**TOPICAL REVIEW**

# Geometry of jet spaces and integrable systems

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**Abstract.** An overview of some recent results on geometry of partial differential equations in application to integrable systems is given. Lagrangian and Hamiltonian formalism both in the free case (on the space of infinite jets) and with constraints (on a PDE) are discussed. Analogs of tangent and cotangent bundles to a differential equation are introduced and the variational Schouten bracket is defined. General theoretical constructions are illustrated by a series of examples.

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## Introduction

The main task of this paper is to overview a series of our results achieved recently in understanding integrability properties of partial differential equations (PDEs) arising in mathematical physics and geometry [1–10]. These results are essentially based on the geometrical approach to PDEs developed since 1970s by A. Vinogradov and his school (see [11–15] and references therein). The approach treats a PDE as an (infinite-dimensional) submanifold in the space $J^\infty(\pi)$ of infinite jets for a bundle $\pi\colon E \to M$ whose sections play the rôle of unknown functions (fields). This attitude allowed to apply to PDEs powerful techniques of differential geometry and homological algebra. The latter, in particular, made it possible to give an invariant and efficient formulation of higher-order Lagrangian formalism with constraints and Calculus of variations (see [14, 16–25] and, as it became clear later, was a bridge to BRST cohomology in gauge theories, anti-field formalism and related topics, [26]; see also [27–29].

Geometrical treatment of differential equation has a long history and originates in the works by Sophus Lie [30–32], as well as in research by A.V. Bäcklund [33], G. Monge [34], G. Darboux [35], L. Bianchi [36] and, later, by Élie Cartan [37]. Note incidentally that Cartan's theory of involutivity for external differential systems was an inspiration for another cohomological theory associated to PDEs and developed in papers by D. Spencer and his school, [38, 39]. Spencer's work (the so-called formal theory) closely relates to earlier and unfairly forgotten results by M. Janet [40] and Ch. Riquier [41]; see also [42] as well as [43–46].

A milestone in geometry of differential equations was introduction by Charles Ehresmann the notion of jet bundles [47, 48] that became a most adequate language for

Lie's theory, but a real revival of the latter came with the works by L.V. Ovsyannikov (see his book [49] on group analysis of PDEs; see also [50–53]).

A new impulse for reappraisal of the Sophus Lie heritage was given by the discovery of integrability phenomena in nonlinear systems [54–62] in the fall of 1960s‡ and Hamiltonian interpretation of this integrability [58, 66]. In particular, it became clear that integrable equations possess infinite series of higher, or generalized, symmetries (see [11, 67]), and classification of evolution equations with respect to this property allowed to discover new, at that time, integrable equation, [68–70]. Later the notion of a higher symmetry was generalized further to that of a nonlocal one [71] and the search for a geometrical background of nonlocality led to the concept of differential covering [72]. The latter proved to play an important rôle in the geometry of PDEs and we discuss it in our review.

It also became clear that the majority of integrable evolutionary systems possess a bi-Hamiltonian structure [73, 74], i.e., can be represented as Hamiltonian flows on the space of infinite jets in at least two different ways and the corresponding Hamiltonian structures are compatible. Bi-hamiltonianity, by Magri's scheme [74], leads to existence of infinite series of commuting symmetries and conservation laws. In addition, it gives rise to a recursion operator for higher symmetries that is an efficient tool for practical construction of symmetry hierarchies. Nevertheless, recursion operators exist for equations possessing no Hamiltonian structure at all (e.g., for the Burgers equation). A self-contained cohomological approach to recursion operators based on Nijenhuis brackets and related to the theory of deformations for PDE structures is exposed in [15].

The literature on the Hamiltonian theory of PDEs is vast and we confine ourselves here to the key references [75–78], but one feature is common to all research: theories and techniques are applicable to evolution equations only. Then a natural question arises: what to do if the equation at hand is not represented in the evolutionary form? We believe that (at least, a partial) answer to this question will be found in this paper.

Of course, one of possible solutions is to transform the equation to the evolutionary form. But:

- Not all equations can rendered to this form.
- How to check independence of Hamiltonian (and other) structures on a particular representation of our equation? In other words, if we found a Hamiltonian operator in one representation what guarantees that it survives when the representation is changed?
- Even if the answer to the previous question is positive how to transform the results when passing to the initial form of the equation?

