists of* a) *all non-phantom differential invariants J^i and I^α coming from the un-normalized zero*$^{\text{th}}$ *order lifted invariants y^i, v^α, and* b) *all non-phantom differential invariants of the form $I^\alpha_{J,i}$ where I^α_J is a phantom differential invariant. In other words, every other differential invariant can, locally, be written as a function of the generating invariants and their invariant derivatives, $\mathcal{D}_K J^i$, $\mathcal{D}_K I^\alpha_{J,i}$. All syzygies among the differentiated invariants are differential consequences of the following three fundamental types:*

(i) $\mathcal{D}_j J^i = \delta^i_j + M^i_j$, *when J^i is non-phantom,*

(ii) $\mathcal{D}_J I^\alpha_K = c + M^\alpha_{K,J}$, *when I^α_K is a generating differential invariant, while $I^\alpha_{J,K} = c$ is a phantom differential invariant, and*

(iii) $\mathcal{D}_J I^\alpha_{LK} - \mathcal{D}_K I^\alpha_{LJ} = M^\alpha_{LK,J} - M^\alpha_{LJ,K}$, *where I^α_{LK} and I^α_{LJ} are generating differential invariants the multi-indices $K \cap J = \varnothing$ are disjoint and non-zero, while L is an arbitrary multi-index.*

A similar algorithm produces the commutation formulae

$$[\mathcal{D}_i, \mathcal{D}_j] = \sum_{i=1}^{p} A^k_{ij}\, \mathcal{D}_k, \qquad i, j = 1, \ldots, p, \tag{7}$$

among the normalized invariant differential operators. Let X^k denote the $r \times p$ matrix whose entries are the total derivatives $X^k_{\kappa i} = D_i \xi^k_\kappa$ of the k^{th} independent variable coefficients. Let $Y^k = X^k \circ I^{(1)}$ denote its invariantization. Let B^k the result of performing the same Gauss–Jordan reduction on Y^k as was done on W; in other words, if $P = E \cdot W$, then $B^k = E \cdot Y^k$. Then the coefficient matrix $A^k = (A^k_{ij})$ in (7) is the skew-symmetric part of the matrix product $C^k = T \cdot B^k$, i.e., $A^k = C^k - (C^k)^T$.

Example 12. Let $M = \mathbb{R}^3$, with coordinates x^1, x^2, u. Let $G = \mathrm{GL}(2)$, and consider the action $(x^1, x^2, u) \mapsto (\alpha x^1 + \beta x^2, \gamma x^1 + \delta x^2, \lambda u)$, where $\lambda = \alpha\delta - \beta\gamma$. This action plays a key role in the classical invariant theory of binary forms, when u is a homogeneous polynomial, cf. [**21**]. The order zero invariants are obtained by inverting the group transformations:

$$y^1 = \lambda^{-1}(\delta x^1 - \beta x^2), \qquad y^2 = \lambda^{-1}(-\gamma x^1 + \alpha x^2), \qquad v = \lambda^{-1} u.$$

The lifted contact-invariant coframe and associated invariant differential operators are

$$\begin{aligned} \eta^1 &= d_H y^1 = \lambda^{-1}(\delta\, dx^1 - \beta\, dx^2), & \mathcal{E}_1 &= \alpha D_1 + \gamma D_2, \\ \eta^1 &= d_H y^2 = \lambda^{-1}(-\gamma\, dx^1 + \alpha\, dx^2), & \mathcal{E}_2 &= \beta D_1 + \delta D_2. \end{aligned}$$

The higher order lifted differential invariants are then $v_{jk} = (\mathcal{E}_1)^j (\mathcal{E}_2)^k v$; in particular

$$v_1 = \frac{\alpha u_1 + \gamma u_2}{\lambda}, \qquad v_2 = \frac{\beta u_1 + \delta u_2}{\lambda}, \qquad v_{11} = \frac{\alpha^2 u_{11} + 2\alpha\gamma u_{12} + \gamma^2 u_{22}}{\lambda},$$
$$v_{12} = \frac{\alpha\beta\delta u_{11} + (\alpha\delta + \beta\gamma) u_{12} + \gamma\delta u_{22}}{\lambda}, \qquad v_{22} = \frac{\beta^2 u_{11} + 2\beta\delta u_{12} + \delta^2 u_{22}}{\lambda}.$$

