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Similarity solutions to the power-law generalized Newtonian fluid

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Abstract

The authors of this paper study a second-order nonlinear parabolic equation with a background describing the motion of the power-law generalized Newtonian fluid. The existence of similarity solutions is obtained and the asymptotic behavior of the solution is studied.

Keywords: Non-Newtonian fluid; Similarity solutions; Nonlinear boundary value problems; Existence; Asymptotic behavior

1. Introduction

This work is concerned with the following flow problem: the mass of an incompressible fluid occupying the half-space $\eta > 0$ is set in motion by imparting a velocity $u(0,\tau) = U(\tau),\ \tau > 0$ on the boundary $\eta = 0$.

In dimensionless variables, the equation of motion of a power-law liquid reads

$$\begin{cases} \dfrac{\partial u}{\partial \tau} = \dfrac{\partial}{\partial \eta}\left(\left|\dfrac{\partial u}{\partial \eta}\right|^{N-1} \dfrac{\partial u}{\partial \eta}\right), & \eta > 0, \tau > 0, \\ u(\eta, 0) = 0, & \eta > 0, \\ u(0,\tau) = U_0 \tau^{\sigma}, & \tau > 0, U_0 > 0, \\ \lim\limits_{\eta\to+\infty} u(\eta,\tau) = 0, & \tau > 0. \end{cases} \tag{1.1}$$

The equation describes the motion of Newtonian fluid in the case $N = 1$ and the motion of non-Newtonian fluid in the case $N \neq 1$. Such problems have been studied extensively by many authors, interested readers may refer to [1–3] and the references therein.

The problem of finding similarity solutions can be formulated by setting $x = X\eta\tau^{-\rho}$ and writing

$$u = U_0\tau^{\sigma} y(x), \qquad \rho = \frac{1 + (N-1)\sigma}{N+1} > 0, \qquad X^{N+1} = \rho U_0^{1-N}.$$

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Noticing that $\tau^\sigma \sim \eta^\lambda x^{-\lambda}$ where $\lambda = \frac{\sigma}{\rho}$, we then obtain that the similarity solution of our problem satisfies

$$\begin{cases} \left(|y'(x)|^{N-1}y'(x)\right)' = \lambda y(x) - xy'(x), \quad x > 0, \\ y(0) = 1, \\ \lim_{x\to+\infty} x^{[-\lambda]_+} y(x) = 0, \end{cases} \tag{1.2}$$

where

$$[-\lambda]_+ = \begin{cases} -\lambda, & \text{if } -\lambda > 0 \\ 0, & \text{otherwise.} \end{cases}$$

The same problem of finding similarity solutions to the above equation was studied by the authors of [2]. They formulated the similarity solution with certain boundary conditions and obtained some numerical results. It is unfortunate that the boundary conditions mentioned there was not the equivalent form of the original problem.

In this paper, we will reformulate the problem of finding a similarity solution to the problem mentioned above and show that there exists a similarity solution to the problem. Furthermore, we will study the asymptotic behavior of the solutions.

The main result in this paper is

Theorem 1. *The problem (1.2) has a nonnegative solution for any $N > 0$ and $\lambda \geq -1$. Also, the solution $y(x)$ has the following asymptotic properties:*

(1) *$y(x)$ has compact support for $N > 1$, that is, there exists a positive number $x_0 < +\infty$ such that $y(x) \equiv 0$ for all $x \in [x_0, +\infty)$;*
(2) *$y(x) = O(\mathrm{e}^{-x^2/2})$ for $N = 1$;*
(3) *$y(x) = O(x^{-\frac{1+N}{1-N}})$ for $0 < N < 1$.*

Remark. It is clear that the conditions $\lambda \geq -1$ and $\rho > 0$ place some restrictions on the parameter σ. It is easy to verify that the allowable range for σ in our theorem is $\sigma > -\frac{1}{2N}$ for $N \geq 1$ and $-\frac{1}{2N} < \sigma < \frac{1}{1-N}$ for $0 < N < 1$.

The outline of this paper is the following. In the next section, we first formally reduce the problem (1.2) to that of a initial value problem of an integro-differential equation, then we will study the reduced problem in Section 3 by utilizing a perturbation technique and the Schauder's fixed point theorem, the proof of the main result is given in the last section of this paper.

