CLASSICAL GEOMETRIES ARISING IN FEEDBACK EQUIVALENCE

Robert B. Gardner[1,2] and George R. Wilkens[2]

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599–3250
and The Fields Institute for Research in Mathematical Sciences, Waterloo, Ontario N2L 5Z5

Department of Mathematics, University of Hawaii at Manoa, Honolulu, HI 96822
and The Fields Institute for Research in Mathematical Sciences, Waterloo, Ontario N2L 5Z5

Abstract

The equivalence problem for control systems under non-linear feedback is recast as a problem involving the determination of the invariants of submanifolds in the tangent bundle of state space under fiber preserving transformations. This leads to a fiber geometry described by the invariants of submanifolds under the general linear group, which is the classical subject of centro-affine geometry. Unfortunately, the invariants of this geometry were known only in low dimensions and the fundamental theorem of such submanifolds needed to be established. Applying the solution to the fiber geometry induced by n-states and $(n-1)$-controls leads in a surprisingly simply way to the solution of the equivalence problem on the whole total space. In particular, mysterious results on the existence of feedback invariant pseudo-Riemannian geometries uncovered in earlier work [3], [7] is clearly explained with a precise geometric meaning. Similar analysis of the general scalar control problem has also been worked out and required a solution of the fundamental theorem of curves in centro-affine n-space, and again gives a solution to the equivalence problem on the total space, which will not be described in this note. The original solution of the equivalence problem for n-states and $(n-1)$-controls, due to Robert Bryant and the first author, was sufficently complicated that a complete proof was never published, although an outline exists in [1]. This approach had the disadvantage that the meanings of the various invariants uncovered were not visible. The new ideas make these results accessable and in this case lead to the Finsler geometry of a generalized variational problem arrising from the variational problem of time optimal control along control trajectories [2].

[1] Partially supported by NSF grants DMS-9102920 and DMS-9204942

[2] Partially supported by the Ontario Ministry of Education and Training and the Natural Sciences and Engineering Research Council of Canada

1. Introduction

A control system

$$\frac{dx}{dt} = f(x,u) \qquad \text{with} \quad x \in \mathbf{R}^n \quad \text{and} \quad u \in \mathbf{R}^m$$

is equivalent to its associated Pfaffian system

$$I = \{dx - f\,dt\}.$$

The problem under consideration is the determination and interpretation of a complete set of invariants of such a system under feedback transformations, which are the diffeomorphisms of the form

$$\Phi(t,x,u) = (t, \phi(x), \psi(x,u)),$$

preserving integral curves of the associated Pfaffian system.

As such we can view the control system as the submanifold of the tangent space of the state space $T(\mathbf{R}^n)$ defined by

$$V : \mathbf{R}^n \times \mathbf{R}^m \longrightarrow T(\mathbf{R}^n),$$

where

$$V(x,u) = \sum_{i=1}^{n} f^i(x,u)\frac{\partial}{\partial x^i}.$$

Now, if we fix a fiber over $x \in \mathbf{R}^n$ and restrict the mapping V to define

$$V_x : \mathbf{R}^m \to T_x(\mathbf{R}^n) \simeq \mathbf{R}^n,$$

where $V_x(u) = f(x,u)$, then a feedback transformation $\overline{x} = \phi(x)$ satisfies

$$\frac{d\overline{x}}{dt} = \frac{dx}{dt}\frac{\partial \overline{x}}{\partial x} = f\frac{\partial \overline{x}}{\partial x}.$$

This means that the action induced on the fiber is $\partial\overline{x}/\partial x$, which is an arbitrary member of the general linear group since ϕ was an arbitrary diffeomorphism. Thus the fiber geometry is the study of invariants of submanifolds under the general linear group, i.e. the venerable subject of centro-affine geometry.

The basic idea and battle plan now is to understand the fiber geometry, in particular using the fundamental theorems characterizing submanifolds of given dimensions up to general linear transformations, and then to investigate the evolution of this fiber geometry over the the entire state space.

The problem which occurs is that although recognized as a classical geometry since minimally the nineteen-thirties, the only published results were for curves and surfaces in the centro-affine plane or space [5], [6], or for curves in affine unimodular n-space [4].

