Advanced Course
on
Integral Geometry

Notes of the course

September 15 – September 23, 1999

Centre de Recerca Matemàtica
Bellaterra (Spain)

Supporting institutions:

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Foreword

These notes correspond to the Advanced Course on Integral Geometry that will take place from September 15 to September 23, 1999 at the Centre de Recerca Matemàtica (CRM) in Bellaterra. This is the first activity of a three months period devoted by the CRM to Differential Geometry, and which is coordinated by Marcel Nicolau.

The Advanced Course is organized in two lecture series that will be delivered by professors Rolf Schneider and Rémi Langevin both very well known speciallists in this area.

We thank the lecturers for their effort in the preparation of these notes and for having them on time to assure that the volume will be ready at the beginning of the course.

The course is complemented with problems sessions and invited conferences by X. Gual, M. Santander, E. García-Barroso and L. M. Cruz-Orive, whom we thank for their collaboration. This volume also contains abstracts of these lectures.

We want express our gratitude to prof. M. Castellet director of the CRM as well as to the staff of the Center, Mrs. Consol Roca and Mrs. Maria Julià who helped us in the organization of this course.

We hope that this course will be profitable to all the participants and that all of us will remember these days with great pleasure.

E. Gallego
A. Reventós

Contents

The lecture notes contained in this booklet were printed directly from files supplied by the authors before the course.

Introduction to Integral Geometry

R. Langevin

1 Introduction

In 1777 Buffon published his *Essai d'arithmétique morale* [Bu], where he describes the needle experiment.

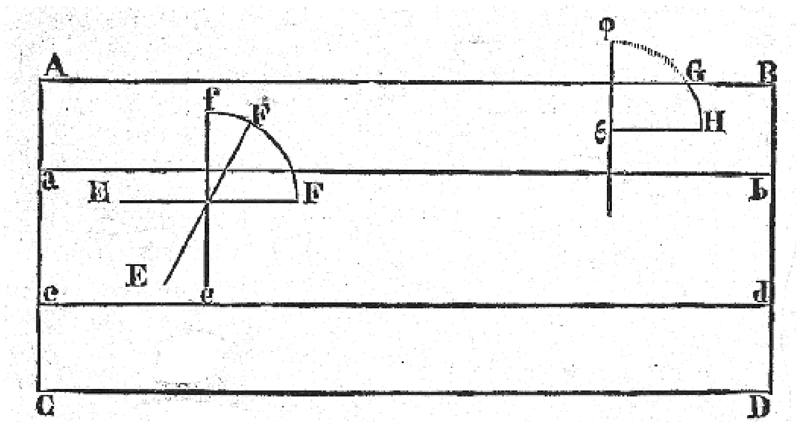


Figure 1: Buffon's calculation

The first paragraph of the essai is:

La mesure des choses incertaines est ici mon objet: je vais tacher de donner quelques règles pour estimer les rapports de vraisemblance, les degrés de probabilité, le poids des témoignages, l'influence des hasards, l'inconvénient des risques, et juger en même temps de la valeur réelle de nos craintes et de nos espérances. After some considerations about a game called "franc-carreau", where the players gamble on the position of a coin thrown on a tiling (entirely in a tile or accross some division line) , Buffon proves that, when a needle is thrown "at random" on the boards of a parquet, if the length of the needle is equal to the width of the boards, the probability it will lay across two boards is $2/\pi$. He admits without the slightest doubt that the right probability measure on the space of positions of the needle is the measure $\frac{1}{2\pi}|dx \wedge d\theta|$ which we shall consider below.

The appearance of the number π hides a circle. The physicist Paul Langevin described in 1908 a way to visualise a proof of Buffon's result.

Let us throw thousands of needles and move them using only translations parallel to the boards or perpendicular to them with length an integer multiple of the width of the board. As all relative positions (angle, distance of the needles to the lines boundary of the boards) are equally likely, we can rearrange the needles along a very large circle as in fig.2 having essentially

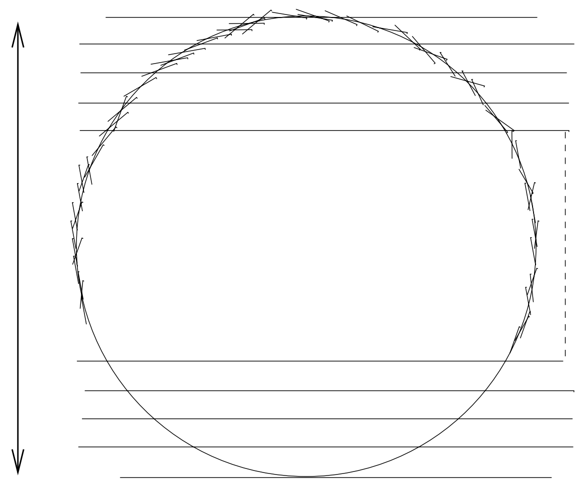


Figure 2: Needles and rearranged needles

the same amount, say N, of needles above any point of the circle. The total amount of needles is close to N.L, where L is the length of the circle and the number of needles crossing the lines is close to

$$N.(number\ of\ intersection\ points\ of\ the\ lines\ with\ the\ circle)$$

that is 2N.D, where D is the diameter of the circle. The required probability is then $2N.D/N.L = \frac{2}{\pi}$.

A hundred years will be needed to clarify the notion of probability involved. Before coming to that, let us give a conventional proof confirming Buffon's result. Locate the position of the needle on the floor by the position of its tip and the angle of the needle with the direction of the lines. Using as before translations parallel to the boards, or multiple of the width of the boards, we can suppose that our needle has its tip on the vertical segment AB of fig 3. We assume that AB has length 1. Call x the distance between the tip of the needle and A.

Therefore the set of all possible positions of the needle is $[AB[.S^1$, (or rather $\mathbb{R}/\mathbb{Z} \cdot S^1$). The needle meets the line L_B if $x + sin\theta \geq 1$ and L_A if $x + sin\theta \leq 0$.

The ratio between the dashed area and the area of the rectangle $]0,1].[0,\pi[$ is $2/\pi$.

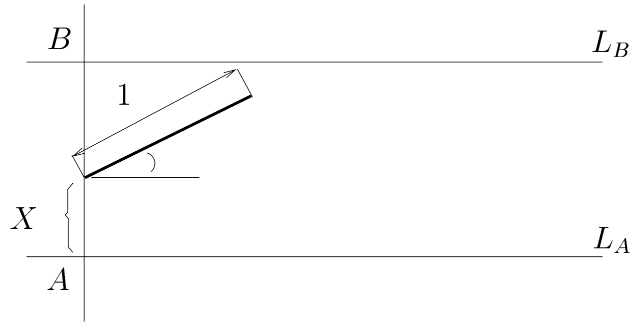


Figure 3: Localization of the needle

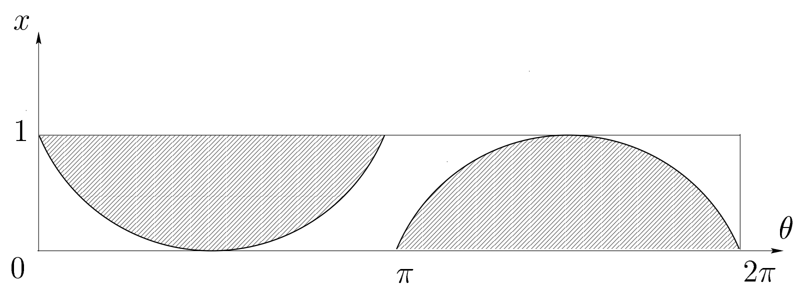


Figure 4: Area in $[0,1] \times [0,2\pi]$ of the needles crossing a line

2 The birth of the notion of geometric measure

2.1 Cauchy and Crofton

In 1832, in a communication to the French Academy of Sciences, Cauchy noticed that the length of a convex curve is the average of the lengths of the orthogonal projection of the convex curve on all lines through the origin. More generally, for any rectifiable planar curve C, denote by m(C,L) the "absolute length" of the orthogonal projection of C on the line L, the length of the projection counted with multiplicity. In modern language:

$$m(C,L) = \int card(p^{-1}(y)dy)\,; y \in L$$

Then:

Theorem 2.1.1 *Cauchy formula [Cau]*

$$\int_{-\pi 2}^{\pi/2} m(C, L_\theta) d\theta = 2(length\, of\, C)$$

Cauchy's proof amounted to prove the formula for a segment, and then approximate any curve by inscribed polygons.

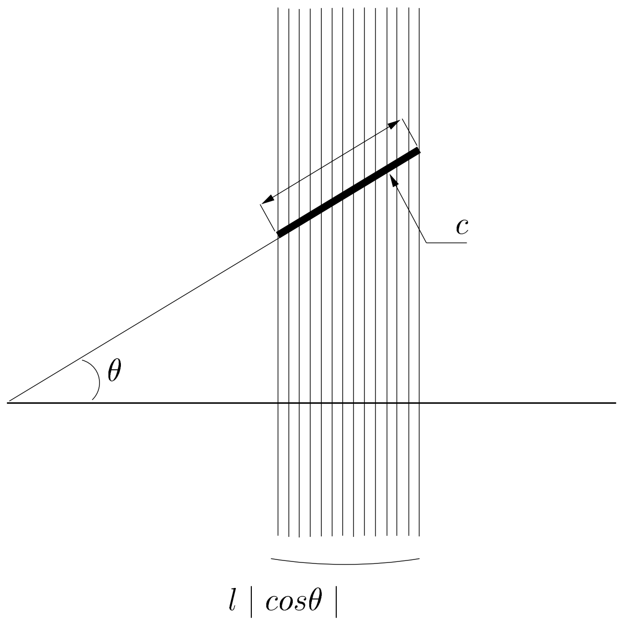


Figure 5: Measure of the lines orthogonal to a given direction and intersecting a segment

From Cauchy's communication to the french academy in 1832 [Cau], to Crofton's mémoire (1868) [Cro] 36 years where needed to clarify the notion of a measure on the set of affine lines. Let us quote Crofton: *The expression "at random" has in common language a very clear and definite meaning; one which cannot be better conveyed than by Mr Wilson's expression "according to no law"... There is always a direct reference to the assemblage of things to which it belongs and from which we take, and not till then, we can proceed to sum up the favorable cases,... But there are several classes or questions in which the totality of cases is not merely infinite, but of an inconceivable nature...We can thus continually suppose variations of the experiment, each variation giving a new infinity of cases.* (then Crofton justifies the choice of the measure on the plane). *What means: an infinity of lines drawn at random on the plane, what is the nature of this aggregate? First, since any direction is as likely as the others, as many of the lines are parallel to any direction as to any other. As this infinite system of parallels is drawn at random, they are as thickly disposed along any part of the perpendicular as along any other...*

Crofton did found the right answer as we will see in next section. Nevertheless, at the turn of the century the choice of a measure on a continuum was not obvious, because there were too many possibilities.

2.2 Bertrand's paradoxes

Let us give three different answers proposed by the probabilist Bertrand to the same problem of elementary geometry. At that point, integral geometry was close to desappear. The question is (see pictures below): what is the probability for a chord of a circle taken at random to be longer than the side of an equilateral inscribed circle? The three different answers Bertrand proposes will come from three different ways to choose the chord.

1) Chose an arbitrary point A on the circle. Using the rotational symetry of the picture we can forget about A and choose now another point B on the circle, endowed with arc length measure.

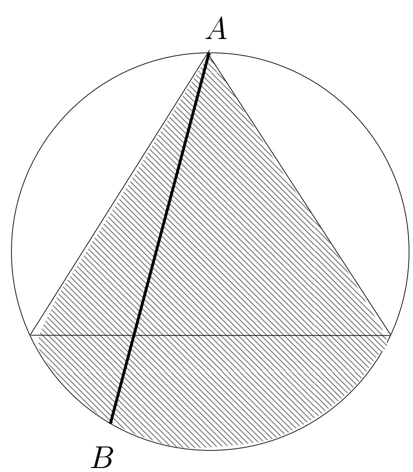


Figure 6: Probability 1/3

The chord is then longer than the side of the inscribed equilateral triangle with probability 1/3.

2) Chose at random the affine line supporting the chord. The rotational symetry of the picture allows us to forget about the direction of the line.

Figure 7: Probability 1/2

As $cos(\pi/3) = 1/2$ the probability is now 1/2.

3) Chose at random the middle of the chord in the disc (the measure is the Lebesgue measure on the disc). We ignore chords through the origin, as they form a set of measure zero.

Then the probability is 1/4.

Poincaré will take integral geometry out of this dead end. For him, (see for example his book published in 1912 [Poin], the most interesting measure is the one which is invariant under the group of affine isometries of the plane. Only isometries preserving the origin are allowed by presentations 1) and 3). In 2) translations also act on the set of affine lines and preserve our measure. It was also Crofton's answer.

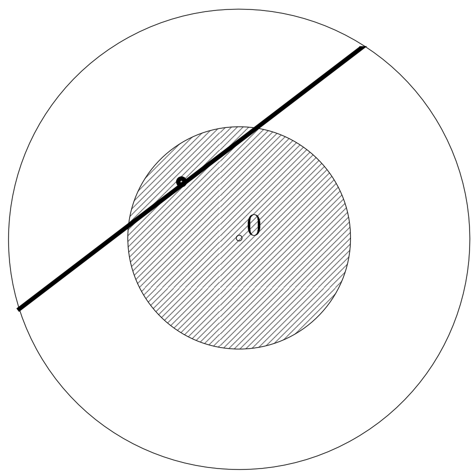


Figure 8: Probability 1/4

3 The euclidean plane

3.1 Geometric measures on sets of lines

We will start with the Euclidean geometry of the plane. The group of Euclidean motions $\mathcal{M}$ acts on the points of $\mathbb{R}^2$. It leaves invariant the Lebesgue measure $dx \wedge dy$. It acts also on the set of affine lines of the plane $\mathcal{A}(2,1)$. The oriented lines through the origin of $\mathbb{R}^2$ form a circle, as any oriented half-line cuts the unit circle in a point. This correspondence defines the topology of the set of oriented lines through the origin. The set of unoriented lines is the quotient of this first circle by the relation $x \sim -x$ We denote this set by $G(2,1)$. We can visualize the latter identifying a line (distinct from the x-axis) with its intersection (different from the origin) with the circle tangent at the origin to the x-axis of next picture.

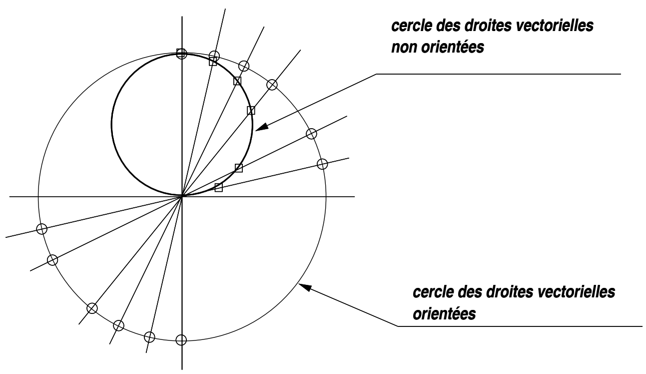


Figure 9: Oriented and non-oriented directions.

A non-oriented affine line corresponds to each point t of a direction D; just take Δ_t to be the perpendicular through t to D. Using oriented directions we would get oriented affine lines $\mathcal{A}^+(2,1)$. In that case we consider an oriented direction D^+ and the affine line perpendicular to a point $t \in D^+$ (which can be identifies with its coordinate on the oriented line D^+). Let θ be the oriented angle of the x-axis and D^+. We see that the oriented affine grassmannian $\mathcal{A}^+(2,1)$ is a cylinder $S^1 \times \mathbb{R}$ on which natural coordinates are θ and t.

Figure 10: coordinates on the set of oriented lines.

From the angular measure $|d\theta|$ on the unit circle and the Lebesgue measure $|dt|$ on the line D, we get a measure $|d\theta \wedge dt|$ on the set of oriented affine lines. This measure is invariant by a rotation of center the origin. A translation of vector v moves the line (θ, t) to the line $(\theta, t+ < e^{i\theta}|v>$; it also leaves invariant the measure $|d\theta \wedge dt|$. As an exercise, let us represent on the cylinder $S^1 \times \mathbb{R}$ the oriented affine lines through the extremity O' of the vector v on the picture below.

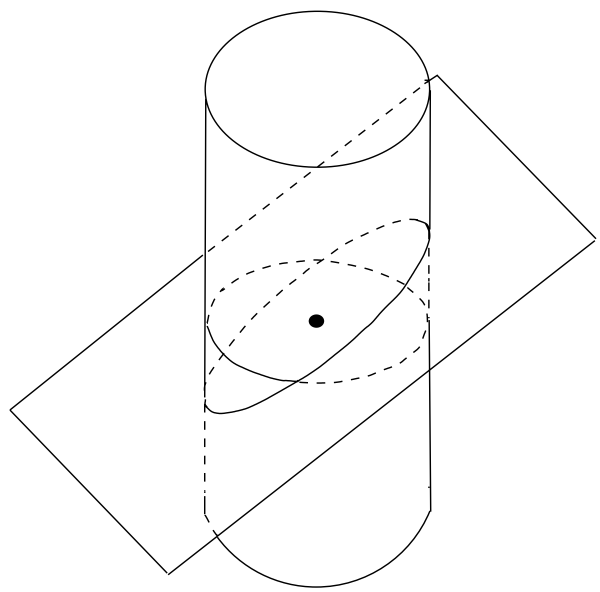


Figure 11: Oriented lines through the origin and through another point.

We will call the family of lines through a point, or the family of lines parallel to a given direction a *linear pencil*. The equation of a line of a linear pencil is a linear combination of the equations of any two different lines of the family.

Remark: The oriented affine lines through the point $m = (a, b)$ are the intersection of the cylinder $x^2 + y^2 = 1$ with the plane of equation $z = ax + by$. Parallel lines are the intersection of the cylinder with a vertical plane through the origin.

The projection (forgetting the orientation) of $\mathcal{A}^+$ on $\mathcal{A}$ defines the measure, still denoted $|d\theta \wedge dt|$, on $\mathcal{A}^+(2, 1)$. this projection also permits us to recognize that $\mathcal{A}(2, 1)$ is the Möbius band obtained from the rectangle $[0, \pi] \times \mathbb{R}$ identifying $(0, t)$ with $(\pi, -t)$. the next picture shows the set of lines corresponding to the small rectangle $[\theta_1, \theta_2].[t_1, t_2]$.

Figure 12: The Möbius band.

3.2 The Gauss map

During this section, curves will be of class $\mathcal{C}^\infty$. An essential tool in the study of hypersurfaces of $\mathbb{R}^n$ and first planar curves, is the Gauss map γ which to each point m of an oriented curve C associates its oriented normal, $N(m) = R_{\pi/2}(T(m))$ where $T(m)$ is the oriented unit tangent at m to the curve.

$$\gamma : C \to S^1$$

The jacobian k(m) of γ at a point $m \in C$ is called the curvature of C at that point. Notice that we can define a Gauss map with value in P_1, forgetting the orientation of $N(m)$. Notice also that the tangent map $T : C \mapsto S^1$ mapping a point $m \in C$ to the oriented unit tangent to C at m, has the same jacobian $k(m)$. We will use the map $C \mapsto \mathcal{A}^+(2,1)$ the paragraph "envelopes" of next section.

Remark: Let $m \in C$ be a noncritical point of the Gauss map. Then the point m is a nondegenerate critical point of the orthogonal projection of C on the oriented line $L(x)$ defined by $N(x)$.

Proof: Locally C has the equation $y = f(x)$ where x is a coordinate on the line generated by $T(x)$ and y a coordinate on the line $L(x)$. We

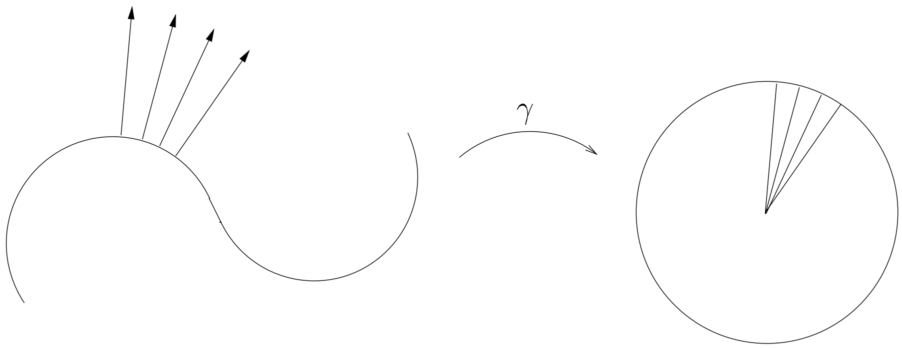


Figure 13: The Gauss map.

can choose the euclidean coordinates x, y such that $f(0) = 0$. We also have $f'(0) = 0$. The curvature $k(m)$ is in that case just $2 \cdot f"(0)$. If the curvature is nonzero, the orthogonal projection of C on L has the nondegenerate hessian $f"(0)$. $\square$

For a direction L, denote by $\mu(C, L)$ the number of critical points of the orthogonal projection of C on L. The change of variable theorem implies then that there exists a neighbourhood v of m such that

$$\int_v |k(m)|dm = \int_{P_{n-1}} |\mu|(v, L)dL$$

The result holds globally on C.

Theorem 3.2.1

$$\int_C |k(m)|dm = \int_{P_{n-1}} |\mu|(C, L)dL$$

Proof: The proof relies on Sard's theorem. The set Σ of critical values of γ is of zero measure. Its inverse image $\gamma^{-1}(\Sigma)$ is the union of critical points of γ, where $k = 0$, and of noncritical points of γ with image in σ, the latter form a set of measure zero. The complement of $\gamma^{-1}(\Sigma)$ is an enumerable union of open sets of C. Discarding at most an enumerable set of points if necessary, we get an enumerable union $\bigcup_i(U_i)$ of open set of C where the restriction of γ is a diffeomorphism on its image. Using the change of variable theorem and summing on i we get:

$$\int_C |k(m)|dm = \int_{\bigcup_i(U_i)} |k(m)|dm = \sum_i(\gamma(U_i) = \int_{P_{n-1}} |\mu|(C, L)dL$$

□

We can also count "most" of critical points with a sign. Assign to the non degenerate critical points of the orthogonal projection of the oriented curve C on the oriented line L^+ the sign $\epsilon(m) = (-1)^{index(m)}$. When the two unit vectors contained in L are non degenerate values of the Gauss map, we can, at each point m such that $\gamma(m) \subset L$ orient the line L using the normal $N(m)$ to define $\epsilon(m)$. Thus we get:

$$\mu(C, L) = \sum_{\gamma(m) \in L} (\epsilon(m))$$

Theorem 3.2.2 *If one of the integrals*

$$\int_C |k(m)|dm = \int_{P_{n-1}} |\mu|(C, L)dL$$

is finite, then:

$$\int_C k(m)dm = \int_{P_{n-1}} \mu(C, L)dL$$

To prove this last theorem, is is enough to track the signs in the proof of the preceeding one.

A classical theorem for embedded closed planar curve states that:

Theorem 3.2.3

$$|\int_C k(m)dm| = 2\pi$$

It is a consequence of the following fact that we will explain below:

$$\mu(C, L) = 2 \cdot degree(Gauss\ map) = \pm 2$$

when C is a simple closed curve, and when $\mu(C, L)$ makes sense.

As a corollary we get the inequality:

$$\int_C |k(m)|dm \geq 2\pi$$

3.3 Volume of the tube around a curve

We will use the previous definitions to compute the volume of a small tubular neighbourhood of a closed planar curve C, and the volume of the thickening on one side of the curve.

Let C_τ be the curve

$$C_\tau = \{C(t) + \tau N(t)\}$$

The tubular neibourhood of C

$$Tub_r(C) = \{m | d(m, C) \leq r\}$$

is the union

$$Tub_r(c) = \{\cup_{-r \leq \tau \leq r} C_\tau\}$$

The tubular neighbourhood lemma tells us that for r small enough the map:

$$(t, \tau) \mapsto \{C(t) + \tau N(t)\}$$

is diffeomorphism. Let us also define the thickening (on the side of N) of C:

$$Th_r(C) = \{\cup_{0 \leq \tau \leq r} C_\tau\}$$

The volume of $Th_r(C)$ is the integral:

$$vol[Th_r(C)] = \int_{0 \leq \tau \leq r} vol(C_\tau)$$

The projection, "counted with multiplicity" of the curve C_τ on a line L is obtained, modifying the projection of C on intervals of length τ with one extremity a critical value of the orthogonal projection of C on L.

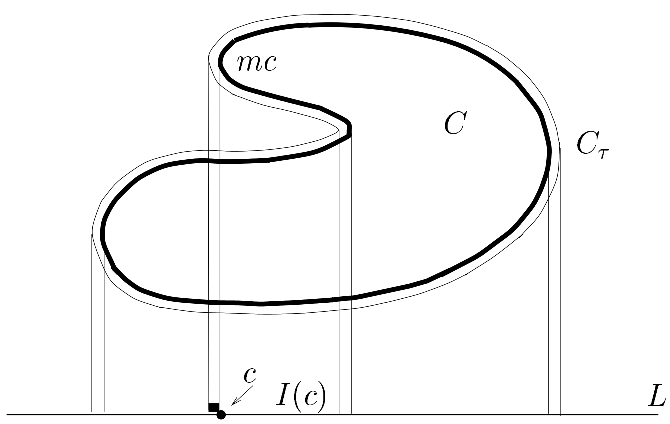


Figure 14: Thickening of a curve.

To give a formula suppose first that the orthogonal projection π_L on L is a *Morse function*, that is, has only non degenerate critical points, which all have different images. Then a critical value $c \in L$ is the image of one critical point $m_c \in C$. The normal $N(m)$ is parallel to L and allows us to define the interval $I(c) = [c, c + \tau N(m)]$. The projection of C on L defines a function (with integer values) on L:

$$\varphi(C, L)(y) = \sharp\, \pi_L^{-1}(y)$$

Depending on the local position of C , $N(m)$ and the line orthogonal to L in c, we define a sign

$$\epsilon(c) = \epsilon(m_c) = \pm 1$$

(this generically makes sense, as the critical value c will, for almost every line, be the image of a unique critical point; see section 7.3 for more precise statements). Then:

$$\epsilon(c) = +1\, if\, C\, is\, locally\, not\, on\, the\, side\, of\, N(m),$$
$$-1\, if\, C\, is\, locally\, on\, the\, side\, of\, N(m)$$

Remark: To change the orientation of N will change the sign of $\epsilon(c)$.

Proposition 3.3.1 *The function $\varphi(C_\tau, L)$ is equal, when π_L is a Morse function, to*

$$\varphi(C_\tau, L) = \varphi(C, L) + \sum_c \epsilon(c) \cdot 2 \cdot 1_{I(c)}$$

In the formula the summation is over all critical values c of π_L, and $1_{I(c)}$ is the characteristic function of the interval $I(c)$.

Observe that the degree of the Gauss map γ can be computed using any *generic* line L, that is, here, any line such that the projection π_L is a Morse function. This degree is

$$\sum_c \epsilon(c)$$

We now also that the set of non-generic lines is of measure 0. We know that, depending of the orientation of the curve, this degree is ± 1. The proof follows from the definition of the function $\varphi(C, L)$ See fig. Rewrite Cauchy's formula for C_τ using the functions φ:

$$2 \cdot length(C_\tau) = \int_{L \in P_1} \int_L \varphi(C_\tau, L)$$

Using the proposition, the remark on the degree of γ, and permuting the order of integration (this makes sense when the curve is compact smooth arc) one gets the:

Theorem 3.3.2

$$vol(Th_r(C)) = r \cdot length(C) + (-1)^{ind\gamma} \cdot \pi \cdot r^2$$

and, using also the previous remark, we get the corollary:

Corollary 3.3.3 *for r small enough,*

$$vol(Tub_r(C)) = 2r \cdot length(C)$$

In the section **higher dimensional convex bodies**, we will generalise this proof to higher dimensions.

4 Two dimensional convex bodies and translations

4.1 Envelopes

We mentioned in the previous section the Gauss map γ. If we retain not only the normal (or tangent direction at a point m but the oriented affine tangent line we get a map $C \mapsto \mathcal{A}(2,1)$. Conversely, to a smooth one-parameter family of affine lines, corresponds in general a curve, which is the envelope of this family of lines. Let $D_t = \{a(t)x + b(t)y + c(t) = 0\}$ be a smooth family of lines where $a(t), b(t), c(t)$ are smooth functions of t. The lines D_t and D_{t+h} have an intersection in the plane if they are not parallel. When h goes to zero this intersection point may have a limit $m(t)$. Let us give a sufficient condition for the points $m(t)$ to exist, and belong to a curve C which admits the tangent D_t at the point $m(t)$.

Theorem 4.1.1 *Let D_t be a smooth family of lines of equations $a(t)x + b(t)y + c(t) = 0; (x,y) \in I\!R^2$. If for all $t \in [\alpha, \beta]$, the determinant det $\begin{pmatrix} a(t) & b(t) \\ a'(t) & b'(t) \end{pmatrix}$ is different from zero, the family envolves a curve C, that is, the curve is the union of the points: $m(t) = D_t \cap D'_t$, where D'_t is the affine line of equation $a'(t)x + b'(t)y + c'(t) = 0$. Moreover if the determinant det $\begin{pmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c't \\ a''(t) & b''(t) & c''(t) \end{pmatrix}$ is also different from zero, the curve is smooth at $m(t)$ and the tangent to C at $m(t)$ is D_t.*

We will note D''_t the line of equation $a''(t)x + b''(t)y + c''(t) = 0$.

Proof: Let us find the intersection point of D_t and D_{t+h}. We need to solve the linear system:

$$\begin{aligned} &a(t)x + b(t)y + c(t) = 0 \\ &a(t+h)x + b(t+h)y + c(t+h) = 0 \end{aligned}$$

A first order Taylor expansion of the second equation gives:

$$\begin{aligned} &a(t)x + b(t)y + c(t) = 0 \\ &(a(t) + a't)h + o(h))x + (b(t) + b'(t)h + o(h))y + (c(t) + c'(t)h + o(h)) = 0 \end{aligned}$$

This is equivalent to the system:

$$\begin{aligned} &a(t)x + b(t)y + c(t) = 0 \\ &[a'(t)h + o(h)]x + [b'(t)h + o(h)]y + [c'(t)h + o(h)] = 0 \end{aligned}$$

If the determinant $det \begin{pmatrix} a(t) & b(t) \\ a'(t) & b'(t) \end{pmatrix} \neq 0$, the limit of the solution, when h goes to zero, is the solution $m(t)$ of the system:

$$a(t)x + b(t)y + c(t) = 0$$
$$a'(t)x + b'(t)y + c'(t) = 0$$

(we shall refer to that system as (*)).

The condition $det \begin{pmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c't \\ a''(t) & b''(t) & c''(t) \end{pmatrix} \neq 0$ guarantees that the three lines D, D' and D'' do not belong to the same linear pencil. Up to terms negligible compared with h the point $m(t+h)$ is the point $D_t \cap D'_{t+h}$, which show that the limit of the line containing the chord $m(t), m(t+h)$ is D_t; See next picture □

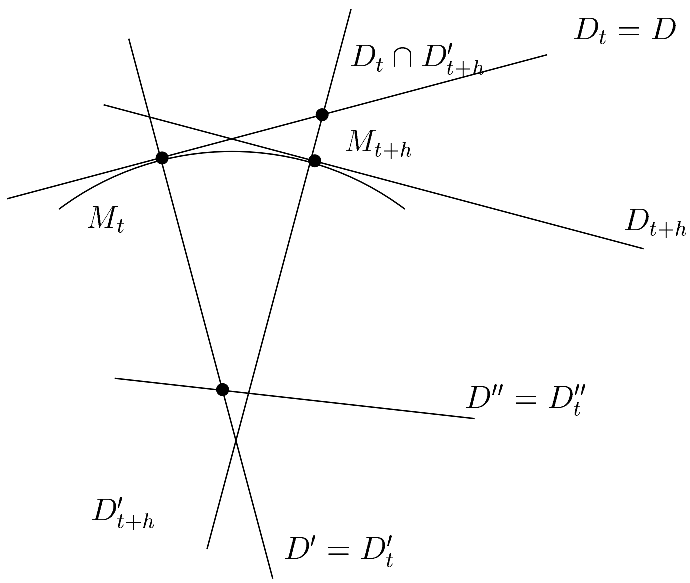


Figure 15: A non degenerate piece of envelope

Linear pencils are in that sense "degenerate" envelopes.

4.2 Support functions and hérissons

The name *hérisson* (french word for hedge hog) has been chosen because the skin of this animal cannot fold to much without inconvenience because of its spikes. We will call *hérisson* the envelope of a family of lines parametrised by their direction. In fact our definition gives oriented affine lines, as we can orient D_u by $R_{\pi/2}(u)$. More precisely, each lines of the family $D_u, u \in S^1$ admits the equation:

$$D_u = \{m | < m|u >= h(u)\}$$

where u is a unit vector, and h(u) a real function. The system(*)which gives the points of the envelope becomes:

$$< m|u >= h(u)$$

$$< m|R_{\pi/2}(u) >= (dh/du)(u)$$

and has automatically a non zero determinant.

Let Q be a compact convex body. We can define a function $h(u)$ by :

$$h(u) = sup[< m|u >; m \in Q]$$

The line D_u of equation $< m|u >= h(u)$ is the support line of Q in the oriented direction u.

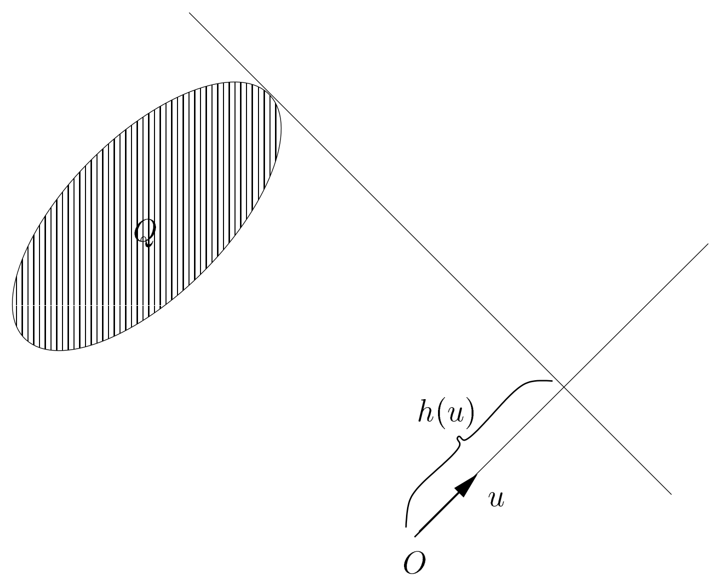


Figure 16: support function

It touches Q and Q stays on one side of D_u. The convex body Q is the intersection of the half spaces $< m|u > \leq h(u)$.

exercise check that the tangents to an hyperbola and the two asymptotes do not satisfy the conditions D_t smooth and

$$det \begin{pmatrix} a(t) & b(t) & c(t) \\ a'(t) & b'(t) & c't \\ a''(t) & b''(t) & c''(t) \end{pmatrix} \neq 0$$

Proposition 4.2.1 *When ∂Q is a smooth curve of nowhere zero curvature, it is the envelope of the family D_u. The radius of curvature of ∂Q at the point where D_u is tangent to the curve is $h(u) + h''(u)$, where h is the support function defining the family D_u. Conversely a bounded smooth support function h such that $h + h''$ is everywhere strictly positive has an envelope which is the boundary of a compact convex body.*

In a generalized sense the boundary ∂Q can always be seen as the envelope of the family D_u. The condition (*) is always satisfied. At a point where ∂Q has a right and a left tangent which are different, the family D_u contains an arc in the pencil of line through that point.

Proof: When it is different from zero, the curvature of the boundary ∂Q, with the (conterclockwise) boundary orientation, is positive. Let us compute the radius of curvature of a hérisson, and prove that, if it is always positive, the hérisson is the boundary of a smooth convex body, with everywhere positive curvature. The caracteristic point $m(u) = (D_u \cap D'(u))$ satisfies the equations:

$$< m|u >= h(u); < m|R_{\pi/2}(u) >= h'(u)$$

Let θ be the angle $((0,1), u)$. The rotation R_θ sends the vector $(1,0)$ to u and $(0,1)$ to $R_{\pi/2}(u)$. The two equations are equivalent to:

$$R_{-\theta}(m) = (h(u), h'(u))$$

The solution is then:

$$m(u) = R_\theta(h(u), h'(u))$$

Therefore, the derivative of the map $G : u \mapsto m(u)$ is:

$$R_{\theta+\pi/2}(h(u), h'(u)) + R_\theta(h'(u), h''(u))$$

Here we identify the derivative with respect to $u \in S^1$ and the derivative with respect to θ. This vector is just $R_{\theta+\pi/2}(h(u) + h''(u))$. Of course the tangent to the envelope is, at least when $h + h'' \neq 0$, the line D_u. The map G is the inverse of the Gauss map γ, and we have just proved that its jacobian is $h(u) + h''(u)$. The radius of curvature ρ of the envelope is then:

$$\rho(u) = \frac{1}{k(m(u))} = h(u) + h''(u)$$

The envelope is locally convex and closed, therefore it is the boundary of a compact convex body. Conversely, the condition $k > 0$ implies that there is only one point $m(u)$ on ∂Q satisfying:

$$< m(u)|u >= h(u) = sup < m|u >; m \in Q$$

Moreover, the Gauss map is invertible because $k \neq 0$. Therefore the tangents to the envelope can be parametrised by $u \in S^1$. Observe that D_u is orthogonal to u. The limits

$$lim_{\delta \to 0; \delta > 0} D_u \cap D_{u+\delta}$$

and

$$lim_{\delta \to 0; \delta > 0} D_u \cap D_{u+\delta}$$

exist because Q is convex. The point $m(u)$ has to be equal to both:

$$lim_{\delta \to 0; \delta > 0} D_u \cap D_{u+\delta}$$

and

$$lim_{\delta \to 0; \delta < 0} D_u \cap D_{u+\delta}$$

as, if any of these limits were different, the tangent at that point would also be D_u, which is impossible, as γ is a bijection. □

Remark: Using standard arguments in singularity theory, one can check that for a generic support function h, a plane hérisson will have only non degenerate cusps (where $R(u) = 0, R'(u) \neq 0$).

As an example the hérisson defined by the support function $h(\theta) = cos3\theta$ is pictured below

4.3 Minkowski sum and mixed volumes

The intersection of a compact convex body with one of its support lines D_u has to be convex, that is has to be a segment. Let us define the *Minkowski sum* of two convex bodies Q_1 and Q_2 by:

$$Q_1 + Q_2 = \{m_1 + m_2 | m_1 \in Q_1, m_2 \in Q_2\}$$

One verifies that the support line of $Q_1 + Q_2$ orthogonal to the vector $u \in S^1$ has the equation:

$$< m|u >= h_1(u) + h_2(u)$$

where $Q_1 + Q_2$ are the support functions of Q_1 and Q_2. In other words, $h_1 + h_2$ is the support function of $Q_1 + Q_2$. Of course scalar multiplication (homothety) is compatible with the Minkowski sum:

$$\lambda Q = \lambda_1 Q + \lambda_2 Q \ when \ \lambda_1 + \lambda_2 = \lambda, \lambda_1 \geq 0, \lambda_2 \geq 0$$

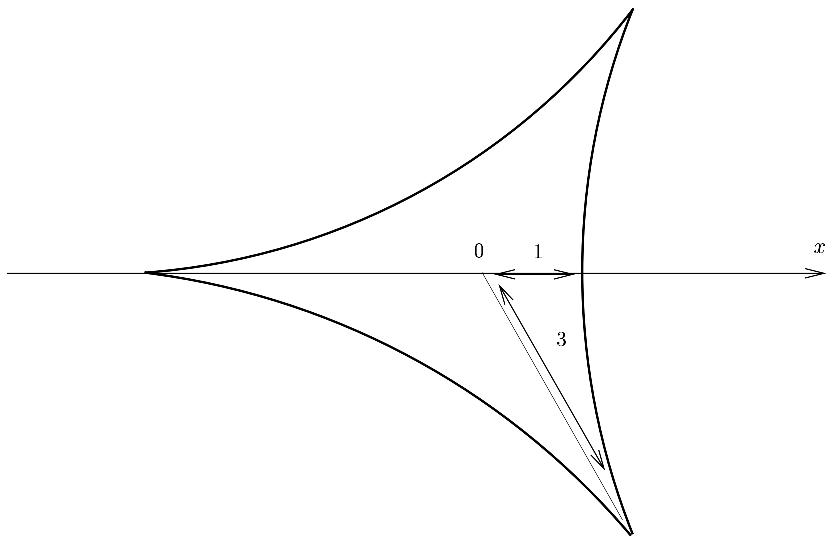


Figure 17: $\cos 3\theta$

Remark: When the two convex bodies have at every point of their boundary, a strictly positive curvature, the boundary of $Q_1 + Q_2$ is the set of points $\{m_1(u) + m_2(u), u \in S^1\}$.

Proposition 4.3.1 *The volume of the Minkowski sum $\lambda Q_1 + \mu Q_2$ is an homegeneous polynomial in λ and μ:*

$$vol(\lambda Q_1 + \mu Q_2) = \lambda^2 vol Q_1 + (\lambda \cdot \mu) V(Q_1, Q_2) + \mu^2 vol Q_2$$

Proof: We will compute the area of a convex body Q in terms of its support function h. Mixing the support function h and the arc length ds of the boundary ∂Q one gets (see fig)

$$vol(Q) = \int_{\partial Q} h ds$$

An unambiguous , but heavier notation would be:

$$vol(Q) = \frac{1}{2} \int_{\partial Q} h(N(c(s))ds$$

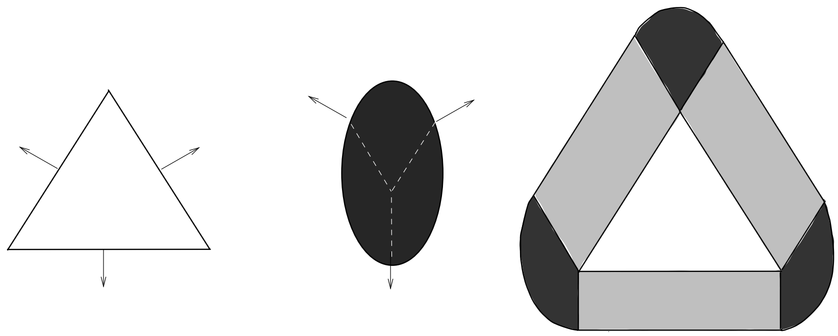


Figure 18: Minkowski sum of a triangle and a convex body of smooth boundary

where $c : S^1_{lenght(\partial Q)} \to \partial Q$ is a parametrisation by arc length of ∂Q, and $N(c(s))$ is the exterior normal at $c(s) \in \partial Q$. We have computed in terms of the support function the ratio between the arc length and the length swept by the normal:

$$\frac{ds}{du} = R(u) = |jac(G)| = h(u) + h''(u)$$

Here $R(u)$ denotes the radius of curvature of the envelope of the lines $< m|u >= h(u)$ at the caracteristic point $m(u)$. We get:

$$vol(Q) = \frac{1}{2} \int_{S^1} h(h + h'')du$$

Recalling that the support function of the Minkowski sum of $\lambda Q_1 + \mu Q_2$ is: $h_1 + h_2$ (h_1 and h_2 being the support functions of Q_1 and Q_2), we get :

$$vol(\lambda Q_1 + \mu Q_2) = \frac{1}{2} \int_{S^1} (\lambda h_1 + \mu h_2)[(\lambda h_1 + \mu h_2) + (\lambda h_1 + \mu h_2)'']du$$

$$= \frac{1}{2} \int_{S^1} (\lambda h_1)(\lambda h_1 + \lambda h_1'') + \frac{1}{2} \int_{S^1} (\mu h_2)(\mu h_2 + \mu h_2'') +$$

$$+ \frac{1}{2} \int_{S^1} (\lambda h_1)(\mu h_2 + \mu h_2'') + \frac{1}{2} \int_{S^1} (\mu h_2)(\lambda h_1 + \lambda h_1'')$$

The first two integrals are respectively $\lambda^2 vol(Q_1)$ and $\mu^2 vol(Q_2)$. The sum of the two last ones is $\lambda\mu$ times an integral mixing the two support

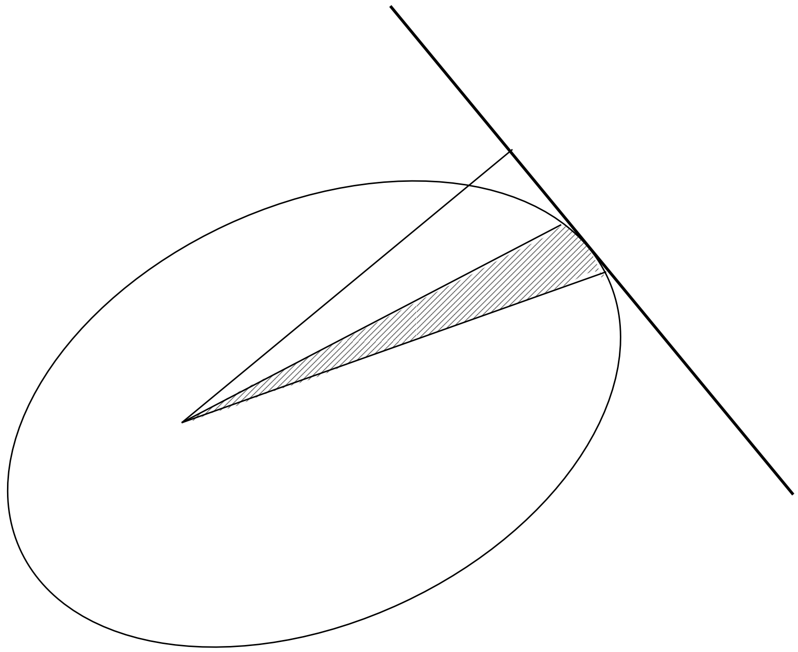


Figure 19: Area of a convex body in terms of the support function

functions and the two radii of curvature. This integral $V(Q_1, Q_2)$ is called the *mixed volume* of Q_1 and Q_2. □

We "see" the mixed volume on fig *Minkowski sum of a triangle and a convex of smooth boundary* above. It has also interesting interpretations in algebraic geometry see [Tei4].

4.4 Inequalities

Inequalities between functions of length, volume, mixed volume of convex bodies is a very rich topic, including isoperimetric inequalities. The interested reader can consult [Bo-Fe], [Schnei] for example.

5 Grassmann manifolds

5.1 Definition of vectorial and affine Grassmann manifolds

Let us now show that the set $G(n,p)$, called *Grassmann manifold*, of vectorial subspaces of dimension p of $\mathbb{R}^n$ has a natural structure of a $(n-p)\cdot p)$-dimensional manifold. Consider a p-dimensional subspace h_0 of $\mathbb{R}^n$. Let us denote by $h_0^\perp$ its othogonal subspace ($h_0^\perp$ has dimension n-p). Any p-dimensional subspace h of $\mathbb{R}^n$ transverse to $h_0^\perp$ is the graph of a linear map L_h from h_0 to $h_0^\perp$, and any such graph is a p-dimensional subspace transverse to $h_0^\perp$. Chosing bases in h_0 and $h_0^\perp$ the matrix of that map is a $p\times(n-p)$ matrix. This procedure defines a chart of $G(n,p)$. Using all the p-dimensional subspaces of $\mathbb{R}^n$, we get an atlas of $G(n,p)$. It is, in fact, enough to consider the $\binom{n}{p}$ p-dimensional coordinate subspaces to get an atlas.

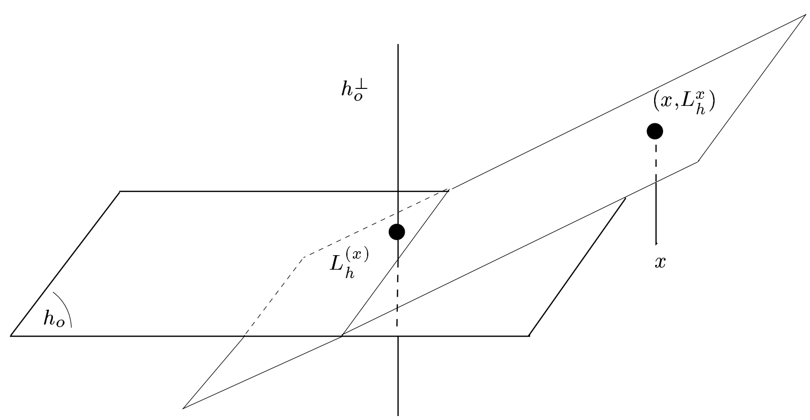


Figure 20: A chart of G(n,p)

Remark: The Grassmann manifold $G(n,1)$, that is the set of lines of $\mathbb{R}^n$, is the projective space $\mathbb{P}_{n-1}$. It is the quotient of the sphere S^{n-1} by the antipodal map, $q\mapsto(-q)$. The Grassmann manifold $G(n,n-1)$ is also diffeomorphic to $\mathbb{P}_{n-1}$ as you can see using the correspondance between a plane and its orthogonal line. Using the same diffeomorphism we can see that the hyperplanes containing a given line form a $\mathbb{P}_{n-2}\subset\mathbb{P}_{n-1}$. After defining a riemannian metric on $G(n,1)$ we shall see that the diffeomorphism $G(n,1)\to G(n,n-1)$ is also an isometry.

Using the action of the group of linear isometries on $G(n,p)$ we will prove that the Grassmann manifolds are compact.

Lemma 5.1.1 *The Group $O(n)$ of linear isometries of $\mathbb{R}^n$ is compact.*

Proof: The product $S^n \times S^n \times ... \times S^n$ (n times) is compact . The set of orthonormal bases of $\mathbb{R}^n$ is a closed subset of $S^n \times S^n \times ... \times S^n$, defined by the equations $< u_i|u_j >= 0$. It can be identified with the linear map which sends the canonical basis $(e_1, e_2, ..., e_n)$ to the orthogonal basis $(u_1, u_2, ..., u_n)$. The group O(n) is therefore a compact set. □

Theorem 5.1.2 *The Grassmann manifold $G(n,p)$ is homeomorphic to the quotient:*

$$SO(n)/SO(p) \times SO(n-p)$$

Proof: Let us first prove that the two sets are the same. The image by an element g of $O(n)$ of the p first vectors $(e_1, e_2, ..., e_p)$ of the canonical basis generate a p-dimensional subspace h of $\mathbb{R}^n$. Let us call

$$E_p : O(n) \to G(n,p)$$

this map. Let us now consider two isometries, $g_1 \in O(p)$ and $g_2 \in O(n-p)$. They determine an isometry $(g_1 \oplus g_2) \in O(n)$. The image of $g \circ (g_1 \oplus g_2)$ is again h. A subspace h of dimension p admits an orthogonal basis $(u_1, u_2, ..., u_p)$; the orthogonal $h^\perp$ admits an orthogonal basis $(u_{(p+1)}, ..., u_n)$ and the basis $(u_1, ..., u_n)$ is the image by a linear map of the form $g \circ (g_1 \oplus g_2)$ of the canonical basis $(e_1, ..., e_n)$. Therefore, the kernel of the map E_p is the subgroup $[(g_1 \oplus g_2)] \in O(n)$. This proves the set equality $G(n,p) = O(n)/[O(p) \times O(n-p)]$. To prove that the topologies coincide, one needs essentially to prove that the map from an orthogonal system $(u_1, ..., u_p)$ to the linear subset h it generates is continuous. This is easy, lengthy and boring, therefore we "leave that proof to the reader". □

Remark: Equally exciting is to prove that the topology on $G(n,p)$ obtained using the Haussdorff distance on the intersections of p-dimensional subspaces with the closed unit ball (or with the unit sphere) is again the same as the manifold topology.

Remark: The orthogonality in $\mathbb{R}^n$ provides a diffeomorphism between $G(n,p)$ and $G(n,(n-p))$. This diffeomorphism is an isometry for the riemannian metrics invariant by the action of the isometries we define below.

The set $\mathcal{A}(n,p)$ of affine p-dimensional subspaces form a fiber space over $G(n,(n-p))$ with fiber $\mathbb{R}^{(n-p)}$. The fibration map associates to a p-dimensional affine subspace H of $\mathbb{R}^n$ its orthogonal complement $h^\perp$. The intersection $H \cap h^\perp$ gives the isomorphism between the fiber and $\mathbb{R}^n$.

5.2 Metrics and measures

The group of linear isometries of $\mathbb{R}^n$ acts on $G(n,p)$. It is natural to look for a metric on $G(n,p)$ which is invariant by this action. To do that, first observe that our charts

$$\{linear\, maps\, h \mapsto h^{\perp}\}$$

give also the tangent space in h to $G(n,p)$. The euclidean metric of $\mathbb{R}^n$ allows us to choose an orthogonal basis in h and in $h^{\perp}$. Let us put on the $(p \times (n-p))$ matrix space the natural euclidean norm:

$$|\mathcal{M}|^2 = \sum (squares\, of\, the\, coefficients)$$

This defines on $G(n,p)$ a riemannian metric invariant by the action of the linear isometries. We leave as an exercise for the reader to check that the covering map from $S^{(n-1)}$ to $G(n,1)$ is a local isometry.

The volume measure associated to this riemannian metric is also invariant by the group of linear isometries.

Remark: The previous results can be rephrased in terms of homogeneous spaces. One then observes that the measure defined above is a quotient of the Haar measure on $O(n)$, and that the metric we defined on $G(n,p)$ is such that the projection $O(n) \to G(n,p)$ is a riemannian submersion. [Sa2]

6 The Gauss map and what can be done in higher dimensions and codimensions

6.1 The Gauss map and the principal curvatures

We consider first the case of an embedded hypersurface M of $\mathbb{R}^n$. It is then oriented (the normal vector $N(m)$ at $m \in M$ points out of the bounded component of $\mathbb{R}^n \setminus M$), we can define the Gauss map:

$$\gamma : M \to S^{n-1}, m \mapsto N(m)$$

We will also consider a projective Gauss map, also denoted by γ when there will be no ambiguity, using the line $L(m)$ normal at m to M:

$$\gamma : M \to \mathbb{P}_{n-1}, m \mapsto L(m)$$

Its critical values are images under the natural projection of the critical values of the (spherical) Gauss map and the critical points of both Gauss maps are the same. The *Gauss (or Gauss-Kronecker) curvature* $K(m)$ at $m \in M$ is the jacobian at m of the (spherical) Gauss map. The eigenvalues of $d\gamma(m)$: $k_1, k_2, ..., k_{n-1}$ (there may be repetitions) are called the *principal curvatures* of M in m. To each corresponds an eigenvector e_i , and these eigenvectors can be chosen to form an orthonormal basis.

On a neighbouhood of a point where the principal direction are non zero and all different, the vector fields defined by the eigenvectors can be integrated, giving lines of curvature. Geometrically, following a line of curvature the tangent hyperplane only rolls. In general it undergoes a mixture of pitch and rolling. On a surface embedded in $\mathbb{R}^3$, on a domain where the Gaussian curvature is strictly negative, the tangent plane only pitches along the asymptotic curves (curves which are everywhere tangent to vectors v such that $< d\gamma(v)|v >= 0$).

The *second fundamental form* $II(m)$ is defined by:

$$II(m)(v) =< d\gamma(m)(v)|v >$$

It can be diagonalised in an orthonormal basis, precisely the one we have chosen before to diagonalise $d\gamma$. The *symmetric functions of curvature* are the coefficients of the polynomial

$$det[Id + td\gamma(m)] = \prod(1 + k_1 t)(1 + k_2 t)...(1 + k_{n-1} t) = \sum_0^{n-1} \sigma_i(m) t^i$$

When possible, we shall drop the point m in $\sigma_i(m)$.

Remark: Consider an i-dimensional subspaces $h \subset T_m M$. In a neighbourhood of the point m the intersection $M \cap (h \oplus L(m))$ is an hypersurface of $h \oplus L(m)$. Denote by $K(m, h)$ the Gauss-Kronecker curvature of this last hypersurface, oriented by $N(m)$.

Proposition 6.1.1

$$\sigma_i(m) = const \cdot \int_{G(T_m M, i)} K(m, h)$$

where $G(T_m M, i)$ is the Grassmann manifold of i-dimensional subspaces $h \subset T_m M$ and const is a constant depending only on dimensions.

The proof amounts to compare the integral of the proposition with a trace of $\wedge^i(\gamma)$ acting on the exterior algebra $\bigwedge^i(T_m M)$.("folklore", [Lan5]).

We can locally write an equation for M:

$$x_n = f(x_1, x_2, ..., x_{n-1})$$

choosing the first (n-1) coordinates to be on axes generated by the vectors e_i and the last on the axis generated by the normal $N(m)$. Then the Hessian of f at m is a diagonal matrix with entries $k_1, k_2, ..., k_{n-1}$. This proves the:

Proposition 6.1.2 *The point m is a degenerate critical point of the orthogonal projection of M on the line $L(m)$ generated by the normal $N(m)$ if and only if m is a critical point of the projective Gauss map.*

Corollary 6.1.3 *The set of lines $L \in \mathbb{P}_{n-1}$ such that the projection p_L on L admits degenerated critical points is of zero measure.*

Proof: By Sard theorem, those lines, which are critical values of the projective Gauss map, form a set of measure zero. □

Generalzing the result of subsection **The Gauss map** about plane curves to hypersurfaces in $\mathbb{R}^n$, in particular surfaces in $\mathbb{R}^2$ amounts again to replace the computation of a curvature integral by a count of critical points. First suppose that M is an oriented hypersurface of $\mathbb{R}^n$. The Gauss map sends $m \in M$ to $N(m)$, the projective Gauss map sends m to the non-oriented normal $L(m) \in \mathbb{P}_{n-1}$. Let us observe that, even if M is not orientable, then the projective Gauss map and the absolute value $|K(m)|$ of the Gauss-Kronecker curvature still make sense.

Definition 6.1.4 *Let$L_0(M, L)$ be the number of critical points of the orthogonal projection p_L of M onto L. The notation emphazizes the zero dimensionality of the counting of critical points.*

(Often in the litterature the notation $|\mu|(M,L)$ is used). If the manifold M is oriented, we can compute the index of each critical point of the orthogonal projection on the line L_z generated and oriented by a vector $z \in S^{n-1}$. Let us define again:

$$L_0^+(M,N) = \sum_{m\, critical} (-1)^{index(m)}$$

If the dimension of M is even, the previous sum does not depend on the orientation of L, and indeed can also be defined without assuming that M is orientable.

The proof of the Exchange theorem of section 3 can be copied to get the:

Theorem 6.1.5 *Exchange theorem in codimension 1*

$$\int_M |K(m)|dm = \int_{\mathbb{P}_{n-1}} L_0(M,L)dL$$

When the previous integrals converge, and if either M is oriented, or M is even dimensional, an analogous equality , keeping track of signs, holds.

Theorem 6.1.6

$$\int_M K(m)dm = \int_{P_n} L_0^+(M,L)dL = \chi(M)$$

6.2 Lipschitz-Killing curvature

Suppose now that M is a submanifold of codimension $p > 1$ of $\mathbb{R}^N$. The dimension of M is n. We denote by $\mathcal{N}(M)$ the normal bundle of M and by $\mathcal{N}(m)$ its fiber: $(T_mM)^\perp \subset T_mM$. We can either

- Define a generalised Gauss map from the unit normal bundle $\mathcal{N}^1(M)$ of M to S^{N-1} by $\gamma(m,v) = w$. Denote by $K(m,v)$ its jacobian at the point $(m,v) \in \mathcal{N}^1(M)$. This makes sense as the unit normal bundle has a natural metric, induced by its embedding in $T\mathbb{R}^N$, which makes the bundle projection a riemannian submersion:

  $$\tilde{g}\mid_{fiber} = \text{restriction of the ambient euclidean metric}$$

  $$\tilde{g}\mid_{horizontal\ space} = \text{pull back of the metric of } M$$

  We also define the projective normal bundle $P\mathcal{N}(M)$ as the quotient of $\mathcal{N}^1(M)$ by the antipodal map on each fiber; we denote by $P\mathcal{N}(m)$ the fiber of this bundle.

1)The Lipschitz-Killing curvature of M at m is:

$$K(m) = 1/2 \int_{\mathcal{N}_1(m)} K(m,v)$$

When the dimension of M is even $K(m,v) = K(m,(-v))$ so we can write:

$$K(m) = \int_{\mathbb{P}\mathcal{N}(m)} K(m,v)$$

2)The absolute curvature of M at m is:

$$|K|(m) = \int_{\mathbb{P}\mathcal{N}(m)} |K(m,v)|$$

Notice that in general $|K|(m) \neq |K(m)|$.

- Consider, for each $v \in \mathcal{N}^1(m)$ the orthogonal projection $p_{m,v}$ of a neighbourhood of m on the subspace $T_m M \oplus \mathbb{R} \cdot v$. At m we can compute the Gauss-Kronecker curvature of the hypersurface $p_{m,v}$ (neighbourhood of m). Let us call it also $K(m,v)$. The Lipschitz-Killing curvature and the absolute curvature are then obtained by the same formula as above:

$$K(m) = 1/2 \int_{\mathcal{N}_1(m)} K(m,v)$$

and:

$$|K|(m) = \int_{\mathbb{P}\mathcal{N}(m)} |K(m,v)|$$

Proposition 6.2.1 *The two definitions of $K(m,v)$ given above coincide.*

Proof: Let us take a point (m,v) of the unit normal bundle. If $K(m,v) \neq 0$, locally, the inverse image by the Gauss map of

$$\mathbb{R} \cdot v \oplus T_m M$$

is an n-dimensional submanifold $\mathcal{V} \in \mathcal{N}^1(M)$ transverse at (m,v) to the fiber $\mathcal{N}^1(m)$. Observe that if (x,w) is a point of $\mathcal{V}$, the vector w is orthogonal to $p_{m,v}(T_x M)$ at $p_{m,v}(x)$. Let $J(x,w)$ be the jacobian of the projection of $T_{x,w}\mathcal{V}$ onto the horizontal space $\mathcal{H}$. Almost by definition of the horizontal space it is also the jacobian of the restriction to $T_{m,v}\mathcal{V}$ of the differential of the projection of the fiber bundle $\mathcal{N}^1(M)$ onto its base space M. Using the splittings :

$$T_{m,v}\mathcal{N}^1(M) = \mathcal{H} \oplus T_{m,v}(\mathcal{N}^1(m))$$

$$\mathbb{R}^N = T_m M \oplus \mathcal{N}(m)$$

the linear map $dG(m,v)$ has the matrix:

$$\begin{pmatrix} (dG(m,v)|_{\mathcal{H}}) & (0) \\ * & Id \end{pmatrix}$$

Therefore, using the first definition of $K(m,v)$:

$$K(m,v) = det(dG(m,v)|_{\mathcal{H}}$$

As

$$G|_{\mathcal{V}}(x,w) = w = \gamma(p_{m,v}(x))$$

One has, using the second definition of $K(m,v)$, which uses the projection $pm, v(M)$:

$$J(m,v)K(m,v) = detd(G|_{\mathcal{V}}(m,v)) = J(m,v).det(dG|_{\mathcal{H}})$$

□

An exchange theorem can now be stated in any dimension and codimension:

Theorem 6.2.2 *General exchange theorem*

$$\int_M |K|(m)|dm = \int_{\mathbb{P}_{n-1}} L_0(M,L)dL$$

Proof: Use the change of variable theorem for the map

$$G : \mathcal{N}^1(M) \to S^{N-1},$$

the first definition of the Lipschitz-Killing curvature, and use Sard's theorem as before. □

Example

Let C be a curve in $\mathbb{R}^3$. We will use the Frenet frame (T,N,B), $T(m)$ unit tangent vector to C in m given by the orientation, $N(m) = \frac{dT}{ds}$ and $B(m) = T(m) \wedge N(m)$. Let θ be the angle between a vector $v \in \mathcal{N}(m)$ and the principal normal $N(m)$ in the normal plane oriented by the base $N(m), B(m)$. then $K(m,v) = k(m) \cdot cos\theta$, where k(m) is the curvature of C at m. This proves:

Proposition 6.2.3 *For a space curve $C \subset \mathbb{R}^3$, the absolute curvature satisfies:*

$$|K|(m) = 2k(m)$$

Remark: Using our second viewpoint we can also associate to each projection $p_{m,v}(M)$ a second fundamental form $II_{m,v}$.

