Acta Applicandae Mathematicae **72:** 1–17, 2002.

Geometry of Differential Equations: A Concise Introduction

JOSEPH KRASIL'SHCHIK
Independent University of Moscow, B. Vlasievskij per., 11, 121002 Moscow, Russia.
e-mail: josephk@diffiety.ac.ru

(Received: 1 February 2001)

Abstract. A short introduction to geometrical theory of nonlinear differential equations is given to provide a unified overview to the collection 'Symmetries of differential equations and related topics'.

Mathematics Subject Classifications (2000): 58A20, 58J70, 58Z05, 34C14, 37K05, 37K10.

Key words: jets, nonlinear differential equations, geometry, symmetries, conservation laws.

The collection of papers below mostly deals with symmetries and conservation laws of (in particular, nonlinear) differential equations or topics closely related to these problems. As an editor, I thought it reasonable to include this short introduction stating all necessary definitions, notation and results of a general nature. More details can be found, for example, in [1, 6], while wider applications and generalizations are contained in [2]. It should be noted that our references here are in no way complete or exhaustive: they reflect only one of several viewpoints concerning the geometry of differential equations.

1. Jets and Lie Transformations ([1])

Let $\pi\colon E \to M$ be a locally trivial smooth bundle over a smooth manifold M. We shall consider vector bundles in the sequel, though this assumption is not essential. Denote by $\Gamma(\pi)$ the $C^\infty(M)$-module of sections $f\colon M \to E$. If necessary, we shall consider local sections. Let $\theta \in E$, $\pi(\theta) = x \in M$, and $f(x) = \theta$. The *k-jet* $[f]_x^k$ of f at x is the class of sections $f' \in \Gamma(\pi)$ such that their graphs are tangent to the graph of f at θ with order k. We use the notation

$$J_x^k(\pi) = \{[f]_x^k \mid f \in \Gamma(\pi)\} \quad \text{and} \quad J^k(\pi) = \bigcup_{x \in M} J_x^k(\pi).$$

The set $J^k(\pi)$ carries a natural structure of a smooth manifold, while $\pi_k\colon J^k(\pi) \to M$, $[f]_x^k \mapsto x$, is a smooth vector bundle. Moreover, the mappings

$$\pi_{k,k-1}\colon J^k(\pi) \to J^{k-1}(\pi), \qquad [f]_x^k \mapsto [f]_x^{k-1}, \quad k \geqslant 1,$$

are affine bundles. The bundle π_k is called the *bundle of k-jets* for the bundle π, while $J^k(\pi)$ is called the *manifold of k-jets*. To any section $f \in \Gamma(\pi)$ one can put into correspondence the section $j_k(f)\colon M \to J^k(\pi)$, $x \mapsto [f]_x^k$, which is called the *k-jet* of f.

If $\mathcal{U} \subset M$ is a local chart with coordinates $x_1, \dots, x_n$ such that π becomes trivial over $\mathcal{U}$ and $e_1, \dots, e_m$ is a basis of local sections over $\mathcal{U}$, then *adapted* (or *canonical*) coordinates $x_1, \dots, x_n, \dots, p_\sigma^j, \dots$ in $\pi_k^{-1}(\mathcal{U})$ arise defined by

$$p_\sigma^j([f]_x^k) = \left.\frac{\partial^{|\sigma|} f^j}{\partial x_\sigma}\right|_x,$$

where σ is multi-index of length $\leqslant k$ and f^j is the jth component of f in the basis $e_1, \dots, e_m$.

Let $\theta_{k+1} = [f]_x^{k+1} \in J^{k+1}(\pi)$ and M_f^k be the graph of the jet $j_k(f)$. Then the point θ_{k+1} is uniquely determined by θ_k and the tangent plane $L_{\theta_{k+1}} = T_{\theta_k}(M_f^k)$. The linear span $\mathcal{C}_{\theta_k} \subset T_{\theta_k} J^k(\pi)$ of all planes $L_{\theta_{k+1}}$, $\pi_{k+1,k}(\theta_{k+1}) = \theta_k$, is called the *Cartan plane* at θ_k. The correspondence $\theta_k \mapsto \mathcal{C}_{\theta_k}$ is called the *Cartan distribution* on $J^k(\pi)$.

PROPOSITION 1. *Let $\pi\colon E \to M$ be a vector bundle and $J^k(\pi)$ be the manifold of its k-jets.*

(1) *For any $\theta_k \in J^k(\pi)$ one has $\mathcal{C}_{\theta_k} = (\pi_{k,k-1})_*^{-1}(L_{\theta_k})$.*
(2) *An n-dimensional manifold $N \subset J^k(\pi)$ nondegenerately projecting to M is a maximal integral manifold of the Cartan distribution on $J^k(\pi)$ if and only if $N = M_f^k$ for some $f \in \Gamma(\pi)$.*

In adapted coordinates, the Cartan distribution is described by the system of 1-forms (the so-called *Cartan forms*)

$$\omega_\sigma^j = \mathrm{d}p_\sigma^j - \sum_{i=1}^n p_{\sigma i}\mathrm{d}x_i, \quad |\sigma| = 0, \dots, k-1,\ j = 1, \dots, m,$$

where $\sigma i = i_1 \dots i_s i$ for $\sigma = i_1 \dots i_s$, $1 \leqslant i, i_\alpha \leqslant n$. In particular, we see that $J^1(\pi)$ is a contact manifold, if $\dim \pi = 1$.

Cartan distribution determines geometry of the manifolds $J^k(\pi)$.

DEFINITION 1. Let $J^k(\pi)$ be the manifold of k-jets.

(1) A diffeomorphism $F\colon J^k(\pi) \to J^k(\pi)$ is called a *Lie transformation*, if it preserves the Cartan distribution, i.e., if $F_*(\mathcal{C}_{\theta_k}) = \mathcal{C}_{F(\theta_k)}$ for any $\theta_k \in J^k(\pi)$.
(2) A vector field X on $J^k(\pi)$ is called a *Lie field*, if the corresponding one-parameter group consists of Lie transformations.

If $F\colon J^k(\pi) \to J^k(\pi)$ is a Lie transformation, then for a point $\theta_{k+1} = (\theta_k, L_{\theta_{k+1}})$ one can set $F^{(1)}(\theta_{k+1}) = (F(\theta_k), F_* L_{\theta_{k+1}})$. The mapping $F^{(1)}$ is defined almost everywhere and is a Lie transformation in its domain. It is called the *first lifting* of F. We set by induction $F^{(l+1)} = (F^{(l)})^{(1)}$. For a Lie field X, we set

$$X^{(l)} = \left.\frac{\mathrm{d}A_t^{(l)}}{\mathrm{d}t}\right|_{t=0},$$

where A_t is the one-parameter group of the field X. Contrary to Lie transformations, the liftings $X^{(l)}$ are defined everywhere.

THEOREM 1 (Lie–Bäcklund theorem). *Any Lie transformation F of the space $J^k(\pi)$ is of the following form*:

(1) *If* $\dim\pi = 1$ *and* $k \geqslant 1$, *then* $F = F_1^{(k-1)}$ *for some contact transformation* $F_1\colon J^1(\pi) \to J^1(\pi)$.
(2) *If* $\dim\pi > 1$ *and* $k \geqslant 0$, *then* $F = F_0^{(k)}$ *for some diffeomorphism* F_0: $J^0(\pi) \to J^0(\pi)$.

A similar theorem is valid for Lie fields.

Remark 1. With natural modifications, the theory above (as well as what follows below) can be constructed in a more general context. Namely, instead of graph of sections in E one can consider jets of arbitrary n-dimensional submanifolds. Note that the manifold $J^k(E, n)$ arising in such a way can be covered by manifolds of the form $J^k(\xi)$, ξ being vector bundles.

2. Differential Equations and Classical Symmetries ([1])

Let $\pi\colon E \to M$ be a vector bundle.

DEFINITION 2. A *differential equation* of order k posed on sections of the bundle π is a submanifold $\mathcal{E} \subset J^k(\pi)$. A section $f \in \Gamma(\pi)$ is a *solution* of $\mathcal{E}$, if $M_f^k \subset \mathcal{E}$.