2. Reduction of the problem

Assume that $y(x)$ is a nonnegative solution of the boundary value problem (1.2) with the property that $y'(x) < 0$, then the function $t = y(x)$ has an inverse function which we denote as $x = w(t)$. It is obvious that $w(1) = 0$ since $y(0) = 1$ and $w(0) = +\infty$ since $\lim_{x\to+\infty} y(x) = 0$. Since for any $t \in (0, 1]$, $t = y(w(t))$, we obtain $y'(w(t)) = 1/w'(t) < 0$. Plugging $x = w(t)$ into (1.2), we get

$$\left(\frac{-1}{|w'(t)|^N}\right)' \cdot \frac{1}{w'(t)} = \left(-|y'(w(t))|^N\right)'_x = \lambda t - \frac{w(t)}{w'(t)}.$$

Therefore

$$\left(\frac{-1}{|w'(t)|^N}\right)' = \lambda t w'(t) - w(t). \tag{2.1}$$

Set $v(t) = 1/|w'(t)|^N$, then

$$w'(t) = -v^{-1/N}(t), \quad \text{and} \quad w(t) = \int_t^1 v^{-1/N}(s)\mathrm{d}s.$$

Hence by (2.1),

$$v'(t) = \lambda t v^{-1/N}(t) + \int_t^1 v^{-1/N}(s)\mathrm{d}s. \tag{2.2}$$

Then $\lim_{x\to+\infty} y(x) = 0$ and the monotonically decreasing property of $y(x)$ imply that

$$\lim_{x\to\infty} y'(x) = 0,$$

and

$$v(0) = \lim_{t\to 0} \frac{1}{|w'(t)|^N} = \lim_{x\to\infty} |y'(x)|^N = 0. \tag{2.3}$$

We then reduce the problem (1.2) to that of (2.2) (2.3) which is an initial value problem of the integro-differential equation. We have

Theorem 2. *The problem (2.2) (2.3) has a unique solution $v(t)$ for $N > 0$ and $\lambda \geq -1$ which is positive in $(0, 1)$ and can be written as*

$$v(t) = \begin{cases} t f(t), & N > 1, \\ t\sqrt{-\ln t}\, g(t), & N = 1, \\ t^{\frac{2N}{1+N}} h(t), & 0 < N < 1, \end{cases}$$

where $f(t)$, $g(t)$ and $h(t)$ are continuous functions defined on $[0, 1]$ which are positive in $(0, 1)$.

3. Study of the reduced problem

To prove the existence of solutions to the problem (2.2) (2.3), we first consider the following approximation problem:

$$\begin{cases} v'(t) = \lambda t v^{-1/N}(t) + \displaystyle\int_t^1 v^{-1/N}(s)\,\mathrm{d}s, & 0 < t < 1, \\ v(0) = h, \quad (h > 0). \end{cases} \tag{3.1}$$

We have the following lemmas.

Lemma 1. *The problem (3.1) has a positive solution for any $h > 0$, $N > 0$ and $\lambda \geq -1$.*

Proof. We define a mapping $\varphi : \Omega \to \Omega$ as

$$\varphi v(t) = h + (\lambda + 1)\int_0^t s v^{-1/N}(s)\,\mathrm{d}s + t\int_t^1 v^{-1/N}(s)\,\mathrm{d}s,$$

where

$$\Omega = \{v(t) \in C[0, 1], v(t) \geq h > 0\}.$$

Then, it is clear that $\varphi v(t) \in \Omega$ for any $v(t) \in \Omega$. Also, Ω is obviously a convex subset of the Banach space $C[0, 1]$. If $\{v_n(t)\} \subset \Omega$ is a sequence of functions converging to $v_0(t)$ uniformly, the Lebesgue Dominated Convergence Theorem implies that

$$\begin{aligned} \lim_{n\to+\infty} \varphi v_n(t) &= \lim_{n\to\infty}\left[h + (\lambda+1)\int_0^t s v_n^{-1/N}(s)\mathrm{d}s + t\int_t^1 v_n^{-1/N}(s)\mathrm{d}s\right] \\ &= h + (\lambda+1)\int_0^t s v_0^{-1/N}(s)\mathrm{d}s + t\int_t^1 v_0^{-1/N}(s)\mathrm{d}s \\ &= \varphi v_0(t). \end{aligned}$$

This shows that φ is a continuous mapping over Ω.

