Journal of Lie Theory
Volume **15** (2005) 89–104

Generalized Nonlinear Superposition Principles for Polynomial Planar Vector Fields*

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Communicated by P. Olver

Abstract. In this paper we study some aspects of the integrability problem for polynomial vector fields $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$. We analyze the possible existence of first integrals of the form $I(x,y) = (y - g_1(x))^{\alpha_1}(y - g_2(x))^{\alpha_2} \cdots (y - g_\ell(x))^{\alpha_\ell} h(x)$, where $g_1(x), \ldots, g_\ell(x)$ are unknown particular solutions of $dy/dx = Q(x,y)/P(x,y)$, α_i are unknown constants and $h(x)$ is an unknown function. We show that for certain systems some of the particular solutions remain arbitrary and the other ones are explicitly determined or are functionally related to the arbitrary particular solutions. We obtain in this way a nonlinear superposition principle that generalize the classical nonlinear superposition principle of the Lie theory. In general, the first integral contains some arbitrary solutions of the system but also quadratures of these solutions and an explicit dependence on the independent variable. In the case when all the particular solutions are determined, they are algebraic functions and our algorithm gives an alternative method for determining such type of solutions.

Keywords: nonlinear differential equations, polynomial planar vector fields, nonlinear superposition principle, Darboux first integral, Liouvillian first integral.
AMS classification: Primary 34C05; Secondary 34C14, 22E05.

1. Introduction

We consider in this paper two–dimensional systems

$$\frac{dx}{dt} = \dot{x} = P(x,y) \ , \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \ , \tag{1}$$

in which $P, Q \in \mathbb{R}[x,y]$ are polynomials in the real variables x and y and the independent variable (the time) t is real. Throughout this paper we will denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of system (1). Obviously, we can also express system (1) as the differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)} \ . \tag{2}$$

* The first and third authors are partially supported by a MCYT grant number BFM 2002-04236-C02-01. The third author is partially supported by DURSI of Government of Catalonia's Acció Integrada ACI2002-24.

ISSN 0949–5932 / \$2.50

We associate to system (1) the vector field $\mathcal{X}$ defined by $\mathcal{X} = P\partial/\partial x + Q\partial/\partial y$.

System (1) is integrable on an open set U of $\mathbb{R}^2$ if there exists a nonconstant C^1 function $H : U \to \mathbb{R}$, called a first integral of the system on U, which is constant on all solution curves $(x(t), y(t))$ in U; i.e. $H(x(t), y(t)) = c$, where c is a constant, for all values of t where the solution is defined. Clearly H is a first integral of the system (1) on U if and only if $\mathcal{X}H \equiv 0$ on U.

The search of first integrals is a classical tool in the classification of all trajectories of a dynamical system. The local existence of first integrals in a neighborhood of a regular point is a consequence of some classical theorems of differential calculus. The interesting point is the search of global first integrals. For a two-dimensional system the existence of a global first integral completely determines its phase portrait. Since for such systems the notion of integrability is based on the existence of a first integral, the natural question is: given a planar system depending on parameters, how to recognize the values of the parameters for which the system has a global first integral ?

The planar integrable systems which are not Hamiltonian are in general very difficult to detect. Many different methods have been used for studying the existence of first integrals for non-Hamiltonian systems.

It is known that there are strong relationships between the integrability of a polynomial differential system like (1) and its number of invariant algebraic curves. Darboux showed in [5] that the existence of a certain finite number of invariant algebraic curves for a system of fixed degree implies the integrability of the system. The first integral is, in this case, a product of the invariant algebraic curves with complex exponents. Jouanolou showed in [10] that the existence of a certain number of invariant algebraic curves (higher than Darboux's bound) for a system of a fixed degree implies the algebraic integrability of the system, i.e. the existence of a rational first integral. In this case, all the solutions of the system are algebraic.

An invariant algebraic curve of system (1) is given by an irreducible algebraic curve, which is defined as the set of points in $\mathbb{C}^2$ satisfying an equation $f(x, y) = 0$, where f is a polynomial in x and y such that

$$\mathcal{X}f = Kf ,$$

for some polynomial $K(x, y) \in \mathbb{C}[x, y]$ with $\deg K \leq m - 1$, see for instance [3]. The polynomial K is termed the cofactor. In 1878, Darboux showed how the first integral of polynomial systems possessing sufficient invariant algebraic curves are constructed. In particular he proved that if a polynomial system of degree m has at least $m(m+1)/2+1$ invariant algebraic curves, then it has a first integral that can be directly constructed from these algebraic curves. More precisely, one of the results of Darboux is the following:

Suppose that a polynomial system (1) of degree m admits q invariant algebraic curves $f_i = 0$ with cofactors K_i for $i = 1, \ldots, q$. If $q \geq m(m+1)/2+1$, then the function $f_1^{\lambda_1} \ldots f_q^{\lambda_q}$ for suitable $\lambda_i \in \mathbb{C}$ not all zero is a first integral and $\sum_{i=1}^{q} \lambda_i K_i = 0$. The method of Darboux turns out to be a very useful and elegant one for proving integrability for some classes of systems depending on parameters.

A systematic search of invariant algebraic curves of a given degree can be carried out with the help of a computer algebra system, but the involved calculations become very difficult when the degree of the polynomial increases.

Generalizations of the Darboux method can be found in [3] where the notion of exponential factor is employed. A function of the form $f_1^{\lambda_1} \dots f_q^{\lambda_q} \exp(h/g)$, where f_i, g and h are polynomials in $\mathbb{C}[x,y]$ and the λ_i's are complex numbers, is called a *generalized Darbouxian function.* System (1) is called *generalized Darbouxian integrable* if the system has a first integral or an integrating factor which is a generalized Darbouxian function.

In the method of Darboux a finite number of implicit particular solutions are sufficient to construct in an algebraic way a first integral and, as we treat with planar systems, the general solution of the system.