Let $\pi'\colon E' \to M$ be another vector bundle. Consider the pullback $\pi^*(\pi')$ and a section $\Delta \in \Gamma(\pi_k^*(\pi')) =_{\text{def}} \mathcal{F}_k(\pi, \pi')$. Then Δ can be identified with a (nonlinear) differential operator acting from $\Gamma(\pi)$ to $\Gamma(\pi')$ by $\Delta(f) = j_k(f)^*(\Delta)$, $f \in \Gamma(\pi)$. Note that $\mathcal{F}_k(\pi, \pi')$ is a module over the ring $C^\infty(J^k(\pi)) =_{\text{def}} \mathcal{F}_k(\pi)$. For any differential equation $\mathcal{E} \subset J^k(\pi)$ there exists a vector bundle π' and a differential operator $\Delta = \Delta_{\mathcal{E}} \in \mathcal{F}(\pi, \pi')$ such that $\mathcal{E} = \{\theta_k \in J^k(\pi) \mid \Delta_{\theta_k} = 0\}$. A section $f \in \Gamma(\pi)$ is a solution of $\mathcal{E}$ if and only if $\Delta_{\mathcal{E}}(f) = 0$. Vice versa, to any operator $\Delta \in \mathcal{F}_k(\pi, \pi')$ one can put in correspondence an equation $\mathcal{E} = \mathcal{E}_\Delta \subset J^k(\pi)$.

DEFINITION 3. Let $\mathcal{E} \subset J^k(\pi)$ be a differential equation.

(1) A Lie transformation $F\colon J^k(\pi) \to J^k(\pi)$ is called a (*finite classical*) *symmetry* of $\mathcal{E}$, if $F(\mathcal{E}) = \mathcal{E}$.
(2) A Lie field X on $\mathcal{E}$ is called an (*infinitesimal classical*) *symmetry* of $\mathcal{E}$, if it is tangent to $\mathcal{E}$.

From definitions it follows that finite symmetries take (local) solutions of $\mathcal{E}$ to local solutions. The same is valid for elements of one-parameter groups of infinitesimal symmetries. A solution f is said to be *invariant* (or *self-similar*) with respect to a finite symmetry F, if $F(f) = f$. It is X-invariant, if X is tangent to M_f^k, X being an infinitesimal symmetry.

Remark 2. Let in an adapted coordinate system a Lie field be expressed by

$$X = \sum_i a_i \frac{\partial}{\partial x_i} + \sum_{j,\sigma} b_\sigma^i \frac{\partial}{\partial p_\sigma^j}.$$

Then $b_{\sigma i}^j = D_i(b_\sigma^j) - \sum_s p_{\sigma s}^j D_i(a_s)$, where D_i are the total derivatives (see below). Thus, to compute the coefficients of the lifting, one only needs to know the functions a_i and $b_\emptyset^j$. In the case $m > 1$ they are arbitrary smooth functions on $J^0(\pi)$, while for $m = 1$ one has

$$a_i = -\frac{\partial f}{\partial x_i}, \qquad b_\emptyset = f - \sum_s p_s \frac{\partial f}{\partial p_s},$$

where f is an arbitrary smooth function on $J^1(\pi)$.

There is an alternative approach to the concept of a symmetry. Namely, let $\theta \in \mathcal{E}$ and $\mathcal{C}_\theta(\mathcal{E}) = \mathcal{C}_\theta \cap T_\theta \mathcal{E}$. Thus we obtain the *Cartan distribution on* $\mathcal{E}$. We say that a diffeomorphism $F\colon \mathcal{E} \to \mathcal{E}$ is an *intrinsic symmetry* of $\mathcal{E}$ if it preserves $\mathcal{C}(\mathcal{E})$. Obviously, any extrinsic symmetry gives rise to an intrinsic one. The following result shows that if the equation at hand is not 'highly overdetermined', all intrinsic symmetries are obtained in such a way.

THEOREM 2. *If $\mathcal{E} \subset J^k(\pi)$ is an equation of order k,* $\dim M = n$, $\dim \pi = m$ *and fibers of the projection $\pi_k|_{\mathcal{E}}$ are connected, then the condition*

$$\operatorname{codim} \mathcal{E} \leqslant \frac{(n+k-2)!}{(k-1)!(n-1)!} - 2$$

is sufficient for any intrinsic symmetry of $\mathcal{E}$ to be the restriction of some extrinsic one.

In particular, if $\mathcal{E}$ is a determined equation (i.e., its codimension coincides with dimension of π), then the condition above is violated in the following *exceptional* cases:

(a) $k = 1$ (equations and systems of 1st order);
(b) $n = 1$ (ordinary differential equations and systems);
(c) $m = 1$, $k = n = 2$ (scalar 2nd-order equations in one dependent and two independent variables).

3. Infinite Prolongations and Higher Symmetries ([1])

Consider the sequence of projections

$$M \xleftarrow{\pi} E \xleftarrow{\pi_{1,0}} J^1(\pi) \leftarrow \cdots \leftarrow J^k(\pi) \xleftarrow{\pi_{k+1,k}} J^{k+1}(\pi) \leftarrow \cdots .$$

Its inverse limit is denoted by $J^\infty(\pi)$ and is called the *manifold of infinite jets* for the bundle π. By definition, the vector fiber bundles $\pi_\infty\colon J^\infty(\pi) \to M$ and affine bundles $\pi_{\infty,k}\colon J^\infty(\pi) \to J^k(\pi)$ exist, satisfying $\pi_\infty = \pi_k \circ \pi_{\infty,k}$, $\pi_{\infty,k-1} = \pi_{k,k-1} \circ \pi_{\infty,k}$. Points of $J^\infty(\pi)$ are identified with classes $[f]_x^\infty$ of sections whose graphs are tangent to each other with infinite order. To any section $f \in \Gamma(\pi)$ the section $j_\infty(f) \in \Gamma(\pi_\infty)$ corresponds, $x \mapsto [f]_x^\infty$, with the graph $M_f^\infty \subset J^\infty(\pi)$, and one has $j_k(f) = \pi_{\infty,k} \circ j_\infty(f)$, $\pi_{\infty,k}(M_f^\infty) = M_f^k$ for any $f \in \Gamma(\pi)$ and $k \geqslant 0$.

The *algebra of smooth functions on* $J^\infty(\pi)$ is the filtered algebra $\mathcal{F}(\pi) =_{\text{def}} \bigcup_{k\geqslant 0} \mathcal{F}_k(\pi)$. If $\pi'\colon E' \to M$ is another vector bundle, we introduce the filtered $\mathcal{F}(\pi)$-module $\mathcal{F}(\pi,\pi') =_{\text{def}} \bigcup_{k\geqslant 0} \mathcal{F}_k(\pi,\pi')$ and identify its elements with nonlinear differential operators $\Gamma(\pi) \to \Gamma(\pi')$ of arbitrary order. A *vector field* on $J^\infty(\pi)$ is a filtered derivation $X\colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$. The module of all these derivations is denoted by $\mathrm{D}(\pi)$. The module of i*-differential forms* on $J^\infty(\pi)$ is also filtered and we define it by $\Lambda^i(\pi) =_{\text{def}} \bigcup_{k\geqslant 0} \Lambda^i(J^k(\pi))$.

Consider a point $\theta \in J^\infty(\pi)$ which may be understood as a sequence of points $\theta_k \in J^k(\pi)$, $\pi_{k+1,k}(\theta_{k+1}) = \theta_k$, $k = 0, 1, \ldots$. For any Cartan plane $\mathcal{C}_{\theta_{k+1}}$ one has $(\pi_{k+1,k})_* \mathcal{C}_{\theta_{k+1}} \subset \mathcal{C}_{\theta_k}$ and the Cartan plane $\mathcal{C}_\theta$ is defined as the corresponding inverse limit. The correspondence $\theta \mapsto \mathcal{C}_\theta$ is the *Cartan distribution* on $J^\infty(\pi)$.

PROPOSITION 2. *Let* $\pi\colon E \to M$ *be a vector bundle. Then*:

(1) *For any* $\theta \in J^\infty(\pi)$ *the Cartan plane* $\mathcal{C}_\theta$ *is* $\dim M$*-dimensional and* π_∞*-horizontal.*
(2) *The distribution* $\mathcal{C}$ *is integrable in formal sense*: *for any two vector fields lying in* $\mathcal{C}$ *their commutator lies in* $\mathcal{C}$ *as well.*
(3) *Manifolds of the form* M_f^∞ *and they only are maximal integral manifolds of* $\mathcal{C}$.

From this proposition it follows that the bundle π_∞ is endowed with a flat connection $\mathcal{C}\colon \mathrm{D}(M) \to \mathrm{D}(\pi)$ called the *Cartan connection*. Moreover, one can show that this connection is generalized to the following construction. Let π' and π'' be two vector bundles over M and $\Delta\colon \Gamma(\pi') \to \Gamma(\pi'')$ be a linear differential operator. Then there exists a unique linear differential operator $\mathcal{C}\Delta\colon \mathcal{F}(\pi,\pi') \to$

$\mathcal{F}(\pi, \pi'')$ satisfying $j_\infty(f)^* \circ \mathcal{C}\Delta = \Delta \circ j_\infty(f)^*$ for any $f \in \Gamma(\pi)$. From the very definition it follows that operators of the form $\mathcal{C}\Delta$ admit restrictions to submanifolds M_f^∞. Operators possessing this properties are called *$\mathcal{C}$-differential* (or *total differential*) operators.