For any $v(t) \in \Omega$,

$$h \le \varphi v(t) \le h + (\lambda+1)h^{-1/N}\int_0^t s\,\mathrm{d}s + h^{-1/N}t(1-t)$$
$$\le h + h^{-1/N}\left[\frac{1}{2}(\lambda+1)+\frac{1}{4}\right].$$

Therefore, $\{\varphi v(t), v(t) \in \Omega\}$ is uniformly bounded. On the other hand,

$$\left|[\varphi v(t)]'\right| \le \left|\lambda t v^{-1/N}(t)\right| + \int_t^1 v^{-1/N}(s)\,\mathrm{d}s$$
$$\le |\lambda| h^{-1/N} + h^{-1/N}$$
$$= (|\lambda|+1)h^{-1/N}.$$

Hence, for any $t_1, t_2 \in [0,1]$ and $v(t) \in \Omega$

$$|\varphi v(t_1) - \varphi v(t_2)| \le \left|[\varphi v(t)]'_{t=\xi}\right| |t_1 - t_2|$$
$$\le (|\lambda|+1)h^{-1/N}|t_1 - t_2|,$$

where ξ is between t_1 and t_2. This inequality shows that $\{\varphi v(t), v(t) \in \Omega\}$ is a family of equicontinuous functions. The Arzela–Ascoli Theorem implies that φ is a compact operator on Ω. By the Schauder fixed point principle, there is a function $v(t;h) \in \Omega$ such that

$$v(t;h) = h + (\lambda+1)\int_0^t s v^{-1/N}(s;h)\mathrm{d}s + t\int_t^1 v^{-1/N}(s;h)\mathrm{d}s, \quad \text{for } 0 \le t \le 1. \tag{3.2}$$

It is easy to verify that $v(t;h)$ is a solution of (3.1). □

Lemma 2. *For any $h > 0$, the solution of (3.1) is unique.*

Proof. Let $v_1(t;h)$ and $v_2(t;h)$ be two solutions of (3.1).

We first show that $v_1(1;h) = v_2(1;h)$. If this were not true, without loss of generality, we may assume that $v_1(1;h) > v_2(1;h)$. By the continuity of v_1 and v_2, we know that there must be a point $t_0 \in [0,1)$ such that $v_1(t_0;h) = v_2(t_0;h)$ and $v_1(t;h) > v_2(t;h)$ for all $t \in (t_0, 1]$. This leads to $v_1'(t_0;h) \ge v_2'(t_0;h)$. On the other hand, since v_1 and v_2 are solutions of (3.1), we have

$$v_1'(t_0;h) = \lambda t_0 v_1^{-1/N}(t_0;h) + \int_{t_0}^1 v_1^{-1/N}(s;h)\mathrm{d}s$$
$$< \lambda t_0 v_2^{-1/N}(t_0;h) + \int_{t_0}^1 v_2^{-1/N}(s;h)\mathrm{d}s$$
$$= v_2'(t_0;h),$$

a contradiction. Now, we set $v_1(1;h) = v_2(1;h) = a \ge h > 0$, then by (3.1),

$$v_1'(1;h) = \lambda a^{-1/N} = v_2'(1;h).$$

It is clear that $v_1(t;h)$ and $v_2(t;h)$ are solutions of the following initial value problem

$$\begin{cases} v''(t) = (\lambda-1)v^{-1/N}(t) - \dfrac{\lambda}{N} t v^{-1-1/N}(t)v'(t), & 0 < t < 1 \\ v(1) = a, \quad v'(1) = \lambda a^{-1/N}, \end{cases}$$

since the right-hand side of this equation is continuous for $t \in [0,1]$ and is continuously differentiable for $t > 0$. The uniqueness of solutions to the initial value problem implies that $v_1(t;h) \equiv v_2(t;h)$. □

Lemma 3. *Let $v(t;h)$ be the solution of the problem (3.1). Then, for any $h_1 > h_2 > 0$, we have*

$$0 \le v(t;h_1) - v(t;h_2) \le h_1 - h_2, \quad 0 \le t \le 1.$$

Proof. The inequality $v(t; h_1) \geq v(t; h_2)$ can be proven in a similar way as that for Lemma 2.