We are lead in this way to the problem of characterizing the systems of differential equations for which a superposition function, allowing to express the general solution in terms of a certain finite number of particular solutions, does exist. As it is well known, this problem has been studied by Lie [11]. Let $\Sigma = \{g_1(x), \dots, g_n(x)\}$ be a set of particular solutions of equation (2). Then $F(y, g_1(x), \dots, g_n(x))$ is defined as a *connecting function* of (2) if $F = 0$ is also an implicitly defined particular solution. Formally, a nonlinear superposition principle is an operation $F : \mathbb{R} \times \mathcal{F}^n \to \mathcal{G}$ where $\mathcal{F}$ and $\mathcal{G}$ are function spaces such that the former properties hold.

Moreover, we will say that Σ is a *fundamental set of solutions* of (2) if a connecting function F exists, and such that F is a first integral or equivalently $F = c$ is the general solution of equation (2), where c is an arbitrary constant. The standard example of nonlinear first order differential equation with a fundamental set of solutions is the Riccati equation $dy/dx = A_0(x) + A_1(x)y + A_2(x)y^2$ for which the general solution is given by the cross ratio of three arbitrary particular solutions $y = g_1(x)$, $y = g_2(x)$ and $y = g_3(x)$

$$F(y, g_1(x), g_2(x), g_3(x)) = \frac{(y - g_1(x))(g_3(x) - g_2(x))}{(y - g_2(x))(g_3(x) - g_1(x))} = c \ ,$$

where c is an arbitrary constant.

It follows from the work of Lie and Scheffers [11] that the real equation (2) with n arbitrary particular solutions defining a fundamental set of solutions is associated with finite dimensional Lie algebras of vector fields on $\mathbb{R}$. In fact, Lie showed that there is a fundamental set of n arbitrary solutions for the differential equation (2) if and only if it can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sum_{i=0}^{s} A_i(x) B_i(y) \ , \tag{3}$$

where the vector fields $\mathcal{X}_i = B_i(y)\partial/\partial y$ with $i = 0, 1, \dots, s$, generate an r-dimensional Lie algebra with $s + 1 \le r \le n$. Unfortunately, no easy way to construct the nonlinear superposition principle is known, see for instance [16]. A modern treatment of the Lie and Scheffers works of fundamental set of arbitrary solutions can be found in [1]. Moreover, the notion of a fundamental set of solutions developed by Lie is extremely restrictive as can be seen from the following theorem proved by Lie [11].

Theorem 1.1. *The only ordinary differential equations of the form $dy/dx = f(x,y)$ allowing a fundamental set of arbitrary solutions are the Riccati equation $dy/dx = A_0(x) + A_1(x)y + A_2(x)y^2$ and any equation obtained from it by a change of dependent and independent variables of the form $\psi = \psi(y)$, $\tau = \tau(x)$.*

Are there any connections between the Darboux method and Lie's theory for polynomial systems (1)? In order to analyze this question we can express the first integral given by the Darboux theory in the following way

$$H(x,y) = f_1^{\lambda_1} \dots f_q^{\lambda_q} = (y - g_1(x))^{\alpha_1}(y - g_2(x))^{\alpha_2} \dots (y - g_\ell(x))^{\alpha_\ell} h(x) \ . \tag{4}$$

Here we have privileged the variable y but an analogous expression can be written interchanging the role of x and y. For obtaining this expression we have employed the algebraic functions $g_j(x)$ for $j = 1, \dots, \ell$, associated to the invariant algebraic curves $f_i(x,y) = 0$ with $i = 1, \dots, q$. A function $g_i(x)$ is an algebraic function if there exists a polynomial in two variables $f(x,y)$ such that $f(x, g_i(x)) \equiv 0$. The number ℓ of factors depends on q and the degree of each polynomial f_i. To give an upper bound for the number ℓ of factors in terms of the degree m of the polynomial system is a very difficult problem. It is directly related to another problem raised by Poincaré: to find an upper bound for the degree of an irreducible invariant algebraic curve in terms of the degree of the system. From Darboux's results, it is known that for every polynomial vector field, there exists an upper bound for the possible degrees of irreducible invariant algebraic curves. However, to explicitly determine when such an upper bound in terms of m exists is an unsolved problem at present. Some bounds have been given under certain conditions on the invariant algebraic curves, see for instance [17]. The α_i are, in general, complex constants. Obviously this expression is formal in nature. In general it is not possible to obtain explicit expressions for the functions $g_j(x)$. The existence of such factorization is ensured by the algebra fundamental theorem. Let us observe also that the functions $g_j(x)$ are particular solutions of (2).

Lemma 1.2. *Let $f(x,y) = 0$ be an invariant algebraic curve of system (1). Then, each algebraic function $g_j(x)$ associated to this curve is a particular solution of equation (2).*

Proof. Let $y = g_j(x)$ be a function defined implicitly by $f(x,y) = 0$. So, we can use the implicit function theorem as follows

$$\left. \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dg_j}{dx} \right|_{y = g_j(x)} = 0 \ . \tag{5}$$

Since $f = 0$ is an invariant algebraic curve of system (1), we also have

$$\left. \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{Q}{P} \right|_{f=0} = 0 \ . \tag{6}$$

Since the former equation is valid when $f = 0$, in particular it is valid when $y = g_j(x)$. Consequently, from equations (5) and (6) we conclude that

$$\frac{dg_j}{dx} = \frac{Q(x, g_j(x))}{P(x, g_j(x))} \ ,$$

that is, $g_j(x)$ is a particular solution of the differential equation (2) ■

In the factorized expression of the Darboux first integral given in (4) the particular solutions $g_i(x)$ are not arbitrary. They are well determined functions obtained by the factorization in the y variable of the polynomials $f_i(x,y)$. By contrary, in the expression of the first integral given by the Lie theory, the particular solutions $g_i(x)$ of the fundamental set are arbitrary. A natural question is: are there intermediate cases of nonlinear superposition principles for which some of the particular solutions remains arbitrary and the others are explicitly determined?