In adapted coordinates, the Cartan connection is expressed by

$$\mathcal{C}\frac{\partial}{\partial x_i} = D_i \stackrel{\text{def}}{=} \frac{\partial}{\partial x_i} + \sum_{\sigma,j} p^j_{\sigma i}\frac{\partial}{\partial p^j_\sigma},$$

where D_i is the *total derivative* along x_i. Total derivatives form a local basis of the Cartan distribution on $J^\infty(\pi)$. An operator $\square\colon \mathcal{F}(\pi, \pi') \to \mathcal{F}(\pi, \pi'')$ is a $\mathcal{C}$-differential operator if and only if it is locally expressed in total derivatives.

Denote by $\mathcal{C}\mathrm{D}(\pi)$ the module of vector fields lying in the Cartan distribution. A vector field $X \in \mathrm{D}(\pi)$ is called an (infinitesimal) automorphism of $\mathcal{C}$, if $[X, \mathcal{C}\mathrm{D}(\pi)] \subset \mathcal{C}\mathrm{D}(\pi)$. These automorphisms form a Lie algebra $\mathrm{D}_\mathcal{C}(\pi)$, and $\mathcal{C}\mathrm{D}(\pi)$ is its ideal consisting of *trivial* automorphisms. Elements of the quotient Lie algebra $\operatorname{sym}\pi = \mathrm{D}_\mathcal{C}(\pi)\big/\mathcal{C}\mathrm{D}(\pi)$ are called *symmetries* of the Cartan distribution.

The Cartan connection splits the module $\mathrm{D}_\mathcal{C}(\pi)$ into the direct sum $\mathrm{D}_\mathcal{C}(\pi) = \mathrm{D}^v_\mathcal{C}(\pi) \oplus \mathcal{C}\mathrm{D}(\pi)$, where $\mathrm{D}^v_\mathcal{C}(\pi)$ consists of *vertical* vector fields $X \in \mathrm{D}_\mathcal{C}(\pi)$, i.e., fields such that $X(C^\infty(M)) = 0$. Hence, any coset $\xi \in \operatorname{sym}\pi$ contains a unique vertical representative and we identify $\operatorname{sym}\pi$ with $\mathrm{D}^v_\mathcal{C}(\pi)$. With this identification, the following result is valid.

THEOREM 3. *There is a one-to-one correspondence between* $\operatorname{sym}\pi$ *and the module* $\mathcal{F}(\pi, \pi)$. *In adapted coordinates this correspondence is given by the formula*

$$\partial\!\!\!/\colon \varphi \mapsto \mathit{Э}_\varphi = \sum_{\sigma,j} D_\sigma(\varphi^j)\frac{\partial}{\partial p^i_\sigma},$$

where φ^j, $j = 1, \dots, \dim\pi$, *are the components of* φ *in local representation and* $D_\sigma = D_{i_1} \circ \dots \circ D_{i_s}$ *for* $\sigma = i_1 \dots i_s$.

The field $\mathit{Э}_\varphi$ is called an *evolutionary vector field* with the *generating section* (or *function*) $\varphi \in \mathcal{F}(\pi, \pi)$. Note that $\varphi^j = \mathit{Э}_\varphi \lrcorner\, \omega^j_\emptyset$, where $\omega^j_\emptyset$ is the Cartan form corresponding to the empty multi-index. Let $\pi'\colon E' \to M$ be a vector bundle. Then any evolutionary vector field $\mathit{Э}_\varphi$ is uniquely extended to a first-order differential operator $\mathit{Э}^{\pi'}_\varphi\colon \mathcal{F}(\pi, \pi') \to \mathcal{F}(\pi, \pi')$ satisfying $\mathit{Э}^{\pi'}_\varphi(f\Delta) = \mathit{Э}_\varphi(f)\Delta + f\mathit{Э}^{\pi'}_\varphi(\Delta)$ for any $f \in \mathcal{F}(\pi)$ and $\Delta \in \mathcal{F}(\pi, \pi')$.

Evolutionary vector fields form a Lie algebra and consequently for any $\varphi, \psi \in \mathcal{F}(\pi, \pi)$ the commutator $[\mathit{Э}_\varphi, \mathit{Э}_\psi]$ is of the form $\mathit{Э}_\xi$ for some section $\xi \in \mathcal{F}(\pi, \pi)$. This section is denoted by $\{\varphi, \psi\}$ and called the *Jacobi bracket* of φ and ψ. This bracket can be computed by the formula $\{\varphi, \psi\} = \mathit{Э}^\pi_\varphi(\psi) - \mathit{Э}^\pi_\psi(\varphi)$ while in adapted coordinates one has

$$\{\varphi, \psi\}^j = \sum_{\sigma,\alpha}\left(D_\sigma(\varphi^\alpha)\frac{\partial\psi^j}{\partial p^\alpha_\sigma} - D_\sigma(\psi^\alpha)\frac{\partial\varphi^j}{\partial p^\alpha_\sigma}\right).$$

Consider now an equation $\mathcal{E} \subset J^k(\pi)$.

DEFINITION 4. The set

$$\mathcal{E}^l = \{[f]_x^{k+l} \mid \theta_k = [f]_k \in \mathcal{E},\ M_f^k \text{ is tangent to } \mathcal{E} \text{ at } \theta_k \text{ with order } l\}$$

is the lth *prolongation* of $\mathcal{E}$, $l = 0, 1, \dots, \infty$. An equation is said to be *formally integrable*, if all $\mathcal{E}^l$ are smooth manifolds and the mappings $\pi_{k+l+1,k+l}\colon \mathcal{E}^{l+1} \to \mathcal{E}^l$ are smooth fiber bundles.

Let in local coordinates $\mathcal{E}$ be given by equations $F^1 = 0, \dots, F^r = 0$, $F^\alpha \in \mathcal{F}(\pi)$. Then its lth prolongation is described by the system $D_\sigma F^\alpha = 0$, $|\sigma| \leqslant l$, $\alpha = 1, \dots, r$.

Our concern now is the infinite prolongation, $\mathcal{E}^\infty$.

DEFINITION 5. An evolutionary derivation ∂_φ (or a section φ) is called a *higher symmetry* of $\mathcal{E}$, if it is tangent to $\mathcal{E}^\infty$.

Higher symmetries of $\mathcal{E}$ form a Lie algebra over $\mathbb{R}$ denoted by $\operatorname{sym}\mathcal{E}$.

To describe higher symmetries in efficient terms, let us note the following. Let $\Delta \in \mathcal{F}(\pi, \pi')$ be a differential operator corresponding to $\mathcal{E}$. Consider the operator $\ell_\Delta\colon \mathcal{F}(\pi, \pi) \to \mathcal{F}(\pi, \pi')$ defined by

$$\ell_\Delta \varphi \stackrel{\text{def}}{=} \partial_\varphi^{\pi'} \Delta, \quad \varphi \in \mathcal{F}(\pi, \pi).$$

The operator ℓ_Δ is called the *universal linearization* of Δ. Let locally Δ be given by its components $F^1, \dots, F^r$. Then ℓ_Δ is a matrix linear differential operator of the form

$$\ell_\Delta = \left\| \sum_\sigma \frac{\partial F^\alpha}{\partial p_\sigma^\beta} D_\sigma \right\|, \quad \alpha = 1, \dots, \dim \pi',\ \beta = 1, \dots, \dim \pi.$$

In particular, it follows that ℓ_Δ is a $\mathcal{C}$-differential operator.

Let us now recall that $\mathcal{C}$-differential operators admit restriction to manifolds of the form $\mathcal{E}^\infty$ and introduce the notation $\ell_\mathcal{E} = \ell_\Delta|_{\mathcal{E}^\infty}$, where $\Delta = \Delta_\mathcal{E}$.

THEOREM 4. *Let $\mathcal{E} \subset J^k(\pi)$ and a section $\Delta = \Delta_\mathcal{E} \in \mathcal{F}(\pi, \pi')$ be chosen in such a way that its graph intersects the graph of the zero section transversally. Then* $\operatorname{sym}\mathcal{E} = \ker \ell_\mathcal{E}$.