Let us choose the cross-section $\mathcal{K}^1 = \{x^1 = 1, x^2 = 0, u_1 = 1, u_2 = 0\}$. The normalization equations

$$y^1 = 1, \qquad y^2 = 0, \qquad v_1 = 1, \qquad v_2 = 0,$$

are then solved for the group parameters, leading to a first order moving frame

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} x^1 & -u_2 \\ x^2 & u_1 \end{pmatrix},$$

which is well-defined on surfaces $u = u(x, y)$ provided $x^1 u_1 + x^2 u_2 \neq 0$. The resulting normalized differential invariants are

$$J^1 = 1, \qquad J^2 = 0, \qquad I = \frac{u}{x^1 u_1 + x^2 u_2}, \qquad I_1 = 1, \qquad I_2 = 0,$$
$$I_{11} = \frac{(x^1)^2 u_{11} + 2x^1 x^2 u_{12} + (x^2)^2 u_{22}}{x^1 u_1 + x^2 u_2}, \qquad I_{12} = \frac{-x^1 u_2 u_{11} + (x^1 u_1 - x^2 u_2) u_{12} + x^2 u_1 u_{22}}{x^1 u_1 + x^2 u_2},$$
$$I_{22} = \frac{(u_2)^2 u_{11} - 2u_1 u_2 u_{12} + (u_1)^2 u_{22}}{x^1 u_1 + x^2 u_2}.$$

The Replacement Theorem 5 shows that each of these can be rewritten as the *identical* function of the lifted invariants; e.g., $I = (x^1 u_1 + x^2 u_2)^{-1} u = (y^1 v_1 + y^2 v_2)^{-1} v$. According to Theorem 11, we can take $I, I_{11}, I_{12}, I_{22}$ as our generating system, meaning that all higher order differential invariants can be obtained by successively applying the invariant differential operators to them. The normalized coframe is

$$\omega^1 = \frac{u_1\, dx^1 + u_2\, dx^2}{x^1 u_1 + x^2 u_2} = \frac{d_H u}{x^1 u_1 + x^2 u_2}, \qquad \omega^2 = \frac{-x^2\, dx^1 + x^1\, dx^2}{x^1 u_1 + x^2 u_2},$$

The associated invariant differential operators are well-known in classical invariant theory: $\mathcal{D}_1 = x^1 D_1 + x^2 D_2$ is the scaling process and $\mathcal{D}_2 = -u_2 D_1 + u_1 D_2$ is the Jacobian process.

Let us now illustrate the algorithm for determining recurrence formulae, syzygies and commutation formulae. The prolonged infinitesimal generator coefficient matrix and its invariantized counterpart are, up to second order,

$$V = \begin{pmatrix} x^1 & 0 & u & 0 & u_2 & u_{11} & 0 & u_{22} \\ x^2 & 0 & 0 & 0 & -u_1 & 0 & -u_{11} & -2u_{12} \\ 0 & x^1 & 0 & -u_2 & 0 & -2u_{12} & -u_{22} & 0 \\ 0 & x^2 & u & u_1 & 0 & u_{11} & 0 & -u_{22} \end{pmatrix},$$

$$W = \begin{pmatrix} 1 & 0 & I & 0 & 0 & -I_{11} & 0 & I_{22} \\ 0 & 0 & 0 & 0 & -1 & 0 & -I_{11} & -2I_{12} \\ 0 & 1 & 0 & 0 & 0 & -2I_{12} & -I_{22} & 0 \\ 0 & 0 & I & 1 & 0 & I_{11} & 0 & -I_{22} \end{pmatrix}.$$

Since we are normalizing x^1, x^2, u_1, u_2, we also need the matrices

$$S = \begin{pmatrix} 1 & 0 & u_{11} & u_{12} \\ 0 & 1 & u_{12} & u_{22} \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 & I_{11} & I_{12} \\ 0 & 1 & I_{12} & I_{22} \end{pmatrix}.$$