We now prove $v(t; h_1) - v(t; h_2) \leq h_1 - h_2$.

By (3.2),

$$v(t; h_1) = h_1 + (\lambda + 1)\int_0^t s v^{-1/N}(s; h_1)\mathrm{d}s + t\int_t^1 v^{-1/N} v(s; h_1)\mathrm{d}s$$
$$v(t; h_2) = h_2 + (\lambda + 1)\int_0^t s v^{-1/N}(s; h_2)\mathrm{d}s + t\int_t^1 v^{-1/N} v(s; h_2)\mathrm{d}s.$$

Then for $0 \leq t \leq 1$

$$v(t; h_1) - v(t; h_2) = h_1 - h_2 + (\lambda + 1)\int_0^t s\left[v^{-1/N}(s; h_1) - v^{-1/N}(s; h_2)\right]\mathrm{d}s + t\int_t^1 \left[v^{-1/N}(s; h_1) - v^{-1/N}(s; h_2)\right]\mathrm{d}s.$$

The inequality $v(t; h_1) \geq v(t; h_2)$ implies that $v^{-1/N}(t; h_1) \leq v^{-1/N}(t; h_2)$. Therefore, the two integrals on the right-hand side of the above equality are all nonpositive and hence $v(t; h_1) - v(t; h_2) \leq h_1 - h_2$. The lemma is proven. □

From Lemma 3, We know that as h approaches 0 monotonically, we can find a function $v(t) \geq 0$ such that $\lim_{h\downarrow 0} v(t; h) = v(t)$ uniformly for $t \in [0, 1]$. Taking limit in (3.2) and using the monotone convergence theorem, we get

$$v(t) = (\lambda + 1)\int_0^t s v^{-1/N}(s)\,\mathrm{d}s + t\int_t^1 v^{-1/N}(s)\,\mathrm{d}s. \tag{3.3}$$

Differentiating (3.3) with respect to t, we have

$$v'(t) = \lambda t v^{-1/N}(t) + \int_t^1 v^{-1/N}(s)\,\mathrm{d}s,$$

and

$$v(0) = \lim_{h\downarrow 0} v(0, h) = \lim_{h\downarrow 0} h = 0.$$

It is clear that $v(t)$ is a solution of the problem (2.2) (2.3). Hence, we have proven

Lemma 4. *The problem (2.2) (2.3) has a solution for each $N > 0$ and $\lambda \geq -1$.*

Lemma 5. *Let $v(t)$ be the solution of (2.2) (2.3), then $v(t) > 0$ for $t \in (0, 1)$.*

Proof. If $v(t_0) = 0$ for some $t_0 \in (0, 1)$, then by (3.3),

$$0 = v(t_0) = (\lambda + 1)\int_0^{t_0} s v^{-1/N}(s)\,\mathrm{d}s + t_0\int_{t_0}^1 v^{-1/N}(s)\,\mathrm{d}s.$$

The nonnegativity of $v(t)$ implies that $v^{-1/N}(t) \equiv 0$ for $t \in (t_0, 1)$, that is $v(t) = \infty$ for $t \in (t_0, 1)$ which is impossible since $v(t)$ is a continuous function on $[0, 1]$. The proof is complete. □

Lemma 6. *Let $v(t)$ be the solution of (2.2) (2.3). Then for any $\lambda \geq -1$, $v(t)$ can be expressed as*

$$v(t) = \begin{cases} t f(t), & N > 1, \\ t\sqrt{-\ln t}\, g(t), & N = 1, \\ t^{\frac{2N}{1+N}} h(t), & 0 < N < 1, \end{cases}$$

where $f(t)$, $g(t)$ and $h(t)$ are continuous functions defined on $[0, 1]$ *which are positive on* $[0, 1)$.