Remark 3. Similar to the case of classical symmetries, one can define the notion of *intrinsic higher symmetry* introducing the Cartan distribution on $\mathcal{E}^\infty$ and considering nontrivial symmetries of this distribution. Contrary to the classical case, we obtain nothing new:

THEOREM 5. *If an equation $\mathcal{E}$ is such that $\pi_{\infty,0}(\mathcal{E}^\infty) = J^0(\pi)$, then any intrinsic symmetry is a restriction to $\mathcal{E}^\infty$ of some extrinsic one.*

To conclude this section, let us note that the theory of classical symmetries is included in the theory of higher ones.

4. Coverings and Nonlocal Symmetries ([1, 9])

Let $\mathcal{E}^\infty \subset J^\infty(\pi) \xrightarrow{\pi_\infty} M$ be an infinitely prolonged equation, $\dim M = n$.

DEFINITION 6. A locally trivial bundle $\tau\colon W \to \mathcal{E}^\infty$ is called a *covering* over $\mathcal{E}$, if the space W is endowed with an n-dimensional integrable distribution $\tilde{\mathcal{C}}$ such that $\tau_*(\tilde{\mathcal{C}}_{\tilde{\theta}}) = \mathcal{C}_{\tau(\theta)}$ for any $\tilde{\theta} \in W$.

From this definition it follows that the bundle $\pi_\tau = \pi_\infty \circ \tau\colon W \to M$ is endowed with a flat connection which we denote by $\tilde{\mathcal{C}}$ and which 'covers' the Cartan connection in the bundle π_∞: for any $X \in \mathrm{D}(M)$ one has $\tau_*(\tilde{\mathcal{C}}X) = \mathcal{C}X$. In an adapted coordinate system such that τ trivializes over the corresponding coordinate neighborhood, this connection is described by the formulas

$$\tilde{\mathcal{C}}\frac{\partial}{\partial x_i} \stackrel{\text{def}}{=} \tilde{D}_i = D_i + X_i,$$

where D_i are the restrictions of the total derivative to $\mathcal{E}^\infty$ and

$$X_i = \sum_\alpha X_i^\alpha \frac{\partial}{\partial w^\alpha}$$

are τ-vertical vector fields, $\{w^\alpha\}$ being local coordinates along the fiber of τ. The condition for τ to be a covering is expressed by

$$[\tilde{D}_i, \tilde{D}_j] = D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \quad 1 \leqslant i < j \leqslant n,$$

where $D_i(X_j)$ denotes the component-wise action of D_i on coefficients of the field X_j.

Two coverings $\tau\colon W \to \mathcal{E}^\infty$, $\tau'\colon W' \to \mathcal{E}^\infty$, are called *equivalent*, if there exists a diffeomorphism $\Phi\colon W \to W'$ satisfying

$$\tau = \tau' \circ \Phi \quad \text{and} \quad \Phi_*(\tilde{\mathcal{C}}_{\tilde{\theta}}) = \tilde{\mathcal{C}}'_{\Phi(\tilde{\theta})}, \quad \tilde{\theta} \in W.$$

Consider the trivial bundle $\tau\colon W = \mathbb{R}^l \times \mathcal{E}^\infty \to \mathcal{E}^\infty$ and define a $\dim M$-dimensional distribution $\tilde{\mathcal{C}}$ on W in such a way that $\tau_*\tilde{\mathcal{C}}_{\tilde{\theta}} = \mathcal{C}_{\tau(\tilde{\theta})}$ for any $\tilde{\theta} \in W$ while the projection of $\tilde{\mathcal{C}}_{\tilde{\theta}}$ on the fiber is trivial. Any covering equivalent to this one is called *trivial*. A covering is called *linear*, if τ is a vector bundle and the fields $\tilde{C}X$ preserve the subset of fiberwise linear functions in $C^\infty(W)$.

Similar to the case of $J^\infty(\pi)$, we can introduce the Lie algebras $\tilde{\mathcal{C}}\mathrm{D}(\tau)$ of vector fields lying in $\tilde{\mathcal{C}}$ and

$$\mathrm{D}_{\tilde{\mathcal{C}}}(\tau) = \{X \in \mathrm{D}(W) \mid [X, \tilde{\mathcal{C}}\mathrm{D}(\tau)] \subset \tilde{\mathcal{C}}\mathrm{D}(\tau)\}.$$

As before, $\tilde{\mathcal{C}}\mathrm{D}(\tau)$ is an ideal of $\mathrm{D}_{\tilde{\mathcal{C}}}(\tau)$ and the elements of the quotient Lie algebra

$$\mathrm{sym}_\tau\,\mathcal{E} \stackrel{\text{def}}{=} \mathrm{D}_{\tilde{\mathcal{C}}}(\tau)/\tilde{\mathcal{C}}\mathrm{D}(\tau)$$

are called *nonlocal τ-symmetries* of $\mathcal{E}$. Any coset $\xi \in \mathrm{sym}_\tau\,\mathcal{E}$ contains a unique $(\pi_\infty \circ \tau)$-vertical representative and $\mathrm{sym}_\tau\,\mathcal{E}$ may be identified with the Lie algebra of such vertical vector fields.

Note now that if π' and π'' are vector bundles over the base M and $\Delta\colon \mathcal{F}(\pi,\pi') \to \mathcal{F}(\pi,\pi'')$ is a $\mathcal{C}$-differential operator, then its restriction $\Delta_{\mathcal{E}}$ to $\mathcal{E}^\infty$ can be naturally lifted to a linear differential operator $\tilde{\Delta}\colon \Gamma(\pi_\tau^*(\pi')) \to \Gamma(\pi_\tau^*(\pi''))$. In particular, we can construct the lifting $\tilde{\ell}_{\mathcal{E}}$ of the operator $\ell_{\mathcal{E}}$.

DEFINITION 7. A section $\varphi \in \Gamma(\pi_\tau^*(\pi))$ is called a *τ-shadow*, if $\tilde{\ell}_{\mathcal{E}}(\varphi) = 0$.

If φ is a τ-shadow, we can define the derivation $\tilde{\mathcal{Э}}_\varphi\colon C^\infty(\mathcal{E}^\infty) \to C^\infty(W)$ by

$$\tilde{\mathcal{Э}}_\varphi = \sum \tilde{D}_\sigma(\varphi^j)\frac{\partial}{\partial p_\sigma^j},$$

where the sum is taken over all *internal* coordinates in $\mathcal{E}^\infty$.

DEFINITION 8. Let $\mu\colon W' \to W$ be a bundle such that

(1) The bundle $\tau' = \tau \circ \mu\colon W' \to \mathcal{E}^\infty$ is endowed with a covering structure.
(2) The connection $\tilde{\mathcal{C}}_{\tau'}$ covers the connection $\tilde{\mathcal{C}}_\tau$.

A τ-shadow φ is said to be *τ'-reconstructable*, if there exists a nonlocal τ'-symmetry S such that $S|_{C^\infty(\mathcal{E}^\infty)} = \tilde{\mathcal{Э}}_\varphi$.

THEOREM 6. *Let $\tau\colon W \to \mathcal{E}^\infty$ be a covering and $\varphi_1,\dots,\varphi_s$ be τ-shadows. Then there exists a covering τ' such that these shadows are τ'-reconstructable.*

5. Horizontal Cohomology ([11])

Consider the de Rham complex

$$0 \to C^\infty(M) \xrightarrow{\mathrm{d}} \Lambda^1(M) \to \cdots \to \Lambda^{n-1}(M) \xrightarrow{\mathrm{d}} \Lambda^n(M) \to 0$$

of the manifold M. Denote by $\wedge^i\colon \bigwedge^i T^*M \to M$ the ith exterior power of the cotangent bundle of the manifold M and by $\bar{\Lambda}^i(\pi)$ the modules $\mathcal{F}(\pi,\wedge^i)$. Since d are linear differential operators, we can construct the operators $\bar{\mathrm{d}} =_{\text{def}} \mathcal{C}\mathrm{d}\colon \bar{\Lambda}^i(\pi) \to \bar{\Lambda}^{i+1}(\pi)$ and obtain the complex

$$0 \to \mathcal{F}(\pi) \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda}^1(\pi) \to \cdots \to \bar{\Lambda}^{n-1}(\pi) \xrightarrow{\bar{\mathrm{d}}} \bar{\Lambda}^n(\pi) \to 0,$$

which is called the *horizontal de Rham complex* on $J^\infty(\pi)$. Elements of $\bar\Lambda^i(\pi)$ can be identified with *horizontal i-forms* on $J^\infty(\pi)$, i.e., the forms $\omega \in \Lambda^i(J^\infty(\pi))$ such that $X \lrcorner\, \omega = 0$ for any π_∞-vertical vector field. Since the operators $\bar{\mathrm{d}}$ are $\mathcal{C}$-differential, one can restrict the above complex to any infinite prolongation $\mathcal{E}^\infty \subset J^\infty(\pi)$ and obtain the complex

$$0 \to \mathcal{F}(\mathcal{E}^\infty) \xrightarrow{\bar{\mathrm{d}}} \bar\Lambda^1(\mathcal{E}^\infty) \to \cdots \to \bar\Lambda^{n-1}(\mathcal{E}^\infty) \xrightarrow{\bar{\mathrm{d}}} \bar\Lambda^n(\mathcal{E}^\infty) \to 0,$$

which is called the *horizontal de Rham complex* of the equation $\mathcal{E}$. Its cohomology is denoted by $\bar{H}^i(\mathcal{E})$.