6.3 Total curvature of submanifolds

As in the previous section, M is an n-dimensional submanifold of codimension p of $\mathbb{R}^N$.

Definition 6.3.1 *The total curvature of M is :*

$$L_0(M) = \frac{1}{2|\mathbb{P}_{N-1}|}\int_M |K|$$

The constant is chosen in a way that round spheres Σ contained in an affine p-space of $\mathbb{R}^N$ satisfy $L_0(\Sigma) = 1$, extending the choice $L_0(point) = 1$,which one may view as the starting point of integral geometry!

Theorem 6.3.2 *Exchange theorem*

$$L_0(M) = \frac{1}{2|\mathbb{P}_{N-1}|}\int_{\mathbb{P}_{N-1}} L_0(M,L) = \frac{1}{2|\mathbb{P}_{N-1}|}\int_M |K|$$

where $L_0(M,L)$ is the number of critical points of the orthogonal projection of M on L.

Remark: ¿From now on the notation $L_0(M,L)$ is more convenient than the usual one: $|\mu|(M,L)$,as it will give a nicer form to the reproductibility property of the p-length functional (see chapter **Blaschke formulas and kinematic formulas**).

Proof: It reduces to an application to the generalised Gauss map:

$$\gamma : \mathbb{P}\mathcal{N}(M) \to \mathbb{P}_{N-1}$$

of the coarea formula:

$$\int_{\mathbb{P}_{N-1}} \sharp(\gamma^{-1}(L) = \int_{\mathbb{P}\mathcal{N}(M)} |jac\gamma|$$

A point $m \in M$ is a critical point of the orthogonal projection p_L on the line L if and only if L is contained in the normal space at m to M, that is, if and only if $L \in \mathbb{P}(\mathcal{N}(m))$. This shows that the number $\sharp\gamma^{-1}(L)$ is just the number $L_0(M,L)$. Finally observe that for almost all lines L, the orthogonal projection on L is a Morse function (see [Mi2]), which implies that

$$L_0(M,L) = \sharp(critical\,values\,of\,p_L)$$

□

In particular, for curves and surfaces immersed in $\mathbb{R}^3$ we get:

Proposition 6.3.3 *Let C be a curve in $\mathbb{R}^3$, then:*

$$L_0(C) = \frac{1}{2|\mathbb{P}^2|} \int_{\mathbb{P}^2} L_0(C, L)$$

This formula is usually written as:

$$\int_C k = \frac{1}{2} \int_{\mathbb{P}^2} |\mu|(C, L)$$

Proposition 6.3.4 *Let M be a surface immersed in $\mathbb{R}^3$, then:*

$$L_0(M) = \frac{1}{2|\mathbb{P}^2|} \int_{\mathbb{P}^2} L_0(M, L) = \frac{1}{2|\mathbb{P}^2|} \int_M |K|$$

This formula is usually written as :

$$\int_M |K| = \int_{\mathbb{P}^2} |\mu|(M, L)$$

7 Integral geometry and topology

The development of this chapter of integral geometry really started in 1949, although Fenchel's results [Fe1] $\int_C |k| \geq 2\pi$ were already proved in 1929.

7.1 Integral geometry of polyhedral surfaces in $\mathbb{R}^3$

The proof of a polyhedral Gauss-Bonnet theorem is easier that the proof of a smooth one, so we will start this chapter by Banchoff's proof of the Gauss-Bonnet theorem for polyedral surfaces, [Ban1]. Let us first define a polyhedral surface of $\mathbb{R}^3$. The basic pieces are closed plane triangles. Any triangle has in its boundary three edges and three vertices. Triangles, edges and vertices will be called *simplices*. A polyhedral surface is a union of triangles σ_i satisfying the following properties:

1. The interiors of the σ_i are disjoint.
2. The union of the σ_i is connected, and homeomorphic to a closed surface.
3. The intersection of two triangles is a simplex.

As the triangles are usual euclidean triangles, given a vertex $v \in \sigma$ we define the segment $e(v, \sigma)$ as the image of the edge of σ opposite to v by the homotethy of center v and ratio $1/2$.

The *link* of v is the union:

$$\mathcal{L}(v) = \cup e(v, \sigma); v \in \sigma$$

If q edges contain the vertex v, the planes containing an edge which contains v form q projective lines in$\mathbb{P}_2$. Let us call $\mathcal{C}$, (for critical) or $\mathcal{C}(v)$ the union of the projective lines defined previously. Any plane through v not belonging to $\mathcal{C}$ cuts $\mathcal{L}(v)$ in a finite number of points. If all the triangles containing the vertex v are in the same plane, for a plane $P \in P_2 \setminus \mathcal{C}$ one has

$$\sharp(\mathcal{L}(v) \cap P) = 2$$

The number $(\mathcal{L}(v) \cap P)$ is always even. It is natural to mesure how "non-trivial" the plane P is with respect to v by:

$$\phi(v, P) = (1/2)[2 - \sharp(\mathcal{L}(v) \cap P)]$$

We can now define the *extrinsic curvature* of v as the integral:

$$k(v) = \int_{P_2} \phi(v, P) dP$$

Intrinsically, that is inside the polyhedral surface $M = \cup\sigma_i$, at each vertex we can compute the intrinsic curvature $k(v)$ as the difference of 2π with the sum of the angles in v of the triangles which contain v.

$$k(v) = 2\pi - \sum_i \alpha(i, v); v \in f_i$$

The ambiguity between the two definitions we gave of $k(v)$ desappears with the following theorem:

Theorem 7.1.1 theorema egregium, (remarkable theorem in latin) *The intrinsic and the extrinsic way of computing $k(v)$ give the same result.*

Proof: Let us compute the measure of the planes which intersect one side e of $\mathcal{L}(v)$. In P_2 the length of the arc formed by the planes through v, intersecting e and orthogonal to the plane containing v and e is the angle α of the triangle containing v and e at e. The measure of the planes through v that intersect e is then $2\alpha_e$. In fig below we draw the corresponding set of oriented planes in S^2.

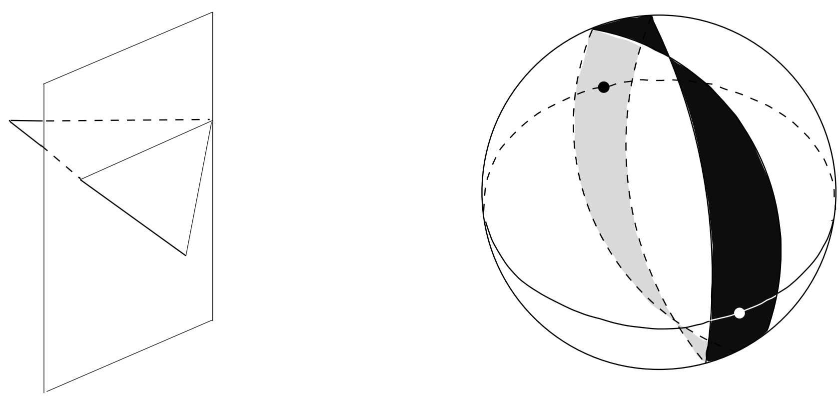


Figure 21: Oriented planes intersecting e as vectors in S^2

Summing on all the edges of $\mathcal{L}(v)$ we get :

$$\int_{P_2} \sharp(\mathcal{L}(v) \cap P)] = 2 \cdot \sum_{e \in \mathcal{L}(v)} \alpha_e$$

or :

$$\int_{P_2} \phi(v,P) = 2\pi - \sum \alpha_e$$

which is the relation we sought after between the extrinsic integral $\int_{P_2} \phi(v,P)$ and the intrinsic defect or excess of angle (compared to a point of the euclidean flat plane): $2\pi - \sum_{e\in\mathcal{L}(v)} \alpha_e$ □

We can now prove the polyhedral version of the Gauss-Bonnet theorem:

Theorem 7.1.2 (Polyhedral Gauss-Bonnet theorem)

Let M be a polyhedral surface embedded (or immersed) in $\mathbb{R}^3$ then its total curvature satisfies:

$$\sum_{v\ vertex\ of\ M} k(v) = 2\pi \cdot \chi(M)$$

Proof: Every triangle (face of M) has three edges, and, as M is a surface, every edge belongs to two faces. Let consider the set $\mathcal{D}$ of all pairs $e \in f$ of an edge contained in a face. There is a map between $\mathcal{D}$ and the set $\mathcal{F}$ of all faces and a map between $\mathcal{D}$ and the set $\mathcal{E}$ of all edges.

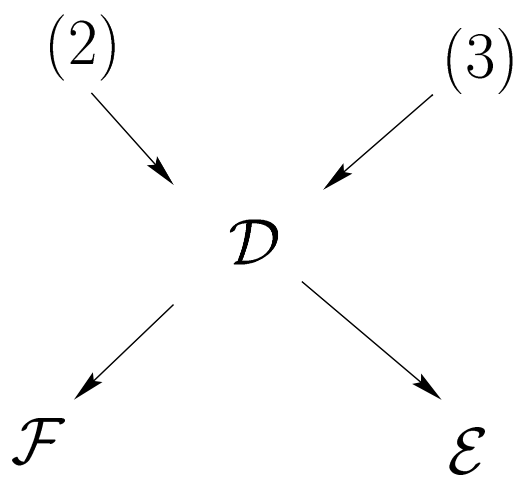


Figure 22: diagramme

By the first map, a face has three inverse images: the pairs formed by one edge of the face and the face itself. By the second an edge has two inverse images: the pairs formed by the edge and one of the two faces which contain it. Then:

$$\sharp\mathcal{D} = 3 \cdot \sharp\mathcal{F} = 2 \cdot \sharp\mathcal{E}$$

Let us denote the set of vertex of M by $\mathcal{V}$. The sum $\sum_{\mathcal{V}} k(v)$ is $2\pi\sharp\mathcal{V}M$ minus the sum of all the angles of the faces of M. It is then equal to $2\sharp\mathcal{V} - \pi \cdot (\sharp\mathcal{F})$. Adding $0 = \pi[3 \cdot \sharp\mathcal{F} - 2 \cdot \sharp\mathcal{E}]$, we get:

$$2\pi \cdot (\sharp\mathcal{F}) - 2\pi \cdot (\sharp\mathcal{E}) + 2\pi\sharp\mathcal{V} = \sum_{\mathcal{V}} k(v)$$

The first term is $2\pi \cdot \chi(M)$. □

7.2 Critical points and Gauss curvature, Chern and Lashoff's theorem

Generalzing the result of subsection **The Gauss map** about plane curves we have proved in section **The Gauss map and what can be done in higher dimensions** an exchange theorem for hypersurfaces in $\mathbb{R}^n$, in particular surfaces in $\mathbb{R}^2$.

The theorem of Chern and Lashoff is now a natural application of the exchange theorem [Ch-La] :

Theorem 7.2.1 *The total curvature $\int_M |K(m)|dm$ of a surface of genus g embedded or immersed in $\mathbb{R}^3$ is bigger or equal to $2\pi(2g+2)$. More generally, if M is a compact hypersurface immersed in $\mathbb{R}^n$, one has:*

$$\int_M |K(m)|dm \geq vol(P_{n-1} \cdot \sum_{i=1,\ldots,n-1} \beta_i(M)$$

Where the numbers β_i are the Betti numbers of M.

First we need a lemma:

Lemma 7.2.2 *For almost any line L (that is except for a measure zero set in P_{n-1}), the orthogonal projection of M on L is a Morse function.*

To prove the lemma the reader will need to check that the hessian of a local equation of M as a graph of a function from the tangent plane at m to the normal line at m coincides with the second fundamental form of M in M. Degenerated critical points of the projection on a line are then critical points of the Gauss map, and the critical values in P_{n-1} of the Gauss map form a subset of measure zero.

To prove the theorem we need only to integrate on P_{n-1} the Morse inequality [Mil2]:

$$L_0(M, L) \geq \sum_{i=1,\ldots,n-1} \beta_i(M)$$

When M is a surface $\sum_{i=1,\ldots,n-1} \beta_i(M) = 2g + 2$.

7.3 Total curvature of closed curves and knots

The first result in this line is Fenchel's theorem.

Theorem 7.3.1 *The total curvature of a closed curve C immersed in $\mathbb{R}^3$ satisfies:*

$$\int_C |k| \geq 2\pi$$

In 1949, independently, Fary, Fenchel and Milnor proved that "more topology implies more geometry". [Far] [Fe2] [Mil1] .

Theorem 7.3.2 *If the curve C is knotted (that is embedded and not the boudary of an embedded disc) then its total curvature satisfies:*

$$\int_C |k| > 4\pi$$

The proof of the first theorem and of the large inequality $\int_C |k| \geq 4\pi$ are a consequence of an easy topological argument.

Lemma 7.3.3 *The orthogonal projection p_L on the line L of an immersed curve C satisfies, if C is not in a plane orthogonal to L:*

$$L_0(C, L) \geq 2$$

If moreover C is knotted, and the projection p_L is a Morse function, then:

$$L_O(C, L) \geq 4$$

Proof: For all lines L (except one if the curve is planar) the projection p_L has at least one maximum and one minimum, so $L_0(C, L) \geq 2$. Let us now suppose that there exist a direction L such that p_L is a Morse function and such that $L_0(C, L) = 2$. Let a and b be the minimal and maximal values of the function p_L; let m_a and m_b be the corresponding critical points of p_L . Any plane P_t orthogonal to L in $a < t < b$ intersects the curve C transversly in exactly two points. Let I_t be the segment joining the two points $C \cap P_t$. The union :

$$x_a \cup \bigcup_{a<t<b} I_t \cup x_b$$

is an embedded disc with boundary C, and C cannot then be knotted. □

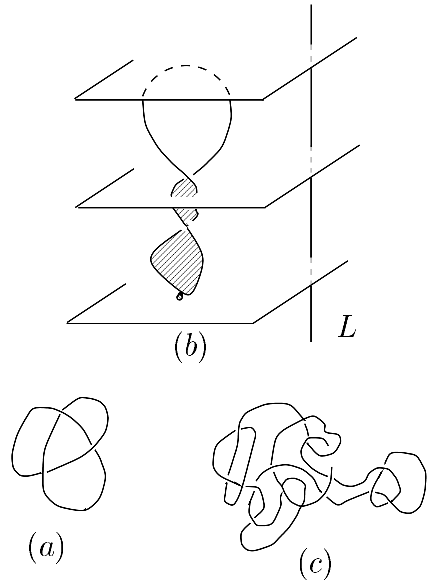


Figure 23: La projection sur une droite générique d'une courbe nouée a plus de deux points critiques

7.4 More theorems involving the topology of an immersion or of an embedding

the next question concerns embeddings of surfaces in $\mathbb{R}^3$: Does the topology of the embedding force "more geometry"? In particular do topological conditions imply a lower bound for the total curvature of a submanifold? The answer is often yes.

Let us state a result of this type about tori in $\mathbb{R}^3$.

We will add a point "at infinity" to $\mathbb{R}^3$ to get the compactification S^3. Naturally an embedding in $\mathbb{R}^3$ can then be considered simultaneously as an embedding in S^3. In $\mathbb{R}^3$ a torus of revolution bounds a thick torus $S^1 \times D^2$. Completing $\mathbb{R}^3$ with a point a infinity, the same torus bounds two thick tori.

To obtain an example analytically , one can see the sphere S^3 as the unit sphere of $\mathbb{C}^2$ (we write $|z_1^2| + |z_2^2| = 1$ the equation of S^3). Then the equation $|z_1^2| = |z_2^2| = 1/2$ defines a torus which bounds the two thick tori:

$$|z_1| \leq 1/2 \, and \, |z_2| \leq 1/2$$

We will call this torus the *Clifford torus.*

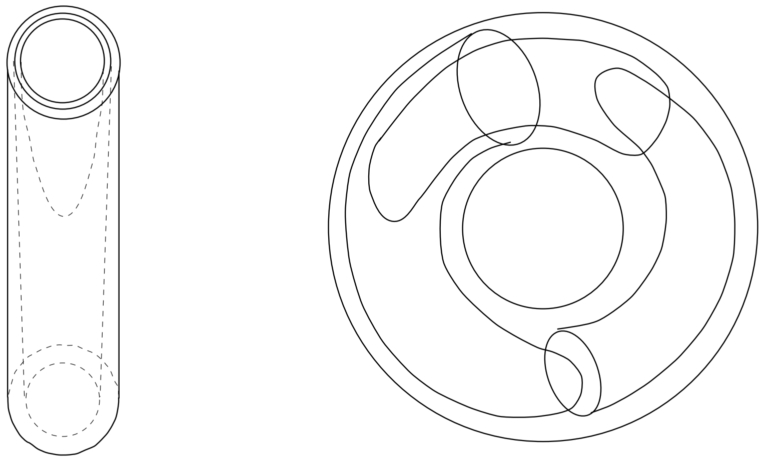


Figure 24: A torus of revolution bounding to thick tori in S^3

We will here define a torus embedded in S^3 as *unknotted* if it bounds to solid tori. One can prove this implies the existence of an isotopy with the Clifford torus.

Recall also that, if cutting a three dimensional manifold N with boundary a torus T by an embedded disc with boundary a circle contained in T, one get a three dimensional ball B^3, N is a thick torus $D^2 \times S^1$.

Figure 25: From a ball to a thick torus.

Theorem 7.4.1 *[La-Ro1]. Let T be a torus embedded in $\mathbb{R}^3$. If T is knotted, then*

$$\int_T |K(m)|dm \geq 16\pi$$

Recall that Chern-Lashoff's theorem proves that for any immersed torus one has:

$$\int_T |K(m)|dm \geq 8\pi$$

Using the exchange theorem, we need to prove the following inequality:

Lemma 7.4.2 *If there exist a generic direction L such that the number of critical points $L_0(T,L)$ of the orthogonal projection of T on L satisfies $L_0(T,L) \leq 6$ then T is not knotted.*

Proof: Suppose there exists a direction L such that $L_0(T,L)$ is 4 or 6. To get indices in $\mathbb{N}$, let us fix the orientation of L. The proof is easier when $L_0(T,L)$ is 4. Let T_t be the set $(p_L)^{-1}(-\infty,t]$. Suppose that the four critical values are $a<b<c<d$, and the corresponding critical points m_a, m_b, m_c, m_d.

Elementary Morse theory [Mil2] shows that

1. For $a<t<b$, T_t is homeomorphic to a disc. The boundary $p_L^{-1}(t)$ of T_t is a circle.

2. For $b<t<c$, T_t is homeomorphic to a cylinder. The boundary $p_L^{-1}(t)$ of T_t is then two circles.

3. For $c<t<d$, T_t is homeomorphic to a torus minus a disc. The boundary of T_t is again a circle.

There is no critical value of p_L between b and c, therefore the interior of $T_c \setminus T_b$ is the product of $p_L^{-1}(t), b<t<c$ by an interval, that is union of two cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$. In the closure of $\mathcal{C}_1$ we can choose a monotonous arc α_1(for the projection on L), joining m_b to m_c, and in the same way an arc α_2 in $\mathcal{C}_2$. The union of these two arcs is a closed curve α such that its orthogonal projection on L has two critical points.

We recalled (case 2) that the intersection of T with the plane $P_t b<t<c$ orthogonal to L at the point t is the union of two disjoint circles $C_{t,1}$ and $C_{t,2}$. In the plane P_t one, at least, is innermost and therefore bounds a disc D_1.

We can then construct in T_t an embedded arc c_t joining the two curves $C_{t,1}$ and $C_{t,2}$ and meeting them only at its end points. It is convenient to start at the critical point m_b and conctruct in each of the connected components of $T_t \setminus T_b$ which are cylinders a monotonous (for the orthogonal projection p_L on L) arc joining m_b to respectively $C_{t,1}$ and $C_{t,2}$. Then the curve c_t, union of those two arcs, intersect each plane $P_\tau b<\tau \leq t$ in two points. At each level, those two points can be joined by an “horizontal” arc contained in the component of $S^3 \setminus T$ which does not contain D_1. Therefore , when t goes from b to c, the arc c_t sweeps a disc D_2 (see picture “construction of discs in the complement of T”) .

The boundary of D_1 and the boundary of D_2 intersect in one point. Both boundary curves then are non zero in $H_1(T)$. So if we cut the component of $S^3 \setminus T$ along D_1 we get a ball B^3, proving that the component was a solid torus $S^1 \times D^2$. Cutting the other component of $S^3 \setminus T$ along D^2 we get another ball B^3 proving that the second component of $S^3 \setminus T$ is also a solid torus.

We need now to find similar discs in the two components of $S^3 \setminus T$ with the weaker hypothesis $L_0(T, L) = 6$. Now the Morse function p_L has six critical points $m_1, m_2, .., m_6$. With no loss of generality we can suppose that the critical values are $1, 2, ..., 6$. The intersection $P_{i+1/2} \cap T$ is a disjoint union of closed curves embedded in T. Let $n(i + 1/2)$ be the number of connected components of the intersection $P_{i+1/2} \cap T$. There are two possibilities for the sequence $n(i + 1/2), 1 \leq i \leq 5$: $(1, 2, 1, 2, 1)$ and $(1, 2, 3, 2, 1)$.

Let us first consider the case $(1, 2, 1, 2, 1)$. Let $C = P_{3+1/2} \cap T$. Since the middle section C is a simple closed curve on T, it separates T into two connected components: A and B. One , say is $A = T \setminus \mathit{open\, disc}$, and the other $B = D^2$ an open disc. Suppose that the level $P_{1+1/2}$ is contained in A, and denote by C_a and C_b the two connected components of $P_{2+1/2} \cap T$. Let α_1 be an arc from a point of $P_{1+1/2}$ to C intersecting C_a which satisfies: $(p_L) \circ \alpha_1)'(t) < 0, 1 + 1/2 < t < 3 + 1/2$. Similarly let α_2 be an arc joining $P_{1+1/2}$ to C and intersecting C_b. Let α be the union $\alpha = \alpha_1 \cup \alpha_2$. See fig below

As in the easy case $L_0(T, L) = 4$, we can construct two embedded discs, the interior of which meet just one component of $S^3 \setminus T$. In the plane $P_{2+1/2}$, one, at least, of the curves C_a and C_b bounds a disc D_1. Suppose then $C_a = \partial D_1$. This disc is contained in one of the components of $S^3 \setminus T$. The curve α, following the same proof as in the case $L_0(T, L) = 4$, bounds a disc D_2 contained in the other component of $S^3 \setminus T$.

Again, the boundary of D_1 and the boundary of D_2 intersect in one point. Both then are non zero in $H_1(T)$. So if we cut the component of $S^3 \setminus T$ along D_1 we get a ball B^3, proving that the component was a solid torus. Cutting the other component of $S^3 \setminus T$ along D^2 we get another ball B^3 proving that the second component of $S^3 \setminus T$ is also a solid torus.

Let us now consider the case $(1, 2, 3, 2, 1)$. Let A be the part of T above $P_{3+1/2}$ and B be the part below. If B were to contain only critical points of index 0, A would contain two critical points of index 1 and one of index 2, to guarantee the connectedness of T. Then T would be a sphere. Therefore,we know that B contains a point of index 1. As $P_{3+1/2} \cap T$ is three closed curves, B has to contain two critical points of index 0 (one is m_1). B has two connected components. If A were not connected, it would contain one critical point of index 1 at most, and inspection will show that T would be one or two spheres. Therefore we know that A is connected and contains two critical points of index 1.

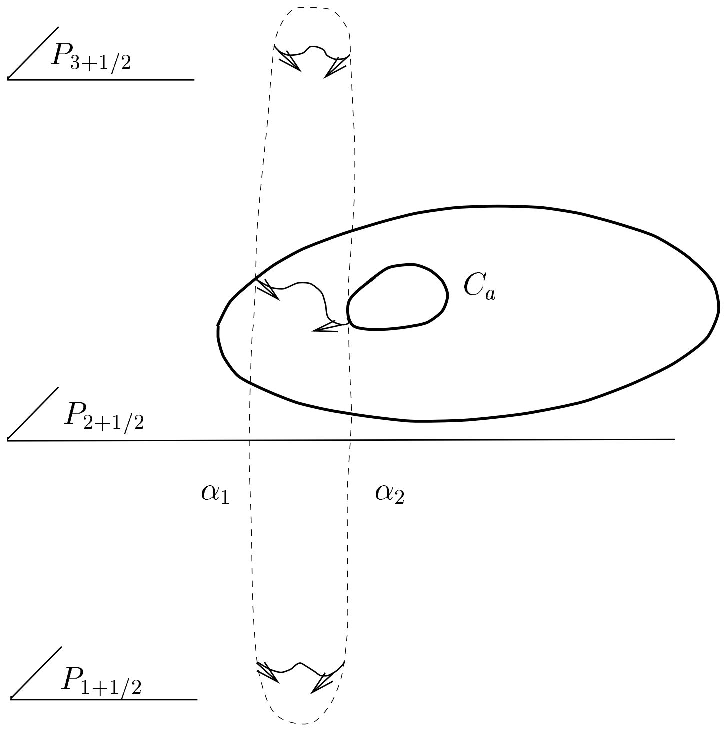
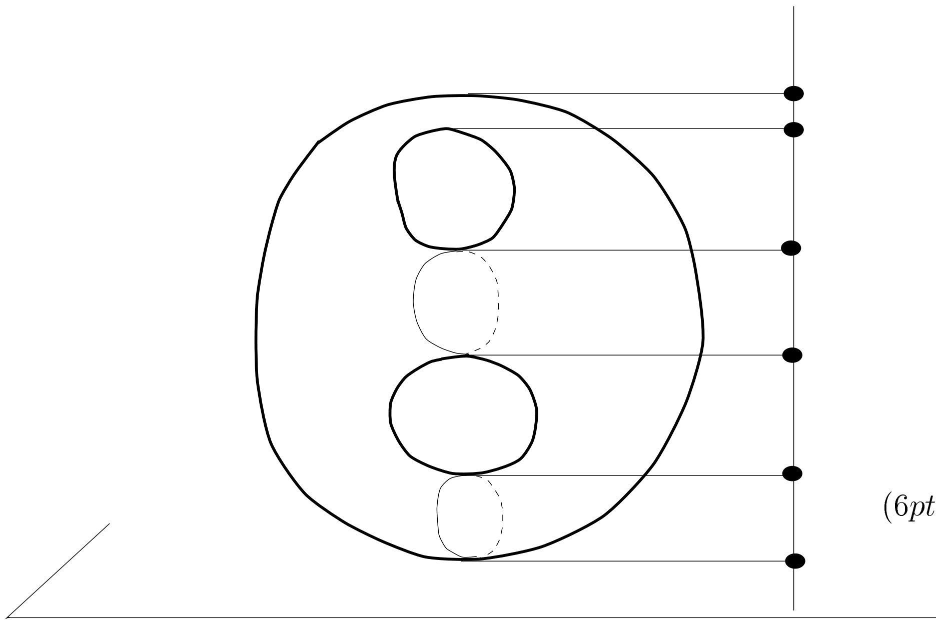


Figure 26: Construction of discs in the complement of T.

Let C_a, C_b, C_c be the three components of $P_{3+1/2} \cap T$, labelled so that C_a and C_b do not bound a disc in B. C_a and C_b are then both generators of $\pi_1(T)$. Let P be the one point compactification of $P_{3+1/2}$. One of the circles C_a, C_b, say C_a, bounds a disc D_1 in P whose interior does not meet $C_b \cup C_c$. Then the connected component of $S^3 \setminus T$ containing D_1 is a solid torus. As before we can construct an arc α such that the restriction of p_L to α has only two critical points, and which meets C_a and C_b in one point. It bounds a disc D_2 which contains an embedded arc joining C_a to C_b in P meeting C_a and C_b only at its endpoints. The disc D_2 is then contained in the other component of $S^3 \setminus T$, bounds the nontrivial curve α on T, so that the other component is also a solid torus, proving that T is unknotted. □

In a similar way we can prove the

Theorem 7.4.3 *Let S be a surface of genus 2, and suppose that one of the Morse projection p_L has six critical points on S, then it is unknotted which* *means that it is isotopic to the surface of fig below.*



$(6pts)$

Figure 27: Standard embedding of a surface of genus two

The proof can be found in [La-Ro1] .

7.5 The equality case: tight immersions

The theory of tight immersions started whith N.H.Kuiper's article [Kui1] in 1960. It was followed by many others. Good references are also [Kui2] [Kui3].

When $g = 0$, we see that the total curvature of a sphere satisfies:

$$\int_M |K| \geq 4\pi$$

Gauss-Bonnet's theorem implies that:

$$\int_M K = 4\pi$$

Therefore, when the total curvature of the sphere M is 4π, the Gaussian curvature has to be everywhere nonnegative, which implies that M is the boundary of a convex body.

Definition 7.5.1 *Tight immersions are immersions which achieve equality in the theorem of Chern and Lashoff:*

$$\int_M |K| = 2\pi(2g+2)$$

where g is the genus of the oriented surface M.

To avoid heavy notation we will denote by M the surface and its image by the immersion. To feel more comfortable the reader may suppose M is embedded. Let us denote $\mathcal{H}(M)$ the *convex hull* of M:

$$\mathcal{H}(M) = \{\lambda_1 m_1 + \lambda_2 m_2\,;\, \lambda_i \geq 0\,;\, \lambda_1 + \lambda_2 = 1\}$$

Let us call the *convex envelope* of M the boundary $\partial\mathcal{H}(M)$ of the convex hull of M.

Let $z \in S^2$ be a unit vector, and let p_z be the orthogonal projection of M on the oriented line L_z generated by z.

Definition 7.5.2 *The topset $Top(M,z)$ of M in the direction z is the intersection of M with the plane of equation:*

$$< z|m >= max_M(< z|m >)$$

Of course we can start with curves in $\mathbb{R}^2$.

In that case, tight closed curves are just convex closed curves. A fancy proof of that give half of the idea that will be used for surfaces.

Affirmation

The top set of a tight curve C is connected.

proof

First notice that :

For almost every $z \in S^1$, the topset $Top(C,z)$ is a point, the only $m \in C$ where $N(m) = z$. If that is not the case the function $|\mu|(C,z)$ would be ≥ 3 for a non zero measure set of S^1 providing, using the exchange theorem,a contradiction.

If $Top(Z,z)$ is not connected, then one can find two disjoint open intervals of finite length on the curve, each containing points of this topset. Let $A =< m, z\, for\, m \in Top(C,z)$. The values of $< \cdot|z >$ on the four extremities of the two intervals are less than $A - \epsilon$, $\epsilon > 0$. Let tilt z by a very smalle angle β. For β small enough, we get a direction z' such that

$$A - \frac{\epsilon}{2} < m|z' > \leq A + \frac{\epsilon}{2}\, for\, m \in Top(C,z).$$

Therefore the functions $< m|z' >$ have at least two local maxima, one in each of the intervals we just defined, and then, $\mu|(C,z') \geq 3$.

Let us now come back to surfaces and discriminate the indices of the critical points of a projection p_z.

Definition 7.5.3

$$\mu_2(z) = \sharp\{critical\, points\, of\, index\, 2\, of\, p_z\}$$

$$\mu_{0,2}(z) = \sharp\{critical\, points\, of\, index\, 0\, or\, 2\, of\, p_z\}$$

$$\mu_1(z) = \sharp\{critical\, points\, of\, index\, 1\, of\, p_z\}$$

When we need to specify the surface M, or nonzero measure subset v of a surface where we count critical points we write:

$$\mu_{0,2}(M,z)\, \mu_{0,2}(v,z), \mu_1(M,z)\, or\, \mu_1(v,z)$$

When a point $m \in M$ has positive curvature, the Gauss map is a diffeomorphism from a neighbourhood v of m on its image $\gamma(v) \subset S^2$. Moreover:

$$\int_{\gamma(v)} \mu_{0,2}(v) = \int_v K = \int_v |K|$$

Similarly we get in a small enough neighbourhood of a point of negative curvature:

$$\int_{\gamma(v)} \mu_1(v) = -\int_v K = \int_v |K|$$

Remark: As by hypothesis M is tight, one has:

$$\frac{1}{2}\int_{S^2} \mu_{0,2}(z) = \int_M K = \int_v |K| = 4\pi$$

where if $K > 0; K^+ = K$, if $K \leq 0; K^+ = 0$ and:

$$\frac{1}{2}\int_{S^2} \mu_1(z) = 4\pi g$$

Proof: Rephrasing the theorem of Chern and Lashoff one gets:

$$\frac{1}{2}\int_{S^2} \mu_{0,2}(z) = \int_M K^+$$

and:

$$\frac{1}{2}\int_{S^2} \mu_1(z) = \int_M K^-$$

where if $K < 0; K^- = -K$, if $K \geq 0; K^- = 0$

We know that the projection p_z should have at least a maximum and a minimum, which have to belong to the convex envelope $\partial H(M)$. There cannot exist more that two critical points of p_z where the Gauss curvature is positive, otherwise:

$$\int_{S^2} \mu_{0,2}(z) > 4\pi$$

which will contradict tightness.

It follows that the point $m \in M$ where $K > 0$ must belong to the intersection of the envelope of M and M as:

$$\int_{\partial H(M)} |K| = 4\pi$$

At a point m of $\partial H(M)$ the Gauss curvature has to be nonnegative , as it is a maximum of the function $p_{N(m)}$. □

Lemma 7.5.4 *For almost every $z \in S^2$, the topset $Top(M, z)$ is a point, the only $m \in M$ where $N(m) = z$*

Proof: If that is not the case the function $\mu_{0,2}(z)$ would be ≥ 3 for a non zero measure set of S^2 providing, using the exchange theorem,a contradiction. □

Lemma 7.5.5 *Let $Top(M, z)$ be the topset of the immersed surface in the direction z, and let h be the maximum of the orthogonal projection p_z on the oriented axis defined by z ($h = p_z(Top(M, z))$).*

Let W be a compact "isolated" subset of the topset in the direction z of an immersion of a compact surface M,that is a piece of $Top(M, z)$ which admits an open neighbourhood U such that, for a positive ϵ

$$sup_{\partial U} p_z \geq h - 3\epsilon$$

Then we can follow the piece W in U when we move z' in a neighbourhood of z. More precisely there exists a neighbourhood $v(z)$ of $z \in S^2$ such that for almost any $z' \in v(z)$

$$\mu_2(v(z), z) \geq 1$$

Proof: The function $p_z(m)$ is continuous on $M \times S^2$, so if we chose $m \in \overline{U}$, there exists a neigbourhood $v(z) \subset S^2$ such that, for $z' \in v(z)$

$$|p_z(m) - p_{z'}(m)| < \epsilon$$

in particular for $m \in W$

$$p'_z(m) \geq p_z(m) - \epsilon = h - \epsilon$$

and for $m \in \partial U$

$$p_{z'}(m) \leq p_z(m) + \epsilon \leq h - 3\epsilon + \epsilon = h - 2\epsilon$$

This implies that the point in $\overline{U}$ where $p_{z'}$ takes its maximum value does not belong to the boundary, but to the interior. The conclusion follows now from the fact that for almost all $z \in S^2$ all critical points of p_z are non degenerate. □

Corollary 7.5.6 *Any topset $Top(M, z)$ of a tight immersed surface is connected.*

The next step of the proof is motivated by the idea that in some sense a topset of a tight immersion has to be tight, in fact a point, a disc bounded by a plane convex curve or a planar domain bounded by convex curves.

To prove such a result it is natural to consider the topset of a topset (*toptopset*).

Let $Top(M, z_1)$ be the topset of the immersion M in the direction z_1. It is contained in a plane orthogonal to z_1. We can construct the toptopset $Top((Top(M, z_1)z_2)$.

Lemma 7.5.7 *If M is a tight immersed surface, then the toptopset associated to two orthogonal vectors (z_1, z_2), $Top((Top(M, z_1)z_2)$ is connected.*

Proof: Let again h be the value $p_{z_1}(Top(M, z_1)$ and let h^* be the value $p_{z_2}(Top(Top(M, z_1), z_2))$.

Suppose that $Top((Top(M, z_1)z_2)$ is the union of two disjoint closed sets W_1 and W_2. Chose two open neigbourhoods $U_1 \subset M$ and $U_2 \subset M$ of W_1 and W_2 in M, with disjoint closure.

If a point $m \in \partial U_i$ is in $Top(M, z_1)$, it satisfies $p_{z_1}(m) = h$. As $Top(Top(M, z_1), z_2)$ is closed in the open set $U_1 \cup U_2$ it does not contain any point of $\partial U_1 \cup \partial U_2$, so a point m of $\partial U_i \cap Top(M, z_1)$ satisfies $p_{z_2}(m) < h^*$. This implies that the function p_{z_1} does not take the value h on the any of the closed sets

$$\partial U_i \cap \{m | p_{z_2}(m) \geq h^*\}$$

(see picture below)

Hence the function p_{z_1} achieves on $(\partial U_1 \cup \partial U_2) \cap \{m | p_{z_2}(m) \geq h^*\}$ a maximal value, strictly smaller than h, that we will note $h - 3\epsilon; \epsilon > 0$

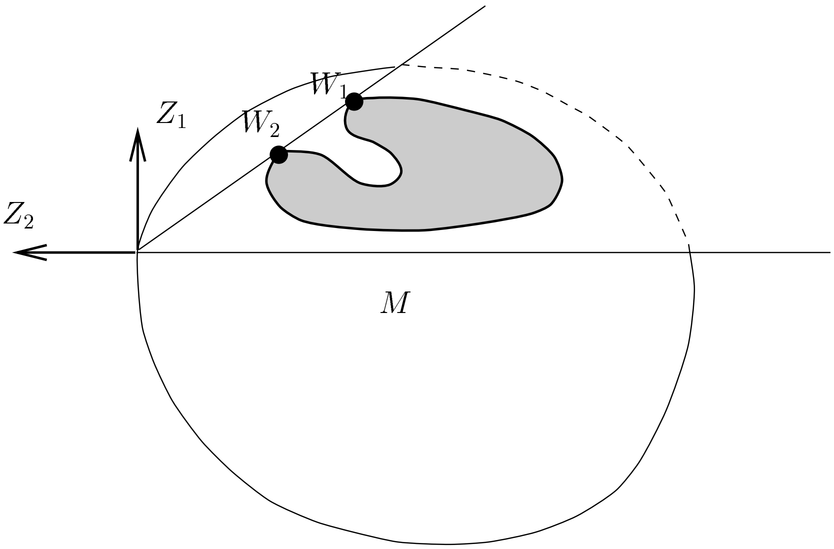


Figure 28: Toptopset, p_{z_1} and p_{z_2}

Let us now tilt the $z_1 - axis$ of a very small angle $0 < \eta < \pi/2$ in the direction z_2. We observe that the function p_z, where $z = z_1 cos\eta + z_2 sin\eta$, has for small enough η two local maxima. We chose η small enough to have, for any $m \in M$, $|p_z(m) - p_{z_1}(m)| < \epsilon$.

Let us now study the function p_z on the open set:

$$U_i^* = \{m | p_{z_2}(m) > h^*\} \cap U_i$$

and on its closure $\overline{U}_i^*$.

Affirmation The function p_z takes its maximal value on $\overline{U}_i^*$ in the interior U_i^*.

Let us now take a point $m \in \partial U_i^* \cap \{m | p_{z_2}(m) \geq h^*\}$, we have:

$$p_z(m) \leq p_{z_1}(m) + \epsilon \leq (h - 3\epsilon + \epsilon = h - 2\epsilon$$

The value of p_z at a point $w \in Top(Top(M, z_1), z_2)$ is $cos\eta h + sin\eta h^*$ (as $p_{z_1}(w) = h$ and $p_{z_2}(w) = h^*$). Then for a point $m \in \partial U_i^* \cap \{m | p_{z_2}(m) \geq h^*\}$ we have, as $p_{z_2}(m) \leq h^*$ and $p_{z_1}(m) \leq h$:

$$p_z(m) \leq p_z(w)$$

Hence the restriction to ∂U_i^* of p_z takes its maximal values at the points of $W_1 \cup W_2$. Take again a point $w \in W_i$, we can construct a differential

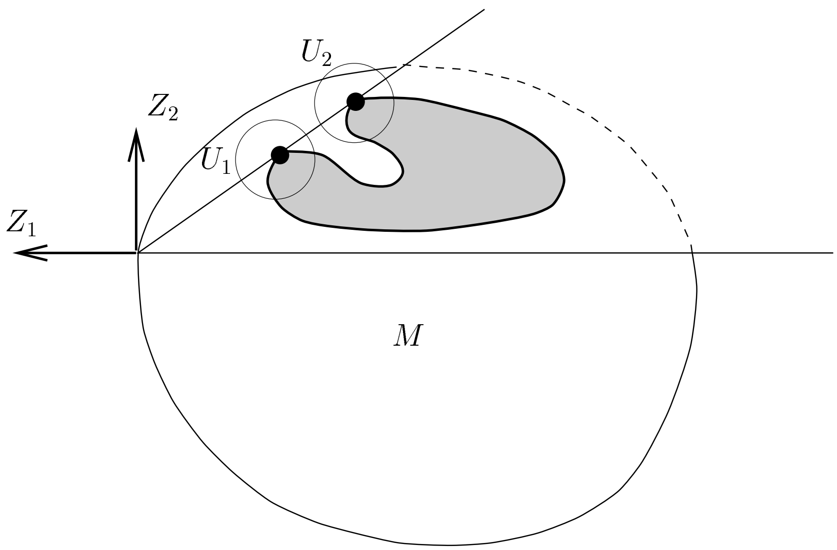


Figure 29: A toptopset is connected.

curve $c(t), c(O) = w, \forall t \in [0,1]\, c(t) \in \overline{U}_i^*$ starting at w tangent to z_2); we can suppose $\frac{dc}{dt}(0) = z_2$. The curve is normal at w to $Top(Top(M, z_1), z_2)$ and tangent to the plane containing $Top(M, z_1)$. As $\frac{dp_{z_2}(c)}{dt} = 1$, we can compute:

$$\frac{dp_z(c)}{dt} = (sin\eta) \cdot 1 + (cos\eta) \cdot 0 = sin\eta > 0$$

As the curve $c(t)$ goes from $w = c(0)$ to the interior of $\overline{U}_i^*$ (which is also the interior of U_i^*), and as the function p_z is strictly increasing along that curve, the function p_z has in U_i^* values which are greater than the maximal value $p_z(w)$ achieved on $\partial\overline{U}_i^*$. Therefore the restriction of p_z to U_i^* has a topset in the interior of U_i^* (for i=1,2). This implies that:

$$\mu_2(z) \geq 2\, and\, \mu_{2,0} \geq 3$$

and again contradicts tightness. □

Corollary 7.5.8 *For any tight immersion of a surface in $\mathbb{R}^3$ the topset in any top plane contains its convex envelope (the boundary of its convex hull) in this plane.*

Using local maxima of the restriction to $Top(M, z_1)$ of p_{z_2} we prove the same way the:

Corollary 7.5.9 *The topset $Top(M, z_1)$ is either a point, a convex closed curve or a planar domain with boundary convex closed curves.*

When the topset $Top(M, z_1)$ is not a point let call *top 1-cycle* the outer convex curve in $\partial Top(M, z_1)$. If the topset is a disc, we will say that the top-cycle is *simple*

Lemma 7.5.10 *Let M be a tight surface and $U \subset M$ a topological disc; we denote by $M \setminus U$ the complement of U in M. Suppose that the boundary $C = \partial U$ is a top 1-cycle associated to the topset $Top(M, z_1)$. Then either U or $M \setminus U$ is the plane interior $Int(conv(C))$ of the plane disc bounded by C.*

Proof: Let us suppose U is a topological disc. If U is $Int(conv(C))$, then:

$$\int_U |K| = 0$$

If not, for z_1 or $-z_1$, U has a topset contained in its interior, providing an open set of direction z such that $Top(U, z)$ is a point contained in the interior of U. Then:

$$\int_U |K| > 0$$

Replacing U by the plane disc of boundary C will then decrease stricly the total curvature of M (the new immersion is a priori only $\mathcal{C}^1$ but we can smooth it increasing as little as we want the total curvature , of less than $\frac{1}{2}\int_U |K|$, and keep the contradiction, even in the smooth frame). □

We have proved the:

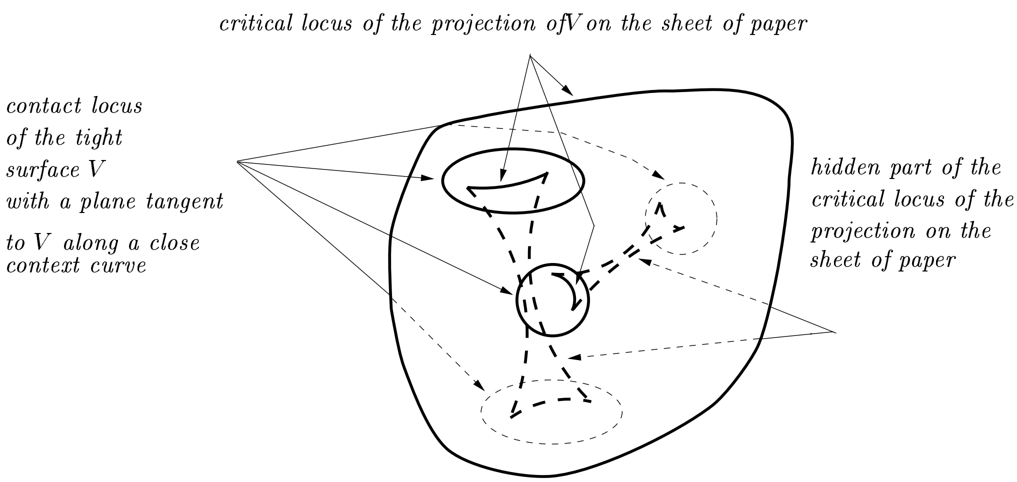
Theorem 7.5.11 *An immersed tight orientable surface is obtained from the boundary N of a convex body by replacing a finite (≥ 2) number of convex plane discs by surfaces of negative curvature contained in the convex hull of N of boundary the convex plane curves boundary of the previous discs.*

Remark: A torus of revolution is a tight embedded torus.

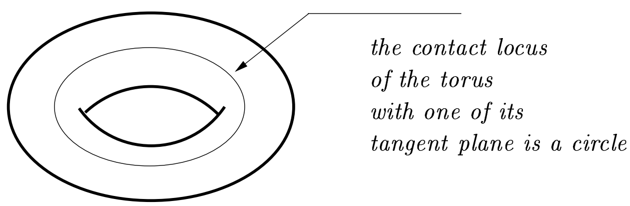
One can also construct immersed and non embedded tight tori. [Lan5] the idea is to construct a ruled surface (with double points) spanned by segments the extremities of which belong to two plane convex curves situated in parallele planes. The end points of the segments are chosen using the two Gauss maps of the curves to spin them properly.

With more topology one can prove that the projective plane and the Klein bottle do not admit tight immersion in $\mathbb{R}^3$. [Kui1].

Remorse We have not said much about polyhedral surfaces. An important difference with smooth surfaces is the fact tightness is not equivalent to the *two piece property.*



$a)$ *Example of a tight genus 3 surface*



$b)$ *The revolution torus* T *is tight*

Figure 30: A tight embedded surface.

Definition 7.5.12 *A compact subset A in $\mathbb{R}^N$ satisfies the two piece property if any affine hyperplane separates A in at most two connected components.*

A good reference to start the study of polyhedral surfaces is Banchoff's article [Ban1].

8 Higher dimensional convex bodies and related matters

8.1 Support function

As in the case of $\mathbb{R}^2$ let us consider a function $H : S^{(n-1)} \to \mathbb{R}$. To such a map corresponds a family (parametrised by $S^{(n-1)}$) of hyperplanes of $\mathbb{R}^n$:

$$u \mapsto h = \{x| < x|u >= H(u)\}$$

We have observed in part 4 (Prop 4.2.1) that a support function on S^1 defines the boundary of a convex body if $(h' + h")$ is everywhere strictly positive. A.D. Alexandrov [Alex] observed that, if a support function $H : S^{n-1} \to \mathbb{R}$ satisfies $det[hessian(H) + Id \cdot H] > 0$ then the hyperplanes of equation $< m|u >= H(u)$ envelope the boundary of a convex body. Alexandrov also observed that, even if this condition is not satisfied, the envelope associated to H is well-defined and keeps some properties of convex bodies. The study of such envelopes is the topic of [3].

The Minkowski sum of convex bodies is defined as in the dimension 2 case:

$$Q_1 + Q_2 = \{m_1 + m_2 | m_1 \in Q_1, m_2 \in Q_2\}$$

In the same way as in the plane case, the mixed volumes $V(p, Q_1, q, Q_2), p + q = n$ appear as coefficients of the homogeneous polynomial $vol(\lambda Q_1 + \mu Q_2)$.

Theorem 8.1.1 *Let Q_1 and Q_2 be two compact convex bodies of $\mathbb{R}^n$. The volume of the convex body $(\lambda Q_1 + \mu Q_2)$ is an homogeneous polynomial of weight n in λ and μ:*

$$vol(\lambda Q_1 + \mu Q_2) =$$

$$= \lambda^n vol Q_1 + \lambda^{n-1}\mu V(n-1, Q_1, 1, Q_2) + ... + \lambda^p \mu^q V(p, Q_1, q, Q_2) + ... + \mu^n vol Q_2$$

Proof: We need, as before to observe that the support function of the Minkowski sum is the sum of the support functions of the convex bodies λQ_1 and μQ_2, and to use the formula:

$$vol Q = \int_{S^{n-1}} H \cdot det[Hess(H) + Id \cdot H]$$

where again H is the support function of Q. □

8.2 Quermassintegrals and Steiner's formula

A particular case is the case where the second convex body is the unit ball $B(0,1)$. The Minkowski sum $Q + rB(0,1)$ is the thickened convex set:

$$Q_r = \{x | d(x,Q) \leq r\}$$

There are two ways to compute $volQ_r$.

Proposition 8.2.1 *$vol(Q_r)$ is a polynomial in r, the coefficients of which are the symmetric functions of curvature defined in the previous paragraph:*

$$vol(Q_r) = volQ + \sum_{p=0}^{n-1} \frac{r^{p+1}}{p+1} \int_{\partial Q} \sigma_p$$

Proof: Let us consider the map ϕ_t from ∂Q to ∂Q_t defined by:

$$\phi_t : m \mapsto m + tN(m), 0 \leq t \leq r$$

which, for a fixed t, maps ∂Q to ∂Q_t. The reader can check that $T_m \partial Q$ and $T_{m+tN(m)} \partial Q_t$ are parallel. We can compute the jacobian $|det(d\phi_t)$:

$$|det(d\phi_t)| = |detId + td\gamma(m)| = \sum_{p=0}^{n-1} \sigma_p t^p$$

Integrating on ∂Q, and for $0 \leq t \leq r$, one gets:

$$volQ_r = volQ + \sum_{p=0}^{n-1} \frac{r^{p+1}}{p+1} \int_{\partial Q} \sigma_p$$

□

To state a second way of computing $volQ_r$ we need first to define the Quermassintegrales of the compact convex body Q.

Definition 8.2.2

$$M_{p+1}(Q) = \int_{G(n,n-1-p)} vol(p_h(Q))$$

where p_h is the orthogonal projection on the (n-1-p)-dimensional space h.

In particular M_1 is the volume of ∂Q. By convention M_n is 2.

Theorem 8.2.3 *Steiner's formula*

$$volQ_r = volQ + \sum_{p=0}^{n-1} \binom{n}{p+1} M_{p+1}(Q) \cdot r^{p+1}$$

Proof: The proof uses induction on the dimension. The convex Q_r is the union of Q and the parallel hypersurfaces $\partial Q_{t_1}, 0 \leq t_1 \leq r$. Therefore:

$$volQ_r = volQ + \int_0^r vol(\partial Q_{t_1}) dt_1$$

Let us compute $vol(\partial Q_{t_1})$ using Cauchy's formula:

$$vol(\partial Q_{t_1}) = \frac{n-1}{\omega_{n-2}} \int_{G(n,n-1)} vol(p_h(\partial Q_{t_1})$$

where ω_{n-2} is the volume of the unit (n-2)-sphere, and h is a hyperplane of $\mathbb{R}^n$. The projection $p_h(\partial Q_{t_1})$ is the Minkowski sum of the projection $p_h(\partial Q)$ and the ball $B(0, t_1)$ of radius t_1 in h. Therefore by the induction hypothesis it is a polynomial in t_1:

$$vol(p_h(\partial Q_{t_1})) = vol(p_h \partial Q) + \sum_{p=0}^{n-2} \binom{n-1}{p+1} M_{p+1}(p_h \partial Q) t_1^{p+1}$$

Integrating the constant term, for $0 \leq t_1 \leq r$ will give the coefficient of r in Steiner's formula. To get the other terms we proceed with the induction. Let $h_q \subset h_{q-1} \subset ... \subset h_2 \subset h_1 \subset \mathbb{R}^n$ be a flag of nested subspaces of codimension (q,...,2,1) of $\mathbb{R}^n$. The projection p_{h_q} satisfies:

$$p_{h_q} = p_{h_q} \circ p_{h_{q-1}} \circ ... \circ p_{h_2} \circ p_{h_1}$$

We call *flaf space* the set of $h_q \subset h_{q-1} \subset ... \subset h_2 \subset h_1 \subset \mathbb{R}^n$. The natural map $h_q \subset h_{q-1} \subset ... \subset h_2 \subset h_1 \subset \mathbb{R}^n \mapsto h_i; 1 \leq i \leq q$ defines , for each i, a fibration of total space the flag space and base the Grassmann manifold $G(n, n-i)$. These fibrations endowed with natural metrics we shall not explicit in general inherit measures invariant by the action of the group of isometries which can be locally decomposed in the product of a measure on the fiber and a measure on the base. This justifies our frequent use of Fubini's theorem, in particular when a given flag space admits two different projections on two different Grassmann manifolds. Integrating on the *flag space*

$$\mathcal{F}(n, n-1, ...n-q) = \{\mathbb{R}^n \supset h_1 \supset h_2 \supset h_{n-q}\}$$

we get :

$$const \cdot \int_{\mathcal{F}(n,n-1,\ldots n-q)} vol(p_{h_q}(\partial Q)) = \int_{G(n,n-q)} vol(p_{h_q}(Q)) = M_q$$

□

Remark: Identifying the coefficient of r^p in the two expressions of $volQ_r$, we get an equality between a quermassintegrale and an integral of a symmetric function of curvature on ∂Q.

8.3 Orthogonal projections, polar varieties, and p-length of an n-dimensional submanifold of $\mathbb{R}^n$

In this paragraph we shall modify the definition of Quermassintegrale so that it can be extended to any submanifold, and will also carry a sign information.

Definition 8.3.1 *Let Γ_h be the set of critical points of the orthogonal projection p_h of M on h, and let $\gamma_h = p_h(\Gamma_h)$ be the critical locus of p_h. We shall call Γ_h a polar variety .*

It is not in general a manifold but is one almost everywhere for almost every h.

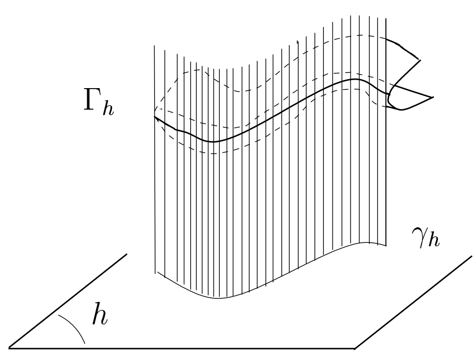


Figure 31: The polar curve Γ_h and its projection γ_h

In this paragraph we shall often use the word *generically* which means: "up to a suitably chosen measure zero set", the measure should be natural, and the choice is often part of a nontrivial theorem involving sometimes a computation in a jet space. A theorem of Thom [Th1] [Th2] implies that generically (for almost every h) γ_h is almost everywhere a (dim(h)-1)-submanifold of h if $dim(M) \geq dim(h) - 1$. If $dim(M) \leq dim(h) - 1$ then γ_h is just $p_h(M)$ and has generically the same dimension as M. Moreover generically the projection of Γ_h on γ_h is one-to-one and a local diffeomorphism.

Polar varieties will appear again in the study of foliations and of complex singularities.

Instead of proving the affirmation, we shall justify it, by describing Γ_h (and its projection γ_h) in a neighbourhood of a point m where M is not flat.

Proposition 8.3.2 *Let h be a linear subspace of $\mathbb{R}^N$ of dimension n and let $M^n \subset \mathbb{R}^N$ be an n-dimensional submanifold. Let m be a critical point of the orthogonal projection p_h on h. Let H be the affine subspace orthogonal to h and containing m. Let v be a unit vector contained in $(T_mM)^\perp \cap h$ and w be a unit vector contained in $T_mM \cap h^\perp$. Then, if $II_{m,v}(w)$ is different from zero, the polar variety Γ_h is transverse to $T_mM \cap h^\perp$.*

Proof: Choose a local parametrisation Φ of M such that

$$\frac{\partial \Phi}{\partial t_1}(m) = w \in (h^\perp \cap T_mM), \frac{\partial \Phi}{\partial t_i}(m) \in (\mathbb{R}w)^\perp for\, i > 1$$

Then at m,

$$det[p_h(\frac{\partial \Phi}{\partial t_1}), p_h(\frac{\partial \Phi}{\partial t_2}), ..., p_h(\frac{\partial \Phi}{\partial t_n})](m) = 0$$

The derivative at m of that determinant is different from 0:

$$\frac{\partial}{\partial t_1} det[p_h(\frac{\partial \Phi}{\partial t_1}), p_h(\frac{\partial \Phi}{\partial t_2}), ..., p_h(\frac{\partial \Phi}{\partial t_n})](m) =$$

$$= det[p_h(\frac{\partial^2 \Phi}{\partial t_1^2}), p_h(\frac{\partial \Phi}{\partial t_2}), ..., p_h(\frac{\partial \Phi}{\partial t_n})](m) +$$

$$+ \sum_{i \geq 1} det[p_h(\frac{\partial \Phi}{\partial t_1}), ..., p_h(\frac{\partial^2 \Phi}{(\partial t_i)^2}), ..., p_h(\frac{\partial \Phi}{\partial t_n})]$$

So:

$$\frac{\partial}{\partial t_1} det[p_h(\frac{\partial \Phi}{\partial t_1}), p_h(\frac{\partial \Phi}{\partial t_2}), ..., p_h(\frac{\partial \Phi}{\partial t_n})] =$$

$$= det[p_h(\frac{\partial^2 \Phi}{(\partial t_1)^2}), p_h(\frac{\partial \Phi}{\partial t_2}), ..., p_h(\frac{\partial \Phi}{\partial t_n})]$$

as we have chosen the coordinates such that $p_h(\frac{\partial \Phi}{\partial t_1})(m) = 0$. It is not difficult now to check that the component of $p_h(\frac{\partial^2 \Phi}{(\partial t_1)^2})$ on w is non zero if and only if $II_{m,v}(w)$ is non zero.

In a similar way, when h is p-dimensional we have the:

Proposition 8.3.3 *Let h be a linear subspace of $\mathbb{R}^N$ of dimension $p \leq n$ and let $M^n \subset \mathbb{R}^N$ be an n-dimensional submanifold. Let m be a critical point of the orthogonal projection p_h on h. Let H be the affine subspace orthogonal to h and containing m. Let v be a unit vector contained in $(T_m M)^{\perp} \cap h$. Then, if $II_{m,v} \mid T_m M \cap h^{\perp}$ is non degenerate, the polar variety Γ_h is transverse to $T_m M \cap h^{\perp}$.*

Proof: As the second fundamental form $II_{m,v} \mid T_m M \cap h^{\perp}$ is symmetric we can choose a basis $(b_1, ..., b_{n-p+1})$ of $T_m M \cap h^{\perp}$ made of eigenvectors. The polar variety Γ_h is the intersection of the polar varieties Γ_{h_j} where the n-dimensional spaces h_j are generated by h and all the vectors of the base $(b_1, ..., b_{n-p+1})$ except b_j. Then we can apply the previous proposition to the projections p_{h_j}. $\square$

Definition 8.3.4 *The p-length of M, $L_p(M)$ is:*

$$L_p(M) = C(N, n, p) \int_{G(N,p+1)} |\gamma_h| dh$$

where $|\gamma_h|$ denotes the volume of γ_h (when $p = O$, γ_h is a finite set and $|\gamma_h$ is the number of points $\sharp(\gamma_h)$ of γ_h). The constant $C(N, n, p)$ is chosen so that if M is the boundary of an ϵ-tubular neighbourhood of a p-dimensional submanifold C of $\mathbb{R}^N$, then:

$$lim_{\epsilon \to O} L_p(M) = |C|$$

If tM denotes an homothetic image of M by an homothety of ratio $t > 0$ then:

$$L_p(tM) = t^p L_p(M)$$

This motivates the choice of the constant $\frac{1}{2|\mathbb{P}_{N-1}|}$ occuring in the definition of L_0, since a sphere of any dimension (≥ 1) satisfies $|\gamma_L| = 2$ for every line $L \in G(N, 1) = \mathbb{P}_{N-1}$, and in particular so does a small sphere of radius ϵ centred at a point p.

The functional L_1 has been applied to measure the ability of an algae to house little mobile marine animals (see [Ja-La]). As an exercise, the reader may check the value of the constant in the definition of L_1 when M is a surface in $\mathbb{R}^3$:

$$L_1(M) = \frac{1}{\pi^2} \int_{G(3,1)} |\gamma_h|$$

Hint: Compare the projections of a round cylinder and of its axis on the plane $h \in G(3,2)$

In section 8 we will show that the functionals L_p satisfy a linear kinematic formula relating them to the functional L_0.

□

8.4 Tubes (2)

The main tool to add a sign information to the varieties γ_h is d'Ocagne's theorem. Let M be an oriented surface of $\mathbb{R}^3$ and let h be a plane. Let m be a critical point of the orthogonal projection of M on h such that $II_m(w) \neq 0$, where w is a unit vector generating $h^\perp$. Then we have seen in the previous subsection that the projection $p_h(\Gamma_h \cap v(m))$ of the critical points of $p_h\,|_M$ contained in a neighbourhood $v(m)$ of m form an oriented curve γ_h in a neighbourhood of $p_h(m)$.

