Siberian Mathematical Journal, Vol. 51, *No.* 1, *pp.* 1–11, 2010

SIMILARITY OF APPROXIMATE TRANSFORMATION GROUPS

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UDC 517.95

Abstract: We propose similarity conditions for isomorphic approximate transformation groups and their Lie algebras. The construction of similarity transformations reduces to solving systems of first-order semilinear partial differential equations with small parameter. We consider the solvability of overdetermined systems of this type and the structure of their general solutions.

Keywords: approximate transformation group, Lie algebra, Lie algebra isomorphism

Introduction

It is well known that the realizations of isomorphic Lie algebras in the space of first-order differential operators can lead to nonsimilar Lie algebras of operators. A classical example of this is the representation of two-dimensional Lie algebras by differential operators with two variables (see [1] for instance). In Lie theory some Lie group is associated to every Lie algebra, and similar problems appear in the theory of continuous transformation groups.

The questions of similarity for Lie transformation groups (and Lie algebras) are essential, for instance, in the classification problems for differential equations from the viewpoint of their symmetry properties. Also, the knowledge of similarity transformations of Lie algebras admitted by two differential equations can be used for constructing a change of variables relating these equations.

Eisenhart studied the similarity of exact groups [2] (also see [3]) by considering the two arbitrary r-parametric transformation groups in $\mathbb{R}^n$:

$$T_a : \bar{x}^\alpha = f^\alpha(x^1, \dots, x^n; a^1, \dots, a^r), \quad T'_{a'} : \bar{x}'^\alpha = h^\alpha(x'^1, \dots, x'^n; a'^1, \dots, a'^r),$$

$\alpha = 1, \dots, n$. He stated necessary and sufficient conditions for their similarity. Those conditions and their proofs are in terms of the corresponding Lie algebras of operators. Namely, studying the similarity of transformation groups reduces to studying the solvability of a system of first-order partial differential equations whose coefficients are ones of the basis operators of the algebras in question. These systems are complete due to the commutation of the basis operators and an isomorphism between the Lie algebras. Their compatibility is equivalent to the solvability of a certain system of algebraic equations relating the variables x^α and x'^α of the Lie algebras.

In this article we solve a similar problem for approximate transformation groups. Considering similarity for approximate groups is equivalent to considering the similarity of approximate Lie algebras [4] and reduces to solving a system of first-order partial differential equations with small parameter, whose construction is given in Section 1. Some particular cases of the solvability of the systems of this type were previously studied in [5], which dealt with the systems linearly disconnected in the principal order by ε in the cases that they are either complete (i.e., the Jacobi brackets yield no new equations linearly disconnected with the original ones) or the completeness condition for the system is a system of algebraic equations in the variables x^α and x'^α whose solution cannot lead to relation between x^α or x'^α. The case was not considered in [5] of the systems of linearly connected equations, which often arises in the problem of similarity for approximate transformation groups and leads to additional differential equations even when the system is complete (see Subsection 3.2).

In contrast to the case of exact Lie algebras, the completeness of approximate Lie algebras with respect to commutation (even in the case of their isomorphism) fails to imply the completeness of the

Ufa. Translated from *Sibirskiĭ Matematicheskiĭ Zhurnal*, Vol. 51, No. 1, pp. 3–15, January–February, 2010.
Original article submitted September 9, 2008.

corresponding system of differential equations (see Section 2), which increases the amount of calculations required to solve the problem. Therefore, it is important to construct necessary conditions for the similarity of approximate Lie algebras, which we obtained in Sections 1 and 3. In Section 4 we give examples of the construction of similarity transformations for some approximate Lie algebras.

The article uses the following notation. The equality $f(x,\varepsilon)=o(\varepsilon)$ means that $\lim_{\varepsilon\to 0}\frac{f(x,\varepsilon)}{\varepsilon}=0$. By an approximate equality $f\approx g$ we mean $f(x,\varepsilon)=g(x,\varepsilon)+o(\varepsilon)$. In the expressions of the form $\xi_a^\alpha(x,\varepsilon)\frac{\partial}{\partial x^\alpha}$ we assume summation over the repeated index.

1. The System of Differential Equations for a Similarity Transformation

In $\mathbb{R}^n$ consider two r-parametric approximate (up to $o(\varepsilon)$) transformation groups: the group $\widetilde{G}_r$ of transformations [4, 6]

$$T_a : \bar{x}^\alpha \approx f^\alpha(x,a,\varepsilon)\equiv f_0^\alpha(x;a)+\varepsilon f_1^\alpha(x;a)+o(\varepsilon),\quad \alpha=1,\dots,n, \tag{1}$$

where the approximate functions f^α are considered at the points $x=(x^1,\dots,x^n)$ and depend on some parameters $a^1,\dots,a^r$, and the group $\widetilde{H}_r$ of transformations

$$T'_{a'} : \bar{x}'^\alpha \approx h^\alpha(x',a',\varepsilon)\equiv h_0^\alpha(x';a')+\varepsilon h_1^\alpha(x';a')+o(\varepsilon),\quad \alpha=1,\dots,n, \tag{2}$$

where the functions h^α are considered at the points $x'=(x'^1,\dots,x'^n)$ and depend on some parameters $a'^1,\dots,a'^r$. We consider the similarity of these groups in the following sense.

Definition 1. Approximate transformation groups $\widetilde{G}_r$ and $\widetilde{H}_r$ are called *similar* whenever there exists a system of r independent functions $\theta^\alpha(a)$ such that we can find a nondegenerate (for $\varepsilon=0$) coordinate transformation

$$x'^\alpha=\psi^\alpha(x,\varepsilon)=\psi_0^\alpha(x)+\varepsilon\psi_1^\alpha(x)+o(\varepsilon),\quad \alpha=1,\dots,n, \tag{3}$$

and the replacement $a'=\theta(a)$, $x'=\psi(x,\varepsilon)$, $\bar{x}'=\psi(\bar{x},\varepsilon)$ carries $T'_{a'}$ into T_a.

The question of similarity for approximate transformation groups is equivalent to the question of similarity for the corresponding approximate Lie algebras [4]. Therefore, instead of the groups $\widetilde{G}_r$ and $\widetilde{H}_r$ we will consider the corresponding approximate Lie algebras L and L' with the basis operators

$$L:\quad \begin{aligned} X_{a_0}&=\xi_{a_0}^\alpha(x,\varepsilon)\frac{\partial}{\partial x^\alpha}\approx\left(\xi_{a_0(0)}^\alpha(x)+\varepsilon\xi_{a_0(1)}^\alpha(x)\right)\frac{\partial}{\partial x^\alpha},\\ \varepsilon X_{a_1}&=\varepsilon\xi_{a_1}^\alpha(x,\varepsilon)\frac{\partial}{\partial x^\alpha}\approx\left(\varepsilon\xi_{a_1(0)}^\alpha(x)\right)\frac{\partial}{\partial x^\alpha},\end{aligned} \tag{4}$$

$$\alpha=1,\dots,n,\quad a_0=1,\dots,r_0,\quad a_1=r_0+1,\dots,r,$$

and

$$L':\quad \begin{aligned} X'_{a_0}&=\xi'^\alpha_{a_0}(x',\varepsilon)\frac{\partial}{\partial x'^\alpha}\approx\left(\xi'^\alpha_{a_0(0)}(x')+\varepsilon\xi'^\alpha_{a_0(1)}(x')\right)\frac{\partial}{\partial x'^\alpha},\\ \varepsilon X'_{a_1}&=\varepsilon\xi'^\alpha_{a_1}(x',\varepsilon)\frac{\partial}{\partial x'^\alpha}\approx\left(\varepsilon\xi'^\alpha_{a_1(0)}(x')\right)\frac{\partial}{\partial x'^\alpha},\end{aligned} \tag{5}$$

respectively.