In the sequel we shall need horizontal cohomology with coefficients. To this end, we give the following

DEFINITION 9. Let $\mathcal{F} = \mathcal{F}(\mathcal{E})$ be the smooth function algebra on $\mathcal{E}^\infty$ and $\mathcal{C}\,\mathrm{Diff}(\mathcal{F})$ be the algebra of $\mathcal{C}$-differential operators acting from $\mathcal{F}$ to $\mathcal{F}$. An $\mathcal{F}$-module P is called a *$\mathcal{C}$-module*, if it is endowed with a left module structure over $\mathcal{C}\,\mathrm{Diff}(\mathcal{F})$ such that any $a \in \mathcal{F} \subset \mathcal{C}\,\mathrm{Diff}(\mathcal{F})$ acts on P by multiplication.

It is useful to note that P is a $\mathcal{C}$-module if and only if it is of the form $P = \Gamma(\tau)$, where τ is linear covering over $\mathcal{E}$.

PROPOSITION 3. *Let Q, Q' be $\mathcal{F}$-modules and $\Delta\colon Q \to Q'$ be a $\mathcal{C}$-differential operator. Then for any $\mathcal{C}$-module P the operator Δ can be naturally extended to a $\mathcal{C}$-differential operator $\Delta_P\colon Q \otimes_{\mathcal{F}} P \to Q' \otimes_{\mathcal{F}} P$ of the same order. If $\Delta'\colon Q' \to Q''$ is another $\mathcal{C}$-differential operator, then $(\Delta' \circ \Delta)_P = \Delta'_P \circ \Delta_P$.*

Applying this result to the horizontal de Rham complex, we obtain the complex

$$0 \to P \xrightarrow{\bar{\mathrm{d}}_P} \bar\Lambda^1(\mathcal{E}^\infty) \otimes P \xrightarrow{\bar{\mathrm{d}}_P} \cdots \xrightarrow{\bar{\mathrm{d}}_P} \bar\Lambda^n(\mathcal{E}^\infty) \otimes P \to 0,$$

whose cohomology is called the *horizontal de Rham cohomology of $\mathcal{E}$ with coefficients in P* and is denoted by $\bar{H}^i(P)$. In particular, $\bar{H}^i(\mathcal{E}) = \bar{H}^i(\mathcal{F})$.

Efficient computation of horizontal cohomologies is based on the notion of compatibility complex. Let Q, Q_1 be $\mathcal{F}$-modules.

PROPOSITION 4. *There exists an $\mathcal{F}$-module $\bar{\mathcal{J}}^k(Q)$ and a $\mathcal{C}$-differential operator $\bar{j}_k\colon Q \to \bar{\mathcal{J}}^k(Q)$ of order k such that for any $\mathcal{C}$-differential operator $\Delta\colon Q \to Q_1$ of order k a homomorphism $\varphi_\Delta\colon \bar{\mathcal{J}}^k(Q) \to Q_1$ satisfying $\Delta = \varphi_\Delta \circ \bar{j}_k$ is uniquely defined.*

By its properties, $\bar{\mathcal{J}}^k(Q)$ is defined up to an isomorphism and is called the *module of horizontal k-jets*. One can also see that for any $l \geqslant k$ there exists a natural homomorphism $\bar{\mathcal{J}}^l(\pi) \to \bar{\mathcal{J}}^k(\pi)$ and thus the module $\bar{\mathcal{J}}^\infty(\pi) = \mathrm{proj\,lim}\, \bar{\mathcal{J}}^k(\pi)$ is defined. For any $\mathcal{C}$-differential operator $\Delta\colon Q \to Q_1$ of order k one can consider the operators $\Delta^{(s)} = \bar{j}_s \circ \varphi_\Delta\colon Q \to \bar{\mathcal{J}}^s(Q_1)$ and the corresponding homomorphism of horizontal jets $\varphi^s_\Delta\colon \bar{\mathcal{J}}^{k+s}(Q) \to \bar{\mathcal{J}}^s(Q_1)$. Passing to the inverse limit, we also obtain the homomorphism $\varphi^\infty_\Delta\colon \bar{\mathcal{J}}^\infty(Q) \to \bar{\mathcal{J}}^\infty(Q_1)$. Let us denote the kernel of this homomorphism by $\mathcal{R}_\Delta$. It can be seen that $\mathcal{R}_\Delta$ is a $\mathcal{C}$-module.

Without loss of generality we can always assume that φ_Δ is an epimorphism. Now choose an integer $k_1 > 0$ and consider the homomorphism $\varphi^{k_1}_\Delta\colon \bar{\mathcal{J}}^{k+k_1}(Q) \to \bar{\mathcal{J}}^{k_1}(Q)$. Let us introduce the module $Q_2 = \operatorname{coker}\varphi^{k_1}_\Delta$ and the operator $\Delta_1\colon Q_1 \to Q_2$ as the composition of $\bar{j}_{k_1}$ with the natural projection $\bar{\mathcal{J}}^{k_1}(Q_1) \to Q_2$. Applying this procedure to Δ_1, we shall obtain the operator $\Delta_2\colon Q_2 \to Q_3$, etc. Thus we get the complex $Q^\Delta_\bullet$

$$0 \to Q \xrightarrow{\Delta=\Delta_0} Q_1 \xrightarrow{\Delta_1} Q_2 \to \cdots \to Q_i \xrightarrow{\Delta_i} Q_{i+1} \to \cdots$$

of $\mathcal{C}$-differential operators satisfying the following property: for any $\mathcal{C}$-differential operator $\nabla\colon Q_i \to P$ of order $\geqslant k_i$ such that $\nabla \circ \Delta_{i-1} = 0$ there exists a $\mathcal{C}$-differential operator $\square\colon Q_{i+1} \to P$ such that $\nabla = \square \circ \Delta_i$. By this reason, we call this complex the *compatibility complex* of the operator Δ. For an *involutive* Δ (see [6]), this complex is *formally exact* which means that the complex of homomorphisms

$$0 \to \bar{\mathcal{J}}^\infty(Q) \xrightarrow{\varphi^\infty_\Delta} \bar{\mathcal{J}}^\infty(Q_1) \xrightarrow{\varphi^\infty_{\Delta_1}} \bar{\mathcal{J}}^\infty(Q_2) \to \cdots$$
$$\cdots \to \bar{\mathcal{J}}^\infty(Q_i) \xrightarrow{\varphi^\infty_{\Delta_i}} \bar{\mathcal{J}}^\infty(Q_{i+1}) \to \cdots$$

is exact in all positive terms.

THEOREM 7. *For any $\mathcal{C}$-module P one has*

$$\bar{H}^i(\mathcal{R}_\Delta \widehat{\otimes} P) = H^i(Q^\Delta_\bullet \otimes P),$$

where $\mathcal{R}_\Delta \widehat{\otimes} P = \operatorname{proj\,lim} \mathcal{R}^s_\Delta \otimes P$ *with* $\mathcal{R}^s_\Delta =_{\mathrm{def}} \ker \varphi^s_\Delta$.

Let us now dualize the above construction. Let $\delta\colon Q \to Q'$ be a $\mathcal{C}$-differential operator. Consider the diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathcal{C}\,\mathrm{Diff}(Q', \bar{\Lambda}^i) & \xrightarrow{w} & \mathcal{C}\,\mathrm{Diff}(Q', \bar{\Lambda}^{i+1}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle\tilde{\delta}} & & \downarrow{\scriptstyle\tilde{\delta}} & \\
\cdots \longrightarrow & \mathcal{C}\,\mathrm{Diff}(Q, \bar{\Lambda}^i) & \xrightarrow{w} & \mathcal{C}\,\mathrm{Diff}(Q, \bar{\Lambda}^{i+1}) & \longrightarrow \cdots
\end{array}$$

where $w(\nabla) = \bar{\mathrm{d}} \circ \nabla$ and $\tilde{\delta}(\nabla) = \nabla \circ \delta$. Denote the cohomologies of these complexes at the term $\mathcal{C}\,\mathrm{Diff}(\bullet, \bar{\Lambda}^n)$ by $\hat{Q}'$ and $\hat{Q}$ respectively. Then $\tilde{\delta}$ induces the mapping $\delta^*\colon \hat{Q}' \to \hat{Q}$ which is called the *adjoint operator* of δ.