To answer this question it is quite natural to introduce the following ansatz for the first integral

$$I(x,y) = (y - g_1(x))^{\alpha_1}(y - g_2(x))^{\alpha_2} \ldots (y - g_\ell(x))^{\alpha_\ell} h(x) \;, \tag{7}$$

where $g_j(x)$ are unknown particular solutions of (2), $h(x)$ is an unknown function of x and the α_i are unknown constants such that $\prod_{i=1}^{\ell} \alpha_i \neq 0$, in order to detect first integrals of (1). Here, the particular solutions $g_j(x)$ are not, in general, algebraic functions.

Let us suppose that $I(x,y)$ given by (7) is a first integral of (1) such that some particular solutions $g_i(x)$ remain arbitrary with $i = 1, \ldots, j$ where $0 \leq j \leq \ell$. Then, the following possibilities arise:

(i) $j = \ell$ and $h(x)$ is a function only of $g_1(x), \ldots, g_\ell(x)$, that is, $h(x) = G(g_1(x), \ldots, g_\ell(x))$. Then the expression of $I(x,y)$ is a particular case of a nonlinear superposition principle given by Lie's theory, i.e., $\{g_1(x), \ldots, g_\ell(x)\}$ is a fundamental set of arbitrary solutions of (2).

(ii) $j < \ell$ and $h(x)$ is a function only of $g_1(x), \ldots, g_j(x)$. In this case $h(x) = G(g_1(x), \ldots, g_j(x))$ and each one of the determined functions $g_i(x)$ is a function only of the arbitrary particular solutions $\{g_1(x), \ldots, g_j(x)\}$. Then the expression of $I(x,y)$ is a particular case of a nonlinear superposition principle given by Lie's theory, i.e., $\{g_1(x), \ldots, g_j(x)\}$ is a fundamental set of arbitrary solutions of (2).

(iii) $j < \ell$ and $h(x)$ is a fixed function of x, , i.e., with explicit expression independent of $g_1(x), \ldots, g_j(x)$ and each one of the determined functions $g_i(x)$ is a function of only the arbitrary particular solutions $\{g_1(x), \ldots, g_j(x)\}$. Then, taking into account that we can construct a new first integral $I_1(x,y)/I_2(x,y)$ independent of $h(x)$ we obtain a particular case of a nonlinear superposition principle included also in Lie's theory, see Example 1.

(iv) $j = 0$; we will see that in this case all the particular solutions $g_i(x)$ are algebraically determined.

(v) In all the other cases we obtain a new type of superposition principle that we call *generalized nonlinear superposition principle*, see Example 4.

It is evident that a first integral $I(x,y)$ of the form (7), where the functions $g_j(x)$ are particular solutions of (2), is more general than the Darboux first integral $H(x,y)$ given by (4). Also, with the exception of the above particular cases (i) (ii) and (iii), the first integral (7) is not included in the Lie's theory. So the problem is: for a given system (1), how to determine whether it exists a first integral of the form (7)?

This question was first studied by Painlevé, see [14] and references therein. He proved that a differential system (1) has a first integral of the form (7) if and only if it has an integrating factor of the form

$$M = \frac{\alpha(x)S(x,y)}{(y-g_1(x))(y-g_2(x))\ldots(y-g_\ell(x))} \ , \tag{8}$$

where $S(x,y)$ is polynomial in the variable y of degree $\ell - m - 1$. Moreover he demonstrated that if the system has two different integrating factors M_1 and M_2 of the form (8) with M_2/M_1 nonconstant, then there exists a change of variable that is rational in the variable y which transforms the equation (2) into a Riccati equation. He also proved that if the differential system has only one integrating factor of the form (8), then the particular solutions $g_i(x)$ from the ansatz (7) are calculated algebraically and $h(x)$ is given by a logarithmic quadrature.

We follow here another approach with respect to Painlevé' works. In particular, we are interested in detecting, in an algorithmic way, the systems which admit a first integral of the form (7) and specially the cases when some of the particular solutions $g_i(x)$ remain arbitrary. In these cases we will obtain an expression of $I(x,y)$ that we call a *generalized nonlinear superposition principle.*

If we introduce the ansatz (7) with a fixed number of factors, we will show that it is possible to decide if the system has a first integral of the proposed form. In some cases either all the functions $g_j(x)$ are determined explicitly or there are particular solutions $g_j(x)$ expressed in terms of other arbitrary solutions $g_i(x)$. In addition, the function $h(x)$ is always determined in terms of the $g_j(x)$ and the α_i, but the expression of $h(x)$ contains, in general, quadratures of the functions $g_j(x)$, see Example 2. When some of the functions $g_j(x)$ remain arbitrary the resulting expression of $I(x,y)$ will contain a double dependence on the variable x. There is a first dependence of x through the arbitrary particular solutions $g_i(x)$. In addition, the x variable can appear in an explicit way, see for instance Example 4. By contrary, in the Lie theory the dependence of $I(x,y)$ on the variable x appears only through the fundamental set of solutions.

Then, in some cases, we will arrive to the conclusion that a given system of the type (1) has a first integral of the form (7) with the proposed number of factors, but the algorithm does not determine all the arbitrary particular solutions $g_j(x)$. In any way, in this case we can arrive to the important conclusion that a first integral of the system can be constructed from a finite number of particular solutions and we can determine this number. When all the particular solutions $g_i(x)$ are determined they are algebraic functions. For these cases, the algorithm introduced in this work represents an alternative method for determining such type of solutions.

Since even a formal invariant curve $f(x,y) = 0$ of system (1) given by a formal power series $f \in \mathbb{R}[[x,y]]$ must satisfy always an equation $\mathcal{X}f = Lf$ where $L(x,y)$ is also a formal power series, see [2] and [15], we introduce the next definition.

Definition 1.3. A quasipolynomial cofactor $M(x,y)$ associated to a non-algebraic invariant curve $f(x,y)=0$ of system (1) is a function that is polynomial in one of the variables x or y, satisfying $\mathcal{X}f = Mf$.