2. The Fiber Geometry of Control Systems of n-States and (n-1)-Controls

We restrict the above discussion to $m = n-1$, and choose a frame $(e_0, e_1, \ldots, e_m)$ on $\mathbf{R}^n$ defined for each point (x,u) in the fiber over x by

$$dx = \sum_{i=0}^{m} \omega^i e_i.$$

As the reader will see shortly, the first leg will have special meaning, which motivates the curious range of indicies. In particular, given an integral curve $\gamma : \mathbf{R} \longrightarrow \mathbf{R}^n \times \mathbf{R}^m$ of the original control system,

$$f dt = \gamma^* dx = \sum_{i=0}^{m} \gamma^* \omega^i e_i,$$

choosing $e_0 = f$ results in the normalizations

$$\gamma^*\omega^0 = dt, \quad \gamma^*\omega^1 = 0, \ldots, \gamma^*\omega^m = 0.$$

Next let us change notation to reflect this choice by defining 1-forms ϕ and $\eta^1, \ldots, \eta^m$ by the equation

$$dx = \phi e_0 + \sum_{\alpha=1}^{m} \eta^\alpha e_\alpha.$$

In particular we note that

$$\{dx\} = \{\phi, \eta^1, \ldots, \eta^m\},$$

and

$$\{dx - f dt\} = \{\phi - dt, \eta^1, \ldots, \eta^m\}.$$

Now if we define $I = \{\eta^1, \ldots, \eta^m\}$ then a diffeomorphism $\Phi(t,x,u)$ is a feedback equivalence if and only if

$$\Phi^* I = I \quad \text{and} \quad \Phi^*\phi \in \phi + I.$$

This follows because these two conditons along with the characterization of $\{dx\}$ and $\{dx - f dt\}$ immediately above, are equivalent to Φ preserving states and integral curves.

The basic problem is to choose frames and coframes such that they are simultaneously adapted to both the centro-affine geometry of the fiber and the feedback geometry of the control system.

If we restrict the consideration to the centro-affine geometry of a hypersurface

$$Y : M_m \longrightarrow \mathbf{R}^{m+1},$$

then the hypersurface is called *non-conical* if Y is normal at every point. This occurs in the fiber geometry of a control-affine system precisely when the control system has non-zero drift. Now given an affine frame $(e_0, \ldots, e_m)$ at the origin, we have the structure equations

$$de_i = \sum_{j=0}^{m} \omega_i^j e_j \quad \text{and} \quad d\omega_i^j = \sum_{k=0}^{m} \omega_i^k \wedge \omega_k^j.$$

If we now adapt the family of frames so that $e_0 = Y$ and $(e_1, \ldots, e_m)$ are tangent, then it follows immediately that $\omega_0^0 = 0$ and differentiation of this normalization and Cartan's lemma guarentees the existence of a symmetric matrix of functions $(h_{\alpha\beta})$ satisfying

$$\omega_\alpha^0 = \sum_{\beta=1}^{m} h_{\alpha\beta}\omega_0^\beta, \qquad (1 \le \alpha \le m).$$

If we package this information in the symmetric quadratic differential form

$$II_{CA} = \sum_{\alpha,\beta=1}^{m} h_{\alpha\beta}\omega_0^\alpha \odot \omega_0^\beta,$$

then we have the analog of the Blaschke form in affine geometry.

The admissible action on this symmetric tensor $(h_{\alpha\beta})$ includes conjugation and hence may be normalized in the usual ways. Let us restrict to the negative definite case, since this involves the simplest notation. Under this hypothesis we may normalize

$$h_{\alpha\beta} = -\delta_{\alpha\beta}.$$

Integrability conditions then imply that

$$\begin{aligned}\Delta_\alpha^\beta &= \frac{1}{2}(\omega_\beta^\alpha + \omega_\alpha^\beta)\\ &= \sum_{\gamma=1}^{m} C_{\alpha\gamma}^\beta \omega_0^\gamma, \quad (1 \le \alpha, \beta \le m)\end{aligned}$$

Where the symbol $(C_{\alpha\gamma}^\beta)$ is symmetric in all three indicies. The resulting cubic form

$$P_{CA} = \sum_{\alpha,\beta,\gamma=1}^{m} C_{\beta\gamma}^\alpha \omega_0^\beta \odot \omega_0^\alpha \odot \omega_0^\gamma$$

is the analog of the Pick cubic form in affine geometry.