Let us now consider the complex $\hat{Q}^\Delta_\bullet$

$$0 \leftarrow \hat{Q} \xleftarrow{\Delta^*} \hat{Q}_1 \xleftarrow{\Delta^*_1} \hat{Q}_2 \leftarrow \cdots \leftarrow \hat{Q}_i \xleftarrow{\Delta^*_i} \hat{Q}_{i+1} \leftarrow \cdots$$

adjoint to the compatibility complex of the operator Δ.

THEOREM 8. *For any* $\mathcal{C}$*-module* P *one has*

$$\bar{H}^i(\mathcal{R}^*_\Delta \otimes P) = H_{n-i}(\hat{Q}^\Delta_\bullet \otimes P),$$

where $\mathcal{R}^*_\Delta = \hom(\mathcal{R}_\Delta, \mathcal{F})$.

Both theorems are proved using techniques of spectral sequences associated to bicomplexes.

6. $\mathcal{C}$-cohomology and Recursion Operators ([5, 10])

Consider an equation $\mathcal{E} \subset J^k(\pi)$ and denote by $\mathrm{D}^{\mathrm{v}}(\Lambda^i(\mathcal{E}))$ the module of π_∞-vertical derivations $\mathcal{F}(\mathcal{E}) \to \Lambda^i(\mathcal{E}^\infty)$. Recall that the module $\mathrm{D}^{\mathrm{v}}(\Lambda^*(\mathcal{E})) = \bigoplus_i \mathrm{D}^{\mathrm{v}}(\Lambda^i(\mathcal{E}))$ carries the following structures:

- the structure of a *graded* $\Lambda^*(\mathcal{E}^\infty)$*-module*

$$\wedge\colon \Lambda^i(\mathcal{E}^\infty) \times \mathrm{D}^{\mathrm{v}}(\Lambda^j(\mathcal{E})) \to \mathrm{D}^{\mathrm{v}}(\Lambda^{i+j}(\mathcal{E}));$$

- the *inner product* operations
  $\lrcorner\colon \mathrm{D}^{\mathrm{v}}(\Lambda^i(\mathcal{E})) \times \mathrm{D}^{\mathrm{v}}(\Lambda^j(\mathcal{E})) \to \mathrm{D}^{\mathrm{v}}(\Lambda^{i+j-1}(\mathcal{E}))$,
  $\lrcorner\colon \mathrm{D}^{\mathrm{v}}(\Lambda^i(\mathcal{E})) \times \Lambda^j(\mathcal{E}^\infty) \to \Lambda^{i+j-1}(\mathcal{E}^\infty)$;
- the *Frölicher–Nijenhuis bracket*

$$[\![\cdot,\cdot]\!]\colon \mathrm{D}^{\mathrm{v}}(\Lambda^i(\mathcal{E})) \times \mathrm{D}^{\mathrm{v}}(\Lambda^j(\mathcal{E})) \to \mathrm{D}^{\mathrm{v}}(\Lambda^{i+j}(\mathcal{E}))$$

  with respect to which $\mathrm{D}^{\mathrm{v}}(\Lambda^*(\mathcal{E}))$ is a graded Lie algebra.

Consider the Cartan connection $\mathcal{C}$ in $\pi_\infty\colon \mathcal{E}^\infty \to M$ and its connection form $U_\mathcal{E} \in \mathrm{D}^{\mathrm{v}}(\Lambda^1(\mathcal{E}))$ (also called the *structural element* of the equation $\mathcal{E}$). By flatness of $\mathcal{C}$, one has $[\![U_\mathcal{E}, U_\mathcal{E}]\!] = 0$. Then $\partial_\mathcal{E} =_{\mathrm{def}} [\![U_\mathcal{E}, \cdot]\!]\colon \mathrm{D}^{\mathrm{v}}(\Lambda^i(\mathcal{E})) \to \mathrm{D}^{\mathrm{v}}(\Lambda^{i+1}(\mathcal{E}))$ is a first-order differential operator and, due to the Jacobi identity for the Frölicher–Nijenhuis bracket, $\partial_\mathcal{E} \circ \partial_\mathcal{E} = 0$. Thus we obtain the complex

$$0 \to \mathrm{D}^{\mathrm{v}}(\mathcal{E}) \xrightarrow{\partial_\mathcal{E}} \mathrm{D}^{\mathrm{v}}(\Lambda^1(\mathcal{E})) \to \cdots \to \mathrm{D}^{\mathrm{v}}(\Lambda^i(\mathcal{E})) \xrightarrow{\partial_\mathcal{E}} \mathrm{D}^{\mathrm{v}}(\Lambda^{i+1}(\mathcal{E})) \to \cdots$$

which is called the $\mathcal{C}$*-complex* of the equation $\mathcal{E}$ and whose cohomology (the $\mathcal{C}$*-cohomology*) is denoted by $H^i_\mathcal{C}(\mathcal{E})$.

THEOREM 9. *For any formally integrable equation* $\mathcal{E} \subset J^k(\pi)$ *one has*:

(1) $H^0_\mathcal{C}(\mathcal{E}) = \mathrm{sym}\,\mathcal{E}$.
(2) $H^1_\mathcal{C}(\mathcal{E})$ *is identified with equivalence classes of nontrivial infinitesimal deformations of the equation structure.*
(3) $H^2_\mathcal{C}(\mathcal{E})$ *contains obstructions for continuation of infinitesimal deformations to formal ones.*

We shall first describe the groups $H^i_{\mathcal{C}}(\mathcal{E})$ for the 'empty' equation $\mathcal{E}^\infty = J^\infty(\pi)$. To do this, let us introduce the mapping $d_{\mathcal{C}}\colon \Lambda^i(\mathcal{E}^\infty) \to \Lambda^{i+1}(\mathcal{E}^\infty)$ defined by $d_{\mathcal{C}}(\omega) = U_{\mathcal{E}} \lrcorner (d\omega) - d(U_{\mathcal{E}} \lrcorner \omega)$. It is called the *Cartan* (or *vertical*) *differential* and it can be easily shown that $d_{\mathcal{C}} \circ d_{\mathcal{C}} = 0$. Let $\mathcal{C}\Lambda(\mathcal{E}) \subset \Lambda^1(\mathcal{E}^\infty)$ be the module generated by the image of $d_{\mathcal{C}}$ and $\mathcal{C}^i\Lambda(\mathcal{E}) \subset \Lambda^i(\mathcal{E}^\infty)$ be its ith external power.

THEOREM 10. *For the 'empty' equation $\mathcal{E}^\infty = J^\infty(\pi)$ the groups $H^i_{\mathcal{C}}(\mathcal{E})$ are isomorphic to $\mathcal{C}^i\Lambda(\mathcal{E}) \otimes_{\mathcal{F}} \mathcal{F}(\pi,\pi)$. In adapted coordinates, this isomorphism takes an element $\Omega \in \mathcal{C}^i\Lambda(\mathcal{E}) \otimes_{\mathcal{F}} \mathcal{F}(\pi,\pi)$ to the class of the vertical derivation*

$$\mathcal{Э}_\Omega = \sum D_\sigma(\Omega^j) \otimes \frac{\partial}{\partial p^j_\sigma},$$

where Ω^j are components of ω in local representation.

To deal with the general case, let us first note that both $\mathcal{C}^p\Lambda(\mathcal{E})$ and $D^v(\mathcal{C}^p\Lambda(\mathcal{E}))$ are $\mathcal{C}$-modules. Hence, we can consider the horizontal cohomology $\bar{H}^q(D^v(\mathcal{C}^p\Lambda(\mathcal{E}))$ of $\mathcal{E}$ with coefficients in $D^v(\mathcal{C}^p\Lambda(\mathcal{E}))$. Let us now take the compatibility complex for the linearization operator $Q^{\ell_{\mathcal{E}}}_\bullet$

$$0 \to \mathcal{F}(\mathcal{E},\pi) \xrightarrow{\ell_{\mathcal{E}}} Q_1 \xrightarrow{\Delta_1} Q_2 \to \cdots .$$

THEOREM 11. *The following isomorphisms are valid*:

(1) $\bar{H}^q(D^v(\mathcal{C}^p\Lambda(\mathcal{E}))) = H^q(Q^{\ell_{\mathcal{E}}}_\bullet \otimes \mathcal{C}^p\Lambda(\mathcal{E}))$.
(2) $H^i_{\mathcal{C}}(\mathcal{E}) = \bigoplus_{p+q=i} \bar{H}^q(D^v(\mathcal{C}^p\Lambda(\mathcal{E})))$.