Theorem 8.4.1 d'Ocagne's theorem *The Gauss curvature of the surface M at m is related to the normal curvature $II_m(w)$ in the direction w and the curvature k_{γ_h} of γ_h at $p_h(m)$ by:*

$$K(m) = II_m(w) \cdot k_{\gamma_h}(p_h(m))$$

Proof: First recall that the orientation of M imposes the choice of the normal vector $N(m)$ used in the definition of the Gauss map and of the second fundamental form. This normal vector $N(m)$ belongs to h, and therefore is normal to γ_h at $p_h(m)$, define the orientation of γ_h. The vector $N(m)$ is also normal at m to the curve $C = M \cap (h^\perp \oplus \mathbb{R}N(m))$. Meusnier's theorem implies in particular that the curvature at m of the curve C is $II_m(w)$. We will now compute $d\gamma(m)$ using at the target the orthogonal basis (w, e), where e is a unit vector tangent to γ_h at $p_h(m)$, and at the source the basis(not orthogonal but of determinant one) (w, ϵ), where ϵ is a tangent vector to Γ_h at m such that $p_h(\epsilon) = e$. The matrix of $d\gamma(m)$ is:

$$\begin{pmatrix} II_m(w) & 0 \\ * & d\tilde{\gamma}(p_h(m)) \end{pmatrix}$$

where $\tilde{\gamma}$ is the Gauss map associated to γ_h. Therefore:

$$K(m) = det \begin{pmatrix} II_m(w) & (0) \\ * & d\tilde{\gamma}(p_h(m)) \end{pmatrix}$$

□

Remark: Suppose m is a point of negative Gauss curvature $K(m) < 0$. If $h^{\perp}$ is an asymptotic direction of $T_m M$, then $II_m(w) = 0$ and the critical curve γ_h will have a cusp at $p_h(m)$. As the curvature goes to infinity when a point approches a cuspidal point, this agrees with d'Ocagne's theorem.

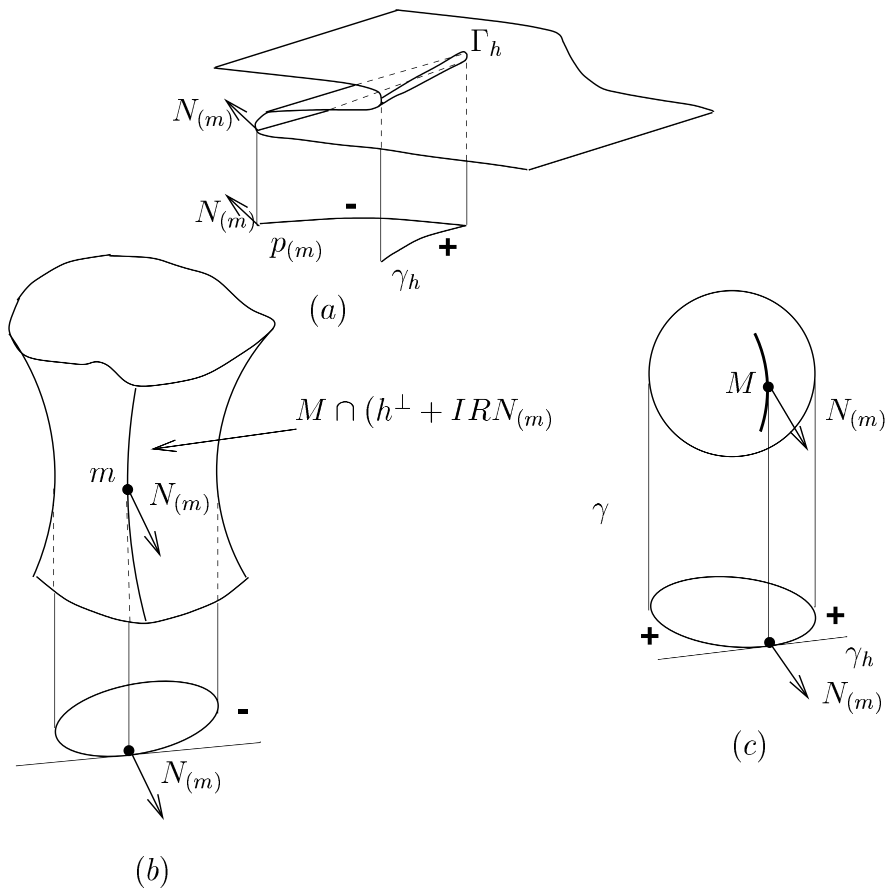


Figure 32: γ_h with a sign

The generalisation of this theorem to higher dimension hypersurfaces $M^{n-1} \subset \mathbb{R}^n$ is straigtforward. The subspace h is now p-dimensional, γ_h has generically dimension p-1, and the intersection $C = M \cap (h^{\perp} \oplus \mathbb{R}N(m))$ is now an hypersurface of the (n-p+1)-dimensional affine space (containing the point $m \in M$), $(h^{\perp} \oplus \mathbb{R}N(m))$. At m, C is oriented by $N(m)$, so we can compute the Gauss-Kronecker curvature $K(C, N(m), M)$ of C at m. As γ_h is also oriented by $N(m)$ at $p_h(m)$, the Gauss-Kronecker curvature $K(\gamma_h, N(m), p_h(m))$ is also well-defined.

Recall that in the previous paragraph we showed that if the restriction to $(h^\perp \oplus \mathbb{R}N(m))$ of the second fundamental form II_m is non degenerate, that is if $K(C, N(m), m)$ is different from zero, then Γ_h is, in a neighbourhood of m, transverse to $h^\perp$.

Theorem 8.4.2 *Let h be a p-dimensional subspace of $\mathbb{R}^n$ and M an hypersurface. If $K(C, N(m), m)$ is different from zero, then :*

$$K(m) = K(C, N(m), m) \cdot K(\gamma_h, N(m), p_h(m))$$

Proof: Use at the target an orthonormal basis

$$(e_1, e_2, ..., e_{n-p}, \epsilon_1, ..., \epsilon_{p-1}),$$

split between $h^\perp$ and $T_{p_h(m)}\gamma_h$, and at the source the basis of determinant one: $(e_1, ..., e_{n-p}, \alpha_1, ..., \alpha_{p-1})$ where α_j is a vector tangent to Γ_h at m satisfying $p_h(\alpha_j) = \epsilon_j$, and repeat the previous computation. □

That way we can see γ_h as a weighted variety (or a chain), weighting generically the points $\omega \in \gamma_h$ with the sign $\epsilon(\omega)$ defined below:

Definition 8.4.3

$$\epsilon(\omega) = sign[K(C, N(m), m)]$$

where m is the (generically unique) point in γ_h which projects on ω and where $C = (h^\perp \oplus \mathbb{R}N(m)) \cap M$ is the oriented "vertical" intersection considered above.

We need now to define the sign $\epsilon(\omega)$ when M is of codimension higher than one. Each generic projection on a p-dimensional space h determines two varieties Γ_h and γ_h. At a generic point $\omega \in \gamma_h$ a normal line ν is well-defined. When the dimension of $C = M \cap (h^\perp \oplus \nu)$ is even, the sign of the Gauss-Kronecker curvature of the orthogonal projection of C on $T_m C \oplus \nu$ does not depend on the choice of the unit vector generating the line ν. So we can still define

$$\epsilon(\omega) = sign(K(C, m, \nu))$$

when the dimension $dim(C) = n - p + 1$ is even.

A d'Ocagne theorem will still be valid, for generic h and $m \in \Gamma_h$, when the dimension of M (and γ_h) will also be even:

$$K(M, m, \nu) = K(C, m, \nu) \cdot K(\gamma_h, p_h(m), \nu)$$

In particular, when h is a line L, γ_L is generically finite and $\epsilon(\omega)$ is well defined if M is even dimensional, or if M is an oriented hypersurface. Then:

$$\epsilon(\omega) = sign(K(M, m, L)) \, or \, sign(K(M, m, N(m)))$$

In the first case it coincides with $(-1)^{index(m)}$, where the index is the Morse index of the critical point m of the Morse function p_L (its parity does not depend on the orientation of L, as $dim(M)$ is even).

Definition 8.4.4 *We will call γ_h^+ the chain obtained by considering along γ_h the almost everywhere defined weight $\epsilon(\omega)$*

Definition 8.4.5 *We will call $\mid \gamma_h^+ \mid$the integral:*

$$\mid \gamma_h^+ \mid = \int_{\gamma_h} \epsilon(\omega) d\omega$$

D'Ocagne's theorem implies that the sign $\epsilon(\omega)$ behaves nicely through compositions of projections. Let us consider a flag $h_1 \subset h_2 \subset ... \subset h_k$ of nested linear subspaces of $\mathbb{R}^N$ such that $dim(h_k) < dim(M)$. Let $m \in \Gamma_h$ be a critical point of p_{h_1} such that $K(M \cap (h_1^\perp \oplus \mathbb{R}N(m))$ is not zero. Let $\omega_1 = p_{h_1}(m), \omega_2 = p_{h_2}(m), ..., \omega_k = p_{h_k}(m)$. Suppose also that the projection of $\gamma_{h_{i+1}}$ on γ_{h_i} is such that the curvature $K(\gamma_{h_{i+1}} \cap [(h_i)^\perp \cap h_{i+1} \oplus \mathbb{R}N(m)], N(m))$ is non zero at ω_{i+1}. Then we can define the sign:

$$\epsilon(i+1, i) = sign(K(\gamma_{h_{i+1}} \cap [(h_i)^\perp \cap h_{i+1} \oplus \mathbb{R}N(m)], N(m)))$$

Similarly, projecting γ_j on γ_i for $j > i$, we can define an index

Definition 8.4.6

$$\epsilon(h_j, h_i) = \epsilon(j, i) = sign(K(\gamma_{h_j} \cap [(h_i)^\perp \cap h_j \oplus \mathbb{R}N(m)], N(m)))$$

Proposition 8.4.7 *The signs $\epsilon(j, i)$ multiply in a nice way:*

$$\epsilon(j, i) = \epsilon(j, l) \cdot \epsilon(l, i) \, if \, j < l < i$$

and in particular:

$$\epsilon(\omega) = \prod_{n-1}^{p} (\epsilon(i+1, i)$$

We can now apply Steiner's method to compute the volume of $Tub_r(M)$, and $Th_r(M)$ when M is of codimension 1, replacing the Quermassintegrales by the signed lengths $\mid \gamma_h^+ \mid$. This is what we have already done for plane curves in 3. Let us prove a theorem for compact surfaces in $\mathbb{R}^3$. Its generalisation to $M^n \subset \mathbb{R}^N$ is natural but cumbersome.

Theorem 8.4.8 *The volume of the thickening $Th_r(M)$ of the compact oriented surface M immersed in $\mathbb{R}^3$ is:*

$$volTh_r(M) = r[vol(\partial M) + r\frac{1}{3\pi}\int_{G(3,2)} \mid \gamma_h \mid + \frac{1}{3}r^2 \int_{\mathbb{P}_2} \mid \gamma_L \mid$$

Proof: To prove the formula, we have to compare two functions on the plane h defined using the vertical (orthogonal to h) affine lines L_y through points $y \in h$:

$$\varphi_{h,0}(y) = \sharp(L_y \cap M)$$

$$\varphi_{h,t}(y) = \sharp(L_y \cap M_t)$$

where M_t is the surface:

$$M_t = \{m + tN(m), m \in M\}$$

Let us also denote by $\gamma_{h,t}$ the critical locus of the orthogonal projection of M_t on h, and by $\gamma_{h,t}^+$ the corresponding weighted curve.

The discontinuity locus of $\varphi_{h,0}$ is contained in the curve γ_h, the distribution derivative of $\varphi_{h,0}$ is γ_h^+. In the same way, the distribution derivative of $\varphi_{h,t}$ is $\gamma_{h,t}^+$.

Lemma 8.4.9 *For a given h, the difference $\varphi_{h,t} - \varphi_{h,0}$ is:*

$$\varphi_{h,t} - \varphi_{h,0} = \int_0^t \mid \gamma_{h,t}^+ \mid$$

Proof: (of the lemma) The curve $\gamma_{h,t}$ is parallel to γ_h:

$$\gamma_{h,t} = \{\omega + tN(m), \omega \in \gamma_h\}$$

where m is the (generically unique) point of Γ_h which projects on $\omega \in \gamma_h$. For almost every h, almost every ω the curve γ_h is smooth in a neighbourhood of ω. Then so is $\gamma_{h,t}$ in a neighbourhood of $\omega + tN(m)$; the vector $N(m) = N(\omega)$ is orthogonal to all the curves $\gamma_{h,\tau}, 0 \leq \tau \leq t$ at the point $\omega + \tau N(m)$.

It is clear that, out of the union of the curves $\gamma_{h,\tau}, 0 \leq \tau \leq t$, the functions $\varphi_{h,0}$ and $\varphi_{h,t}$ are equal. In a neighbourhood of a small smooth arc α_1 of $\gamma_{h,t}$, itself of the form $\alpha_1 = \{\omega + tN(\omega), \omega \in \alpha \subset \gamma_h\}$, we can take a patch of the form:

$$\{\omega_1 + \theta \cdot N(\omega), \omega_1 = \omega + tN(\omega), \omega \in \alpha\}$$

On this patch the difference $(\varphi_{h,t+t_1} - \varphi_{h,t}$ is $2 \cdot \epsilon(\omega)$. The area of the patch is $\int_t^{t+t_1} |\gamma_{h,\theta}|$. Then the functions $\varphi_{h,t}$ and $\varphi_{h,0}$ may have different values

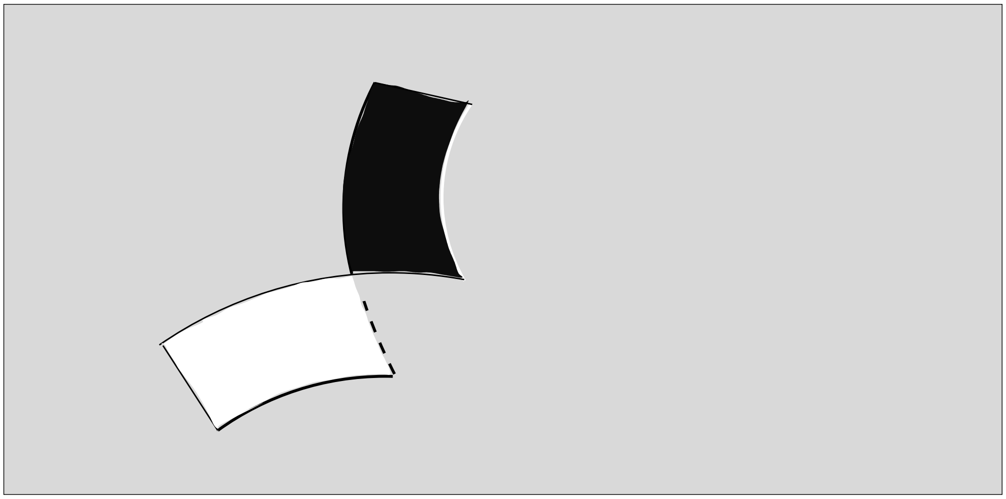


Figure 33: $\int | \gamma_{h,t}^{+} |$

in $y \in h, y \notin \gamma_h, y \notin \gamma_{h,t}$ only if y belongs to some curve $\gamma_{h,\theta}, 0 < \theta < t$; more precisely, if y is not a center of curvature of γ_h, then:

$$\varphi_{h,t}(y) - \varphi_{h,0} = \sum_{a \in A} \epsilon(a)$$

where A is the set:

$$A = \{a \in h \mid y = a + \tau_a N(m_a), \tau_a < t, p_h(m_a) = a \in \gamma_h$$

□

Let $p_{h,L}$ be the projection of the curve γ_h on a line $L \subset h$. We get a function $\varphi_{h,L,0}$ defined by:

$$\varphi_{h,L,0} = \sum_{u \in p_{h,l}^{-1}(z)} \epsilon(z)$$

Cauchy's formula implies that:

$$| \gamma_h^{+} | = \int_{\mathbb{P}_1} \varphi_{h,L,O} dL$$

As the same is true for the curves $\gamma_{h,t}^{+}$, we need now to compare the functions $\varphi_{h,L,t}$ and $\varphi_{h,L,0}$

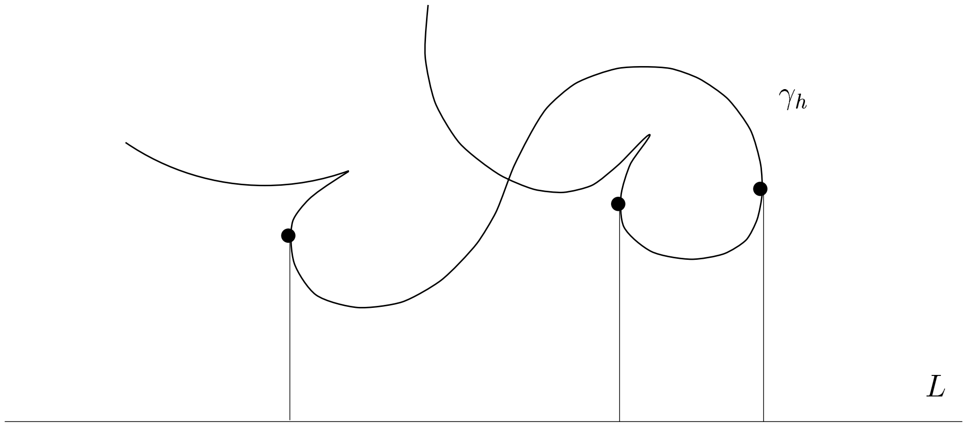


Figure 34: critical points of $p_{h,L}$

Notice that the projection of the cusp of γ_h is not critical for the projection $p_L : M \to L$, as the tangent to γ_h at that point is not orthogonal to L. We can compute the integral on L :

$$\int_L (\varphi_{h,l,t} - \varphi_{h,L,0}) = t[\sum_{critical\ points\ of\ p_L} \epsilon(h,L)(z)] \cdot \epsilon(\omega)$$

where we define the sign $\epsilon_{h,L}$ using the curve γ_h oriented by $p_h(N(m)) = N(m) = N(\omega)$; D'Ocagne's theorem proves that this integral is:

$$\int_L (\varphi_{h,l,t} - \varphi_{h,L,0}) = t[\sum_{critical\ points\ of\ p_l} \epsilon(z)]$$

We can now perform the same induction as for convex bodies to get:

$$volTh_r(M) = r[vol(M) + \frac{1}{\pi} r \int_{G(3,2)} \mid \gamma_h \mid + \frac{1}{3} r^2 \int_{\mathbb{P}_2} \mid \gamma_L \mid]$$

and integrating from $-r$ to $+r$

$$volTub_r(M) = 2r[vol(M) + \frac{1}{3} r^2 \int_{\mathbb{P}_2} \mid \gamma_L \mid$$

this formula gives the "usual" one:

$$volTub_r(M) = 2rvol(M) + \frac{4\pi}{3} \cdot \chi(M)$$

as $| \gamma_L |= \mu(M,L) = \chi(M)$. □

The universal constants in the general formulas are more complicated, but we can conclude that, up to universal constants depending only on the dimensions involved, the volume of $Tub_r(M)$ and of $Th_r(M)$ when M is an oriented hypersurface, are polynomials in r whose coefficients are the oriented p-lengthes $L_p^+(M) = \int_{G(N,p+1)} | \gamma_h^+ |$.

8.5 The localization of the p-lenghts L_p

In 1939 H.Weyl [Wey] has computed the volume of the tube $Tub_r(M)$ in another way, proving of course it is a polynomial in r, the coefficients of which are integrals on M of functions that can be computed from the curvature tensor. From the previous result we get equalities between Weyl's integrals of curvature and the oriented p-lengths.

A natural question is: is it possible to "localize" the (non-oriented) p-lengths $L_p(M)$?

The answer is positive. Let us first define the function $h_1(m)$ on a surface $M \subset \mathbb{R}^3$. In the chapter **The Gauss map and what can be done in higher dimensions**) we expressed the symmetric functions of curvature $\sigma_i(m)$ of an hypersurface as integrals of Gauss curvature of properly chose sections. Now define:

Definition 8.5.1

$$h_1(m) = \frac{1}{vol \mathbb{P}_1} \int_{\mathbb{P}_1(T_m M)} |k(m,l)|$$

Where $\mathbb{P}_1(T_m M) = G(T_m M, 1) = \{lines\ in\ T_m M\}$*, and* $| k(m,l) |$ *is the absolute value of the curvature at* m *of the curve* $M \cap (l \oplus L(m))$.

For future calculations it is useful to introduce the following notation. Let $p : E \to B$ be a riemannian fibration and $V \subset E$ a submanifold transverse to the fibers $F(y) = p^{-1}(y)$, $y \in B$. Let $\mathcal{H} = \{\mathcal{H}(x)\}$ be the horizontal plane field of the fibration.

The normal bundle $N \to M$ is endowed with a metric turning it into a riemannian fibration. At $x \in N$, $T_x N$ is the orthogonal sum $t_x(N \cap F_{p(x)} \oplus V(x)$ where $V(x)$ is a subspace transverse to the fibers of complementary dimension as $\mathcal{H}(x)$. Denote by $Jacp_{\mathcal{H}(x)}$ the jacobian of the orthogonal projection of $V(x)$ to $\mathcal{H}(x)$. Then the coarea formula ([Bu-Za]) yields:

$$\int_N |Jacp_{\mathcal{H}(x)}| dx = \int_B |F(y) \cap N| dy$$

and more generally, if

$$\phi : M \to E$$

is an immersion transverse to the fibers, $N = \phi(M)$, then:

$$\int_M |Jac\phi||Jacp_{\mathcal{H}(x)}|dx = \int_B |F(y) \cap N|dy.$$

Now we can "localize" $L_1(M)$.

Proposition 8.5.2 *For M a surface in $\mathbb{R}^3$,*

$$L_1(M) = \frac{1}{\pi}\int_M h_1$$

Proof: Let $\pi : E = E(3,2) \to G(3,2) = G$ be the tautological line bundle, $E = \{l \in G, m \in l\}$.

Define also the projective tangent bundle of M: $\mathbb{P}_1(M) = \bigcup_{m \in M} \mathbb{P}(T_m M)$.

Let $\phi : \mathbb{P}_1(M) \to E$ be the map:

$$\phi(m,l) = (h = l^\perp, p_h(m))$$

where p_h is the orthogonal projection on the plane h, and let $\phi(\mathbb{P}_1(M)) = N$. We have just recalled that:

$$\int_G |\gamma_h| = \int_{\mathbb{P}_1(M)} |Jac\phi||Jacp_{\mathcal{H}}|,$$

so we compute the jacobians. Let l be a line through m in $T_m M$, $L(m) \subset T_m M$ denotes the line normal to M at m, $h = l^\perp$ the subspace of $\mathbb{R}^3$ orthogonal to l and W the orthogonal to $L(m)$ in h; see next picture. We choose a basis of $T_{(m,l)}(\mathbb{P}_1(M))$ as follows:

-U_f is a unit vector tangent to the circle fiber of $\mathbb{P}_1(M)$ at m

-U_Γ is a horizontal lift of a unit vector tangent to the polar curve Γ_h at m.

-U_l is a horizontal lift of a unit vector tangent to $(l \oplus L(m)) \cap M$ at m.

Also, let U_γ be a horizontal lift (in E) of a unit vector tangent to the critical locus γ_h at $y = p_h(m)$.

The volume of the parallelepiped generated by the first three vectors is $|cos\theta|$, where θ is the angle between $T_m\Gamma_h$ and h.

The image $d\phi(U_\Gamma)$ is the vector $\pm cos\theta \cdot U_\gamma$. The vectors $d\phi(U_f)$ and $d\phi(U_l)$ are projected by the differential $d\pi$ of the projection $\pi : E(3,2) \to G(3,2)$ on two orthogonal vectors of $T_{\pi\phi(m)}G(3,2)$, the first unitary and the second of norm $|k(m,l)|$.

Hence

$$|Jac\phi(m)||Jacp_{\mathcal{H}}| = |k(m,l)|,$$

and the proposition follows by integrating over the fibers of $\mathbb{P}_1(M)$.

Remark: A different proof of the proposition can be found in [La-Shi] based on a Meusnier's formula.

□

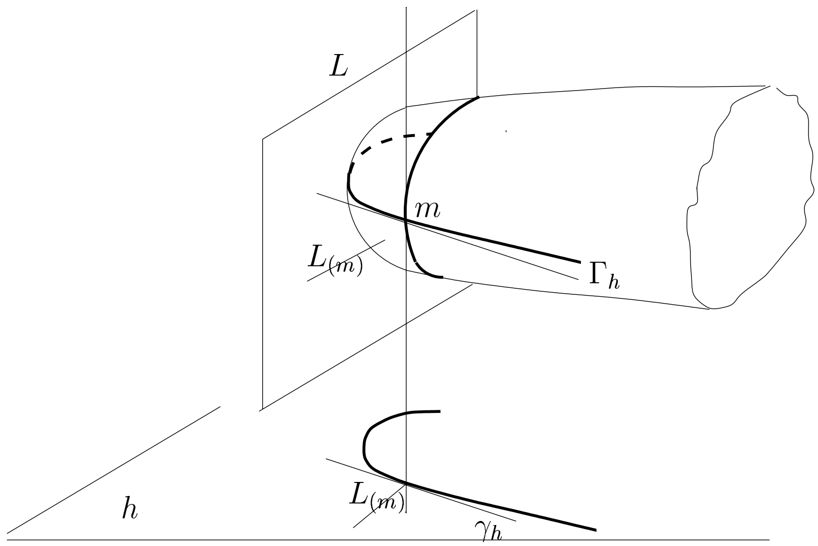


Figure 35: Localization of $L_(m)$

More generally we can define the functions $h_i(m)$ on an hypersurface $M \subset \mathbb{R}^n$. Let h be an i-dimensional subspace of $T_m M$, and $L(m)$ be the normal line to M at m. Denote by $|K|(x,h)$ the absolute value of the Gauss-Kronecker curvature at m of the hypersurface $M \cap (h \oplus L(m))$ of $h \oplus L(m)$.

Definition 8.5.3

$$h_i(m) = \frac{1}{volG(n-1,i)} \int_{G(T_m M,i)} |K|(m,h)dh,$$

where again $G(T_m M, i)$ is the set of i-dimensional subspaces of $T_m M$.

The next proposition is now natural:

Proposition 8.5.4 *The functions $h_{n-i}(m)$ localize the i-length $L_i(M)$; more precisely,*

$$\int_M h_{n-i}(m) = const \cdot L_i(M),$$

where the constant const depends only on dimensions.

The proof can be found in [La-Ro2].

The definitions of the function $h_i(m)$ in higher codimensions can also be found in [La-Ro2].

9 Blaschke's formulas and kinematic formulas

It is not by chance that the name "integral geometry" was used (and probably invented by) Blaschke.[Bla]. One essential tool will now be a measure on the group of affine isometries invariant by left and right composition by an element of the group. Choosing an origin 0 of the euclidean plane we can write the group of affine isometries as the semidirect product:

$$\mathcal{G} = \mathbb{R}^2| \times SO(2)$$

The invariant measure is then $dg = |dv \wedge d\theta|$, where dv is the volume of $\mathbb{R}^2$, and θ the angle of the rotation. The existence of such an invariant volume on a Lie group is a more general phenomenon; see [Sa2].

9.1 Poincaré's formulas

The first directly generalises Cauchy's:

Theorem 9.1.1 *Poincaré's formula Let C_1 and C_2 be two compact arcs, then:*

$$\int_{\mathcal{G}} \sharp(C_1 \cap C_2) = 4length(C_1) \cdot length(C_2)$$

Proof: Let us consider the map

$$\Phi : C_1 \times C_2 \times S^1 \to \mathcal{G}$$

$$(m_1, m_2, \theta) \mapsto (translation\, m_1 \mapsto m_2) \circ R_\theta$$

to compute the jacobian the choice of the origin is irrelevant, so we can choose m_1, and see that it is: $|sin\phi|$, the angle at m_2 of $(translation\, m_1 \mapsto m_2) \circ R_\theta(C_1)$ and C_2. The coarea formula gives:

$$\int_{\mathcal{G}} \sharp(C_1 \cap C_2) = \int_{C_1 \times C_2 \times S^1} |sin\phi|$$

Integrating the left term on S^1 give the theorem. □

Remark: We can reformulate that proof, saying that the kinematic density satisfies locally

$$|dg| = |sin\theta| ds_1 \wedge ds_2 \wedge d\theta|$$

where θ is the angle at a point $P \in C_1 \cap g(C_2)$ of the two curves.

In the same vein is the:

Theorem 9.1.2 *Let $-\pi \leq \theta < \pi$ be the angle at an intersection point of the oriented curves C_1 and C_2. Then:*

$$\int_{\mathcal{G}} \sum_{C_1 \cap C_2} |\theta| = 2\pi length(C_1) \cdot length(C_2)$$

the only difference with the previous proof is that we need to compute $\int_0^\pi |\theta| \cdot |sin\theta| d\theta$.

9.2 Blaschke formulas

As usual in this book we will presentonly the simplest cases of the theory. A comprehensive reference is Santalò's book [Sa2] . Blaschke formulas compute averages of Euler characteristics of intersections of a compact domain with boundary of $\mathbb{R}^2$ and the image of another by all the isometries. The "miracle" is that averaging the Euler characteristic of the intersection $D_1 \cap D_2$ of two such domains on all affine isometries, the result can be calculated using only integrals defined separately using D_1 and D_2. Let us attribute weight zero to area, weight one to length and weight two to integrals of curvature along curves. Just observe that the weight is related to their place in the formula giving the volume of a tube or in Steiner's formula. As often it is easier to prove first a formula "with no sign". So, let us first prove a formula for the total curvature of the boundary of a domain:

definition The total curvature of an arc piecewise of class $\mathcal{C}^{\in}$ is:

$$\mathcal{T}(C) = \int_C |k| + \sum |\theta_i|$$

where the angles θ_i at the corners are oriented, defined by the oriented tangents to the two curves.

Then one has the :

Theorem 9.2.1

$$\int_{\mathcal{G}} \mathcal{T}\partial(D_1 \cap g \cdot D_2) = \int_{\mathcal{G}} \mathcal{T}\partial(D_2 \cap g \cdot D_1) =$$

$$= 2\pi[vol(D_1)\mathcal{T}\partial(D_2) + length(\partial D_1) length(\partial D_2) + \mathcal{T}\partial(D_1) vol(D_2)]$$

Proof: Let us first compute:

$$I_1 = \int_{\mathcal{G}} [\int_{\partial(D_1 \cap g D_2)} |k|] dg$$

The map $g \mapsto g^{-1}$ is an isometry of $\mathcal{G}$. So the integral I_1 is equal to:

$$I_1 = \int_{\mathcal{G}} [\int_{\partial(gD_1 \cap D_2)} |k|] dg$$

The integral I_1 is split into two pieces: one taking care of pieces of $\partial(D_1 \cap gD_2)$ images of arcs of ∂D_1, the other taking care of pieces of $\partial(D_1 \cap D_2)$ images of arcs of ∂D_2. For the first piece we use the second expression of I_1, for the second piece, the first expression. The measure of the set of isometries which send an infinitesimal arc ds of ∂D_1 centered in $m_1 \in \partial D_1$ into D_2 is $2\pi vol(D_2)$. In the same way, the measure of the set of isometries which send an infinitesimal arc ds of ∂D_2 centered in $m_2 \in \partial D_2$ into D_1 is $2\pi vol(D_1)$

We then get

$$I_1 = 2\pi [\int_{\partial D_1} |k| \cdot vol(D_2) + vol(D_1) \cdot \int_{\partial D_2} |k|]$$

The angles of $\partial(D_1 \cap D_2)$ are of two kinds:

the angles $\theta_b, b \in B$ between an arc of ∂D_1 and an arc of $g(\partial D_2)$, and the angles $\theta_i^j, j = 1, 2$ of ∂D_1 or $g(\partial D_2)$ (here all angles are between $-\pi$ and π). Let

$$I_2 = \int_{\mathcal{G}} \sum_{i \in I_1} |\theta_i^1|$$

$$I_3 = \int_{\mathcal{G}} \sum_{i \in I_2} |\theta_i^2|$$

$$I_4 = \int_{\mathcal{G}} \sum_{b \in B} |\theta_b|$$

Inverting as above the orders of integration we get:

$$I_2 + I_3 = 2\pi [\sum_{i \in I_1} |\theta_i^1| \cdot vol(D_2) + vol(D_1) \cdot \sum_{i \in I_2} |\theta_i^2|]$$

Summing with I_1 we get :

$$I_1 + I_2 + I_3 = 2\pi [\mathcal{T}\partial(D_1) \cdot vol(D_2) + vol(D_1) \cdot \mathcal{T}\partial(D_2)]$$

The integral I_4 is an avatar of Poincaré's formula (proved in previous subsection) for all the pairs of curves, one contained in ∂D_1 and the other in ∂D_2. We conclude that:

$$I_4 = 2\pi length(\partial D_1) \cdot length(\partial D_2)$$

□

Taking care of the signs of the curvature and the angles in the previous formula we get:

Theorem 9.2.2 (Blaschke's formula) *The following weighted homogeneous formula holds:*

$$\int_{\mathcal{G}} \chi(D_1 \cap g \cdot D_2) = \int_{\mathcal{G}} \chi(D_2 \cap g \cdot D_1) =$$

$$= 2\pi[vol(D_1)\chi(D_2) + length(\partial D_1)length(\partial D_2) + \chi(D_1)vol(D_2)]$$

Proof: The Gauss-Bonnet theorem for a compact domain D of $\mathbb{R}^2$ with boundary a piecewise $\mathcal{C}^2$ boundary is:

$$\chi(D) = \int_{\partial D} k + \sum \theta_i$$

where the sign of the curvature is defined using the boundary orientation of ∂D and where θ_i are exterior angles at corner points counted with the appropriate sign; see do Carmo's book [dCa].

Let us compute first, exactly as in the previous theorem:

$$I_1 = \int_{\mathcal{G}} [\int_{\partial D_1 \cap g D_2} k] dg$$

We then get

$$I_1 = 2\pi[\int_{\partial D_1} k \cdot vol(D_2) + vol(D_1) \cdot \int_{\partial D_2} k]$$

As in the previous theorem, consider: the angles $\theta_b, b \in B$ between an arc of ∂D_1 and an arc of $g(\partial D_2)$, and the angles $\theta_i^j, j = 1, 2$ of ∂D_1 or $g(\partial D_2)$ (again all angles are between $-\pi$ and π). Let

$$I_2 = \int_{\mathcal{G}} \sum_{i \in I_1} \theta_i^1$$

$$I_3 = \int_{\mathcal{G}} \sum_{i \in I_2} \theta_i^2$$

$$I_4 = \int_{\mathcal{G}} \sum_{b \in B} \theta_b$$

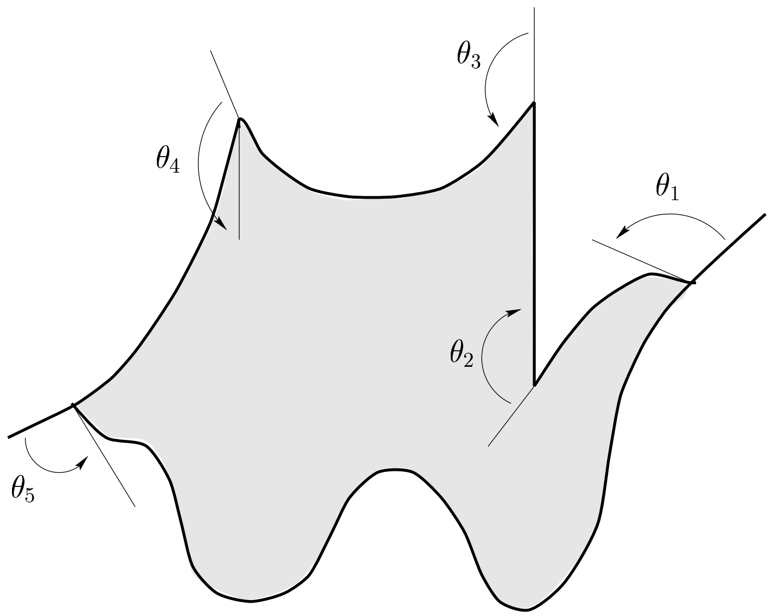


Figure 36: Gauss-Bonnet for a planar domain with boundary(and corner)

Exactly as above we get:

$$I_2 + I_3 = 2\pi[\sum_{i\in I_1} \theta_i^1 \cdot vol(D_2) + vol(D_1) \cdot \sum_{i\in I_2} \theta_i^2]$$

Summing with I_1 we get :

$$I_1 + I_2 + I_3 = 2\pi[\chi(D_1) \cdot vol(D_2) + vol(D_1) \cdot \chi(D_2)]$$

Now observe that if we take care simultaneously of the sign of the angles θ_b and of the orientation $\epsilon(F)$ of the frame F made of the tangent vectors to ∂D_1 and ∂D_2 we get to compute:

$$\int_{\mathcal{G}} \sum_{C_1\cap C_2} \theta \cdot (\epsilon(F) = 2\pi lenth(C_1) \cdot length(C_2)$$

Notice that the density $|sin\theta| ds_1 \cdot ds_2 \cdot d\theta$ coincide with the differential form $sin\theta ds_1 \wedge ds_2 \wedge d\theta$. The integral above is equal to the integral of theorem 8.1.2 as $\theta\epsilon(F) = |\theta|$.

We conclude that:

$$I_4 = 2\pi length(\partial D_1) \cdot length(\partial D_2)$$

□

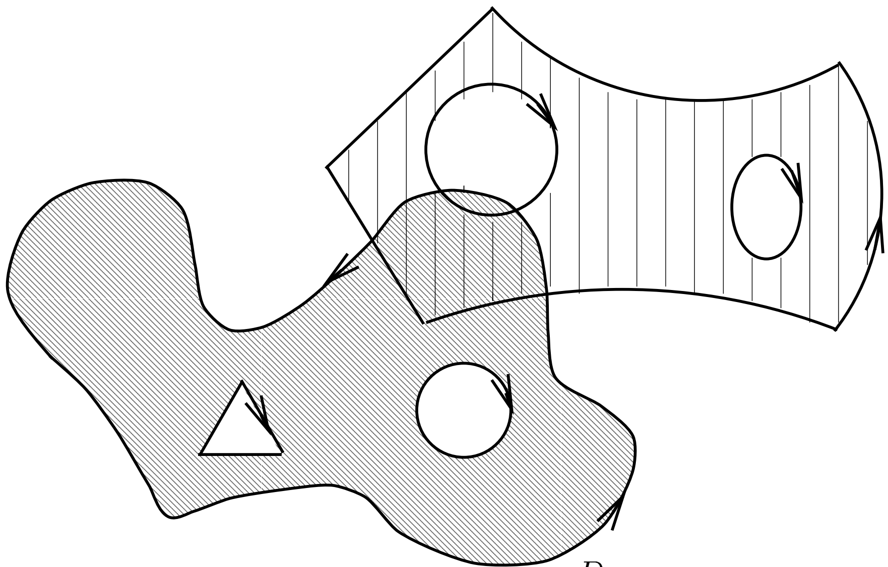


Figure 37: Blaschke formula

9.3 Linear kinematic formulas, variation of a functional

In 1950, at the beginning of his book "multidimensional variation" [Vi] Vitushkin proposes the following general construction: Let M be a compact submanifold of $\mathbb{R}^n$. The intersections of M with almost all affine subspaces are also submanifolds. More precisely, the intersection of M with an affine subspace of dimension $N-p$ achieves its maximal dimension $n+p-N$ and is a transverse intersection on an open subset of $\mathcal{A}(N, N-p)$ if $(n+p \geq N$. It is void on another open subset of $\mathcal{A}(N, N-p)$. the other cases form a subset of measure zero of $\mathcal{A}(N, N-p)$. Consider any functional F defined on submanifolds of euclidean space like

- The Euler characteristic $\chi(M)$
- The total curvature $L_0(M)$

- The number of connected components of $M : \mathcal{N}(M)$

Then intersecting M with affine subspaces H such that the dimension of $M \cap H$ is p, and averaging we get:

Definition 9.3.1 *The variation F_p of the functional F is the integral:*

$$\int_{\mathcal{A}(N,N-p)} F(M \cap H)$$

We can now state a theorem relating the variation of the total curvature and integrals of functions locally defined on the manifold:

Theorem 9.3.2 *let M be a compact connected smooth (at least $\mathcal{C}^2$) n-dimensional submanifold of $\mathbb{R}^N$; then:*

$$(L_0)_p(M) = const \cdot L_p(M)$$

Notice that:

$$(L_0)_n(M) = L_n(M) = vol(M)$$

The variations of the Euler characteristic are linked to the symmetric functions of curvature σ_i, the key result is Gauss-Bonnet's formula:

$$\chi(M) = \frac{2}{vol S^n} \int_M K$$

when M is an n-dimensional hypersurface. We have observed that Weyl's formula computing the volume of the tube $Tub_r(M)$ implies that:

$$L_p^+(M) = const \cdot \int_M \sigma_{n-p}$$

The following *reproductibility formulas* are equivalent to Chern's linear Kinematic formulas.

Theorem 9.3.3 *(Reproductibility formulas) Let M be a compact connected smooth (at least $\mathcal{C}^2$) n-dimensional submanifold of $\mathbb{R}^N$; then:*

$$L_p^+(M) = const.L_0^+)_p(M)$$

For $n = dim(M)$, for $i = n$, $(L_0^+)_n(M) = vol(M)$

The $p^t h$ variation of L_0 is the integral $\int_{\mathcal{A}(N,N-p)} \int_{M \cap H} \sigma_{n-p}$, so we get

Theorem 9.3.4 *(Chern's linear kinematic formulas)*

$$\int_M \sigma_i = const \cdot \int_{\mathcal{A}(N,N-p)} \int_{M \cap H} \sigma_p$$

In this form the name "reproductibility" given to that property of the symmetric functions of curvature becomes clear. Chern was asking if this was a characteristic property of those functions. The theorem concerning the p-length function and the fact they are integrals of the locally defined functions h_{n-p} proves that the functions h_{n-p} also are reproductible.

Proof: (of the reproductibility formulas)

Let $\mathcal{GA}(N, p+1, 1)$ be the flag space of all couples $L \subset h$; h a $(p+1)$ dimensional vector subspace of $\mathbb{R}^N$ and L an affine subspace in h. Let H be the affine subspace of $\mathbb{R}^N$:

$$H = L \oplus h^{\perp}$$

Lemma 9.3.5 *If the line L is transverse to γ_h, the intersection $L \cap \gamma_h$ is the set of critical values of the orthogonal projection of $(M \cap H)$ on L.*

Proof: A critical point ω of the projection of $(M \cap H$ on L is a critical point of the projection of M on h, as L cannot belong to the image $p_h(T_m(M)$ of the tangent space to M at m by p_h.

Conversely, the projection of the tangent space $T_m(M)$ is orthogonal in h to $(T_m(M)) \cap h$ and is the tangent space at ω to γ_h when γ_h is smooth. If L is transverse to γ_h then:

$$p_n(T_m(M \cap H)) = p_h(h^{\perp} + (T_m(M) \cap h)^{\perp}) = \{0\}$$

which implies that

$$p_h(T_m(M \cap H)) = \{0\}$$

□

Observe now that the flag space $\mathcal{GA}(N, p+1, 1)$ can be identified with the flag space $\mathcal{AG}(N, N-p, 1)$ of vectorial lines contained in affine (N-p)-spaces.

By definition

$$L_p(M) = const \cdot \int_{G(n,p+1} |\gamma_h|$$

Using Cauchy's formula for γ_h we get:

$$L_p(M) = const \cdot \int_{\mathcal{GA}(N,p+1,1)} \sharp(\gamma_h \cap L)$$

Using the lemma and the previous identification between flag spaces we get:

$$L_p(m) = const \cdot \int_{\mathcal{AG}(N,N-p,1)} \sharp(\gamma_L(M \cap H))$$

Integrating on the fibers of the fibration

$$\mathcal{AG}(N, N-p, 1) \to \mathcal{A}(N, N-p)$$

we get the desired equality. □

To get the result concerning signed length it is enough to observe that the sign $\epsilon(\omega)$ is precisely the sign of the Gauss-Kronecker curvature of the projection of $M \cap h$ on $(T_m M \cap H) + L$. This last sign is also equal to $(-1)^{index(m)}$, where $index(m)$ is the Morse index of the projection of $M \cap H$ on the line L , oriented by $N(m)$, if M is an odd dimensional codimension one submanifold.

As an exercise, juggling with flag spaces, the reader can prove that a variation of one of the previous variations is a variation, that is:

Proposition 9.3.6 *For $i < p$, one has:*

$$L_p(M) = const \cdot \int_{\mathcal{A}(N,N-p+i)} L_i(M \cap H)$$

$$L_p^+(M) = const \cdot \int_{\mathcal{A}(N,N-p+i)} L_i^+(M \cap H)$$

9.4 General kinematic formulas

We have described a natural path leading from Blaschke's formula to Chern's kinematic formulas: Consider two objects in $\mathbb{R}^n$, move the second, integrate some curvature function on the intersection, and average on $\mathcal{G}$. The result is a weighted homogeneous polynomial in curvatures integrals on the two initial objects.

Theorem 9.4.1 *Chern's kinematic formulas [Che] . If one of the integrals is absolutely convergent, then both following integrals are finite and equal:*

$$\int_{\mathcal{G}} L_i^+(M_1 \cap g(M_2) = \sum_{p+q=i} const \cdot L_p^+(M_1) \cdot L_q^+(M_2)$$

As before const replaces constants depending only on dimensions.

The reader who needs the constants will find them in Santalò's book [Sa2].

9.5 Pohl's, Banchoff-Pohl's formulas and other formulas involving linking numbers

The ancestor of the linking number is the index $i_C(m)$ of a point m with respect to an oriented closed plane curve C. When the curve is also simple the isoperimetric inequality is:

$$L^2 - 4\pi A \geq 0$$

where L is the length of the curve and A is the area it bounds. Equality holds if and only if the curve is a circle.

For non simple closed curves we have ([Po1] [Ba-Po])

Theorem 9.5.1

$$L^2 - 4\pi \int_{\mathbb{R}^2} (i_C(x))^2 \geq 0$$

Equality holds for a circle, or a multiple circle (a circle traversed several times or several coïncident circles each traversed in the same direction any number of times).

This can be generalized to higher dimensions. For example let C be a closed space curve, then the linking numbers of affine lines with the curve also satisfy an analogous inequality [Ba-Po]

Theorem 9.5.2

$$L^2 - 2 \int_{\mathcal{A}(3,1)} link(C, D) \geq 0$$

Equality holds here only for C a circle, which may be multiple.

Kinematic-like formulas using the linking number of two curves can also be obtained and have been applied to obtain a better estimate of the osmotic pressure of a solution of circle-shaped molecules as a function of the concentration [Po2], [Edw1], [Edw2], [Del], [Dup] .

10 Integral geometry of foliations

A foliation $\mathcal{F}$ of a manifold M is a partition of M by connected submanifolds called *leaves* in a way such that locally the connected components of the intersection of a leaf with open sets of a suitable family, the *distinguished charts*, have a product structure. See [Ca-Li] for a rigorous definition and basic properties of foliations; another more riemannian viewpoint can be found in [To], a very complete reference is [Go].

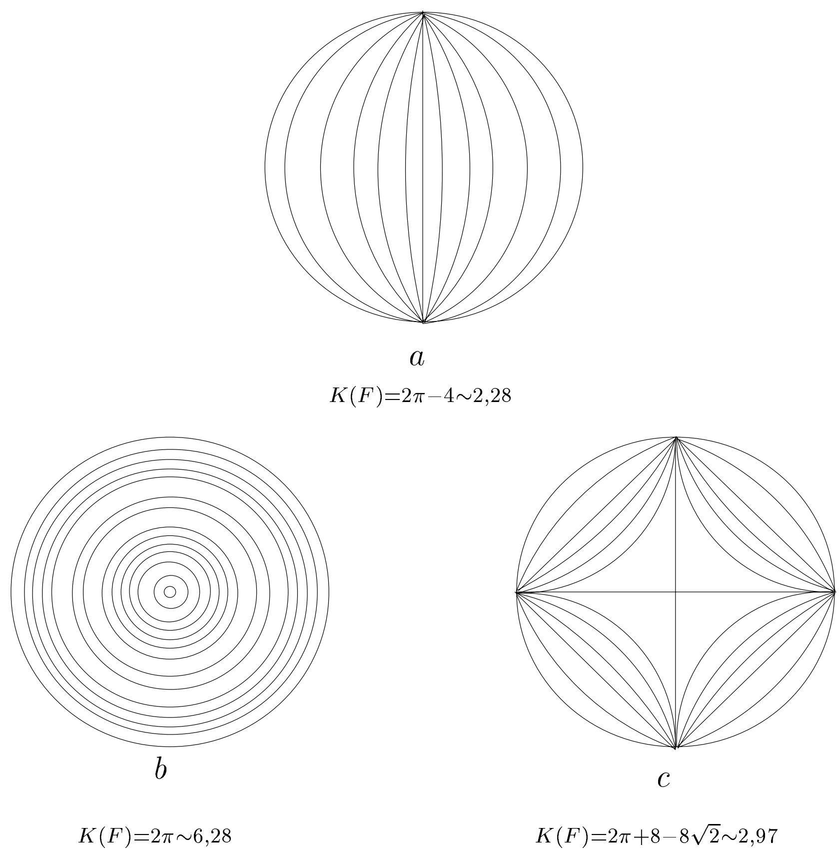


Figure 38: A few examples of foliations

We will soon need to relax a little the definition, accepting a singular locus Σ, a stratified set of codimension bigger than one. The foliated manifold in this case is $M \setminus \Sigma$.

Many results will still be valid if we suppose only the existence of a p-plane field, dropping the *integrability* condition, (a plane field $\mathcal{P}$ is integrable if there exists a foliation such that it is tangent to it).

10.1 Codimension 1 foliations of a domain of $\mathbb{R}^{n+1}$.

Let $W \subset \mathbb{R}^n$ be an open subset, and let $\mathcal{F}$ be a codimension 1 orientable foliation of W. As $\mathcal{F}$ is orientable, a unit normal $N(m)$ is defined at each point $m \in W$.

Symmetric functions of curvature associated to a foliation

As, through every point m of the foliated space there is a leaf L_m of $\mathcal{F}F$. The symmetric functions of curvature of the leaf L_m at the point m are defined by:

$$detId + t(d\gamma)(m) = \sum t^i \cdot \sigma_i^+.$$

This defines the functions σ_i^+ on W. The first result about the integrals $\int \sigma_i^+$ was obtained by D. Asimov :

Theorem 10.1.1 *[Asi] Let $\mathcal{F}$ be an oriented codimension 1 foliation of the flat torus T^{n+1}. The integrals of the symmetric functions of curvature satisfy:*

$$\int_{T^{n+1}} \sigma_i^+ = O, i \geq 1$$

Proof: Note $N(m)$ the unit vector normal in m to the leaf of $\mathcal{F}$ through m, defined by the orientation of $\mathcal{F}$. (The torus T is the quotient $\mathbb{R}^{n=1}/\mathbb{Z}^{n+1}$ or $\mathbb{R}^{n+1}/\Lambda$ for an (n+1)-dimensional lattice Λ).

The covering $\mathbb{R}^{n+1}$ is naturally foliated. Let $\tilde{m}$ be a point of $\mathbb{R}^{n+1}$ of image m, $\tilde{\mathcal{F}}$ be the covering foliation, and $\tilde{L_m}$ be the leaf of the covering foliation through $\tilde{m}$. There exists a fundamental domain $W \subset \mathbb{R}^{n+1}$, (the unit cube for the "square" torus $\mathbb{R}^{n+1}/\mathbb{Z}^{n+1}$) of the covering projection $\mathbb{R}^{n+1} \to T$. So we can identify the normal at m to L_m and the normal at $\tilde{m} \in \mathbb{R}^{n+1}$ to $\tilde{L_m}$. Consider the map

$$m \mapsto m + tN(m)$$

When t is small enough this map is a diffeomorphism. Its differential computed using an orthonormal basis split between $T_m\mathcal{F}$ and $T_m\mathcal{F}^\perp$ is:

$$\begin{pmatrix} t \cdot d\gamma(m) + Id & O \\ & 1 \end{pmatrix}$$

Its jacobian is

$$det(Id + t(d\gamma)(m)) = \sum t^i \cdot \sigma_i$$

The integral:

$$\int_T det(Id + t(d\gamma)(m) = \int_T 1 + \sum \sigma_i^+ t^i$$

is equal to the volume of the torus. Therefore the integrals of the coefficients of the monomials $t^i, 1 \leq i \leq n$ are all zero. □

Asimov, and after Brito Langevin and Rosenberg computed integrals of curvature associated to foliations of compact manifolds of constant curvature using carefully chosen differential forms.[B-L-R].

Here we will prove first euclidean results and sketch their extensions to constant curvature spaces using again an exchange theorem.

Contacts with affine hyperplanes and the exchange theorem

Let H be an affine hyperplane of $\mathbb{R}^{n+1}$. The trace $\mathcal{F}|_H$ of $\mathcal{F}$ on H is generically a foliation of $(W \cap H)$ with only isolated singularities. We call this finite set of singular points $\Sigma(\mathcal{F}|_H)$.

In fact generically those singularities are hyperbolic.

When the ambient space is the plane the singularities are of one of the two following types : center or saddle. We attribute signs to those singular points:

$$\epsilon(saddle) = -1 \, and \, \epsilon(center) = +1$$

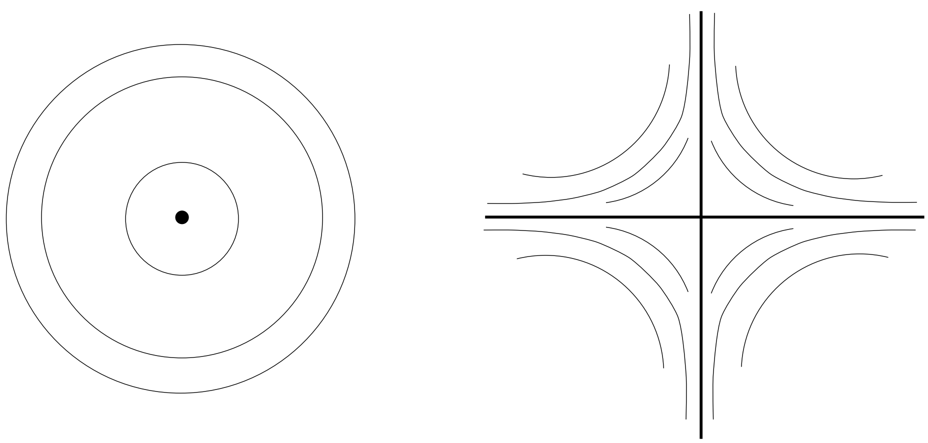


Figure 39: Center and saddle.

When the foliation is of codimension one and transversely oriented, The normals N to the leaves define a vector field with an isolated singularity at m. The sign $\epsilon(m)$ is:

$$\epsilon(m) = (-1)^{index_N(m)}$$

Definition 10.1.2 *The number $|\mu|(\mathcal{F}, H)$ is the number of singular points of $\mathcal{F}|_H$.*

When $|\mu|(\mathcal{F}, H)$ is finite,and the singularities are all hyperbolic, the number $\mu^+(\mathcal{F}, H)$ is:

$$\mu^+(\mathcal{F}, H) = \sum_{m \in \Sigma(\mathcal{F}|_H)} \epsilon(m)$$

Remark: A singular point m of $\mathcal{F}|_H$ is a point where the leaf L_m is tangent to H. We can also locally project L_m on the normal in m to H (and to L_m). We get a function which is in general a Morse function, for which the Morse index of m satisfies:

$$(-1)^{Morse\,index\,of\,m} = \epsilon(m).$$

The sign $\epsilon(m)$ is, when the dimension of the leaves of $\mathcal{F}$ is even, the sign of the Gauss curvature of L_m at m.

We will call the integral $\int_W |K|$ (or $\int_W |k|$ when W is of dimension 2) the *total curvature of $\mathcal{F}$*.

Theorem 10.1.3 foliated exchange theorem.

$$\int_W |K| = \int_{\mathcal{A}(3,2)} |\mu|(\mathcal{F}, H)$$

Moreover, if one of the previous integrals are finite:

$$\int_W K = \int_{\mathcal{A}(3,2)} \mu^+(\mathcal{F}, H)$$

To prove this theorem, we will define the polar curves of the foliation and a foliated Gauss map.

Polar curves

The critical points of the orthogonal projection of a leaf L of $\mathcal{F}$ on a line Δ are in general isolated on L.

Definition 10.1.4 *The closure of the union of those critical points :*

$$\Gamma(\mathcal{F}, \Delta) = \overline{\bigcup_L crit(p_\Delta|_L)}$$

is generically almost everywhere a smooth curve (it may have singular points).

Proposition 10.1.5 *[Th2] Generically the polar curve $\Gamma(\mathcal{F}, \Delta)$ is transverse to $\Delta^\perp$*

Remark: When $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp)$ is tangent to $T_m\mathcal{F}$ the Gauss curvature of the leaf L_m is zero, as, in that case, the differential of the Gauss map of the leaf L_m restricted to $T_m\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp)$ is zero.

To prove the foliated exchange theorem we need to introduce a foliated Gauss map with values in $\mathcal{A}(3,2)$:

Definition 10.1.6

$$\gamma_{\mathcal{F}}(m) = the\ affine\ plane\ tangent\ at\ m\ to\ \mathcal{F}$$

Proof: To compute the jacobian of the foliated Gauss map $\gamma_{\mathcal{F}}$ at a point $m \in W$ we will use, when $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp$ is transverse to $T_m\mathcal{F}$, in the domain, the frame $u_1, u_2, ..., u_n$, $u_1, u_2, ..., u_n$ orthogonal basis of $T_m\mathcal{F}$, u_n unit vector tangent at m to $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp$. In $\mathcal{A}(3,2)$ we use at $\gamma_{\mathcal{F}}(m)$ the frame v_1, v_2, v_3, where v_1, v_2 form an orthogonal basis of the horizontal space at $\gamma_{\mathcal{F}}(m)$ of the riemannian fiber bundle $\mathcal{A}(3,2) \to \mathbb{P}_2$, and where v_3 is a unit vector tangent to the fiber of $\mathcal{A}(3,2) \to \mathbb{P}_2$. In these bases, the matrix of $d\gamma_{\mathcal{F}}$ is:

$$\begin{pmatrix} d\gamma_{\mathcal{F}}|_{L_m} & 0 \\ * & |cos\phi| \end{pmatrix}$$

where ϕ is the angle between $T_m\Gamma_{\mathcal{F}}$ and $T_m\mathcal{F}^\perp$

As the volume of the parallelogram determined by the frame u_1, u_2, u_n is also $|cos\phi|$, and as the map $d\gamma_{\mathcal{F}}|_{L_m}$ is just the Gauss-Kronecker map of the leaf L_m, the jacobian we are looking for is just $|K|$.

On one hand, when $\Gamma(\mathcal{F}, T_m\mathcal{F}^\perp$ is tangent to $T_m\mathcal{F}$ the Gauss-Kronecker curvature K is zero. On the other hand using a frame split between $T_m\mathcal{F}$ and $T_m\mathcal{F}^\perp$ we see that at such a point the matrix of $d\gamma_{\mathcal{F}}$ is:

$$\begin{pmatrix} d\gamma(m) & * \\ 0 & 1 \end{pmatrix}$$

where in the formula $d\gamma$ is the Gauss map of the leaf L_m. As the rank of $d\gamma(m)$ is one the point m is critical for $\gamma_{\mathcal{F}}$, by Sard's theorem the measure of the images by $\gamma_{\mathcal{F}}$ of these points is zero. □

Let us first give some applications of the foliated exchange theorem in dimension 2. We note $|k|(m)$ the absolute value of the curvature of the leaf L_m of $\mathcal{F}$ through m.

Theorem 10.1.7 *[La-Le2] Let $D \in \mathbb{R}^2$ be the unit disc and $\mathcal{F}$ be an orientable foliation with isolated singularities, tangent to ∂D. Then:*

$$\int_D |k| \geq 4\pi - 2$$

the minimal value is achieved by the foliation (a) of the next picture.

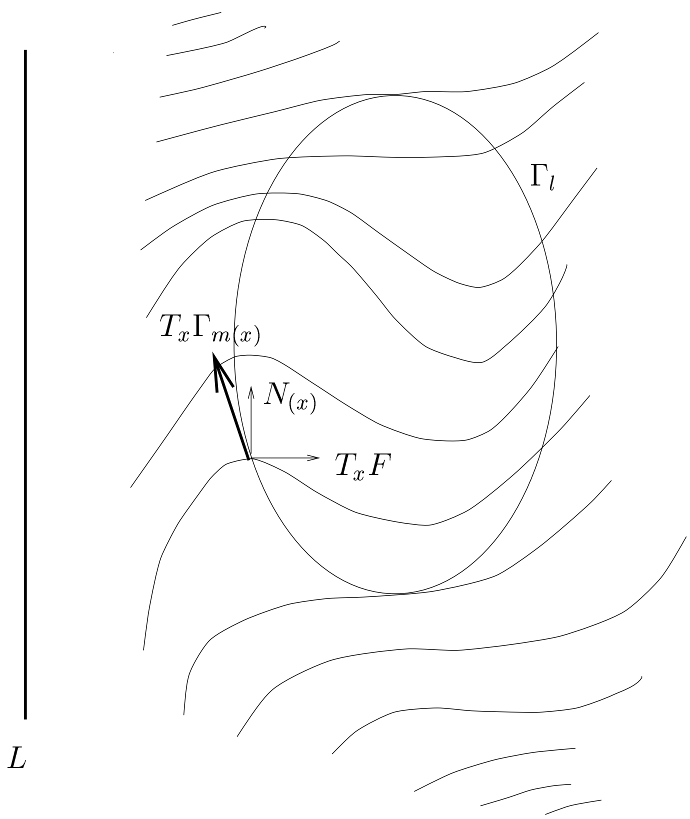


Figure 40: Computation of the jacobian of $\gamma_{\mathcal{F}}$.