Definition 2. The approximate Lie algebra L' of operators $\langle X'_{a_0},\varepsilon X'_{a_1}\rangle$ is isomorphic to the Lie algebra L of the operators $\langle X_{a_0},\varepsilon X_{a_1}\rangle$, $a_0=1,\dots,r_0$, $a_1=r_0+1,\dots,r$, whenever there exists a bijective linear mapping $\phi: L\to L'$ such that $\phi([\varepsilon^i X_{a_i},\varepsilon^j X_{b_j}])\approx[\phi(\varepsilon^i X_{a_i}),\phi(\varepsilon^j X_{b_j})]$, where $i,j=0,1$.

It is obvious that if some approximate algebras are similar then they are isomorphic. By analogy with exact Lie algebras [7] we can show that in isomorphic approximate Lie algebras we can choose bases to achieve the equalities of their structure constants. By Definition 1 the similarity of two approximate algebras with the operators (4) and (5) is equivalent to the existence of transformations of the form (3), which carries the operators of one algebra into the operators of the other. Then by the change-of-variables formula in the operators $X' = X(x')\frac{\partial}{\partial x'}$, taking the form of (4) and (5) into account, we obtain

$$\xi'^{\alpha}_{a_0(0)}(x') + \varepsilon\xi'^{\alpha}_{a_0(1)}(x') \approx \big(\xi^{\beta}_{a_0(0)}(x) + \varepsilon\xi^{\beta}_{a_0(1)}(x)\big)\frac{\partial x'^{\alpha}}{\partial x^{\beta}}, \tag{6}$$

$$\varepsilon\xi'^{\alpha}_{a_1(0)}(x') \approx \varepsilon\xi^{\beta}_{a_1(0)}(x)\frac{\partial x'^{\alpha}}{\partial x^{\beta}}, \tag{7}$$

$$\alpha,\beta = 1,\dots,n, \quad a_0 = 1,\dots,r_0, \quad a_1 = r_0+1,\dots,r.$$

Consequently, the construction of similarity transformations reduces to solving the system $n\cdot r$ of first-order semilinear partial differential equations with a small parameter of the form (6) and (7) in the unknowns x'^{α} for $\alpha = 1,\dots,n$. It is convenient to consider (6) and (7) as a set of n systems (resulting for fixed α), each of which is determined by the same r differential operators X_{a_0}, εX_{a_1}, and which are connected one to another with the free terms (the left-hand sides of (6) and (7)).

We study the completeness and compatibility conditions of the system (6), (7) in the following sections. Observe that the nondegeneracy condition for the similarity transformation (3), expressed as

$$\operatorname{rg}\left\|\frac{\partial\psi_0^{\alpha}}{\partial x^{\beta}}\right\| = n,$$

yields the equality of the ranks of the matrices

$$\operatorname{rg}\big\|\xi'^{\alpha}_{a_0(0)}\big\| = \operatorname{rg}\big\|\xi^{\alpha}_{a_0(0)}\big\|, \quad \operatorname{rg}\big\|\xi'^{\alpha}_{a(0)}\big\| = \operatorname{rg}\big\|\xi^{\alpha}_{a(0)}\big\|, \tag{8}$$

where $\big\|\xi^{\alpha}_{a(0)}\big\| = \left\|\begin{matrix}\xi^{\alpha}_{a_0(0)}\\ \xi^{\alpha}_{a_1(0)}\end{matrix}\right\|$, $\alpha = 1,\dots,n$, $a_0 = 1,\dots,r_0$, and $a_1 = r_0+1,\dots,r$.

Lemma 1. *If two algebras L and L' with basis operators* (4) *and* (5) *are similar then* (8) *holds.*

Moreover, the similarity of approximate algebras L and L' implies the similarity of exact algebras $L_{0(0)}$ and $L'_{0(0)}$ with the basis operators $X_{a_0(0)}$ and $X'_{a_0(0)}$ respectively and the similarity of algebras $L_{(0)}$ and $L'_{(0)}$ generated by the operators $X_{a_0(0)}, X_{a_1(0)}$ and $X'_{a_0(0)}, X'_{a_1(0)}$ respectively. Here

$$X_{a_i(0)} = \xi^{\alpha}_{a_i(0)}(x)\frac{\partial}{\partial x^{\alpha}}, \quad X'_{a_i(0)} = \xi'^{\alpha}_{a_i(0)}(x')\frac{\partial}{\partial x'^{\alpha}}, \quad i = 0,1.$$

2. The Completeness Condition for the System

Rewrite (6) and (7) for a fixed α as

$$X_{a_0}(x'^{\alpha}) \approx \xi'^{\alpha}_{a_0}(x',\varepsilon), \quad \varepsilon X_{a_1}(x'^{\alpha}) \approx \varepsilon\xi'^{\alpha}_{a_1}(x')$$

and verify its completeness. In order to calculate the Jacobi brackets (see [8]) introduce the differential operators $\overline{X}_{a_0} = \big(\xi^{i}_{a_0(0)}(x) + \varepsilon\xi^{i}_{a_0(1)}(x)\big)D_i$ corresponding to (6), and $\overline{X}_{a_1} = \xi^{i}_{a_1(0)}(x)D_i$ corresponding to (7), where $D_i = \frac{\partial}{\partial x^i} + \frac{\partial x'^{\alpha}}{\partial x^i}\frac{\partial}{\partial x'^{\alpha}} + \dots$ is the total derivative operator ($i = 1,\dots,n$). Then determine the Jacobi brackets $\{\,,\}$ for the equations of the system (6), (7) as

$$\begin{aligned}&\big\{\varepsilon^t X_{a_t}(x'^{\beta}) - \varepsilon^t\xi'^{\beta}_{a_t}(x',\varepsilon), \varepsilon^s X_{b_s}(x'^{\beta}) - \varepsilon^s\xi'^{\beta}_{b_s}(x',\varepsilon)\big\}\\ &\approx \varepsilon^m\big(\overline{X}_{a_t}\big(X_{b_s}x'^{\beta} - \xi'^{\beta}_{b_s}(x',\varepsilon)\big) - \overline{X}_{b_s}\big(X_{a_t}x'^{\beta} - \xi'^{\beta}_{a_t}(x',\varepsilon)\big)\big)\\ &\approx \varepsilon^m\left([X_{a_t}, X_{b_s}]x'^{\beta} + \left(\xi'^{\mu}_{b_s}\frac{\partial\xi'^{\beta}_{a_t}}{\partial x'^{\mu}} - \xi'^{\mu}_{a_t}\frac{\partial\xi'^{\beta}_{b_s}}{\partial x'^{\mu}}\right)\right),\end{aligned} \tag{9}$$

where $s,t = 0,1$, $m = \max\{s,t\}$, and $[X_{a_t}, X_{b_s}] \approx X_{a_t}(X_{b_s}) - X_{b_s}(X_{a_t})$ is the commutator of X_{a_t} and X_{b_s}.