This definition is a generalization of the so called generalized cofactor introduced in [6] where a generalization of the Darboux integrability theory in order to find non-Liouvillian first integrals of system (1) was presented. For the special invariant curve $f(x,y)=y-g(x)=0$ of (1), where $g(x)$ is a particular solution of (2), a quasipolynomial cofactor always exists as we will see in the next proposition.

Proposition 1.4. *A particular solution $g(x)$ of equation (2) has always an unique associated quasipolynomial cofactor of the form $M(x,y)=K_{m-1}(x)y^{m-1}+\cdots+K_1(x)y+K_0(x)$ where m is the degree of system (1).*

Proof. Let $g(x)$ be a particular solution of (2). Then $f(x,y):=y-g(x)=0$ is an invariant curve of system (1). Denote by $M(x,y)$ its associated cofactor, i.e., $\mathcal{X}f = Mf$. So we have

$$Q - Pg' = M(y-g)\ . \tag{9}$$

Let us suppose that $M(x,y)$ is a quasipolinomial cofactor $M(x,y)=K_{m-1}(x)y^{m-1}+\cdots+K_1(x)y+K_0(x)$. We will prove that such kind of cofactor always exists and it is unique. Substituting the expressions $P(x,y)=\sum_{j=0}^{m}P_j(x)y^j$ and $Q(x,y)=\sum_{j=0}^{m}Q_j(x)y^j$, where P_j and Q_j are polynomials, the former equation reads for

$$\left(\sum_{j=0}^{m}Q_j(x)y^j\right)-g'(x)\left(\sum_{j=0}^{m}P_j(x)y^j\right)=\left(\sum_{j=0}^{m-1}K_j(x)y^j\right)(y-g(x))\ .$$

Equating the $m+1$ coefficients of the same powers of y in both members of this equation, the next linear system for the unknowns $K_j(x)$ is obtained

$$\begin{pmatrix} 1 & 0 & & & & \\ -g & 1 & 0 & & & \\ 0 & -g & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 0 \\ & & & 0 & -g & 1 \\ & & & & 0 & -g \end{pmatrix}\begin{pmatrix} K_{m-1} \\ K_{m-2} \\ K_{m-3} \\ \vdots \\ K_1 \\ K_0 \end{pmatrix}=\begin{pmatrix} Q_m - g'P_m \\ Q_{m-1}-g'P_{m-1} \\ Q_{m-2}-g'P_{m-2} \\ \vdots \\ Q_1 - g'P_1 \\ Q_0 - g'P_0 \end{pmatrix}.$$

From routine linear algebra it is possible to show that the above system is equivalent to

$$\begin{pmatrix} 1 & 0 & & & & \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & 0 & 1 \\ & & & & 0 & 0 \end{pmatrix}\begin{pmatrix} K_{m-1} \\ K_{m-2} \\ K_{m-3} \\ \vdots \\ K_1 \\ K_0 \end{pmatrix}=\begin{pmatrix} Q_m - g'P_m \\ (gQ_m+Q_{m-1})-g'(gP_m+P_{m-1}) \\ \sum_{j=0}^{2}Q_jg^j - g'\sum_{j=0}^{2}P_jg^j \\ \vdots \\ \sum_{j=0}^{m-1}Q_jg^j - g'\sum_{j=0}^{m-1}P_jg^j \\ \sum_{j=0}^{m}Q_jg^j - g'\sum_{j=0}^{m}P_jg^j \end{pmatrix}$$

Finally, since $\sum_{j=0}^{m} Q_j g^j - g' \sum_{j=0}^{m} P_j g^j = 0$ because g is a particular solution of (2), the above linear system reduces to a system with matrix coefficients given by the identity matrix of order m. Therefore we obtain a unique solution and the proposition is proved. ■

Therefore, a unique quasipolynomial cofactor is always associated to any particular solution of (2). Let us give an example of quasipolynomial cofactor. The differential equation $dy/dx = (x-y^4)/(2y)$ has the particular solution $g(x) = -\sqrt{\mathrm{Ai}'(x)/\mathrm{Ai}(x)}$, where $\mathrm{Ai}(x)$ is one of the pair of linearly independent solutions of the Airy equation $w'' = xw$. The particular solution $y = g(x)$ posesses the quasipolynomial cofactor $M(x,y) = -y^3 + y^2\sqrt{\mathrm{Ai}'(x)/\mathrm{Ai}(x)} - y\mathrm{Ai}'(x)/\mathrm{Ai}(x) + x\sqrt{\mathrm{Ai}(x)/\mathrm{Ai}'(x)}$. This example was obtained in [7].

Proposition 1.5. *Let $I(x,y)$ be the ansatz function defined in (7). $I(x,y)$ is a first integral of system (1) if and only if*

$$0 \equiv P_m(x)h'(x) \,, \tag{10}$$

$$0 \equiv P_j(x)h'(x) + h(x)\sum_{i=1}^{\ell} \alpha_i K_j^{(i)}(x) \,, \; j = 0,1,\ldots,m-1, \tag{11}$$

where $M_i(x,y) = \sum_{j=0}^{m-1} K_j^{(i)}(x) y^j$ is the quasipolynomial cofactor associated to the particular solution $g_i(x)$.

Proof. It follows by straightforward calculations from $\mathcal{X}I \equiv 0$ taking into account that, from Proposition 1.4, each particular solution $g_i(x)$ of equation (2) has associated the quasipolynomial cofactor $M_i(x,y)$. ■

Theorem 1.6. *System (1) with $P(x,y) \not\equiv 0$ has ℓ particular solutions $\{y = g_i(x)\}_{i=1}^{\ell}$ with associated linearly dependent quasipolinomial cofactors $\{M_i(x,y)\}_{i=1}^{\ell}$ if and only if system (1) has the first integral $I(x,y) = \prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i}$ with $\prod_{i=1}^{\ell} \alpha_i \neq 0$.*

Proof. The sufficient condition is proved as follows. If $\{M_i(x,y)\}_{i=1}^{\ell}$ are linearly dependent then there exist constants α_i with $\prod_{i=1}^{\ell} \alpha_i \neq 0$ such that, from (11) and (10), we have $P_j(x)h'(x) \equiv 0$ for $j = 0,1,\ldots,m$. Therefore the function $h(x)$ becomes constant since $P(x,y) = \sum_{j=0}^{m} P_j(x)y^j \not\equiv 0$. So, by Proposition 1.5 we have that $I(x,y) = \prod_{i=1}^{\ell}(y - g_i(x))^{\alpha_i}$ is a first integral of system (1).