We use Gauss–Jordan reduction on the invariantized coefficient matrix W making the chosen normalization columns — in the present case columns 1, 2, 4 and 5 — into an identity matrix, and then premultiply the resulting matrix by T. The entries of the resulting matrix product

$$\begin{pmatrix} 1 & 0 & I_{11} & I_{12} \\ 0 & 1 & I_{12} & I_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & I & 0 & 0 & -I_{11} & 0 & I_{22} \\ 0 & 1 & 0 & 0 & 0 & -2I_{12} & -I_{22} & 0 \\ 0 & 0 & I & 1 & 0 & I_{11} & 0 & -I_{22} \\ 0 & 0 & 0 & 0 & 1 & 0 & I_{11} & 2I_{12} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & I(1+I_{11}) & I_{11} & I_{12} & (I_{11}-1)I_{11} & I_{11}I_{12} & 2I_{12}^2 - (I_{11}-1)I_{22} \\ 0 & 1 & II_{12} & I_{12} & I_{22} & (I_{11}-2)I_{12} & (I_{11}-1)I_{22} & I_{12}I_{22} \end{pmatrix}$$

are minus the required correction terms:

$$\begin{aligned} \mathcal{D}_1 J^1 &= \delta^1_1 - 1 = 0, & \mathcal{D}_2 J^1 &= \delta^1_2 - 0 = 0, \\ \mathcal{D}_1 J^2 &= \delta^2_1 - 0 = 0, & \mathcal{D}_2 J^2 &= \delta^2_2 - 1 = 0, \\ \mathcal{D}_1 I &= I_1 - I(1+I_{11}) = 1 - I(1+I_{11}), & \mathcal{D}_2 I &= I_2 - I\,I_{12} = -I\,I_{12}, \\ \mathcal{D}_1 I_1 &= I_{11} - I_{11} = 0, & \mathcal{D}_2 I_1 &= I_{12} - I_{12} = 0, \\ \mathcal{D}_1 I_2 &= I_{12} - I_{12} = 0, & \mathcal{D}_2 I_2 &= I_{22} - I_{22} = 0, \\ \mathcal{D}_1 I_{11} &= I_{111} + (1-I_{11})I_{11}, & \mathcal{D}_2 I_{11} &= I_{112} + (2-I_{11})I_{12}, \\ \mathcal{D}_1 I_{12} &= I_{112} - I_{11}I_{12}, & \mathcal{D}_2 I_{12} &= I_{122} + (1-I_{11})I_{22}, \\ \mathcal{D}_1 I_{22} &= I_{122} + (I_{11}-1)I_{22} - 2I_{12}^2, & \mathcal{D}_2 I_{22} &= I_{222} - I_{12}I_{22}. \end{aligned}$$

The formulae for $\mathcal{D}_1 I$ and $\mathcal{D}_2 I$ provide the syzygies of the second type, and show that we can use I to generate I_{11} and I_{12}. (There are no syzygies of the first type since we normalized all the lifted independent variables.) There are three fundamental syzygies of the third type:

$$\begin{aligned} \mathcal{D}_1 I_{12} - \mathcal{D}_2 I_{11} &= -2I_{12}, \\ \mathcal{D}_1 I_{22} - \mathcal{D}_2 I_{12} &= 2(I_{11}-1)I_{22} - 2I_{12}^2, \\ (\mathcal{D}_1)^2 I_{22} - (\mathcal{D}_2)^2 I_{11} &= 2I_{22}\mathcal{D}_1 I_{11} + (5I_{12}-2)\mathcal{D}_1 I_{12} + (3I_{11}-5)\mathcal{D}_1 I_{22} - \\ &\qquad - (2I_{11}-5)(I_{11}-1)I_{12} + 4(I_{11}-1)I_{12}^2. \end{aligned}$$

Finally, the commutation formulae can be determined directly:

$$[\mathcal{D}_1, \mathcal{D}_2] = -I_{12}\mathcal{D}_1 + (I_{11}-1)\mathcal{D}_2. \tag{8}$$