In what follows, we treat any concrete equation "as is" and try to uncover those objects and constructions that are naturally associated to this equation. In particular, we do not assume existence of any additional structures that enrich the equation. Such

‡ Though discussions on "what is integrability" continued [63] and hold now still (see, e.g. quite recent papers [64] or [65]).

structures are by all means extremely interesting and lead to very nontrivial classes of equations (e.g., equations of hydrodynamical type [79], Monge-Ampére equations [80] or equations associated to Lie groups [81]), but here we look for internal properties of an arbitrary PDE.

As it was said in the very beginning, an equation (or, to be more precise, its infinite prolongation, i.e., equation itself together with all its differential consequences) is a submanifold in a jet space $J^\infty(\pi)$. To escape technical difficulties, we consider the simplest case, when $\pi\colon E \to M$ is a locally trivial vector bundle, though all the results remain valid in a more complicated situation (e.g., for jets of submanifolds). The reader who is interested in local results only may keep in mind the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ instead of $\pi$.

Understood in such a way, any equation $\mathcal{E} \subset J^\infty(\pi)$ is naturally endowed with a $(\dim M)$-dimensional integrable distribution $\mathcal{C}$ (the Cartan distribution) which consists, informally speaking, of planes tangent to formal solutions to $\mathcal{E}$. This is the main and essentially the only geometric structure that we use.

In the research of the PDE differential geometry we use somewhat informal but quite productive guidelines which were originally introduced in [82] (see also [83, 84]) and may be formulated as

**The structural principle:** Any construction and concept must take into account the Cartan distribution on $\mathcal{E}$.

**The correspondence principle:** "Physical dimension" of $\mathcal{E}$ is $n = \dim \mathcal{C} = \dim M$ and differential geometry of $\mathcal{E}$ reduces to the finite-dimensional one when passing to the "classical limit $n \to 0$".

**The invariance principle:** All constructions must be independent of the embedding $\mathcal{E} \to J^\infty(\pi)$ and defined by the equation $\mathcal{E}$ itself.

Below we accompany our exposition by toy dictionaries that illustrate the correspondence between two languages, those of the geometry of PDEs and classical differential geometry.

The paper consists of three sections. In Section 1 we describe the geometry of the "empty equation", i.e., of the jet space $J^\infty(\pi)$. In particular, we define the tangent and cotangent bundles to $J^\infty(\pi)$, introduce variational differential forms and multivectors, define the variational Schouten bracket. We discuss geometry of Hamiltonian flows on the space of infinite jets (i.e., Hamiltonian evolutionary equations) and Lagrangian formalism without constraints. Section 2 deals with the same matters, but in the context of a differential equation $\mathcal{E} \subset J^\infty(\pi)$. Although the exposition in this part is quite general, the result on the Hamiltonian theory (the definition of the cotangent bundle, in particular) are valid for the so-called 2-line equations only (we call such equations normal in Section 2). This notion is related to the cohomological length of the compatibility complex for the linearization operator of $\mathcal{E}$ and manifests itself, for example, in the number of nontrivial lines in Vinogradov's $\mathcal{C}$-spectral sequence, [16] (see also [14, 18, 85] and [13, 86]). From this point of view, jet spaces are 1-line equations and this is the

reason why they have to be treated separately. Finally, in Section 3 we briefly overview the theory of differential coverings for PDEs and some of its applications: nonlocal symmetries and shadows, Bäcklund transformations, etc.

We did our best to illustrate the general theory by a reasonable number of examples and really do hope that the result will be interesting to the readers.

## 1. Jet spaces

Jet spaces constitute a natural geometric environment for differential equations and for equations of mathematical physics, in particular. But these spaces are themselves an interesting geometric object that contains information on Lagrangian and Hamiltonian formalisms without constraints. Thus, we begin our exposition with a description of these spaces and structures related to them.