Now let us introduce a matrix notation to compactify the information in the normalized structure equations. Thus define

$$\Omega = (\omega_\beta^\alpha), \quad \Phi = \frac{1}{2}(\Omega - {}^t\Omega), \quad \Delta = \frac{1}{2}(\Omega + {}^t\Omega),$$
$$\omega = (\omega_0^\alpha), \quad e = {}^t(e_1, \ldots, e_m),$$

so that

$$d\begin{pmatrix} e_0 \\ e \end{pmatrix} = \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Phi + \Delta \end{pmatrix}\begin{pmatrix} e_0 \\ e \end{pmatrix}$$

Differentiation of this last set of equations gives

$$\begin{aligned} &d\begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Phi + \Delta \end{pmatrix}\\ &\qquad = \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Phi + \Delta \end{pmatrix} \wedge \begin{pmatrix} 0 & \omega \\ -{}^t\omega & \Phi + \Delta \end{pmatrix}\end{aligned}$$

and hence

$$d\omega = \omega \wedge \Phi + \omega \wedge \Delta \ \text{ with } \ {}^t\Phi = -\Phi.$$

Now this last set of equations determines Φ uniquely. This is seen by using an algebraic theorem similar to that used to prove the characterization of the Levi-Civita connection in Riemannian geometry.

3. Evolution of the Fiber Geometry across the State Space

Next we extend the choice of n-frames to the principal $SO(m, R)$ bundle over the image of V in $T(\mathbf{R}^n)$ where the group $SO(m, R)$ is the stabilizer of the Blaschke form normalized as above. Thus we have frames

$$(e_0(x, u), e_1(x, u, S), \ldots, e_m(x, u, S))$$

with $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $S \in SO(m, R)$, $m = n-1$. As above these have dual 1-forms

$$(\phi(x, u, S), \eta^1(x, u, S), \ldots, \eta^m(x, u, S))$$

which satisfy $dx = \phi e_0 + \sum \eta^\alpha e_\alpha$.

On each fiber we have complementary forms $(\mu^1, \ldots, \mu^m)$ satisfying

$$d_{fiber}\begin{pmatrix} e_0 \\ e \end{pmatrix} = \begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi + \Delta \end{pmatrix}\begin{pmatrix} e_0 \\ e \end{pmatrix},$$

and from the representation of the exterior derivatives restricted to the fibers we have

$$d_{fiber}(\phi, \eta) = (\phi, \eta) \wedge \begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi + \Delta \end{pmatrix}.$$

As a consequence the full exterior derivative has the form

$$d(\phi,\eta) = (\phi,\eta)\wedge \begin{pmatrix} 0 & \mu \\ -{}^t\mu & \Phi+\Delta \end{pmatrix} + \text{terms quadratic in the states.}$$

We note that the terms quadratic in the states are precisely the terms quadratic in ϕ, η. Now a careful absorption argument can be used to establish the existence of a unique extension of the 1-forms μ, Δ in such a way that the extensions absorb all the forms quadratic in the base and a extension of the 1-forms Φ is uniquely forced by requiring that the structure equations

$$d\phi = -\eta \wedge {}^t\mu \quad \text{and} \quad d\eta = \phi\wedge\mu + \eta\wedge(\Phi+\Delta),$$

with ${}^t\Phi = -\Phi$, be satisfied.

These equations and the resulting integrability conditions obtained by differentiation result in the structure equations of the Finsler Geometry associated to the general variational problem

$$\delta \int_{\eta=0} \phi = 0,$$

with integrand ϕ restricted to integral curves of $\eta = 0$, which are just the integral curves of the original control system. Thus the integrand ϕ is the generalization of the Cartan form [1, p.52] in the usual calculus of variations corresponding to the problem of time optimal control.

In summary this analysis shows the geometric meaning of the vanishing of invariants discovered by ad hoc methods in our past work [1], [7] and in particular give a geometric explaination of the existence of feedback invariant pseudo-riemannian metrics which have closed loop time optimal trajectories as geodesics [3], [7]. If the Blaschke form is just non-singular instead of positive definite, then everything goes through in the context of pseudo-Riemannian geometry. The complete details of the computations outlined here will appear elsewhere (see [8] for the lowest dimensional example).

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