The alternative method is to first construct the matrices

$$X^1 = Y^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^2 = Y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

where X^i are obtained by differentiating the coefficients ξ^i of ∂_{x^i} in the infinitesimal generators; Y^i are their invariantizations, which are the same because the X^i happen to be constant, and B^i is obtained from Y^i by applying the same Gauss–Jordan row operations as were done to W above. Multiplying B^i by T to obtain C^i, and then skew symmetrizing to obtain A^i yields

$$\begin{aligned} C^1 &= T \cdot B^1 = \begin{pmatrix} 1 & -I_{12} \\ 0 & I_{22} \end{pmatrix}, & A^1 &= C^1 - (C^1)^T = \begin{pmatrix} 0 & -I_{12} \\ I_{12} & 0 \end{pmatrix}, \\ C^2 &= T \cdot B^2 = \begin{pmatrix} 0 & I_{11} \\ 1 & I_{12} \end{pmatrix}, & A^2 &= C^2 - (C^2)^T = \begin{pmatrix} 0 & I_{11}-1 \\ 1-I_{11} & 0 \end{pmatrix}. \end{aligned}$$

The $(1,2)$ entry of A^i provides the coefficient of $\mathcal{D}_i$ in (8).

In applications to equivalence problems and geometry, we restrict the moving frame and associated invariants to a submanifold of the appropriate dimension.

Definition 13. A p-dimensional submanifold parametrized by $\iota: X \to S \subset M$ is called *regular* with respect to a moving frame $\rho^{(n)}: \mathrm{J}^n \to G$ if its n-jet $\mathrm{j}_n S$ lies in the domain of definition of $\rho^{(n)}$. In this case, the restricted *moving frame* on the submanifold is defined as the composition $\lambda^{(n)} = \rho^{(n)} \circ \mathrm{j}_n \iota : X \to G$.

Theorem 14. *A submanifold $S \subset M$ admits an n^{th} order moving frame if and only if S is regular of order n, i.e., $\mathrm{j}_n S \subset \mathcal{V}^n$. Thus, in the analytic category, a submanifold S admits a moving frame (of some sufficiently high order) if and only if its isotropy subgroup G_S acts freely on S itself.*

Let S be a regular submanifold for a moving frame $\rho^{(n)}$. For any $k \geq n$, the k^{th} order *differential invariant classifying manifold* $\mathcal{C}^{(k)}(S)$ associated with a submanifold $\iota: X \to M$ is the manifold parametrized by the normalized differential invariants of order k, namely $J^{(k)} = I^{(k)} \circ \mathrm{j}_k \iota$. For simplicity, let us assume that $\mathcal{C}^{(k)}(S)$ is an embedded submanifold of its classifying space $Z^{(k)} \simeq \mathrm{J}^k$ of dimension t_k for $k \geq n$. Note that t_k equals the number of functionally independent invariants obtained by restricting the normalized k^{th} order differential invariants to S. In the fully regular case, then, we have

$$t_n < t_{n+1} < t_{n+2} < \cdots < t_s = t_{s+1} = \cdots = t \leq p,$$

where t is the *differential invariant rank* and s is the *differential invariant order* of S. We can now state the fundamental equivalence and symmetry theorems.

Theorem 15. *Let $S, \overline{S} \subset M$ be regular p-dimensional submanifolds with respect to a moving frame map $\rho^{(n)}$. Then S and $\overline{S}$ are (locally) congruent, $\overline{S} = g \cdot S$, if and only if they have the same differential invariant order s and their classifying manifolds of order $s+1$ are identical: $\mathcal{C}^{(s+1)}(\overline{S}) = \mathcal{C}^{(s+1)}(S)$.*

Theorem 16. *Let $S \subset M$ be a regular p-dimensional submanifold of differential invariant rank t with respect to a moving frame $\rho^{(n)}$. Then its isotropy group G_S is an $(r-t)$-dimensional subgroup of G acting locally freely on S.*

A submanifold S is *order k congruent* to a submanifold $\overline{S}$ at $z \in S$ if there is a group transformation $g \in G$ such that S and $g \cdot \overline{S}$ have order k contact at z. Note that the group transformation $g = g(z)$ may vary from point to point. The *rigidity order* of S is the minimal k for which order k congruence implies congruence, so $\overline{S} = g \cdot S$ for fixed $g \in G$. It turns out that this also means that the only congruent submanifold $\overline{S} = g \cdot S$ which has k^{th} order contact with S at a point is S itself.