### 1.1. Definition of jet spaces

Let $\pi\colon E \to M$ be a locally trivial smooth vector bundle§ over a smooth manifold $M$, $\dim M = n$, $\dim E = m + n$. In what follows, $M$ will be the manifold of independent variables while sections of $\pi$ will play the rôle of unknown functions (fields). The set of all sections $s\colon M \to E$ will be denoted by $\Gamma(\pi)$ and it forms a module over the algebra $C^\infty(M)$. Two sections $s$, $s' \in \Gamma(\pi)$ are said to be *k-equivalent* at a point $x \in M$ if their graphs are tangent to each other with order $k$ at the point $s(x) = s'(x) \in E$. The equivalence class of $s$ with respect to this relation is denoted by $[s]_x^k$ and is called the *k-jet* of $s$ at $x$. The set

$$J^k(\pi) = \{\, [s]_x^k \mid x \in M, s \in \Gamma(\pi) \,\}$$

is endowed with a natural structure of a smooth manifold; the latter is called the *manifold of k-jets* of sections of $\pi$. Moreover, the maps

$$\pi_k\colon J^k(\pi) \to M, \quad [s]_x^k \mapsto x, \tag{1.1}$$

and

$$\pi_{k,l}\colon J^k(\pi) \to J^l(\pi), \quad [s]_x^k \mapsto [s]_x^l, \qquad k \geq l, \tag{1.2}$$

are smooth fibre bundles, $\pi_k$ being vector bundles. For any section $s \in \Gamma(\pi)$ the map

$$j_k(s)\colon M \to J^k(\pi), \quad x \mapsto [s]_x^k, \tag{1.3}$$

is a smooth section of $\pi_k$ that is called the *k-jet* of $s$.

Here we are mostly interested in the case $k = \infty$, i.e., in the space $J^\infty(\pi)$. It can be understood as the inverse limit of the chain

$$\cdots \to J^{k+1}(\pi) \xrightarrow{\pi_{k+1,k}} J^k(\pi) \to \cdots \to J^1(\pi) \xrightarrow{\pi_{1,0}} J^0(\pi) = E \xrightarrow{\pi} M. \tag{1.4}$$

Due to projections (1.4) there exist monomorphisms of function algebras

$$C^\infty(M) \subset \mathcal{F}_0(\pi) \subset \ldots \subset \mathcal{F}_k(\pi) \subset \mathcal{F}_{k+1}(\pi) \subset \ldots, \tag{1.5}$$

§ For a definition of jets in a more general setting see, e.g., [11, 14, 87, 88].

where $\mathcal{F}_k(\pi) = C^\infty(J^k(\pi))$, and we define the *algebra of smooth functions* on $J^\infty(\pi)$ as the filtered algebra $\mathcal{F}(\pi) = \cup_k \mathcal{F}_k(\pi)$. Elements of $\mathcal{F}(\pi)$ are identified with nonlinear scalar differential operators acting on sections of $\pi$ by the following rule:

$$\Delta_f(s) = j_\infty(s)^*(f), \qquad s \in \Gamma(\pi), f \in \mathcal{F}(\pi). \tag{1.6}$$

More general, let $\pi' \colon E' \to M$ be another vector bundle and $\pi^*(\pi')$ be its pull-back to $J^\infty(\pi)$. Introduce the notation $\mathcal{F}(\pi, \pi') = \Gamma(\pi^*(\pi'))$. Then any section $f \in \mathcal{F}(\pi, \pi')$ is identified, by a formula similar to (1.6), with a nonlinear differential operator that acts from $\Gamma(\pi)$ to $\Gamma(\pi')$.

## 1.2. Vector fields and differential forms

A *vector field* on $J^\infty(\pi)$ is a derivation of the function algebra $\mathcal{F}(\pi)$, i.e., an $\mathbb{R}$-linear map $X \colon \mathcal{F}(\pi) \to \mathcal{F}(\pi)$ such that

$$X(fg) = f X(g) + g X(f)$$

for all $f, g \in \mathcal{F}(\pi)$. The set of all vector fields is denoted by $\mathcal{X}(\pi)$ and it is a Lie algebra with respect to commutator (the *Lie bracket*).

The definition of a *differential form* of degree $r$ on $J^\infty(\pi)$ is similar to that of smooth functions. Using projections (1.4) we consider the embeddings $\Lambda^r(J^k(\pi)) \subset \Lambda^r(J^{k+1}(\pi))$ and set $\Lambda^r(\pi) = \cup_k \Lambda^r(J^k(\pi))$. We shall also consider the Grassmann algebra of all forms $\Lambda^*(\pi) = \oplus_{r \geq 0} \Lambda^r(\pi)$ with respect to the wedge product.