Proof. For $N > 1$, set $v(t) = tf(t)$. Then, it is easily seen from (3.3) that $f(t)$ is a continuous function on (0, 1] which is positive in (0, 1). We only have to show that $f(0) = \lim_{t\to 0} f(t) > 0$. Using (3.3), we get, for $0 \le t \le 1$,

$$tf(t) = (\lambda+1)\int_0^t s^{1-1/N} f^{-1/N}(s)\mathrm{d}s + t\int_t^1 s^{-1/N} f^{-1/N}(s)\mathrm{d}s.$$

Therefore

$$f(t) = (\lambda+1)\frac{1}{t}\int_0^t s^{1-1/N} f^{-1/N}(s)\mathrm{d}s + \int_t^1 s^{-1/N} f^{-1/N}(s)\mathrm{d}s.$$

In particular, this implies that for $0 < t \le 1/2$,

$$f(t) > \int_{1/2}^1 s^{-1/N} f^{-1/N}(s)\mathrm{d}s > 0.$$

The claim of $f(t) > 0$ is proved. Furthermore, we have

$$\begin{aligned} f(0) &= \lim_{t\to 0} f(t) \\ &= \lim_{t\to 0}\left[(\lambda+1)\frac{1}{t}\int_0^t s^{1-1/N} f^{-1/N}(s)\mathrm{d}s + \int_t^1 s^{-1/N} f^{-1/N}(s)\mathrm{d}s\right] \\ &= \lim_{t\to 0}\left[(\lambda+1)t^{1-1/N} f^{-1/N}(t) + \int_t^1 s^{-1/N} f^{-1/N}(s)\mathrm{d}s\right] \\ &= \int_0^1 s^{-1/N} f^{-1/N}(s)\mathrm{d}s. \end{aligned}$$

In the case of $N = 1$, set $v(t) = t\sqrt{-\ln t}\, g(t)$. We need to show that $g(0) > 0$.
By (3.3)

$$t\sqrt{-\ln t}\, g(t) = (\lambda+1)\int_0^t \frac{\mathrm{d}s}{\sqrt{-\ln s}\, g(s)} + t\int_t^1 \frac{\mathrm{d}s}{s\sqrt{-\ln s}\, g(s)}.$$

That is

$$g(t) = \frac{(\lambda+1)}{t\sqrt{-\ln t}}\int_0^t \frac{\mathrm{d}s}{\sqrt{-\ln s}\, g(s)} + \frac{1}{\sqrt{-\ln t}}\int_t^1 \frac{\mathrm{d}s}{s\sqrt{-\ln s}\, g(s)}.$$

Therefore

$$\begin{aligned} g(0) &= \lim_{t\to 0}\left[\frac{(\lambda+1)}{t\sqrt{-\ln t}}\int_0^t \frac{\mathrm{d}s}{\sqrt{-\ln s}\, g(s)} + \frac{1}{\sqrt{-\ln t}}\int_t^1 \frac{\mathrm{d}s}{s\sqrt{-\ln s}\, g(s)}\right] \\ &= \lim_{t\to 0}\left[\frac{2(\lambda+1)}{(2(-\ln t)-1)g(t)} + \frac{2}{g(t)}\right] = \frac{2}{g(0)}. \end{aligned}$$

This proves that $g(0) = \sqrt{2} > 0$.

For $0 < N < 1$, set $v(t) = t^{\frac{2N}{1+N}} h(t)$. By (3.3) and a similar argument as that above, we can show that

$$h(0) = \left[\frac{1+N}{2N}(\lambda+1) + \frac{N+1}{1-N}\right]^{\frac{N}{N+1}} > 0.$$

The details are omitted. The proof is complete. □

In the following, we study the uniqueness of solutions to the problem (2.2) (2.3). We have

Lemma 7. *The problem (2.2) (2.3) has at most one solution for $N > 0$ and $\lambda \ge -1$.*

Proof. We first consider the case $\lambda > -1$.

Let $v_1(t)$ and $v_2(t)$ be two solutions of the problem (2.2) (2.3). We claim that $v_1(1) = v_2(1)$. Since if this were not true, then without loss of generality, we may assume that $v_1(1) > v_2(1)$. By the continuity of the functions, we can find a point $t_0 \in [0, 1)$ such that $v_1(t_0) = v_2(t_0)$ and $v_1(t) > v_2(t)$ for all $t \in (t_0, 1]$.

If $t_0 > 0$, then $v_1'(t_0) \geq v_2'(t_0)$, but (2.2) implies that

$$\begin{aligned} v_1'(t_0) &= \lambda t_0 v_1^{-1/N}(t_0) + \int_{t_0}^{1} v_1^{-1/N}(s)\,\mathrm{d}s \\ &< \lambda t_0 v_2^{-1/N}(t_0) + \int_{t_0}^{1} v_2^{-1/N}(2)\,\mathrm{d}s = v_2'(t_0), \end{aligned}$$

which is impossible.