Since X_{a_0}, εX_{a_1} and X'_{a_0}, $\varepsilon X'_{a_1}$ constitute bases for approximate Lie algebras L and L', it follows from (9) with $t=0$, $s=0,1$ that

$$\{X_{a_0}(x'^{\beta}) - \xi'^{\beta}_{a_0}(x',\varepsilon), X_{b_0}(x'^{\beta}) - \xi'^{\beta}_{b_0}(x',\varepsilon)\} \approx c^{j}_{a_0 b_0}\xi^{\mu}_{j}(x',\varepsilon)\frac{\partial x'^{\beta}}{\partial x^{\mu}} - c'^{j}_{a_0 b_0}\xi'^{\beta}_{j}(x',\varepsilon),$$

$$\varepsilon\{X_{a_0(0)}(x'^{\beta}) - \xi'^{\beta}_{a_0(0)}(x'), X_{b_1}(x'^{\beta}) - \xi'^{\beta}_{b_1}(x')\} \approx \varepsilon c^{j}_{a_0 b_1}\xi^{\mu}_{j}(x')\frac{\partial x'^{\beta}}{\partial x^{\mu}} - \varepsilon c'^{j}_{a_0 b_1}\xi'^{\beta}_{j}(x'),$$

where $c^{j}_{a_0 b_s}$ and $c'^{j}_{a_0 b_s}$ for $j=1,\dots,r$ are the structure constants of L and L' respectively. By the isomorphism of the original algebras ($c'^{j}_{a_0 b_s} = c^{j}_{a_0 b_s}$ for $s=0,1$) these brackets amount to linear combinations of equations in (6) and (7) and cannot lead to new equations. Observe that if the original algebras are not isomorphic then algebraic equations on x'^{β} result, which means that the system in question is incompatible.

The Jacobi bracket of two equations of the first order in ε (the equality (9) for $t=s=1$) in the general case can lead to a new equation of the same form, while the commutator of the corresponding approximate operators is always equal to zero since it is of the second order in ε. This follows since calculating the Jacobi brackets involves calculating commutators in $X_{a_0(0)}$, $X_{a_1(0)}$ and $X'_{a_0(0)}$, $X'_{a_1(0)}$ which in the general case do not form Lie algebras. If some commutators of the operators of the form $X_{a_1(0)}$ are not representable as linear combinations of $X_{a_0(0)}$ and $X_{a_1(0)}$ then we add them to these operators. Make similar operations for the corresponding operators in $X'_{a_0(0)}$ and $X'_{a_1(0)}$. Eventually we obtain exact Lie algebras $L_{(0)}$ and $L'_{(0)}$ with the operators $X_{a_0(0)}$, $X_{a_1(0)}$, $\widetilde{X}_{d_1(0)}$ and $X'_{a_0(0)}$, $X'_{a_1(0)}$, $\widetilde{X}'_{d_1(0)}$. The operators $\widetilde{X}_{d_1(0)}$ and $\widetilde{X}'_{d_1(0)}$ generate new equations in (7). If the algebras $L_{(0)}$ and $L'_{(0)}$ have different structure constants then (9) implies that the system

$$X_{a_0}(x'^{\alpha}) \approx \xi'^{\alpha}_{a_0}(x',\varepsilon), \quad \varepsilon X_{a_1}(x'^{\alpha}) \approx \varepsilon\xi'^{\alpha}_{a_1}(x'), \quad \varepsilon\widetilde{X}_{d_1}(x'^{\alpha}) \approx \varepsilon\tilde{\xi}'^{\alpha}_{d_1}(x'), \tag{10}$$

$\alpha=1,\dots,n$, $a_0=1,\dots,r_0$, $a_1=r_0+1,\dots,r$, $d_1=r+1,\dots,\tilde{r}$, $r\le\tilde{r}\le n$, is incompatible.

Consequently, the necessary conditions for the similarity of approximate Lie algebras are not only the conditions of their isomorphism, but also the condition of the isomorphism of some exact Lie algebras $L_{(0)}$ and $L'_{(0)}$, as well as the *concordance* of the algebra structures of L with $L_{(0)}$ and of L' with $L'_{(0)}$. This means that it is possible to choose the basis operators in L and $L_{(0)}$ so that the structure constants in the commutators of operators in L of type X_{a_0} among each other and those of type X_{a_0} with εX_{a_1} in the principal order by ε coincide with the structure constants in the commutators of the corresponding operators in $L_{(0)}$. Assume henceforth that the structures of exact and approximate Lie algebras are concordant.

The above implies that the construction of similarity transformations in the Lie algebras with X_{a_0}, εX_{a_1} and X'_{a_0}, $\varepsilon X'_{a_1}$ is equivalent to solving the same problem for approximate algebras with X_{a_0}, εX_{a_1}, $\varepsilon\widetilde{X}_{d_1}$ and X'_{a_0}, $\varepsilon X'_{a_1}$, $\varepsilon\widetilde{X}'_{d_1}$. In these algebras the operators $X_{a_0(0)}$, $X_{a_1(0)}$, $\widetilde{X}_{d_1(0)}$ and $X'_{a_0(0)}$, $X'_{a_1(0)}$, $\widetilde{X}'_{d_1(0)}$ constitute bases for exact Lie algebras. Precisely these algebras usually arise in applications while we consider approximate algebras resulting as perturbations of some exact algebras. Therefore, henceforth we consider only the algebras of this type and assume that (6), (7) is a complete system.

3. The Compatibility Condition

We will consider a complete system (6), (7), in which all equations are linearly independent. However, the equations of the system can turn out linearly connected: some of them result as linear combinations of others with coefficients which are functions of x^{α} or x'^{α}.

3.1. The case of linearly disconnected operators. Suppose that all operators of the approximate algebra L (and, by (8), of L' as well) are not linearly connected: $\mathrm{rg}\left\|\xi^{\alpha}_{a(0)}\right\| = r$. Then the system

(6), (7) is integrable (see [5]) and its general solution in the zeroth order by ε depends on n arbitrary functions of $n-r$ variables, and in the first order by ε on n arbitrary functions of $n-r_0$ variables. Moreover, the arguments in the arbitrary functions are invariants of approximate transformation groups.