*Coordinates.* Let $\mathcal{U} \subset M$ be a coordinate neighbourhood such that the bundle $\pi$ becomes trivial over $\mathcal{U}$. Choose local coordinates $x^1, \ldots, x^n$ in $\mathcal{U}$ and $u^1, \ldots, u^m$ along the fibres of $\pi$ over $\mathcal{U}$. Then *adapted* coordinates in $\pi^{-1}(\mathcal{U}) \subset J^\infty(\pi)$ naturally arise. These coordinates are denoted by $u_I^j$, $I$ being a multi-index, and are defined by

$$j_\infty(s)^*(u_I^j) = \frac{\partial^{|I|} s^j}{\partial x^I},$$

where $s = (s^1, \ldots, s^m)$ is a local section of $\pi$ over $\mathcal{U}$. In other words, the coordinate functions $u_I^j$ correspond to partial derivatives of local sections.

In these coordinates, smooth function on $J^\infty(M)$ are of the form

$$f = f(x^i, u_I^j),$$

where the number of arguments is *finite*. Vector fields are represented as *infinite* sums

$$X = \sum_i a_i \frac{\partial}{\partial x^i} + \sum_{I,j} a_I^j \frac{\partial}{\partial u_I^j}, \qquad a_i, a_I^j \in \mathcal{F}(\pi),$$

while differential forms of degree $r$ are *finite* sums

$$\omega = \sum b_{i_1,\ldots,i_c,j_{c+1},\ldots,j_r}^{I_{c+1},\ldots,I_r} \, dx^{i_1} \wedge \ldots \wedge dx^{i_c} \wedge du_{I_{c+1}}^{j_{c+1}} \wedge \ldots \wedge du_{I_r}^{j_r}.$$

*1.3. Main structure: the Cartan distribution*

Let $\theta \in J^\infty(\pi)$. Then graphs of all sections $j_\infty(s)$, $s \in \Gamma(\pi)$, passing through the point $\theta$ have a common $n$-dimensional tangent plane $\mathcal{C}_\theta$ (the *Cartan plane*). The correspondence $\mathcal{C}: \theta \mapsto \mathcal{C}_\theta$ is an integrable‖ $n$-dimensional distribution on $J^\infty(\pi)$ that is called the *Cartan distribution*. This distribution is the basic geometric structure on the manifold $J^\infty(\pi)$. In particular, the following result is valid:

**Proposition 1.1.** *A submanifold of $J^\infty(M)$ is a maximal integral manifold of $\mathcal{C}$ if and only if it is a graph of $j_\infty(s)$, where $s$ is a local section of $\pi$.*

Moreover, since the planes $\mathcal{C}_\theta$ project to $M$ non-degenerately, any vector field $X$ on $M$ can be uniquely lifted up to a field $\mathcal{C}X$ on $J^\infty(\pi)$. In such a way one obtains a connection in the bundle $\pi_\infty$ called the *Cartan connection*. This connection is flat, i.e.,

$$\mathcal{C}[X, Y] = [\mathcal{C}X, \mathcal{C}Y] \tag{1.7}$$

for all vector fields $X$, $Y$ on $M$. Due to (1.7), the space $\mathcal{CX}(\pi)$ of all vector fields lying in the Cartan distribution is a Lie subalgebra in $\mathcal{X}(\pi)$. Vector fields belonging to $\mathcal{X}(\pi)$ will be called *Cartan fields*.

Any vector field $Z \in \mathcal{X}(\pi)$ can be uniquely decomposed to its *vertical* and *horizontal* components,

$$Z = Z^v + Z^h, \tag{1.8}$$

where $Z^v$ is the projection of $X$ to the fibre of the bundle $\pi_\infty$ along Cartan planes, while $Z^h$ lies in the Cartan distribution. Thus, one has

$$\mathcal{X}(\pi) = \mathcal{X}^v(\pi) \oplus \mathcal{CX}(\pi), \tag{1.9}$$

where $\mathcal{X}^v(\pi)$ is the Lie algebra of vertical vector fields.