Proof: Let us choose an orientation of $\mathcal{F}$; that induces an orientation of $\partial D \setminus sing(\mathcal{F})$. Among the singularities of $\mathcal{F}$ on ∂D let A be those where the orientation of ∂D changes. The set $A = a_1, a_2, ..., a_{2n}$ is finite and has an even number of points. Let G_e be the set of lines which meet D, do not meet A, and cut A in two subsets containing an even number of points; let G_o be the similar set of lines cutting A in two subsets of odd cardinality. the formula of Cauchy and Crofton implies that the sum of the measures of G_e and G_o is 2π (the length of ∂D). If a line L is in G_e, then, if it contains no singularity of $\mathcal{F}$, $|\mu|(\mathcal{F}, L) \geq 1$ (see next picture)

Using the exchange theorem, we get the inequality:

$$\int_D |k| \geq measure(G_e) = 2\pi - measure(G_o)$$

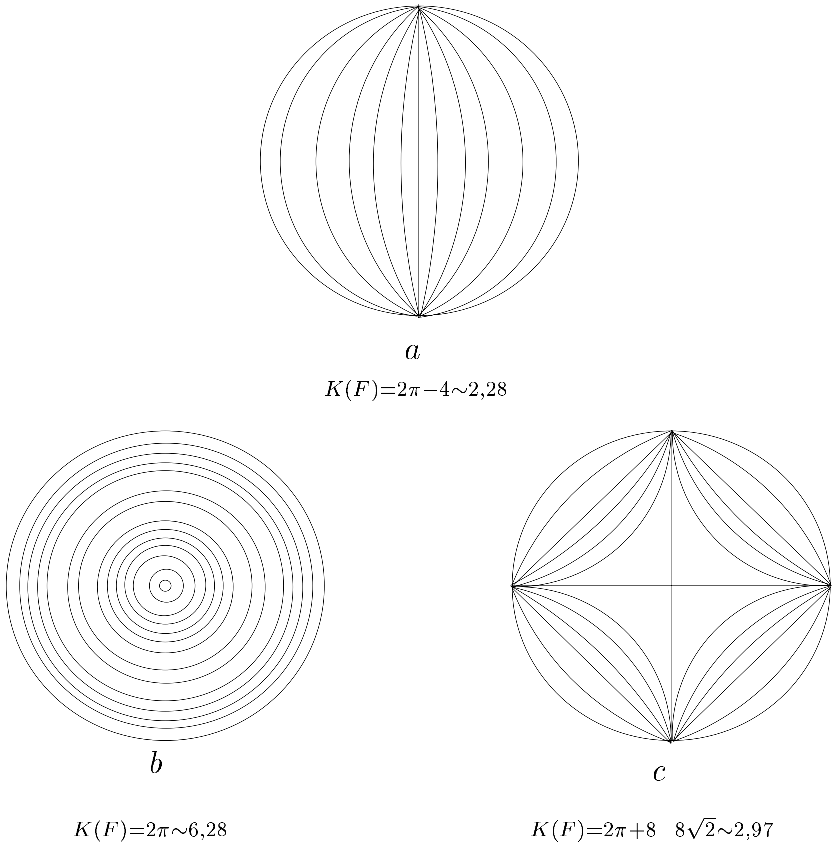


Figure 41: 3 exemples of foliations of the disc

In order to finish the proof we need a lemma:

Lemma 10.1.8 *for any finite subset A of the unit circle ∂D the measure of the set G_o of lines cutting A in two odd subsets satisfies:*

$$measure(G_o) \leq 4$$

$\square$

Remark: When $A = \{a, -a\}$ is made of two opposite points, $measure(G_o) = 4$, when $A = \emptyset$, $measure(G_o) = 0$, when A is the union of the vertices of a regular 2n-gon, $measure(G_o)$ goes to π when n goes to infinity.

The proof of the lemma is elementary but technical and can be found in [La-Le2] .

Let now $D \subset \mathbb{R}^2$ be a domain homeomorphic to a disc and with a piecewise $\mathcal{C}^2$ boundary ∂D.

Definition 10.1.9 *The internal distance $d(m_1, m_2)$ of two points m_1 and m_2 is:*
$d(m_1, m_2) =$

$$= inf\{length(\gamma) | \gamma : [a, b] \to D \text{ a regular curve }, \gamma(a) = m_1, \gamma(b) = m_2$$

where $length(\gamma)$ is the length of the curve γ

We get that way a metric on D. In fact the assumptions made on D imply that given the two end points, there exists exactly one minimizing curve joining them. Such a curve will be called a *geodesic of D*.

Definition 10.1.10 *The diameter of D is defined as:*

$$d = sup\{d(m_1, m_2) | m_1 \in D, m_2 \in D\}$$

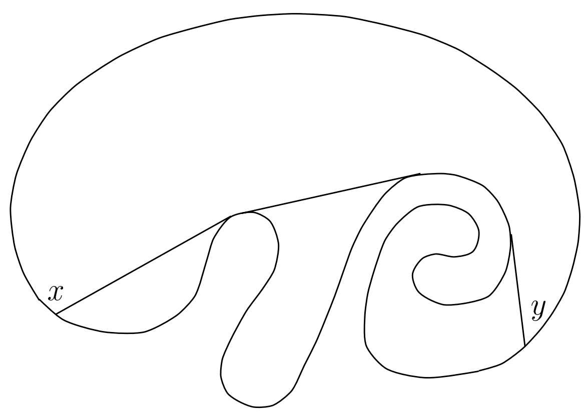


Figure 42: Diameter of a topological disc.

Theorem 10.1.11 *[La-Po] Let $\mathcal{F}$ be a foliation (by curves) of D, tangent to ∂D, with isolated singularities of positive index, not necessarily orientable. Then:*

$$\int_D |K| \geq length(\partial D) - 2d$$

Definition 10.1.12 *The index of an isolated singularity m of a non-orientable foliation of the plane is a half integer $\iota(m) \in \frac{1}{2}\mathbb{Z}$ which is half of the degree of the map*

$$\Phi_\epsilon : S_\epsilon(m) \to \mathbb{P}_1$$

associating to a point q of a small enough circle centered at m the direction of the line $T_q\mathcal{F}$, (if the singularity is orientable, the index is the usual one).

Proof: Let us first show that we can eliminate the case when $\mathcal{F}$ has a singularity of index one, studying only the case where $\mathcal{F}$ has two singularities of index $\frac{1}{2}$. which are of sunset type (see next picture).

All singularities can be substituted by a source/sink or a sunset singularity without increasing the total curvature of the foliation by more than a given ϵ. This can be done by considering on the boundary of a small disc D_r of radius r an homotopy between the "angle" function determined by $\mathcal{F}$ and the "angle" function of one of the models of the next picture.

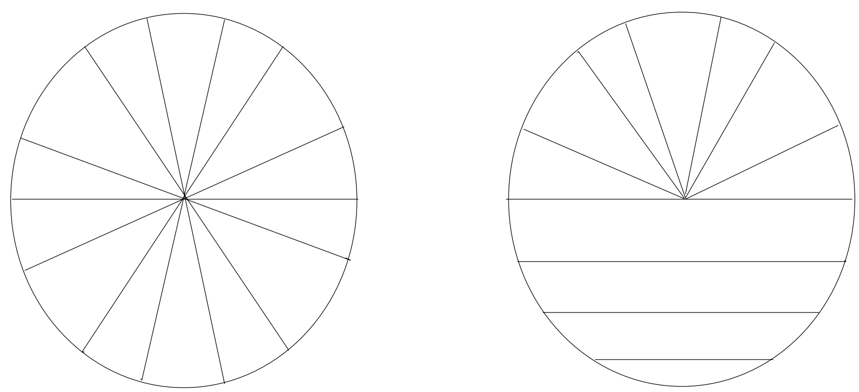


Figure 43: Source/sink and sunset.

A source/sink can be replaced by two sunsets using the modification indicated in the next figure:

Let P and Q be two sunsets of $\mathcal{F}$, and γ be a geodesic of D joining P to Q. We need to estimate the number of contact points of $\mathcal{F}$ with an affine line L. All lines, but a measure zero set, meet the disc D in a finite number of segments.

Let $[a, b]$ be a connected component of $L \cap D$ such that $[a, b] \cap \gamma = \emptyset$. Then $[a, b]$ divides D into two discs, one of them containing P and Q.In

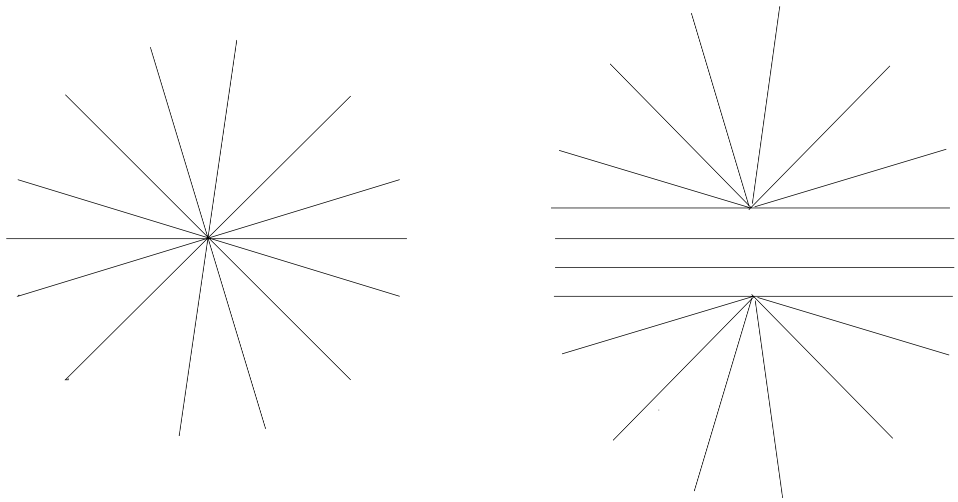


Figure 44: Transformation of a source/sink into two sunsets.

the other disc, $\mathcal{F}$ is orientable, and therefore there is at least one point of contact between $\mathcal{F}$ and the segment $[a, b]$. See next figure:

Let $n(L)$ be the number of segments of $L \cap D$ in which L meets γ, and $c(L)$ the number of segments of $D \cap L$ which do not. Then we have:

$$|\mu|(\mathcal{F}, L) \geq c(L)$$

Cauchy's formula yields:

$$\int_{\mathcal{A}(2,1)} \sharp\{(components\, of\, L \cap D\} = \frac{1}{2} \int_{\mathcal{A}(2,1)} \sharp\{L \cap \partial D\} = length(\partial D)$$

Applying Cauchy-Crofton's formula to the arc γ we get $length(\gamma) = \frac{1}{2} \int_{\mathcal{A}(2,1)} \sharp(L \cap \gamma)$. Then we have:

$$length(\partial D) = \int_{\mathcal{A}(2,1)} n(L) + c(L) = \int_{\mathcal{A}(2,1)} \sharp(L \cap \gamma) + \int_{\mathcal{A}(2,1)} c(L).$$

Using the exchange theorem and the inequality on $|\mu|(\mathcal{F}, L)$ we get:

$$length(\partial\gamma) \leq 2 \cdot length(\gamma) + \int_{\mathcal{A}} (2,1)|\mu|(\mathcal{F}, L) = 2 \cdot length(\gamma) + \int_{D} |k|$$

$\square$

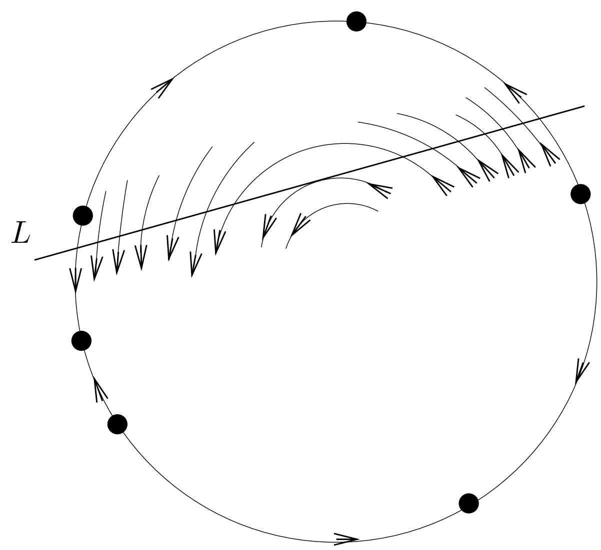


Figure 45: forced

With the same techniques, one can obtain inequalities for foliations of a compact flat annulus, and for foliations of a disc extending a given line field defined on the boundary. In the second case a sort of "length" of the envelope of the one parameter family of affine lines defined by the boundary condition will play a role [La-Po].

When a foliation achieves equality in the inequality of the previous theorem, we call it *tight*.

When the disc D is not convex we can show there do not exists tight foliations tangent to ∂D with singulaties of positive index. This comes from the fact that if $P \in \partial D$ is a point of inflexion, and a regular point of $\mathcal{F}$, then there is an open set of affine lines which have more than one contact point with $\mathcal{F}$ in a neighbourhood of P. But we can exhibit a sequence $\mathcal{F}_n$ of foliations of D satisfying the hypothesis of our theorem such that:

$$lim_{n\to\infty} \int_D |k| = length(\partial D) - 2d$$

We can think of the limit of this sequence of foliations as a foliation all leaves of which have corners along ∂D, in order to force on ∂D all the critical points of the orthogonal projections of the leaves on lines; see next picture.

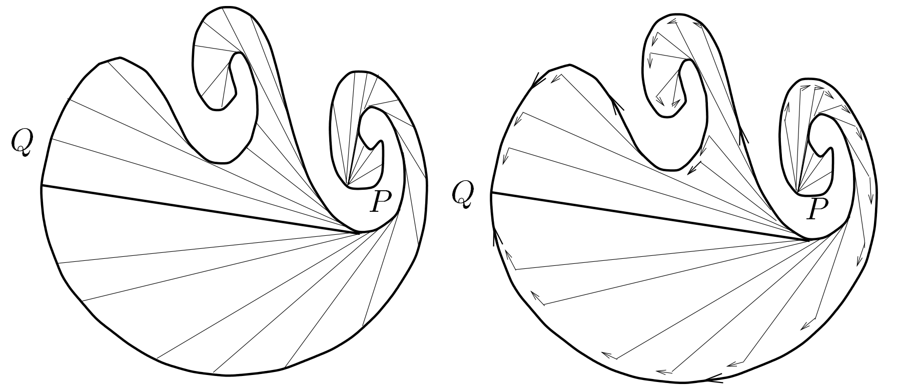


Figure 46: A tight singular foliation $\mathcal{F}$; a non-singular foliation $\mathcal{F}_n$ close to $\mathcal{F}$.

10.2 Codimension one foliations of spaces of constant curvature.

When the foliated space is a domain W in S^n or $\mathbb{H}^n$, one can also prove an exchange theorem, replacing Gauss-Kronecker curvature by the determinant of the second fundamental form (that we will still denote by K) obtained from the normal vector given by the orientation (in an orthonormal basis), and replacing the euclidean affine hyperplanes by codimension one totally geodesic subspaces $H \in \mathcal{A}$. The form of the theorem is the same for $W \subset \mathbb{H}^{n+1}$, $W \subset \mathbb{R}^{n+1}$, $W \subset S^{n+1}$. In each case the set $\mathcal{A}$ admits a measure invariant by the action of the isometries of the space [Sa2] p.28 and 307.

In dimension 2, we can chose (locally in the case of S^2) coordinates $(r, \theta); r \in \mathbb{R}^+, \theta \in \mathbb{P}_1$ on a neighbourhood of a geodesic γ_0. Chose a point $m \in \gamma_0$; the geodesics rays through m form a circle S^1, identifying them with their unit tangent vectors at m. A geodesic γ of $\mathbb{H}^2$ or $\mathbb{R}^2$ which does not contain m is orthogonal to exactly one geodesic ray starting at m and intersects it at a point q. This is true for all geodesics of S^2 different from the "equator" conjugated to m, and not containing m. This defines the coordinates $\theta(\gamma), r(\gamma) = d(m, q)$.

The measures are:

- $m = |dr \wedge d\theta|$ if $W \subset \mathbb{R}^2$
- $m = |cosr \cdot dr \wedge d\theta|$ if $W \subset S^2$
- $m = |coshr \cdot dr \wedge d\theta|$ if $W \subset \mathbb{H}^2$

We have seen the first measure in the chapter **the euclidean plane**; for the other two, see [Sa2]. The (natural) formulas for the measures on the set $\mathcal{A}$ of totally geodesic hypersurfaces in $\mathbb{R}^{n+1}$, $S^{n=1}$ and $\mathbb{H}^{n+1}$ can also be found in [Sa2] .

Theorem 10.2.1

$$\int_W |K| = \int_{\mathcal{A}} |\mu|(\mathcal{F}, H)$$

Proof: We need to replace the orthogonal projections on lines. A geodesic $\mathcal{L}$ defines a one-parameter family, called a *pencil* $\mathcal{P}_{\mathcal{L}}$ of totally geodesic hypersurfaces: those orthogonal to it. In $\mathbb{H}^{n+1}$ a pencil is a foliation and defines a projection on the geodesic $\mathcal{L}$. In S^{n+1} a pencil defines a foliation of $S^{n+1} \setminus S^n$ and a projection of $S^{n+1} \setminus S^n$ on $\mathbb{P}_1$.

Definition 10.2.2 *The polar curve $\Gamma_{\mathcal{P}}$ is the closure of the set of points where a hypersurface of the pencil $\mathcal{P}$ is tangent to the foliation.*

Remark: As in the euclidean case, $\Gamma_{\mathcal{P}}$ is, for almost all $\mathcal{P}$, almost everywhere a smooth curve.

Definition 10.2.3 *The foliated Gauss map $\gamma_{\mathcal{F}} : W \to \mathcal{A}$ associates to a point $m \in W$ the totally geodesic hypersurface tangent at m to the leaf L_m of $\mathcal{F}$ through m.*

The computation of the jacobian of $\gamma_{\mathcal{F}}$ is the same as in the euclidean case, observing that the totally geodesic hypersurfaces orthogonal to the geodesic $\mathcal{L}(m)$ through m orthogonal to L_m, and the totally geodesic hypersurfaces through m, form two submanifolds of $\mathcal{A}$ othogonal in $\mathcal{A}$ for the natural riemannian metric of $\mathcal{A}$. □

The following theorem is now a consequence of the fact that the intersection of a foliation of S^3 with a generic totally geodesic S^2 has at least two singular points.

Theorem 10.2.4 *Let $\mathcal{F}$ be a foliation of S^3 having a finite number of singularities, then*

$$\int_{S^3} |K| \geq 2\pi^2$$

Using the Poincaré-Hopf theorem on all the generic S^2's we prove also the following theorem:

Theorem 10.2.5 *If one of the previous integrals is finite,then:*

$$\int_{S^3} K = 2\pi^2$$

foliations of hyperbolic surfaces. Let us now state a theorem for foliations with only saddle-like singularities of compact surfaces of constant curvature (-1) [La-Le1]. It is similar to the result of $La - Ro1$ in the sense that it translates in terms of total curvature a topological property of those foliations.

Theorem 10.2.6 *Let M be a compact surface without boundary endowed with a hyperbolic metric (that is a metric of constant curvature (-1)) and $\mathcal{F}$ a foliation the only singularities of which are saddles. The total curvature of $\mathcal{F}$ satisfies:*

$$\int_M |k| \geq (12Log2 - 6Log3)|\chi(M)|$$

Remark:

- We will give below examples of foliations which achieve the minimal value given by the theorem.
- If all the saddles have an even number of separatrices (in particular if $\mathcal{F}$ is orientable), one can show that the total curvature of $\mathcal{F}$ satisfies:

$$\int_M |K| \geq 4Log2 \cdot |\chi(M)|$$

- It is hopeless to look for a generalisation to all surfaces; see [La-Le1]

We will need a few facts from hyperbolic geometry. The hyperbolic plane $\mathbb{H}^2$ is identified with the interior of the unit disc (Poincaré's model). The boundary S_∞ of this disc is the *circle at infinity* of $\mathbb{H}^2$. The geodesics of $\mathbb{H}^2$ are the arcs of circles orthogonal to S_∞ contained in $\mathbb{H}^2$. Recall that by analogy with the notation $\mathcal{A}(3,1)$ used for the set of affine lines of $\mathbb{R}^3$we denote by $\mathcal{A}$ the set of all geodesics of $\mathbb{H}^2$. It has a measure invariant by the action of the hyperbolic isometries.

Two distinct points m and m' of $\mathbb{H}^2$ are "joined" by a unique geodesic; it is also the case if m and m' are in S_∞; in that case we say that the points are the *points at infinity* of the geodesic. Three distinct points of S_∞ define that way an *asymptotic triangle* and all asymptotic triangles are isometric (there is a global isometry of $\mathbb{H}^2$ sending one on the other). An asymptotic triangle has, as one can check using the Gauss-Bonnet theorem, area π.

Let $p : \mathbb{H}^2 \to M$ be the universal covering map. If the restriction of p to the interior of an asymptotic triangle is injective, we will also call its image in M an asymptotic triangle.

In order to get foliations minimising total curvature, we need first to construct a foliation $\mathcal{F}_a$ on an asymptotic triangle $\mathcal{T}$ (see next picture)

Let b be the center of symmetry of $\mathcal{T}$. The foliation $\mathcal{F}_a$ has just one singularity, at b, a three prong saddle. The separatrices starting at b are geodesic rays joining b to the points at infinity of $\mathcal{T}$; they intersect in b in equal angles (equal to $2\pi/3$). To get $\mathcal{F}_a$ just fill each sector with geodesically convex curves, in such a way that the boundary of $\mathcal{T}$ is the union of three leaves. If the projection p is injective on $\mathcal{T}$, we can project $\mathcal{F}_a$ on M; see next picture

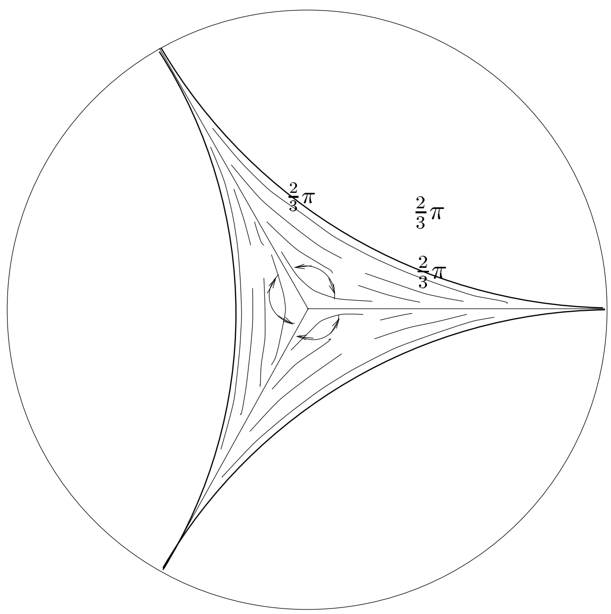


Figure 47: "Standard" foliation of an asymptotic triangle.

The total curvature of that foliation of the asymptotic triangle is 2Log2-Log3) as we will see below. Let now M be a closed orientable hyperbolic surface of genus g. Choose on M a family of $3g-3$ compact disjoint geodesics slicing M into g pairs of pants (each pair of pants is topologically a disc with

Figure 48: How to fit an asymptotic triangle on a hyperbolic pair of pants.

two holes). Choose in each pair of pants three disjoint geodesics spiraling towards the boundary (see the picture above).

We can then fill the surfaces with copies of the model foliation constructed above, achieving the lower bound given by the theorem. Using Whitehead transformations we can split the saddles with more than two separatrices into three prong saddles without increasing the curvature by more than ϵ, see picture below and [F-L-P] for a carefull construction.

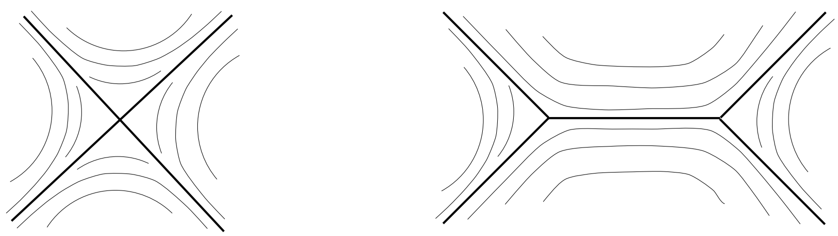


Figure 49: A Whitehead transformation

As the singularities of $\mathcal{H}$ are all saddles, one cannot find in $\mathcal{H}$ Whitehead discs, that is discs with boundary made either of a finite number of arcs of leaves, or of a finite number of arcs of leaves and one arc transverse to $\mathcal{F}$.

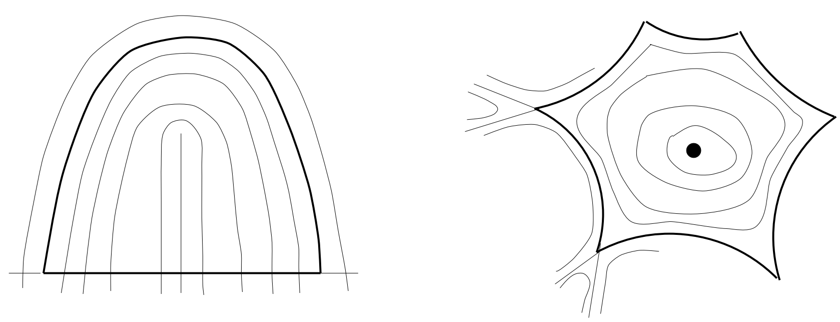


Figure 50: Whitehead discs.

We can also, without increasing the total curvature by more than ϵ, brake all the saddle connections. The foliation $\mathcal{F}$ of M lifts to a foliation $\mathcal{H}$ of $\mathbb{H}^2$.

Affirmation We can associate to each saddle s of $\mathcal{H}$ a set of geodesics A_s of measure bigger or equal to (6Log2-3Log3), and an injection of A_s in $\mathbb{H}^2$ sending each geodesic to a point where it is tangent to $\mathcal{H}$. Moreover the respective images $B_s \subset \mathbb{H}^2$ and $B_{s'} \subset \mathbb{H}^2$ of the sets of geodesics A_s and $A_{s'}$ associated to different saddles are disjoint.

The fact that $\mathcal{F}$ has $2|\chi(M)|$ saddles, and a carefull application of the foliated exchange theorem will end the proof of our result about hyperbolic surfaces.

Lemma 10.2.7 *Any half-leaf δ of $\mathcal{H}$ which does not end at a saddle goes to a point of the circle at infinity S_∞*

Proof: First observe that the behaviour at infinity of the half leaves of $\mathcal{H}$ does not change if we change $\mathcal{F}$ by an isotopy (if $\tilde{\phi}$ is an homeomorphism of $\mathbb{H}^2$ lifting of a homeomorphism of M isotopic to the identity, then $sup_{m\in\mathbb{H}^2}[d(m,\tilde{\phi}(m))]$ is finite). This proves the lemma if the half leaf $p(\delta)$ of $\mathcal{F}$ is compact or spirals towards a compact leaf: a compact leaf of $\mathcal{F}$ cannot be null-homotopic in M , as it cannot bound a disc, and then is (free)homotopic to a closed geodesic.

If the closure $\overline{p}(\delta)$ does not contain a compact leaf, we can choose a leaf $\delta_1 \in \overline{p}(\delta)$ and a closed curve c transverse to $\mathcal{F}$ and intersecting δ_1. The curve C meets δ infinitely many times, as it cannot bound a foliated disc, it is also homotopically not null-homotopic, so its lift to $\mathbb{H}^2$ will stay at bounded distance from the closed geodesic in the same free homotopy class. As the foliation $\mathcal{H}$ of $\mathbb{H}^2$ does not admit Whitehead discs, the half-leaf δ meets a component of $p^{-1}(C)$ in at most one point. The intersection in $\mathbb{H}^2 \cup S_\infty$ of the sequence of nested half-spaces which δ enters (see next picture) is exactly one point of S_∞, because it cannot contain any point of $\mathbb{H}^2$, as the distance between two different lifts of C is bounded below (it cannot contain two points of S_∞ without containing the geodesic joining them). $\square$

Remark: Two separatrices δ and δ' starting at the same saddle s of $\mathcal{H}$ converge to distinct points of S_∞.

Proof: This is true when the union $p(\delta) \cup p(\delta')$ meets at least twice a closed simple curve C transverse to the foliation $\mathcal{F}$, as, again, $\mathcal{H}$ has no Whitehead discs, so any component $p^{-1}(C)$ meeting δ or δ' separates the points at infinity of δ and δ'. If such a curve C does not exist, then $p(\delta)$ and $p(\delta')$ spiral towards compact leaves δ_0 and δ_0' of $\mathcal{F}$. If δ and δ' where isotopic, the compact leaves δ_0 and δ_0' should also be, as two geodesics which have compact projections cannot share a point at infinity if they do not coincide. If $\delta_0 = \delta_0'$ the union of two arcs starting at s of respectively $p(\delta)$ and $p(\delta')$, with an arc transverse to $\mathcal{F}$ joining their endpoints, will bound a Whitehead disc, providing a contradiction. If δ_0 and δ_0' were distinct, they should bound an anulus. This annulus cannot contain singularities of $\mathcal{F}$ because the singulaties of $\mathcal{F}$, all saddles, will give to the anulus a negative Euler characteristic.

Looking at the same time at $\mathcal{H}$ and $\mathcal{F}$ the reader will check that the only remaining possibility is that $p(\delta)$ and $p(\delta')$ are spiraling toward the same leaf of $\mathcal{F}$, on the same side, which again will allow the construction of a Whitehead disc.

□

So we can associate to each saddle of $\mathcal{H}$ three points of S_∞ which define an asymptotic triangle Δ_s (see next picture).

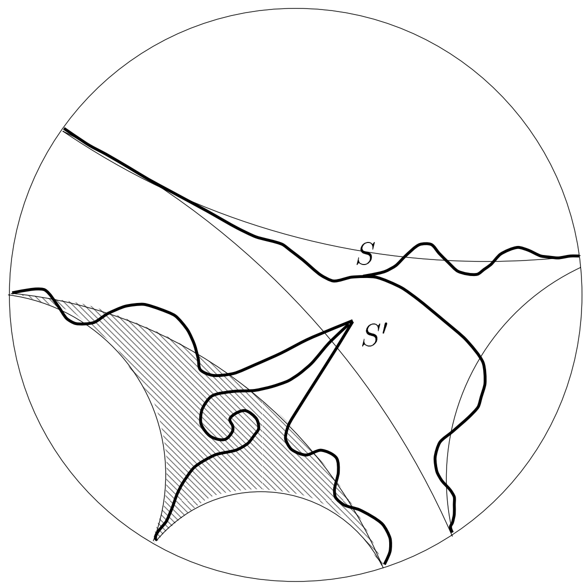


Figure 51: Asymptotic triangle associated to a saddle s

We will call the three geodesics joining these points at infinity the *asymptotes* of s. Two asymptotes starting at distinct saddles cannot intersect in $\mathbb{H}^2$ (as it will force an intersection of some of the separatrices), so the asymptotic triangles associated to distinct saddles have disjoint interiors.

Fix now a geodesic $\mathcal{L}$ of $\mathbb{H}^2$ which does not contain any saddle of $\mathcal{F}H$, is not asymptotic to any separatrix of $\mathcal{H}$, and is not tangent to any separatrix of $\mathcal{H}$ (these conditions are generic).

Definition 10.2.8 *Given a generic geodesic $\mathcal{L}$, the couple (s, D), s a saddle of $\mathcal{H}$, and D one of its three asymptotes , is called Ladmissible if it satisfies the following conditions :*
-$s \notin D$
-s and Δ_s are on the same side of D
-$\mathcal{L}$ does not intersect D and separates s from D

To each L-admissible couple (s, D) we will associate a compact domain $T_{s,D}$ see picture below).

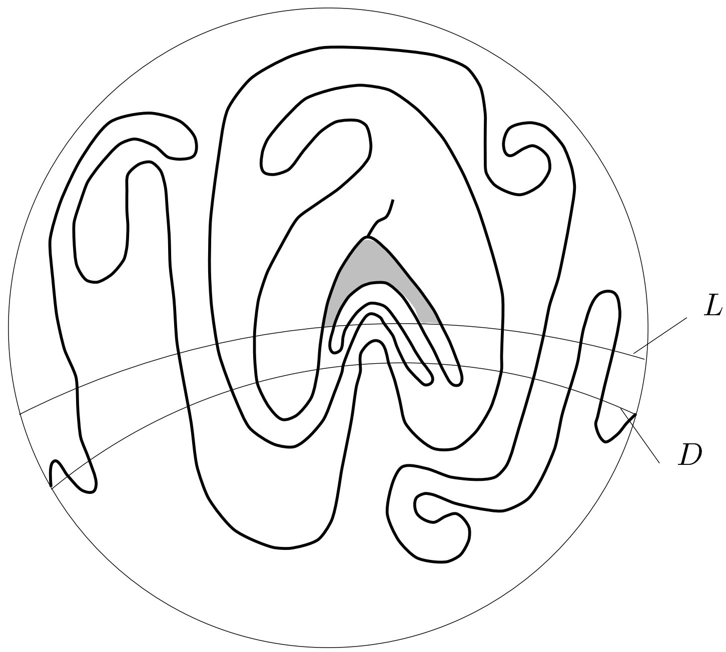


Figure 52: The domain $T_{s,d}$.

The two separatrices starting at s and asymptotic to D cut $\mathbb{H}^2$ into two domains. We will call $\mathcal{D}_{s,D}$ the closure of the one which does not contain the points at infinity of $\mathcal{L}$. Let us call $H^+(\mathcal{L}, s)$ the closed half plane of boundary $\mathcal{L}$ which contains s, and $T_{s,D}$ the connected component of $H^+(\mathcal{L}, s) \cap \mathcal{D}_{s,D}$ which contains s. The domain $T_{s,D}$ is compact and homeomorphic to a disc (see picture above).

If (s, D) and (s', D') are two L-admissible couples, only the four following situations are possible:

-$T_{s,D}$ is contained in $T_{s',D'}$

-$T_{s',D'}$ is contained in $T_{s,D}$

-$T_{s,D}$ and $T_{s',D'}$ are disjoint

-$T_{s,D}$ and $T_{s',D'}$ have disjoint interiors and $s = s'$

In particular the situation of the next picture is impossible.

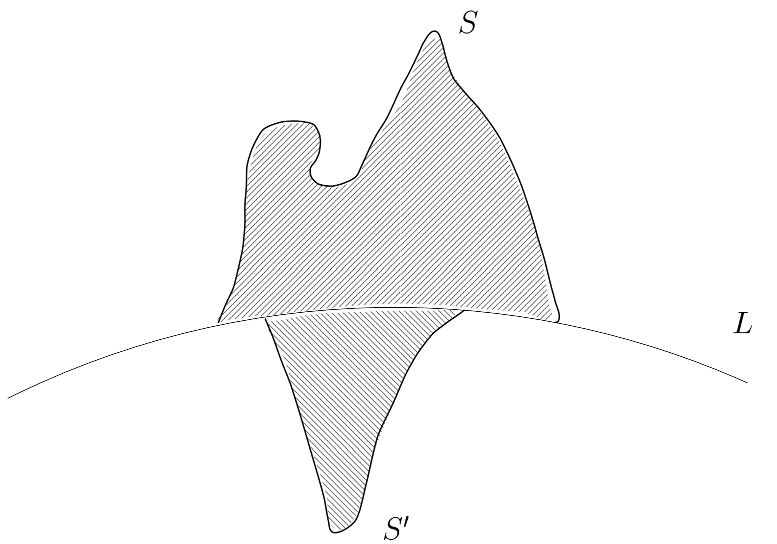


Figure 53: Impossible position of the two domains $T_{s,d}$ and $T_[s', d']$.

Lemma 10.2.9 *For any L-admissible couple (s, D), the collections of arcs $\mathcal{L} \cap T_{s,D}$ is tangent to $\mathcal{H}$ at at least one point.*

Proof: The compacity of $T_{s,D}$ and the fact that the set of saddles of $\mathcal{H}$ is discrete implies that $T_{s,D}$ can contain at most a finite number of domains $T_{s',D'}$. It is then enough to prove the lemma for a minimal (for the inclusion) domain $T_{s,D}$ (see next picture).

If the lemma is false, $T_{s,D}$ is a disc which does not contain in its interior any singularity of $\mathcal{H}$ and the boundary of which is made alternatively of arcs of leaves of $\mathcal{H}$ and arcs transverse to $\mathcal{H}$. Moreover, the definition of an L-admissible couple implies that, in a neighbourhood of s the third separatrix starting at s (the one which is not asymptotic to D) is not contained in $T_{s,D}$ (see next picture)

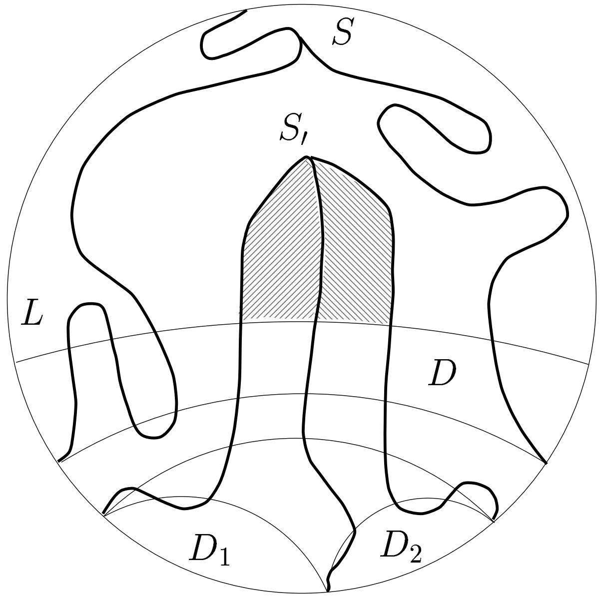


Figure 54: A domain $T_{s,d}$ which is not minimal.

The only possibility for $T_{s,D}$ is to be a "rectangle" (see picture above). The arc of the leaf between the points 2 and 3 on the above picture belongs to one of the separatrices starting at s and asymptotic to D, say the one which contains the point 1. Let us now consider the arc of separatrix joining 1 to 2. this arc does not meet the segment of $\mathcal{L}$ of extremities 1 and 2, and, with this segment, bounds a Whitehead disc, providing a contradiction. $\square$

We will call *strongly L-admissible* a couple (s, D) if it is L-admissible and if $\mathcal{L}$ meets Δ_s (and then the two sides of Δ_s different from D). Then, given $\mathcal{L}$, a saddle s cannot belong to more than one couple strongly L-admissible, and the domains $T_{s,D}$ corresponding to different couples strongly L-admissible are disjoint.

To a saddle s of $\mathcal{H}$ let us now associate the set A_s of geodesics $\mathcal{L}$ such that there exists an asymptote D of s such that the couple (s, D) is strongly L-admissible. We obtain the required injection $i_s : A_s \to \mathbb{H}^2$ associating to a geodesic $\mathcal{L}$ one of the points of $\mathcal{L} \cap T_{s,D}$ where $\mathcal{L}$ is tangent to $\mathcal{H}$ (see lemma 2). We can choose the injection i_s in an equivariant way, that is, if σ is an automorphism of the universal covering $\mathbb{H}^2 \to M$, and σ^* the induced transformation on the set of geodesics $\mathcal{A}$, then, for all saddles s of $\mathcal{H}$, $i_{\sigma(s)} \circ \sigma^* = \sigma \circ i_s$. Let us call B_s the image $i_s(A_s) \subset \mathbb{H}^2$. As, for a fixed

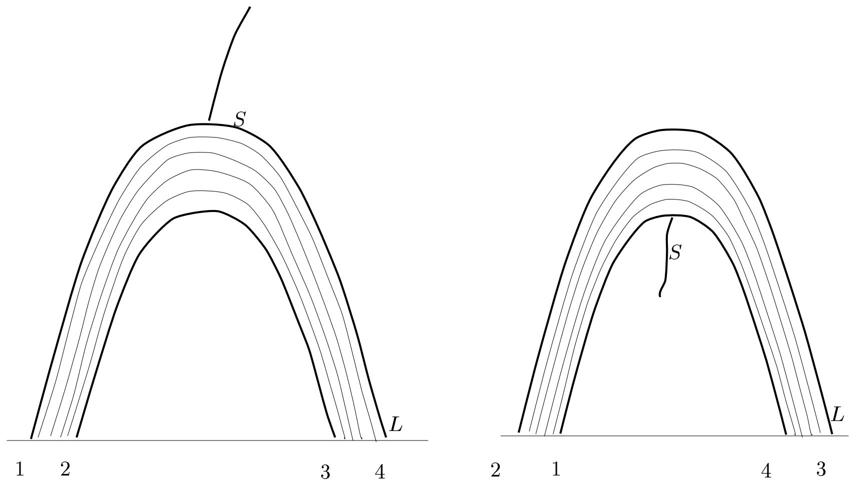


Figure 55: position of the third separatrix

$\mathcal{L}$, the domains $T_{s,D}$ corresponding to distinct strongly L-admissible couples are disjoint, we have: $B_s \cap B_{s'} = \emptyset$ if $s \neq s'$. To prove the affirmation stated above, we need now to check the inequality $m(A_s) \geq (6Log2 - 3Log3)$. Let us first proof a lemma of hyperbolic geometry.

Lemma 10.2.10 *Let $0 < \alpha \leq \pi$ be the angle between two geodesic rays starting at a point $s \in \mathbb{H}^2$ and asymptotic to a geodesic D (see picture below), and let $f(\alpha)$ be the measure of the set of geodesics which do not intersect D but separate D and s. Then:*

a)$f(\alpha) = -2Log\, sin(\alpha/2)$

b)if $0 < \alpha \leq \pi$, $0 < \beta \leq \pi$ and $0 < \gamma \leq \pi$ are three angles such that $\alpha + \beta + \gamma = 2\pi$, then

$$f(\alpha) + f(\beta) + f(\gamma) = 6Log2 - 3Log3$$

Proof: a) As $f(\pi) = 0$ it is enough to prove that $f'(\alpha) = -cotg(\alpha/2)$. Let $h(\alpha)$ be the (hyperbolic) distance between s and D, the quantity $f'(\alpha) \cdot d\alpha$ is equal to the measure of the set of geodesics intersecting a geodesic segment of infinitesimal length $dh = h'(\alpha) \cdot d\alpha$ with an angle bigger or equal to $\alpha/2$. This measure is proportional to dh and the coefficient $(2cos(\alpha/2)$, can be computed using the "euclidean" formula, tangent to the hyperbolic one if the origin is in dh. Then $f'(\alpha) = 2cos(\alpha/2) \cdot h'(\alpha)$. Hyperbolic trigonometry provides the formula $cosh(h(\alpha) = \frac{1}{sin(\alpha/2}$ (see for example [Thu2] formula 2.6.12). After checking that $h'(\alpha) = 1/2\, sin(\alpha/2)$, we get the required formula $f'(\alpha) = -cotg(\alpha/2)$.

b) Triples (α, β, γ) of angles between 0 and π parametrise the vertices of an asymptotic triangle. For example, if the point s is on the boundary of the triangle, one of the angles, say $\gamma = \pi$, and

$$f(\alpha) + f(\beta) + f(\gamma) = -2Log[sin(\alpha 2) \cdot sin((\pi - \alpha)2)] = 2Log(2/sin\alpha)$$

Then:

$$f(\alpha) + f(\beta) + f(\gamma) \geq 2Log2 > 6Log2 - 3Log3$$

If the point s tends to a vertex of the asymptotic triangle, then one of the angles goes to 0 and $f(\alpha) + f(\beta) + f(\gamma)$ goes to $+\infty$. to prove assertion (b) it is enough to check that the only extremum of $f(\alpha) + f(\beta) + f(\gamma)$ in the triangle is achieved when s is a center of symmetry and $\alpha = \beta = \gamma = 2\pi/3$. This is true, as the differential of the function $f(\alpha) + f(\beta) + f(\gamma)$ is $-cotg(\alpha/2) \cdot d\alpha$ is zero only if $cotg(\alpha/2) = cotg(\beta/2) = cotg(\gamma/2)$, that is if $\alpha = \beta = \gamma = 1\pi/3$ □

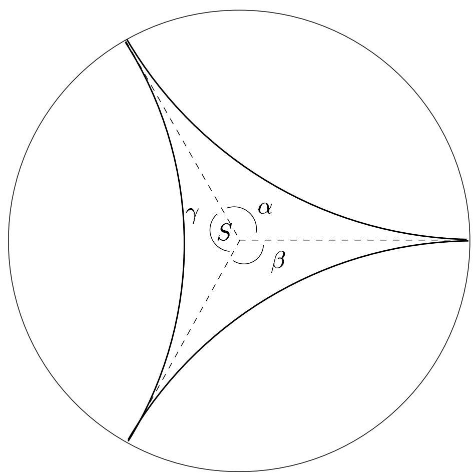


Figure 56: The only extremum of $f(\alpha) + f(\beta) + f(\gamma)$

Coming back to a saddle s of $\mathcal{H}$, two cases are possible:

-s belongs to the asymptotic triangle Δ_s (or to its boundary), then the previous lemma implies that:

$$m(A_s) \geq 6Log2 - 3Log3$$

-s is exterior to Δ_s (see next picture) then the couple (s, D_i), $(i = 1, 2)$ is strongly L-admissible for m-almost all geodesics $\mathcal{L}$ which does not intersect D_i and separating the point $t \in \partial\delta_s$ (see picture below) from D_i.

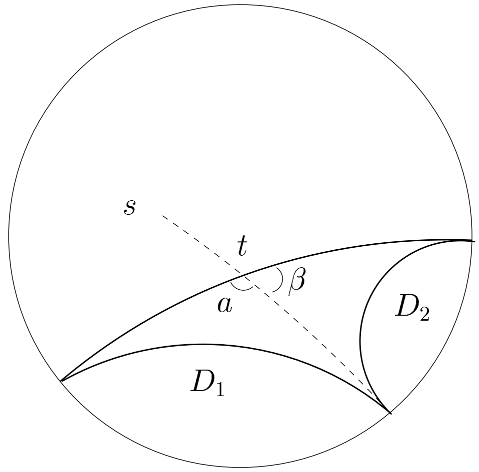


Figure 57: Outside of the asymptotic triangle δ_s.

Then

$$m(A_s) > f(\alpha) + f(\beta) = f(\alpha) + f(\beta) + f(\pi) \geq 6Log2 - 3Log3$$

We proved the affirmation. Let us now deduce the theorem from the affirmation. For each saddle $\overline{s}_i$ of $\mathcal{F}$ we choose a lift s_i in $\mathbb{H}^2$. Recall that the number of (three prong) saddles of $\mathcal{F}$ is $h = 2\chi(M)$. Let B be the disjoint union of the sets B_{s_i}. As for any automorphism σ of the covering, we have: $\sigma B = \cup_{i=1}^{h} A_{\sigma s_i}$ the sets B and σB are disjoint if $\sigma \neq Id$, and this implies that the restriction to B of the covering projection p is injective.

Suppose first there exists a neighbourhood U of B such that the restriction of p to U is also injective. Then the total curvature of $\mathcal{F}$ is bigger than or equal to $\sum_{i=1}^{h} m(A_{s_i}) \geq 2 \cdot |\chi(M)| \cdot (6Log2 - 3Log3)$. If such a neighbourhood U would exist, the theorem would be proven.

In general it is impossible to find the neighbourhood U of B, but we will construct, for each small $\epsilon > 0$, subsets $A^{\epsilon}_{s_i}$ such that $m(A_{s_i} \setminus A^{\epsilon}_{s_i})$ goes to 0 with ϵ, and such that we can find an open neighbourhood U^{ϵ} of the corresponding set B^{ϵ} to which the restriction of p is injective. The foliated exchange theorem implies that the theorem is a consequence of the existence of the sets $A^{\epsilon}_{s_i}$.

Let us fix $\epsilon > 0$ and let $\mathcal{L}$ be a geodesic of A_{s_i}. There exists then an asymptote D of s_i such that the couple (s_i, D) is strongly L-admissible. From s let us consider the geodesic ray orthogonal to D. It intersects D

at a point t. The geodesic D' is orthogonal to that ray at a point situated between s and t, at distance ϵ from t (see the picture below).

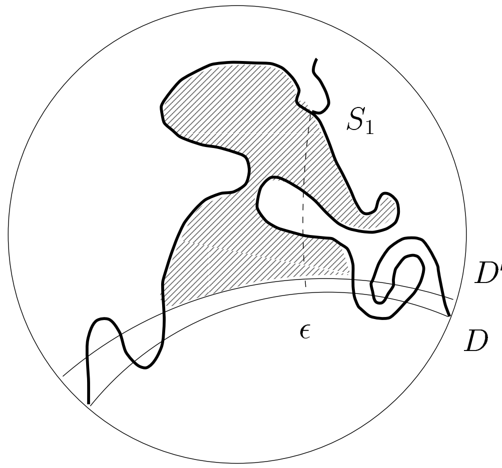


Figure 58: construction of A^ϵ

We can suppose that D' is transverse to the two separatrices starting at s_i and asymptotic to D and define as with s, D and $\mathcal{L}$ a compact domain $T^\epsilon_{s_i,D'} = \mathcal{D}_{s,D} \cap H^+_{s,D}$ (shaded on previous picture). Let n be the number of saddles contained in $T^\epsilon_{s_i,D'}$; we can choose a neighbourhood v_ϵ of the boundary $\partial T^\epsilon_{s_i,D'}$ of the domain $T^\epsilon_{s_i,D'}$, such that the total curvature of $\mathcal{H}|_{v_\epsilon}$ is bounded by ϵ.

We keep in $A^\epsilon_{s_i}$ a geodesic $\mathcal{L} \in A_{s_i}$ if and only if:

i) $\mathcal{L}$ does not intersect D' and separates s_i from D'

ii) the distance from $\mathcal{L}$ to each saddle $s \in T^\epsilon_{s_i,D'}$ is at least ϵ/n

iii) $\mathcal{L}$ is transverse to $\mathcal{H}$ in the neighbourhood v_ϵ of $\partial T^\epsilon_{s_i,D'}$

The exchange theorem and the definition of v_ϵ show that the measure $m(A_{s_i} \setminus A^\epsilon_{s_i})$ goes to zero with ϵ. Let $B^\epsilon_{s_i} \subset B_{s_i}$ be the image of $A^\epsilon_{s_i}$ in $\mathbb{H}^2$, and let $B^\epsilon = \cup_{i=0}^h B^\epsilon_{s_i}$.

To finish the proof we will show that for fixed ϵ, i and j, the distance from $B^\epsilon_{s_i}$ to the union of the conjugates of $B^\epsilon_{s_j}$ is strictly positive (if $i = j$ we use only conjugation of the covering different from the identity). Let then $Q \in B^\epsilon_{s_i}$ and $Q' \in B^\epsilon_{s_j}$ be such that Q and $\sigma Q'$ are very close (supposing again that σ is not the identity if $i = j$). The condition (iii) above implies that $\sigma Q'$ is in $T^\epsilon_{s_i,D}$ (see next picture).

The asymptotic triangle associated to the saddle σs_j is then on the side of D which does not contain s_i (the analogous condition interverting the

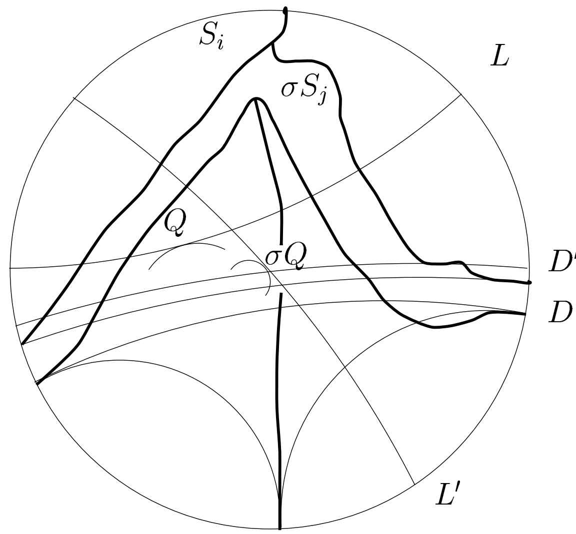


Figure 59: Position of $B^\epsilon_{s_i}$ and $B^\epsilon_{s_j}$.

roles of s_i and s_j may also happen). The geodesic $\mathcal{L}'$ tangent to $\mathcal{H}$ at $\sigma Q'$ should then meet D and D'.

One cannot define a reasonable metric on the set of all geodesics of $\mathbb{R}^2$ or $\mathbb{H}^2$. Two geodesics intersecting with a small angle should be close. Then it is impossible to separate parallel geodesics ($\mathbb{R}^2$) or asymptotic geodesics (geodesics with one point at infinity in common, in the $\mathbb{H}^2$ case). But it is possible to define a distance on the set of geodesics which intersect a given compact $K \subset \mathbb{H}^2$ by:

$$d_K(\mathcal{L}, \mathcal{L}') = sup[(inf_{x \in \mathcal{L} \cap K, y \in \mathcal{L}' \cap K} d(x,y)\, ;\, angle(\mathcal{L}, \mathcal{L}')]$$

if $\mathcal{L} \cap \mathcal{L}' \cap K = \emptyset$ (just forget the angle term if $\mathcal{L} \cap \mathcal{L}' = \emptyset$)

$$= angle(\mathcal{L}, \mathcal{L}')\, if\, \mathcal{L} \cap \mathcal{L}' = m \in K$$

The geodesics $\mathcal{L}$ and $\mathcal{L}'$ constructed above satisfy $d_K(\mathcal{L}, \mathcal{L}') \geq \eta > 0$ taking $K = T^\epsilon_{s_i, D'}$, where η does not depend on Q, Q' and σ. If $\mathcal{L}$ and $\mathcal{L}'$ do not intersect, or intersect far from K, they cannot be close in K and satisfy the required conditions. Otherwise , as our conditions (ii) guarantees $\mathcal{L}$ does not pass by to close to the saddles, this implies the distance between Q and $\sigma Q'$ is bounded below by a positive constant independent of Q, Q' and σ. We use the following fact: given $\theta > 0$ and a compact $K \subset M$ containing no saddle of $\mathcal{F}$, there exist $\alpha > 0$ such that, if two geodesics $\mathcal{L}_1$ and $\mathcal{L}_2$ tangent to $\mathcal{H}$ at two points a_1 and a_2 belonging to $p^{-1}(K)$ intersect at an angle bigger than θ, then the distance between a_1 and a_2 is at least α.

In [La-Le1], the reader can find an application of the foliated exchange theorem to pairs of orthogonal foliations of S^2.

10.3 Tight foliations

We have seen that the foliated exchange theorem and some topological analysis of the foliation provide inequalities. Do there exist foliations achieving the equality case? We had called tight such foliations. An example of a positive result is the following:

Theorem 10.3.1 *Let A be a plane annulus limited by two convex curves C_1 of length δ_1 and C_2 of length δ_2. We suppose that C_2 is the "inner" one (Cauchy-Crofton's formula implies that $\delta_1 > \delta_2$). Then the leaves of the tight foliation of the annulus (tangent to the boundary) are either closed convex curves isotopic in A to C_1 (and C_2) or locally convex curves spiraling towards convex curves isotopic to C_1. (see picture below). the total curvature of the foliation is, in that case:*

$$\int_A |k| = \delta_1 - \delta_2$$

Proof: Using Cauchy-Crofton's formula, we know that the set B of affine lines intersecting C_1 and not intersecting C_2 has measure $\delta_1 - \delta_2$. Such a line $\mathcal{L}$ intersect the annulus in a segment I. The foliation $\mathcal{F}$ is not transverse to the interior of I, otherwise the boundary of C_1 and I would form a Whitehead disc for $\mathcal{F}$, which is impossible as $\mathcal{F}$ has no singularity. Then

$$|\mu|(\mathcal{F}, \mathcal{L}) \geq 1$$

so the total curvature of $\mathcal{F}$ is bigger or equal than the measure of B. The equality is achieved for the foliations described in the theorem, as they satisfy:

$$\mathcal{L} \in B \Rightarrow |\mu|(\mathcal{F}, \mathcal{L}) = 1$$

$$\mathcal{L} \notin B \Rightarrow |\mu|(\mathcal{F}, \mathcal{L}) = 0$$

□

In [Lan2], the reader will find a study of tight (in their isotopy class) foliations of the torus T^2.

Let us now consider the same question for (nonsingular) foliations of S^3.

Theorem 10.3.2 *Their does not exist any tight foliation of the sphere S^3.*

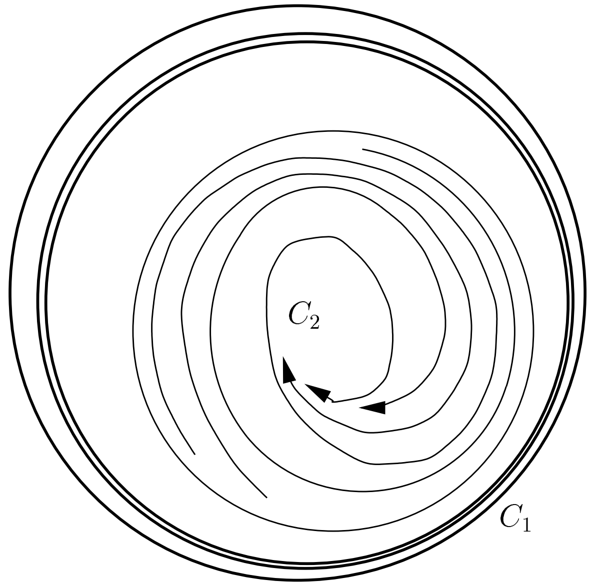


Figure 60: Tight foliation of a plane annulus with convex boundary curves.

Proof: We have seen before that the total curvature of a foliation $\mathcal{F}$ of S^3 satisfies:

$$\int_{S^3} |K| \geq 2\pi^2$$

because for a generic totally geodesic sphere $\Sigma \subset S^3$ one has $|\mu|(\mathcal{F},\Sigma) \geq 2$. We have also seen that

$$\int_{S^3} K = 2\pi^2$$

If a foliation $\mathcal{F}$ of S^3 satisfy $\int_{S^3} |K| = \int_{S^3} K$, then the curvature function should satisfy $K \geq 0$. In S^3 the intrinsic curvature K_e of an embedded surface satisfy $K_e = K + 1$ (one can perform the computation using the exponential map (see [Spi]). Novikov's theorem states that the foliation has a Reeb component ([Ca-Li]) with boundary a torus leaf L. The Gauss-Bonnet theorem applied to L states that $\int_L K_e = 0$. Then $\int_L K = -vol(L) < 0$ so the leaf has a point of negative (extrinsic) curvature K, contradicting the hypothesis.

The theorem will then be proved if we can show that:

$$inf \int_{S^3} |K| = 2\pi^2$$

Let us consider the singular foliation $\mathcal{P}$ of S^3 defined by a pencil of geodesic 2-spheres. It has a one dimensional singular locus: a geodesic circle C. The trace of $\mathcal{P}$ on a geodesic sphere Σ transverse to C is a foliation with two singular points of index 1 (of type sink/source).

The next object we need is the model Reeb foliation of the thick torus $D^2 \times S^1$. To obtain it we will construct a foliation of $D^2 \times \mathbb{R}$ invariant by unit translations in $\mathbb{R}$ (we can visualise $D^2 \times \mathbb{R}$ as a vertical thick cylinder). In the vertical band $[-1, 1] \times \mathbb{R}$ of the $(x, z) - plane$ consider a convex curve asymptotic to both sides of the band.

The equation $z = tg(\pi/2)x^2$ should provide such a curve. by revolution around the $z - axis$ we obtain a convex surface asymptotic to the boundary of the cylinder (on the $z \to +\infty$ side. Translating it vertically, we foliate the thick cylinder. By construction the foliation is invariant by vertical translation and then gives a foliation of the thick torus $T = (D^2 \times \mathbb{R}/(2\pi \cdot \mathbb{Z})$. (see picture below)

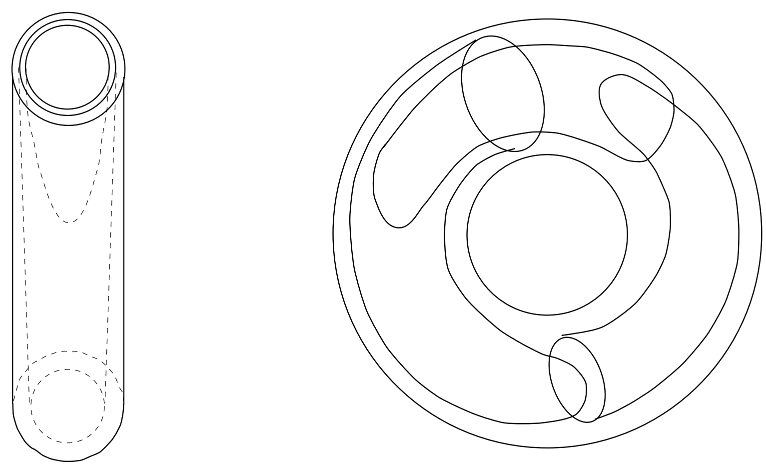


Figure 61: Reeb component.

We will now shadow the foliation $\mathcal{P}$ by non singular ones, introducing a very thin Reeb component in a tubular neighbourhood of C.

To construct the foliation in a tubular neigbourhood $Tub_{2r}(C)$ of radius 2r of C, we will first construct a model in the cylinder $D^2_{2r} \times \mathbb{R}$, invariant by vertical translations.

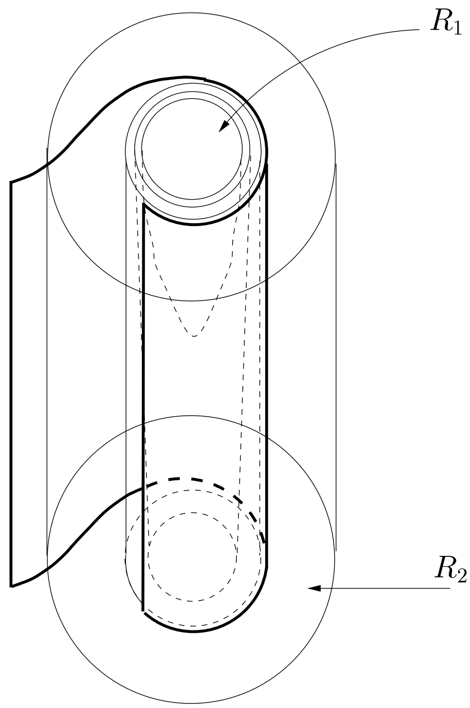


Figure 62: A piece of a thin Reeb component and how the other leaves wrap around it.

In the cylinder $D_r^2 \times \mathbb{R}$ just put a Reeb component defined as above. In the anulus $D_{2r}^2 \setminus D_r^2$, seen as a subset of the $(x, y) - plane$, consider a curve entering, normally to the boundary, into D_{2r}^2 and spiraling towards the circle ∂D_r^2 (see picture below).

The product of that curve by the vertical line is a surface of $\mathbb{R}^3$ entering normally the cylinder $D_r^2 \times \mathbb{R}$ and spiraling toward the inner cylinder $D_r^2 \times \mathbb{R}$. By rotation around the z-axis we foliate the set $(D_{2r}^2 \setminus D_r^2) \times \mathbb{R}$. So we get the desired foliation of the thick cylinder $D_r^2 \times \mathbb{R}$.

The quotient by the vertical translations by vectors of lenth 2π is a foliation of $D_{2r}^2 \times S^1$. Let us now map $D_{2r}^2 \times S^1$ to the tubular neighbourhood of (geodesic) radius 2r of C, mapping isometrically S^1 on C and using the exponential map to map the discs D_{2r}^2 centered on points $(0, 0, z) \in S^1$ onto totally geodesic discs normal to C. We obtain a foliation $\mathcal{F}_r$ which fits with $\mathcal{P}|_{S^3 \setminus Tub_{2r}(C)}$. The reader will now believe that :

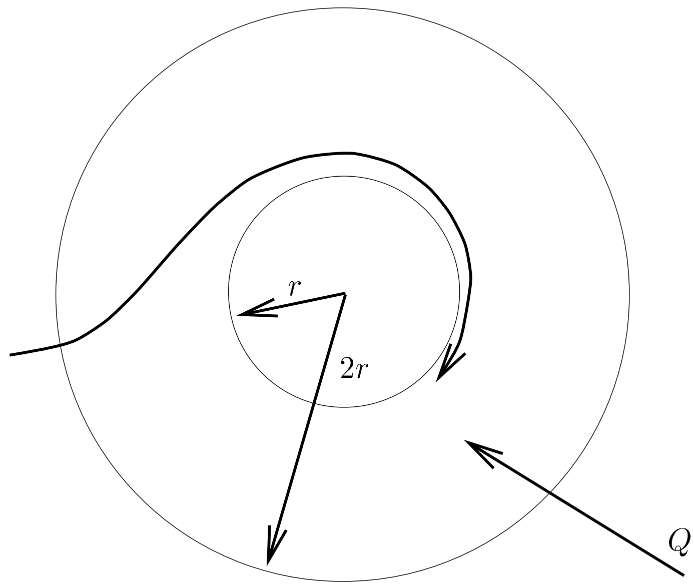


Figure 63: Horizontal section of the foliation $(D^2_{2r} \setminus D^2_r) \times \mathbb{R}$.