Therefore, we have

Theorem 1. *Suppose that two r-dimensional approximate Lie algebras L and L' are isomorphic and the following hold:*

(1) *the structures of Lie algebras with the operators $\langle X_{a_0(0)}, X_{a_1(0)}\rangle$ and $\langle X'_{a_0(0)}, X'_{a_1(0)}\rangle$ are concordant with the structures of the corresponding approximate Lie algebras with the operators* (4) *and* (5) *respectively;*

(2) $\mathrm{rg}\|\xi^{\alpha}_{a_0(0)}\| = \mathrm{rg}\|\xi'^{\alpha}_{a_0(0)}\|$, $\mathrm{rg}\|\xi^{\alpha}_{a(0)}\| = \mathrm{rg}\|\xi'^{\alpha}_{a(0)}\|$;

(3) $\mathrm{rg}\|\xi^{\alpha}_{a(0)}\| = r$.

Then L and L' are similar and, moreover, so are the corresponding approximate groups $\widetilde{G}_r$ and $\widetilde{H}_r$.

Remark 1. If approximate Lie algebras are not perturbations of exact algebras then by analogy with Section 2 the Jacobi brackets can yield new differential equations. Then Theorem 1 remains valid if conditions 1 and 2 hold for (10):

$$\mathrm{rg}\begin{Vmatrix} \xi^{\alpha}_{a(0)} \\ \xi^{\alpha}_{d_1(0)} \end{Vmatrix} = \mathrm{rg}\begin{Vmatrix} \xi'^{\alpha}_{a(0)} \\ \xi'^{\alpha}_{d_1(0)} \end{Vmatrix} = \tilde{r}.$$

3.2. The case of linearly connected operators. Suppose that the essential operators of the algebra are linearly connected: part of the operators $X_{a_0(0)}$ and $X_{a_1(0)}$ is a linear combination $\varphi(x)$ of the remaining operators with some functions as coefficients. (By (8) we also have this for the operators in L'.) If we put $\mathrm{rg}\|\xi^{\alpha}_{a(0)}\| = q$ and $\mathrm{rg}\|\xi^{\alpha}_{a_0(0)}\| = q_0$ then linear connectedness means that $q < r$ and $q_0 \le r_0$, and consequently the two cases are possible: $r_0 \le q$ or $q < r_0$. Consider the case $q_0 < r_0 < q$; the remaining cases are similar.

In order to determine the form of arbitrary functions in the similarity transformation, construct approximate invariants of L as solutions to

$$\begin{gathered}\big(\xi^{\alpha}_{a_0(0)}(x) + \varepsilon\xi^{\alpha}_{a_0(1)}(x)\big)\frac{\partial f}{\partial x^{\alpha}} = 0, \quad \alpha = 1,\dots,n,\ a_0 = 1,\dots,r_0, \\ \varepsilon\xi^{\alpha}_{a_1(0)}(x)\frac{\partial f}{\partial x^{\alpha}} = 0, \quad a_1 = r_0+1,\dots,r.\end{gathered} \tag{11}$$

Assuming that a solution is of the form $f(x,\varepsilon) \approx f_{(0)}(x) + \varepsilon f_{(1)}(x) + o(\varepsilon)$, for the unknown functions $f_{(0)}(x)$ and $f_{(1)}(x)$ we obtain from (11) the systems Ω_0 and Ω_1 of the form

$$\begin{gathered}\Omega_0: \quad \xi^{\alpha}_{a_0(0)}(x)\frac{\partial f_{(0)}}{\partial x^{\alpha}} = 0, \quad \xi^{\alpha}_{a_1(0)}(x)\frac{\partial f_{(0)}}{\partial x^{\alpha}} = 0, \\ \Omega_1: \quad \xi^{\alpha}_{a_0(0)}(x)\frac{\partial f_{(1)}}{\partial x^{\alpha}} + \xi^{\alpha}_{a_0(1)}(x)\frac{\partial f_{(0)}}{\partial x^{\alpha}} = 0.\end{gathered}$$

The system Ω_0 of homogeneous equations is complete and has $n-q$ independent solutions. The system Ω_1 can be regarded as a system of inhomogeneous equations for determining $f_{(1)}(x)$ under the condition that $f_{(0)}(x)$ are available and satisfy Ω_0. The completeness of Ω_1 follows from the completeness of (6) and (7), and its compatibility generates additional equations for $f_{(0)}(x)$. Indeed, since $q_0 < r_0$, the $r_0 - q_0$ operators $X_{a_0(0)}$ are represented as linear combinations of the remaining q_0 disconnected operators:

$$\xi^{\alpha}_{p(0)} = \varphi^{h}_{p}(x)\xi^{\alpha}_{h(0)}$$

for some functions $\varphi^{h}_{p}(x)$, where $p = q_0+1,\dots,r_0$ and $h = 1,\dots,q_0$. Inserting these relations into Ω_1, we obtain $r_0 - q_0$ differential equations

$$\big(\xi^{\alpha}_{p(1)}(x) - \varphi^{h}_{p}(x)\xi^{\alpha}_{h(1)}(x)\big)\frac{\partial f_{(0)}}{\partial x^{\alpha}} = 0 \quad \text{for } f_{(0)}.$$

Suppose that the system Ω_0^* obtained by adding these equations to Ω_0 and completing the extended system includes q^* ($\geq q$) linearly disconnected equations. Then it has $s_0 = n - q^*$ independent solutions, and the original system (11) has $s_1 = n - q_0$ independent solutions of the form (see [9] for instance)

$$f^{\vartheta}(x,\varepsilon) = f^{\vartheta}_{(0)}(x) + \varepsilon f^{\vartheta}_{(1)}(x), \quad \vartheta = 1,\dots,s_0,$$
$$f^{\rho}(x,\varepsilon) = \varepsilon f^{\rho}_{(0)}(x), \quad \rho = s_0+1,\dots,s_1,$$

which determine invariants of L.

Using the solutions just found, construct the change of variables

$$\begin{aligned} \bar{x}^{\lambda} &= x^{\lambda}, \quad \lambda = 1,\dots,q_0, \\ \bar{x}^{\mu} &= f^{\mu}_{(0)}(x), \quad \mu = q_0+1,\dots,q^*, \\ \bar{x}^{\sigma} &= f^{\sigma}_{(0)}(x) + \varepsilon f^{\sigma}_{(1)}(x), \quad \sigma = q^*+1,\dots,n. \end{aligned} \tag{12}$$

Then by (6) and (7) the basis operators become

$$\overline{X}_{a_0} = \left(\bar{\xi}^{\lambda}_{a_0(0)} + \varepsilon\bar{\xi}^{\lambda}_{a_0(1)}\right)\frac{\partial}{\partial\bar{x}^{\lambda}} + \varepsilon\bar{\xi}^{\mu}_{a_0(1)}\frac{\partial}{\partial\bar{x}^{\mu}}, \quad \varepsilon\overline{X}_{a_1} = \varepsilon\bar{\xi}^{\lambda}_{a_1(0)}\frac{\partial}{\partial\bar{x}^{\lambda}} + \varepsilon\bar{\xi}^{\mu}_{a_1(0)}\frac{\partial}{\partial\bar{x}^{\mu}}, \tag{13}$$

$$\lambda = 1,\dots,q_0, \quad \mu = q_0+1,\dots,q^*, \quad a_0 = 1,\dots,r_0, \quad a_1 = r_0+1,\dots,r.$$