Dually to (1.9), the module of differential forms $\Lambda^1(\pi)$ splits into the direct sum

$$\Lambda^1(\pi) = \Lambda_c^1(\pi) \oplus \Lambda_h^1(\pi), \tag{1.10}$$

where $\Lambda_c^1(\pi)$ consists of 1-forms that annihilate the Cartan distribution (they will be called *Cartan forms*), while elements of $\Lambda_h^1(\pi)$ are *horizontal forms*.

*Coordinates.* Choose an adapted coordinate system $(x^i, u_I^j)$ in $J^\infty(\pi)$. Then one has

$$\mathcal{C}: \frac{\partial}{\partial x^i} \mapsto D_i = \frac{\partial}{\partial x^i} + \sum_{I,j} u_{Ii}^j \frac{\partial}{\partial u_I^j}. \tag{1.11}$$

The fields $D_i$ are called *total derivatives* and they span the Cartan distribution.

For the basis in the module $\Lambda_c^1(\pi)$ one can choose the forms

$$\omega_I^j = \mathrm{d}u_I^j - \sum_i u_{Ii}^j \, \mathrm{d}x^i, \tag{1.12}$$

while horizontal forms are

$$\omega = \sum_i a_i \, \mathrm{d}x^i, \qquad a_i \in \mathcal{F}(\pi). \tag{1.13}$$

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‖ Integrability is understood formally here and means that if $X$ and $Y$ are two vector fields lying in $\mathcal{C}$ then their bracket $[X, Y]$ lies in $\mathcal{C}$ as well.

*Remark* 1. It should be noted that all results and constructions below are valid not for the entire jet space only, but for an arbitrary open domain in $J^\infty(\pi)$. Everywhere below, when speaking about $J^\infty(\pi)$, we actually mean an open domain.

### 1.4. Evolutionary vector fields and linearizations

We shall now describe infinitesimal symmetries of the Cartan distribution on $J^\infty(\pi)$. A vector field $X \in \mathcal{X}(\pi)$ is a *symmetry* if $[X, Z] \in \mathcal{CX}(\pi)$ as soon as $Z \in \mathcal{CX}(\pi)$. The space $\mathcal{CX}(\pi)$ is an ideal in the Lie algebra $\mathcal{X}_c(\pi)$ of symmetries. Due to integrability of the Cartan distribution, any $Z \in \mathcal{CX}(\pi)$ is a symmetry, and we call such symmetries *trivial*. Thus, we introduce the Lie algebra of *nontrivial symmetries* as

$$\text{sym}\,\pi = \mathcal{X}_c(\pi)/\mathcal{CX}(\pi).$$

By (1.9), $\text{sym}\,\pi$ is identified with the vertical part of $\mathcal{X}_c(\pi)$.

Take a vector field $X \in \text{sym}\,\pi$ and restrict it to the subalgebra $\mathcal{F}_0(\pi) \subset \mathcal{F}(\pi)$. Then this restriction can be identified with an element $\varphi_X \in \mathcal{F}(\pi, \pi)$. For shortness, we shall use the notation $\mathcal{F}(\pi, \pi) = \varkappa(\pi)$.

**Theorem 1.2.** *The correspondence* $X \mapsto \varphi_X$ *defines a bijection between* $\text{sym}\,\pi$ *and* $\varkappa(\pi)$.

The element $\varphi_X$ is called the *generating section* of a symmetry $X$, while the symmetry corresponding to a section $\varphi \in \varkappa(\pi)$ is called an *evolutionary vector field* and is denoted by $\boldsymbol{E}_\varphi$.

Theorem 1.2 allows to introduce an $\mathcal{F}(\pi)$-module structure into the Lie algebra $\text{sym}\,\pi$ by setting

$$f \cdot \boldsymbol{E}_\varphi = \boldsymbol{E}_{f\varphi}.$$

This multiplication differs from the usual multiplication of vector field by functions and does not survive when passing from the space of jets to equations (see Section 2) below.

