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Application of He's homotopy perturbation method for solving $K(2,2)$, KdV, burgers and cubic boussinesq equations

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Abstract. In this study, homotopy perturbation method (HPM) is used for solving , KdV, $K(2,2)$, Burgers and cubic Boussinesq equations. Contrary to classical perturbation techniques, homotopy perturbation method does not require small or large parameters so that it can be used to obtain formulae uniformly valid for both small and large parameters in nonlinear problems. The results reveal that the present method is very effective and convenient, and it is suggested that HPM can be found widely applications in engineering and physics.

Keywords: homotopy perturbation method (HPM), KdV equation, $K(2,2)$ equation, burgers equation, cubic boussinesq equation

1 Introduction

Nonlinear phenomena play an important role in applied physics. Exact or approximate solutions of these nonlinear problems can guide us to know the described processes more deeply[22]. KdV equation is the pioneering equation that gives rise to solitary wave solutions. Solitons are localized waves that propagate without any changes in their shape and velocity properties, and are stable against mutual collisions.

The $K(n,n)$ equation developed in [21] is the pioneering equation for compactons. In the theory of solitary waves, compactons are defined as solitons with finite wavelengths or as solitons that are free of exponential tails.

Burgers equation appears in fluid mechanics. This equation incorporates both convection and diffusion in fluid dynamics, and is used to describe the structure of shock waves. The Cubic Boussinesq equation gives rise to solitons and appeared in the work of Priestly and Clarkson[20]. This equation has been investigated for solitary waves and rational solutions as well. Solitons and compactons with and without exponential wing, respectively, are termed by using the suffix on, to indicate that it has the property of a particle, such as phonon, peakon, cuspon, and photon.

In this work, the authors will apply a kind of analytical technique called HPM to approximately solve equations. This method was first proposed by He[6, 8, 10, 12–18], and has been shown to be capable of solving a large class of nonlinear problems effectively and easily with approximations converging rapidly to accurate solutions. This method was proposed to search for limit cycles and bifurcation curves of nonlinear equations[10, 12–14].

In [10], a heuristic example was given to illustrate the basic idea of the homotopy perturbation method and its advantages over the δ-method. This method was also applied to solve boundary value problems[16] and heat radiation equations[2, 4]. In [11], a comparison of HPM and homotopy analysis method was made, revealing that the former is more powerful than the latter. Other applications of HPM can also be found in [2–5].

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There are other approaches to analytical solutions for nonlinear equations, such as the variational iteration method[7, 19] and $tanh$-function method[1], for which a complete review is available in [9].

In this work, we have compared the solutions of KdV, $K(2,2)$, Burgers and cubic Boussinesq equations with those of variational iteration method (VIM), previously obtained by other[22].

2 Basic idea of homotopy perturbation method

To illustrate the basic ideas of this method, we consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, r \in \Omega \tag{1}$$

Considering the boundary conditions of:

$$B(u, \partial u/\partial n) = 0, r \in \Gamma \tag{2}$$

where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω.

The operator A can generally be divided into two parts of L and N, where L is linear but N is nonlinear. Eq. (2) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0 \tag{3}$$

Using the homotopy technique, we construct a homotopy $\nu(r,p) : \Omega \times [0,1] \to R$, which satisfies:

$$H(\nu, p) = (1-p)[L(\nu) - L(u_0)] + p[A(\nu) - f(r)] = 0, p \in [0,1], r \in \Omega \tag{4}$$

or

$$H(\nu, p) = L(\nu) - L(u_0) + pL(u_0) + p[N(\nu) - f(r)] = 0 \tag{5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of Eq. (1), which satisfies the boundary conditions. Obviously, considering Eqs. (4) and (5), we will have:

$$H(\nu, 0) = L(\nu) - L(u_0) = 0 \tag{6}$$
$$H(\nu, 1) = A(\nu) - f(r) = 0 \tag{7}$$

The changing process of p from zero to unity is just that of $\nu(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation, and $L(\nu) - L(u_0)$ and $A(\nu) - f(r)$ are called homotopy. According to HPM, we can first use the embedding parameter as an expanding parameter, and assume that the solution of Eqs. (4) and (5) can be written as a power series in p:

$$\nu = \nu_0 + p\nu_1 + p^2\nu_2 + \cdots \tag{8}$$

Setting $p = 1$ results in the approximate solution of Eq. (1):

$$u = \lim_{p \to 1} \nu = \nu_0 + p\nu_1 + p\nu_2 + \cdots \tag{9}$$

The combination of perturbation method and homotopy method is called the homotopy perturbation method (HPM), which lacks the limitations of the traditional perturbation methods, while, it can take full advantage of them.