Likewise, construct invariants of L' and make the corresponding change of variables, upon which the operators in L' reduce to the form (13). In this case for the similarity of L and L' it is necessary that the number of invariants of these algebras be the same. This is possible if $\operatorname{rg}\|\xi\| = \operatorname{rg}\|\xi'\|$, where [10]

$$\|\xi\| = \begin{Vmatrix} \xi^{\alpha}_{a_0(0)} & \xi^{\alpha}_{a_0(1)} \\ 0 & \xi^{\alpha}_{a_0(0)} \\ 0 & \xi^{\alpha}_{a_1(0)} \end{Vmatrix}, \qquad \|\xi'\| = \begin{Vmatrix} \xi'^{\alpha}_{a_0(0)} & \xi'^{\alpha}_{a_0(1)} \\ 0 & \xi'^{\alpha}_{a_0(0)} \\ 0 & \xi'^{\alpha}_{a_1(0)} \end{Vmatrix},$$

while the basis minors of these matrices must be constructed on using the corresponding isomorphic operators. Then, omitting the bar, we write (6), (7) in the new variables as

$$\left(\xi^{\lambda}_{a_0(0)}(x) + \varepsilon\xi^{\lambda}_{a_0(1)}(x)\right)\frac{\partial x'^{\kappa}}{\partial x^{\lambda}} + \varepsilon\xi^{\mu}_{a_0(1)}(x)\frac{\partial x'^{\kappa}}{\partial x^{\mu}} \approx \xi'^{\kappa}_{a_0(0)}(x') + \varepsilon\xi'^{\kappa}_{a_0(1)}(x'), \tag{14}$$

$$\varepsilon\xi^{\lambda}_{a_1(0)}(x)\frac{\partial x'^{\kappa}}{\partial x^{\lambda}} + \varepsilon\xi^{\mu}_{a_1(0)}(x)\frac{\partial x'^{\kappa}}{\partial x^{\mu}} \approx \varepsilon\xi'^{\kappa}_{a_1(0)}(x'), \tag{15}$$

$$\left(\xi^{\lambda}_{a_0(0)}(x) + \varepsilon\xi^{\lambda}_{a_0(1)}(x)\right)\frac{\partial x'^{\nu}}{\partial x^{\lambda}} + \varepsilon\xi^{\mu}_{a_0(1)}(x)\frac{\partial x'^{\nu}}{\partial x^{\mu}} \approx \varepsilon\xi'^{\nu}_{a_0(1)}(x'), \tag{16}$$

$$\varepsilon\xi^{\lambda}_{a_1(0)}(x)\frac{\partial x'^{\nu}}{\partial x^{\lambda}} + \varepsilon\xi^{\mu}_{a_1(0)}(x)\frac{\partial x'^{\nu}}{\partial x^{\mu}} \approx \varepsilon\xi'^{\nu}_{a_1(0)}(x'), \tag{17}$$

$$\lambda,\kappa = 1,\dots,q_0, \quad \mu,\nu = q_0+1,\dots,q^*, \quad a_0 = 1,\dots,r_0, \quad a_1 = r_0+1,\dots,r.$$

The resulting systems of partial differential equations in x'^{κ} and x'^{ν} are connected to each other with coefficients and free terms. In order to solve them it is necessary to verify the completeness and compatibility conditions. In Section 2 we established that the system is complete. However, since it includes linearly connected equations, we can choose an order q_0 minor of $\left\|\xi'^{\alpha}_{a_0(0)}\right\|$ and renumber the indices so that

$$\xi'^{\alpha}_{p_0(0)} = \varphi'^{h_0}_{p_0}(x')\xi'^{\alpha}_{h_0(0)}, \quad p_0 = q_0+1,\dots,r_0, \; h_0 = 1,\dots,q_0. \tag{18}$$

Similarly we can choose an order q minor of $\left\|\xi'^{\alpha}_{a(0)}\right\|$, while by the construction of this matrix the chosen minor will include the already available minor of order q_0, and the rows of this matrix satisfy

$$\xi'^{\alpha}_{p_1(0)} = \varphi'^{h_0}_{p_1}(x')\xi'^{\alpha}_{h_0(0)} + \varphi'^{h_1}_{p_1}(x')\xi'^{\alpha}_{h_1(0)}, \tag{19}$$
$$h_0 = 1,\dots,q_0, \quad h_1 = r_0+1,\dots,m, \quad m = r_0+q-q_0,$$
$$p_1 = m+1,\dots,r, \quad \alpha = 1,\dots,q^*.$$

Choosing the corresponding basis rows in the matrices $\left\|\xi^{\alpha}_{a_0(0)}\right\|$ and $\left\|\xi^{\alpha}_{a(0)}\right\|$, in which the rank q_0 and q minors occur, we can write down similar relations for the unprimed variables as well. Inserting the expressions in (18) into (14) with $a_0 = q_0+1,\dots,r_0$, we obtain the equations of the form

$$\varepsilon\left[\big(\xi^{\lambda}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\lambda}_{h_0(1)}\big)\frac{\partial x'^{\kappa}}{\partial x^{\lambda}} + \big(\xi^{\mu}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\mu}_{h_0(1)}\big)\frac{\partial x'^{\kappa}}{\partial x^{\mu}}\right]$$
$$= \big(\varphi'^{h_0}_{p_0} - \varphi^{h_0}_{p_0}\big)\xi'^{\kappa}_{h_0(0)} + \varepsilon\big[\big(\varphi^{h_0}_{p_0}\xi'^{\kappa}_{h_0(1)} - \xi'^{\kappa}_{p_0(1)}\big)\big], \tag{20}$$

which for $\varepsilon = 0$ amount to algebraic equations in x' and x, and for ε in the first power they are differential equations. Inserting (18) into (16) with $a_0 = q_0+1,\dots,r_0$, we obtain the differential equations

$$\varepsilon\big(\xi^{\lambda}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\lambda}_{h_0(1)}\big)\frac{\partial x'^{\nu}}{\partial x^{\lambda}} + \varepsilon\big(\xi^{\mu}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\mu}_{h_0(1)}\big)\frac{\partial x'^{\nu}}{\partial x^{\mu}} = \varepsilon\big(\xi'^{\nu}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi'^{\nu}_{h_0(1)}\big). \tag{21}$$

Finally, inserting (19) into (15) and (17) with $a_1 = r_0+q-q_0+1,\dots,r$, we obtain the system of algebraic equations

$$\varepsilon\big(\varphi'^{h_0}_{p_1} - \varphi^{h_0}_{p_1}\big) \approx 0, \quad \varepsilon\big(\varphi'^{h_1}_{p_1} - \varphi^{h_1}_{p_1}\big) \approx 0, \tag{22}$$

where $h_0 = 1,\dots,q_0$, $h_1 = r_0+1,\dots,m$, $p_1 = m+1,\dots,r$.