Theorem 17. *If $S \subset M$ is a regular submanifold of differential invariant order s with respect to a moving frame, then S has rigidity order at most $s+1$.*

We now describe the method of moving coframes, which provides an alternative approach, based on invariant differential forms, that also extends to pseudo-group actions. For simplicity, let us assume that G acts transitively on M. Choose a base point $z_0 \in M$. A smooth map $\rho^{(0)}: M \to G$ is called a *compatible lift* with base point z_0 if it satisfies

$$\rho^{(0)}(z) \cdot z_0 = z. \tag{9}$$

We will call the general compatible lift $\rho^{(0)}(z,h)$ the *moving frame of order zero*. It is computed by solving the system of m equations (9) for m of the group parameters in terms of the coordinates z on M and the remaining $r - m = \dim G - \dim M$ group parameters, which we denote by h. Unlike the preceding moving frames, unless G acts locally freely on M, the order zero moving frame *will* depend on some of the group parameters. We can use $\rho^{(0)}$ to pull-back the left-invariant Maurer–Cartan forms on G, leading to the *moving coframe* of order zero. We can determine lifted invariants by analyzing the linear dependencies among the horizontal components of the moving coframe forms. Group-dependent invariants can be normalized to convenient constant values by solving for some of the unnormalized parameters. We successively eliminate parameters by substituting the normalization formulae into the moving coframe and recomputing dependencies. After the parameters have all been normalized, the differential invariants will appear through any remaining dependencies among the final moving coframe elements. Let us illustrate the basic method by a classical example; see [**9**] for more details and applications.

Example 18. The equiaffine geometry of curves in the plane is governed by the special affine group $\mathrm{SA}(2) = \mathrm{SL}(2) \ltimes \mathbb{R}^2$, acting on $M = \mathbb{R}^2$ according to

$$g: \quad \mathbf{x} \longmapsto A\mathbf{x} + \mathbf{a}, \qquad \mathbf{x} \in M, \quad A \in \mathrm{SL}(2), \quad \mathbf{a} \in \mathbb{R}^2.$$

We adopt a vector notation for the matrix $A = (\boldsymbol{\alpha}\,\boldsymbol{\beta}) \in \mathrm{SL}(2)$, where $\det A = \boldsymbol{\alpha} \wedge \boldsymbol{\beta} = 1$. The Maurer–Cartan forms on SA(2) are

$$\mu_1 = \boldsymbol{\alpha} \wedge d\boldsymbol{\alpha}, \quad \mu_2 = \boldsymbol{\beta} \wedge d\boldsymbol{\alpha} = \boldsymbol{\alpha} \wedge d\boldsymbol{\beta}, \quad \mu_3 = \boldsymbol{\beta} \wedge d\boldsymbol{\beta}, \quad \nu_1 = \boldsymbol{\alpha} \wedge d\mathbf{a}, \quad \nu_2 = \boldsymbol{\beta} \wedge d\mathbf{a}.$$

Choose the base point to be $\mathbf{x}_0 = 0$. Solving the compatible lift equations $\mathbf{x} = g \cdot \mathbf{x}_0 = \mathbf{a}$ yields the zero$^{\text{th}}$ order moving frame $\mathbf{a} = \mathbf{x}$. Substituting into the Maurer–Cartan forms, we find that, for a parametrized curve $\mathbf{x}(t)$, the forms ν_1, ν_2 restrict to the following two horizontal forms:

$$\nu_1 = (\boldsymbol{\alpha} \wedge \mathbf{x}_t)\, dt, \qquad \nu_2 = (\boldsymbol{\beta} \wedge \mathbf{x}_t)\, dt.$$