In order to prove the necessary condition, let us suppose that $I(x,y) = \prod_{i=1}^{\ell}(y-g_i(x))^{\alpha_i}$ is a first integral of system (1). Then $h(x) = 1$ and therefore conditions (11) and (10) reduce to $\sum_{i=1}^{\ell} \alpha_i K_j^{(i)}(x) \equiv 0$ for $j = 0,1,\ldots,m-1$. In fact, since $M_i(x,y) = \sum_{j=0}^{m-1} K_j^{(i)}(x)y^j$, the former conditions give $\sum_{i=1}^{\ell} \alpha_i M_i(x,y) \equiv 0$ and the theorem is proved. ■

The following proposition is related to the result of Painlevé [14] presented in the introduction, when the differential system has a unique integrating factor of the form (8). In this case our proof follows directly from the existence of the first integral.

Proposition 1.7. *Suppose that system* (1) *admits the first integral* $I(x,y)$ *defined in* (7) *where all the particular solutions* $g_i(x)$ *of equation* (2) *are determined. Then* $g_i(x)$ *are algebraic functions for* $i = 1, \ldots, \ell$.

Proof. The condition $\mathcal{X}I = P(x,y)\partial_x I + Q(x,y)\partial_y I \equiv 0$ implies

$$\prod_{i=1}^{\ell}(y-g_i(x))^{\alpha_i-1}\left\{\left[h(x)\sum_{j=1}^{\ell}\alpha_j\prod_{i\neq j}^{\ell}(y-g_i(x))\right]Q(x,y)\right.$$

$$\left.+\left[h'(x)\prod_{i=1}^{\ell}(y-g_i(x))-h(x)\sum_{j=1}^{\ell}\alpha_j\,\frac{Q(x,g_j(x))}{P(x,g_j(x))}\prod_{i\neq j}^{\ell}(y-g_i(x))\right]P(x,y)\right\}\equiv 0,$$

where we have replaced $g_j'(x)$ by $Q(x,g_j(x))/P(x,g_j(x))$. We can rewrite this expression as follows

$$\prod_{i=1}^{\ell}(y-g_i(x))^{\alpha_i}\left[h'(x)P(x,y)\right.$$

$$\left.+\,h(x)\sum_{j=1}^{\ell}\alpha_j\frac{P(x,g_j(x))Q(x,y)-Q(x,g_j(x))P(x,y)}{(y-g_j(x))P(x,g_j(x))}\right]\equiv 0. \tag{12}$$

The expression inside the brackets is a polynomial in the variable y because $P(x,g_j(x))\ \ Q(x,y) - Q(x,g_j(x))P(x,y)|_{y=g_j(x)} \equiv 0$. The identically vanishing of the coefficients of every power of the variable y gives an algebraic system where the number of equations is $\leq m+1$. This system has $\ell+1$ unknown functions $g_i(x)$ and $h(x)$, and the arbitrary constants α_i. Let us notice that the number of equations is independent of the number of factors ℓ that appear in the expression of the first integral (7). The vanishing of the coefficient of the highest power of y gives the equation $P_m(x)h'(x) = 0$ which is condition (10) found in Proposition 1.5. This algebraic system is always compatible, because it has the trivial solution $\alpha_1 = \ldots = \alpha_\ell = 0$ and $h(x)$ an arbitrary constant.

When we solve the algebraic system there are two possibilities:

(i) The only solution is the trivial one and then there is no first integral of the proposed form.

(ii) The system has non-trivial solutions. In this case some of the functions $g_i(x)$ (or all of them) are determined. The algebraic relations obtained must be compatible with the fact that the functions $g_i(x)$ are particular solutions of system (2). Therefore, the next step is to derive these algebraic relations respect to the variable x and replace $g_i'(x)$ by $Q(x,g_i(x))/P(x,g_i(x))$. After this we obtain a new set of algebraic equations which must be satisfied. The algorithm finishes when the derivation with respect to the variable x and the substitutions of $g_i'(x)$ give expressions which are identically satisfied when the algebraic equations that have been derivated are satisfied. At the end of the process if all the functions $g_i(x)$ are determined then they will be algebraic functions because all the equations are algebraic. If some of the functions $g_i(x)$ remains arbitrary we obtain a generalized nonlinear superposition principle. Also, it is possible that at the end of the process we obtain a constant function $I(x,y)$ as the only possibility. In this case the system does not admit a first integral of the form (7).

The arbitrary coefficients α_i are chosen in order to ensure the existence of non-trivial solutions, when they exist. ■

2. Examples

Let us now analyze some examples in order to show the application of the method above described.

Example 1. Consider the following polynomial differential system

$$\dot{x} = -y + x^2, \quad \dot{y} = x. \tag{13}$$

The equation for the orbits is

$$\frac{dx}{dy} = \frac{-y + x^2}{x} \ . \tag{14}$$

The quadratic system (13) has a center at the origin because it is monodromic and ϕ_0-time-reversible with respect to the involution $\phi_0(x,y) = (-x,y)$. We propose the following first integral

$$I(x,y) = (x - g_1(y))^{\alpha_1} (x - g_2(y))^{\alpha_2} h(y) \ ,$$

where $\alpha_1 \alpha_2 \neq 0$ and the functions $g_1(y)$ and $g_2(y)$ are particular solutions for equation (14) with quasipolynomial cofactors M_1 and M_2, respectively. Imposing that $\mathcal{X} I \equiv 0$ we obtain

$$\alpha_1 M_1(x,y) h(y) + \alpha_2 M_2(x,y) h(y) + h'(y) x = 0. \tag{15}$$

The equation $\mathcal{X} f_i = M_i f_i$, with $f_i(x,y) = x - g_i(y)$ takes the form

$$\mathcal{X} f_i = \dot{x} \frac{\partial f_i}{\partial x} + \dot{y} \frac{\partial f_i}{\partial y} = -y + x^2 - x g_i'(y) = (K_1^{(i)}(y) x + K_0^{(i)}(y))(x - g_i(y)) \ , \tag{16}$$

which gives the following system of equations

$$\begin{aligned} 1 &= K_1^{(i)}(y), \\ -g_i'(y) &= K_0^{(i)}(y) - g_i(y) \ K_1^{(i)}(y), \\ -y &= -K_0^{(i)}(y) \ g_i(y). \end{aligned}$$