The series (9) is convergent for most cases. However, the convergent rate depends on the nonlinear operator $A(\nu)$.

3 Applications

In this section, we will apply the proposed method to some nonlinear equations.
Example 1:
We consider first KdV equation in the form:

$$u_t - 3(u^2)_x + u_{xxx} = 0 \tag{10}$$

with the following initial condition:

$$u(x,0) = 6x \tag{11}$$

To solve Eq. (10) using HPM, we consider:

$$u(x,t) = \nu(x,t) \tag{12}$$

$$\nu(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + p^3u_3(x,t) \tag{13}$$

According to Eq. (11), the initial approximations for $\nu(x,t)$ is as follows:

$$u_0(x,t) = \nu_0(x,t) = 6x \tag{14}$$

A homotopy for Eq. (10) can be constructed as follows:

$$(1-p)(\frac{\partial\nu}{\partial t} - \frac{\partial\nu_0}{\partial t}) + p(\frac{\partial\nu}{\partial t} - 6\nu\frac{\partial\nu}{\partial x} + \frac{\partial^2\nu}{\partial x^2}) = 0 \tag{15}$$

Substituting Eq. (14) into Eq. (15) yields:

$$(1-p)(\frac{\partial\nu}{\partial t}) + p(\frac{\partial\nu}{\partial t} - 6\nu\frac{\partial\nu}{\partial x} + \frac{\partial^2\nu}{\partial x^2}) = 0 \tag{16}$$

Substituting Eq. (13) into Eq. (16) and collecting the results up to p^3 and also having the following initial conditions:

$$u_i(x,0) \quad i = 1,2,3 \tag{17}$$

the results are as follows:

$$\begin{array}{l} u_1(x,t) = 216xt \\ u_2(x,t) = 7776xt^2 \\ u_3(x,t) = 279936xt^3 \end{array} \tag{18}$$

According to the HPM, we can conclude that:

$$\nu(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + p^3u_3(x,t) \tag{19}$$

Substituting Eqs. (14) and (18) into Eq. (45) yields:

$$\nu(x,t) = 6x + 216xt + 7776xt^2 + 279936xt^3 \tag{20}$$

Or:

$$\nu(x,t) = 6x(1 + 216t + 7776t^2 + 279936t^3) \tag{21}$$

Example 2:
We next consider the $K(2,2)$ equation in the form:

$$u_t + (u^2)_x + (u^2)_{xxx} = 0 \tag{22}$$

with the following initial condition:

$$u(x, 0) = x \tag{23}$$

To solve Eq. (22) using HPM, we consider:

$$u(x, t) = \nu(x, t) \tag{24}$$

$$\nu(x, t) = u_0(x, t) + pu_1(x, t) + p^2u_2(x, t) + p^3u_3(x, t) \tag{25}$$

According to Eq. (23), the initial approximations for $\nu(x, t)$ is as follows:

$$u_0(x, t) = \nu_0(x, t) = x \tag{26}$$

A homotopy for Eq. (22) can be constructed as follows:

$$(1 - p)(\frac{\partial \nu}{\partial t} - \frac{\partial \nu_0}{\partial t}) + p(\frac{\partial \nu}{\partial t} + 2\nu\frac{\partial \nu}{\partial x} + 6\frac{\partial \nu}{\partial x}\frac{\partial^2 \nu}{\partial x^2} + 2\nu\frac{\partial^3 \nu}{\partial x^3}) = 0 \tag{27}$$