Add (20) and (21) to the remaining linearly independent equations of (14)–(17). The resulting system of differential equations can turn out incomplete since the Jacobi bracket of, for instance, (17) and (21) includes the commutator of the operators in (13) and

$$\widetilde{X}_{p_0} = \big(\xi^{\lambda}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\lambda}_{h_0(1)}\big)\frac{\partial}{\partial x^{\lambda}} + \big(\xi^{\mu}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\mu}_{h_0(1)}\big)\frac{\partial}{\partial x^{\mu}},$$

which in the general case creates new equations. The same commutators are calculated in constructing the invariants; hence, the resulting complete system will have exactly q^* linearly disconnected differential equations with the unknowns $x'^1,\dots,x'^{q^*}$. However, then the compatibility conditions can yield additional algebraic equations of the form (22).

Seek a solution to the new complete system in the form (3). Then the unknown functions $\psi^{\alpha}_0(x)$ and $\psi^{\alpha}_1(x)$ ($\alpha = 1,\dots,q^*$) satisfy the equations of the system $\overline{\Omega}_0$ and $\overline{\Omega}_1$ of the following form: the system $\overline{\Omega}_0$

$$\begin{cases} \xi^{\lambda}_{h_0(0)}(x)\frac{\partial\psi^{\kappa}_0}{\partial x^{\lambda}} = \xi'^{\kappa}_{h_0(0)}(\psi_0), \\ \xi^{\lambda}_{h_1(0)}(x)\frac{\partial\psi^{\kappa}_0}{\partial x^{\lambda}} + \xi^{\mu}_{h_1(0)}(x)\frac{\partial\psi^{\kappa}_0}{\partial x^{\mu}} = \xi'^{\kappa}_{h_1(0)}(\psi_0), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{23}$$

$$\begin{cases} \xi^{\lambda}_{h_0(0)}(x)\frac{\partial\psi^{\nu}_0}{\partial x^{\lambda}} = 0, \\ \xi^{\lambda}_{h_1(0)}(x)\frac{\partial\psi^{\nu}_0}{\partial x^{\lambda}} + \xi^{\mu}_{h_1(0)}(x)\frac{\partial\psi^{\nu}_0}{\partial x^{\mu}} = \xi'^{\nu}_{h_1(0)}(\psi_0), \\ \big(\xi^{\lambda}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\lambda}_{h_0(1)}\big)\frac{\partial\psi^{\nu}_0}{\partial x^{\lambda}} + \big(\xi^{\mu}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\mu}_{h_0(1)}\big)\frac{\partial\psi^{\nu}_0}{\partial x^{\mu}} = \big(\xi'^{\nu}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi'^{\nu}_{h_0(1)}\big), \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases} \tag{24}$$

$$\begin{cases} \varphi'^{h_0}_{p_0}(\psi_0) - \varphi^{h_0}_{p_0}(x) = 0, \\ \varphi'^{h_0}_{p_1}(\psi_0) - \varphi^{h_0}_{p_1}(x) = 0, \\ \varphi'^{h_1}_{p_1}(\psi_0) - \varphi^{h_1}_{p_1}(x) = 0, \\ \dots\dots\dots\dots\dots \end{cases} \tag{25}$$

and the system $\overline{\Omega}_1$

$$\begin{cases} \xi^{\lambda}_{h_0(0)}(x)\frac{\partial\psi^{\kappa}_1}{\partial x^{\lambda}} + \xi^{\lambda}_{h_0(1)}(x)\frac{\partial\psi^{\kappa}_0}{\partial x^{\lambda}} + \xi^{\mu}_{h_0(1)}(x)\frac{\partial\psi^{\kappa}_0}{\partial x^{\mu}} = \frac{\partial\xi'^{\kappa}_{h_0(0)}}{\partial\psi^{\alpha}_0}\psi^{\alpha}_1 + \xi'^{\kappa}_{h_0(1)}(\psi_0), \\ \big(\xi^{\lambda}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\lambda}_{h_0(1)}\big)\frac{\partial\psi^{\kappa}_0}{\partial x^{\lambda}} + \big(\xi^{\mu}_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^{\mu}_{h_0(1)}\big)\frac{\partial\psi^{\kappa}_0}{\partial x^{\mu}} = \frac{\partial\varphi'^{h_0}_{p_0}}{\partial\psi^{\alpha}_0}\psi^{\alpha}_1 + \big(\varphi^{h_0}_{p_0}\xi'^{\kappa}_{h_0(1)} - \xi'^{\kappa}_{p_0(1)}\big), \end{cases} \tag{26}$$

$$\xi^{\lambda}_{h_0(0)}(x)\frac{\partial\psi^{\nu}_1}{\partial x^{\lambda}} + \xi^{\lambda}_{h_0(1)}(x)\frac{\partial\psi^{\nu}_0}{\partial x^{\lambda}} + \xi^{\mu}_{h_0(1)}(x)\frac{\partial\psi^{\nu}_0}{\partial x^{\mu}} = \xi'^{\nu}_{h_0(1)}(\psi_0), \tag{27}$$

$$\lambda,\kappa = 1,\ldots,q_0, \quad \mu,\nu = q_0+1,\ldots,q^*, \quad h_0 = 1,\ldots,q_0, \quad p_0 = q_0+1,\ldots,r_0,$$

$$h_1 = r_0+1,\ldots,m, \quad p_1 = m+1,\ldots,r, \quad m = r_0+q-q_0.$$

Here the dots stand for the additional equations adding in the completeness. Observe that all these equations include derivatives with respect to $x^{q_0+1},\ldots,x^{q^*}$.

If in the system $\overline{\Omega}_0$ the algebraic equations (25) are compatible and yield no relations among just x^{α} or ψ^{α}_0, and $\mathrm{rg}\big\|\frac{\partial(\varphi'^{h_0}_{p_0},\varphi'^{h_0}_{p_1},\varphi'^{h_1}_{p_1},\ldots)}{\partial\psi^{\alpha}_0}\big\| = \hat{q}$, then we can express $\hat{q}$ functions $\psi^1_0,\ldots,\psi^{\hat{q}}_0$ in terms of x and the unknown functions $\psi^{\hat{q}+1}_0,\ldots,\psi^{q^*}_0$:

$$\psi^{\beta}_0 = \psi^{\beta}_0\big(x,\psi^{\hat{q}+1}_0,\ldots,\psi^{q^*}_0\big), \quad \beta = 1,\ldots,\hat{q}. \tag{28}$$

Suppose that $\hat{q} < q_0$. The homogeneous equations (24_1) yield

$$\psi^{\nu}_0 = \psi^{\nu}_0\left(x^{q_0+1},\ldots,x^n\right), \quad \nu = q_0+1,\ldots,q^*. \tag{29}$$