Their ratio produces the lifted invariant $(\boldsymbol{\alpha} \wedge \mathbf{x}_t)/(\boldsymbol{\beta} \wedge \mathbf{x}_t)$, which is normalized to 0 by setting $\boldsymbol{\alpha} = \lambda\, \mathbf{x}_t$ for some scalar parameter λ. This implies that $\mu_1 = \lambda^2 (\mathbf{x}_t \wedge \mathbf{x}_{tt})\, dt$. Assuming $\mathbf{x}_t \wedge \mathbf{x}_{tt} \neq 0$, the latter form can be normalized to equal $-\nu_2$ by setting

$$-\boldsymbol{\beta} \wedge \mathbf{x}_t = \lambda^2 (\mathbf{x}_t \wedge \mathbf{x}_{tt}), \qquad \text{or} \qquad \boldsymbol{\beta} = \lambda^2\, \mathbf{x}_{tt} + \mu\, \mathbf{x}_t,$$

for some scalar μ. Unimodularity implies $\lambda = (\mathbf{x}_t \wedge \mathbf{x}_{tt})^{-1/3}$. Therefore

$$-\nu_2 = ds = \sqrt[3]{\mathbf{x}_t \wedge \mathbf{x}_{tt}}\; dt$$

reproduces the equi-affine arc length element. Furthermore, $\mu_2 = \boldsymbol{\beta} \wedge d\boldsymbol{\alpha} = J\, ds$, where the lifted invariant

$$J = \mu (\mathbf{x}_t \wedge \mathbf{x}_{tt})^{1/3} + \frac{\mathbf{x}_t \wedge \mathbf{x}_{ttt}}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{4/3}}$$

is normalized to zero in the obvious manner. Therefore, the final moving frame is given by

$$\boldsymbol{\alpha} = \frac{d\mathbf{x}}{ds} = \frac{\mathbf{x}_t}{\sqrt[3]{\mathbf{x}_t \wedge \mathbf{x}_{tt}}}, \qquad \boldsymbol{\beta} = \frac{d^2\mathbf{x}}{ds^2} = \frac{\mathbf{x}_{tt}}{(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{2/3}} - \frac{\mathbf{x}_t}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{5/3}}, \qquad \mathbf{a} = \mathbf{x}.$$

Finally, $\mu_3 = \kappa\, ds$, where

$$\kappa = \mathbf{x}_{ss} \wedge \mathbf{x}_{sss} = \frac{(\mathbf{x}_t \wedge \mathbf{x}_{tttt}) + 4(\mathbf{x}_{tt} \wedge \mathbf{x}_{ttt})}{3(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{5/3}} - \frac{5(\mathbf{x}_t \wedge \mathbf{x}_{ttt})^2}{9(\mathbf{x}_t \wedge \mathbf{x}_{tt})^{8/3}}$$

defines the equi-affine curvature. All higher order differential invariants are obtained by differentiating κ with respect to the equi-affine arc length ds. This reproduces the basic invariants of the equi-affine geometry of curves, [**13**]; see also [**3**], [**8**], for applications in computer vision. The classical Frenet equations are a simple reformulation of the final moving frame formulae. We identify the linear part $A = (\mathbf{e}_1, \mathbf{e}_2) = (\mathbf{x}_s, \mathbf{x}_{ss})$ of the final moving frame with the equi-affine frame at a point $\mathbf{x}(t)$ on the curve, so that $\mathbf{e}_1 = \mathbf{x}_s$ is the unit affine tangent vector, whereas $\mathbf{e}_2 = \mathbf{x}_{ss}$ is the unit equi-affine normal. Combining this with the SL(2) Maurer–Cartan matrix $A^{-1}\,dA = \begin{pmatrix} 0 & 1 \\ \kappa & 0 \end{pmatrix} ds$ leads to the complete Frenet equations of planar equi-affine geometry:

$$\frac{d\mathbf{x}}{ds} = \mathbf{e}_1, \qquad \frac{d\mathbf{e}_1}{ds} = \mathbf{e}_2, \qquad \frac{d\mathbf{e}_2}{ds} = \kappa\,\mathbf{e}_1.$$

Note that the chosen normalizations are governed by the cross-section

$$\mathcal{K}^3 = \{x = u = u_x = 0, u_{xx} = 1, u_{xxx} = 0\}$$

to the group orbits on J^3. In fact, it is not hard to apply the regularized method directly in this example. In general, the more complicated the group action, the more efficient the moving coframe approach becomes. Curves whose 2-jets pass through the singular locus $\mathbf{x}_t \wedge \mathbf{x}_{tt} = 0$ can be covered by higher order moving frames, except for the straight lines, which are the totally singular curves in equi-affine geometry.