-the geodesic spheres Σ satisfy $|\mu|(\mathcal{F}_r, \Sigma) = 2$ if Σ intersects C with not too small an angle.

-There exists a uniform bound, independent of r, for the number $|\mu|(\mathcal{F}_r, \Sigma)$ when it is finite.

As the measure of the geodesic spheres which intersect C with an angle smaller than ϵ goes to zero with ϵ, we proved, using the foliated exchange theorem, that :

$$lim_{r \to 0} \int_{S^3} |K| = 2\pi^2$$

where $|K|$ is the curvature function defined by the leaves of $\mathcal{F}_r$. □

Foliations of codimension higher than one and diverging integrals.

We will present here without proofs particular cases of the results of [La-Ni].

Theorem 10.3.3 *Let $\mathcal{F}$ be a smooth foliation by curves of a domain $W \subset \mathbb{R}^3$. Let C_H be the contact set (in general a curve) of $\mathcal{F}$ and the affine hyperplane H:*

$$C_H = \{m \in W | T_m \mathcal{F} \subset H\}$$

Then

$$\int_W |k| = const \int_{\mathcal{A}(3,1)} \int_{C_H} |sin\varphi|$$

Where at a smooth point $m \in C_H$, φ *is the angle between* C_H *and the leaf of* $\mathcal{F}$ *through* m.

11 Integral geometry in spheres

The results of this paragraph come from [La-Ro2]. When C is a submanifold of dimension p of S^N,we shall use the notation $|C|$ for the p-volume of C . We sometimes for aesthetic reasons shall use the notation $L_p(M)$.

11.1 The spherical formula of Cauchy and Crofton

We shall prove it for surfaces $M \in S^3$, the proof for hypersurfaces of S^n is identical. The proof for higher codimension submanifolds is more technical; see [Sa2] [La-Ro2]. We denote by $L_2(M)$ the area of the surface $M \subset S^3$.

Theorem 11.1.1

$$L_2(M) = \frac{1}{\pi} \int_{G(4,2)} |M \cap l| dl,$$

where l is a geodesic circle of S^3 which we can think of as a 2-plane through the origin of $\mathbb{R}^4$; $|M \cap l|$ is the number of points of $M \cap l$.

Proof: Denote by $P(E)$ the projective space of vectorial lines of the vector space E. From the restriction to M of the tangent bundle to S^3 we construct the fiber bundle $\mathbb{P}(TS^3|_M)$ replacing the fibers $\mathbb{R}^3$ by projective planes $\mathbb{P}_2$. Denote by $\mathbb{P}_m(TS^3|_M)$ its fiber above the point $m \in M$; it is a riemannian fiber bundle on M. Consider the map

$$\phi : \mathbb{P}(TS^3|_M) \to G(4,2),\ \phi(m, L) = l$$

where l is the geodesic circle whose tangent at m is $L \in \mathbb{P}(T_m M)$.

Write the tangent space to $G(4,2)$ at l_0 as an orthogonal sum:

$$T_{l_0} G(4,2) = T_{l_0}\{l | m \in l\} \oplus T_{l_0}\{l \perp \Sigma_{l_0,m}\},$$

where $\Sigma_{l_0,m}$ is the geodesic 2-sphere orthogonal to l_0 at m.

Write $T_{(m,L)}(\mathbb{P}(TS^3|_M) = V \oplus H$, where V is the tangent space to the fiber and $H = V\perp$. Then $d\phi$ is given by the matrix:

$$\begin{pmatrix} Id & * \\ 0 & p_{L^\perp} \end{pmatrix},$$

where $p_{L^\perp}$ is the orthogonal projection of $T_m M$ to $T_m(\Sigma_{l,m}) = L^\perp$. Then:

$$\int_{L \in \mathbb{P}_m(TS^3|_M)} |Jac(d\phi)| = \int_{\mathbb{P}_2} |cos(angle(L^\perp, T_m M))| = \pi$$

Since

$$\int_{G(4,2)} |(\phi)^{-1}(l)| = \int_{G(4,2)} |l \cap M|,$$

we have:

$$\int_{G(4,2)} |l \cap M| = \pi |M| = \pi L_2(M)$$

□

11.2 Flags

A flag in a vector space is a nested sequence of subspaces

$$(h_1 \subset h_2 \subset ... \subset h_k)$$

We call it complete if it contains a subspace in each dimension.

Let us denote by $|\mu|(M, \mathcal{F})$ the number of contact points of the submanifold M and the codimension one foliation $\mathcal{F}$. The notion makes sense even if the foliation admits a singular locus, as far as it is of codimension higher than one.

In S^2 a complete flag is just a pair $\Sigma_0 \subset \Sigma_1$, where σ_0 is a pair of antipodal points $(x, -x)$, intersection of S^2 with a vectorial line and Σ_1 a geodesic circle intersection of S^2 with a vectorial plane.

In S^3 a complete flag is a a sequence

$$\Sigma_0 \subset \Sigma_1 \subset \Sigma_2$$

of spheres, intersection of S^3 with vectorial subspaces of $\mathbb{R}^4$ of dimension 1,2,3. Replacing 1,2,3 by 1,2,...k and $\mathbb{R}^4$ by $\mathbb{R}^{k+1}$, we get the definition of a complete flag of S^k.

Definition 11.2.1 *We denote by $\mathcal{D}_k$ the set of complete flags of S^k*

We start with curves $C \subset S^2$ to give the flavour of the proofs, although the significant results start in S^3. We can define the total number of contact points of a curve C with the foliations associated to a complete flag Δ:

$$Geom(C, \Delta) = \sharp(C \cap \Sigma_1) + |\mu|(C, \mathcal{F}(\Sigma_0)$$

In Bourbaki style, the first number would be the number of contact points of $(C \cap \Sigma_1)$ with the point foliation of Σ_1. We can now define:

Definition 11.2.2

$$Geom(C) = \frac{1}{vol(\mathcal{C}_2)} \int_{\mathcal{D}_2} Geom(C, \Delta)$$

The number $Geom(C,\Delta)$ plays the same role as the total number of critical points of the orthogonal projection of the curve on a line in plane geometry. Let us first construct a sequence of foliations by curves in S^2 associated to a complete flag Δ, which will better and better follow the foliations $\mathcal{F}_0$ of Σ_1 by points and the foliation $\mathcal{F}(\Sigma_0)$ of S^2.

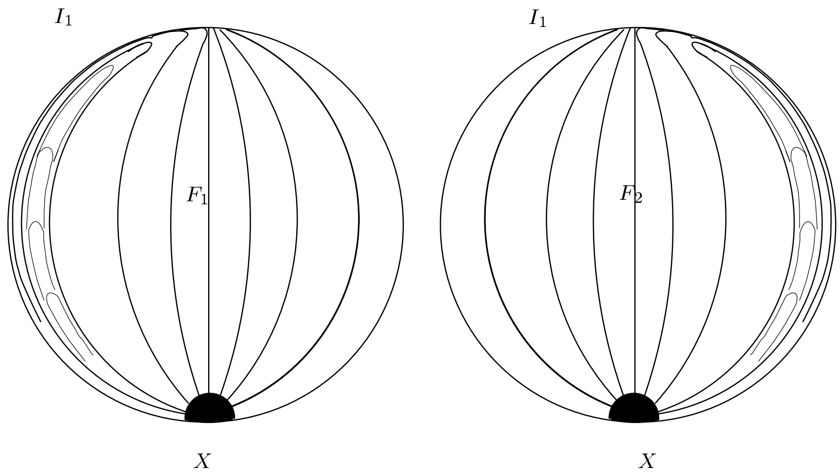


Figure 64: Foliation close to a pencil.

Chose a point $x \in \Sigma_0$ and delete from S^2 a small disc $B(x,\epsilon)$ of radius ϵ centered at x. The circle Σ_1 is divided in two arcs of length π, δ_1 and δ_2 by the two antipodal points $(x,-x)$ of Σ_O Now follow, starting near x, δ_1 with very thin nested arcs with boudary on the boundary of the small disc up to the $(-x) \in \Sigma_0$. Then continue the construction of the foliation with arcs, the left side of which will sneak along δ_1 from $(-x)$ to x and the right part of which will sweep half of the sphere Σ_2 by curves mostly equal to arcs (geodesic arcs) of the foliation $\mathcal{F}(\Sigma_0)$. The last leaf is $\Sigma_1 \cap (complement\, of\, the\, small\, disc)$. Proceed symetrically to fill up the other half of Σ_2. We shall call $\mathcal{F}_\epsilon$ the foliations associated to Δ. Do not ask the author what exactly means ϵ in the construction!

Observe that the foliation we have constructed is a product foliation by intervals of $S^2 \setminus B(x,\epsilon)$. This gives a diffeomorphism sending $S^2 \setminus B(x,\epsilon)$ to the plane, the leaves of the foliation to the horizontal affine lines and C to another closed curve.

As the projection of this image curve on the vertical has at least two critical points, we know that C has at least two points of contact with the foliation.

Corollary 11.2.3 *Any closed curve in S^2 satisfies:*

$$Geom(C) \geq 2$$

In the sphere S^3, $\Sigma_0 \subset \Sigma_1 \subset \Sigma_2$ allows us to define a pencil of circles $\mathcal{F}(\Sigma_0)$ in Σ_2: the circles of Σ_2 which contain Σ_0 In the same way $\Sigma_1 \subset \Sigma_2 \subset S^3$ allows us to define a pencil of 2-dimensional geodesic spheres $\mathcal{F}(\Sigma_1)$: the geodesic spheres which contain Σ_2.

We define:

Definition 11.2.4

$$Geom(M, \Delta) = \sharp(M \cap \Sigma_1) + |\mu|((M \cap \Sigma_2), \mathcal{F}(\Sigma_0)) + |\mu|(M, \mathcal{F}(\Sigma_1))$$

and:

$$Geom(M) = \frac{1}{vol(\mathcal{C}_3)} \int_{\mathcal{C}_3} Geom(M, \Delta)$$

Let us now construct the foliations $\mathcal{F}_\epsilon$ approximating the foliations defined by the complete flag Δ. The point x is disjoint from M and we choose ϵ such that the ball $B(x, \epsilon)$ does not meet M. Let H_1 and H_2 be the hemispheres of Σ_2 bounded by Σ_1. Let $\mathcal{F}_\epsilon^2$ be the one dimensional foliation of $\Sigma_2 \setminus (B(x, \epsilon) \cap \Sigma_2)$ defined above. The trace on Σ_2 of $\mathcal{F}_\epsilon$ will be $\mathcal{F}_\epsilon^2$. Each leaf α of $\mathcal{F}_\epsilon^2$ (more precisely, each leaf α of $\mathcal{F}_\epsilon^2$, together with an arc (we choose one of length $\leq \pi \cdot \epsilon$) on $\partial B(x, \epsilon) \cap \Sigma_2$ joining the extremities of α) bounds a disc in Σ_2 Let $D(\alpha)$ be the "small" one; there will be only one ambiguous case: when α is an arc of σ_1. Starting with the small arcs α emerging near x which sneak along δ_1 we obtain discs $D(\alpha)$ which are thin flat tongues. Now inflate those to obtain thin glove fingers following δ_1. When the discs $D(\alpha)$ spead over H_1, inflating them slightly provides thin pancakes, foliating a thickening of H_1. Next step fills one of the half spheres , say B_1 of boundary Σ_2, inflating the last pancake of the previous step dissimetrically. One of the sides will sweep B_1 following the pencil of geodesic spheres $\mathcal{F}(\Sigma_1)$, the other side will just move slightly . We are in fact sweeping the ball B_1 exactly as we swept a disc of S^2, bounded by a geodesic circle Σ_1 . We proceed symmetrically to fill the other half of S^3. The foliations $\mathcal{F}_\epsilon^3$ we have constructed prove the following lemma:

Lemma 11.2.5 *For any flag Δ in general position with respect to M, there exists a sequence of foliations $\mathcal{F}_\epsilon^3$ by discs of $S^3 \setminus B(x, \epsilon)$ such that:*

$$lim_{\epsilon \to 0} |\mu|(M, \mathcal{F}_\epsilon^3) = Geom(M, \Delta)$$

Moreover the foliations $\mathcal{F}_\epsilon^3$ are product foliations defining a diffeomorphism

$$\Phi_\epsilon : S^3 \setminus B(x, \epsilon) \to I\!R^3$$

Proof: The reader should to check that the contact points of $\mathcal{F}^3_\epsilon$ and M, for ϵ small enough, correspond to points counted in $Geom(M, \Delta)$. □

Morse theory applied to the ℝ-valued function defined by the foliation $\mathcal{F}^3_\epsilon$ implies that

$$|\mu|(M, \mathcal{F}^3_\epsilon) \geq 2g + 2,$$

so we get,using the considerations of the chapter **Integral geometry and topology** the theorem :

Theorem 11.2.6 *Let M be a surface embedded in S^3, then*

$$Geom(M) \geq 2g + 2$$

If M is a knotted torus, then

$$Geom(M) \geq 8$$

and if M is a knotted (oriented) surface of genus g then:

$$Geom(M) \geq 2g + 4$$

Instead of integrating $Geom(M, \Delta)$ we could have integrated separately the different terms

$$\sharp(M \cap \Sigma_1), \ |\mu|((M \cap \Sigma_2), \mathcal{F}(\Sigma_0)), \ |\mu|(M, \mathcal{F}(\Sigma_1)).$$

Integrating on $\mathcal{C}_3$ a geometric term which depends only on one of the constituents of the complete flags Δ just multiply by a constant depending only on dimensions the corresponding integral on the set of geodesic k-spheres of S^3.

We can now recognize spherical versions of the p-lengths defined in section **higher dimensional convex bodies and related matters**:

$$L_p(M) = C(N, n, p) \int_{G(N,p+1)} |\gamma_h| dh$$

where $|\gamma_h|$ denotes the volume of γ_h (when $p = O$, γ_h is a finite set and $|\gamma_h|$ is the number of points $\sharp(\gamma_h)$ of γ_h). Recall that, in the euuclidean case, the constant $C(N, n, p)$ has been chosen so that if M is the boundary of an ϵ-tubular neighbourhood in a (n+1)-dimensional space h of a p-dimensional submanifold C of h, then:

$$lim_{\epsilon \to O} L_p(M) = |C|$$

First observe that the set of antipodal pairs in S^3 is the Grassmann manifold $G(4, 1)$, the set of geodesic circles is $G(4, 2)$ and the set of geodesic spheres is $G(4, 3)$.

The reader will easily believe that the integral:

$$\int_{G(4,2)} \sharp(M \cap \Sigma_1)$$

is proportional to the area of M. Define in S^3:

Definition 11.2.7

$$L_2(M) = \frac{1}{\pi} \int_{G(4,2)} \sharp(\Sigma_1 \cap M)$$

To unify notations we will note:

$$|\Sigma_1 \cap M| = \sharp(\Sigma_1 \cap M)$$

A pencil $\mathcal{F}(\Sigma_1)$ of geodesic 2-spheres of axis a geodesic circle Σ_1 defines a projection $p_{\mathcal{F}(\Sigma_1)}$ of $S^3 \setminus \Sigma_1$ on the set $\{leaves\ of(\mathcal{F}(\Sigma_1))\}$ which is a circle. Restricted to $M \setminus (M \cap \Sigma_1)$ this projection has in general a discrete critical locus γ_{Σ_1} and a finite number of critical values $|\gamma_{\Sigma_1}|$. Define:

Definition 11.2.8

$$L_0(M) = \frac{1}{2volG(4,2)} \int_{G(4,2)} |\gamma_{\Sigma_1}|$$

As the function $p_{\mathcal{F}(\Sigma_1)}$ is generically a Morse function on $M \setminus (M \cap \Sigma_1)$ the number $|\gamma_{\Sigma_1}|$ is generically equal to the number $|\mu|(M, \mathcal{F}(\Sigma_1))$. So the integral of the last term of $Geom(M, \Delta)$ is proportional to $L_0(M)$.

To define the 1-lenth $L_1(M)$, project M on a geodesic sphere Σ_2 following the geodesic arcs orthogonal to it. These arcs are contained in the geodesic circles containing the two points $h^{bot} \cap S^3 = (x, -x)$ where h is the subspace of $\mathbb{R}^4$ such that $h \cap S^3 = \Sigma_2$. We say that the points $(x, -x) = h^{bot} \cap S^3$ are conjugate to Σ_2. The arcs are of the form $\Sigma_1 \setminus (x, -x); (x, -x) \subset \Sigma_1$. Loosing only a measure zero set of spheres,we can suppose that none of the conjugate points $x, -x$ to geodesic spheres Σ_2 are on M. Denote by p_{Σ_2} this projection on Σ_2 and by γ_{Σ_2} its critical locus.

Definition 11.2.9

$$L_1(M) = \frac{1}{\pi^2} \int_{G(4,3)} |\gamma_{\Sigma_2}|$$

It is also true, but less straitforward to prove, that the integral of the middle term of $Geom(M, \Delta)$ is proportional to $L_1(M)$. This last result is a consequence of the following kinematic-type formula:

Theorem 11.2.10 *Let M be a surface in S^3 Then:*

$$L_1(M) = \frac{1}{\pi}\int_{G(4,3)} L_0(M \cap \Sigma)$$

where Σ runs over the set of all geodesic 2-spheres of S^3.

Proof: First observe that the constant is obtained considering small spheres of geodesic radius t. Then $L_1(S_t) \approx 4t$ and $\int_{G(4,2)} L_o(S_t \cap \Sigma) \approx 4\pi t$. Recall that by definition

$$L_1(M) = \frac{1}{2\pi^2}\int_{G(4,3)} |\gamma_\Sigma|.$$

The Cauchy-Crofton formula in S^2 says:

$$|\gamma_\Sigma| = \frac{1}{2}\int_{G(3,2)} |\gamma_\Sigma \cap l|$$

where l runs over the set of geodesic circles in Σ.

The inverse image of the orthogonal projection onto Σ of the geodesic circle l is a sphere Σ_l. the points of $\gamma_\Sigma \cap l$ are the critical points of the orthogonal projection of $\Sigma_l \cap M$ onto l. The reader is invited to compare this argument with the argument proving the linear reproductibility formula in section **Blashke's formulas and kinematic formulas**. Hence:

$$L_1(M) = \frac{1}{4\pi^2}\int_{G(4,3)}\int_{G(3,2)} |\gamma_\Sigma \cap l| = \frac{1}{4\pi^2}\int_{D(4,3,2)} |\mu|(\Sigma_l \cap M, \mathcal{F}(l)),$$

where $\mathcal{F}(l)$ is the (singular) foliation of the 2-sphere Σ_l by geodesic circles orthogonal to l. Here $D = D(4,3,2)$ is the space of flags $(\Sigma, l), \Sigma \supset l$. The flag space D fibers over $G(4,3)$ and over $G(4,2)$, so using Fubini's theorem for both fibrations, we get:

$$L_1(M) = \frac{1}{4\pi^2}\int_{G(4,3)} 4\pi L_0(\Sigma \cap M) = \int_{G(4,3)} L_0(\Sigma \cap M)$$

□

Gathering our results we can express $Geom(M)$ in terms of the p-lengths or of integrals of the functions h_i.

Theorem 11.2.11 *[La-Ro2] Let M be a compact surface in S^3; then:*

$$Geom(M) = \pi^2 L_2(M) + 4\pi^3 L_1(M) + 2\pi^2 volG(4,2) L_0(M)$$

$$Geom(M) = \int_M [\pi^3 + 2\pi h_1 + \frac{\pi}{2} volG(4,2)|K|]$$

11.3 Functions h_i

In this subsection we construct functions on M the integral of which are the spherical p-lengths $L_p(M)$ analogous to the euclidean p-lengths defined in section **higher dimensional convex bodies and related matter** and define the functions which localize them.

Let Σ_{p+1} be a (p+1)-dimensional geodesic sphere of S^N; it is the intersection of a (p+2) plane h_1 of $\mathbb{R}^{n+1}$ with S^n. The intersection $(h_1)^\perp \cap S^N$ is called the (geodesic) sphere conjugate to Σ_{p+1}; we denote it by Σ^*_{p+1}. The set of geodesic spheres of dimension (N-p-1) containing the (N-p-2) gedesic sphere Σ^*_{p+1} foliate $S^N \setminus \Sigma^*_{p+1}$. Moreover each leaf of the foliations meets Σ_{p+1} in two antipodal points. The foliation then defines a projection $p_{\Sigma_{p+1}}$ of $S^N \setminus \Sigma^*_{p+1}$ on $\mathbb{P}_{p+1}$. Consider the restriction of this projection to $M \setminus (M \cap \Sigma^*_{p+1})$.

Definition 11.3.1 *The polar variety $\Gamma_{\Sigma_{p+1}}$ is the closure of the set of critical point of the restriction $p_{\Sigma_{p+1}}|_{M\setminus(M\cap\Sigma^*_{p+1})}$.*

The critical locus $\gamma_{\Sigma_{p+1}}$ is the closure of the inverse image by the covering map

$$\pi : S^{p+1} \to \mathbb{P}^{p+1}$$

of the critical locus of $p_{\Sigma_{p+1}}|_{M\setminus(M\cap\Sigma^*_{p+1})}$.

To define the p-lenth we need just to integrate the p-volume $|\gamma_{\Sigma_{p+1}}|$ of $\gamma_{\Sigma_{p+1}}$.

Definition 11.3.2

$$L_p(M) = const \cdot \int_{G(N+1,p+2)} |\gamma_{\Sigma_{p+1}}|$$

where the constant depends only on the dimensions involved and is chosen in such a way that:

$$lim_{r\to 0} L_p(Tub_r(M)) = p - volume(M)$$

if M is p-dimensional.

When M is of codimension 1 the functions $h_i(m)$ are defined exactly as in the euclidean case using the second fundamental form of $M \subset S^n$. The numbers $|k(m,h)|$ are absolute values of the determinant of the restriction of this second fundamental form to $h \subset T_m M$, expressed in an orthonormal basis.

Remark: The inverse image $(exp_m)^{-1}(M) \subset T_m S^n$ has at $m \in (exp_m)^{-1}(M)$ the same fundamental form as $M \subset S^n$ at $m \in M$.

We can know state a localization theorem:

Theorem 11.3.3 *Let M be a codimension 1 submanifold of S^n. The functions $h_{n-i}(m)$ localize the i-lenghts $L_i(M)$; more precisely:*

$$\int_M h_{n-1-i} = const \cdot L_i(M)$$

The proof is technical.

12 The space of spheres

Let L be the Lorentz quadratic form defined on the n-dimensional space E by:

$$L(x_1, x_2, ..., x_n) = (x_1)^2 + (x_2)^2 + ... - (x_n)^2$$

We will call *light cone* the isotropic cone of L. We note also L the associated bilinear form, and call L-orthogonal vectors a, b such that $L(a, b) = 0$.

Let us prove that the set $\mathcal{S}$ of oriented $(n-3)$-spheres of the sphere S^{n-2} admits a bijection with the set of points of the quadric Λ of equation $L = 1$.

The points at ∞ of the light cone form two (n-2)-dimensional spheres. We retain the "positive" one S^3_∞, that is the points at ∞ of the light cone in the upper half space $x_n > 0$.

Definition 12.0.4 *A vector of E is called space-like if L(v)¿0. It is called time-like if $L(v) < 0$. A line is called space-like (resp time-like) if it contains a space-like (resp time-like) vector.*

Any space-like line L intersects the quadric Λ in exactly two points. The hyperplane orthogonal (for the Lorentz quadratic form) to a space-like line L (notation $L^{L\perp}$), intersects the light cone transversly and therefore intersects its positive sphere at ∞ in a sphere Σ_L. This gives a correspondance between the set $\mathcal{S}$ of oriented (n-3)-dimensional spheres of S^{n-2} and the quadric Λ.

Proposition 12.0.5 *Let c be a path in Λ. If at each point c(t) of the path, the tangent vector v(t) satisfies:*

$$L(v) > 0, \ (space - like \ curve),$$

the corresponding family of spheres Σ_t admits an envelope;
if

$$L(v) < 0, \ (time - like \ curve),$$

at any point of the path, the spheres Σ_t are nested.

Proof: As c(t) belongs to Λ, that is satisfies $L(c(t)) = 1$, one has $L(c(t), v) = 0$. The condition for a 1-parameter family of spheres to admit an envelope is that the L-orthogonal space $car(t)$ to the plane generated by c(t) and v(t) intersects the light cone. The intersection of $car(t)$ with the sphere S^3_∞ is a caracteristic circle of the envelope, that is the limit

$$lim_{h \to 0} \Sigma_t \cap \Sigma_{t+h}$$

As c(t) and v(t) are L-orthogonal, it is equivalent to $L(v(t)) > 0$ □

Let us also observe that $L = -1$ endowed with the restriction to each tangent space of the quadratic form L, restriction which is positive definite, is a model of H, the hyperbolic space. Each sphere Σ of S^{n-2} is the "boundary at infinity" of a totally geodesic subspace h of H.

Let $\mathcal{G}$ be the group of linear isomorphisms of R^n leaving L invariant. Its restriction to H is the group of isometries of the hyperbolic space H. To chose a point z in H determines a metric on the sphere S^{n-2}. This metric is the projection on S^{n-2}, sphere at infinity of H, of the metric on $T_z(H)$ using the geodesic rays of origin z.

Different choices of the point z determine conformally equivalent metrics on the sphere S^{n-2}. The sphere does not even admit a measure invariant by the conformal group. Fortunately the sets of spheres of S^{n-2} do. In particular, Λ is endowed with a measure m invariant by $\mathcal{G}$. That measure can also be seen as the measure, invariant by the isometries of H, defined on the set of totally geodesic hyperplanes of H. Let us project the sphere S^{n-2} stereographically on an affine space $\mathbb{R}^{n-2}$. There, a sphere Σ is located with its center $x_1, x_2, ..., x_{n-2}$ and its radius r. the measure m is expressed by:

$$m = |[1/(r^{n-1})]dx_1 \wedge dx_2 \wedge ... \wedge dx_{n-2} \wedge dr|$$

Remark: Let $(v_1, v_2, ..., v_{n-1})$ be $n-1$ vectors of $T_{v_0}\Lambda$. The volume of the parallelepiped constructed on these vectors is

$$|det(v_0, v_1, v_2, ..., v_{n-1})| = \sqrt{(} - det(\mathcal{L}(v_i, v_j))$$

Remark: This measure can also be seen as a measure on the set of hyperplanes of the hyperbolic space $\mathbb{H}$ which is invariant by the action of the hyperbolic isometries [Sa2].

12.1 Spheres of dimension 0

We will start with spheres of dimension 0 in S^1, and study their positions with respect to a "torus" T made of 4 distinct points. An oriented sphere σ disjoint from T bounds an interval I. We will say that σ is trivial if I contains two points of T. Informally we may say that the small enough spheres will all be trivial.

Proposition 12.1.1 *The torus T which minimises the measure of the set of non trivial spheres is the torus made of the four vertices of a square (or its image by the conformal group of the circle).*

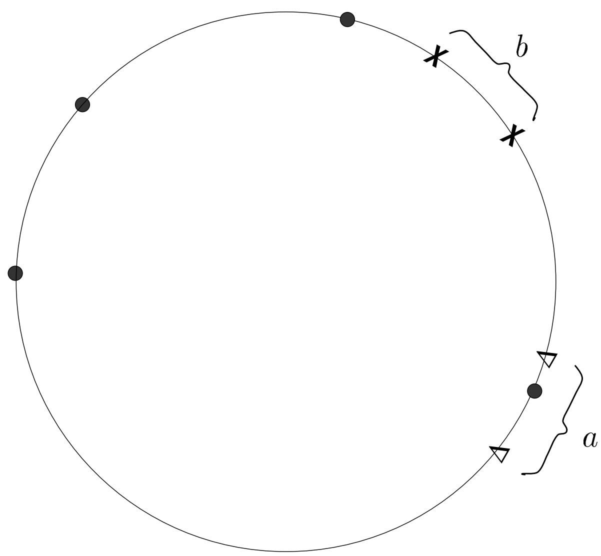


Figure 65: a nontrivial (a), and a trivial (b) 0-sphere

The domain Z of $\mathcal{S}$ formed by the non trivial spheres is bounded by segments of light rays formed by the spheres containing one of the four points of T.

As the only conformal invariant of a set of four points is their cross-ratio, The measure $m(Z)$ is a function of this cross-ratio.

Proof: The proof of the proposition is a computation. Using the stereographic projection of S^1 on $\mathbb{R}$ the measure on $\mathcal{S} = \{\sqrt{\ } \dashv\rangle\nabla\int\wr\{\sqrt{\ } \wr\rangle\backslash\sqcup\int\wr\{\mathcal{S}^\infty\}$ is $\frac{1}{(y-x)^2}|dx \wedge dy|$. Without loss of generality, we can suppose that the four points of the "torus" T are $\{\infty\,,\, 0\,,\, 1\,,\, z\}$. We will make the computation of the measure of "half" of the points of Z, that is $\{\infty < x < 0; 1 < y < z\}$, supposing $z > 1$. The other cases are analogous. One has $m(Z_z) = m(\{\infty < x < 0; 1 < y < z\}) + m(\{0 < x < 1; z < y < \infty\})$. One has :

$$m(\{\infty < x < 0; 1 < y < z\}) = \int_{1<y<z} \int_{-infty<x<0} \frac{1}{(y-x)^2} |dx \wedge dy| = log(z).$$

In the same way we compute:

$$m(\{0 < x < 1; z < y < \infty\}) = log(z) - log(z-1)$$

The minimum of $m(Z_z)$ is achieved for $z = 2$, $m(Z_2) = 2log(2)$. This correspond to the "square" torus $T = \{e^{ik\pi/2}\}$.

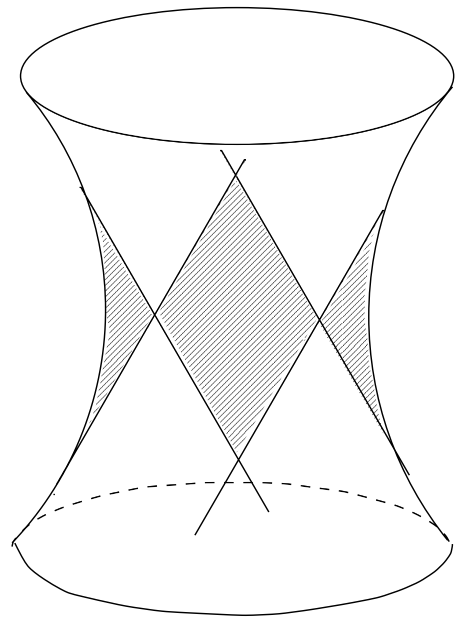


Figure 66: the set of non-trivial 0-spheres

□

12.2 The circles of S^2

The set of circles of S^2 is identified with the points of the 3-dimensional quadric $L = 1 \in \mathbb{R}^4$.

The circle two piece property

Definition 12.2.1 *A top circle $C \subset S^2$ for a closed subset $A \subset S^2$ is a circle intersecting A, bounding two discs D_1 and D_2 such that the closure of one , say $\overline{D}_1$ contains A, and the interior of D_2 is disjoint from A.*

We will call the intersection $C \cap A$ of a top circle with A a *topset* of A.

Definition 12.2.2 *A simple closed curve of S^2 has the circle two piece property , if it is divided by any circle in at most two connected components.*

Proposition 12.2.3 *A simple closed curve satisfying the circle two piece property is a (round) circle.*

The result is clear as, for any other simple closed curve, rotating through the tangency point an osculating circle with generic contact with the curve will give circles which contradict the circle two piece property. The circle two piece property is also meaningful for 2-dimensional submanifolds with smooth boundary of S^2.

Remark: Notice that if a curve C does not satisfy the circle two piece property, one can find a top circle intersecting it in at least two connected components. Suppose that a disc D_1 intersects C in two components. Let us call D_2 the interior of the complement of D_1. Chose a point a in $D_2 \setminus C$. The circles of the pencil generated by a and ∂D_1 , ordered from ∂D_1 by the inclusion of the discs they bound, starting with D_1, have a first tangency with C. That circle C_t is tangent to C in at least a point b, and the intersection $C_t \cap C$ still has at least two components. It is the boundary of a disc D_t containing D_1. Consider now the pencil of the circles tangent to C_t at b ordered from C_t using as before the inclusion of discs that they bound. One of them is a top circle, and the corresponding topset has two connected components at least.

Proposition 12.2.4 *The only 2-dimensional manifolds W with smooth boundary of S^2 having the circle two piece property are obtained by removing from S^2 a finite number of disjoint closed discs D_i with boundary (round) circles.*

Lemma 12.2.5 *The top sets of a closed set A satisfying the circle two piece property are connected , that is they are either a point or an arc of the corresponding top circle. Conversely if all the topsets of a closed set are connected, then it satisfies the circle two piece property.*

Proof: Consider a sequence of increasing discs D_1^i converging to the disc D_1 of boundary C such that $A \subset \overline{D}_1$. If all the intersections $A \cap D_2^i$ of A whith the complement of D_1^i in Σ are void or connected the intersections $\partial D_1^i \cap A$ are also void or connected and would converge to at most one interval or a point of C, contradicting the hypothesis that the topset in C is not connected. If a circle cuts A in more than two pieces, a disc D_2 of boundary C will intersect the closed set A in at least two connected components. We can decrease D_2 , keeping two connected components in $\overline{D}_2^t$ till its boundary is a top circle (first reduce one component of $A \cap \overline{D}_2^t$ to a point p or an interval containing a point p, then proceed using circles tangent at p). Therefore ∂D_2^t provide the top circle intersecting a in two connected components. □

Proof: (of the proposition) If the boundary of W is not a union of circles, then , consider one component of ∂W which is not a circle. Performing a suitable inversion, this component can be seen as the outer boundary component of $W \subset \mathbb{R}^2$. Some circle bounding a disc in $\mathbb{R}^2$ containing W will be tangent to the outer component of ∂W defining a non-connected topset. The previous lemma provides a contradiction. The conditions of the proposition are sufficient because any circle C tranverse to ∂W intersects each circle ∂D_i in zero or two points. Then C and the discs D_i it crosses form a necklace the complement of which has two open connected components which are the components of $W \setminus C \cap W$. □

12.3 Spheres of dimension two

They form a 4-dimensional manifold. We observed that time-like curves in $\mathcal{S}$ correspond to nested spheres, space-like curves to spheres enveloping a canal surface. A limit case is the family of osculating spheres to a surface M of R^3, along a line of curvature.The corresponding curve of $\mathcal{S}$ is everywhere tangent to the light cone.

12.4 The spherical two-piece property

Definition 12.4.1 *A closed surface $M \subset S^3$ satisfies the sperical two piece property, S.T.P.P. if for any sphere Σ the difference $M \setminus (M \cap \Sigma)$ has at most two connected components. Such a surface is called taut*

In 1970 T.Banchoff proved the following theorem:

Theorem 12.4.2 *[Ban2] A surface embedded in S^3 satisfying the spherical two piece property is either a embedded round sphere or a Dupin cyclide, that is the conformal image of a torus of revolution of (complex) equation*

$$|z_1| = a, |z_2| = b; a^2 + b^2 = 1; (z_1, z_2) \in \mathbb{C}^2$$

Remark: The Dupin cyclides are in two different ways the envelopes of one dimensional families of spheres tangent to three spheres bounding three disjoint balls.

The proof of this theorem is analogous to the proof of Kuiper's result about tight immersions. One needs to consider *spherical topsets* and *top spheres*.

Definition 12.4.3 *A sphere Σ is a top sphere if it bounds two balls B_1 and B_2 such that:*

-the interior of say, B_2 does not meet M

-both $\overline{B}_2$ and $\overline{B}_1$ do meet M.

We can weaken that definition:

Definition 12.4.4 *A sphere Σ is said to be a local topsphere of M at $m \in M$ if m belongs to $\Sigma \cap M$ and if m has a neighbourhood $U \subset M$ which is contained in one, say B_1 of the balls B_1 and B_2 of boundary Σ. If the neighbourhood $U \subset M$ can be chosen to intersect Σ only in m then we say that the sphere Σ is a strict local topsphere.*

Proposition 12.4.5 *A surface $M \subset S^3$ has the spherical two piece property if and only if every local topsphere is a topsphere.*

Proof: Suppose it is not the case, then there exists a point $q \in int(B_2)$. For a sphere Σ' tangent to M at m, but bounding a closed ball B_1' which strictly contains B_1. It is a strict local topsphere of M at m, and the intersection $\overline{B}_2' \cap M$ has at least two connected components, one reduced to m, and one containing q. A third sphere Σ'' tangent in p to Σ', very close to Σ_1' and contained in B_1' contradicts the spherical two piece property. □

At a point m, we can consider the pencil of spheres tangent to M at m which, with the point m is a circle $\mathcal{P}(m)$. The support spheres of M form, if M is not a (round) sphere an interval of this pencil. Let us call $\Sigma^+(M,m)$ and $\Sigma^-(M,m)$ the boundary spheres of this interval. Applying this construction to nested neighbourhoods $U_i \subset M; i \in \mathbb{N}$ such that $\cap_{i\in\mathbb{N}} U_i = m$ we get spheres $\Sigma_i^+(M,m)$ and $\Sigma_i^-(M,m)$ which converge to the two osculating spheres of M at m: $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$. We can also define them using a stereographic projection of center different from m and the principal curvatures of $stereo(M)$ at $stereo(m)$. This last observation implies that, when $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$ are different, the intersection $\Sigma_1(M,m) \cap M$ is tangent to a line $L_1(m) \subset T_m M$ and the intersection $\Sigma_2(M,m) \cap M$ is tangent to a line $L_2(m) \subset T_m M$. We call these directions *principal directions.* A point where $\Sigma_1(M,m) = \Sigma_2(M,m)$ is called an *umbilic.*

Lemma 12.4.6 *If M is a taut smooth surface of S^3 then $\Sigma^+(M,m)$ and $\Sigma^-(M,m)$ coincide with $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$.*

Proof: The interval of $\mathcal{P}(m)$ containing the point sphere m and bounded by $\Sigma_1(M,m)$ and $\Sigma_2(M,m)$ is in that case equal to the set of topspheres. □

We are ready to prove the:

Theorem 12.4.7 *A smooth taut surface embedded in S^3 is either a (round) sphere or a smooth torus*

Proof:

First notice that a (round) circle of S^3 has the spherical two piece property.

If M has an umbilic m, then it "lies" between identical spheres $\Sigma_1(M,m) = \Sigma_2(M,m)$, and is therefore a sphere. If it does not have any umbilical point, then there exist two transverse line fields on M, $L_1(m)$ and $L_2(m)$. As M is embedded in S^3 it is orientable, and therefore is a torus. □

Proposition 12.4.8 *A Dupin cyclide is taut.*

Proof: The envelope of a time like curve in Λ is a canal surface, union of the caracteristic circles of the family. The directions tangent to this family of circles are principal directions. A Dupin cyclide is in two different ways a canal surface, and therefore admits two transverse foliations by circles (tangent to the principal directions). The components of $M \setminus \Sigma$ are the union of plaques of these two foliations. The circle two piece property applied to the leaves of the two foliations imply that they are cut in at most two intervals, and can match in at most two connected components. □

Then an essential lemma is:

Lemma 12.4.9 *A spherical top set of a taut embedded torus M satisfies the circle two piece property.*

Proof: As before B_1 is the ball of boundary a topsphere Σ which contains M in its closure and B_2 the other ball of boundary Σ. If the topset does not satisfy the two piece property,in the topsphere Σ we can find a circle C which is a topcircle of $\Sigma \cap M$ such that the intersection $C \cap \partial\Sigma \cap M = C \cap M$ is not connected. As before the intersection $M \cap \Sigma$ is contained in $\overline{D}_1$, a disc of boundary C, and the other disc D_2 of boundary C does not meet $M \cap \Sigma$. Choose a and c on different components of $C \cap M$ and b and d in different components of $C \setminus (C \cap M)$, so that these points are in cyclic order on C. Let γ be a geosdesic arc from b to d in D_2 and V a neighbourhood of γ in S^3 disjoint from M. Turning Σ around C we get a family Σ^t. We chose the rotation sign to leave γ out of the component, but chose the rotation small enough to garantee the existence of a continuous family of paths γ^t joining a to c in $\Sigma^t \cap V$ (B_1^t obtained by continuity from B_1). Then the points a and c will be in different components of $B_2^t \cap M$,as there is no path connecting a and c in $M \cap C = M \cap \Sigma \cap \Sigma^t$, and as any path in the union of the hemispheres Σ^t containing the arcs γ^t joining a and b should cross V. Therefore, for t small enough, (with the right sign), Σ^t cuts M in at least three connected components. (This last argument is quite analogous to Kuiper's for tight surfaces). □

Proposition 12.4.10 *If M is a taut torus in S^3 then for any topsphere Σ, $\Sigma \cap M$ is a point or a circle.*

Proof: We know by the previous proposition that the top set satisfies the circle two piece property. It cannot be Σ as M is a torus, nor contain interior points, which would be umbilical points of M, and imply again the equality $M = \Sigma$. The topset could apriori also be $\Sigma \setminus \{non\ finite\ family\ of\ round\ discs\}$. The boundary of those discs cannot bound a disc in M without contradicting tautness (consider a Poncelet pencil of spheres containing Σ), but then these boundary curves would be disjoint simple closed curves on M ; three disjoint simple curves on a torus always disconnect it into more than two pieces, so $M \cap int(B_1)$ would have at least two components. Moving Σ slightly into B_1 provides a sphere Σ' bounding a ball B_1' such that $M \cap \overline{B}_1'$ has at least two connected components. The only possibilities left are a point and a circle. □

The interval of topspheres tangent at $m \in M$ to the taut torus M is bounded by the two osculating spheres at m, Σ_1 and Σ_2. Let us consider a sphere Σ tangent at m to M close to Σ_1 which is not a topsphere. It intersects M in a neighbourhood of m into two transverse arcs crossing at m the tangents of which are form a very acute angle and are close to the principal direction $L_1 \subset T_m M$. Suppose that the intersection $\Sigma_1 \cap M$ is the point m. Choose a neighbourhood $U \subset S^3$ of m such that the intersection $M \cap U$ is a small disc. For non topsphere Σ^t tangent to M at m close enough to Σ_1 the intersection $\Sigma^t \cap M$ is contained in U. As, at m there are four arcs of $\Sigma^t \cap M$ with distinct tangents, we can find two points p and q in $\Sigma^t \cap M$ such that any path from p to q in D_1^t passes through m. Choose in $U \cap \Sigma_1$ a very small circle σ centered at m, such that that the small disc $\delta_\sigma$it bounds does not contain any of the points p and q. In the pencil of spheres containing Σ_1 ,and following by continuity the ball B_1, some interval of spheres Σ^τ starting at $\Sigma_1$will be such that $\overline{B}_1^\tau$ contains p and q but does not contain m . For τ small enough and with the right sign, Σ^τ does not satisfy the two piece property. Then we can conclude that the osculating spheres intersect a taut torus M in circles. Those circles are necessarily lines of curvature, so M is a Dupin cyclide [Dar] . This ends the proof of the theorem giving the list of taut surfaces in S^3.

12.5 Intersection of surfaces and curves of the sphere S^3 with spheres

Let us now show that we can associate to a closed surface or a closed curve of S^3 a subset of $\mathcal{S}$ the measure of which is a conformal invariant of the surface or curve.

Let M be a compact surface embedded in S^3. There exists a radius ϵ (depending on M) such that any sphere $\Sigma \subset S^3$ of radius smaller than ϵ

either does not meet M or meets M in a point or a closed curve bounding a disc in M. Then the measure of the set of *nontrivial spheres*, that is the spheres which meet M in more than one curve, or in a curve which is not the boundary of a disc in M, is a conformal invariant. A smaller conformal invariant is the measure of the spheres which intersection with M contains a nontrivial component in the homology of M.

Definition 12.5.1

$$nt(M) = measure\{non\,trivial\,spheres\,for\,M\}$$

$$ntop(M) = measure\{\sigma \text{ intersecting } M \text{ non trivially in } H^1(M)\}$$

Let γ be a compact closed curve embedded in S^3. There exists a radius ϵ (depending on γ) such that any sphere $\Sigma \subset S^3$ of radius smaller than ϵ either does not meet γ or meets it in one or two points. Then the measure of the set of *nontrivial spheres for* γ , here spheres which meet γ in at least four points, is a conformal invariant of the curve γ We can define:

Definition 12.5.2

$$nt(\gamma) = measure\{non\,trivial\,spheres\,for\,\gamma\}$$

$$NT(\gamma) = \int_{\mathcal{S}} (\sharp\gamma \cap \Sigma - 2)^+$$

where φ^+ *is the function equal to* φ *when* $\varphi \geq 0$ *and equal to 0 when* $\varphi \leq 0$

12.6 Conjectures

- **conjecture** There exists a positive constant α such that, when the closed embedded curve $\gamma \subset S^3$ is knotted,

$$nt(\gamma) \geq \alpha$$

- **conjecture** There exists a positive constant β such that, when the closed embedded surface $M \subset S^3$ is not a sphere,

$$nt(M) \geq \beta$$

- **The Willmore conjecture** The following 2-form on a surface M embedded or immersed in S^3 is invariant by the action of the conformal group on S^3:

$$dw = (k_1 - k_2)^2 \cdot dv$$

where k_1 and k_2 are the principal curvatures and dv the area form of M.

The integral on M of this form:

$$W(M) = \int_M dw$$

is then a conformal invariant of the immersed surface. Looking first at revolution tori of equation

$$|z_1| = a, |z_2| = b; a^2 + b^2 = 1; (z_1, z_2) \in \mathbb{C}^2$$

Conjecture [Wil1] [Wil2] When M is a torus:

$$W(M) \geq 2\pi^2$$

The value of $W(M)$ can be interpreted as an area in the quadric Λ [Bry] . View M as embeddedd in $\mathbb{R}^3$. Consider at each point of $m \in M$ the sphere $\Sigma_b(m)$, tangent at m to M, and with mean curvature the mean curvature of M in M.

Remark: First observe that this is a conformal property, equivalent to impose that the intersection curves of M and $\Sigma_b(m)$ intersect at right angles in m. There may be an inequality linking $W(M)$ and the measure of the spheres with non trivial intersection with M. **Proof:** We can write local equations of M and $\Sigma_b(m)$ in the neighbourhood of m, using axis tangent to the principal directions of M at m and to the normal to M at m (k_1 and k_2 are the principal curvatures of M at m) :

$$z = \frac{1}{2}[k_1 x^2 + k_2 y^2] + \textit{higher order}$$

$$z = \frac{1}{2}[\frac{k_1 + k_2}{2} x^2 + \frac{k_1 + k_2}{2} y^2] + \textit{higher order}$$

The equation of the intersection is

$$\frac{k_2 - k_1}{2} x^2 + \frac{k_1 - k_2}{2} y^2 + \textit{higher order} = 0$$

proving that the projection of the intersection $M \cap \Sigma_b(m)$ on the tangent plane is, in the neighbourhood of m, two curves intersecting at m with right angles.

□

This defines a map $\gamma_b M \to \Lambda$. The image of this map is a surface which is space-like (that is the tangent plane to it, in every point, contains only space-like non-zero vectors).

The measure of the image of this map is then again $W(M)$.

This conjecture has proved to be particulary rich in connection with other problems see [Wil1] [Wil2] [Li-Ya].

Theorem 12.6.1 *Let M be a two dimensional flat torus in $\mathbb{R}^3$ with lattice generated by $\{(0,1),(x,y)\}$ where $0 \leq x \leq \frac{1}{2}$ and $\sqrt{1-x^2} \leq y \leq 1$, then*

$$\int_M |H^2| \geq 2\pi^2$$

12.6.1 Conformal structures on tori.

First recall the possible conformal structures on a torus. [Bri-Kno] and [Jo-Si].

Let ω be a complex number with positive imaginary part, and let Γ_ω be the lattice in $\mathbb{C}$ consisting of all complex numbers $n + m\omega$, where m and n are integers. Then $\mathbb{C}/\Gamma_\omega$ has the structure of a one-dimensional complex manifold. As a topological manifold, $\mathbb{C}/\Gamma_\omega$ is homeomorphic to $S^1 \times S^1$. We are interested in the possible conformal structures. Then (z_1, z_2) and $(az_1, az_2, a \in \mathbb{C}^*)$ define the same conformal structure. Chosing the proper orientation we may also chose between $(1,\omega)$ and $(1,\overline{\omega})$. Brieskorn and Knörrer's book contains also an algebraic geometry interpretation; they prove that all complex tori can be interpreted as a cubics. One can prove (see [Jo-Si] p.273) that the space of moduli describing the conformal structures on a torus is a quotient of:

$$\mathcal{D} = \{\omega,\, |\omega| \geq 1,\, |Re\omega| \leq 1/2\}$$

identifying the corresponding sides by the maps $z \to z+1$, $z \to -1/z$.

Allowing the reversing orientation conformal map $z \to -1/\overline{z}$ one reduces the modulus to:

$$\{\omega,\, |\omega| \geq 1,\, 0 \leq Re\omega \leq 1/2\}$$

12.6.2 Conformal volume.

Definition 12.6.2 *Let M be a surface or a curve embedded in S^3. Define*

$$Volconf(M) = sup_{g \in \mathcal{M}} vol(g(M))$$

where $\mathcal{M}$ is, as before the Möbius group of conformal diffeomorphisms of S^3.

Proposition 12.6.3 – *The conformal length of a curve in at least 2π, the length of a geodesic circle.*

- *The conformal area of a surface is at least 4π, the area of a geodesic sphere.*
- *The conformal area of an immersed surface admitting double points is at least 8π.*
- *The conformal area of an immersed surface admitting triple points is at least 12π.*

Let $proj$ be the steoreographic projection with "south pole" a point m of M. The maps $(proj)^{-1} \bullet \mathcal{H}_R \bullet proj$, when $R \to \infty$ expand, up to cover almost all the sphere, any small neigbourhood of m, and expand the small arc or disc of M contained in that neighbourhood up to a curve or surface very close to a geodesic circle or sphere.

Proposition 12.6.4 *If M is a minimal tori embeddededin S^3 then $vol(M) = Volconf(M)$*

We use again the relation between the extrinsinc Gaussian curvature K_{ext} and the intrinsic Gaussian curvature K_{int} of a surface in S^3: $K_{int} = 1 + K_{ext}$ and the fact that in R^3or S^3, the 2-form $(k_1 - k_2)^2 dv$ and its integral on M are conformally invariant. Denote also H_S the mean curvature of this surface $M \in S^3$.

Let $proj$ be again a stereographic projection. Denote by H and K the mean curvature and Gauss curvature of $proj(M) \in \mathbb{R}^3$, and by k_1 and k_2 its principal curvatures. Then the quantity

$$\int_{proj(M)} (k_1 - k_2)^2 dv = \int_{proj(M)} H^2 - K$$

is equal to:

$$\int_M (H_S)^2 - K_{ext} = \int_M (H_S)^2 - K_{int} + 1 = Vol(M)$$

as M is minimal and is a torus. Using again the fact M is a torus and Gauss Bonnet theorem, one gets that $\int_{proj(M)} H^2$ is a conformal invariant, and therefore that

$$\int_{proj(M)} H^2 = \int_{g(M)} H_S^2 + vol(g(M), \forall g \in \mathcal{M}$$

Then

$$Volconf(M) \leq vol(M)$$

The other inequality is immediate.

The study of the first eigenvalue of the Laplacian on M [Li-Ya], implies the inequality:

Proposition 12.6.5 *A flat torus with lattice generated by* $\{(1,0),(x,y)\}$ *where* $0 \leq x \leq \frac{1}{2}$ *and* $\sqrt{1-x^2} \leq y \leq 1$ *satisfies:*

$$Volconf(M) \geq 2\pi^2$$

The theorem now follows directly from the inequality: $\int_M H_S^2 + 1 \geq \int_M 1 = vol(M)$ applied to a sequence of embedding the volume of which approximates the conformal volume.

- **Möbius energy** The author thanks D.Rolfsen for pointing out the reference [F-H-W] to him. Recently M.F.Freedman, Z-X.He and Z.Wang using an previous work of O'Hara, [5] defined the Möbius energy of a rectifiable curve embedded in $\mathbb{R}^3$ by:

$$E(\gamma) = \int\int_{\gamma\times\gamma\setminus\Delta} \frac{1}{|\gamma(v)-\gamma(u)|^2} - \frac{1}{[dist_{\mathbb{R}^3}(\gamma(v),\gamma(u))]^2}$$

where Δ is the diagonal of the product $\gamma \times \gamma$. Separately the integral of the two fractions would diverge, but the sum converges. They prove that this function is invariant by the Möbius group.

Let $c([\gamma])$ be the crossing number of the knot type $[\gamma]$ (the infimum of the number of crossings of the projection of the knot $\gamma \in [\gamma]$ on a plane, when γ describes the isotopy class.

Theorem 12.6.6 *[F-H-W] The energy $E(\gamma)$ of a simple closed curve $\gamma \subset \mathbb{R}^3$ satisfies the inequality:*

$$E(\gamma) \geq 2\pi c([\gamma]) + 4$$

The equality $E(\gamma) = 4$ is achieved only when the curve is a (plane, round) circle.

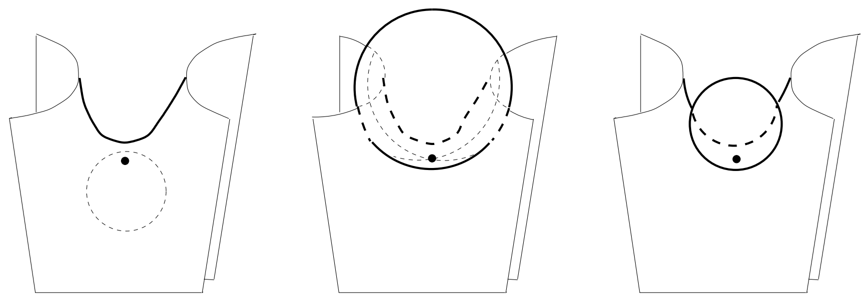


Figure 67: Possible generic contacts of a sphere and a foliation.

Conjecture The Möbius energy and the measure of the set of spheres intersecting the curve γ in at least four points are linked by inequalities. One may have to take a multiplicity involving the number of intersection points into account.

12.7 Conformal integral geometry of foliations.

Curiously, the first result of conformal integral geometry [La-Ni], was obtained for codimension one foliations of $\mathbb{R}^3$ or S^3.

Let now $\mathcal{F}$ be a codimension 1 foliation of a domain $W \subset \mathbb{R}^3$. The number $N^-(\Sigma)$ of negative contact points of Σ with $\mathcal{F}$ is the number of saddle tangencies of Σ and $\mathcal{F}$. It is clear that the number $N^-(\Sigma)$ is conformally well-defined.

A measure on the set $\mathcal{S}$ of spheres of $\mathbb{R}^3$, considered as a subset of the set of spheres of S^3 is constructed in the chapter **The space of spheres**. Using that measure we have the

Theorem 12.7.1 *Let $\mathcal{F}$ be a smoothe foliation of a domain $W \subset \mathbb{R}^3$. Then*

$$\frac{1}{6}\int_W |k_1 - k_2|^3 = \int_{\mathcal{S}} N^-(\Sigma)dm(\Sigma)$$

where k_i are the principal curvatures of the leaves.

Remark: We could have stated the theorem in S^3 as the form $|k_1 - k_2|^3 dv$, where dv is the volume element of W, is a conformal invariant.

13 Complex integral geometry

The n-dimensional complex space $\mathbb{C}^n$ has a natural hermitian structure, and an associated scalar product:

$$< u|v >_e = \mathcal{R}e(< u|v >)$$

The n-dimensional complex space $\mathbb{C}^n$ endowed with the quadratic form $|v|^2 = \mathcal{R}e(< v|v >$ is just a euclidean space of dimension 2n. But, among the real euclidean planes, some have an extra property: they are globally invariant by multiplication by complex numbers. The complex integral geometry will deal with those particular real planes: the complex lines. To compensate the relatively few partial datas given by the projections on the complex lines and complex subspaces only, and by the section by the affine complex subspaces only, we need to suppose that the submanifolds studied have some extra structure. So in this chapter the submanifolds are local images of $\mathbb{C}^p$ by a locally defined holomorphic map.

13.1 Critical points of projections on complex lines

The orthogonal projection of $\mathbb{C}^n$ onto a complex line of $\mathbb{C}^n$ is a holomorphic map.

Many interesting consequences can be deduced from the properties of the complex curve C of equation $y = ax^2$ in a neighbourhood of the origin.

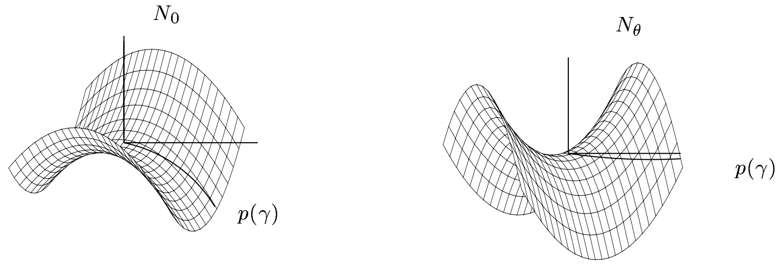
The tangent space to C at $(0,0)$ is the x-line and the normal space at $(0,0)$ is the y-line. Let D_θ be the oriented real line of the y complex line making the angle θ with the oriented real axis. The orthogonal projection C_θ of C on the sum :

$$E_\theta = (x \ complex \ line) \oplus D_\theta$$

has equation :

$$z = Re(e^{i\theta}.x^2) \ ,$$

z being the real coordinate on D_θ determined by the euclidean structure of $\mathbb{C}^2$ and the orientation cf D_θ,.



projection of the curve $y=x^2$ *on* $T_m M+N_0$

projection of the curve $y=x^2$ *on* $T_m M+N_\theta$

Figure 68: Saddle and turned saddle

Performing the change of variable $x' = e^{i\theta/2}.x$, we see that the projections C_θ are all isometric, more precisely that C_θ is deduced from C_0 (of equation $z = Re(x^2)$ by a rotation with a vertical axis and angle $-\theta/2$.

A section of C_0 by the vertical plane F_φ containing the real line Δ_φ of the x-complex line has the equation:

$$z = Re(a\rho^2.e^{2i\theta}) = \mid a \mid \ \rho^2 cos\ (2\varphi + \varphi_a)$$

where φ_a is the argument of a, therefore the maximal and minimal values of the curvature in $(0,0)$ of those curves are opposite and of absolute value $2. \mid a \mid$. This implies that at (0.0), C_0 has zero mean curvature and Gaussian curvature $4 \mid a \mid^2$.

Remark: The projections of the complex curve of equation $z = ax^n$ on the 3-spaces E_θ are obtained from the projection on E_0 by rotations of angle $-\theta/n$. As curvatures depend only on 2-jets at the point where they are computed, we have proved the following proposition:

Proposition 13.1.1 *Let C be a holomorphic curve of $\mathbb{C}^2$ then the orthogonal projections of C on the 3-spaces $E_\theta = T_m + D_\theta$, where T_m is the complex line tangent at m to C and where D_θ is a real line normal to C in m, have all the same Gaussian curvature at m and have all zero mean curvature at m.*

13.2 Complex Gauss map and critical points.

The normal space $N(m)$ of C at m is a complex line. This allows us to define a map $\gamma_{\mathbb{C}}$ of C to $\mathbb{C}P_1$ by $\gamma_{\mathbb{C}}(m) = N(m)$. At the point (1,0), the

Fubini-Study metric of $\mathbb{C}P_1$ is the euclidean metric of the chart given by the map x/y.

Let $K(\Delta, m)$ be the gaussian curvature of the projection M_Δ of M on the space $E_\Delta = T_m M \oplus \Delta$.

The Lipschitz-Killing curvature at m, $K(m)$, of an even dimensional submanifold M of R^N is, up to a constant depending only on the dimensions involved, equal to the integral on the projective space on (real) lines of the normal space, of $K(\Delta, m)$:

$$K(m) = const \int_{\mathbb{P}_{N(m)}} K(\Delta, m)$$

where const indicates a constant depending only on the dimensions involved.

Proposition 13.2.1 *The jacobian of the complex Gauss map satisfies :*

$$\mid det\ D\gamma_{\mathbb{C}}(m) \mid^2 = -K(\Delta, m) = const\ K(m)$$

where const is a universal constant.

Proof: It is enough to prove the proposition for the curve C of equation $y = ax^2$ at the origin as the numbers we shall compute depend only on 2-jets. Let $m(x)$ be the point (x, x^2). The complex normal line is generated by the vector $(-2a\bar{x}, 1)$, therefore, using the map x/y, the differential of the complex Gaussmap is $-2a.J$, where J is conjugation.

One gets $\mid det\ D\gamma_{\mathbb{C}}(0) \mid = 4 \mid a \mid^2$. □

Let us now see what the counterpart of the existence of a complex Gauss map is when one looks at projections on complex lines. We will note $\pi_{L_{\mathbb{C}}}$ the orthogonal projection on the complex line $L_{\mathbb{C}}$. Let C be a holomorphic local parametrisation of the curve C. The differential $D(\pi_{L_{\mathbb{C}}}.C)$ is a linear complex map which implies that its real rank (as a real linear map) can be only 0 or 2. This implies that a point is a critical point of $\pi_{\mathbb{C}}.C$ if and only if it is a critical point of $\pi_D.C$, where D is a real line contained in $L_{\mathbb{C}} -$.

Corollary 13.2.2 *Let* $\mid \mu \mid (C, D)$ *denote the number of critical points of the orthogonal projection of* C *on the real line* D *and* $\mid \mu \mid (C, L_{\mathbb{C}})$ *be the number of critical points of the projection of* C *on the complex line* $L_{\mathbb{C}}$*. For every real line* D *contained in a complex line* $L_{\mathbb{C}}$ *one has:*

$$\mid \mu \mid (C, D) = \mid \mu \mid (C, L_{\mathbb{C}}).$$

Remark: The critical values of the projection of a complex curve on a real 2-plane which is not a complex line may contain arcs. A nice study of this critical locus for a family of planes containing a complex line in the neighbourhood of non degenerate critical value of the projection on the complex line can be found in the book by Arnold, Gusein-Zade et Varchenko [A-G-V] p. 20-21, see fig. I.2.

Figure 69: Projection of the complex curve $y = x^2$ on a real plane which is close to the complex y-axis

13.3 Polar curves.

We have already met polar varieties Γ_h and γ_h respectively the critical points and the critical locus of the orthogonal projection of a submanifold on the subspace h. They are equally important in the complex frame; (see for example Teissier [Tei2]. Slightly more generally, a polar variety is always (the closure of) the set of points where an incidence relation between the tangent subspaces to a certain object and a fixed subspace satisfy a given incidence relation. Let us now give the examples we shall use later.