Substituting Eq. (26) into Eq. (27) yields:

$$(1 - p)(\frac{\partial \nu}{\partial t}) + p(\frac{\partial \nu}{\partial t} + 2\nu\frac{\partial \nu}{\partial x} + 6\frac{\partial \nu}{\partial x}\frac{\partial^2 \nu}{\partial x^2} + 2\nu\frac{\partial^3 \nu}{\partial x^3}) = 0 \tag{28}$$

Substituting Eq. (25) into Eq. (28) and collecting the results up to p^3 and also considering the following initial conditions:

$$u_i(x, 0) = 0, \quad i = 1, 2, 3 \tag{29}$$

the results are as follows:

$$\begin{aligned} u_1(x, t) &= -2xt \\ u_2(x, t) &= 4xt^2 \\ u_3(x, t) &= -8xt^3 \end{aligned} \tag{30}$$

According to HPM, we can conclude that:

$$\nu(x, t) = u_0(x, t) + pu_1(x, t) + p^2u_2(x, t) + p^3u_3(x, t) \tag{31}$$

Substituting Eqs. (26) and (30) into Eq. (31) yields

$$\nu(x, t) = x - 2xt + 4xt^2 - 8xt^3 \tag{32}$$

Or:

$$\nu(x, t) = 6x(1 - 2t + 4t^2 - 8t^3) \tag{33}$$

Example3:
We next consider the Burgers equation as:

$$u_t - \frac{1}{2}(u^2)_x + u_{xxx} = 0 \tag{34}$$

with the initial condition of:

$$u(x, 0) = x \tag{35}$$

To solve Eq. (34) using HPM, we consider:

$$u(x,t) = \nu(x,t) \tag{36}$$

$$\nu(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + p^3u_3(x,t) \tag{37}$$

According to Eq. (35), the initial approximations for is as follows:

$$u_0(x,t) = \nu_0(x,t) = x \tag{38}$$

A homotopy for Eq. (34) can be constructed as follows:

$$(1-p)(\frac{\partial \nu}{\partial t} - \frac{\partial \nu_0}{\partial t}) + p(\frac{\partial \nu}{\partial t} + \nu\frac{\partial \nu}{\partial x} - \frac{\partial^2 \nu}{\partial x^2}) = 0 \tag{39}$$

Substituting Eq. (38) into Eq. (39) yields:

$$(1-p)(\frac{\partial \nu}{\partial t}) + p(\frac{\partial \nu}{\partial t} + \nu\frac{\partial \nu}{\partial x} - \frac{\partial^2 \nu}{\partial x^2}) = 0 \tag{40}$$

Substituting Eq. (37) into Eq. (40) and collecting the results up to and also having the following initial conditions:

$$u_i(x,0) \quad i = 1,2,3 \tag{41}$$

the results are as follows:

$$\begin{aligned} u_1(x,t) &= -xt \\ u_2(x,t) &= xt^2 \\ u_3(x,t) &= -xt^3 \end{aligned} \tag{42}$$

According to the HPM, we can conclude that:

$$\nu(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + p^3u_3(x,t) \tag{43}$$

Substituting Eqs. (38) and (42) into Eq. (43) yields

$$\nu(x,t) = x - xt + xt^2 - xt^3 \tag{44}$$

Or:

$$\nu(x,t) = x(1 - t + t^2 - t^3) \tag{45}$$

Example 4:
We next consider the cubic Boussinesq equation as:

$$u_{tt} - u_{xx} + 2(u^3)_{xx} + u_{xxx} = 0 \tag{46}$$

with the following initial condition:

$$u(x,0) = \frac{1}{x}, \quad u_i(x,0) = -\frac{1}{x^2} \tag{47}$$

To solve Eq. (46) using HPM, we consider:

$$u(x,t) = \nu(x,t) \tag{48}$$

$$\nu(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + p^3u_3(x,t) \tag{49}$$