Finally, we demonstrate how the moving coframe method can be adapted to the case of infinite Lie pseudo-groups. By definition, a Lie pseudo-group consists of an infinite-dimensional family of invertible (local) transformations that form the general solution to an involutive system of partial differential equations, cf. [**17**], [**6**]. We can always characterize the pseudo-group transformations $\psi\colon M \to M$ as the projections of bundle maps $\Psi\colon \mathcal{B} \to \mathcal{B}$, defined on a principal fiber bundle $\mathcal{B} \to M$, that preserve a system of one-forms $\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_k\}$, so that $\Psi^*\boldsymbol{\zeta} = \boldsymbol{\zeta}$. The forms $\boldsymbol{\zeta}$ will play the role of the moving coframe forms for the pseudo-group, and the fiber coordinates of the bundle $\mathcal{B}$ will play the role of the undetermined group parameters. Of course, in this case $\boldsymbol{\zeta}$ does not form a full coframe on $\mathcal{B}$. A compatible lift, or moving frame of order zero, is just an arbitrary section $\sigma^{(0)}\colon M \to \mathcal{B}$. With these provisos, the normalization and reduction procedure is implemented as in the finite-dimensional situation.

Example 19. Consider the intransitive pseudo-group consisting of (local) diffeomorphisms on $M = \mathbb{R}^3$ of the form

$$\overline{x} = f(x), \qquad \overline{y} = y, \qquad \overline{u} = \frac{u}{f'(x)}. \tag{10}$$

This pseudo-group was introduced by Lie, [**18**; p. 373], in his study of second order partial differential equations integrable by the method of Darboux, and was further investigated by Medolaghi, [**19**], Vessiot, [**26**], and Kumpera, [**16**]. Following a general procedure presented in [**15**], a zero$^{\text{th}}$ order moving coframe consists of the one-forms

$$\zeta_1 = u\,dx, \qquad \zeta_2 = \alpha\,dx + \frac{du}{u}, \qquad \zeta_3 = dy,$$

which are defined on a rank one bundle $\mathcal{B} \to M$ with fiber coordinate α. Indeed, any transformation that satisfies $\Psi^*\zeta_i = \zeta_i$, $i = 1, 2, 3$, projects to a pseudo-group transformation (10). For surfaces $u = u(x, y)$, the linear dependency $\zeta_2 = -(u\alpha + u_x)\,\zeta_1 - (u_y/u)\,dy$ produces the normalization $\alpha = -u_x/u$, along with the basic first order differential invariant $I = u_y/u$. The final invariant moving coframe is

$$\zeta_1 = u\,dx, \qquad \zeta_2 = \frac{du - u_x\,dx}{u} \qquad \zeta_3 = dy.$$

The invariant total differential operators are thus $\mathcal{D}_1 = u^{-1} D_x$, $\mathcal{D}_2 = D_y$. A complete system of differential invariants consists of y, I, and the higher order invariant derivatives $(\mathcal{D}_1)^j (\mathcal{D}_2)^k I$.

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References

[**1**] Anderson, I.M., Introduction to the variational bicomplex, *Contemp. Math.* **132** (1992), 51–73.

[**2**] Anderson, I.M., and Torre, C.G., Two component spinors and natural coordinates for the prolonged Einstein equation manifolds, preprint, Utah State University, 1997.

[**3**] Calabi, E., Olver, P.J., Shakiban, C., Tannenbaum, A., and Haker, S., Differential and numerically invariant signature curves applied to object recognition, *Int. J. Computer Vision*, to appear.

[**4**] Cartan, É., *La Méthode du Repère Mobile, la Théorie des Groupes Continus, et les Espaces Généralisés*, Exposés de Géométrie No. 5, Hermann, Paris, 1935.