Definition 13.3.1 *Let $\mathcal{F}$ be the foliation defined by an algebraic 1-form of $\mathbb{C}^2 : \omega = P.dx + Q.dy$. The tangent plane at a point (x, y) to the leaf of the foliation through (x, y) is the kernel of ω, when P and Q are not both zero. Let L be a complex line. The polar curve $\Gamma^L(\mathcal{F})$ is defined by:*

$$\Gamma^L(\mathcal{F}) = \{\overline{(x, y) \mid \omega(x, y)(L) = 0 \text{ and } \omega \neq 0}\}.$$

Observe the choice of upper indices; to be consistent with the previous chapters we need to define:

Definition 13.3.2

$$\Gamma_{(L^\perp)}(\mathcal{F}) = \{\overline{(x,y) \mid \omega \neq 0 \text{ and } p_{L^\perp}|_{L_{(x,y)}} \text{ has a critical point at } (x,y)}\}.$$

Here $L_{(x,y)}$ is the leaf of $\mathcal{F}$ trough the regular point (x,y).

Of course, it is the curve $\Gamma^L(\mathcal{F})$

As in the real case,the name polar curve comes from the fact it is generically a curve except for a set of lines of measure zero. Again, we shall extensively use generic properties. In the algebraic context the measure zero bad set we should avoid is often a closed algebraic subset.Except for degenerate cases which we ignore, Γ_L is an algebraic curve whose equation is $P.a + Q.b = 0$, where (a,b) is a vector generating L.

A particular case is when $\mathcal{F}$ is the level foliation of a polynomial $f : \mathbb{C}^n \to \mathbb{C}$. The intersection of the polar curve $\Gamma_L(\mathcal{F})$ with a nonsingular level $f = \lambda$ of the polynomial is the set of critical points $\Gamma_L(f = \lambda)$.

Theorem 13.3.3 Exchange theorem. *Let V be an open piece of a holomorphic curve, its total curvature satisfies:*

$$\int_V \mid K \mid = const. \int_{\mathbb{CP}(1)} \mid \mu \mid (V, L).$$

Proof: The theorem is a consequence of the exchange theorem proved before for codimension p submanifolds of $\mathbb{R}^N$, and of the corollary about numbers of critical points of the projection on real or complex lines proved above. □

A global consequence is the :

Proposition 13.3.4 *Let f be a polynomial of two complex variables of degree d. The total curvature of the algebraic curve C of equation $f = 0$ is less than of equal to $d(d-1)$.*

Proof: Let $\mathcal{F}$ be the foliation defined by the levels of the polynomial f. To each generic complex line L is associated a polar curve Γ_L which has degree less than or equal to $d - 1$. By Bezout's theorem the intersection $\Gamma_L \cap C$ has at most $d.(d-1)$ points; these points are precisely the critical points of the projection of the curve on the complex line L. One deduces now the proposition from the exchange theorem. □

13.4 Isolated singularities

We shall show that when a sequence of smooth objects tends towards a singular one a distribution of curvature with support on the singular locus often arises naturally. The singularity will appear as a condensation at a point of the behaviour of compact submanifolds.

Let us first give a real algebraic example. The plane curve C of equation:

$$x^3 + y^2 = 0$$

is the limit of the family of curves C_λ of equations:

$$x^3 + y^2 = \lambda.$$

Let us consider the total curvature of the arc of C_λ contained in a small ball centered at the singular point.

Proposition 13.4.1 *The following limit :*

$$\lim_{\varepsilon -> 0} \lim_{\lambda -> 0} \int_{C_\lambda \cap B_\varepsilon} | k |$$

exists and is equal to π

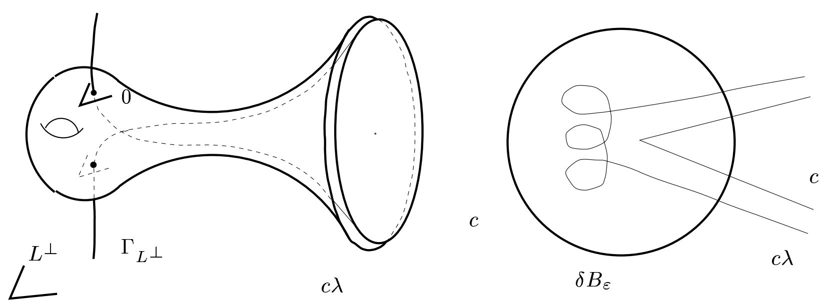


Figure 70: One dimension-faithfull picture and one codimension-faithfull one of C_λ

We shall show that such a phenomenon always occurs when one studies a sequence of levels of a complex polynomial having an isolated singularity or more generally of a polynomial map to $\mathbb{C}^p$ having an isolated singularity such that the zero level is a complete intersection. Let us first recall the topological and algebraic facts we will need. The study in the neighbourhood $B(0, \varepsilon)$ of an isolated singularity of the topology of the level $f = \lambda$ of a complex hypersurface has been done by Milnor [Mil3] .

Theorem 13.4.2 *[Mil3] . Let 0 be an isolated singularity of the complex polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$. Then for ε small enough and λ (chosen after ε) small enough, the intersection $B_\varepsilon \cap (f = \lambda)$ of the level $f = \lambda$ with the ball of radius ε has the homotopy type of a wedge of μ spheres of real dimension n.*

Following Teissier we shall pose $\mu^{(n+1)} = \mu(f)$. The notation is justified by the following theorem :

Theorem 13.4.3 *[Tei1] or [Tei2]. There exists a measure zero analytic closed set of the Grassmann manifold $G_{n+1,1}$ such that, if $H \in G_{n+1,i} \backslash \gamma_i$, the Milnor number $\mu(f_{|H})$ takes the generic value $\mu^{(i)}$ independently of H.*

Let us first consider the case of a polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ The levels of f form a foliation $\mathcal{F}$. At each regular point m of $\mathcal{F}$, $T_m\mathcal{F}$ is the tangent hyperplane to the level of f through m. Let us now fix a (vectorial) complex hyperplane h.

Definition 13.4.4 *The polar curve Γ^h is the closure of the set of regular points m such that $T_m\mathcal{F} = h$ (here we identify the affine space $T_m\mathcal{F}$ and the vector subspace which is parallel to it).*

Proposition 13.4.5 *[Le2] p. 263 and [Tei2] p. 269 (the polynomial f does not need to have in this proposition isolated singularities). The polar curve Γ^h is contained in an algebraic curve Γ'^h more precisely, if Σ is the singular locus of f, one has :*

$$\overline{\Gamma^h} = \overline{\Gamma'^h} \backslash \Sigma.$$

Proof: when the singularity is isolated It is enough to choose a base $e_1, \cdots, e_n$ of h The equations of Γ^h are in this case :

$$df(e_1) = df(e_2) = \cdots = df(e_n) = 0.$$

□

The following theorem about the total curvature is now a mere translation of the previous one, using the complex exchange theorem , [Lan1]:

Theorem 13.4.6 *[Lan1].Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial.*

$$\lim_{\varepsilon -> 0} \lim_{\lambda -> 0} \int_{\mathbb{C}_\lambda \cap B_\varepsilon} | K | = const\ (-1)^n (\mu^{n+1} + \mu^n)$$

where K is the Lipschitz-Killing curvature of the level C_λ and const a positive constant.

Remark: The first study of the curvature of levels $f=\lambda$ of the polynomial f near an isolated singular point is done in the thesis of L. Ness [Ne]. She shows in particular that the curvature of the levels is unbounded in the neighbourhood of the singularity.

Using more information about the polar curves than just the intersection number $\Gamma^h \cdot C_0$ we can give a more precise description of how the curvature of C_λ concentrates near the singular point. The geometric picture is that of concentrations of curvature near the vertices of regular polygons inscribed on circles whose radian are fractional powers of λ. The precise statement for non irreducible curves and the analysis of the phenomenon in terms of the contact of the branches of the generic polar curves and C_0 was done by Teissier [Tei3], after previous results in the irreducible case by the author [Lan4].

The seminal example is $f = x^3 - y^2$. Let us consider the polar curves $\Gamma_{a,b} = \{df(a,b)=0\}$. Their equation is $3ax^2 - 2by = 0$. The intersection points of C_λ and $\Gamma_{a,b}$ satisfy :

$$\begin{cases} x^3 - y^2 = \lambda \\ 3ax^2 - 2by = 0. \end{cases}$$

Their absciss therefore satisfies : $x^3 - (3a/2b)^2 x^4 = \lambda$. The three intersection points of the polar curve and C_λ have abscissas close to $3\sqrt{\lambda}$ and ordinate of principal term $(3a/2b)x^2$. This is true, provided λ is small enough, for any point of $\mathbb{C}P_1$ different from $(0,b)$. Notice first that the cubic root of λ is much larger than the square root of λ, which is the order of the distance of the origin to the curve C_λ. In other words with a lens of strength $(\lambda)^{-1/2}$, one sees two parallel lines at finite distance from the origin :

Figure. $\lim \lambda^{-\frac{1}{2}}(C_\lambda)$

as the lines $ax+by$ cut C_λ at points of ordinate of principal part $(\lambda)^{1/2}$ for all generic values of (a,b). With a weaker lens of strength $(\lambda)^{1/3}$, one sees three branch points :

$$\lim(\lambda)^{-1/3}(C_\lambda) = \left\{ \quad \begin{matrix} & & \cdot & \\ & \cdot & & \cdot \end{matrix} \quad \right\}$$

Each branch point of order two carries a distribution of gaussian curvature of total mass -2π. One can, applying the Gauss Bonnet theorem to the surface $C_\lambda \cap B((\lambda^{\frac{1}{3}}, O), \lambda^{\frac{1}{3}+\eta}$, for a suitably small positive η, check that its total curvature is very close to 2π. This property is true because this ball contains exactly one point of intersection with the polar curves $\Gamma_{a,b}$ for (a, b) not in a neighbourhood shrinking with λ, of the non generic direction $(0, b)$ of $\mathbb{C}P_1$.

Remark also that the Gauss Bonnet theorem applied to $C_\lambda \cap B((0,0), \varepsilon$ implies that the total curvature of this intersection, for a suitably small positive ε is very close to 6π.

The previous calculations prove that the picture of the real levels of $x^3 - y^2$ should look much more acute than usually drawn, as the turn should occur in a very small neighbourhood of the cubic root of λ. Rescaling we see a parabola. See figure below.

Figure 71: $x^3 = y^2$ at two different scales

The general case needs more lenses, the strenth of which are determined using a theorem of Smith and Merle [Sm] et [Me]. See [Tei3].

Let us now give an intuitive justification of this multiscale phenomenon of concentration of curvature. For that consider a family of branches $\Gamma^q_{a,b}$ of the polar curves $\Gamma_{a,b} = \{m | < gradf(m) \subset \mathbb{R}(a, b) = 0 \ (a, b) \in A$ where A is the complement of small open discs centered on non generic-directions of $\mathbb{C}P_1$ with a given contact order with C_0 which is larger than one. Among those non-generic directions are the lines L such that the polar curve Γ_L has $L^\perp$ among its tangents at zero. See [Tei2].

Affirmation. Any complete complex curve, the complex Gauss image of which is contained in A should cross all the curves $\Gamma^q_{a,b}$, $(a,b) \in A$ provided it crosses one of them close enough to the origin.

Proof. The condition $(a,b) \in A$ implies that the angle of the curve and the polar branches is bounded away from 0, since in a small enough neighbourhood of the origin the tangent space to $\Gamma^q_{a,b}$, $(a,b) \in A$, is very close to the set of non generic directions. The curve C_λ through a point close enough from the origin has then to cross the family of branches, and this implies the Gauss image of the intersection of C_λ with the family of branches contains A.

The existence of a positive bound to the angle between the branches considered above of the polar curves and C_λ implies also that the size of the piece of intersection should be of the order of the "transverse size" of the family of branches (the transverse distance makes sense in the neighbourhood of a first intersection point of the curve with one of the branches of the polar curves considered above). See fig. $x^3 = y^2$ at two different scales.

Let us finally observe that, in the non-irreducible case, part of the curvature of C_λ may be spread over a ball of radius $C.(\lambda)^{1/m}$, for m large enough, and C a large enough constant. For example m is the multiplicity at 0 of C_0, if the polynomial f is homogeneous.

The study of P.Rouillé [Rou] of the geometry of a neighbourhood of an isolated complex singularity of a foliation by level curves of a polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ goes beyond integral geometry as he can even describe the shape of the renormalisation of $f = \lambda$ at a concentration of curvature.

Let us now consider a surjective polynomial map $f : \mathbb{C}^{n+p-} \to \mathbb{C}^p$. The levels of f form a singular foliation of $\mathbb{C}^{n+p}$ with singular locus Σ.

Definition 13.4.7 *The polar variety Γ^h is the closure of the set of regular points m such that $T_m\mathcal{F} \subset h$.*

Proposition 13.4.8 *There exists an algebraic variety Γ'^h such that :*

$$\overline{\Gamma}^h = \overline{\Gamma'^h} \backslash \Sigma.$$

Proof: when the singularity is isolated and the intersection is *complete* . Let u be a vector of $\mathbb{C}^p$. The equation $< f \mid u > = < \lambda \mid u >$ defines a hypersurface which contains the level $f = \lambda$. The level $f = \lambda$ is the intersection of the hypersurfaces $< f \mid u > = < \lambda \mid u >$ where u takes all values in $\mathbb{C}^p \backslash 0$. The set of hyperplanes tangent at m to the hypersurfaces containing $T_m\mathcal{F}$. Let us associate to each polynomial $< f \mid u >$ with value in $\mathbb{C}$ a polar curve $\Gamma(< f \mid u >, h)$.

The previous remark shows that the polar variety Γ^h is the closure of the intersection of the union of the polar curves $\Gamma(< f \mid u >, h)$ with the set of regular points of the foliation $\mathcal{F}$. Let us choose coordinates on $\mathbb{C}^{n+p}$ and $\mathbb{C}^p$, and let J be the jacobian matrix :

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial f_1} & \cdots & \dfrac{\partial f_1}{\partial z_{n+p}} \\ \dfrac{\partial f_p}{\partial z_1} & \cdots & \dfrac{\partial f_p}{\partial z_{n+p}} \end{pmatrix}$$

Let $e_1, \cdots, e_{n+p-1}$ be a basis of h. As the function $< f \mid u >$ can be written in the matrix form $\tilde{\bar{u}}. \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ f_n \end{pmatrix}$, the equations of $\Gamma(< f \mid u >, h)$ are :

$$0 =< \tilde{\bar{u}}.J \mid e_1 >=< \tilde{\bar{u}}.J \mid e_2 >= \cdots =< \tilde{\bar{u}}.J \mid e_{n+p-1} > .$$

or :

$$(*) \qquad \tilde{\bar{u}}.J.\bar{e}_1 = \tilde{\bar{u}}.J.\bar{e}_2 = \cdots = \tilde{\bar{u}}.J.\bar{e}_{n+p-1}.$$

The regular point m belongs to Γ^h if and only if there exists a vector u satisfying (*). This amounts to say that the system of vectors of $\mathbb{C}^p$:

$$g_1 = J.\bar{e}_1,\ g_2 = J.\bar{e}_2,\ \cdots,\ g_{n+p-1} = J.\bar{e}_{n+p-1}$$

is of rank smaller or equal to $(p-1)$. The equations of Γ^h are obtained by equating to zero the set of determinants which guarantees this rank condition.

The points of $\Gamma^h \cap [(f = \lambda)\backslash\Sigma]$ are exactly the critical points of the restriction to the smooth part of the leaf $f = \lambda$ of the orthogonal projection $p_{h^\perp}$ on the complex line $h^\perp$. □

Milnor's codimension 1 results were generalised by Hamm [Ha] and Giusti and Henry [G-H] for complete intersections.

Let now $f : \mathbb{C}^{n+1} \to \mathbb{C}^p$ be a surjective algebraic map such that the origin is an isolated singular point of f and such that the level $(f = 0)$ is a complete intersection. We will denote by C_λ the level variety $(f = \lambda)$. Let us state the algebraic results that we will need.

Theorem 13.4.9 *[Ha]. For ε small enough and $\lambda \neq 0$ (chosen after ε) small enough, the manifold with boundary $(C_\lambda \cap B_\varepsilon)$ has the homotopy type of a wedge of μ spheres of real dimension n.*

Theorem 13.4.10 *[G-H]. There exists a measure zero analytical closed set* γ_i *of the Grassmann manifold* $G_{n+p,i+p-1}$ *such that, if*

$H \in G_{n+p,i+p-1} \backslash \gamma_i$, *the Milnor number* $\mu(f_{|H})$ *takes the generic value* $\mu^{(i)}$, *independently of* H.

Generalising the codimension 1 case, see [Tei1], Greuel [Gre] and Lê [Le2] independently proved :

Theorem 13.4.11 *The intersection multiplicity at 0 of the complete intersection and a generic polar variety* Γ^h *satisfies :*

$$\lim_{\varepsilon->0} \lim_{\lambda->0} \sharp(C_\lambda \cap B_\varepsilon \cap \Gamma_h) = (C_0.\Gamma_h) = \mu^{n+1} + \mu^n.$$

The following theorem about the total curvature is now a mere translation, using an exchange theorem in codimension p, of the previous one, extending the codimension 1 result of [Lan1]:

Theorem 13.4.12 *[Lan5].Let* $f : \mathbb{C}^{n+1} \to \mathbb{C}^p$ *be a polynomial such that the level* $f = 0$ *is a complete intersection, then:*

$$\lim_{\varepsilon->0} \lim_{\lambda->0} \int_{\mathbb{C}_\lambda \cap B_\varepsilon} | K |= const\ (-1)^n(\mu^{n+1} + \mu^n)$$

where K *is the Lipschitz-Killing curvature of the level* C_λ *and const a positive constant depending only on dimensions.*

Remark: The study of other symmetric functions of curvature , in the codimension 1 case, was started by Griffiths [Gr], and continued by Kennedy [Ke] and Loeser [Lo].

Remark:(integral geometry in $\mathbb{CP}_n$) In this paragraph f will be a homogeneous polynomial map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ of degree greater or equal to two having only isolated singular points in $\mathbb{CP}_n$. Using a pencil of projective lines, one can define polar curves (see the chapter **spheres** and the chapter **foliation** for the construction of the curves of contact of a foliation with a pencil). Then adding the previous result: (there $f : \mathbb{C}^n \to \mathbb{C}$)

$$\lim_{\varepsilon->0} \lim_{\lambda->0} \int_{C_\lambda \cap B_\varepsilon} | K |= const\ (-1)^n(\mu^n + \mu^{n-1})$$

with Bezout's theorem one gets a geometric proof of Laumon's results [Lau] [Lan6]:

Proposition 13.4.13

$$degree(C^*) = [d(d-1)]^{n-1} - \sum_{m\ singular} (\mu^n + \mu^{n-1})(m)$$

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Integral geometry – measure theoretic approach and stochastic applications

Rolf Schneider

Preface

Integral geometry, as it is understood here, deals with the computation and application of geometric mean values with respect to invariant measures. In the following, I want to give an introduction to the integral geometry of polyconvex sets (i.e., finite unions of compact convex sets) in Euclidean spaces. The invariant or Haar measures that occur will therefore be those on the groups of translations, rotations, or rigid motions of Euclidean space, and on the affine Grassmannians of k-dimensional affine subspaces. However, it is also in a different sense that invariant measures will play a central role, namely in the form of finitely additive functions on polyconvex sets. Such functions have been called additive functionals or valuations in the literature, and their motion invariant specializations, now called intrinsic volumes, have played an essential role in Hadwiger's [2] and later work (e.g., [8]) on integral geometry. More recently, the importance of these functionals for integral geometry has been rediscovered by Rota [5] and Klain-Rota [4], who called them 'measures' and emphasized their role in certain parts of geometric probability. We will, more generally, deal with local versions of the intrinsic volumes, the curvature measures, and derive integral-geometric results for them. This is the third aspect of the measure theoretic approach mentioned in the title. A particular feature of this approach is the essential role that uniqueness results for invariant measures play in the proofs.

As prerequisites, we assume some familiarity with basic facts from measure and integration theory. We will also have to use some notions and results from the geometry of convex bodies. These are intuitive and easy to grasp, and we will apply them without proof. In order to understand the applications to stochastic geometry that we intend to explain, the knowledge of fundamental notions from probability theory will be sufficient.

The material is taken from different sources, essentially from the lecture notes on “Integralgeometrie” [8] and “Stochastische Geometrie” [9], both written together with Wolfgang Weil. Another source is the fourth chapter of the book [7] on convex bodies.

Contents

1 Introduction

It will be one aim of the following lectures to develop some integral geometric formulae for sets in Euclidean space and to show how they can be applied in parts of stochastic geometry. In particular, I want to emphasize the role that integral geometry can play in the theoretical foundations of stereology. By stereology one understands a collection of procedures which are used to estimate certain parameters of real materials by means of measurements in small probes and plane sections. Stereology is applied in biology and medicine as well as in material sciences (e.g., metallography, mineralogy).

Since much of the motivation for the later theoretical investigations comes from these practical procedures, let me first explain the underlying ideas by two typical examples.

In geology, one may be interested in determining the volume proportion of some mineral in a rock. Thus one assumes that for the material in question there is a well-defined parameter, traditionally denoted by V_V, that specifies the volume of the investigated mineral per unit volume of the total material. In order to determine this specific volume V_V, one will first take a probe of the material "at random". As a second step, Delesse (1847) proposed to produce a (polished) plane section of the probe, possibly again "at random", and to determine the specific area A_A of the investigated mineral in that section. On the basis of heuristic arguments, Delesse asserted that

$$V_V = A_A,$$

or rather that the measured value A_A is a good estimate for the unknown parameter V_V.

A second example is taken from medicine. One may be interested in the gas exchange of a mammal lung, and this depends on the alveolar surface of the lung. To measure this specific area, denoted by S_V, only a small probe of the lung tissue will be available, and usually only a thin slice can be observed under the microscope. Tomkeieff (1945) proposed to determine the specific boundary length L_A of the tissue in the section and then to estimate the unknown specific area S_V by means of the formula

$$S_V = \frac{4}{\pi} L_A,$$

again supported by heuristic arguments.

Scientists working in practice have developed similar formulae. The so-called 'fundamental equations of stereology' are

$$V_V = A_A, \quad S_V = \frac{4}{\pi} L_A, \quad M_V = 2\pi\chi_A.$$

Here M denotes the integral of the mean curvature, and χ is the Euler characteristic.

It is evident that such heuristic procedures are implicitly based on many tacit assumptions. A theoretical justification has to begin by analyzing these assumptions, it has to provide suitable models and must finally lead to exactly proven formulae of the type used in practice. The first assumption is that the parameter of the material to be determined, like volume or surface area per unit volume, exists and can be estimated with sufficient accuracy from taking randomly placed probes and averaging. A solid foundation and justification can be achieved if the material under investigation is modelled as the realization of a random set. Taking a probe at random can then be modelled as follows. We fix a shape for the probe or 'observation window', say a compact convex set K with positive volume. Inside K we observe a realization $Z(\omega)$ of our random set Z. We assume that for the intersection $Z(\omega) \cap K$ we are able to measure a geometric functional φ of interest, like volume or surface area. Instead of placing K in a random position, one assumes that the random set Z has a suitable invariance property, meaning that Z and its image under any translation or rigid motion are stochastically equivalent. Under suitable model assumptions, the mathematical expectation $\mathbb{E}\varphi(Z \cap K)$ will exist, and the measured value $\varphi(Z(\omega) \cap K)$ can be considered as an unbiased estimator. If the model is such that the random set Z has a well-defined φ-density, the next question is then how this is related to the local expectation $\mathbb{E}\varphi(Z \cap K)$, depending on the test body K. Similar considerations will be necessary to justify the determination of parameters from randomly placed lower-dimensional sections.

This program, of which we have merely given a rough sketch, will obviously require the development of

- a theory of random sets with suitable invariance properties, admitting densities of geometric functionals, like volume, surface area, Euler characteristic,
- a theory of mean values of geometric functionals, evaluated at intersections of fixed and moving geometric objects.

2 Elementary mean value formulae

We begin with the second part of the program, the development of mean value formulae for fixed and moving geometric objects. By "moving" we mean here that the geometric objects, which are in Euclidean space, undergo translations or rigid motions. The mean values will be taken with respect to invariant measures on the groups of translations or rigid motions. The present section is still part of the introduction and will discuss a few elementary examples of such mean value formulae.

We work in n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$). The subsets of $\mathbb{R}^n$ which will later (in dimensions two and three) be used to model real material, should not be too complicated, in order that functionals like surface area or Euler characteristic are defined (locally). It is sufficient for practical applications to consider only sets which can locally be represented as finite unions of convex bodies (non-empty, compact convex sets). We begin by considering only convex bodies; it will later be easy to extend the results to more general sets of the type just described. By $\mathcal{K}^n$ we denote the set of convex bodies in $\mathbb{R}^n$.

The following is a basic example of the type of questions that we will have to answer. Let $K, M \in \mathcal{K}^n$ be two convex bodies. Let M undergo translations, that is, we consider $M + t$ for $t \in \mathbb{R}^n$. What is the mean value of the volume of $K \cap (M + t)$, taken over all t with $K \cap (M + t) \neq \emptyset$? The mean value here refers to the invariant measure on the translation group, which can be identified with the Lebesgue measure λ on $\mathbb{R}^n$. For convex bodies K, we write $V_n(K) = \lambda(K)$ for the volume. Thus we are asking for the mean value

$$\frac{\int_{\mathbb{R}^n} V_n(K \cap (M + t))\, d\lambda(t)}{\int_{\mathbb{R}^n} \chi(K \cap (M + t))\, d\lambda(t)}. \tag{1}$$

Note that $\chi(K') = 1$ for a non-empty convex body K' and $\chi(\emptyset) = 0$, so that the denominator is indeed the total measure of all translation vectors t for which $K \cap (M + t) \neq \emptyset$. Thus we have to determine integrals of the type

$$\int_{\mathbb{R}^n} \varphi(K \cap (M + t))\, d\lambda(t)$$

for different functionals φ. Extensions of this problem will be our main concern in these lectures.

It is not difficult to determine the numerator in (1). Denoting the indicator function of a set $A \subset \mathbb{R}^n$ by $\mathbf{1}_A$, we have

$$V_n(K \cap (M + t)) = \int_{\mathbb{R}^n} \mathbf{1}_{K \cap (M+t)}(x)\, d\lambda(x)$$

and

$$\mathbf{1}_{K \cap (M+t)}(x) = \mathbf{1}_K(x)\mathbf{1}_{M+t}(x)$$

with

$$\mathbf{1}_{M+t}(x) = 1 \;\Leftrightarrow\; x \in M+t \;\Leftrightarrow\; t \in M^* + x \;\Leftrightarrow\; \mathbf{1}_{M^*+x}(t) = 1.$$

Here we have denoted by

$$M^* := \{y \in \mathbb{R}^n : -y \in M\}$$

the set obtained from M by reflection in the origin. Now Fubini's theorem gives

$$\begin{aligned}
&\int_{\mathbb{R}^n} V_n(K \cap (M+t))\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_{K\cap(M+t)}(x)\, d\lambda(x)\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_K(x)\mathbf{1}_{M^*+x}(t)\, d\lambda(t)\, d\lambda(x) \\
&\quad = \int_{\mathbb{R}^n} \mathbf{1}_K(x) V_n(M^*+x)\, d\lambda(x) \\
&\quad = V_n(M^*) \int_{\mathbb{R}^n} \mathbf{1}_K(x)\, d\lambda(x)
\end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} V_n(K \cap (M+t))\, d\lambda(t) = V_n(K)V_n(M). \tag{2}$$

Note that we have used the invariance of the volume under translations and reflections.

The denominator in (1) is of a different type. We have

$$\begin{aligned}
\chi(K \cap (M+t)) = 1 \quad &\Leftrightarrow \quad K \cap (M+t) \neq \emptyset \\
&\Leftrightarrow \quad \exists\, k \in K\, \exists\, m \in M : k = m + t \\
&\Leftrightarrow \quad t = k - m \text{ with } k \in K, m \in M \\
&\Leftrightarrow \quad t \in K + M^* \\
&\Leftrightarrow \quad \mathbf{1}_{K+M^*}(t) = 1
\end{aligned}$$

and hence

$$-\int_{\mathbb{R}^n} \chi(K \cap (M+t))\, d\lambda(t) = V_n(K+M^*). \tag{3}$$

Convex geometry tells us that

$$V_n(K+M^*) = \sum_{i=0}^{n} \binom{n}{i} V(\underbrace{K,\dots,K}_{i},\underbrace{M^*,\dots,M^*}_{n-i}),$$

where the function $V : (\mathcal{K}^n)^n \to \mathbb{R}$ is the so-called *mixed volume.* The essential observation for us is here that the obtained expression cannot be simplified further. In particular, there is no separation of the roles of K and M on the right-hand side, as it occurred in (2). Such a separation is only achieved if we integrate, not only over the translations of M as in (3), but over all rigid motions of M. This will be one of the fundamental results of integral geometry to be obtained later.

For the moment, however, we stay with the translation group alone. The idea leading to (2) can be extended, to give a first general formula of translative integral geometry.

When we talk of a *measure* on a locally compact space E, we always mean a non-negative, countably additive, extended real-valued function on the σ-algebra $\mathcal{B}(E)$ of Borel sets of E. Such a measure is called *locally finite* if it is finite on compact sets.

2.1 Theorem. *Let α be a locally finite measure on $\mathbb{R}^n$, and let $A, B \in \mathcal{B}(\mathbb{R}^n)$. Then*

$$\int_{\mathbb{R}^n} \alpha(A \cap (B+t))\, d\lambda(t) = \alpha(A)\lambda(B). \tag{4}$$

Proof. Using Fubini's theorem, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} \alpha(A \cap (B+t))\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_{A\cap(B+t)}(x)\, d\alpha(x)\, d\lambda(t) \\
&\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \mathbf{1}_{A}(x)\mathbf{1}_{B+t}(x)\, d\lambda(t)\, d\alpha(x)
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \mathbf{1}_A(x) \int_{\mathbb{R}^n} \mathbf{1}_{B^*+x}(t)\, d\lambda(t)\, d\alpha(x) \\
&= \int_{\mathbb{R}^n} \mathbf{1}_A(x) \lambda(B^* + x)\, d\alpha(x) \\
&= \alpha(A)\lambda(B).
\end{aligned}
$$

■

This can be used to obtain a counterpart to the translative integral formula (2), with volume replaced by surface area. First we have to explain what we mean by the surface area of a general convex body, which need not satisfy any smoothness assumptions. For that purpose, let us first recall the notion of the p-dimensional Hausdorff measure, for $p \geq 0$.

We equip $\mathbb{R}^n$ with the usual scalar product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. For a subset $G \subset \mathbb{R}^n$, the *diameter* is defined by

$$D(G) := \sup\{\|x - y\| : x, y \in G\}.$$

Now for an arbitrary subset M and for $\delta > 0$ one defines

$$
\begin{aligned}
\mathcal{H}^p_\delta(M) \quad := \quad & \frac{\pi^{p/2}}{2^p \Gamma(1 + \frac{p}{2})} \inf \Bigg\{ \sum_{i=1}^{\infty} D(G_i)^p : (G_i)_{i \in \mathbb{N}} \text{ sequence of open sets} \\
& \qquad \text{with } D(G_i) \leq \delta \text{ and } M \subset \bigcup_{i=1}^{\infty} G_i \Bigg\}.
\end{aligned}
$$

The limit

$$\mathcal{H}^p(M) := \lim_{\delta \to 0+} \mathcal{H}^p_\delta(M) = \sup_{\delta > 0} \mathcal{H}^p_\delta(M)$$

exists in $\mathbb{R} \cup \{\infty\}$ and is called the p-dimensional (outer) Hausdorff measure of M. The restriction of $\mathcal{H}^p$ to the σ-algebra $\mathcal{B}(\mathbb{R}^n)$ of Borel sets is a measure. One can show that $\mathcal{H}^n(A) = \lambda(A)$ for $A \in \mathcal{B}(\mathbb{R}^n)$.

Now the *surface area* of a convex body $K \in \mathcal{K}^n$ with interior points is defined by

$$\mathcal{H}^{n-1}(\partial K) =: 2V_{n-1}(K),$$

where ∂ denotes the boundary. The notation $2V_{n-1}$ is chosen with respect to later developments. For $K \in \mathcal{K}^n$ without interior points, we define $V_{n-1}(K) := \mathcal{H}^{n-1}(K)$. This is zero if K is of dimension less than $n - 1$.

2.2 Theorem. *Let $K, M \in \mathcal{K}^n$ be convex bodies with interior points. Then*

$$\int_{\mathbb{R}^n} V_{n-1}(K \cap (M+t))\, d\lambda(t) = V_{n-1}(K)V_n(M) + V_n(K)V_{n-1}(M). \tag{5}$$

Proof. The boundary of the intersection $K \cap (M+t)$ consists of two parts:

$$\partial(K \cap (M+t)) = [\partial K \cap (M+t)] \cup [K \cap (\partial M + t)].$$

The intersection of the two sets on the right satisfies

$$[\partial K \cap (M+t)] \cap [K \cap (\partial M + t)] \subset \partial K \cap (\partial M + t).$$

We define

$$\alpha(A) := \mathcal{H}^{n-1}(\partial K \cap A) \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^n).$$

Then α is a finite measure on $\mathbb{R}^n$. From (4) (with $A = \partial K$ and $B = \partial M$) we get

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-1}(\partial K \cap (\partial M + t))\, d\lambda(t) = \mathcal{H}^{n-1}(\partial K)\lambda(\partial M) = 0.$$

Since the integrand is nonnegative, it follows that

$$\mathcal{H}^{n-1}(\partial K \cap (\partial M + t)) = 0 \qquad \text{for } \lambda\text{-almost all } t,$$

that is, for all $t \in \mathbb{R}^n \setminus N$, with some set N satisfying $\lambda(N) = 0$. Hence, for all $t \in \mathbb{R}^n \setminus N$ we have

$$\mathcal{H}^{n-1}(\partial(K \cap (M+t)) = \mathcal{H}^{n-1}(\partial K \cap (M+t)) + \mathcal{H}^{n-1}(K \cap (\partial M + t)). \tag{6}$$

Using (4) with $A = \partial K$ and $B = M$, we further obtain

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-1}(\partial K \cap (M+t))\, d\lambda(t) = \mathcal{H}^{n-1}(\partial K)\lambda(M).$$

Moreover,

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-1}(K \cap (\partial M + t))\, d\lambda(t)$$

$$= \int_{\mathbb{R}^n} \mathcal{H}^{n-1}((K-t) \cap \partial M)\, d\lambda(t)$$
$$= \int_{\mathbb{R}^n} \mathcal{H}^{n-1}(\partial M \cap (K+t))\, d\lambda(t)$$
$$= \mathcal{H}^{n-1}(\partial M)\lambda(K).$$

Here we have used the facts that $\mathcal{H}^{n-1}$ is translation invariant and that the Lebesgue measure is invariant under the inversion $t \mapsto -t$. Finally we have used (4) again.

Since equation (6) holds for all $t \in \mathbb{R}^n \setminus N$ and since the null set N can be neglected in the integration, we deduce that

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-1}(\partial(K \cap (M+t))\, d\lambda(t) = \mathcal{H}^{n-1}(\partial K)\lambda(M) + \mathcal{H}^{n-1}(\partial M)\lambda(K).$$

This is precisely the assertion (5). ■

Instead of intersecting a fixed convex body with a translated one, we now briefly consider the intersections with a translated hyperplane. We parameterize hyperplanes in the form

$$H(u,\tau) := \{x \in \mathbb{R}^n : \langle x, u\rangle = \tau\}$$

with a unit vector $u \in \mathbb{R}^n$ and a real number $\tau \in \mathbb{R}$. Thus u is one of the two unit normal vectors of the (unoriented) hyperplane $H(u,\tau)$.

For a convex body $K \in \mathcal{K}^n$, Fubini's theorem immediately gives

$$\int_{\mathbb{R}} V_{n-1}(K \cap H(u,\tau))\, d\tau = V_n(K).$$

Can we obtain the surface area of a convex body $K \in \mathcal{K}^n$ with interior points in a similar way, that is, by a formula of type

$$\int_{\mathbb{R}^n} \mathcal{H}^{n-2}(\partial K \cap H(u,\tau))\, d\tau = c_n V_{n-1}(K)$$

with some constant c_n? Simple examples (balls and cubes in $\mathbb{R}^3$) show that such a formula does not hold with a constant independent of K. However, we shall later see that

$$\int_{S^{n-1}} \int_{\mathbb{R}^n} \mathcal{H}^{n-2}(\partial K \cap H(u,\tau))\, d\tau\, d\sigma(u) = c_n V_{n-1}(K) \tag{7}$$

does hold with a constant c_n. Here the outer integration is over the unit sphere S^{n-1} with respect to the rotation invariant measure σ.

Both integrations in (7) together can be interpreted as one integration over the space of hyperplanes, with respect to a rigid motion invariant measure on that space. Thus we have now two examples for the simplifying effect in obtaining mean values when the integrations are performed with respect to motion invariant measures. This observation will be considerably elaborated in the following.

3 Invariant measures of Euclidean geometry

Integral geometry is based on the notion of invariant measure. Here invariance refers to a group operation and thus to a homogeneous space. Invariant measures on homogeneous spaces are also known as Haar measures. We do not presuppose here any knowledge of the theory of Haar measure. In the present section, we give an elementary introduction to the invariant measures on the groups and homogeneous spaces that are used in the integral geometry of Euclidean space.

A *topological group* is a group G together with a topology on G such that the map from $G \times G$ to G defined by $(x, y) \mapsto xy$ and the map from G to G defined by $x \mapsto x^{-1}$ are continuous. Let G be a group and X a non-empty set. An *operation* of G on X is a map $\varphi : G \times X \to X$ satisfying

$$\varphi(g, \varphi(g', x)) = \varphi(gg', x), \qquad \varphi(e, x) = x$$

for all $g, g' \in G$, the unit element e of G and all $x \in X$. One also says that G *operates on* X, by means of φ. For $\varphi(g, x)$ one usually writes gx, provided that the operation is clear from the context. The group G *operates transitively* on X if for any $x, y \in X$ there exists $g \in G$ so that $y = gx$. If G is a topological group, X is a topological space, and the operation φ is continuous, one says that G *operates continuously* on X.

The following situation often occurs: X is a nonempty set and G is a group of transformations (bijective mappings onto itself) of X, with the composition as group multiplication; the operation of G on X is given by $(g, x) \mapsto gx :=$ image of x under g. When transformation groups occur in the following, multiplication and operation are always understood in this sense.

We consider three groups of bijective affine maps of $\mathbb{R}^n$ onto itself, the *translation group* T_n, the *rotation group* SO_n, and the *rigid motion group* G_n. The *translations* $t \in T_n$ are the maps of the form $t = t_x$ with $x \in \mathbb{R}^n$, where $t_x(y) := y + x$ for $y \in \mathbb{R}^n$. The mapping $\tau : x \mapsto t_x$ is an isomorphism of the additive group $\mathbb{R}^n$ onto T_n. Hence, we can identify T_n with $\mathbb{R}^n$, which we shall often do tacitly. In particular, T_n carries the topology inherited from $\mathbb{R}^n$ via τ. Since $t_x \circ t_y = t_{x+y}$ and $t_x^{-1} = t_{-x}$, composition and inversion are continuous, hence T_n is a topological group. In view of the topological properties of $\mathbb{R}^n$ we can thus state the following.

3.1 Theorem. *The translation group T_n is an abelian, locally compact topological group with countable base. The operation of T_n on $\mathbb{R}^n$ is continuous.*

The elements of the rotation group SO_n are the linear mappings $\vartheta : \mathbb{R}^n \to \mathbb{R}^n$ that preserve scalar product and orientation; they are called *(proper) rotations*. With respect to the standard (orthonormal) basis of $\mathbb{R}^n$, every rotation ϑ is represented by an orthogonal matrix $M(\vartheta)$ with determinant 1. The mapping $\mu : \vartheta \mapsto M(\vartheta)$ is an isomorphism of the group SO_n onto the group $\mathcal{SO}(n)$ of orthogonal (n,n)-matrices with determinant 1 under matrix multiplication. If we identify an (n,n)-matrix with the n^2-tuple of its entries (in lexicographic order, say), we can consider $\mathcal{SO}(n)$ as a subset of $\mathbb{R}^{n^2}$. This set is bounded, since the rows of an orthogonal matrix are normalized, and it is closed in $\mathbb{R}^{n^2}$, hence compact. The mappings $(M,N) \mapsto MN$ and $M \mapsto M^{-1}$ are continuous, and so is the mapping $(M,x) \mapsto Mx$ (where x is considered as an $(n,1)$-matrix) from $\mathcal{SO}(n) \times \mathbb{R}^n$ into $\mathbb{R}^n$. Using the mapping μ^{-1} to transfer the topology from $\mathcal{SO}(n)$ to SO_n, we thus obtain the following.

3.2 Theorem. *The rotation group SO_n is a compact topological group with countable base. The operation of SO_n on $\mathbb{R}^n$ is continuous.*

The elements of the motion group G_n are the affine maps $g : \mathbb{R}^n \to \mathbb{R}^n$ that preserve distances between points and the orientation; they are called *(rigid) motions*. Every rigid motion $g \in G_n$ can be represented uniquely as the composition of a rotation ϑ and a translation t_x, that is, $g = t_x \circ \vartheta$, or $gy = \vartheta y + x$ for $y \in \mathbb{R}^n$. The mapping

$$\begin{array}{rccc} \gamma : & \mathbb{R}^n \times SO_n & \to & G_n \\ & (x,\vartheta) & \mapsto & t_x \circ \vartheta \end{array}$$

is bijective. We use it to transfer the topology from $\mathbb{R}^n \times SO_n$ to G_n. Using Theorems 3.1 and 3.2, it is then easy to show the following.

3.3 Theorem. *G_n is a locally compact topological group with countable base. Its operation on $\mathbb{R}^n$ is continuous.*

After these topological groups, we now consider the homogeneous spaces that will play a role in the following. Let $q \in \{0, \dots, n\}$, let $\mathcal{L}_q^n$ be the set of all q-dimensional linear subspaces of $\mathbb{R}^n$, and let $\mathcal{E}_q^n$ be the set of all q-dimensional affine subspaces of $\mathbb{R}^n$. The natural operation of SO_n on $\mathcal{L}_q^n$ is given by $(\vartheta, L) \mapsto \vartheta L :=$ image of L under ϑ. Similarly, the natural operation of G_n on $\mathcal{E}_q^n$ is given by $(g, E) \mapsto gE :=$ image of E under g. We introduce suitable topologies on $\mathcal{L}_q^n$ and $\mathcal{E}_q^n$. For this, let $L_q \in \mathcal{L}_q^n$ be fixed and let $L_q^\perp$ be its orthogonal complement. The mappings

$$\begin{array}{cccc} \beta_q: & SO_n & \to & \mathcal{L}_q^n \\ & \vartheta & & \vartheta L_q \end{array}$$

and

$$\begin{array}{cccc} \gamma_q: & L_q^\perp \times SO_n & \to & \mathcal{E}_q^n \\ & (x, \vartheta) & \mapsto & \vartheta(L_q + x) \end{array}$$

are surjective (but not injective). We endow $\mathcal{L}_q^n$ with the finest topology for which β_q is continuous, and $\mathcal{E}_q^n$ with the finest topology for which γ_q is continuous. Thus a subset $A \in \mathcal{E}_q^n$, for example, is open if and only if $\gamma_q^{-1}(A)$ is open. It is an elementary task to prove the following.

3.4 Theorem. *$\mathcal{L}_q^n$ is compact and has a countable base, the map β_q is open, and the operation of SO_n on $\mathcal{L}_q^n$ is continuous and transitive.*

3.5 Theorem. *$\mathcal{E}_q^n$ is locally compact and has a countable base, the map γ_q is open, and the operation of G_n on $\mathcal{E}_q^n$ is continuous and transitive.*

It should be remarked that the topologies on $\mathcal{L}_q^n$ and $\mathcal{E}_q^n$, as well as the invariant measures on these spaces to be introduced below, do not depend on the special choice of the subspace L_q. This follows easily from the fact that SO_n operates transitively on $\mathcal{L}_q^n$, and G_n operates transitively on $\mathcal{E}_q^n$.

The topological spaces $\mathcal{L}_q^n$ are called *Grassmann manifolds*; a common notation for $\mathcal{L}_q^n$ is $G(n, q)$. The spaces $\mathcal{E}_q^n$ are also called *affine Grassmannians.*

Occasionally, we have talked of homogeneous spaces; it seems, therefore, appropriate here to give the general definition. If G is a topological group, a *homogeneous G-space* is, by definition, a pair (X, φ), where X is a topological space and φ is a transitive continuous operation of G on X with the additional property that the map $\varphi(\cdot, p)$ is open for $p \in X$. In this sense, $\mathcal{L}_q^n$ is a homogeneous SO_n-space (with respect to the standard operation), and $\mathcal{E}_q^n$ is a homogeneous G_n-space. Also with the standard operations, $\mathbb{R}^n$ is a homogeneous T_n-space and G_n-space, and the unit sphere

$$S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$$

is a homogeneous SO_n-space.

We shall now introduce invariant measures on the groups and homogeneous spaces considered. We begin with some general definitions and remarks. All topological spaces occurring here are locally compact and second countable. By a *Borel measure* ρ on X we understand a measure on the σ-algebra $\mathcal{B}(X)$ of Borel sets of X satisfying $\rho(K) < \infty$ for every compact set $K \subset X$. Every such measure is regular. Instead of 'Borel measure' we often say 'measure' for short. The notion 'measurable', without extra specification, means 'Borel measurable'.

Let the topological group G operate continuously on the space X. A measure ρ on X is called *G-invariant* (or briefly *invariant*, if G is clear from the context) if

$$\rho(gA) = \rho(A) \qquad \text{for all } A \in \mathcal{B}(X) \text{ and all } g \in G.$$

This definition makes sense: for each $g \in G$, the mapping $x \mapsto gx$ is a homeomorphism, hence $A \in \mathcal{B}(X)$ implies $gA \in \mathcal{B}(X)$. Invariant regular Borel measures on locally compact homogeneous spaces are called *Haar measures*, if they are not identically zero.

From basic measure theory, we assume familiarity with Lebesgue measure on $\mathbb{R}^n$, in particular with the following result. Here we use the unit cube $C^n := [0,1]^n$ for normalization.

3.6 Theorem and Definition. *There is a unique translation invariant measure λ on $\mathcal{B}(\mathbb{R}^n)$ satisfying $\lambda(C^n) = 1$. It is called the Lebesgue measure.*

It is easy to see that λ is also rotation invariant (SO_n-invariant). If $\vartheta \in SO_n$ and if one defines $\rho(A) := \lambda(\vartheta A)$ for $A \in \mathcal{B}(\mathbb{R}^n)$, then ρ is a translation invariant measure on $\mathcal{B}(\mathbb{R}^n)$. By Theorem 3.6, $\rho = c\lambda$ with $c = \rho(C^n)$. The unit ball B^n satisfies $c\lambda(B^n) = \rho(B^n) = \lambda(\vartheta B^n) = \lambda(B^n)$, hence $c = 1$.

Since the Lebesgue measure λ is thus rigid motion invariant, it is the Haar measure on the homogeneous G_n-space $\mathbb{R}^n$, normalized in a special way.

We mention the special value

$$\kappa_n := \lambda(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(1 + \frac{n}{2})},$$

which will play a role in many later formulae. We put $\kappa_0 := 1$.

The Haar measure on the homogeneous SO_n-space S^{n-1}, the unit sphere, is easily derived from the Lebesgue measure. For $A \in \mathcal{B}(S^{n-1})$ we define

$$\hat{A} := \{\alpha x \in \mathbb{R}^n : x \in A,\ 0 \leq \alpha \leq 1\}.$$

A standard argument shows that $\hat{A} \in \mathcal{B}(\mathbb{R}^n)$, hence we can define $\sigma(A) := n\lambda(\hat{A})$. This yields a finite measure σ on $\mathcal{B}(S^{n-1})$ for which

$$\sigma(S^{n-1}) =: \omega_n = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

The rotation invariance of λ implies the rotation invariance of σ. We call σ, with the normalization specified above, the *spherical Lebesgue measure*. Up to a constant factor, σ is the only rotation invariant Borel measure on $\mathcal{B}(S^{n-1})$. This follows from Corollary 3.12 below.

Our next aim is the introduction of an invariant measure on the rotation group SO_n. For a measure on a group, several notions of invariance are natural. A topological group G operates on itself by means of the mapping $(g, x) \mapsto gx$ (multiplication in G). The corresponding invariance on G is called *left invariance*. More generally, for $g \in G$ and $A \subset G$ we write

$$gA := \{ga : a \in A\}, \quad Ag := \{ag : a \in A\}, \quad A^{-1} := \{a^{-1} : a \in A\}.$$

If $A \in \mathcal{B}(G)$, then also gA, Ag, A^{-1} are Borel sets. A measure ρ on G is called *left invariant* if $\rho(gA) = \rho(A)$, and *right invariant* if $\rho(Ag) = \rho(A)$, for all $A \in \mathcal{B}(G)$ and all $g \in G$. The measure ρ is *inversion invariant* if $\rho(A^{-1}) = \rho(A)$ for all $A \in \mathcal{B}(G)$. If ρ has all three invariance properties, it is just called *invariant*.

With these definitions we connect two general remarks. Let ρ be a left invariant measure on the topological group G. Then each measurable function $f \geq 0$ on G satisfies

$$\int_G f(ag)\, d\rho(g) = \int_G f(g)\, d\rho(g) \tag{8}$$

for all $a \in G$. This follows immediately from the definition of the integral. Vice versa, if (8) holds for all measurable functions $f \geq 0$, then the left invariance of ρ is obtained by applying (8) to indicator functions. Similarly, the right invariance of ρ is equivalent to

$$\int_G f(ga)\, d\rho(g) = \int_G f(g)\, d\rho(g) \tag{9}$$

for $a \in G$, and the inversion invariance of ρ is equivalent to

$$\int_G f(g^{-1})\, d\rho(g) = \int_G f(g)\, d\rho(g), \tag{10}$$

in each case for all measurable functions $f \geq 0$.

The following theorem on invariant measures on compact groups will be needed for the rotation group only, but can be proved without additional effort in a more general setting.

3.7 Theorem. *Every left invariant Borel measure on a compact group with countable base is invariant.*

Proof. Let ν be a left invariant Borel measure on the group G satisfying the assumptions. Since it is finite on compact sets, we may assume $\nu(G) = 1$, without loss of generality. For measurable functions $f \geq 0$ on G and for $x \in G$ we have

$$\int f(y^{-1}x)\, d\nu(y) = \int f((x^{-1}y)^{-1})\, d\nu(y) = \int f(y^{-1})\, d\nu(y). \tag{11}$$

Here the integrations extend over all of G; similar conventions will be adopted in the following. Fubini's theorem gives

$$\begin{aligned} \int f(y^{-1})\, d\nu(y) &= \int \int f(y^{-1}x)\, d\nu(y)\, d\nu(x) \\ &= \int \int f(y^{-1}x)\, d\nu(x)\, d\nu(y) = \int f(x)\, d\nu(x). \end{aligned}$$

Hence, the measure ν is inversion invariant. Using this fact and (11), we get for $x \in G$ that

$$\begin{aligned} \int f(yx)\, d\nu(y) &= \int f(y^{-1}x)\, d\nu(y) \\ &= \int f(y^{-1})\, d\nu(y) = \int f(y)\, d\nu(y), \end{aligned}$$

which shows that ν is also right invariant. ∎

Concerning the application of Fubini's theorem here and later, we remark the following. All topological spaces occurring in our considerations are locally compact and second countable, thus they are σ-compact. Moreover, all the measures that occur are finite on compact sets. Therefore, all measure spaces under consideration are σ-finite, so that Fubini's theorem can be applied in its usual form.

The following uniqueness result for invariant measures makes special assumptions, but in this form it is sufficient for our purposes and is easy to prove.

3.8 Theorem. *Let G be a locally compact group with a countable base, let $\nu \neq 0$ be an invariant and μ a left invariant Borel measure on G. Then $\mu = c\nu$ with a constant $c \geq 0$.*

Proof. For measurable functions $f, g \geq 0$ on G we have

$$\begin{aligned} &\int f \, d\nu \int g \, d\mu = \int \int f(xy) g(y) \, d\nu(x) \, d\mu(y) \\ &= \int \int f(xy) g(y) \, d\mu(y) \, d\nu(x) = \int \int f(y) g(x^{-1}y) \, d\mu(y) \, d\nu(x) \\ &= \int f(y) \int g(x^{-1}y) \, d\nu(x) \, d\mu(y) = \int g \, d\nu \int f \, d\mu. \end{aligned}$$

Here we have used, besides Fubini's theorem, the right and inversion invariance of ν and the left invariance of μ.

Since $\nu \neq 0$, there is a compact set $A_0 \subset G$ with $\nu(A_0) > 0$. For arbitrary $A \in \mathcal{B}(G)$ we put $f := \mathbf{1}_{A_0}$ and $g := \mathbf{1}_A$ and obtain $\nu(A_0)\mu(A) = \nu(A)\mu(A_0)$, hence $\mu = c\nu$ with $c := \mu(A_0)/\nu(A_0)$. ■

The notation $\mathbf{1}_A$ used here for the indicator function of a set A will also be employed in the following.

Now we turn to the existence of some invariant measures. First we describe a direct construction of the invariant measure on the rotation group, without recourse to the general theory of Haar measure.

3.9 Theorem. *On the rotation group SO_n, there is an invariant measure ν with $\nu(SO_n) = 1$.*

Proof. By LI_n we denote the set of linearly independent n-tuples of vectors from the unit sphere S^{n-1}. We define a map $\psi : LI_n \to SO_n$ in the following way. Let $(x_1, \ldots, x_n) \in LI_n$. By Gram-Schmidt orthonormalization, we transform $(x_1, \ldots, x_n)$ into the n-tuple $(z_1, \ldots, z_n)$; then we denote by $(\overline{z}_1, \ldots, \overline{z}_n)$ the positively oriented n-tuple for which $\overline{z}_i := z_i$ for $i = 1, \ldots, n-1$ and $\overline{z}_n := \pm z_n$. If $(e_1, \ldots, e_n)$ denotes the standard basis of $\mathbb{R}^n$, there is a unique rotation $\vartheta \in SO_n$ satisfying $\vartheta e_i = \overline{z}_i$ for $i = 1, \ldots, n$. We define $\psi(x_1, \ldots, x_n) := \vartheta$.

Explicitly, we have $z_i = y_i/\|y_i\|$ with $y_1 = x_1$ and

$$y_k = x_k - \sum_{j=1}^{k-1} \langle x_k, y_j \rangle \frac{y_j}{\|y_j\|^2}, \qquad k = 2, \ldots, n.$$

From this representation, the following is evident. If $\rho \in SO_n$ is a rotation and if the n-tuple $(x_1, \ldots, x_n) \in LI_n$ is transformed into $(z_1, \ldots, z_n)$ and then into $(\overline{z}_1, \ldots, \overline{z}_n)$, then the n-tuple $(\rho x_1, \ldots, \rho x_n)$ is transformed into $(\rho z_1, \ldots, \rho z_n)$ and subsequently into $(\rho \overline{z}_1, \ldots, \rho \overline{z}_n)$. Thus we have $\psi(\rho x_1, \ldots, \rho x_n) = \rho \psi(x_1, \ldots, x_n)$.

For $(x_1, \ldots, x_n) \in (S^{n-1})^n \setminus LI_n$ we define $\psi(x_1, \ldots, x_n) := \mathrm{id}$. For the product measure

$$\sigma^{\otimes n} := \underbrace{\sigma \otimes \cdots \otimes \sigma}_{n},$$

the set $(S^{n-1})^n \setminus LI_n$ has measure zero; hence for any $\rho \in SO_n$ the equality $\psi(\rho x_1, \ldots, \rho x_n) = \rho \psi(x_1, \ldots, x_n)$ holds $\sigma^{\otimes n}$-almost everywhere. The mapping $\psi : (S^{n-1})^n \to SO_n$ is measurable, since LI_n is open and ψ is continuous on LI_n and constant on $(S^{n-1})^n \setminus LI_n$.

Now we define $\overline{\nu}$ as the image measure of $\sigma^{\otimes n}$ under ψ, thus $\overline{\nu} = \psi(\sigma^{\otimes n})$. Then $\overline{\nu}$ is a finite measure on SO_n, and for $\rho \in SO_n$ and measurable $f \geq 0$ we obtain

$$\begin{aligned}
&\int_{SO_n} f(\rho\vartheta)\, d\overline{\nu}(\vartheta) \\
&\quad = \int_{(S^{n-1})^n} f(\rho\psi(x_1, \ldots, x_n))\, d\sigma^{\otimes n}(x_1, \ldots, x_n) \\
&\quad = \int_{(S^{n-1})^n} f(\psi(\rho x_1, \ldots, \rho x_n))\, d\sigma^{\otimes n}(x_1, \ldots, x_n) \\
&\quad = \int_{S^{n-1}} \cdots \int_{S^{n-1}} f(\psi(\rho x_1, \ldots, \rho x_n))\, d\sigma(x_1) \cdots d\sigma(x_n) \\
&\quad = \int_{S^{n-1}} \cdots \int_{S^{n-1}} f(\psi(x_1, \ldots, x_n))\, d\sigma(x_1) \cdots d\sigma(x_n) \\
&\quad = \int_{SO_n} f(\vartheta)\, d\overline{\nu}(\vartheta).
\end{aligned}$$

Here we have used the rotation invariance of the spherical Lebesgue measure. We have proved that the measure $\overline{\nu}$ is left invariant and thus invariant, by Theorem 3.7. The measure $\nu := \overline{\nu}/\overline{\nu}(SO_n)$ is invariant and normalized. ■

From now on, ν will always denote the normalized invariant measure on SO_n.
Now we turn to the motion group G_n. Since it is not compact, an invariant measure μ on G_n cannot be finite. In order to normalize μ, we specify the compact set $A_0 := \gamma(C^n \times SO_n)$ and require that $\mu(A_0) = 1$.

3.10 Theorem. *On the motion group G_n, there is an invariant measure μ with $\mu(A_0) = 1$. Up to a constant factor, it is the only left invariant measure on G_n.*

Proof. We define μ as the image measure of the product measure $\lambda \otimes \nu$ under the homeomorphism $\gamma : \mathbb{R}^n \times SO_n \to G_n$ defined by (3). Then μ is a Borel measure on G_n with $\mu(\gamma(C^n \times SO_n)) = \lambda(C^n)\nu(SO_n) = 1$.

To show the left invariance of μ, let $f \geq 0$ be a measurable function on G_n and let $g' \in G_n$. With $g' = \gamma(t', \vartheta')$ we have

$$\begin{aligned}
&\int_{G_n} f(g'g)\, d\mu(g) \\
&\quad = \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t', \vartheta')\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta) \\
&\quad = \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t' + \vartheta' t, \vartheta'\vartheta))\, d\lambda(t)\, d\nu(\vartheta) \\
&\quad = \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta) \\
&\quad = \int_{G_n} f(g)\, d\mu(g),
\end{aligned}$$

where we have used the motion invariance of λ and the left invariance of ν. Hence, μ is left invariant. Similarly, the right invariance of ν implies via

$$\begin{aligned}
\int_{G_n} f(gg')\, d\mu(g) &= \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t + \vartheta t', \vartheta\vartheta'))\, d\lambda(t)\, d\nu(\vartheta) \\
= \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta) &= \int_{G_n} f(g)\, d\mu(g)
\end{aligned}$$

the right invariance of μ. The inversion invariance of μ is obtained from

$$\int_{G_n} f(g^{-1})\, d\mu(g) = \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(-\vartheta^{-1}t, \vartheta^{-1}))\, d\lambda(t)\, d\nu(\vartheta)$$
$$= \int_{SO_n} \int_{\mathbb{R}^n} f(\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta) = \int_{G_n} f(g)\, d\mu(g),$$

where the inversion invariance of ν was used.

The uniqueness assertion is a special case of Theorem 3.8. ■

Having constructed invariant measures on the groups SO_n and G_n, we next turn to the introduction of invariant measures on the homogeneous spaces $\mathcal{L}_q^n$ and $\mathcal{E}_q^n$. First we prove a formula of integral-geometric type, extending Theorem 2.1, which will be useful for obtaining uniqueness results.

3.11 Theorem. *Suppose that the compact group G operates continuously and transitively on the Hausdorff space X, and that G and X have countable bases. Let ν be an invariant measure on G with $\nu(G) = 1$, let $\rho \neq 0$ be a G-invariant Borel measure on X and α an arbitrary Borel measure on X. Then*

$$\int_G \alpha(A \cap gB)\, d\nu(g) = \alpha(A)\rho(B)/\rho(X)$$

for all $A, B \in \mathcal{B}(X)$.

Proof. If φ denotes the operation of G on X and if $x \in X$, the mapping $\varphi(\cdot, x) : G \to X$ is continuous and surjective, hence X is compact. Therefore, the Borel measures α and ρ are finite. Let $A, B \in \mathcal{B}(X)$ and $g \in G$ be given, then

$$\alpha(A \cap gB) = \int_X \mathbf{1}_{A \cap gB}\, d\alpha(x) = \int_X \mathbf{1}_A(x) \mathbf{1}_B(g^{-1}x)\, d\alpha(x).$$

Fubini's theorem yields

$$\int_G \alpha(A \cap gB)\, d\nu(g) = \int_X \mathbf{1}_A(x) \int_G \mathbf{1}_B(g^{-1}x)\, d\nu(g)\, d\alpha(x). \tag{12}$$

The integral $\int_G \mathbf{1}_B(g^{-1}x)\, d\nu(g)$ does not depend on x, since for $y \in X$ there exists $\tilde{g} \in G$ with $y = \tilde{g}x$ and, therefore,

$$\int_G \mathbf{1}_B(g^{-1}y)\, d\nu(g) = \int_G \mathbf{1}_B((\tilde{g}^{-1}g)^{-1}x)\, d\nu(g) = \int_G \mathbf{1}_B(g^{-1}x)\, d\nu(g).$$

Hence we obtain

$$\rho(X) \int_G \mathbf{1}_B(g^{-1}x)\, d\nu(g) = \int_X \mathbf{1}_B(g^{-1}x)\, d\nu(g)\, d\rho(x)$$

$$= \int_G \int_X \mathbf{1}_B(g^{-1}x)\, d\rho(x)\, d\nu(g) = \int_G \rho(gB)\, d\nu(g) = \rho(B).$$

Inserting this into (12), we complete the proof. ■

3.12 Corollary. *Suppose that the compact group G operates continuously and transitively on the Hausdorff space X and that G and X have countable bases. Let ν be an invariant measure on G with $\nu(G) = 1$.*

Then there exists a unique G-invariant measure ρ on X with $\rho(X) = 1$. It can be defined by

$$\rho(B) := \nu(\{g \in G : gx_0 \in B\}), \qquad B \in \mathcal{B}(X),$$

with arbitrary $x_0 \in X$.

Proof. Let ρ be a G-invariant measure on X with $\rho(X) = 1$. We choose $x_0 \in X$ and let α be the Dirac measure on X concentrated in x_0. Theorem 3.11 with $A := \{x_0\}$ gives

$$\rho(B) = \nu(\{g \in G : g^{-1}x_0 \in B\})$$

for $B \in \mathcal{B}(X)$. Thus ρ is unique. Vice versa, if ρ is defined in this way, it is clear that it is a G-invariant normalized measure. ■

Now we turn to invariant measures on the space $\mathcal{L}_q^n$ of q-dimensional linear subspaces and on the space $\mathcal{E}_q^n$ of q-dimensional affine subspaces. By an *invariant measure* on $\mathcal{L}_q^n$ we understand an SO_n-invariant measure on $\mathcal{L}_q^n$, and an *invariant measure* on $\mathcal{E}_q^n$ is defined as a G_n-invariant measure on $\mathcal{E}_q^n$.