The resolution of this system of equations gives

$$\begin{aligned} K_1^{(i)}(y) &= 1, \\ K_0^{(i)}(y) &= \frac{y}{g_i(y)}, \\ g_i'(y) &= \frac{-y + g_i^2(y)}{g_i(y)}, \end{aligned} \tag{17}$$

where the equation (17) indicates that $g_i(y)$ is a particular solution of equation (14). Hence, the quasipolynomial cofactors M_1 and M_2 are of the form $M_1(x,y) = x + y/g_1(y)$ and $M_2(x,y) = x + y/g_2(y)$. Therefore, from equation (15) we obtain the following system of equations

$$\begin{aligned} \alpha_1 h(y) + \alpha_2 h(y) + h'(y) &= 0, \qquad (18)\\ \frac{\alpha_1\ y\ h(y)}{g_1(y)} + \frac{\alpha_2\ y\ h(y)}{g_2(y)} &= 0. \qquad (19)\end{aligned}$$

These equations can also be directly obtained from equation (12). From equation (18) we obtain $h(y) = e^{-(\alpha_1+\alpha_2)y}$ where we have taken the arbitrary integration constant equal to 1. On the other hand, from (19) we obtain $g_2(y) = \alpha g_1(y)$, where $\alpha := -\alpha_2/\alpha_1$. Then, we have solved equations (18) and (19) but the function $g_1(y)$ and the coefficients α_i remain arbitrary. However, we must impose that the algebraic relation $g_2(y) = \alpha g_1(y)$ to be compatible with fact that $g_1(y)$ and $g_2(y)$ are particular solutions of (14). So, deriving this expression we obtain $g_2'(y) = \alpha g_1'(y)$. Substituting here the expression of the derivatives of $g_1(y)$ and $g_2(y)$ we obtain

$$\frac{-y + g_2^2(y)}{g_2(y} = \alpha \frac{-y + g_1^2(y)}{g_1(y)}. \qquad (20)$$

Taking into account that $g_2(y) = \alpha g_1(y)$ we have

$$\frac{-y + \alpha^2 g_1^2(y)}{\alpha g_1(y)} = \alpha \frac{-y + g_1^2(y)}{g_1(y)},$$

which implies that $\alpha = \pm 1$. For these values of α the algebraic relation obtained after derivation of $g_2(y) = \alpha g_1(y)$ is identically satisfied and the algorithm is finished for this example. For $\alpha = 1$ we obtain $\alpha_2 = -\alpha_1$, $g_2(y) = g_1(y)$ and $h(y) \equiv 1$, i.e., $I(x,y) = 1$, a constant function. For $\alpha = -1$ we obtain $\alpha_2 = \alpha_1$, $g_2(y) = -g_1(y)$ and $h(y) = e^{-2\alpha_1 y}$. For this case we obtain a generalized nonlinear superposition principle given by the first integral

$$I(x,y) = (x^2 - g_1^2(y))e^{-2y} \ ,$$

where $g_1(y)$ is an arbitrary particular solution of equation (14). For this example our method enables to conclude that system (13) admits a generalized nonlinear superposition principle constructed from only one particular solution of the equation of the orbits (14). This nonlinear superposition principle is not a particular case of the Lie theory, owing to the explicit dependence of $I(x,y)$ on the variable y contained in the exponential factor. On the other hand, system (13) has a generalized Darboux first integral given by $H(x,y) = (x^2 - y - 1/2)e^{-2y}$. Therefore, two very simple solutions of equation (14) are $g_1(y) = \sqrt{1/2 + y}$ and $g_2(y) = -\sqrt{1/2 + y}$. Replacing $g_1(y)$ or $g_2(y)$ in the generalized superposition principle we obtain the generalized Darboux first integral. If we employ another particular solution $g_2(y)$ of equation (14) we have a different first integral $I_2(x,y) = (x^2 - g_2^2(y))e^{-2y}$. The quotient of these first integrals gives a new first integral

$$I(x,y) = \frac{x^2 - g_1^2(y)}{x^2 - g_2^2(y)} \ ,$$

which is a nonlinear superposition principle in the classical sense of Lie because $g_1(y)$ and $g_2(y)$ are arbitrary particular solutions. In fact, system (13) has also the first integral $H(x,y) = (x^2 - y - 1/2 + e^{2y})/(x^2 - y - 1/2)$.

We see that for this example we have a generalized nonlinear superposition principle constructed from only one particular solution and also a classical one constructed from two particular solutions of equation (14). Moreover, from the existence of the generalized Darboux first integral we can conclude nothing about the existence of a nonlinear superposition principle.

Let us notice that equation (14) can be written in the form $dx/dy = -y/x + x$. Hence, it has the form (3), interchanging x with y, with associated vector fields $\mathcal{X}_1 = -1/x\partial/\partial x$ and $\mathcal{X}_2 = x\partial/\partial x$. Since the Lie bracket $[\mathcal{X}_1, \mathcal{X}_2] = -2/x\partial/\partial x = 2\mathcal{X}_1$, then $\{\mathcal{X}_1, \mathcal{X}_2\}$ generates a 2-dimensional Lie algebra $\mathcal{L}_2$. Therefore, from the standard Lie theory, there exists a change of the dependent variable $z = \phi(x)$ such that $\mathcal{X}_1$ is transformed to its canonical form $\bar{\mathcal{X}}_1 = \partial/\partial z$. In short, such change of variable is given by $z = -x^2/2$. In this new dependent variable, we obtain the linear equation $dz/dy = y + 2z$, a special case of a Riccati equation, in according with Theorem 1.1. Consequently, in the new variables there exists a linear superposition principle of the linear equation.