[**5**] Cartan, É., *La Théorie des Groupes Finis et Continus et la Géométrie Différentielle Traitées par la Méthode du Repère Mobile*, Cahiers Scientifiques, Vol. 18, Gauthier–Villars, Paris, 1937.

[**6**] Cartan, É., Sur la structure des groupes infinis de transformations, *in*: *Oeuvres Complètes*, Part. II, Vol. 2, Gauthier–Villars, Paris, 1953, pp. 571–624.

[**7**] Chern, S.-S., Moving frames, *in*: *Élie Cartan et les Mathématiques d'Aujourn'hui*, Soc. Math. France, Astérisque, numéro hors série, 1985, pp. 67–77.

[**8**] Faugeras, O., Cartan's moving frame method and its application to the geometry and evolution of curves in the euclidean, affine and projective planes, *in*: *Applications of Invariance in Computer Vision*, J.L. Mundy, A. Zisserman, D. Forsyth (eds.), Springer–Verlag Lecture Notes in Computer Science, Vol. 825, 1994, pp. 11–46.

[**9**] Fels, M., and Olver, P.J., Moving coframes. I. A practical algorithm, preprint, University of Minnesota, 1996.

[**10**] Fels, M., and Olver, P.J., Moving coframes. II. Regularization and theoretical foundations, preprint, University of Minnesota, 1997.

[**11**] Green, M.L., The moving frame, differential invariants and rigidity theorems for curves in homogeneous spaces, *Duke Math. J.* **45** (1978), 735–779.

[**12**] Griffiths, P.A., On Cartan's method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry, *Duke Math. J.* **41** (1974), 775–814.

[**13**] Guggenheimer, H.W., *Differential Geometry*, McGraw–Hill, New York, 1963.

[**14**] Jensen, G.R., *Higher order contact of submanifolds of homogeneous spaces*, Lecture Notes in Math., No. 610, Springer–Verlag, New York, 1977.

[**15**] Kamran, N., Contributions to the study of the equivalence problem of Elie Cartan and its applications to partial and ordinary differential equations, *Mém. Cl. Sci. Acad. Roy. Belg.* **45** (1989), Fac. 7.

[**16**] Kumpera, A., Invariants différentiels d'un pseudogroupe de Lie, *J. Diff. Geom.* **10** (1975), 289–416.

[**17**] Lie, S., Die Grundlagen für die Theorie der unendlichen kontinuierlichen Transformationsgruppen, *Leipzig. Ber.* **43** (1891), 316–393; also *Gesammelte Abhandlungen*, Vol. 6, B.G. Teubner, Leipzig, 1927, pp. 300–364.

[**18**] Lie, S., Zur allgemeinen Theorie der partiellen Differentialgleichungen beliebeger Ordnung, *Leipz. Berich.* **47** (1895), 53–128; also *Gesammelte Abhandlungen*, Vol. 4, B.G. Teubner, Leipzig, 1929, 320–384.

[**19**] Medolaghi, P., Classificazione delle equazioni alle derivate parziali del secondo ordine, che ammettono un gruppo infinito di trasformazioni puntuali, *Ann.Mat. Pura Appl.* **1** (3) (1898), 229–263.

[**20**] Olver, P.J., *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics, vol. 107, Springer–Verlag, New York, 1993.

[**21**] Olver, P.J., *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, 1995.

[**22**] Olver, P.J., Singularities of prolonged group actions on jet bundles, preprint, University of Minnesota, 1997.

[**23**] Ovsiannikov, L.V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[**24**] Sternberg, S., *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[**25**] Thomas, T.Y., *The Differential Invariants of Generalized Spaces*, Chelsea Publ. Co., New York, 1991.

[**26**] Vessiot, E., Sur l'intégration des systèmes différentiels qui admettent des groupes continues de transformations, *Acta. Math.* **28** (1904), 307–349.

[**27**] Weyl, H., Cartan on groups and differential geometry, *Bull. Amer. Math. Soc.* **44** (1938), 598–601.

[**28**] Willmore, T.J., *Riemannian Geometry*, Oxford University Press, Oxford, 1993.