If $t_0 = 0$, then $v_1(t) > v_2(t)$ for all $t \in (0, 1]$. Thus

$$\begin{aligned} v_1(1) &= (\lambda + 1) \int_{0}^{1} s v_1^{-1/N}(s)\,\mathrm{d}s \\ &< (\lambda + 1) \int_{0}^{1} s v_2^{-1/N}(2)\,\mathrm{d}s = v_2(1), \end{aligned}$$

this contradicts to $v_1(1) > v_2(1)$. The claim is proven. Then the same argument as that in the proof of Lemma 2 shows that $v_1(t) \equiv v_2(t)$.

Now, we study the case $\lambda = -1$.

If $N > 1$, Lemma 6 implies that $v(t) = t f(t)$ for some continuous function f. By (3.3), we get

$$f(t) = \int_{t}^{1} s^{-1/N} f^{-1/N}(s)\mathrm{d}s, \quad 0 \leq t \leq 1. \tag{3.4}$$

Hence $f(1) = 0$. Set $p(t) = f^{\frac{1+N}{N}}(t)$. Using the fact that $f'(t) = -t^{-1/N} f^{-1/N}(t)$ for $0 < t < 1$, we know that $p(t)$ is the solution of the problem

$$\begin{cases} p'(t) = -\dfrac{1+N}{N} t^{-1/N}, & 0 < t < 1 \\ p(1) = 0. \end{cases} \tag{3.5}$$

The existence and uniqueness results to the initial value problem imply that

$$p(t) = \frac{N+1}{N-1}\left(1 - t^{\frac{N-1}{N}}\right), \quad 0 \leq t \leq 1.$$

Then the only function which satisfies (3.4) is

$$f(t) = \left[\frac{N+1}{N-1}\left(1 - t^{\frac{N-1}{N}}\right)\right]^{N/(N+1)}, \quad 0 \leq t \leq 1.$$

Therefore, the problem (2.2) (2.3) has a unique solution $v(t)$ for $N > 1$ where

$$v(t) = t\left[\frac{N+1}{N-1}\left(1 - t^{\frac{N-1}{N}}\right)\right]^{N/(N+1)}, \quad 0 \leq t \leq 1.$$

For $N = 1$, let $v(t)$ be the solution of the problem (2.2) (2.3), we have by (3.3),

$$v(t) = t \int_{t}^{1} \frac{1}{v(s)}\,\mathrm{d}s, \quad 0 \leq t \leq 1.$$

Then,

$$v'(t) = \frac{v(t)}{t} - \frac{t}{v(t)}, \quad 0 < t < 1. \tag{3.6}$$

Set $v(t) = tg(t)$, then

$$tg'(t) = -\frac{1}{g(t)}, \quad 0 < t < 1. \tag{3.7}$$

If we write $q(t) = g^2(t)$, then by (3.7), $q(t)$ satisfies

$$\begin{cases} q'(t) = -\dfrac{2}{t}, & 0 < t < 1, \\ q(1) = 0. \end{cases} \tag{3.8}$$

The uniqueness of solutions to the initial value problem implies that $q(t) = -2\ln t$. Hence, the function which makes (3.7) hold is unique, that is $g(t) = \sqrt{-2\ln t}$. Therefore, the solution to the problem (2.2) (2.3) is unique when $N = 1$ and

$$v(t) = \sqrt{2}t\sqrt{-\ln t}.$$

In the case $0 < N < 1$, Lemma 6 implies that $v(t) = t^{2N/(1+N)}h(t)$. We get from (3.3) that

$$h(t) = t^{\frac{1-N}{1+N}} \int_t^1 s^{-\frac{2}{1+N}} h^{-1/N}(s)\mathrm{d}s, \quad 0 \le t \le 1.$$

Hence

$$h'(t) = t^{-1}\left[\frac{1-N}{1+N}h(t) - h^{-1/N}(t)\right], \quad 0 \le t \le 1. \tag{3.9}$$

Set $R(t) = h^{(1+N)/N}(t)$. Then it satisfies

$$\begin{cases} R'(t) = \dfrac{1+N}{N}\left[\dfrac{1-N}{1+N}R(t) - 1\right]t^{-1}, & 0 < t < 1. \\ R(1) = 0. \end{cases}$$