Example 2. Let us consider the following polynomial differential system

$$\dot{x} = -y + x^4, \qquad \dot{y} = x. \tag{21}$$

The equation for the orbits is

$$\frac{dx}{dy} = \frac{-y + x^4}{x} \, . \tag{22}$$

The quartic system (21) has a center at the origin because it is monodromic and ϕ_0-time-reversible with respect to the involution $\phi_0(x,y) = (-x,y)$. We propose the following first integral

$$I(x,y) = (x - g_1(y))^{\alpha_1}(x - g_2(y))^{\alpha_2}(x - g_3(y))^{\alpha_3}(x - g_4(y))^{\alpha_4} h(y) \ ,$$

where the functions $g_i(y)$ for $i = 1, \ldots, 4$ are particular solutions for equation (22). Applying the same method as in Example 1 we obtain that $g_2(y) = -g_1(y)$, $g_4(y) = -g_3(y)$, $\alpha_2 = \alpha_1$, $\alpha_3 = \alpha_4 = -\alpha_1$ and $h(y) = e^{2\alpha_1 \int (g_3^2(y) - g_1^2(y))dy}$. Therefore, a generalized nonlinear superposition principle is given by the first integral

$$I(x,y) = \frac{(x^2 - g_1^2(y))}{(x^2 - g_3^2(y))} e^{2\int (g_3^2(y) - g_1^2(y))dy} \ ,$$

where $g_1(y)$ and $g_3(y)$ are arbitrary particular solutions of equation (22). The algebraic calculations involved in the obtention of this first integral have been made with an algebraic manipulator. This nonlinear superposition principle is not a particular case of the Lie theory, because the expression of the first integral contains quadratures of the particular solutions $g_1(y)$ and $g_3(y)$.

This example has neither a Darboux or Darboux generalized first integral nor a Darboux or Darboux generalized inverse integrating factor, see for instance [6]. Therefore, system (21) has no Liouvillian first integral. In fact, system (21) possesses the non-Liouvillian first integral $H(x,y) = f_1 f_2^{-1}$ where $f_1(x,y) = 2x^2\mathrm{Ai}(4^{1/3}y) + \mathrm{Ai}'(4^{1/3}y) = 0$ and $f_2(x,y) = 2x^2\mathrm{Bi}(4^{1/3}y) + \mathrm{Bi}'(4^{1/3}y) = 0$ are non-algebraic invariant curves. Here $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ is a pair of linearly independent solutions of the Airy equation $w'' = zw$. Let us notice that system (21) can be written in the form $dx/dy = -y/x + x^3$. Hence, it has the form (3), interchanging x with y, with associated vector fields $\mathcal{X}_1 = -1/x\partial/\partial x$ and $\mathcal{X}_2 = x^3\partial/\partial x$. Since the Lie bracket $[\mathcal{X}_1, \mathcal{X}_2] = \mathcal{X}_3 = -4x\partial/\partial x$, it is easy to see that $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3\}$ generates a 3-dimensional Lie algebra $\mathcal{L}_3$. Therefore, from the standard Lie theory, there exists a change of the dependent variable $z = \phi(x)$ such that $\mathcal{X}_1$ is transformed to its canonical form $\bar{\mathcal{X}}_1 = \partial/\partial z$. In short, such change of variable is given by $z = -x^2/2$. In this new dependent variable, equation (22) becomes the Riccati equation $dz/dy = y - 4z^2$, in according with Theorem 1.1. Consequently, in the new variables, there exists a classical nonlinear superposition principle given by the cross ratio of three particular solutions. However, by applying our algorithm we have shown that this system admits a generalized nonlinear superposition principle constructed with only two arbitrary solutions.

Example 3. Let us consider the polynomial differential systems called Liénard system

$$\dot{x} = y, \qquad \dot{y} = -f_s(x)y - g_n(x), \tag{23}$$

where $f_s(x) = \sum_{i=0}^{s} a_i x^i$ and $g_n(x) = \sum_{i=0}^{n} b_i x^i$ are polynomials of degrees s and n, respectively. Odani in [12] investigated invariant algebraic curves of (23). He also studied the example of Wilson [18]

$$\dot{x} = y, \qquad \dot{y} = -\mu(x^2-1)y - x - \frac{\mu^2 x^3}{16}(x^2-4), \tag{24}$$

with $\mu \neq 0$. System (24) has the invariant algebraic curve

$$\phi_1 := [y + (\mu/4)x(x^2-4)]^2 + x^2 - 4 = 0,$$

as a limit cycle when $0 < |\mu| < 2$. For $|\mu| \geq 2$ the invariant algebraic curve turns out to contain a singular point, and so it cannot be a limit cycle, see for instance [13]. Moreover, system (24) has two additional invariant algebraic curves

$$\phi_2, \phi_3 := y + (\mu/4)x(x^2-2) \pm (\lambda/2)x = 0, \qquad \lambda := \sqrt{\mu^2-4},$$

and the Darboux first integral

$$H(x,y) = \frac{\phi_1}{\phi_2\phi_3}\left(\frac{\phi_2}{\phi_3}\right)^{\frac{\mu}{\lambda}}. \tag{25}$$

The equation for the orbits of system (24) is

$$\frac{dy}{dx} = \frac{-\mu(x^2-1)y - x - \frac{\mu^2 x^3}{16}(x^2-4)}{y}. \tag{26}$$