According to Eq. (47), the initial approximations for is as follows:

$$u_0(x,t) = \nu_0(x,t) = \frac{1}{x} - \frac{t}{x^2} \tag{50}$$

A homotopy for Eq. (46) can be constructed as follows:

$$(1-p)(\frac{\partial^2 \nu}{\partial t^2} - \frac{\partial^2 \nu_0}{\partial t^2}) + p(\frac{\partial^2 \nu}{\partial t^2} - \frac{\partial^2 \nu}{\partial x^2} + 12\nu(\frac{\partial \nu}{\partial x})^2 + 6\nu\frac{\partial^2 \nu}{\partial x^2} - \frac{\partial^4 \nu}{\partial x^4}) = 0 \tag{51}$$

Substituting Eq. (50) into Eq. (51) yields:

$$(1-p)(\frac{\partial^2 \nu}{\partial t^2}) + p(\frac{\partial^2 \nu}{\partial t^2} - \frac{\partial^2 \nu}{\partial x^2} + 12\nu(\frac{\partial \nu}{\partial x})^2 + 6\nu\frac{\partial^2 \nu}{\partial x^2} - \frac{\partial^4 \nu}{\partial x^4}) = 0 \tag{52}$$

Substituting Eq. (49) into Eq. (52) and collecting the results up to and also considering the following initial conditions:

$$u_i(x,0) \quad i = 1,2,3 \tag{53}$$

the results are as follows:

$$\begin{array}{l} u_1(x,t) = \frac{t^2}{x^3} - \frac{t^3}{x^4} \\ u_1(x,t) = \frac{t^4}{x^5} - \frac{t^5}{x^6} \end{array} \tag{54}$$

According to the HPM, we can conclude that:

$$\nu(x,t) = u_0(x,t) + pu_1(x,t) + p^2u_2(x,t) + p^3u_3(x,t) \tag{55}$$

Substituting Eqs. (50) and (54) into Eq. (55) yields

$$\nu(x,t) = \frac{1}{x} - \frac{t}{x^2} + \frac{t^2}{x^3} - \frac{t^3}{x^4} + \frac{t^4}{x^5} \tag{56}$$

Or:

$$\nu(x,t) = \frac{1}{x}(1 - \frac{t}{x} + \frac{t^2}{x^2} - \frac{t^3}{x^4}) \tag{57}$$

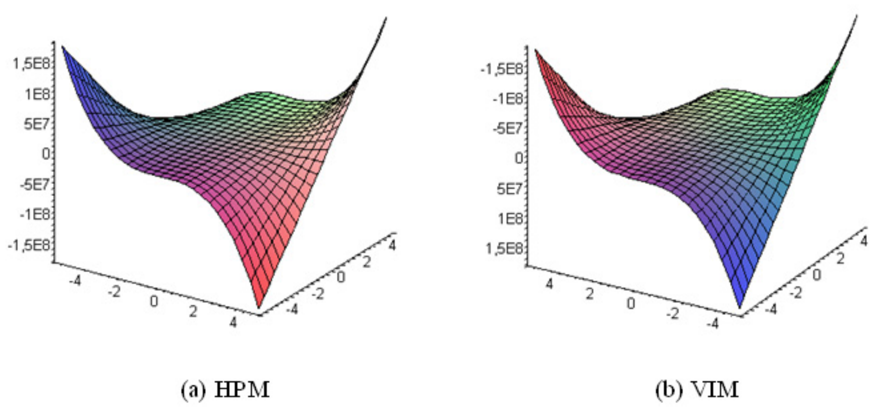
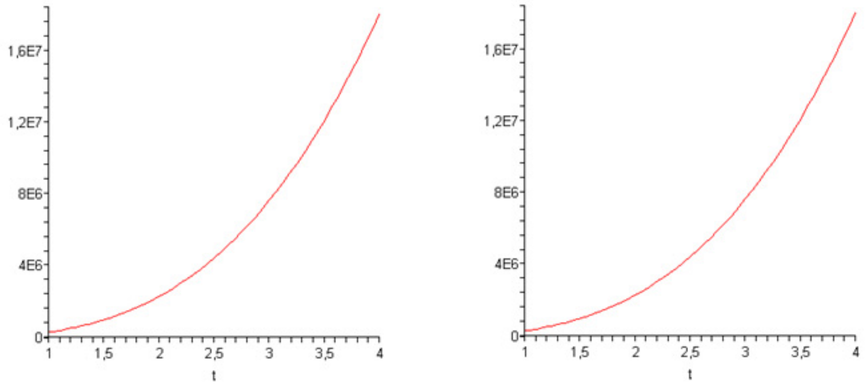
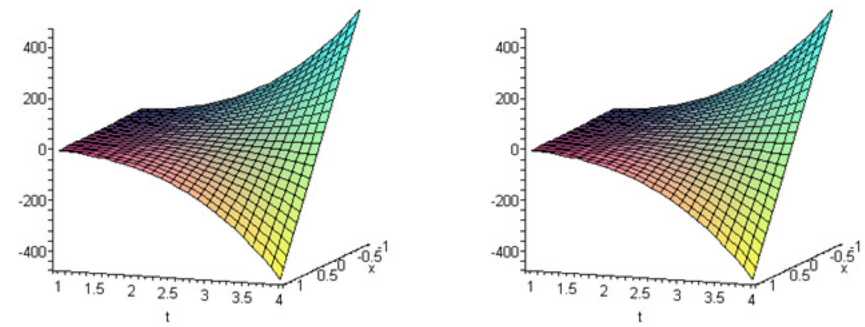