Then, taking (28) and (29) into account, we obtain from (23_1) for each function $\psi^{\hat{q}+1}_0,\ldots,\psi^{q_0}_0$ a system of first-order q_0 linearly disconnected semilinear partial differential equations with q_0 independent variables $x^1,\ldots,x^{q_0}$. According to [8] we can write solutions to the systems of this type as

$$\psi^{\gamma}_0 = \psi^{\gamma}_0\big(x^1,\ldots,x^{q_0},\psi^{q_0+1}_0,\ldots,\psi^{q^*}_0,\theta^{\hat{q}+1}_0,\ldots,\theta^{q_0}_0\big), \quad \gamma = \hat{q}+1,\ldots,q_0,$$

with arbitrary functions $\theta^{\gamma}_0 = \theta^{\gamma}_0\left(x^{q_0+1},\ldots,x^n\right)$. By analogy, inserting this solution into the remaining q^*-q_0 linearly disconnected equations of (24) in q^*-q_0 independent variables $x^{q_0+1},\ldots,x^{q^*}$, by [8] we can write (29) as

$$\psi^{\nu}_0 = \psi^{\nu}_0\big(x^{q_0+1},\ldots,x^{q^*},\theta^{\hat{q}+1}_0,\ldots,\theta^{q_0}_0,\sigma^{q_0+1}_0,\ldots,\sigma^{q^*}_0\big), \quad \nu = q_0+1,\ldots,q^*,$$

where $\sigma^{\nu}_0 = \sigma^{\nu}_0\left(x^{q^*+1},\ldots,x^n\right)$ are arbitrary functions.

Proceed to solving $\overline{\Omega}_1$. Taking the constructed solutions to $\overline{\Omega}_0$ into account, for each of the unknown functions $\psi^{q_0+1}_1,\ldots,\psi^{q^*}_1$ we obtain from (27) a system of q_0 linearly disconnected differential equations with q_0 independent variables $x^1,\ldots,x^{q_0}$. Therefore, by [8] its solution is

$$\psi^{\nu}_1 = \psi^{\nu}_1\big(x^1,\ldots,x^{q_0},\theta^{\hat{q}+1}_0,\ldots,\theta^{q_0}_0,\sigma^{q_0+1}_0,\ldots,\sigma^{q^*}_0,\sigma^{q_0+1}_1,\ldots,\sigma^{q^*}_1\big), \quad \nu = q_0+1,\ldots,q^*,$$

where $\sigma^{\nu}_1 = \sigma^{\nu}_1\big(x^{q_0+1},\ldots,x^n\big)$ are arbitrary functions. Inserting these solutions into the right-hand side of (26_1) we obtain for the unknown functions $\psi^1_1,\ldots,\psi^{q_0}_1$ a system of q_0 linear inhomogeneous equations with q_0 independent variables $x^1,\ldots,x^{q_0}$. According to [8] its solution is

$$\psi^{\kappa}_1 = \psi^{\kappa}_1\big(x^1,\ldots,x^{q_0},\theta^{\hat{q}+1}_0,\ldots,\theta^{q_0}_0,\sigma^{q_0+1}_0,\ldots,\sigma^{q^*}_0,\sigma^1_1,\ldots,\sigma^{q^*}_1\big), \quad \kappa = 1,\ldots,q_0.$$

We can show that inserting all functions $\psi_1^1, \dots, \psi_1^{q^*}$ thus found into the differential equations of the system (26_2) for ψ_0^κ and adding the unsolved $q - q_0$ equations of the system (23_2), we obtain the system of $q^* - q_0$ equations

$$\xi^\mu_{h_1(0)}(x)\frac{\partial\psi_0^\gamma}{\partial\theta_0^\iota}\frac{\partial\theta_0^\iota}{\partial x^\mu} = \xi'^\gamma_{h_1(0)}(\psi_0) - \xi^\lambda_{h_1(0)}(x)\frac{\partial\psi_0^\gamma}{\partial x^\lambda} - \xi'^\nu_{h_1(0)}(x)\frac{\partial\psi_0^\gamma}{\partial\psi_0^\nu},$$

$$\big(\xi^\mu_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^\mu_{h_0(1)}\big)\frac{\partial\psi_0^\gamma}{\partial\theta_0^\iota}\frac{\partial\theta_0^\iota}{\partial x^\mu} = \frac{\partial\varphi'^{h_0}_{p_0}}{\partial\psi_0^\alpha}\psi_1^\alpha + \big(\varphi^{h_0}_{p_0}\xi'^\gamma_{h_0(1)} - \xi'^\gamma_{p_0(1)}\big)$$
$$-\big(\xi^\lambda_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^\lambda_{h_0(1)}\big)\frac{\partial\psi_0^\gamma}{\partial x^\lambda} - \big(\xi^\nu_{p_0(1)} - \varphi^{h_0}_{p_0}\xi^\nu_{h_0(1)}\big)\frac{\partial\psi_0^\gamma}{\partial\psi_0^\nu}$$

for the unknown functions $\theta_0^{\hat q+1}, \dots, \theta_0^{q_0}$ with $q^* - q_0$ independent variables $x^{q_0+1}, \dots, x^{q^*}$.

Since $\det\big\|\frac{\partial\psi_0^\gamma}{\partial\theta_0^\iota}\big\| \neq 0$, we can multiply both sides by the inverse matrix to $\big\|\frac{\partial\psi_0^\gamma}{\partial\theta_0^\iota}\big\|$ and obtain a system for the unknown functions $\theta_0^{\hat q+1}, \dots, \theta_0^{q_0}$ as we had for ψ_0^ν. According to [8] the solution to this system becomes

$$\theta_0^\gamma = \theta_0^\gamma\big(x^{q_0+1}, \dots, x^{q^*}, \sigma_0^{\hat q+1}, \dots, \sigma_0^{q^*}, \sigma_1^1, \dots, \sigma_1^{q^*}\big), \quad \gamma = \hat q + 1, \dots, q_0.$$

Therefore, we have proved the following statement.

Theorem 2. *Suppose that two r-parametric groups $\widetilde G_r$ and $\widetilde H_r$ are isomorphic and the following hold:*

(1) *the group structures with the operators $\langle X_{a_0(0)}, X_{a_1(0)}\rangle$ and $\langle X'_{a_0(0)}, X'_{a_1(0)}\rangle$ are concordant with the group structures of $\widetilde G_r$ and $\widetilde H_r$;*

(2) $\operatorname{rg}\|\xi_{a_0(0)}\| = \operatorname{rg}\|\xi'_{a_0(0)}\|$, $\operatorname{rg}\|\xi_{a(0)}\| = \operatorname{rg}\|\xi'_{a(0)}\| = q$, $\operatorname{rg}\|\xi\| = \operatorname{rg}\|\xi'\| = q^*$, $q \le q^* < r$*;*

(3) *the system of algebraic equations in $\overline\Omega_0$ is compatible and implies no new relations among x'^α or x^α.*

Then $\widetilde G_r$ and $\widetilde H_r$ are similar.

If we can express $\hat q$ functions from the algebraic equations then the similarity transformations depend on $q^* - \hat q$ arbitrary functions $\sigma_0^{\hat q+1}(x^{q^*+1}, \dots, x^n), \dots, \sigma_0^{q^*}(x^{q^*+1}, \dots, x^n)$ of $(n - q^*)$ variables and q^* arbitrary functions $\sigma_1^1(x^{q_0+1}, \dots, x^n), \dots, \sigma_1^{q^*}(x^{q_0+1}, \dots, x^n)$ of $(n - q_0)$ variables.