As a consequence, we get the following result:

THEOREM 12 (the s-line theorem). *If the compatibility complex of the linearization operator is of length s, then*

(1) $\bar{H}^q(D^v(\mathcal{C}^p\Lambda(\mathcal{E}))) = 0$ *for $q \geqslant s$.*
(2) $\bar{H}^0(D^v(\mathcal{C}^p\Lambda(\mathcal{E}))) = \ker \ell^{[p]}_{\mathcal{E}}$.
(3) *In the case $s = 2$ one also has* $\bar{H}^1(D^v(\mathcal{C}^p\Lambda(\mathcal{E})) = \operatorname{coker} \ell^{[p]}_{\mathcal{E}}$.

Here $\ell^{[p]}_{\mathcal{E}}$ is the extension of $\ell_{\mathcal{E}}$ to $\mathcal{C}^p\Lambda(\mathcal{E})$.

Remark 4. An equation $\mathcal{E}$ satisfies the conditions of 2-line theorem, if the functions $F^1, \ldots, F^r$ determining this equation are differentially independent, i.e., there exists no nontrivial relation of the form $\sum \Delta_j F^j = 0$, where Δ_j are $\mathcal{C}$-differential operators. 'Almost all' equations possess this property and we call such equations *ℓ-normal*.

To conclude this section, we shall describe relations between $\mathcal{C}$-cohomology and recursion operators. Recall that the module $D^v(\Lambda^*(\mathcal{E}))$ is endowed with the inner product operation. It can be seen that this operation is inherited by the $\mathcal{C}$-cohomology groups and the following fact is valid:

PROPOSITION 5. *The group $H^1_{\mathcal{C}}(\mathcal{E})$ forms and associative algebra with respect to inner product, the class $U_{\mathcal{E}}$ being its unit. This algebra acts on $H^0_{\mathcal{C}}(\mathcal{E})$ by $R_\Omega(X) = X \lrcorner \Omega$, $\Omega \in H^1_{\mathcal{C}}(\mathcal{E})$, $X \in H^0_{\mathcal{C}}(\mathcal{E})$.*

Note that the above action is trivial for $\Omega \in \bar{H}^1(\mathcal{C}^0\Lambda(\mathcal{E})) \subset H^1_{\mathcal{C}}(\mathcal{E})$. We call elements of $\bar{H}^0(\mathcal{C}^1\Lambda(\mathcal{E})) \subset H^1_{\mathcal{C}}(\mathcal{E})$ *recursion operators* for symmetries of the equation $\mathcal{E}$. Thus to find a recursion operator it needs to solve the equation $\ell^{[1]}_{\mathcal{E}}\Omega = 0$ for $\Omega \in \mathcal{C}\Lambda(\mathcal{E}) \otimes \mathcal{F}(\mathcal{E}, \pi)$ and this operator will act on symmetries by $R_\Omega(\varphi) = \mathcal{Э}_\varphi \lrcorner \Omega$.

Let now $\tau\colon W \to \mathcal{E}^\infty$ be a covering over $\mathcal{E}$. Then the $\mathcal{C}$-cohomology theory can be literary repeated for the bundle $\pi_\infty \circ \tau$. An element $\tilde{\Omega} \in \mathcal{C}\Lambda(W) \otimes \mathcal{F}(W, \pi)$ is called a *τ-shadow of a recursion operator*, if $\tilde{\ell}^{[1]}_{\mathcal{E}}\tilde{\Omega} = 0$. For applications the following result is important:

PROPOSITION 6. *If $\tilde{\Omega}$ is a τ-shadow of a recursion operator and $\tilde{\varphi}$ is a nonlocal τ-symmetry, then $R_{\tilde{\Omega}}\tilde{\varphi}$ is a τ-shadow of a symmetry.*

7. $\mathcal{C}$-spectral Sequence and Conservation Laws ([15])

Consider a differential equation $\mathcal{E} \subset J^k(\pi)$ and the submodule $\mathcal{C}\Lambda(\mathcal{E}) \subset \Lambda^1(\mathcal{E}^\infty)$. Let $\mathcal{I}_{\mathcal{E}} \subset \Lambda^*(\mathcal{E}^\infty)$ be the ideal generated by $\mathcal{C}\Lambda(\mathcal{E})$. Since the Cartan distribution on $\mathcal{E}^\infty$ is integrable, this ideal is closed with respect to the de Rham differential $\mathrm{d}\colon \Lambda^*(\mathcal{E}^\infty) \to \Lambda^*(\mathcal{E}^\infty)$ and the filtration

$$\Lambda^*(\mathcal{E}^\infty) = \mathcal{I}^0_{\mathcal{E}} \supset \mathcal{I}_{\mathcal{E}} \supset \cdots \supset \mathcal{I}^i_{\mathcal{E}} \supset \mathcal{I}^{i+1}_{\mathcal{E}} \supset \cdots$$

is in agreement with d. The corresponding spectral sequence converges to the de Rham cohomology of $\mathcal{E}^\infty$ and is called the *Vinogradov spectral sequence* (or *$\mathcal{C}$-spectral sequence*). Denote its terms by $E^{p,q}_r(\mathcal{E})$ and the corresponding differentials by $\mathrm{d}^{p,q}_r$. For the empty equation $\mathcal{E}^\infty = J^\infty(\pi)$ we use the notation $E^{p,q}_r(\pi)$.

Remark 5. Consider the Cartan differential $\mathrm{d}_{\mathcal{C}}$. Then it can be shown that the difference $\mathrm{d} - \mathrm{d}_{\mathcal{C}}$ is also a differential and its restriction to $\bar{\Lambda}^*(\mathcal{E})$ coincides with the horizontal differential. Let us denote this difference also by $\bar{\mathrm{d}}$. It can be seen that the module $\Lambda^*(\mathcal{E}^\infty)$ is bigraded, $\Lambda^*(\mathcal{E}^\infty) = \bigoplus_{p,q} \bar{\Lambda}^q(\mathcal{E}) \otimes \mathcal{C}^p\Lambda(\mathcal{E})$, and the triple $(\Lambda^*(\mathcal{E}^\infty), \bar{\mathrm{d}}, \mathrm{d}_{\mathcal{C}})$ is a bicomplex. It is called the *variational bicomplex* and the spectral sequence associated to it is isomorphic to the $\mathcal{C}$-spectral sequence.

We start with a description of the $\mathcal{C}$-spectral sequence for $J^\infty(\pi)$.

PROPOSITION 7. *Let π be a vector bundle over an n-dimensional manifold M. Then $E_r^{p,q}(\pi) = 0$, $1 \leqslant r \leqslant \infty$, if $p > 0$, $q \neq n$ or $p = 0$, $q > n$.*

Note now that the 0th column of the term $E_0(\pi)$ coincides with the horizontal complex and consider the sequence

$$\begin{aligned} 0 \;\to\; \mathcal{F}(\pi) \xrightarrow{\bar{d}} \bar{\Lambda}^1(\pi) \xrightarrow{\bar{d}} \cdots \\ \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n(\pi) \xrightarrow{\mathbf{E}} E_1^{1,n}(\pi) \xrightarrow{d_1^{1,n}} E_1^{2,n}(\pi) \xrightarrow{d_1^{2,n}} \cdots, \end{aligned}$$

where $\mathbf{E}$ is the composition of the natural projection $\bar{\Lambda}^n(\pi) \to \bar{H}^n(\pi)$ with the differential $\mathrm{d}_1^{0,n}$: $\bar{H}^n(\pi) \to E_1^{1,n}(\pi)$. This sequence is a complex called the *variational complex* and its cohomology coincides with the cohomology of M. In particular, if M is homologically trivial, the variational complex exact.

Note now that the elements of $\bar{\Lambda}^n(\pi)$ are *Lagrangians* depending on sections of the bundle π and their derivatives while $\bar{\mathrm{d}}$: $\bar{\Lambda}^{n-1}(\pi) \to \bar{\Lambda}^n(\pi)$ is the operator of total divergence. Using an adapted coordinate system one can also see that $\mathbf{E}$ is the *Euler operator* (or *variational derivative*) assigning to a Lagrangian (more exactly, to an equivalence class of Lagrangians) the corresponding *Euler–Lagrange equation*.