Fig. 1. The 3D behavior of $u(x,t)$ obtained by both HPM and VIM, Example 1.



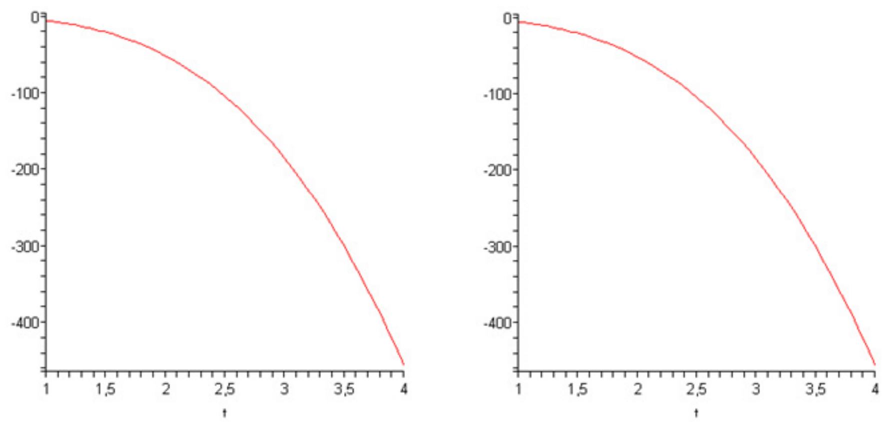
(a) HPM (b)VIM

Fig. 2. The 2D behavior of $u(x,t)$ obtained by both HPM and VIM at $x = 1$, Example 1.



(a)HPM (b) VIM

Fig. 3. The 3D behavior of $u(x,t)$ obtained by both HPM and VIM, Example 2.

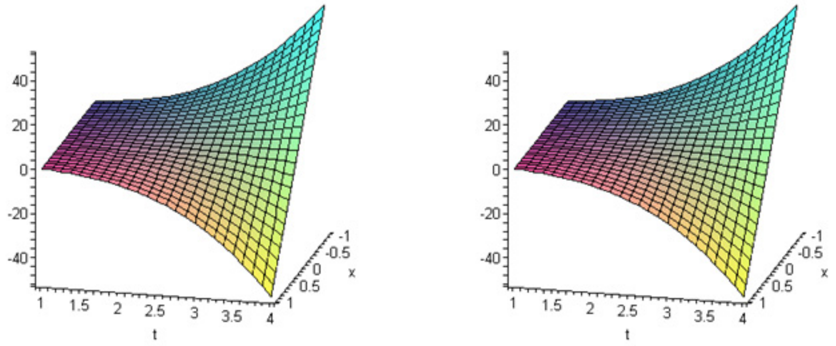


(a) HPM (b) VIM

Fig. 4. The 2D behavior of $u(x,t)$ obtained by both HPM and VIM at $x = 1$, Example 2.