Theorems 1 and 2 remain valid for $r = r_0$ and $q = q_0$, when the algebra includes only the operators of the form

$$X_a = \big(\xi^\alpha_{a(0)}(x) + \varepsilon\xi^\alpha_{a(1)}(x)\big)\frac{\partial}{\partial x^\alpha}.$$

In this case the conditions in Theorems 1 and 2 completely coincide with the analogous similarity criteria for exact groups (see [2]).

4. Examples

Our theorems can be used in solving the classification problem for nonsimilar approximate Lie algebras. These classifications are based on the available classifications of nonsimilar 2- and 3-dimensional Lie algebras on the plane (see [1] for instance) and in space (see [11]). The examples below illustrate that the choice of an isomorphism is essential.

1. Consider the two-dimensional abelian Lie algebra on the plane with the operators

$$L_{(0)}: \quad X_{1(0)} = \frac{\partial}{\partial y}, \quad X_{2(0)} = x\frac{\partial}{\partial y},$$

and two perturbations of it with operators of the form

$$L: \quad X_1 = X_{1(0)}, \quad X_2 = X_{2(0)} + \varepsilon\alpha(x)\frac{\partial}{\partial x},$$
$$L': \quad X'_1 = X'_{1(0)}, \quad X'_2 = X'_{2(0)} + \varepsilon\beta(x')\frac{\partial}{\partial x'} + \varepsilon\gamma(x')\frac{\partial}{\partial y'}.$$

In this case the concordance conditions for the algebra structures are fulfilled, and the ranks of the matrices $\|\xi_{a(0)}\|$ and $\|\xi_{a_0(0)}\|$ coincide and are equal to $q=1$. Consequently, this is the case of linearly connected operators with $q<r$.

An isomorphism between the approximate Lie algebras L and L' can be constructed either by the rule $X_1 \to X_1'$, $X_2 \to X_2'$ or the rule $X_1 \to X_2'$, $X_2 \to X_1'$. Construct the similarity transformation generated by the first isomorphism. The system (6), (7) in this case becomes

$$\begin{cases} \frac{\partial x'}{\partial y} = 0, \\ \varepsilon\alpha(x)\frac{\partial x'}{\partial x} = \varepsilon\beta(x'), \end{cases} \qquad \begin{cases} \frac{\partial y'}{\partial y} = 1, \\ \varepsilon\alpha(x)\frac{\partial y'}{\partial x} = x' - x + \varepsilon\gamma(x'). \end{cases}$$

If we seek the similarity transformation in the form (3) then upon the separation by the powers of ε and solving the system $\overline{\Omega}_0$, we obtain a similarity transformation of the exact algebra $L_{(0)}$. The compatibility conditions of $\overline{\Omega}_1$ imply that the approximate algebras L and L' are similar only for $\alpha(x)=\beta(x)$. The corresponding similarity transformation (up to inessential terms) is of the form $x' = x - \varepsilon\gamma(x)$ and $y'=y$.

While constructing by a similar scheme the similarity transformation generated by the second isomorphism we find that the approximate algebras L and L' are similar only if $\alpha(x) = -x^3\beta(1/x)$, and the corresponding change of variables (up to inessential terms) is of the form

$$x' = \frac{1}{x} + \varepsilon\left\{y\beta\left(\frac{1}{x}\right) - \gamma\left(\frac{1}{x}\right)\right\}, \quad y' = \frac{y}{x} + \varepsilon\left\{\frac{y^2}{2}\beta\left(\frac{1}{x}\right)\right\}.$$

For instance, in this case the algebras in whose operators $\alpha(x) = x$ and $\beta(x') = -x'^2$ are similar.

2. Consider the 3-dimensional Lie algebra in the space of three variables with the operators

$$L_{(0)}: \quad X_{1(0)} = e^{-z}\frac{\partial}{\partial x}, \quad X_{2(0)} = ye^{-z}\frac{\partial}{\partial x}, \quad X_{3(0)} = \frac{\partial}{\partial z}.$$

This algebra is a realization of the Lie algebra with the commutation relations

$$[X_{2(0)},\ X_{3(0)}] = X_{2(0)}, \quad [X_{3(0)},\ X_{1(0)}] = -X_{1(0)}, \quad [X_{1(0)},\ X_{2(0)}] = 0$$

in the space of first-order differential operators with three variables. Consider the two perturbations preserving the commutation relations:

$$L: \quad X_1 = X_{1(0)}, \quad X_2 = X_{2(0)} + \varepsilon ye^{-z}\frac{\partial}{\partial y}, \quad X_3 = X_{3(0)},$$

$$L': \quad X_1' = X_{1(0)}', \quad X_2' = X_{2(0)}' + \varepsilon y'^2 e^{-z'}\frac{\partial}{\partial y'}, \quad X_3' = X_{3(0)}'.$$

It is obvious that the concordance conditions for the algebra structures are fulfilled, the matrices $\|\xi_{a(0)}\|$ and $\|\xi_{a_0(0)}\|$ coincide and have rank 2.

As in the first example, seek the similarity transformation generated by the isomorphism $X_1 \to X_1'$, $X_2 \to X_2'$, and $X_3 \to X_3'$ in the form (3). Then the solution of Ω_0 leads to the change of variables

$$x' = xe^{-\theta_0(y)} + \varphi_0(y) + \varepsilon\varphi_1(x,y,z),\ y' = y + \varepsilon\psi_1(x,y,z),\ z' = z + \theta_0(y) + \varepsilon\theta_1(x,y,z),$$

and the system Ω_1:

$$\begin{cases} \frac{\partial\varphi_1}{\partial x} = -\theta_1 e^{-\theta_0(y)}, \\ \frac{\partial\psi_1}{\partial x} = 0, \\ \frac{\partial\theta_1}{\partial x} = 0, \end{cases} \qquad \begin{cases} y\frac{\partial\varphi_1}{\partial x} - yxe^{-\theta_0(y)}\frac{\partial\theta_0}{\partial y} + y\frac{\partial\varphi_0}{\partial y} = \psi_1 e^{-\theta_0(y)} - y\theta_1 e^{-\theta_0(y)}, \\ \frac{\partial\psi_1}{\partial x} = ye^{-\theta_0(y)}, \\ \frac{\partial\theta_1}{\partial x} + \frac{\partial\theta_0}{\partial x} = 0 \end{cases}$$

turns out incompatible.

If we seek the change of variables generated by the isomorphism $X_1 \to X_2'$, $X_2 \to X_1'$, and $X_3 \to X_3'$; then both systems Ω_0 and Ω_1 are compatible, and the similarity transformation is

$$x' = \frac{x}{y} + \varepsilon\frac{x^2}{2y^2}, \quad y' = \frac{1}{y} + \varepsilon\frac{x}{y^2}, \quad z' = z.$$

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