Using the uniqueness result to the initial value problem again, we know that

$$R(t) = \frac{1+N}{1-N}\left(1 - t^{(1-N)/N}\right).$$

Therefore the function $h(t)$ which satisfies (3.9) is unique and

$$h(t) = \left[\frac{1+N}{1-N}\left(1 - t^{(1-N)/N}\right)\right]^{N/(1+N)}.$$

This proves that for $0 < N < 1$, the solution of (2.2) (2.3) is unique and furthermore

$$v(t) = t^{2N/(1+N)}\left[\frac{1+N}{1-N}\left(1 - t^{(1-N)/N}\right)\right]^{N/(1+N)}.$$

The proof of Lemma 7 is complete. □

The conclusion of Theorem 2 follows from the above lemmas.

4. Proof of the main results

We have proved in the previous section that the problem (2.2) (2.3) has a unique solution for $\lambda \ge -1$ and $N > 0$. The solution is closely related to the solution of (2.1). We will prove Theorem 1 in this section by using the results in Theorem 2.

Set

$$y(x) = \begin{cases} \left(1 + \dfrac{1-N}{1+N}x^{(1+N)/N}\right)^{-N/(1-N)}, & 0 < N < 1, \\ \exp(-x^2/2), & N = 1, \\ \left[1 - \dfrac{N-1}{N+1}x^{(1+N)/N}\right]_+^{N/(N-1)}, & N > 1. \end{cases}$$

Then, it is easily seen that the function $y(x)$ is a solution of the problem (2.1) for $\lambda = -1$. Therefore, we only have to prove Theorem 1 for $\lambda > -1$ and verify the asymptotic behavior of the solutions.

Let $v(t)$ be the solution of (2.2) (2.3). We know that $v(t) > 0$ for $t \in (0, 1)$. We define

$$w(t) = \int_t^1 v^{-1/N}(s)\,\mathrm{d}s, \quad 0 < t \le 1. \tag{4.1}$$

Then

$$w'(t) = -v^{-1/N}(t) < 0, \quad \text{for } 0 < t < 1. \tag{4.2}$$

It is clear that $w(t) = x$ is a strictly decreasing function in $(0, 1]$ and hence the inverse function exists. We denote the inverse function as $t = y(x)$. Then $y(0) = 1$. Set $x_0 = w(0) = \int_0^1 v^{-1/N}(s)\,\mathrm{d}s$, then $x_0 \le +\infty$ and $y(x_0) = 0$. The function $t = y(x)$ is a continuous function defined on $[0, x_0)$. If $x_0 < +\infty$, we define $y(x) = 0$ for $x \in [x_0, +\infty)$. We will prove that $y(x)$ is a solution of the boundary value problem (2.1) with the asymptotic behavior as stated in the theorem.

The asymptotic behavior of $y(x)$: For $N > 1$, Lemma 6 says that $v(t) = tf(t)$ for $0 \le t \le 1$. Thus

$$x_0 = w(0) = \int_0^1 s^{-1/N} f^{-1/N}(s)\mathrm{d}s.$$

It is clear that $x_0 < +\infty$. This implies that $t = y(x)$ is a continuous function defined on $[0, \infty)$ with a compact support.

For $N = 1$, Lemma 6 and its proof imply that $v(t) = t\sqrt{-\ln t}\,g(t)$ for $0 \le t \le 1$ and $g(0) = \sqrt{2}$. By the continuity of $g(t)$, we know that for any $e > 0$, there exists a $\delta > 0$ such that for $0 < t < \delta$, $\sqrt{2} - e < g(t) < \sqrt{2} + e$. Then

$$\begin{aligned} w(t) &= \int_t^1 \frac{\mathrm{d}s}{s\sqrt{-\ln s}\,g(s)} \\ &= \int_t^\delta \frac{\mathrm{d}s}{s\sqrt{-\ln s}\,g(s)} + \int_\delta^1 \frac{\mathrm{d}s}{s\sqrt{-\ln s}\,g(s)} \\ &= A + \int_t^\delta \frac{\mathrm{d}s}{s\sqrt{-\ln s}\,g(s)}, \end{aligned}$$

where the constant

$$A = \int_\delta^1 \frac{\mathrm{d}s}{s\sqrt{-\ln s}\,g(s)}.$$

For $0 < t < \delta$

$$A + \frac{1}{\sqrt{2}+e}\int_t^\delta \frac{\mathrm{d}s}{s\sqrt{-\ln s}} < w(t) < A + \frac{1}{\sqrt{2}-e}\int_t^\delta \frac{\mathrm{d}s}{s\sqrt{-\ln s}}.$$