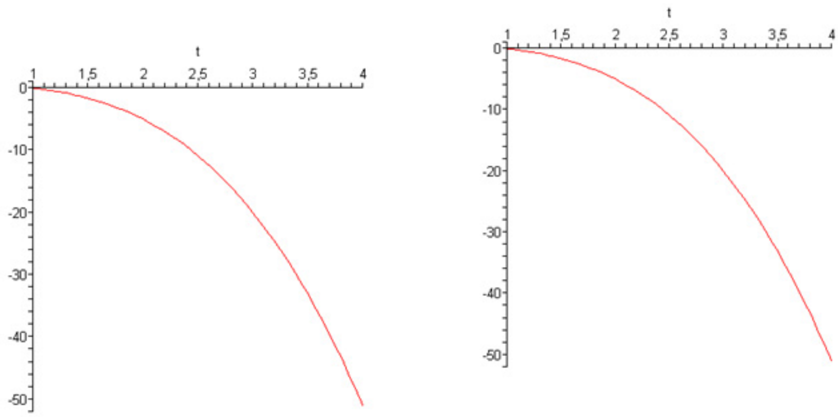
4 Conclusions

In this paper, homotopy perturbation method (HPM) has been successfully applied to find the solutions of some nonlinear partial differential equations. This method has been used for solving KdV, $K(2,2)$, Burgers and Cubic Boussinesq equations. The obtained solutions are shown graphically and are compared with those obtained by VIM[22]. In HPM, the obtained approximations are valid not only for small parameters, but also for larger ones. So HPM does not require small parameters in the equations; and therefore, the limitations of



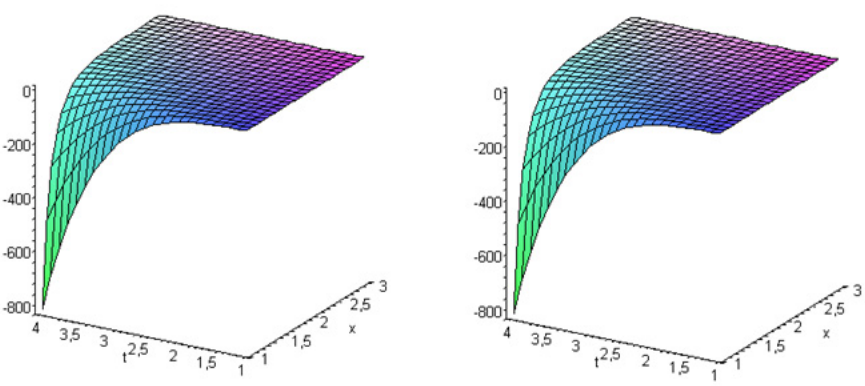
(a)HPM (b) VIM

Fig. 5. The 3D behavior of $u(x, t)$ obtained by both HPM and VIM, Example 3.



(a) HPM (b) VIM

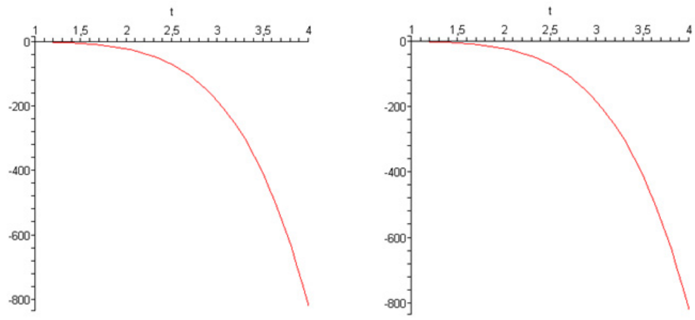
Fig. 6. The 2D behavior of $u(x, t)$ obtained by both HPM and VIM at $x = 1$, Example 3.



(a) HPM (b)VIM

Fig. 7. The 3D behavior of $u(x, t)$ obtained by both HPM and VIM, Example 4.

the traditional perturbation methods can be eliminated. It is proven here that HPM is a powerful mathematical tool for solving nonlinear differential systems that are widely applied in engineering and physics.



(a)HPM (b) VIM

Fig. 8. The 2D behavior of $u(x,t)$ obtained by both HPM and VIM at $x = 1$, Example 4.

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