We propose the following first integral

$$I(x,y) = (y - g_1(x))^{\alpha_1}(y - g_2(x))^{\alpha_2}(y - g_3(x))^{\alpha_3}(y - g_4(x))^{\alpha_4} h(x) \ ,$$

where the functions $g_i(x)$ for $i = 1, \ldots, 4$ are particular solutions of equation (26). Applying the method we obtain that $h(x)$ is an arbitrary constant that we have taken equal to 1, $\alpha_2 = \alpha_1 = 1$ and $\alpha_4 = -\alpha_3 - 2$. Moreover, all the functions $g_i(x)$ for $i = 1, \ldots, 4$ are determined by the relations $(y - g_1(x))(y - g_2(x)) = \phi_1$, $y - g_3(x) = \phi_2$ and $y - g_4(x) = \phi_3$. In addition, α_3 must satisfy the equation $-4 - 8\alpha_3 - 4\alpha_3^2 + 2\alpha_3\mu^2 + \alpha_3^2\mu^2 = 0$. In fact, the method gives the first integral (25), i.e., $I \equiv H$. In this example the method does not give a nonlinear superposition principle, on the contrary all the particular solutions are completely determined by a "one shot procedure". Our algorithm represents for this case an alternative technique to construct a Darboux first integral.

To construct a Darboux first integral for a polynomial system it is necessary to determine a sufficient number of algebraic invariant curves of the system. Each one of these curves must be independently determined. The existence of one of them does not give information, in principle, about the possible existence of others invariants algebraic curves. By contrary, for this example, our algorithm enables to determine in a "one shot procedure" all the invariant algebraic curves of the system that are necessary to construct a Darboux first integral.

Example 4. Let us consider the following cubic system

$$\dot{x} = y, \qquad \dot{y} = -x + Q(x, y). \tag{27}$$

where $Q(x, y) = a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3$. The conditions for the origin of (27) to be a center were given by Kukles [9] in 1944. Two particular cases of (27), i.e. the cases in which $a_2a_7 = 0$ were recently considered in [4, 8]. For $a_2 = 0$, the example given in [8] suggests that Kukles' conditions are incomplete. In fact, it is proved in [4] that the conditions are indeed incomplete by showing that the origin is a center in this case. The example is the following

$$\dot{x} = y, \quad \dot{y} = -x + x^2 - \frac{x^3}{3} - \frac{x^2 y}{\sqrt{2}} - 2y^2 + \frac{y^3}{3\sqrt{2}}. \tag{28}$$

The equation for the orbits is

$$\frac{dy}{dx} = \frac{-x + x^2 - \frac{x^3}{3} - \frac{x^2 y}{\sqrt{2}} - 2y^2 + \frac{y^3}{3\sqrt{2}}}{y} \ . \tag{29}$$

This system has a Darboux generalized inverse integrating factor given by

$$V(x, y) = e^{-x(1-\frac{x}{2})}(3\sqrt{2}(1 - x) + x(\sqrt{2}\, x + y))^3$$

and the following Liouvillian first integral

$$\begin{aligned} H(x, y) \quad &= \quad \frac{y^2(x + 1) + 2\sqrt{2}xy(x - 2) + 6(3x - 2) + 2x^3 - 10x^2}{(x(y + \sqrt{2}x) + 3\sqrt{2}(1 - x))^2}\, e^{x(1-\frac{x}{2})} \\ &\quad + \int e^{x(1-\frac{x}{2})} dx \ . \end{aligned}$$

We propose the following first integral

$$I(x, y) = (y - g_1(x))^{\alpha_1}(y - g_2(x))^{\alpha_2}(y - g_3(x))^{\alpha_3}(y - g_4(x))^{\alpha_4} h(x) \ ,$$

where the functions $g_i(x)$ for $i = 1, \ldots, 4$ are particular solutions of equation (29). Applying the method we obtain that $g_4(x) = g_3(x) = -\sqrt{2}\,(3 - 3x + x^2)/x$, $\alpha_2 = \alpha_1$, $\alpha_3 = \alpha_4 = -2\alpha_1$. Moreover, $g_2(x)$ and $h(x)$ are given in terms of $g_1(x)$ by the following expressions

$$g_2(x) = -\frac{(3 - 3x + x^2)\; g_1(x)}{3 - 3x + x^2 + \sqrt{2}\; x g_1(x)}\,,$$

$$h(x) = e^{-\frac{1}{3}\int \frac{(xg_1(x)+\sqrt{2}\;(3-3x+x^2))^2}{x(3-3x+x^2+\sqrt{2}\;xg_1(x))}dx}\,.$$

Therefore, a generalized nonlinear superposition principle is obtained from the first integral

$$I(x,y) = \frac{(y - g_1(x))\left(y + \frac{(3-3x+x^2)\; g_1(x)}{3-3x+x^2+\sqrt{2}\; xg_1(x)}\right)}{(y + \frac{\sqrt{2}\;(3-3x+x^2)}{x})^2}\; e^{-\frac{1}{3}\int \frac{(xg_1(x)+\sqrt{2}\;(3-3x+x^2))^2}{x(3-3x+x^2+\sqrt{2}\;xg_1(x))}dx},$$

where $g_1(x)$ is an arbitrary particular solution of equation (29). We conclude that from the knowledge of only a particular solution we can construct the general solution of system (28). On the other hand, since $g_1(x)$ is arbitrary we can construct two different integrating factors of (28) of the form (8). Hence, taking into account the results of Painlevé [14] described in the introduction, there exists a change of variable rational in y which transforms equation (29) into a Riccati equation. In this case the change of variable is given by $u = \frac{(3-3x+x^2)y^2}{3-3x+x^2+\sqrt{2}\;xy}$ which transforms equation (29) into the Riccati equation

$$\frac{du}{dx} = \frac{u^2x + u(-36 + (x-6)^2x) - 2x(3 + (x-3)x)^2}{3(3 + (x-3)x)}\,.$$

However, the existence of this change of variables that transforms equation (29) into a Riccati equation does not enable to conclude in an easy way that it is possible to construct a generalized nonlinear superposition principle for equation (29) that contains only one arbitrary particular solution of the system. Our algorithm allows the determination of the form of this generalized nonlinear superposition principle in a direct way employing the original variables of the system.

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Received November 25, 2003
and in final form March 3, 2004