That is

$$A_1 + \frac{2\sqrt{-\ln t}}{\sqrt{2}+e} < w(t) < A_2 + \frac{2\sqrt{-\ln t}}{\sqrt{2}-e}$$

with

$$A_1 = A - \frac{2\sqrt{-\ln\delta}}{\sqrt{2}+e} \quad \text{and} \quad A_2 = A - \frac{2\sqrt{-\ln\delta}}{\sqrt{2}-e}.$$

It is easy to conclude from this inequality that

$$\lim_{t\to 0} \frac{w(t)}{\sqrt{2}\sqrt{-\ln t}} = 1.$$

Therefore, as $t \to 0$

$$w(t) \sim \sqrt{2}\sqrt{-\ln t}$$

and hence

$$t \sim \mathrm{e}^{-w^2(t)/2}.$$

This implies that $y(x) = O(\mathrm{e}^{-x^2/2})$ as $x \to +\infty$.

In the case of $0 < N < 1$, by Lemma 6, $v(t) = t^{2N/(1+N)}h(t)$. Then

$$x = w(t) = \int_t^1 s^{-2/(1+N)} h^{-1/N}(s)\mathrm{d}s.$$

Therefore,

$$\begin{aligned}\lim_{x\to\infty} \frac{y(x)}{x^{-(1+N)/(1-N)}} &= \lim_{t\to 0} \frac{t}{\left[\int_t^1 s^{-2/(1+N)} h^{-1/N}(s)\mathrm{d}s\right]^{-(1+N)/(1-N)}} \\ &= \lim_{t\to 0} \left[\frac{t^{-(1-N)/(1+N)}}{\int_t^1 s^{-2/(1+N)} h^{-1/N}(s)\mathrm{d}s}\right]^{-(1+N)/(1-N)} \\ &= \left[\frac{1-N}{1+N} h^{1/N}(0)\right]^{-(1+N)/(1-N)} > 0,\end{aligned}$$

which shows that $y(x) = O(x^{-(1+N)/(1-N)})$.

From the above, it can be seen that the asymptotic behavior of the function $y(x)$ is exactly what we have in Theorem 1.

The proof that $y(x)$ is a solution of (2.1): We first prove that the function $y(x)$ satisfies the equation.

Inserting (4.1) and (4.2) into the Eq. (2.2), we get

$$v'(t) = -\lambda t w'(t) + w(t).$$

Replacing t with $y(x)$ in this equation, we have

$$v'(y(x)) = -\lambda y(x) w'(y(x)) + w(y(x)), \quad 0 \le x < x_0.$$

Using the fact that

$$x = w(y(x)), \quad \text{and} \quad y'(x) = \frac{1}{w'(y(x))} < 0,$$

we obtain

$$v'(y(x)) = -\lambda y(x) \cdot \frac{1}{y'(x)} + x, \quad 0 \le x < x_0. \tag{4.3}$$

On the other hand, the equation

$$v(y(x)) = \frac{1}{|w'(y(x))|^N} = \left|y'(x)\right|^N$$

and

$$v'(y(x))y'(x) = \left(\left|y'(x)\right|^N\right)'$$

together with (4.3) imply that

$$\left(\left|y'(x)\right|^N\right)' = -\lambda y(x) + x y'(x), \quad 0 \le x < x_0,$$

that is

$$\left(\left|y'(x)\right|^{N-1} y'(x)\right)' = \lambda y(x) - x y'(x), \quad 0 \leq x < x_0.$$

By our definition, if $x_0 < \infty$, $y(x) \equiv 0$ for $x \in [x_0, +\infty)$ which clearly satisfies the equation.

$y(0) = 1$ was proven while constructing the function $y(x)$.

The asymptotic behavior of the function $y(x)$ implies that

$$\lim_{x \to +\infty} x^{[-\lambda]_+} y(x) = 0.$$

The proof of Theorem 1 is complete.

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