3.13 Theorem. *On $\mathcal{L}_q^n$ there is a unique invariant measure ν_q, normalized by $\nu_q(\mathcal{L}_q^n) = 1$.*

This is just a special case of Corollary 3.12. We also notice that ν_q is the image measure of ν under the map β_q defined by (3).

3.14 Theorem. *On $\mathcal{E}_q^n$ there is an invariant measure μ_q. It is unique up to a constant factor.*

Proof. We recall that we have chosen a subspace $L_q \in \mathcal{L}_q^n$ and defined the map $\gamma_q : L_q^\perp \times SO_n \to \mathcal{E}_q^n$ by (3). Let $\lambda^{(n-q)}$ be Lebesgue measure on $L_q^\perp$. We define

$$\mu_q := \gamma_q(\lambda^{(n-q)} \otimes \nu), \tag{13}$$

so that μ_q is the image measure of the product measure $\lambda^{(n-q)} \otimes \nu$ under the map γ_q. If $A \subset \mathcal{E}_q^n$ is compact, the sets

$$\gamma_q(\{x \in L_q^\perp : \|x\| < k\} \times SO_n), \qquad k \in \mathbb{N},$$

constitute an open covering of A, hence A is included in one of these sets. It follows that $\mu_q(A) < \infty$.

By the definition of μ_q, integrals with respect to μ_q can be expressed in the following way. For a nonnegative measurable function f on $\mathcal{E}_q^n$,

$$\begin{aligned}\int\limits_{\mathcal{E}_q^n} f \, d\mu_q &= \int\limits_{SO_n} \int\limits_{L_q^\perp} f(\rho(L_q + x)) \, d\lambda^{(n-q)}(x) \, d\nu(\rho) \\ &= \int\limits_{SO_n} \int\limits_{(\rho L_q)^\perp} f(\rho L_q + y) \, d\lambda^{(n-q)}(y) \, d\nu(\rho).\end{aligned}$$

Since the invariant measure ν_q on $\mathcal{L}_q^n$ is the image measure under the map β_q, this can be written as

$$\int\limits_{\mathcal{E}_q^n} f \, d\mu_q = \int\limits_{\mathcal{L}_q^n} \int\limits_{L^\perp} f(L + y) \, d\lambda^{(n-q)}(y) \, d\nu_q(L). \tag{14}$$

From this representation we infer that μ_q does not depend on the choice of the subspace L_q.

To show the invariance of μ_q, let $g = \gamma(x, \vartheta) \in G$ and let $f \geq 0$ be a measurable function on $\mathcal{E}_q^n$. Denoting by Π the orthogonal projection onto $L_q^\perp$, we have

$$\begin{aligned}&\int\limits_{\mathcal{E}_q^n} f(gE) \, d\mu_q(E) \\ &= \int\limits_{SO_n} \int\limits_{L_q^\perp} f(g\rho(L_q + y)) \, d\lambda^{(n-q)}(y) \, d\nu(\rho)\end{aligned}$$

$$= \int\limits_{SO_n} \int\limits_{L_q^\perp} f(\vartheta\rho(L_q + y + \Pi(\rho^{-1}\vartheta^{-1}x)))\, d\lambda^{(n-q)}(y)\, d\nu(\rho)$$

$$= \int\limits_{SO_n} \int\limits_{L_q^\perp} f(\vartheta\rho(L_q + y))\, d\lambda^{(n-q)}(y)\, d\nu(\rho)$$

$$= \int\limits_{SO_n} \int\limits_{L_q^\perp} f(\rho(L_q + y))\, d\lambda^{(n-q)}(y)\, d\nu(\rho)$$

$$= \int\limits_{\mathcal{E}_q^n} f(E)\, d\mu_q(E),$$

where we have used the invariance properties of $\lambda^{(n-q)}$ and ν. This shows the invariance of μ_q.

To prove the uniqueness (up to a factor), we assume that τ is another invariant Borel measure on $\mathcal{E}_q^n$. Let $\tilde{\mathcal{L}}_q^n$ (respectively $\tilde{\mathcal{E}}_q^n$) be the open set of all $L \in \mathcal{L}_q^n$ (respectively $E \in \mathcal{E}_q^n$) that intersect $L_q^\perp$ in precisely one point. The mapping

$$\begin{array}{rccc} \delta_q: & L_q^\perp \times \tilde{\mathcal{L}}_q^n & \to & \tilde{\mathcal{E}}_q^n \\ & (x, L) & \mapsto & L + x \end{array}$$

is a homeomorphism. For fixed $B \in \mathcal{B}(\tilde{\mathcal{L}}_q^n)$ and arbitrary $A \in \mathcal{B}(L_q^\perp)$ we define $\eta(A) := \tau(\delta_q(A \times B))$. Then η is a Borel measure on $L_q^\perp$, which is invariant under the translations of $L_q^\perp$ into itself. Theorem 3.6 implies that $\eta(A) = \lambda^{(n-q)}(A)\alpha(B)$ with a constant $\alpha(B) \geq 0$. Hence we have

$$\tau(\delta_q(A \times B)) = \lambda^{(n-q)}(A)\alpha(B)$$

for arbitrary $A \in \mathcal{B}(L_q^\perp)$ and $B \in \mathcal{B}(\tilde{\mathcal{L}}_q^n)$. Obviously this equation defines a finite measure α on $\mathcal{B}(\tilde{\mathcal{L}}_q^n)$, and $\delta_q^{-1}(\tau) = \lambda^{(n-q)} \otimes \alpha$. For a measurable function $f \geq 0$ on $\tilde{\mathcal{E}}_q^n$ we obtain

$$\begin{aligned} \int\limits_{\tilde{\mathcal{E}}_q^n} f\, d\tau &= \int\limits_{\tilde{\mathcal{L}}_q^n} \int\limits_{L_q^\perp} f(L + x)\, d\lambda^{(n-q)}(x)\, d\alpha(L) \\ &= \int\limits_{\tilde{\mathcal{L}}_q^n} \int\limits_{L^\perp} f(L + y)\, d\lambda^{(n-q)}(y)\, d\varphi(L) \qquad (15) \end{aligned}$$

with a measure φ on $\tilde{\mathcal{L}}_q^n$ defined by $d\varphi(L)/d\alpha(L) = D(L_q^\perp, L^\perp)^{-1}$, where $D(L_q^\perp, L^\perp)$ is the absolute determinant of the orthogonal projection from $L_q^\perp$ onto $L^\perp$.

Now let $B \in \mathcal{B}(\mathcal{L}_q^n)$ and

$$B' := \{L + y : L \in B,\ y \in L^\perp \cap B^n\}.$$

By $\beta(B) := \tau(B')$ we define a rotation invariant finite measure β on $\mathcal{L}_q^n$. According to Theorem 3.13 it is a multiple of ν_q. On the other hand, (15) gives $\tau(B') = \kappa_{n-q}\varphi(B)$ for $B \subset \tilde{\mathcal{L}}_q^n$. Hence, there is a constant c with $\varphi(B) = c\nu_q(B)$ for all Borel sets $B \subset \tilde{\mathcal{L}}_q^n$. From (15) and (14) we deduce that $\tau(A) = c\mu_q(A)$ for all Borel sets $A \subset \tilde{\mathcal{E}}_q^n$. Since μ_q does not depend on the choice of the subspace $L_q \in \mathcal{L}_q^n$, it is easy to see that $\tau = c\mu_q$. ■

By its definition, the measure μ_q comes with a particular normalization. We want to determine the measure of all q-flats meeting the unit ball B^n. Since

$$\{E \in \mathcal{E}_q^n : E \cap B^n \neq \emptyset\} = \gamma_q((B^n \cap L_q^\perp) \times SO_n),$$

we get

$$\mu_q(\{E \in \mathcal{E}_q^n : E \cap B^n \neq \emptyset\}) = \kappa_{n-q}.$$

For $r > 0$ we have

$$\mu_q(\{E \in \mathcal{E}_q^n : E \cap rB^n \neq \emptyset\}) = r^{n-q}\kappa_{n-q}.$$

4 Additive functionals

Beside special Haar measures, another type of invariant measures that we will use are finitely additive measures on certain systems of subsets of Euclidean space.

We begin with some general definitions. Let φ be a function on a family $\mathcal{S}$ of sets with values in some abelian group. The function φ is called *additive* or a *valuation* if

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L) \tag{16}$$

holds whenever $K, L \in \mathcal{S}$ are sets such that also $K \cup L \in \mathcal{S}$ and $K \cap L \in \mathcal{S}$. If $\emptyset \in \mathcal{S}$, one also assumes that $\varphi(\emptyset) = 0$. We say that the system $\mathcal{S}$ is $\cap$-stable (intersection stable) if $K, L \in \mathcal{S}$ implies $K \cap L \in \mathcal{S} \cup \{\emptyset\}$. In this case, we denote by $U(\mathcal{S})$ the system of all finite unions of sets in $\mathcal{S}$ (including the empty set). The system $U(\mathcal{S})$ is closed under finite unions and intersections and thus is a lattice.

Now let φ be an additive function on $\mathcal{S}$. One may ask whether it has an extension to an additive function on the lattice $U(\mathcal{S})$. Suppose that such an extension exists, and denote it also by φ. Then for $K_1, \ldots, K_m \in U(\mathcal{S})$ the formula

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{r=1}^{m} (-1)^{r-1} \sum_{i_1 < \cdots < i_r} \varphi(K_{i_1} \cap \cdots \cap K_{i_r}) \tag{17}$$

holds. For $m = 2$, this is just the equation (16) defining additivity. The general case of (17) is easily obtained by induction. This formula is called the *inclusion-exclusion principle*.

Formula (17) shows that an additive extension from the $\cap$-stable system $\mathcal{S}$ to the generated lattice $U(\mathcal{S})$, if it exists, is uniquely determined. Conversely, however, one cannot just use (17) for the definition of such an extension, since the representation of an element of $U(\mathcal{S})$ in the form $K_1 \cup \cdots \cup K_m$ with $K_i \in \mathcal{S}$ is in general not unique. Hence, the existence of an additive extension, if there is one, must be proved in a different way.

We will write (17) in a more concise form. For $m \in \mathbb{N}$, let $S(m)$ denote the set of all non-empty subsets of $\{1, \ldots, m\}$. For $v \in S(m)$, let $|v| :=$ card v. If $K_1, \ldots, K_m$ are given, we write

$$K_v := K_{i_1} \cap \cdots \cap K_{i_m} \qquad \text{for } v = \{i_1, \ldots, i_r\} \in S(m).$$

With these conventions, the inclusion-exclusion principle (17) can be written in the form

$$\varphi(K_1 \cup \cdots \cup K_m) = \sum_{v \in S(m)} (-1)^{|v|-1} \varphi(K_v). \tag{18}$$

Of considerable importance in the following is the lattice $U(\mathcal{K}^n)$ generated by the $\cap$-stable family $\mathcal{K}^n \cup \{\emptyset\}$. Thus the system $U(\mathcal{K}^n)$ consists of all subsets of $\mathbb{R}^n$ that can be represented as finite unions of convex bodies. We call such sets *polyconvex* (following Klain-Rota [4], who in turn followed E. de Giorgi). Hadwiger [2] used for $U(\mathcal{K}^n)$ the name 'Konvexring', which has been translated (perhaps not so luckily) into *convex ring*.

The simplest non-zero valuation on $\mathcal{K}^n$ is given by $\chi(K) = 1$ for all $K \in \mathcal{K}^n$. We show that it has an additive extension to $U(\mathcal{K}^n)$.

4.1 Theorem. *There is a unique valuation χ on the convex ring $U(\mathcal{K}^n)$ satisfying*

$$\chi(K) = 1 \qquad \textit{for } K \in \mathcal{K}^n.$$

Proof. The proof uses induction with respect to the dimension. For $n = 0$, the existence is trivial. Suppose that $n > 0$ and the existence has been proved in Euclidean spaces of dimension $n-1$. We choose a unit vector $u \in \mathbb{R}^n$ and define

$$\chi(K) := \sum_{\lambda \in \mathbb{R}} \left[\chi(K \cap H(u,\lambda)) - \lim_{\mu \downarrow \lambda} \chi(K \cap H(u,\mu)) \right] \tag{19}$$

for $K \in U(\mathcal{K}^n)$. On the right-hand side, χ denotes the additive function that exists by the induction hypothesis in spaces of dimension $n-1$. It is obvious that $\chi(K) = 1$ for $K \in \mathcal{K}^n$. If $K = K_1 \cup \cdots \cup K_m$ with $K_i \in \mathcal{K}^n$, then the inclusion-exclusion principle gives

$$\chi(K \cap H(u,\lambda)) = \sum_{v \in S(m)} (-1)^{|v|-1} \chi(K_v \cap H(u,\lambda)),$$

since χ is additive on the polyconvex sets in $H(u,\lambda)$. Now the function $\lambda \mapsto \chi(K_v \cap H(u,\lambda))$ is the indicator function of a compact interval, hence it is clear that the limit in (19) exists for every $\lambda \in \mathbb{R}$ and is non-zero only for finitely many values of λ. Thus χ is well-defined on $U(\mathcal{K}^n)$. It follows from (19) and the induction hypothesis that χ is additive on $U(\mathcal{K}^n)$. This proves the existence of χ. The uniqueness is clear from the inclusion-exclusion principle. ■

The function χ is called the *Euler characteristic*. It coincides, on $U(\mathcal{K}^n)$, with the Euler characteristic as defined in algebraic topology.

Another simple example of a valuation on $U(\mathcal{K}^n)$ is given by the indicator function. For $K \in U(\mathcal{K}^n)$, let

$$\mathbf{1}_K(x) := \begin{cases} 1 & \text{for } x \in K, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus K. \end{cases}$$

For $K, L \in U(\mathcal{K}^n)$ we trivially have

$$\mathbf{1}_{K \cup L}(x) + \mathbf{1}_{K \cap L}(x) = \mathbf{1}_K(x) + \mathbf{1}_K(x)$$

for $x \in \mathbb{R}^n$. Hence, the mapping

$$\begin{array}{rccc} \varphi : & U(\mathcal{K}^n) & \rightarrow & V \\ & K & \mapsto & \mathbf{1}_K \end{array}$$

is an additive function on $U(\mathcal{K}^n)$ with values in the vector space V of finite linear combinations of indicator functions of polyconvex sets. Since $K \mapsto \mathbf{1}_K$

is additive, for $K \in U(\mathcal{K}^n)$ with $K = K_1 \cup \cdots \cup K_m$, $K_i \in \mathcal{K}^n$, the inclusion-exclusion principle gives

$$\mathbf{1}_K = \sum_{v \in S(m)} (-1)^{|v|-1} \mathbf{1}_{K_v}.$$

Thus V consists of all linear combinations of indicator functions of convex bodies.

We will now prove a general extension theorem for valuations on $\mathcal{K}^n$, which is due to Groemer [1]. We endow the set $\mathcal{K}^n$ of convex bodies with the Hausdorff metric δ, which is defined by

$$\begin{aligned} \delta(K, L) &:= \max\{\max_{x \in K} \min_{y \in L} \|x - y\|, \ \max_{x \in L} \min_{y \in K} \|x - y\|\} \\ &= \min\{\epsilon > 0 : K \subset L + \epsilon B^n, \ L \subset K + \epsilon B^n\}, \end{aligned}$$

and with the induced topology. A general extension theorem holds for continuous valuations with values in a topological vector space. This theorem will imply Theorem 4.1, but the short proof of the latter is of independent interest.

4.2 Theorem. *Let X be a topological vector space, and let $\varphi : \mathcal{K}^n \to X$ be a continuous additive mapping. Then φ has an additive extension to the convex ring $U(\mathcal{K}^n)$.*

Proof. An essential part of the proof is the following

PROPOSITION. The equality

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i} = 0$$

with $m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $K_i \in \mathcal{K}^n$ implies

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i) = 0.$$

Assume this proposition were false. Then there is a smallest number $m \in \mathbb{N}$, necessarily $m \geq 2$, for which there exist numbers $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ and convex bodies $K_1, \ldots, K_m \in \mathcal{K}^n$ such that

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i} = 0, \tag{20}$$

but

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i) =: a \neq 0. \tag{21}$$

Let $H \subset \mathbb{R}^n$ be a hyperplane with $K_1 \subset \operatorname{int} H^+$, where H^+, H^- are the two closed halfspaces bounded by H. By (20) we have

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i \cap H^-} = 0, \quad \sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i \cap H} = 0.$$

Since $K_1 \cap H^- = \emptyset$ and $K_1 \cap H = \emptyset$, each of these two sums has at most $m-1$ non-zero summands. From the mimimality of m (and from $\varphi(\emptyset) = 0$) we get

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i \cap H^-) = 0, \quad \sum_{i=1}^{m} \alpha_i \varphi(K_i \cap H) = 0.$$

The additivity of φ on $\mathcal{K}^n$ yields

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i \cap H^+) = a, \tag{22}$$

whereas (20) gives

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i \cap H^+} = 0. \tag{23}$$

A standard separation theorem for convex bodies implies the existence of a sequence $(H_j)_{j \in \mathbb{N}}$ of hyperplanes with $K_1 \subset \operatorname{int} H_j^+$ for $j \in \mathbb{N}$ and

$$K_1 = \bigcap_{j=1}^{\infty} H_j^+.$$

If the argument that has led us from (20), (21) to (23), (22) is applied k-times, we obtain

$$\sum_{i=1}^{m} \alpha_i \varphi \left(K_i \cap \bigcap_{j=1}^{k} H_j^+ \right) = a.$$

For $k \to \infty$ this yields

$$\sum_{i=1}^{m} \alpha_i \varphi(K_i \cap K_1) = a, \tag{24}$$

since

$$\lim_{k\to\infty} K_i \cap \bigcap_{j=1}^{k} H_j^+ = K_i \cap K_1$$

in the sense of the Hausdorff metric (if $K_i \cap K_1 \neq \emptyset$, otherwise use $\varphi(\emptyset) = 0$) and φ is continuous. Equality (20) implies

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i\cap K_1} = 0. \tag{25}$$

The procedure leading from (20) and (21) to (25) and (24) can be repeated, replacing the bodies K_i and K_1 by $K_i \cap K_1$ and K_2, then by $K_i \cap K_1 \cap K_2$ and K_3, and so on. Finally one obtains

$$\sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_1\cap\cdots\cap K_m} = 0$$

and

$$\sum_{i=1}^{m} \alpha_i \varphi(K_1 \cap \cdots \cap K_m) = a$$

(because of $K_i \cap K_1 \cap \cdots \cap K_m = K_1 \cap \cdots \cap K_m$). Now $a \neq 0$ implies $\sum_{i=1}^{m} \alpha_i \neq 0$ and hence $\mathbf{1}_{K_1\cap\cdots\cap K_m} = 0$ by the first relation, but this yields $\varphi(K_1 \cap \cdots \cap K_m) = 0$, contradicting the second relation. This completes the proof of the proposition.
Now we consider the real vector space V of all finite linear combinations of indicator functions of elements of $\mathcal{K}^n$. For $K \in U(\mathcal{K}^n)$ we have $\mathbf{1}_K \in V$, as noted earlier. For fixed $f \in V$ we choose a representation

$$f = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{K_i}$$

with $m \in \mathbb{N}$, $\alpha_i \in \mathbb{R}$, $K_i \in \mathcal{K}^n$ and define

$$\tilde{\varphi}(f) := \sum_{i=1}^{m} \alpha_i \varphi(K_i).$$

The proposition proved above shows that this definition is possible, since the right-hand side does not depend on the special representation chosen for f. Evidently, $\tilde{\varphi} : V \to X$ is a linear map satisfying $\tilde{\varphi}(\mathbf{1}_K) = \varphi(K)$ for $K \in \mathcal{K}^n$. We can now extend φ from $\mathcal{K}^n$ to $U(\mathcal{K}^n)$ by defining

$$\varphi(K) := \tilde{\varphi}(\mathbf{1}_K) \qquad \text{for } K \in U(\mathcal{K}^n).$$

By the linearity of $\tilde{\varphi}$ and the additivity of the map $K \mapsto \mathbf{1}_K$ we obtain, for $K, M \in U(\mathcal{K}^n)$,

$$\begin{aligned}\varphi(K \cup M) + \varphi(K \cap M) &= \tilde{\varphi}(\mathbf{1}_{K\cup M}) + \tilde{\varphi}(\mathbf{1}_{K\cap M}) \\ &= \tilde{\varphi}(\mathbf{1}_{K\cup M} + \mathbf{1}_{K\cap M}) \\ &= \tilde{\varphi}(\mathbf{1}_K + \mathbf{1}_M) \\ &= \tilde{\varphi}(\mathbf{1}_K) + \tilde{\varphi}(\mathbf{1}_M) \\ &= \varphi(K) + \varphi(M).\end{aligned}$$

Thus φ is additive on $U(\mathcal{K}^n)$. ■

5 Local parallel sets and curvature measures

One of our aims will be to compute integrals such as

$$I(K, M) := \int_{G_n} \chi(K \cap gM)\, d\mu(g) \tag{26}$$

for convex bodies $K, M \in \mathcal{K}^n$, where μ is the invariant measure on the motion group G_n; thus $I(K, M)$ is the total Haar measure of the set of rigid motions which bring M into a hitting position with K. We get a first hint to what the result will involve if we choose for M a ball ρB^n of radius $\rho > 0$. In that case,

$$I(K, \rho B^n) = \int_{\mathbb{R}^n} \chi(K \cap (\rho B^n + t))\, d\lambda(t) = V_n(K + \rho B^n),$$

as obtained in Section 2. The set $K + \rho B^n$ is known as the *outer parallel set* of K at distance ρ. It can also be represented as

$$K + \rho B^n = \{x \in \mathbb{R}^n : d(K, x) \le \rho\},$$

where

$$d(K, x) := \min\{\|x - y\| : y \in K\}$$

is the distance of x from K. A fundamental result in the geometry of convex bodies, the *Steiner formula*, says that the volume $V_n(K + \rho B^n)$ of the parallel body, as a function of the parameter ρ, is a polynomial of degree n, thus

$$V_n(K + \rho B^n) = \sum_{i=0}^{n} \rho^{n-i} \kappa_{n-i} V_i(K). \tag{27}$$

The reason for introducing the normalizing factors κ_{n-i} will become clear later in this section. The coefficients $V_0(K), \dots, V_n(K)$ appearing in (27) define important functionals of K. We have just seen that they inevitably appear when we want to compute the integral $I(K, \rho B^n)$. As it turns out, also the general integral $I(K, M)$ given by (26) can be expressed in terms of these functionals alone, evaluated for the bodies K and M.

In the present section, a more general version of the Steiner formula (27) will be obtained. Namely, we replace the parallel body $K + \rho B^n$ by a local version of it. The polynomial expansion generalizing (27) then defines a series of measures on $\mathbb{R}^n$, the *curvature measures* of the convex body K. These measures will appear in very general versions of the kinematic formula of integral geometry.

We need a simple device from convex geometry. Let $K \in \mathcal{K}^n$ be a convex body. For $x \in \mathbb{R}^n$, there is a unique point $p(K, x)$ in K nearest to x, that is,

$$\|p(K,x) - x\| = \min\{\|y - x\| : y \in K\} = d(K, x).$$

This defines a continuous map $p(K, \cdot) : \mathbb{R}^n \to K$, which is called the *nearest-point map* of K, or the *metric projection* onto K. Also the map

$$\begin{aligned} p : \mathcal{K}^n \times \mathbb{R}^n \quad &\to \quad \mathbb{R}^n, \\ (K, x) \quad &\mapsto \quad p(K, x) \end{aligned}$$

is continuous.

Now for $K \in \mathcal{K}^n$, a Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ and a number $\rho \geq 0$, we define the *local parallel set* of (K, A) at distance ρ by

$$M_\rho(K, A) := \{x \in \mathbb{R}^n : d(K, x) \leq \rho,\ p(K, x) \in A\}.$$

This is a Borel set, since $p(K, \cdot)$ is continuous. We can, therefore, define

$$\mu_\rho(K, A) := \lambda(M_\rho(K, A)) \quad \text{for } A \in \mathcal{B}(\mathbb{R}^n).$$

In other words, $\mu_\rho(K, \cdot)$ is the image measure of the Lebesgue measure, restricted to the parallel body $K_\rho = K + \rho B^n$, under the nearest point map of K. In particular, $\mu_\rho(K,\cdot)$ is a finite measure on $\mathcal{B}(\mathbb{R}^n)$. We call it the *local parallel volume* of K at distance ρ.

This measure is concentrated on K, that is, $\mu_\rho(K,A) = \mu_\rho(K, A\cap K)$.

We first prove some fundamental properties of the mapping $\mu_\rho : \mathcal{K}^n \times \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}$. In the following, $\stackrel{w}{\to}$ denotes weak convergence of finite measures.

5.1 Theorem. *Let $(K_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{K}^n$ satisfying $K_j \to K$ for $j\to\infty$. Then*

$$\mu_\rho(K_j,\cdot) \stackrel{w}{\to} \mu_\rho(K,\cdot) \quad \textit{for } j\to\infty, \tag{28}$$

for every $\rho > 0$.

Proof. By a well-known characterization of weak convergence, we have to show that

$$\liminf_{j\to\infty} \mu_\rho(K_j, A) \geq \mu_\rho(K,A) \tag{29}$$

for every open set A, and

$$\lim_{j\to\infty} \mu_\rho(K_j,\mathbb{R}^n) = \mu_\rho(K,\mathbb{R}^n). \tag{30}$$

Let $A \subset \mathbb{R}^n$ be open. Let $x \in M_\rho(K,A)$ be a point with $d(K,x) < \rho$. Since ρ is continuous, we have $p(K_j,x)\to p(K,x)$ and $d(K_j,x)\to d(K,x)$ for $j\to\infty$. Hence, for all sufficiently large j we deduce that $p(K_j,x)\in A$ and $d(K_j,x) < \rho$, hence $x\in M_\rho(K_j,A)$. Thus we have

$$M_\rho(K,A)\setminus \partial K_\rho \subset \liminf_{j\to\infty} M_\rho(K_j,A)$$

and, therefore,

$$\begin{aligned}
\mu_\rho(K,A) &= \lambda(M_\rho(K,A)\setminus\partial K_\rho)\\
&\leq \lambda\left(\liminf_{j\to\infty} M_\rho(K_j,A)\right)\\
&\leq \liminf_{j\to\infty} \lambda(M_\rho(K_j,A))\\
&= \liminf_{j\to\infty} \mu_\rho(K_j,A),
\end{aligned}$$

which proves (29). The relation (30) follows from standard results of convex geometry. ■

5.2 Theorem. *For any Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ and any $\rho > 0$, the function $\mu_\rho(\cdot, A) : \mathcal{K}^n \to \mathbb{R}$ is measurable.*

Proof. For an open set A, the preceding proof shows that the function $\mu_\rho(\cdot, A)$ is lower semicontinuous, hence it is measurable.

Denote by $\mathcal{A}$ the system of all sets $A \in \mathcal{B}(\mathbb{R}^n)$ for which $\mu_\rho(\cdot, A)$ is measurable. We show that $\mathcal{A}$ is a Dynkin system. For $A_1, A_2 \in \mathcal{A}$ with $A_2 \subset A_1$ we have $M_\rho(K, A_2) \subset M_\rho(K, A_1)$ and

$$M_\rho(K, A_1 \setminus A_2) = M_\rho(K, A_1) \setminus M_\rho(K, A_2),$$

hence

$$\mu_\rho(K, A_1 \setminus A_2) = \mu_\rho(K, A_1) - \mu_\rho(K, A_2)$$

for all $K \in \mathcal{K}^n$, which shows that $A_1 \setminus A_2 \in \mathcal{A}$. If $(A_j)_{j \in \mathbb{N}}$ is a disjoint sequence in $\mathcal{A}$, then

$$\mu_\rho\left(K, \bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu_\rho(K, A_j)$$

for $K \in \mathcal{K}^n$, since $\mu_\rho(K, \cdot)$ is a measure. It follows that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$. Thus $\mathcal{A}$ is a Dynkin system. Since it contains the open sets, it also contains the σ-algebra generated by the open sets and thus all Borel sets, as asserted. ■

5.3 Theorem. *For any Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ and for $\rho > 0$, the function $\mu_\rho(\cdot, A) : \mathcal{K}^n \to \mathbb{R}$ is additive.*

Proof. Let $K, L \in \mathcal{K}^n$ be convex bodies with $K \cup L \in \mathcal{K}^n$. Let $x \in \mathbb{R}^n$, and put $y := p(K \cup L, x)$. We assume $y \in K$, without loss of generality. Then

$$p(K \cup L, x) = p(K, x). \tag{31}$$

Let $z := p(L, x)$. Since $K \cup L$ is convex, there is a point $a \in [z, y]$ (the segment with endpoints z and y) with $a \in K \cap L$. From $y = p(K \cup L, x)$ it follows that $\|y - x\| \leq \|z - x\|$ and hence $\|a - x\| \leq \|z - x\|$. From $a \in L$ and the definition of z we conclude that $a = z$ and thus $z \in K \cap L$. This shows that

$$p(K \cap L, x) = p(L, x). \tag{32}$$

For $K' \in \mathcal{K}^n$, let $\mathbf{1}_\rho(K', A, \cdot)$ be the indicator function of the local parallel set $M_\rho(K', A)$. From (31) and (32) it follows that

$$\mathbf{1}_\rho(K \cup L, A, x) = \mathbf{1}_\rho(K, A, x), \quad \mathbf{1}_\rho(K \cap L, A, x) = \mathbf{1}_\rho(L, A, x).$$

Since x was arbitrary, this yields

$$\mathbf{1}_\rho(K \cup L, A, \cdot) + \mathbf{1}_\rho(K \cap L, A, \cdot) = \mathbf{1}_\rho(K, A, \cdot) + \mathbf{1}_\rho(L, A, \cdot).$$

Integrating this equation with the Lebesgue measure, we obtain

$$\mu_\rho(K \cup L, A) + \mu_\rho(K \cap L, A) = \mu_\rho(K, A) + \mu_\rho(L, A),$$

which shows that $\mu_\rho(\cdot, A)$ is additive on $\mathcal{K}^n$. ■

We will now explicitly compute the local parallel volume in the case of a convex polytope. For this, we need some elementary facts about polytopes, which we will use without proof.

A *polyhedral set* in $\mathbb{R}^n$ is a set which can be represented as the intersection of finitely many closed halfspaces. A bounded non-empty polyhedral set is called a *convex polytope* or briefly a *polytope*. Let P be a polytope. If H is a supporting hyperplane of P, then $P \cap H$ is again a polytope. The set $F := P \cap H$ is called a *face* of P, and an *m-face* if $\dim F = m$, $m \in \{0, \ldots, n-1\}$. If $\dim P = n$, we consider P as an n-face of itself. By $\mathcal{F}_m(P)$ we denote the set of all m-faces of P. For $F \in \mathcal{F}_m(P)$ we define

$$\lambda_F(B) := \lambda^{(m)}(B \cap F) \qquad \text{for } B \in \mathcal{B}(\mathbb{R}^n),$$

where $\lambda^{(m)}$ denotes m-dimensional Lebesgue measure. For $F \in \mathcal{F}_m(P)$, $m \in \{0, \ldots, n-1\}$ and a point $x \in \operatorname{relint} F$ (the relative interior of F), let $N(P, F)$ be the *normal cone* of P at F; this is the cone of outer normal vectors of supporting hyperplanes to P at x. It does not depend upon the choice of x. The number

$$\gamma(F, P) := \frac{\lambda^{(n-m)}(N(P, F) \cap B^n)}{\kappa_{n-m}}$$

is called the *external angle* of P at its face F. We also put $\gamma(P, P) = 1$ and $\gamma(F, P) = 0$ if either $F = \emptyset$ or F is not a face of P.

Now let a polytope P, a Borel set $A \in \mathcal{B}(\mathbb{R}^n)$ and a number $\rho > 0$ be given. For $x \in \mathbb{R}^n$, the nearest point $p(P, x)$ lies in the relative interior of a unique face of P. Therefore,

$$M_\rho(P, A) = \bigcup_{m=0}^{n} \bigcup_{F \in \mathcal{F}_m(P)} \left[P_\rho \cap p(P, \cdot)^{-1}(A \cap \operatorname{relint} F)\right] \qquad (33)$$

is a disjoint decomposition of the local parallel set $M_\rho(P, A)$. It follows from the properties of the nearest point map that

$$P_\rho \cap p(P, \cdot)^{-1}(A \cap \operatorname{relint} F) \qquad (34)$$

$$= (A \cap \operatorname{relint} F) \oplus (N(P,F) \cap \rho B^n), \tag{35}$$

where $\oplus$ denotes direct sum. An application of Fubini's theorem gives

$$\begin{aligned}
&\lambda(P_\rho \cap p(P,\cdot)^{-1}(A \cap \operatorname{relint} F)) \\
&\quad = \lambda^{(m)}(A \cap F)\lambda^{(n-m)}(N(P,F) \cap \rho B^n) \\
&\quad = \lambda^{(m)}(A \cap F)\rho^{n-m}\kappa_{n-m}\gamma(F,P).
\end{aligned}$$

Together with (33), this yields

$$\mu_\rho(P,A) = \sum_{m=0}^{n} \rho^{n-m}\kappa_{n-m} \sum_{F \in \mathcal{F}_m(P)} \lambda^{(m)}(A \cap F)\gamma(F,P).$$

Hence, if we define a measure $\Phi_m(P,\cdot)$ on $\mathcal{B}(\mathbb{R}^n)$ by

$$\Phi_m(P,\cdot) := \sum_{F \in \mathcal{F}_m(P)} \gamma(F,P)\lambda_F,$$

then

$$\mu_\rho(P,A) = \sum_{m=0}^{n} \rho^{n-m}\kappa_{n-m}\Phi_m(P,A).$$

This gives the desired polynomial expansion of the local parallel volume in the case of polytopes. The following theorem extends this result to general convex bodies.

5.4 Theorem. (Local Steiner formula) *For every convex body $K \in \mathcal{K}^n$, there exist finite measures $\Phi_0(K,\cdot),\dots,\Phi_n(K,\cdot)$ on $\mathcal{B}(\mathbb{R}^n)$ such that the local parallel volume satisfies*

$$\mu_\rho(K,A) = \sum_{j=0}^{n} \rho^{n-j}\kappa_{n-j}\Phi_j(K,A)$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$ and all $\rho \geq 0$.

Proof. If P is a polytope, we have seen above that

$$\mu_\rho(P,A) = \sum_{j=0}^{n} \rho^{n-j}\kappa_{n-j}\Phi_j(P,A) \tag{36}$$

with

$$\Phi_j(P,\cdot) = \sum_{F\in\mathcal{F}_j(P)} \gamma(F,P)\lambda_F. \tag{37}$$

Now let $K \in \mathcal{K}^n$ be an arbitrary convex body. As one knows from convex geometry, there is a sequence $(P_i)_{i\in\mathbb{N}}$ of polytopes converging to K in the Hausdorff metric. In (36), we replace P by P_i and ρ by each of the numbers $1,\dots,n+1$. The resulting system of linear equations,

$$\mu_k(P,A) = \sum_{j=0}^{n} k^{n-j}\kappa_{n-j}\Phi_j(P_i,A), \qquad k=1,\dots,n+1,$$

can be solved for the 'unknowns' $\kappa_{n-j}\Phi_j(P_i,A)$ (it has a Vandermonde determinant), which yields representations

$$\Phi_j(P_i,A) = \sum_{k=1}^{n+1} \alpha_{jk}\mu_k(P_i,A), \qquad j=0,\dots,n.$$

Here the coefficients α_{jk} do not depend on P_i or A, thus we have

$$\Phi_j(P_i,\cdot) = \sum_{k=1}^{n+1} \alpha_{jk}\mu_k(P_i,\cdot) \qquad \text{for } i\in\mathbb{N}.$$

By Theorem 5.1, for each fixed $\rho \geq 0$ the measures $\mu_\rho(P_i,\cdot)$ converge weakly to $\mu_\rho(K,\cdot)$. Hence, if we define a finite signed measure by

$$\Phi_j(K,\cdot) := \sum_{k=1}^{n+1} \alpha_{jk}\mu_k(K,\cdot),$$

then the measures $\Phi_j(P_i,\cdot)$ converge, for $i\to\infty$, weakly to the signed measure $\Phi_j(K,\cdot)$ ($j=0,\dots,n$). It follows that the latter is nonnegative, and it also follows that

$$\mu_\rho(K,\cdot) = \sum_{j=0}^{n} \rho^{n-j}\kappa_{n-j}\Phi_j(K,\cdot),$$

using (36) and weak convergence. ■

One calls $\Phi_j(K,\cdot)$ the jth *curvature measure* of the body $K\in\mathcal{K}^n$. The reason for this name becomes clear if one considers a convex body K whose boundary is a regular hypersurface of class C^2. In that case, the local parallel volume can be computed by differential-geometric means, and one obtains for $j = 0, \dots, n-1$ the representation

$$\Phi_j(K, A) = \frac{\binom{n}{j}}{n\kappa_{n-j}} \int_{A\cap\partial K} H_{n-1-j}\, dS.$$

Here H_k denotes the kth normalized elementary symmetric function of the principal curvatures of ∂K, and dS is the volume form on ∂K. Thus the curvature measures are (up to constant factors) indefinite integrals of curvature functions, and they replace the latter in the non-smooth case.

For $j = n$, we simply have

$$\Phi_n(K, A) = \lambda(K \cap A) \qquad \text{for } A \in \mathcal{B}(\mathbb{R}^n),$$

as follows immediately from the definition of the local parallel set and the local Steiner formula. For a general convex body K it is clear that the measures $\Phi_0(K, \cdot), \dots, \Phi_{n-1}(K, \cdot)$ are concentrated on ∂K, since $\mu_\rho(K, A) - \lambda(K \cap A)$ depends only on $A \cap \partial K$.

For polytopes P, we have the explicit representation (37) of the curvature measures. The external angle appearing in it does not depend on the dimension of the surrounding space, as follows easily from Fubini's theorem. In other words, if $\dim P < n$, it makes no difference if the external angle $\gamma(F, P)$ is computed in $\mathbb{R}^n$ or in the affine hull of P. This independence of dimension extends to the curvature measures $\Phi_j(P, \cdot)$ and then, by approximation and weak convergence, to the curvature measures $\Phi_j(K, \cdot)$ of arbitrary convex bodies.

We mention without proof that for arbitrary convex bodies K the measures $\Phi_0(K, \cdot)$ and $\Phi_{n-1}(K, \cdot)$ have simple intuitive interpretations. Namely, if $\dim K \neq n-1$, then

$$\Phi_{n-1}(K, A) = \frac{1}{2}\mathcal{H}^{n-1}(A \cap \partial K).$$

For $\dim K = n-1$, one trivially has $\Phi_{n-1}(K, A) = \mathcal{H}^{n-1}(A \cap \partial K)$. The measure Φ_0 is the normalized area of the spherical image. Let $\sigma(K, A) \subset S^{n-1}$ denote the set of all outer unit normal vectors of K at points of $A \cap \partial K$, then

$$\Phi_0(K, A) = \frac{1}{n\kappa_n}\mathcal{H}^{n-1}(\sigma(K, A)).$$

We can use the relation

$$\Phi_j(K, \cdot) = \sum_{k=1}^{n+1} \alpha_{jk}\mu_k(K, \cdot), \tag{38}$$

which was obtained in the proof of Theorem 5.4, to transfer properties of the local parallel volumes $\mu_\rho(K,\cdot)$ to the curvature measures $\Phi_j(K,\cdot)$. In this way Theorems 5.1, 5.2, 5.3, together with some easily obtained additional properties of the local parallel volumes, yield a series of properties of the curvature measures, which we list in the following theorem.

5.5 Theorem. *Let $j \in \{0,\dots,n\}$.*
(a) *$\Phi_j(K,\cdot)$ depends weakly continuously on K, that is, $K_i \to K$ implies the weak convergence $\Phi_j(K_i,\cdot) \stackrel{w}{\to} \Phi_j(K,\cdot)$ for $i \to \infty$.*
(b) *For every $A \in \mathcal{B}(\mathbb{R}^n)$, the function $\Phi_j(\cdot,A)$ is measurable on $\mathcal{K}^n$.*
(c) *Φ_j is motion covariant, that is,*

$$\Phi_j(gK,gA) = \Phi_j(K,A)$$

for every rigid motion $g \in G_n$ and all $K \in \mathcal{K}^n$ and $A \in \mathcal{B}(\mathbb{R}^n)$.
(d) *Φ_j is homogeneous of degree j, that is,*

$$\Phi_j(\alpha K,\alpha A) = \alpha^j \Phi(K,A)$$

for every $\alpha > 0$ and all $K \in \mathcal{K}^n$ and $A \in \mathcal{B}(\mathbb{R}^n)$.
(e) *Φ_j is defined locally, that is, for every open set $A \subset \mathbb{R}^n$ and all convex bodies $K, M \in \mathcal{K}^n$ with $K \cap A = M \cap A$, one has*

$$\Phi_j(K,B) = \Phi_j(M,B)$$

for every Borel set $B \subset A$.
(f) *$\Phi_j(\cdot,A)$ is additive for every $A \in \mathcal{B}(\mathbb{R}^n)$, that is,*

$$\Phi_j(K \cup L,A) + \Phi_j(K \cap L,A) = \Phi_j(K,A) + \Phi_j(L,A)$$

holds for all convex bodies $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$.

The final property, together with Theorem 4.2, has the important consequence that the curvature measures have an additive extension to polyconvex sets. This means that one can define signed measures on the convex ring $U(\mathcal{K}^n)$ in the following way. Let $K \in U(\mathcal{K}^n)$ and choose a representation $K = \bigcup_{i=1}^m K_i$ with $m \in \mathbb{N}$ and $K_i \in \mathcal{K}^n$. Then

$$\Phi_j(K,\cdot) := \sum_{v \in S(m)} (-1)^{|v|-1} \Phi_j(K_v,\cdot)$$

does not depend on the special choice of the representation; in particular, this is consistent with the already defined value $\Phi_j(K,\cdot)$ for convex K.

This follows from Theorem 4.2, since the weak convergence of curvature measures can be interpreted as convergence in the vector space of finite signed measures, on a suitable compact subset of $\mathbb{R}^n$, with respect to a suitable topology.

We have now everything at hand to formulate a central result of integral geometry. This is the *principal kinematic formula*, in a version for curvature measures on the convex ring. Let $K, M \in U(\mathcal{K}^n)$ be polyconvex sets, let $A, B \in \mathcal{B}(\mathbb{R}^n)$ be Borel sets, and let $j \in \{0, \ldots, n\}$. Then

$$\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(g) = \sum_{k=j}^{n} \alpha_{njk} \Phi_j(K, A) \Phi_{n+j-k}(M, B)$$

holds, with certain explicit constants α_{njk}.

We will indicate a proof of this result in Section 9. Before that, however, we will prove a global version of this formula in a different way. The method of proof is of independent interest and leads to further results for which no other access is known.

The global result refers to the total measures

$$V_j(K) := \Phi_j(K, \mathbb{R}^n), \qquad j = 0, \ldots, n.$$

The number V_j is called the jth *intrinsic volume* of K. These important functionals are defined by the classical Steiner formula

$$V_n(K + \rho B^n) = \sum_{j=0}^{n} \rho^{n-j} \kappa_{n-j} V_j(K),$$

of which Theorem 5.4 is the local generalization. As a function on $\mathcal{K}^n$, each intrinsic volume V_j is continuous, additive and rigid motion invariant. In the next section we shall prove that the intrinsic volumes are essentially characterized by these properties.

The additive extensions of the intrinsic volumes to the convex ring $U(\mathcal{K}^n)$ will be denoted by the same symbols. In the following cases, they have simple intuitive interpretations. It is clear that

$$V_n(K) = \lambda(K) \qquad \text{for } K \in U(\mathcal{K}^n),$$

since this holds true for convex bodies K and both functions, V_n and λ, are additive on $U(\mathcal{K}^n)$. It also remains true for polyconvex sets that

$$2V_{n-1}(K) = \mathcal{H}^{n-1}(\partial K)$$

if K is the closure of its interior, but this requires an extra proof. Finally,

$$V_0(K) = \chi(K) \qquad \text{for } K \in U(\mathcal{K}^n),$$

so that V_0 is nothing but the Euler characteristic. For a convex polytope P we have

$$V_0(P) = \Phi_0(P, \mathbb{R}^n) = \sum_{F \in \mathcal{F}_0(P)} \gamma(F, P) = 1,$$

since the normal cones $N(P, F)$ of P at its vertices F cover $\mathbb{R}^n$ and have pairwise no common interior points. By additivity, the equation $V_0(K) = \chi(K)$ extends from $\mathcal{K}^n$ to $U(\mathcal{K}^n)$.

6 Hadwiger's characterization theorem

The jth intrinsic volume $V_j : \mathcal{K}^n \to \mathbb{R}$ is an additive, continuous and rigid motion invariant function. A celebrated theorem due to Hadwiger (see [2]) says that any function on $\mathcal{K}^n$ with these properties is a linear combination of the intrinsic volumes $V_0, \ldots, V_n$. This result can be used to prove some formulae of the integral geometry of convex bodies in a very elegant way. Whereas Hadwiger's original proof was quite long, one has now a shorter proof due to Klain [3]. We will present his proof here, except that at one point we take a certain analytical result for granted.

The crucial step for a proof of the characterization theorem is the following result.

6.1 Theorem. *Suppose that $\psi : \mathcal{K}^n \to \mathbb{R}$ is an additive, continuous, motion invariant function satisfying $\psi(K) = 0$ whenever either* $\dim K < n$ *or K is a unit cube. Then $\psi = 0$.*

Proof. The proof proceeds by induction with respect to the dimension. For $n = 0$, there is nothing to prove. If $n = 1$, ψ vanishes on (closed) segments of unit length, hence on segments of length $1/k$ for $k \in \mathbb{N}$ and therefore on segments of rational length. By continuity, ψ vanishes on all segments and thus on $\mathcal{K}^1$.

Now let $n > 1$ and suppose that the assertion has been proved in dimensions less than n. Let $H \subset \mathbb{R}^n$ be a hyperplane and I a closed segment of length 1, orthogonal to H. For convex bodies $K \subset H$ define $\varphi(K) := \psi(K + I)$. Clearly φ has, relative to H, the properties of ψ in the Theorem, hence the induction hypothesis yields $\varphi = 0$. For fixed $K \subset H$, we thus have $\psi(K + I) = 0$, and a similar argument as used above for $n = 1$

shows that $\psi(K+S)=0$ for any closed segment S orthogonal to H. Thus μ vanishes on right convex cylinders.

Let $K \subset H$ again be a convex body and let $S = \text{conv}\,\{0,s\}$ be a segment not parallel to H. If $m \in \mathbb{N}$ is sufficiently large, the cylinder $Z := K + mS$ can be cut by a hyperplane H' orthogonal to S so that the two closed halfspaces H^-, H^+ bounded by H' satisfy $K \subset H^-$ and $K + ms \subset H^+$. Then $\overline{Z} := [(Z \cap H^-) + ms] \cup (Z \cap H^+)$ is a right cylinder, and we deduce that $m\mu(K+S) = \mu(Z) = \mu(\overline{Z}) = 0$. Thus ψ vanishes on arbitrary convex cylinders.

By Theorem 4.2, the continuous additive function ψ has an additive extension to the convex ring; this extension is also denoted by ψ. It follows that

$$\psi\left(\bigcup_{i=1}^{k} K_i\right) = \sum_{i=1}^{k} \psi(K_i)$$

whenever $K_1, \dots, K_k$ are convex bodies such that $\dim(K_i \cap K_j) < n$ for $i \neq j$.

Let P be a polytope and S a segment. The sum $P+S$ has a decomposition $P + S = \bigcup_{i=1}^{k} P_i$, where $P_1 = P$, the polytope P_i is a convex cylinder for $i > 1$, and $\dim(P_i \cap P_j) < n$ for $i \neq j$. It follows that $\psi(P+S) = \psi(P)$. By induction, we obtain $\psi(P+Z) = \psi(P)$ if Z is a finite sum of segments. By continuity, $\psi(K+Z) = \psi(K)$ for arbitrary convex bodies K and zonoids Z, that is, limits of sums of segments.

Now we have to use an analytic result, for which we do not give a proof. Let K be a centrally symmetric convex body which is sufficiently smooth (say, its support function is of class C^∞). Then there exist zonoids Z_1, Z_2 so that $K + Z_1 = Z_2$ (this can be seem from Section 3.5 in [7], especially Theorem 3.5.3). We conclude that $\psi(K) = \psi(K+Z_1) = \psi(Z_2) = 0$. Since every centrally symmetric convex body K can be approximated by bodies which are centrally symmetric and sufficiently smooth in the above sense, it follows from the continuity of ψ that $\psi(K) = 0$ for all centrally symmetric convex bodies.

Now let Δ be a simplex, say $\Delta = \text{conv}\,\{0, v_1, \dots, v_n\}$, without loss of generality. Let $v := v_1 + \cdots + v_n$ and $\Delta' := \text{conv}\,\{v, v - v_1, \dots, v - v_n\}$, then $\Delta' = -\Delta + v$. The vectors $v_1, \dots, v_n$ span a parallelotope P. It is the union of Δ, Δ' and the part of P lying between the hyperplanes spanned by $v_1, \dots, v_n$ and $v - v_1, \dots, v - v_n$, respectively. The latter, say Q, is a centrally symmetric polytope, and $\Delta \cap Q$, $\Delta' \cap Q$ are of dimension $n-1$. We deduce that $0 = \psi(P) = \psi(\Delta) + \psi(Q) + \psi(\Delta')$, thus $\psi(-\Delta) = \psi(\Delta)$. If the dimension n is even, then $-\Delta$ is obtained from Δ by a proper rigid motion, and the motion invariance of ψ yields $\psi(\Delta) = 0$. If the dimension $n > 1$

is odd, we decompose Δ as follows. Let z be the centre of the inscribed ball of Δ, and let p_i be the point where this ball touches the facet F_i of Δ ($i = 1, \ldots, n+1$). For $i \neq j$, let Q_{ij} be the convex hull of the face $F_i \cap F_j$ and the points z, p_i, p_j. The polytope Q_{ij} is invariant under reflection in the hyperplane spanned by $F_i \cap F_j$ and z. If $Q_1, \ldots, Q_m$ are the polytopes Q_{ij} for $1 \leq c < j \leq n+1$ in any order, then $P = \bigcup_{r=1}^{m} Q_r$ and $\dim (Q_r \cap Q_s) < n$ for $r \neq s$. Since $-Q_r$ is the image of Q_r under a proper rigid motion, we have $\psi(-\Delta) = \sum \psi(-Q_r) = \sum \psi(Q_r) = \psi(\Delta)$. Thus $\psi(\Delta) = 0$ for every simplex Δ.

Decomposing a polytope P into simplices, we obtain $\psi(P) = 0$. The continuity of ψ now implies $\psi(K) = 0$ for all convex bodies K. This finishes the induction and hence the proof of Theorem 6.1. ■

Hadwiger's characterization theorem is now an easy consequence.

6.2 Theorem. *Suppose that $\psi : \mathcal{K}^n \to \mathbb{R}$ is an additive, continuous, motion invariant function. Then there are constants $c_0, \ldots, c_n$ so that*

$$\psi(K) = \sum_{i=0}^{n} c_i V_i(K)$$

for all $K \in \mathcal{K}^n$.

Proof. We use induction on the dimension. For $n = 0$ the assertion is trivial. Suppose that $n > 0$ and the assertion has been proved in dimensions less than n. Let $H \subset \mathbb{R}^n$ be a hyperplane. The restriction of ψ to the convex bodies lying in H is additive, continuous and invariant under motions of H into itself. By the induction hypothesis, there are constants $c_0, \ldots, c_{n-1}$ so that $\psi(K) = \sum_{i=0}^{n-1} c_i V_i(K)$ holds for convex bodies $K \subset H$ (note that the intrinsic volumes do not depend on the dimension of the surrounding space). By the motion invariance of ψ and V_i, this holds for all $K \in \mathcal{K}^n$ of dimension less than n. It follows that the function ψ' defined by

$$\psi'(K) := \psi(K) - \sum_{i=0}^{n} c_i V_i(K)$$

for $K \in \mathcal{K}^n$, where c_n is chosen so that ψ' vanishes at a fixed unit cube, satisfies the assumptions of Theorem 6.1. Hence $\psi' = 0$, which completes the proof of Theorem 6.2. ■

The late Gian-Carlo Rota, in a Colloquium Lecture at the Annual Meeting of the AMS in 1997, called Hadwiger's characterization theorem the 'Main

Theorem of Geometric Probability'. The reason is that it can be used to derive kinematic formulae of integral geometry, which can in turn be interpreted in terms of geometric hitting probabilities. We shall see this, in more elaborate versions, in the next two sections.

7 Kinematic and Crofton formulae

Our aim in this section will be to compute the integrals

$$\int_{G_n} V_j(K \cap gM)\, d\mu(g)$$

and

$$\int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E)$$

for convex bodies K, M, where V_j is an intrinsic volume. For that, we use Hadwiger's characterization theorem. From this result, we first deduce a more general kinematic formula, involving a functional on convex bodies that need not have any invariance property.

7.1 Theorem. *If $\varphi : \mathcal{K}^n \to \mathbb{R}$ is an additive continuous function, then*

$$\int_{G_n} \varphi(K \cap gM)\, d\mu(g) = \sum_{k=0}^{n} \varphi_{n-k}(K) V_k(M) \tag{39}$$

for $K, M \in \mathcal{K}^n$, where the coefficients $\varphi_{n-k}(K)$ are given by

$$\varphi_{n-k}(K) = \int_{\mathcal{E}_k^n} \varphi(K \cap E)\, d\mu_k(E). \tag{40}$$

Proof. In order that the integral in (39) makes sense, we first have to show that for given convex bodies K, M the function $g \mapsto \varphi(K \cap gM)$ is μ-integrable. Let $G_n(K, M)$ denote the set of all rigid motions $g \in G_n$ for which K and gM touch, that is, $K \cap gM \neq \emptyset$ but K and gM can be separated weakly by a hyperplane. Using the map γ from (3), it is easy to see that $\gamma(t, \vartheta) \in G_n(K, M)$ holds if and only if $t \in \partial(K - \vartheta M)$; hence

$$\mu(G_n(K, M)) \quad = \quad \int_{SO_n} \int_{\mathbb{R}^n} \mathbf{1}_{G_n(K,M)}(\gamma(t, \vartheta))\, d\lambda(t)\, d\nu(\vartheta)$$

$$= \int_{SO_n} \lambda(\partial(K - \vartheta M))\, d\nu(\vartheta) = 0.$$

On $G_n \setminus G_n(K,M)$, the map $g \mapsto \varphi(K \cap gM)$ is continuous. Since the continuous function φ is bounded on the compact set $\{K' \in \mathcal{K}^n : K' \subset K\}$, it follows that the integral in (39) is well-defined and finite.

Now we fix a convex body $K \in \mathcal{K}^n$ and define

$$\psi(M) := \int_{G_n} \varphi(K \cap gM)\, d\mu(g) \quad \text{for } M \in \mathcal{K}^n.$$

Then $\psi : \mathcal{K}^n \to \mathbb{R}$ is obviously additive and motion invariant. It follows from the bounded convergence theorem that ψ is continuous. Theorem 6.2 yields the existence of constants $c_0(K), \dots, c_n(K)$ so that

$$\psi(M) = \sum_{k=0}^{n} c_k(K) V_{n-k}(M)$$

for all $M \in \mathcal{K}^n$. The constants depend, of course, on the given body K, and we have now to determine them.

Suppose first that $1 \le k \le n-1$ and let $L_k \in \mathcal{L}_k^n$. We choose a k-dimensional cube $W \subset L_k$ with $0 \in W$ and $\lambda^{(k)}(W) = 1$. For $r \ge 1$ we have

$$\psi(rW) = \int_{G_n} \varphi(K \cap grW)\, d\mu(g) = \sum_{i=1}^{n} \varphi_{n-i}(K) V_i(rW).$$

The intrinsic volumes have the easily established properties

$$V_i(rW) = \begin{cases} 0 & \text{for } i > k, \\ r^k & \text{for } i = k, \\ r^i V_i(W) & \text{for } i < k. \end{cases}$$

This yields

$$\psi(rW) = \varphi_{n-k}(K) r^k + o(r^{k-1}) \tag{41}$$

for $r \to \infty$. On the other hand,

$$\psi(rW) \quad = \quad \int_{G_n} \varphi(K \cap grW)\, d\mu(g)$$

$$= \int_{SO_n} \int_{\mathbb{R}^n} \varphi(K \cap (\vartheta r W + x))\, d\lambda(x)\, d\nu(\vartheta)$$

$$= \int_{SO_n} \int_{\vartheta L_k^\perp} \int_{\vartheta L_k} \varphi(K \cap (\vartheta r W + x_1 + x_2))\, d\lambda^{(k)}(x_2)\, d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta).$$

For fixed $\vartheta \in SO_n$ and $x_1 \in L_k^\perp$ we put

$$\begin{aligned} X &:= \{x_2 \in \vartheta L_k : K \cap (\vartheta r W + x_1 + x_2) = K \cap (\vartheta L_k + x_1)\}, \\ Y &:= \{x_2 \in \vartheta L_k : \emptyset \neq K \cap (\vartheta r W + x_1 + x_2) \neq K \cap (\vartheta L_k + x_1)\}. \end{aligned}$$

Then

$$\begin{aligned} &\int_{\vartheta L_k} \varphi(K \cap (\vartheta r W + x_1 + x_2))\, d\lambda^{(k)}(x_2) \\ &\quad = \varphi(K \cap (\vartheta L_k + x_1)) \int_X d\lambda^{(k)}(x_2) \\ &\quad\quad + \int_Y \varphi(K \cap (\vartheta r W + x_1 + x_2))\, d\lambda^{(k)}(x_2). \end{aligned}$$

For $r \to \infty$, we get

$$\int_X d\lambda^{(k)}(x_2) = r^k + O(r^{k-1}).$$

Since φ is bounded on compact sets,

$$\int_Y \varphi(K \cap (\vartheta r W + x_1 + x_2))\, d\lambda^{(k)}(x_2) = O(r^{k-1}).$$

We deduce that

$$\begin{aligned} \psi(rW) &= r^k \int_{SO_n} \int_{\vartheta L_k^\perp} \varphi(K \cap (\vartheta L_k + x_1))\, d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta) + O(r^{k-1}) \\ &= r^k \int_{SO_n} \int_{L_k^\perp} \varphi(K \cap \vartheta(L_k + x_1))\, d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta) + O(r^{k-1}) \\ &= r^k \int_{\mathcal{E}_k^n} \varphi(K \cap E)\, d\mu_k(E) + O(r^{k-1}). \end{aligned}$$

If we compare this with (41) and let r tend to infinity, we obtain the asserted formula (40) for the coefficients.

In the cases $k = 0$ and $k = n$, simpler versions of the proof, with the obvious changes, give the same result. This completes the proof of Theorem 7.1. ■

In Theorem 7.1, we can choose for φ the intrinsic volume V_j and get

$$\int_{G_n} V_j(K \cap gM)\, d\mu(g) = \sum_{k=0}^{n} V_{j,n-k}(K) V_k(M)$$

with

$$V_{j,n-k}(K) = \int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E).$$

By

$$\psi(K) := \int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E) \qquad \text{for } K \in \mathcal{K}^n$$

we again define a functional $\psi : \mathcal{K}^n \to \mathbb{R}$ which is additive, continuous and motion invariant. This is proved similarly as above. Hadwiger's characterization theorem yields a representation

$$\psi(K) = \sum_{r=0}^{n} c_r V_r(K).$$

Here only one coefficient is non-zero. In fact, from

$$\psi(K) = \int_{\mathcal{L}_k^n} \int_{L^\perp} V_j(K \cap (L + y))\, d\lambda^{(n-k)}(y)\, d\nu_k(L)$$

one sees that ψ has the homogeneity property

$$\psi(\alpha K) = \alpha^{n-k+j} \psi(K)$$

for $\alpha > 0$. Since V_k is homogeneous of degree k, we deduce that $c_r = 0$ for $r \neq n - k + j$. Thus we have obtained

$$\int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E) = \alpha_{njk} V_{n+j-k}(K)$$

with some constant α_{njk}. In order to determine this constant, we choose for K the unit ball B^n. For $\epsilon \geq 0$, the Steiner formula gives

$$\sum_{j=0}^{n} \epsilon^{n-j} \kappa_{n-j} V_j(B^n) = V_n(B^n + \epsilon B^n) = (1+\epsilon)^n \kappa_n = \sum_{j=0}^{n} \epsilon^{n-j} \binom{n}{j} \kappa_n,$$

hence

$$V_j(B^n) = \frac{\binom{n}{j}\kappa_n}{\kappa_{n-j}} \qquad \text{for } j = 0, \ldots, n.$$

Choosing $L \in \mathcal{L}_k^n$, we obtain

$$\begin{aligned}
\alpha_{njk} V_{n+j-k}(B^n) &= \int_{\mathcal{E}_k^n} V_j(B^n \cap E)\, d\mu_k(E) \\
&= \int_{SO_n} \int_{L^\perp} V_j(B^n \cap \vartheta(L+x))\, d\lambda^{(n-k)}(x)\, d\nu(\vartheta) \\
&= \int_{L^\perp \cap B^n} (1 - \|x\|^2)^{j/2} V_j(B^n \cap L)\, d\lambda^{(n-k)}(x) \\
&= \frac{\binom{k}{j}\kappa_k}{\kappa_{k-j}} \int_{L^\perp \cap B^n} (1 - \|x\|^2)^{j/2}\, d\lambda^{(n-k)}(x).
\end{aligned}$$

Introducing polar coordinates, the latter integral is transformed into a Beta integral, and one obtains

$$\begin{aligned}
&\int_{L^\perp \cap B^n} (1 - \|x\|^2)^{j/2}\, d\lambda^{(n-k)}(x) \\
&= (n-k)\kappa_{n-k} \int_0^1 (1-r^2)^{j/2} r^{n-k-1}\, dr \\
&= \frac{1}{2}(n-k)\kappa_{n-k} \int_0^1 (1-t)^{j/2} t^{\frac{n-k-2}{2}}\, dt \\
&= \frac{1}{2}(n-k)\kappa_{n-k} \mathrm{B}\left(\frac{j+2}{2}, \frac{n-k}{2}\right) \\
&= \frac{1}{2}(n-k)\kappa_{n-k} \frac{\Gamma(\frac{j+2}{2})\Gamma(\frac{n-k}{2})}{\Gamma(\frac{n+j-k+2}{2})} = \frac{\kappa_{n+j-k}}{\kappa_j}.
\end{aligned}$$

Altogether this yields

$$\alpha_{njk} = \frac{\binom{k}{j}\kappa_k\kappa_{n+j-k}}{V_{n+j-k}(B^n)\kappa_{k-j}\kappa_k} = \frac{\binom{k}{j}\kappa_k\kappa_{n+j-k}}{\binom{n}{k-j}\kappa_n\kappa_j}.$$

This can be put in still a different form by using the identity

$$n!\kappa_n = 2^n\pi^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right).$$

We collect what we have obtained.