THEOREM 13. *Let the manifold M be homologically trivial. Then*:

(1) $\ker \mathbf{E} = \operatorname{im} \bar{\mathrm{d}}$, *i.e., a Lagrangian with vanishing variational derivative is a total divergence.*
(2) $\bar{\mathrm{d}}\omega = 0$ *if and only if* $\omega = \bar{\mathrm{d}}\theta$ *which means that all zero total divergences are total curls.*
(3) $\psi = \mathbf{E}(\omega)$ *if and only if* $\ell_\psi = \ell_\psi^*$ *which gives the solution to the inverse problem in the calculus of variations.*

Let $\mathcal{E} \subset J^k(\pi)$ and consider the complex $\bar{Q}_\bullet^{\ell_\mathcal{E}}$

$$0 \leftarrow \hat{Q}_0 \xleftarrow{\ell_\mathcal{E}^*} \hat{Q}_1 \xleftarrow{\Delta_1^*} \hat{Q}_2 \leftarrow \cdots$$

adjoint to the compatibility complex for $\ell_\mathcal{E}$ (here $Q_0 = \mathcal{F}(\mathcal{E}, \pi)$). Taking into account the results of Section 5 together with the fact that $\mathcal{C}^p\Lambda(\mathcal{E})$ is a $\mathcal{C}$-module, we obtain

THEOREM 14. *The following facts are valid*:

(1) *For any $\mathcal{F}(\mathcal{E})$-module P one has $\bar{H}^{n-i}(\mathcal{C}\Lambda(\mathcal{E}) \otimes P) = H_i(\bar{Q}_\bullet^{\ell_\mathcal{E}} \otimes P)$.*
(2) $E_1^{p,q}(\mathcal{E}) = \bar{H}^q(\mathcal{C}^p\Lambda(\mathcal{E}))$.
(3) $E_1^{p,q}(\mathcal{E})$ *is a direct summand in* $H_{n-q}(\bar{Q}_\bullet^{\ell_\mathcal{E}} \otimes \mathcal{C}^{p-1}\Lambda(\mathcal{E}))$.

As a consequence, we get

THEOREM 15 (the s-line theorem). *If the compatibility complex of the linearization operator is of length s, then*

(1) $E_1^{p,q}(\mathcal{E}) = 0$ *for* $p > 0$ *and* $q \leqslant n - s$.
(2) $E_1^{p,n}(\mathcal{E}) \subset \operatorname{coker}(\ell_{\mathcal{E}}^{[p]})^*$ *for* $p > 0$.
(3) *In the case* $s = 2$ *one also has* $E_1^{p,n-1}(\mathcal{E}) \subset \ker(\ell_{\mathcal{E}}^{[p]})^*$ *for* $p > 0$.

In conclusion, we shall discuss the theory of conservation laws for ℓ-normal equations. We also assume that equations in question are formally integrable. In this case from the 2-line theorem one has the exact sequence

$$0 \to H^{n-1}(\mathcal{E}) \to \bar{H}^{n-1}(\mathcal{E}) \xrightarrow{\mathrm{d}_1^{0,n-1}} \ker \ell_{\mathcal{E}}^*.$$

DEFINITION 10. Elements of $H^{n-1}(\mathcal{E})$ are called *topological* (or *rough*) *conservation laws* of the equation $\mathcal{E}$. The quotient

$$\mathrm{cl}(\mathcal{E}) \stackrel{\text{def}}{=} \bar{H}^{n-1}(\mathcal{E})/H^{n-1}(\mathcal{E})$$

is called the group of *proper* conservation laws.

The 2-line theorem implies

THEOREM 16. *If* $\mathcal{E}$ *is an* ℓ*-normal equation, then* $\mathrm{cl}(\mathcal{E}) \subset \ker \ell_{\mathcal{E}}^*$. *If, in addition,* $H^n(\mathcal{E}) \subset \bar{H}^n(\mathcal{E})$ (*in particular, if* $H^n(\mathcal{E}) = 0$), *then* $\mathrm{cl}(\mathcal{E}) = \ker \mathrm{d}_1^{1,n-1}$.

An element $\psi \in \ker \ell_{\mathcal{E}}^*$ corresponding to a conservation law is called its *generating function*.

Let ψ satisfy the equation $\ell_{\mathcal{E}}^* \psi = 0$ and $\mathcal{E}$ be given by a section F. Then $\ell_F^*(\psi) = \Delta(F)$ for some $\mathcal{C}$-differential operator Δ.

PROPOSITION 8. *An element* $\psi \in \ker \ell_{\mathcal{E}}^*$ *is the generating function of a conservation law, if there exists a* $\mathcal{C}$*-differential operator* ∇ *such that* $\nabla^* = \nabla$ *and* $\ell_{\psi}^* + (\Delta|_{\mathcal{E}^\infty})^* = \nabla|_{\mathcal{E}^\infty} \circ \ell_{\mathcal{E}}$.

The last two results provide an efficient method for computation of conservation laws.

Acknowledgements

It is our real pleasure to express deep gratitude to Professor Michiel Hazewinkel without whom this collection, as well as [4, 7, 8, 12], could hardly be published.

I am also grateful to A. Verbovetsky for reading the manuscript and for valuable remarks.

References

1. Bocharov, A. V., Chetverikov, V. N., Duzhin, S. V., Khor'kova, N. G., Krasil'shchik, I. S., Samokhin, A. V., Torkhov, Yu. N., Verbovetsky, A. M. and Vinogradov, A. M.: *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, Amer. Math. Soc., Providence, RI, 1999. Edited and with a preface by Krasil'shchik and Vinogradov.
2. Henneaux, M., Krasil'shchik, I. S. and Vinogradov, A. M. (eds): *Secondary Calculus and Cohomological Physics*, Contemp. Math. 219, Amer. Math. Soc., Providence, RI, 1998.
3. Henneaux, M., Krasil'shchik, I. S. and Vinogradov, A. M. (eds): *The International Conference on Secondary Calculus and Cohomological Physics, Moscow* (*communications*), The Electronic Library of European Math. Soc. (ElibEMS), 1997; URL `http://www.emis.de/proceedings/SCCP97/`
4. Kersten, P. H. M. and Krasil'shchik, I. S. (eds): *Geometrical and Algebraic Structures in Differential Equations*, Reprinted from *Acta Appl. Math.* **1–3** (1995), Kluwer Acad. Publ., Dordrecht 1995.
5. Krasil'shchik, I. S., Cohomology background in geometry of PDE, in [2], pp. 121–140; URL `http://diffiety.ac.ru/preprint/98/01_98abs.htm`
6. Krasil'shchik, I. S. and Verbovetsky, A. M.: Homological methods in equations of mathematical physics, Open Education and Sciences, Opava (Czech Rep.), 1998; `arXive:math.DG/9808130`
7. Krasil'shchik, I. S. and Vinogradov, A. M. (eds): *Algebraic Aspects of Differential Calculus, Acta Appl. Math.* **49**(3) (1997).
8. Krasil'shchik, I. S. and Vinogradov, A. M. (eds): *Geometrical Aspects of Nonlinear Differential Equations*, Acta Appl. Math. **56**(2–3) (1999).
9. Krasil'shchik, I. S. and Vinogradov, A. M.: Nonlocal trends in the geometry of differential equations: Symmetries, conservation laws, and Bäcklund transformations, in [12], pp. 161–209.
10. Krasil'shchik, I. S. and Kersten, P. H. M.: *Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations*, Kluwer Acad. Publ., Dordrecht, 2000, 384 pp.
11. Verbovetsky, A. M.: Notes on the horizontal cohomology, in [2], pp. 211–232; `arXive:math.DG/9803115`
12. Vinogradov, A. M. (ed.): *Symmetries of Partial Differential Equations. Conservation Laws – Applications – Algorithms*, Reprinted from *Acta Appl. Math.* **15** & **16** (1989), Kluwer Acad. Publ., Dordrecht, 1989.
13. Vinogradov, A. M.: An informal introduction to the geometry of jet spaces, *Rend. Sem. Fac. Sci. Univ. Cagliari* **58** (1988), 301–333.
14. Vinogradov, A. M.: Local symmetries and conservation laws, *Acta Appl. Math.* **2** (1981), 21–78.
15. Vinogradov, A. M.: The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws, I. The linear theory. II. The nonlinear theory, *J. Math. Anal. Appl.* **100** (1984), 1–129.