7.2 Theorem. *Let $K, M \in \mathcal{K}^n$ be convex bodies and let $j \in \{0, \ldots, n\}$. Then the principal kinematic formula*

$$\int_{G_n} V_j(K \cap gM)\, d\mu(g) = \sum_{k=j}^{n} \alpha_{njk} V_{n+j-k}(K) V_k(M)$$

holds. For $k \in \{1, \ldots, n-1\}$ and $j \leq k$ the Crofton formula

$$\int_{\mathcal{E}_k^n} V_j(K \cap E)\, d\mu_k(E) = \alpha_{njk} V_{n+j-k}(K)$$

holds. The coefficients are given by

$$\alpha_{njk} = \frac{\binom{k}{j}\kappa_k\kappa_{n+j-k}}{\binom{n}{k-j}\kappa_j\kappa_n} = \frac{\Gamma(\frac{k+1}{2})\Gamma(\frac{n+j-k+1}{2})}{\Gamma(\frac{j+1}{2})\Gamma(\frac{n+1}{2})}.$$

Finally, the results are easily extended to polyconvex sets. Let $K \in U(\mathcal{K}^n)$. We choose a representation

$$K = \bigcup_{i=1}^{m} K_i$$

with convex bodies $K_1, \ldots, K_m$. Since V_{n+j-k} is additive on $U(\mathcal{K}^n)$, the inclusion-exclusion principle gives

$$V_{n+j-k}(K) = \sum_{v \in S(m)} (-1)^{|v|-1} V_{n+j-k}(K_v).$$

Now let $M \in \mathcal{K}^n$ be a convex body. Since the principal kinematic formula holds for convex bodies, we obtain

$$\begin{aligned}
&\int\limits_{G_n} V_j(K \cap gM)\, d\mu(g) \\
&\quad = \int\limits_{G_n} V_j\left(\bigcup_{i=1}^{m}(K_i \cap gM)\right)\, d\mu(g) \\
&\quad = \int\limits_{G_n} \sum_{v \in S(m)} (-1)^{|v|-1} V_j(K_v \cap gM)\, d\mu(g) \\
&\quad = \sum_{v \in S(m)} (-1)^{|v|-1} \sum_{k=j}^{n} \alpha_{njk} V_{n+j-k}(K_v) V_k(M) \\
&\quad = \sum_{k=j}^{n} \alpha_{njk} V_{n+j-k}(K) V_k(M).
\end{aligned}$$

Hence, the kinematic formula holds for $K \in U(\mathcal{K}^n)$ and $M \in \mathcal{K}^n$. In a similar way, it can now be extended to $K \in U(\mathcal{K}^n)$ and $M \in U(\mathcal{K}^n)$. An analogous extension is possible for the Crofton formula.

8 Extension to random sets

It has been announced in the introduction that we want to use integral-geometric results to give a theoretical foundation for some formulae used in stereology. To achieve this goal, we shall now extend the kinematic and Crofton formulae to certain random sets.

First we have to explain what one understands by a closed random set in $\mathbb{R}^n$. Let $\mathcal{F}$ denote the system of all closed subsets of $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$ one writes

$$\begin{aligned}
\mathcal{F}_A &:= \{F \in \mathcal{F} : F \cap A \neq \emptyset\}, \\
\mathcal{F}^A &:= \{F \in \mathcal{F} : F \cap A = \emptyset\}.
\end{aligned}$$

The system

$$\{\mathcal{F}_G : G \subset \mathbb{R}^n \text{ open}\} \cup \{\mathcal{F}^C : C \subset \mathbb{R}^n \text{ compact}\}$$

is a subbasis of a topology on $\mathcal{F}$; this topology is called the *topology of closed convergence*. By $\mathcal{B}(\mathcal{F})$ we denote the corresponding σ-algebra of Borel sets.

Now a *random closed set* in $\mathbb{R}^n$, briefly a RACS, is defined as a random variable with values in $\mathcal{F}$. More precisely, a RACS is a measurable map $Z : \Omega \to \mathcal{F}$ from some probability space $(\Omega, \mathcal{A}, \mathbb{P})$ into the measurable space $(\mathcal{F}, \mathcal{B}(\mathcal{F}))$. For $\omega \in \Omega$, the closed set $Z(\omega)$ is called a *realisation* of Z. The image measure $\mathbb{P}_Z := Z(\mathbb{P})$ of the probability measure $\mathbb{P}$ under the map Z is called the *distribution* of Z. Thus, this is a measure on $\mathcal{B}(\mathcal{F})$, and for $A \in \mathcal{B}(\mathcal{F})$ one has

$$\mathbb{P}_Z(A) = \mathbb{P}(Z^{-1}(A)) = \mathbb{P}(\{\omega \in \Omega : Z(\omega) \in A\}) =: \mathbb{P}(Z \in A),$$

which is the probability that Z has a realization in the prescribed set A.

The random closed set Z is called *stationary* if for each vector $t \in \mathbb{R}^n$ the random closed sets Z and $Z + t$ have the same distribution, in other words, if the distribution of Z is invariant under translations. If the distribution of Z is invariant under rotations, then Z is called *isotropic*.

For a measurable nonnegative or $\mathbb{P}$-integrable function $f : \Omega \to \mathbb{R}$, the *expectation* is

$$\mathbb{E}f := \int_\Omega f \, d\mathbb{P}.$$

We will often have a random closed set $Z : \Omega \to \mathcal{F}$ and a measurable function $f : \mathcal{F} \to \mathbb{R}$. If the expectation of $f \circ Z$ exists, it is given by

$$\mathbb{E}f(Z) := \int_\Omega f \circ Z \, d\mathbb{P} = \int_\mathcal{F} f \, d\mathbb{P}_Z,$$

by the transformation formula for integrals.

For our envisaged applications, we have to restrict the admitted random closed sets. The *extended convex ring* is defined by

$$LU(\mathcal{K}^n) := \{F \subset \mathbb{R}^n : F \cap K \in U(\mathcal{K}^n) \text{ for } K \in \mathcal{K}^n\}.$$

The elements of $LU(\mathcal{K}^n)$ will also be called *locally polyconvex sets*. Thus a locally polyconvex set has the property that its intersection with any convex body is a finite union of convex bodies.

If $M \in U(\mathcal{K}^n)$ is a non-empty polyconvex set, there are a number $m \in \mathbb{N}$ and convex bodies $K_1, \dots, K_m \in \mathcal{K}^n$ such that $M = K_1 \cup \dots \cup K_m$. The smallest number m with this property will be denoted as $N(M)$. We also put $N(\emptyset) = 0$. This defines a function $N : U(\mathcal{K}^n) \to \mathbb{N}_0$, which can be shown to be measurable. Now we are in a position to define the random closed sets which will be admitted in the following.

Definition. A *standard random set* in $\mathbb{R}^n$ is a closed random set Z in $\mathbb{R}^n$ with the following properties:
(a) The realizations of Z are locally polyconvex,
(b) Z is stationary,
(c) Z satisfies the integrability condition

$$\mathbb{E}2^{N(Z\cap C^n)} < \infty.$$

Here, as before, $C^n := [0,1]^n$ is the unit cube in $\mathbb{R}^n$.

For a standard random set, one can define a volume density, a surface area density and, more generally, the density of the jth intrinsic volume. Let Z be a standard random set. We choose a 'test body' (or 'observation window') $K \in \mathcal{K}^n$ with $V_n(K) > 0$. For a given realization $Z(\omega)$, the intersection $Z(\omega) \cap K$ is polyconvex, hence the (additively extended) jth intrinsic volume $V_j(Z(\omega) \cap K)$ is defined. One can show that the function $\omega \mapsto V_j(Z(\omega) \cap K)$ is measurable, hence it defines a real random variable. Its expectation,

$$\mathbb{E}V_j(Z \cap K),$$

depends on both, the random set Z and the test body K. However, we shall see that the limit

$$\overline{V}_j(Z) := \lim_{r\to\infty} \frac{\mathbb{E}V_j(Z \cap rK)}{V_n(rK)}$$

exists and is independent of K. This number $\overline{V}_j(Z)$ is called the *density of the jth intrinsic volume* of the random set Z.

The existence proof for the limit, which is a bit technical, is preceded by two lemmas. Recall that C^n is the unit cube given by

$$C^n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : 0 \le x_i \le 1 \text{ for } i = 1, \dots, n\}.$$

The set

$$\partial^+ C^n := \{x = (x_1, \dots, x_n) \in C^n : \max_{1\le i\le n} x_i = 1\}$$

is called the *right upper boundary* of C^n. It is polyconvex. We need the set

$$C_0^n := C^n \setminus \partial^+ C^n$$

as a 'fundamental domain'; the space $\mathbb{R}^n$ can be represented as a disjoint union of translates of C_0^n:

$$\mathbb{R}^n = \bigcup_{z\in\mathbb{Z}^n} (C_0^n + z).$$

We write $\mathbb{Z}^n$ as a sequence $(z_i)_{i\in\mathbb{N}}$ (in any order) and put

$$C_i := C^n + z_i, \qquad \partial^+ C_i := \partial^+ C^n + z_i.$$

The set C_0^n belongs to $U(\mathcal{P}_{r0}^n)$, the class of all finite unions of relatively open convex polytopes. Below we shall use the fact that every additive functional on the class of polytopes has an additive extension to $U(\mathcal{P}_{r0}^n)$. We do not give a proof here, but refer to [6].

8.1 Lemma. *If $\varphi : U(\mathcal{K}^n) \to \mathbb{R}$ is an additive function and $K \in U(\mathcal{K}^n)$ is a polyconvex set, then*

$$\varphi(K) = \sum_{i\in\mathbb{N}} [\varphi(K \cap C_i) - \varphi(K \cap \partial^+ C_i)]$$

Proof. Let $K \in U(\mathcal{K}^n)$. For a polytope $P \in \mathcal{K}^n$ we define

$$\psi(P) := \varphi(K \cap P).$$

Then ψ is an additive functional on convex polytopes and hence has a unique extension to an additive function on $U(\mathcal{P}_{r0}^n)$, also denoted by ψ. Without loss of generality we may assume that

$$K \subset Q := \bigcup_{i=1}^{m} (C_0^n + z_i)$$

and that $\overline{Q}$ is convex (where $\overline{Q}$ denotes the closure of Q). Then

$$\begin{aligned}
\varphi(K) &= \varphi(K \cap \overline{Q}) = \psi(\overline{Q}) = \psi(Q) \\
&= \sum_{i\in\mathbb{N}} \psi(C_0^n + z_i) \\
&= \sum_{i\in\mathbb{N}} [\psi(C_i) - \psi(\partial^+ C_i)] \\
&= \sum_{i\in\mathbb{N}} [\varphi(K \cap C_i) - \varphi(K \cap \partial^+ C_i)].
\end{aligned}$$

Here we have used the additivity of ψ on $U(\mathcal{P}_{r0}^n)$ and the fact that $\psi(P) = 0$ for all convex polytopes P with $K \cap P = \emptyset$. ■

We call a function $\varphi : U(\mathcal{K}^n) \to \mathbb{R}$ *conditionally bounded* if, for each $K' \in \mathcal{K}^n$, the function φ is bounded on the set $\{K \in \mathcal{K}^n : K \subset K'\}$. When φ is translation invariant and additive, it is sufficient for this to assume that φ is bounded on the set $\{K \in \mathcal{K}^n : K \subset C^n\}$.

8.2 Lemma. *Let the function $\varphi : U(\mathcal{K}^n) \to \mathbb{R}$ be translation invariant, additive and conditionally bounded. Then*

$$\lim_{r\to\infty} \frac{\varphi(rK)}{V_n(rK)} = \varphi(C^n) - \varphi(\partial^+ C^n)$$

for every $K \in \mathcal{K}^n$ with $V_n(K) > 0$.

Proof. Let $K \in \mathcal{K}^n$ and $0 \in \operatorname{int} K$, without loss of generality. For $z \in \mathbb{R}^n$ we put

$$\varphi(K, z) := \varphi(K \cap (C^n + z)) - \varphi(K \cap (\partial^+ C^n + z)). \tag{42}$$

Lemma 8.1 shows that

$$\varphi(rK) = \sum_{z\in\mathbb{Z}^n} \varphi(rK, z) \qquad \text{for } r > 0.$$

Define

$$Z_r^1 := \{z \in \mathbb{Z}^n : (C^n + z) \cap rK \neq \emptyset,\ C^n + z \not\subset rK\}$$

and

$$Z_r^2 := \{z \in \mathbb{Z}^n : C^n + z \subset rK\}.$$

Then

$$\lim_{r\to\infty} \frac{|Z_r^1|}{V_n(rK)} = 0, \qquad \lim_{r\to\infty} \frac{|Z_r^2|}{V_n(rK)} = 1, \tag{43}$$

where $|A|$ denotes the number of elements of a set A. The limit relations follow from the fact that one easily shows the existence of numbers $r_0 > s, t > 0$ such that

$$z \in Z_r^1 \quad \Rightarrow \quad C^n + z \subset (r+s)K \setminus (r-s)K,$$
$$(r-t)K \subset \bigcup_{z\in Z_r^2} (C^n + z)$$

for $r \geq r_0$.

By assumption,

$$|\varphi(rK, z)| = |\varphi(rK - z, 0)| \leq b$$

with some constant b independent of z, K and r. This gives

$$\frac{1}{V_n(rK)}\left|\sum_{z\in Z_r^1}\varphi(rK,z)\right| \le b\frac{|Z_r^1|}{V_n(rK)} \to 0 \qquad \text{for } r\to\infty.$$

From this we deduce

$$\begin{aligned}\lim_{r\to\infty}\frac{\varphi(rK)}{V_n(rK)} &= \lim_{r\to\infty}\frac{1}{V_n(rK)}\sum_{z\in\mathbb{Z}^n}\varphi(rK,z)\\ &= \lim_{r\to\infty}\frac{1}{V_n(rK)}\sum_{z\in Z_r^2}\varphi(rK,z)\\ &= [\varphi(C^n)-\varphi(\partial^+C^n)]\lim_{r\to\infty}\frac{|Z_r^2|}{V_n(rK)}\\ &= \varphi(C^n)-\varphi(\partial^+C^n).\end{aligned}$$

■

We are now in a position to prove the existence of the densities of intrinsic volumes for standard random sets.

8.3 Theorem. *For a standard random set Z and for $j \in \{0,\dots,n\}$, the limit*

$$\overline{V}_j(Z) := \lim_{r\to\infty}\frac{\mathbb{E}V_j(Z\cap rK)}{V_n(rK)}$$

exists, and it satisfies

$$\overline{V}_j(Z) = \mathbb{E}[V_j(Z\cap C^n) - V_j(Z\cap\partial^+C^n)].$$

Hence, $\overline{V}_j(Z)$ is independent of K.

Proof. Let $K \in \mathcal{K}^n$ and $V_n(K) > 0$. Without loss of generality, we can assume that $K \subset C^n$. For given $\omega \in \Omega$, there is a representation

$$Z(\omega)\cap K = \bigcup_{i=1}^{N_K(\omega)} K_i(\omega) \qquad \text{with } K_i(\omega)\in\mathcal{K}^n,$$

where $N_K(\omega) := N(Z(\omega)\cap K)$. By the inclusion-exclusion principle,

$V_j(Z(\omega)\cap K)$

$$= \sum_{k=1}^{N_K(\omega)} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le N_K(\omega)} V_j(K_{i_1}(\omega) \cap \cdots \cap K_{i_k}(\omega)),$$

hence, by the monotoneity of the intrinsic volumes,

$$\begin{aligned} \mathbb{E}|V_j(Z \cap K)| &\le V_j(C^n)\mathbb{E}\sum_{k=1}^{N_K} \binom{N_K}{k} \\ &\le V_j(C^n)\mathbb{E}2^{N(Z\cap K)} \\ &\le V_j(C^n)\mathbb{E}2^{N(Z\cap C^n)}, \end{aligned}$$

since $N(Z(\omega) \cap K) \le N(Z(\omega) \cap C^n)$. By assumption, the right-hand side is finite, hence $V_j(Z \cap K)$ is integrable. For a polyconvex set $M \in U(\mathcal{K}^n)$, the integrability of $V_j(Z \cap M)$ then follows from additivity, using the inclusion-exclusion principle again. This shows that all expectations appearing in the theorem exist and are finite. Therefore, we can define a functional $\varphi : U(\mathcal{K}^n) \to \mathbb{R}$ by

$$\varphi(M) := \mathbb{E}V_j(Z \cap M) \qquad \text{for } M \in U(\mathcal{K}^n).$$

Then φ is additive, translation invariant (as follows from the stationarity of Z) and conditionally bounded (as follows from the last estimate above). Now the assertion of the theorem follows from Lemma 8.2. ■

After these preliminaries, we are now able to answer questions of the following kind. Suppose that the realisations $Z(\omega)$ of a closed standard set Z can be observed in a 'window', that is, in a compact convex set K with $V_n(K) > 0$.

By 'observation' we mean that, in principle, the values $V_j(Z(\omega) \cap K)$ can be measured. We want to use the values $V_j(Z(\omega) \cap K)/V_n(K)$ to estimate the densities $\overline{V}_j(Z)$. But in general, $V_j(Z \cap K)/V_n(K)$ will depend on K and thus will not be an unbiased estimator for $\overline{V}_j(Z)$. To control the error, we would have to determine the expectation of $V_j(Z \cap K)$. If Z is an isotropic standard random set, this can be achieved by means of integral geometry. From the obtained set of expectations, one can then also derive unbiased estimators for the densities of the intrinsic volumes.

The next theorem extends the principal kinematic formula to isotropic standard random sets.

8.4 Theorem. *Let Z be an isotropic standard random set in $\mathbb{R}^n$, let $K \in \mathcal{K}^n$*

and $j \in \{0, \dots, n\}$*. Then*

$$\mathbb{E}V_j(Z \cap K) = \sum_{k=j}^{n} \alpha_{njk} V_k(K) \overline{V}_{n+j-k}(Z).$$

(*The coefficients are those of Theorem* 7.2.)

Proof. First we denote that the function

$$\begin{array}{ccc} \mathbb{R}^n \times SO_n \times \Omega & \to & \mathbb{R} \\ (x, \vartheta, \omega) & \mapsto & V_j(Z(\omega) \cap K \cap (\vartheta B^n + z)) \end{array}$$

is integrable with respect to the product measure $\lambda \otimes \nu \otimes \mathbb{P}$. Since $\mathbb{E}2^{N(Z \cap C^n)} < \infty$, this follows as in the proof of Theorem 8.3, if we additionally assume that $K \subset C^n$. For general $K \in \mathcal{K}^n$ it then follows from

$$\begin{aligned} &\int \int \int |V_j(Z(\omega) \cap K \cap (C^n + z) \cap (\vartheta B^n + x))| \, d\lambda(x) d\nu(\vartheta) d\mathbb{P}(\omega) \\ &= \int \int \int |V_j(Z(\omega) \cap (K - z) \cap C^n \cap (\vartheta B^n + x - z))| \, d\lambda(x) d\nu(\vartheta) d\mathbb{P}(\omega) \\ &= \int \int \int |V_j(Z(\omega) \cap (K - z) \cap C^n \cap (\vartheta B^n + x))| \, d\lambda(x) d\nu(\vartheta) d\mathbb{P}(\omega) \\ &< \infty \end{aligned}$$

and the inclusion-exclusion formula.

For $\vartheta \in SO_n$, $x \in \mathbb{R}^n$ and $r > 0$ we deduce from the motion invariance of V_j and the stationarity and isotropy of Z that

$$\begin{aligned} &\mathbb{E}V_j(Z \cap K \cap (\vartheta r B^n + x)) \\ &\quad = \mathbb{E}V_j(\vartheta^{-1}(Z - x) \cap \vartheta^{-1}(K - x) \cap r B^n) \\ &\quad = \mathbb{E}V_j(Z \cap \vartheta^{-1}(K - x) \cap r B^n). \end{aligned}$$

From Fubini's theorem (and the invariance properties of λ and ν) we get

$$\begin{aligned} &\mathbb{E} \int_{SO_n} \int_{\mathbb{R}^n} V_j(Z \cap K \cap (\vartheta r B^n + x)) \, d\lambda(x) \, d\nu(\vartheta) \\ &\quad = \mathbb{E} \int_{SO_n} \int_{\mathbb{R}^n} V_j(Z \cap (\vartheta K + x) \cap r B^n) \, d\lambda(x) \, d\nu(\vartheta). \end{aligned}$$

We apply the principal kinematic formula (Theorem 7.2) to both sides and obtain

$$\sum_{k=j}^{n} \alpha_{njk} \mathbb{E}V_k(Z \cap K) V_{n+j-k}(r B^n) = \sum_{k=j}^{n} \alpha_{njk} V(K) \mathbb{E}V_{n+j-k}(Z \cap r B^n).$$

Now we divide both sides by $V_n(rB^n)$ and let r tend to infinity. Because of $V_m(rB^n) = r^m V_m(B^n)$ and $\alpha_{njj} = 1$, the left side tends to

$$\mathbb{E}V_j(Z \cap K)$$

and by Theorem 8.3, the right side tends to

$$\sum_{k=j}^{n} \alpha_{njk} V_k(K) \overline{V}_{n+j-k}(Z).$$

This completes the proof. ■

The special cases

$$\begin{aligned} \mathbb{E}V_n(Z \cap K) &= V_{n-1}(K)\overline{V}_n(Z), \\ \mathbb{E}V_{n-1}(Z \cap K) &= V_{n-1}(K)\overline{V}_n(Z) + V_n(K)\overline{V}_{n-1}(Z) \end{aligned}$$

of Theorem 8.4 can be obtained without the assumption of isotropy, since corresponding translative integral-geometric formulae are available.

Now we interpret Theorem 8.4. As one application, it describes the error which is made if the measured value $V_j(Z(\omega) \cap K)/V_n(K)$ is used as an estimator for the density $\overline{V}_j(Z)$. Writing the formula of Theorem 8.4 in the form

$$\frac{\mathbb{E}V_j(Z \cap K)}{V_n(K)} = \overline{V}_j(Z) + \frac{1}{V_n(K)} \sum_{k=j}^{n-1} \alpha_{njk} V_j(K) \overline{V}_{n+j-k}(Z),$$

we see that the mean error tends to 0 for increasing windows K, thus the estimator

$$V_j(Z(\omega) \cap K)/V_n(K)$$

is asymptotically unbiased. However, one can also obtain an unbiased estimator. The system of equations given by Theorem 8.4,

$$\mathbb{E}V_j(Z \cap K) = \sum_{k=j}^{n} \alpha_{njk} V_k(K) \overline{V}_{n+j-k}(Z), \qquad j = 0, \ldots, n,$$

can be solved for $\overline{V}_0(Z), \ldots, \overline{V}_n(Z)$, since the coefficient matrix is triangular. This yields formulae of the form

$$\overline{V}_i(Z) = \mathbb{E}\left(\sum_{m=0}^{n} \beta_{nim}(K) V_m(Z \cap K) \right), \qquad i = 0, \ldots, n,$$

hence

$$\sum_{m=0}^{n} \beta_{nim}(K) V_m(Z \cap K)$$

is an unbiased estimator for $\overline{V}_i(Z)$. As an example, we write down the two-dimensional case, using the notations A, L, χ for area, perimeter and Euler characteristic, respectively:

$$\begin{aligned}
\overline{A}(Z) &= \mathbb{E}\frac{A(Z \cap K)}{A(K)}, \\
\overline{L}(Z) &= \mathbb{E}\left[\frac{L(Z \cap K)}{A(K)} - \frac{L(K)A(Z \cap K)}{A(K)^2}\right], \\
\overline{\chi}(Z) &= \mathbb{E}\left[\frac{\chi(Z \cap K)}{A(K)} - \frac{1}{2\pi}\frac{L(K)L(Z \cap K)}{A(K)^2}\right. \\
&\quad \left. + \left(\frac{1}{2\pi}\frac{L(K)^2}{A(K)^3} - \frac{1}{A(K)^2}\right) A(Z \cap K)\right].
\end{aligned}$$

Theorem 8.4 also immediately yields a Crofton formula for random sets. If we talk of a standard random set Z in some affine subspace E, the stationarity and isotropy of Z refer to E, and densities of intrinsic volumes have to be computed in E.

8.5 Theorem. *Let Z be an isotropic standard random set in $\mathbb{R}^n$, let $E \in \mathcal{E}_k^n$ be a k-dimensional flat, where $k \in \{1, \dots, n-1\}$, and let $j \in \{0, \dots, k\}$. Then $Z \cap E$ is an isotropic standard random set in E, and*

$$\overline{V}_j(Z \cap E) = \alpha_{njk} \overline{V}_{n+j-k}(Z).$$

Proof. We omit the (not difficult) proof that $Z \cap E$ is, with respect to E, again an isotropic standard random set. For that reason, the density $\overline{V}_j(Z \cap E)$ exists. Now let $K \in \mathcal{K}^n$, $K \subset E$ and $V_k(K) > 0$. Theorem 8.4 yields

$$\mathbb{E}V_j(Z \cap K) = \sum_{m=j}^{k} \alpha_{njm} V_m(K) \overline{V}_{n+j-m}(Z), \tag{44}$$

where only terms with $m \leq k$ appear since $V_m(K) = 0$ for $m > k$. Since Z is stationary, we can assume that $0 \in E$ and hence $rK \subset E$ for $r > 0$. In (44), we replace K by rK and divide the equation by $V_k(rK)$. For $r \to \infty$, the left side tends to $\overline{V}_j(Z \cap E)$, since $V_j(Z \cap rK) = V_j(Z \cap E \cap rK)$ (and the intrinsic volumes do not depend on the dimension of the surrounding space). The right side tends to $\alpha_{njk}\overline{V}_{n+j-k}(Z)$. ■

The implications of this theorem are clear. After Theorem 8.4, we had seen how the densities $\overline{V}_j(Z)$ of an isotropic standard random set admit asymptotically unbiased or even unbiased estimators. If Z is observed in a k-dimensional section $Z \cap E$, then we can obtain estimators for $\overline{V}_j(Z \cap E)$. Theorem 8.5 tells us that these are at the same time (asymptotically) unbiased estimators for the densities $\alpha_{njk}\overline{V}_{n+j-k}(Z)$.

As an example, we consider the practically relevant case where $n = 3$ and $k = 2$. We deal with the three-dimensional densities $\overline{V}$ (volume), $\overline{S}$ (surface area), $\overline{M}$ (integral of mean curvature) and with the two-dimensional densities $\overline{A}$ (area), $\overline{L}$ (boundary length), $\overline{\chi}$ (Euler characteristic). The equations of Theorem 8.5 now read

$$\overline{V}(Z) = \overline{A}(Z \cap E), \tag{45}$$

$$\overline{S}(Z) = \frac{4}{\pi}\overline{L}(Z \cap E), \tag{46}$$

$$\overline{M}(Z) = 2\pi\overline{\chi}(Z \cap E). \tag{47}$$

These equations, finally, provide an exact theoretic foundation for the 'fundamental equations of stereology', which are traditionally written in the form

$$\begin{aligned} V_V &= A_A, \\ S_V &= \frac{4}{\pi}L_A, \\ M_V &= 2\pi\chi_A. \end{aligned}$$

Concluding we can say that Theorems 8.4 and 8.5 provide theoretical justifications for some practical procedures of stereology, at least in those cases where it is reasonable to model probes of real materials by realisations of isotropic standard random sets. From the practical point of view, the consideration of only locally polyconvex sets does not seem very restrictive. Of the invariance properties, stationarity is always unrealistic, requiring unbounded sets, but it may well be satisfied approximately at close range. The most critical assumption is that of isotropy. For that reason, the applicability of motion invariant stereology is limited, and translative integral geometry is under investigation.

9 The kinematic formula for curvature measures

We shall now prove the local version of the principal kinematic formula, that is, the equation

$$\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(g) = \sum_{k=j}^{n} \alpha_{njk} \Phi_j(K, A) \Phi_{n+j-k}(M, B) \tag{48}$$

for the curvature measures Φ_i. It holds for polyconvex sets $K, M \in U(\mathcal{K}^n)$ and Borel sets $A, B \in \mathcal{B}(\mathbb{R}^n)$. As for the global version, involving the intrinsic volumes V_i, it is sufficient to prove (48) for convex bodies $K, M \in \mathcal{K}^n$, since the general case of polyconvex sets is then easily obtained, using additivity and the inclusion-exclusion principle.

For the proof of (48), we first consider the case where K and M are n-dimensional convex polytopes. We also consider only translations instead of rigid motions, thus we have to investigate the integral

$$I := \int_{\mathbb{R}^n} \Phi_j(K \cap (M + x), A \cap (B + x))\, d\lambda(x).$$

By (37), the jth curvature measure of a polytope P is given by

$$\Phi_j(P, \cdot) = \sum_{F \in \mathcal{F}_j(P)} \gamma(F, P) \lambda_F.$$

It follows that

$$I = \int_{\mathbb{R}^n} \sum_{F' \in \mathcal{F}_j(K \cap (M+x))} \gamma(F', K \cap (M + x)) \lambda_{F'}(A \cap (B + x))\, d\lambda(x). \tag{49}$$

The faces $F' \in \mathcal{F}_j(K \cap (M + x))$ are precisely the j-dimensional sets of the form $F' = F \cap (G + x)$ with a face $F \in \mathcal{F}_k(K)$ and a face $G \in \mathcal{F}_i(M)$, where $k, i \in \{j, \ldots, n\}$. In computing the integral (49), only those translation vectors x need to be considered for which a pair F, G with $F \cap (G + x) \neq \emptyset$ also satisfies $\operatorname{relint} F \cap \operatorname{relint}(G + x) \neq \emptyset$, since the remaining vectors x make up a set of Lebesgue measure zero. Moreover, the pairs F, G for which $k \neq i < n$ or which are in special position, do not contribute to the integral, since for them we have

$$\lambda(\{x \in \mathbb{R}^n : F \cap (G + x) \neq \emptyset\}) = \lambda(F + G^*) = 0.$$

In the remaining cases, we have $\dim F' = \dim F + \dim G - n$ and hence $k + i = n + j$. Therefore, we obtain

$$\begin{aligned} I &= \sum_{k=j}^{n} \sum_{F\in\mathcal{F}_k(K)} \sum_{G\in\mathcal{F}_{n+j-k}(M)} \\ &\quad \int_{\mathbb{R}^n} \gamma(F\cap(G+x), K\cap(M+x))\lambda_{F\cap(G+x)}(A\cap(B+x))\,d\lambda(x). \end{aligned}$$

In the integrand, we may assume that $\operatorname{relint} F \cap \operatorname{relint}(G+x) \neq \emptyset$, and in this case the external angle

$$\gamma(F\cap(G+x), K\cap(M+x)) =: \gamma(F,G,K,M)$$

does not depend on x. Putting

$$J(F,G) := \int_{\mathbb{R}^n} \lambda_{F\cap(G+x)}(A\cap(B+x))\,d\lambda(x),$$

we thus have

$$I = \sum_{k=j}^{n} \sum_{F\in\mathcal{F}_k(K)} \sum_{G\in\mathcal{F}_{n+j-k}(M)} \gamma(F,G,K,M)J(F,G).$$

To compute the integral $J(F,G)$ for given faces $F \in \mathcal{F}_k(K)$ and $G \in \mathcal{F}_{n+j-k}(M)$, we decompose the space $\mathbb{R}^n$ in a way adapted to these faces. We may assume that

$$0 \in L_1 := \operatorname{aff} F \cap \operatorname{aff} G,$$

where aff denotes the affine hull. Let

$$L_2 := L_1^\perp \cap \operatorname{aff} F, \qquad L_3 := L_1^\perp \cap \operatorname{aff} G,$$

and let $\lambda^{(j)}, \lambda^{(k-j)}, \lambda^{(n-k)}$ denote the Lebesgue measures on L_1, L_2, L_3, respectively. With respect to the direct sum decomposition $\mathbb{R}^n = L_1 \oplus L_2 \oplus L_3$, every $x \in \mathbb{R}^n$ has a unique decomposition $x = x_1 + x_2 + x_3$ with $x_i \in L_i$ for $i = 1, 2, 3$. Writing

$$A' := A \cap F, \qquad B' := B \cap G,$$

we get

$$J(F,G) = [F,G] \int_{L_3}\int_{L_2}\int_{L_1} \lambda_{F\cap(G+x_1+x_2+x_3)}(A' \cap (B' + x_1 + x_2 + x_3)$$

$$d\lambda^{(j)}(x_1)\, d\lambda^{(k-j)}(x_2)\, d\lambda^{(n-k)}(x_3).$$

Here the factor $[F, G]$ is an absolute determinant, defined by

$$d\lambda(x) = [F, G]\, d\lambda^{(j)}(x_1)\, d\lambda^{(k-j)}(x_2)\, d\lambda^{(n-k)}(x_3).$$

It can be described as follows, in a more general version. Let $L, L' \subset \mathbb{R}^n$ be two linear subspaces. We choose an orthonormal basis of $L \cap L'$ and extend it to an orthonormal basis of L and also to an orthonormal basis of L'. Let P denote the parallelepiped that is spanned by the vectors obtained in this way. We define $[L, L'] := V_n(P)$. Then $[L, L']$ depends only on the subspaces L and L'. If $L + L' \neq \mathbb{R}^n$, then $[L, L'] = 0$. We extend this definition to faces F, G of polytopes by putting $[F, G] := [L, L']$, where L and L' are the linear subspaces which are translates of the affine hulls of F and G, respectively.

To compute now the inner integral over L_1, we observe that

$$(A' \cap (B' + x_1 + x_2 + x_3)) - x_2 = (A' - x_2) \cap (B' + x_1 + x_3) \subset L_1$$

and hence

$$\begin{aligned}
&\int_{L_1} \lambda_{F \cap (G + x_1 + x_2 + x_3)}(A' \cap (B' + x_1 + x_2 + x_3))\, d\lambda^{(j)}(x_1) \\
&\quad = \int_{L_1} \lambda^{(j)}((A' - x_2) \cap (B' + x_3 + x_1))\, d\lambda^{(j)}(x_1) \\
&\quad = \lambda^{(j)}((A' - x_2) \cap L_1) \lambda^{(j)}((B' + x_3) \cap L_1),
\end{aligned}$$

where we have used Theorem 2.1. The integrations over L_2 and L_3 now require only Fubini's theorem, and we get

$$\begin{aligned}
\int_{L_2} \lambda^{(j)}((A' - x_2) \cap L_1)\, d\lambda^{(k-j)}(x_2) &= \lambda^{(j)} \otimes \lambda^{(k-j)}(A') = \lambda_F(A), \\
\int_{L_3} \lambda^{(j)}((B' + x_3) \cap L_1)\, d\lambda^{(n-k)}(x_3) &= \lambda^{(j)} \otimes \lambda^{(n-k)}(B') = \lambda_G(B).
\end{aligned}$$

Together this yields

$$J(F, G) = [F, G] \lambda_F(A) \lambda_G(B).$$

Inserting this in the integral I, we end up with the following translative integral-geometric formula for polytopes.

9.1 Theorem. *If $K, M \in \mathcal{K}^n$ are polytopes and $A, B \in \mathcal{B}(\mathbb{R}^n)$ are Borel sets, then for $j \in \{0, \dots, n\}$,*

$$\int_{G_n} \Phi_j(K \cap (M+x), A \cap (B+x))\, d\lambda(x)$$
$$= \sum_{k=j}^{n} \sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{n+j-k}(M)} \gamma(F, G, K, M)[F, G] \lambda_F(A) \lambda_G(B).$$

The kinematic formula at which we are aiming requires, for polytopes, the computation of

$$\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(g)$$
$$= \int_{SO_n} \int_{G_n} \Phi_j(K \cap (\vartheta M + x), A \cap (\vartheta B + x))\, d\lambda(x)\, d\nu(\vartheta)$$
$$= \sum_{k=j}^{n} \sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{n+j-k}(M)} \lambda_F(A) \lambda_G(B)$$
$$\int_{SO_n} \gamma(F, \vartheta G, K, \vartheta M)[F, \vartheta G]\, d\nu(\vartheta).$$

Here we have used the fact that $\lambda_{\vartheta G}(\vartheta B) = \lambda_G(B)$. The summands with $k = j$ or $k = n$ are easily determined, since for $k = j$ we get

$$\sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{n+j-k}(M)} \gamma(F, G, K, M)[F, G] \lambda_F \otimes \lambda_G$$
$$= \sum_{F \in \mathcal{F}_j(K)} \gamma(F, M, K, M)[F, M] \lambda_F \otimes \lambda_M$$
$$= \sum_{F \in \mathcal{F}_j(K)} \gamma(F, K) \lambda_F \otimes \lambda_M$$
$$= \Phi_j(K, \cdot) \otimes \Phi_n(M, \cdot),$$

and similarly for $k = n$,

$$\sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{n+j-k}(M)} \gamma(F, G, K, M)[F, G] \lambda_F \otimes \lambda_G$$
$$= \Phi_n(K, \cdot) \otimes \Phi_j(M, \cdot).$$

The remaining integrals over the rotation group are determined in the following theorem.

9.2 Theorem. *Let $K, M \in \mathcal{K}^n$ be polytopes, let $j \in \{0, \ldots, n-2\}$, $k \in \{j+1, \ldots, n-1\}$, $F \in \mathcal{F}_k(K)$ and $G \in \mathcal{F}_{n+j-k}(M)$. Then*

$$\int_{SO_n} \gamma(F, \vartheta G, K, \vartheta M)[F, \vartheta G]\, d\nu(\vartheta) = \alpha_{njk}\gamma(F, K)\gamma(G, M),$$

where α_{njk} is as in Theorem 7.2.

Proof. In order to avoid difficult direct computations, we will give a proof based on the uniqueness of spherical Lebesgue measures. This is possible since external angles are defined in terms of such measures.

By definition,

$$\gamma(F, \vartheta G, K, \vartheta M) = \gamma(F \cap (\vartheta G + x), K \cap (\vartheta M + x))$$

with suitable $x \in \mathbb{R}^n$. As before, let $N(P, F)$ denote the normal cone of a polytope P in a relatively interior point of its face F. From the definition of the external angle we get

$$\gamma(F, \vartheta G, K, \vartheta M) = \frac{\sigma^{(L)}(N(K \cap (\vartheta M + x), F \cap (\vartheta G + x)) \cap S^{n-1})}{\sigma^{(L)}(L \cap S^{n-1})},$$

where $L \in \mathcal{L}^n_{n-j}$ is the orthogonal space of $F \cap (\vartheta G + x)$ (i.e., the orthogonal complement of the linear subspace parallel to the affine hull of $F \cap (\vartheta G + x)$). For a linear subspace $L \subset \mathbb{R}^n$, we have denoted by $\sigma^{(L)}$ the spherical Lebesgue measure on $L \cap S^{n-1}$.

A general property of normal cones of convex bodies gives

$$N(K \cap (\vartheta M + x), F \cap (\vartheta G + x)) = N(K, F) + \vartheta N(M, G).$$

Therefore, we have to evaluate the integral

$$\int_{SO_n} \sigma^{(L_1 + \vartheta L_2)}((N(K, F) + \vartheta N(M, G)) \cap S^{n-1})[F, \vartheta G]\, d\nu(\vartheta),$$

where L_1 is the orthogonal space of F and L_2 is the orthogonal space of G.

More generally, we define the integral

$$I(A, B) := \int_{SO_n} \sigma^{(L_1 + \vartheta L_2)}(C(A) + \vartheta C(B)) \cap S^{n-1})[F, \vartheta G]\, d\nu(\vartheta)$$

for arbitrary Borel sets $A \subset L_1 \subset S^{n-1}$ and $B \subset L_2 \subset S^{n-1}$, where

$$C(A) := \{\alpha x : x \in A,\ \alpha \geq 0\}$$

denotes the cone spanned by A. Concerning the measurability of the integrand, we give the following hints for a proof. The function $\vartheta \mapsto [F, \vartheta G]$ is continuous, hence measurable. Let U denote the set of all rotations $\vartheta \in SO_n$ for which L_1 and ϑL_2 are not in special position. Then it can be shown that $\nu(SO_n \setminus U) = 0$. For $\vartheta \in U$ we have

$$\dim L_1 + \dim L_2 = (n-k) + (k-j) = n - j \leq n,$$

hence the sum $L_1 + \vartheta L_2$ is direct. From this one can deduce that $C(A) + \vartheta C(B)$ is a Borel set (in general, the sum of two Borel sets need not be a Borel set). For different $\vartheta \in U$, the sets $C(A) + \vartheta C(B)$ are connected by linear transformations. All this together is sufficient to show that the mapping

$$\vartheta \mapsto \sigma^{(L_1 + \vartheta L_2)}((C(A) + \vartheta C(B)) \cap S^{n-1})$$

is measurable on U.

For fixed $B \in \mathcal{B}(L_2 \cap S^{n-1})$ we now define

$$\varphi(A) := I(A, B) \qquad \text{for } A \in \mathcal{B}(L_1 \cap S^{n-1}).$$

If $\bigcup_{i=1}^{\infty} A_i$ is a disjoint union of sets $A_i \in \mathcal{B}(L_1 \cap S^{n-1})$, then

$$\left(C\left(\bigcup_{i=1}^{\infty} A_i \right) + \vartheta C(B) \right) \cap S^{n-1} = \bigcup_{i=1}^{\infty} ((C(A_i) + \vartheta C(B)) \cap S^{n-1})$$

for $\vartheta \in U$, and this union is disjoint up to a set of $\sigma^{(L_1+\vartheta L_2)}$-measure zero. We deduce that

$$\sigma^{(L_1+\vartheta L_2)} \left(\left(C\left(\bigcup_{i=1}^{\infty} A_i \right) + \vartheta C(B) \right) \cap S^{n-1} \right) = \sum_{i=1}^{\infty} \sigma^{(L_1+\vartheta L_2)}((C(A_i) + \vartheta C(B)) \cap S^{n-1})$$

for $\vartheta \in U$ and thus

$$\varphi\left(\bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \varphi(A_i),$$

by the theorem of monotone convergence. It follows that φ is a finite measure on $L_1 \cap S^{n-1}$. Let $\rho \in SO_n(L_1)$ be a rotation mapping the subspace L_1 into itself. Then

$$C(\rho A) + \vartheta C(B) = \rho(C(A) + \rho^{-1}\vartheta C(B))$$

and

$$[F, \vartheta G] = [\rho F, \vartheta G] = [F, \rho^{-1}\vartheta G],$$

hence

$$\begin{aligned}
&\varphi(\rho A) \\
&= \int_{SO_n} \sigma^{(L_1+\vartheta L_2)}((C(\rho A) + \vartheta C(B)) \cap S^{n-1})[F, \vartheta G]\, d\nu(\vartheta) \\
&= \int_{SO_n} \sigma^{(L_1+\rho^{-1}\vartheta L_2)}((C(A) + \rho^{-1}\vartheta C(B)) \cap S^{n-1})[F, \rho^{-1}\vartheta G]\, d\nu(\vartheta) \\
&= \varphi(A).
\end{aligned}$$

Since spherical Lebesgue measure is uniquely determined, up to a factor, by its rotation invariance (and finiteness), the measure φ must be a constant multiple of $\sigma^{(L_1)}$. Analogously we deduce that for fixed $A \in \mathcal{B}(L_1 \cap S^{n-1})$ the measure $I(A, \cdot)$ must be a constant multiple of $\sigma^{(L_2)}$. Both results together yield that

$$I(A, B) = \alpha(L_1, L_2)\sigma^{(L_1)}(A)\sigma^{(L_2)}(B)$$

for all $A \in \mathcal{B}(L_1 \cap S^{n-1})$, $B \in \mathcal{B}(L_2 \cap S^{n-1})$; here $\alpha(L_1, L_2)$ is a constant depending only on L_1 and L_2. If we choose $A = L_1 \cap S^{n-1}$, $B = L_2 \cap S^{n-1}$ and observe the invariance properties of the functional I following from its definition, we see that $\alpha(L_1, L_2)$ depends only on the dimensions n, j, k. Therefore, there is a constant β_{njk} so that

$$I(A, B) = \beta_{njk}\sigma^{(L_1)}(A)\sigma^{(L_2)}(B).$$

In particular, this shows that

$$I(N(K, F) \cap S^{n-1}, N(M, G) \cap S^{n-1})) = \beta_{njk}\gamma(F, K)\gamma(G, M).$$

This is the assertion of Theorem 9.2, except that it remains to show that $\beta_{njk} = \alpha_{njk}$.

Collecting the results obtained so far, we have proved the following kinematic formula for polytopes $K, M \in \mathcal{K}^n$:

$$\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(g)$$
$$= \sum_{k=j}^{n} \beta_{njk} \sum_{F \in \mathcal{F}_k(K)} \sum_{G \in \mathcal{F}_{n+j-k}(M)} \gamma(F,K)\gamma(G,M)\lambda_F(A)\lambda_G(B)$$
$$= \sum_{k=j}^{n} \beta_{njk} \Phi_k(K,A)\Phi_{n+j-k}(M,B).$$

If we choose $A = B = \mathbb{R}^n$, the obtained formula must coincide with that of Theorem 7.2, for all polytopes K, M. This shows that $\beta_{njk} = \alpha_{njk}$ and thus completes the proof of Theorem 9.2. ■

For arbitrary convex bodies K, M, the general kinematic formula (48) is now obtained by approximation, using the weak continuity of the curvature measures. An extension to polyconvex sets K, M is easily achieved by additivity, as in the case of Theorem 7.2.

Also the Crofton formula of Theorem 7.2 has a local counterpart. We collect both results in the following theorem.

9.3 Theorem. *Let $K, M \subset U\mathcal{K}^n$ be polyconvex sets, let $j \in \{0, \ldots, n\}$, and let $A, B \in \mathcal{B}(\mathbb{R}^n)$ be Borel sets. Then the principal kinematic formula*

$$\int_{G_n} \Phi_j(K \cap gM, A \cap gB)\, d\mu(G) = \sum_{k=j}^{n} \alpha_{njk} \Phi_j(K,A)\Phi_{n+j-k}(M,B) \qquad (50)$$

holds. For $k \in \{1, \ldots, n-1\}$ and $j \leq k$ the Crofton formula

$$\int_{\mathcal{E}_k^n} \Phi_j(K \cap E, A \cap E)\, d\mu_k(E) = \alpha_{njk} \Phi_{n+j-k}(K,A) \qquad (51)$$

holds. In both cases, the coefficients α_{njk} are those given in Theorem 7.2.

Proof. It remains to prove formula (51). Here we can assume that K is a convex body, since the general case is then obtained by additivity. We deduce (51) from (50), by a similar but simpler argument as used in the proof of Theorem 7.1.

Let $L_k \in \mathcal{L}_k^n$ be a fixed subspace; then $\mu_k = \gamma_k(\lambda^{(n-k)} \otimes \nu)$, as in Section 3. Let W be a unit cube in L_k. Let $A \in \mathcal{B}(\mathbb{R}^n)$. By (50) we have

$$\begin{aligned} J &:= \int_{G_n} \Phi_j(L_k \cap gK, W \cap gA)\, d\mu(g) \\ &= \sum_{m=j}^{n} \alpha_{njm} \Phi_m(L_k, W) \Phi_{n+j-m}(K, A) \end{aligned}$$

with

$$\Phi_m(L_k, W) = \begin{cases} \lambda_{L_k}(W) = 1 & \text{for } m = k, \\ 0 & \text{for } m \neq k, \end{cases}$$

hence

$$J = \alpha_{njk} \Phi_{n+j-k}(K, A).$$

On the other hand,

$$\begin{aligned} J &= \int_{SO_n} \int_{\mathbb{R}^n} \Phi_j(L_k \cap (\vartheta K + x), W \cap (\vartheta A + x))\, d\lambda(x)\, d\nu(\vartheta) \\ &= \int_{SO_n} \int_{L_k^\perp} \int_{L_k} \Phi_j(L_k \cap (\vartheta K + x_1 + x_2), W \cap (\vartheta A + x_1 + x_2)) \\ &\quad d\lambda^{(k)}(x_2)\, d\lambda^{(n-k)}(x_1)\, d\nu(\vartheta). \end{aligned}$$

For the computation of the inner integral, we put

$$\Phi_j(L_k \cap (\vartheta K + x_1), \cdot) =: \varphi, \qquad \vartheta A + x_1 =: A'.$$

Then

$$\begin{aligned} &\int_{L_k} \Phi_j(L_k \cap (\vartheta K + x_1 + x_2), W \cap (\vartheta A + x_1 + x_2))\, d\lambda^{(k)}(x_2) \\ &= \int_{L_k} \varphi((W - x_2) \cap A')\, d\lambda^{(k)}(x_2) \\ &= \varphi(A') \lambda^{(k)}(W) \end{aligned}$$

$$= \Phi_j(L_k \cap (\vartheta K + x_1), L_k \cap (\vartheta A + x_1)),$$

where Theorem 2.1 was used. This yields

$$\begin{aligned} J &= \int_{SO_n} \int_{L_k^\perp} \Phi_j(L_k \cap (\vartheta K + x_1), L_k \cap (\vartheta A + x_1)) \, d\lambda^{(n-k)}(x_1) \, d\nu(\vartheta) \\ &= \int_{SO_n} \int_{L_k^\perp} \Phi_j(K \cap \vartheta(L_k + x), A \cap \vartheta(L_k + x)) \, d\lambda^{(n-k)}(x) \, d\nu(\vartheta) \\ &= \int_{\mathcal{E}_k^n} \Phi_j(K \cap E, A \cap E) \, d\mu_k(E), \end{aligned}$$

where we have used the rigid motion covariance of the curvature measures as well as the inversion invariance of the measures $\lambda^{(n-k)}$ and ν. The two representations obtained for J together prove the assertion. ■

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Stereology: Integral Geometry 'under the Microscope'

Luis M. Cruz-Orive

Stereology can be regarded as geometric sampling. It is a blend of integral geometry, geometric probability, and statistics, to serve the scientist who wants to estimate geometric measures associated with a solid object (e.g. an organ, a tumor, a rock, a sausage...), or with the internal microstructure of it. Actually, the classical Buffon's needle problem (1777) already contains much of the art and spirit of stereology. We propose a brief, lively review of the state-of-the-art in stereology.

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Concentration multi-échelles de courbure dans des fibres de Milnor

Evelia García Barroso

Nous étudions le comportement asymptotique de la courbure de la fibre de Milnor

$$C(\lambda)_\epsilon = f^{-1}(\lambda) \cap \mathbf{B}_\epsilon \subset \mathbf{C}^2$$

d'un germe de courbe plane réduite $(C,0) \subset (\mathbf{C}^2,0)$ défini par une équation $f(x,y)=0$ lorsque ϵ et λ tendent vers 0. Il s'agit de la courbure de Lipschitz-Killing associée à la métrique induite sur $C(\lambda)$ par celle de $\mathbf{C}^2$. On connait déjà, grâce au travail de Langevin [3] la valeur limite de l'intégrale de cette courbure :

$$\mathrm{Lim}_{\epsilon,\lambda\to 0,|\lambda|<<\epsilon} \int_{C(\lambda)_\epsilon} |K| dv = 2\pi(\mu^{(2)}(C) + \mu^{(1)}(C)),$$

où $\mu^{(2)}$ est le nombre de Milnor en 0 de la singularité $C(0) = C$ et $\mu^{(1)}$ sa multiplicité en 0 diminuée de 1. Il faut souligner que le terme de droite ne dépend que de la topologie du plongement dans $\mathbf{C}^2$ du germe de courbe plane réduite $(C,0)$. D'après [4], le nombre $\mu^{(2)}(C) + \mu^{(1)}(C)$ est le nombre d'intersection à l'origine de C avec une de ses courbes polaires relatives génériques, qui sont définies par les équations

$$\frac{\partial f}{\partial y} + \tau \frac{\partial f}{\partial x} = 0.$$

C'est aussi le nombre des points d'intersection (transverses) d'une telle courbe polaire avec une fibre de Milnor $f(x,y) - \lambda = 0$ qui tendent vers 0 avec λ. Ce résultat est donc au fond de la nature d'un résultat de théorie de l'intersection, c'est à dire que l'on compte des points, ou le degré de cycles, sans se préoccuper de leur position.
Nous allons obtenir une information plus précise sur la géométrie de la fibre de Milnor en essayant de localiser les régions de $C(\lambda)$ où se concentre asymptotiquement la courbure, et ce faisant mettre en évidence le fait que la concentration de courbure est un phénomène multi-échelles : la courbure se

concentre dans les intersections avec $C(\lambda)$ de boules, dont les centres $\xi^{Q}_{q,l}(\lambda)$ peuvent être décrits, mais surtout dont les rayons sont de la forme $|\lambda|^{\rho(Q)}$ où les $\rho(Q)$ sont des nombres rationnels dont la collection ne dépend que de la topologie du plongement dans $\mathbf{C}^2$ du germe de courbe plane réduite $(C, 0)$. Il nous semble intéressant que cette description multi-échelles elle même ne dépende que de la topologie.
Lorsque le germe C est analytiquement irréductible en 0, i.e., est une branche, la donnée des exposants $\rho(Q)$, qui sont alors en nombre égal à celui des exposants de Puiseux, et celle de la quantité de courbure qui se concentre dans les boules de rayon $\rho(Q)$ permettent de déterminer les exposants caractéristiques de Puiseux de C, et l'on peut donc dire que le comportement asymptotique de la courbure de la fibre de Milnor permet de déterminer la classe d'équisingularité de la fibre limite singulière. Le cas réductible est plus compliqué, en particulier parce qu'une partie de la courbure reste "diffuse", comme dans le cas extrême de $x^n + y^n = 0$, où il n'y a pas de concentration de courbure dans des boules de centre différent de l'origine (et donc dépendant de λ). Nous savons mesurer quelle est la partie diffuse.
Notre technique de preuve est basée sur l'analyse du contact avec les branches de C des différentes branches des courbes polaires génériques qui a été faite dans [1], [2].
L'idée heuristique part du fait que par définition de la courbe polaire $\frac{\partial f}{\partial y} + \tau \frac{\partial f}{\partial x} = 0$, ses points d'intersection avec $C(\lambda)$ sont les points de $C(\lambda)$ où la tangente a la direction correspondant au paramètre τ. Pour prouver le théorème de Langevin, on compte le nombre de ces points et on applique la formule d'échange. Nous observons que ces points ont répartis sur les différentes branches de la courbe polaire, et si le contact avec C en 0 d'une de ces branches est fort, elle varie peu lorsque l'on varie le paramètre τ et par conséquent ses points d'intersection avec $C(\lambda)$ bougent peu, ce qui signifie que beaucoup de courbure se concentre au voisinage de ces points. Rappelons que le vocable *branche* désigne un germe analytiquement irréductible de courbe, et en particulier une composante irréductible d'un germe de courbe.
Cette travail est fait en collaboration avec Bernard Teissier et va à paraître dans *Commentarii Mathematici Helvetici.*

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Total curvatures and Euler-Poincaré characteristic: Stereological estimation

Ximó Gual-Arnau

1 Introduction

Chern and Lashof defined the total absolute curvatures of immersed submanifolds in the euclidean space by integration, on the submanifold, the absolute value of certain local curvatures. Their work, which relates the theory of total curvatures with Morse theory of critical points of functions defined over the submanifold, has been extended for immersions into spaces of constant curvature (in particular the sphere) and for holomorphic immersions into complex projective spaces.

On the other hand, using techniques of Integral Geometry, which generalize the Quermassintegrale of convex sets, Santaló introduced some global definitions of total absolute curvatures for compact manifolds immersed in a euclidean space and he showed that one of these curvatures coincides with the Chern-Lashof's curvature. However we have not found in the literature a generalization of these tecniques for immersions in the sphere.

Moreover, the concepts used to define the total absolute curvatures from the Integral Geometry viewpoint (Santaló's approach) and those used to obtain a local interpretation of these curvatures (Chern-Lashof's approach) are similar to the ideas presented by several authors in Stereology to estimate the Euler-Poincaré characteristic for $n-$dimensional sets in $\mathbb{R}^n$. These ideas have been adapted from the definition of the Euler-Poincaré characteristic given by Hadwiger and allow to define the Euler number of an $n-$dimensional set in terms of what happens in an $(n-1)-$dimensional plane that sweeps through the set. However, these ideas have not been applied to obtain the Euler number of domains which are not $n-$dimensional sets in $\mathbb{R}^n$, for instance, domains in a surface of $\mathbb{R}^3$.

2 Total absolute curvature of plane curves and Euler-Poincaré characteristic of plane domains

In this first section we summarize some wellknown results refered to curves and domains in $\mathbb{R}^2$. Firstly, we remember the definition of the total absolute curvature of a plane curve given in Integral geometry. Afterwards, we give a characterization of this definition from the theory of critical points of height functions and we prove that this definition coincides with the definition of total abolute curvature given in Classical Differential Geometry.

In the second part of this section we adapt the preceding concepts to obtain the Euler number of a plane domain. This method to obtain the Euler-Poincaré characteristic of a plane domain has been used in different stereological applications.

3 Total absolute curvature of curves in the sphere and Euler-Poincaré characteristic of domains in the sphere

We first particularize some results of [1] to spherical curves. In particular, we define three different total absolute curvatures for spherical curves in terms of what happens in a circle (geodesic) or small circle that sweeps through the curve. Afterwards we prove that local versions of these total absolute curvatures allow us to obtain the different total absolute curvatures studied in Differential Geometry and Topology for curves in spheres.

Secondly, we consider two methods to define the Euler-Poincaré characteristic of a domain with boundary in the sphere. With the first method we consider geodesics that sweeps through the domain and in the second method we use parallel small circles which are tangent to the boundary of the domain.

4 A stereological version of the Euler-Poincaré characteristic for domains in a surface

Here we adapt the theory of critical points of functions to height functions in $\mathbb{R}^3$ to obtain the Euler-Poincaré characteristic of a domain with boundary in a surface of $\mathbb{R}^3$, [2]. In such a manner that the results exposed in the preceding sections for plane and spheric domains are particular cases of the formulae presented in this section.

Finally, we will give a geometrical interpretation of these formulas which can be considered as a stereological version of the Gauss-Bonnet formula.

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A geometrical meaning for Action from Integral Geometry in Space-Time

Mariano Santander

The talk aims to introduce some elementary results on Integral Geometry in space-time. These are presented from a point of view which includes in a natural way the geometries of homogeneous space-times as well as the classical riemannian geometries of constant curvature. It will be organized by merging two only apparently disparate threads: one belongs to Physics while the other comes from Geometry.

Action is the most important single quantity in Classical Mechanics. Absolute time and space-length, the two other basic quantities have a clear geometrical interpretation in terms of the (degenerate) time metric and space metric in classical physics. However, Action appears as an apparently rather *ad-hoc* concept without any known geometrical meaning. I will also give a brief glimpse to the seminal role of action in Quantum Mechanics —through Feynmann's formulation—, and I will comment how quantum mechanics points to the *relative* actions for closed space-time paths as the important quantities.

The second thread comes from geometry. Starting from the prehistory and history of non-euclidean geometry I will comment how the study of the *three* classical geometries of constant curvature could (or better should) be considered as only a part of a more comprehensive and symmetrical scheme, where they are *nine* two dimensional real geometries. The six remaining geometries turns out to provide the right geometrical language to discuss the six possible kinematics of an homogeneous space-time with constant space-time curvature and either non-relativistic (with absolute time) or relativistic (without absolute time). This scheme is better couched in terms of the Lie groups and algebras of these geometries. As an example of this approach having a clear direct relevance for Integral geometry, two mutually dual Gauss-Bonnet like identities will be presented which afford a direct derivation of trigonometry for these nine geometries in a single run. This common frame, which historically first appeared as the Cayley–Klein theory of 'projective metrics' was also considered by Poincaré. It includes the (nine) geometries with a quadratic metric (either riemannian, degenerate or pseudoriemannian) and constant curvature (either positive, zero or negative) in a surprisingly symmetrical arrangement, which suggests relationships between different geometries (as a most important duality) and allows many properties to be studied for all the nine geometries simultaneously.

The first thread contributes to the whole picture as: *"Action is essential in classical mechanics, Classical Mechanics can be recasted as a geometric theory, yet Action has no a known geometrical meaning"*. The second one tells us: *"Study and classification of possible two-dimensional geometries turns out to furnish the actual observed space-time structure, whether in a rather rough approximation (the galilean geometry of classical physics), or as in more refined and accurate ones (as the Lorentz-Minkowski geometry of relativistic physics)"*. The third part of the talk will merge these two threads together, providing a the geometrical interpretation for Action in terms of Integral geometry in space-time.

The three classical space geometries (spherical, euclidean and hyperbolic) have a *compact* isotopy subgroup of rotations (in 2d $SO(2)$) around a point. This leads to a *finite* total measure of the sets of lines through a point and underlies most elementary integral geometry results as the Cauchy-Crofton formula for the length of a curve in euclidean plane by integrating the intersection counting with straight lines:

$$\int N_\Gamma(l)dl = 2L_\Gamma.$$

In contradistinction, the geometries which describe *space-time* have a *non-compact* isotopy subgroup of 'rotations' around an event (or space-time point); physically these are inertial transformations, and in $1+1$ dimensions, the isotopy subgroup is $SO(1,1)$. The non-compact nature of inertial transformations should hold if one wants space-time to be *causal*, a basic requirement deeply involved in all physical theories. Therefore the total measure of space-time time-like lines through an event is infinite. At a first sight this precludes extension of Cauchy-Crofton and similar relations from (locally) euclidean spaces to space-time, which is either locally galilean or locally lorentzian.

However, and this will be the main result presented in the talk, this is not so. The loss of a *finite* total measure for the set of lines through a point is accompanied by the existence of a *time orientation* which allows a classification of time-like lines into future and past, and which is invariant under the kinematical group. In the *flat* space-times —the Galilei space-time of classical mechanics and the Lorentz-Minkowski space-time of special relativity—, this suggests to modify the euclidean intersection counting and to replace it by an *oriented* intersection counting; this is also invariant under the kinematical group.

Let us consider a closed time-like curve (a circuit in space-time) Γ as two possible future-pointing paths, say Γ_1 and Γ_2. These paths go from an initial event A to a final one B. Now two natural questions are:

- first, are the integrals

$$\int N_\Gamma(l)dl$$

of the oriented intersection numbers with Γ, taken over the space-time of all time-like lines well defined? and,

- second, if they are, what is their meaning?

Both in classical and in the relativistic case these integrals are well defined, even if $\int dl_0$ diverges, and also in both cases, they are proportional to the *difference* of actions for a particle going from A to B along the two paths Γ_1 and Γ_2. This could be expected in the relativistic case, where the action for a free particle is just the Lorentz-Minkowski length of the particle worldline, but comes as a surprising result in classical physics. Therefore this gives an unexpected interpretation for Action in terms of Integral Geometry in Space-time. As usually happens with mathematics in/and physics, the right mathematics strongly *suggests* the quantities which *are* the physically important ones; in this case even the fundamental importance of relative actions, as opossed to the unobservable action along an open path, is clearly captured. The interpretation does not circunscribe to the free case, but is also valid for particles in *any* potential; this however will not be discussed here.

A general reference explaining in a descriptive, yet authoritative way the role of Action and space-time in Physics as well as in Mathematics is the book by Yu. I. Manin, *Mathematics and Physics*, Birkhauser, Boston, (1981). This book makes a very stimulating reading. The talk is based mainly on the following papers:

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