On the other hand, the same theorem determines a Lie algebra structure in $\varkappa(\pi)$: the *Jacobi bracket* $\{\varphi, \psi\}$ is uniquely defined by the equation

$$\boldsymbol{E}_{\{\varphi,\psi\}} = [\boldsymbol{E}_\varphi, \boldsymbol{E}_\psi]. \tag{1.14}$$

The Jacobi bracket can also be computed using the formula

$$\{\varphi, \psi\} = \boldsymbol{E}_\varphi(\psi) - \boldsymbol{E}_\psi(\varphi). \tag{1.15}$$

*Coordinates.* Let in an adapted coordinate system a section $\varphi \in \varkappa(\pi)$ be of the form $\varphi = (\varphi^1, \ldots, \varphi^m)$. Then the corresponding evolutionary vector field is

$$\boldsymbol{E}_\varphi = \sum_{I,j} D_I(\varphi^j) \frac{\partial}{\partial u_I^j}, \tag{1.16}$$

where $D_I = D_{i_1} \ldots D_{i_l}$ is the composition of total derivatives corresponding to the multi-index $I = i_1 \ldots i_l$.

If $\psi = (\psi^1, \ldots, \psi^m)$ is another element of $\varkappa(\pi)$ then the components of the Jacobi bracket are

$$\{\varphi, \psi\}^j = \sum_{I,\alpha} \left( D_I(\varphi^\alpha) \frac{\partial \psi^j}{\partial u_I^\alpha} - D_I(\psi^\alpha) \frac{\partial \varphi^j}{\partial u_I^\alpha} \right), \qquad j = 1, \ldots, m. \qquad (1.17)$$

Fix a section $\psi \in \varkappa(\pi)$ and consider the map

$$\ell_\psi \colon \varkappa(\pi) \to \varkappa(\pi), \qquad \ell_\psi(\varphi) = \boldsymbol{E}_\varphi(\psi). \qquad (1.18)$$

This map is called the *linearization* of the element $\psi$ (recall that $\psi$ may be identified with a nonlinear differential operator acting from $\pi$ to $\pi$).

More generally, let $\pi' \colon E' \to M$ be a vector bundle. Then the action of an evolutionary vector field $\boldsymbol{E}_\varphi$ can be extended to

$$\boldsymbol{E}_\varphi \colon \mathcal{F}(\pi, \pi') \to \mathcal{F}(\pi, \pi')$$

in a well defined way. Consider a section $\psi \in \mathcal{F}(\pi, \pi')$, i.e., a nonlinear differential operator from $\Gamma(\pi)$ to $\Gamma(\pi')$. Its linearization is the map

$$\ell_\psi \colon \varkappa(\pi) \to \mathcal{F}(\pi, \pi') \qquad (1.19)$$

defined similar to (1.18).

*Coordinates.* Let in adapted coordinates $\varphi = (\varphi^1, \ldots, \varphi^m)$ and $\psi = (\psi^1, \ldots, \psi^{m'})$. Then the $j$th component of $\ell_\psi(\varphi)$ is

$$\sum_{I,\alpha} \frac{\partial \psi^j}{\partial u_I^\alpha} D_I(\varphi^\alpha),$$

i.e., the linearization is a matrix operator of the form

$$\ell_\psi = \left\| \sum_I \frac{\partial \psi^j}{\partial u_I^\alpha} D_I \right\|_{\alpha=1,\ldots,m.}^{j=1,\ldots,m'} \qquad (1.20)$$

Using linearizations, formula (1.15) can be rewritten as

$$\{\varphi, \psi\} = \ell_\psi(\varphi) - \ell_\varphi(\psi). \qquad (1.21)$$

### 1.5. $\mathcal{C}$-differential operators

From (1.20) we see that linearizations are differential operators in total derivatives. We call such operators $\mathcal{C}$-*differential operators*. More precisely, let $\xi$ and $\xi'$ be two vector bundles over $J^\infty(\pi)$ and $P$, $P'$ be the $\mathcal{F}(\pi)$-modules of their sections. An $\mathbb{R}$-linear map $\Delta \colon P \to P'$ is a $\mathcal{C}$-differential operator of order $k$ if for any point $\theta \in J^\infty(\pi)$ and a section $p \in P$ the value of $\Delta(p)$ at $\theta$ is completely determined by the values of $D_I(p)$, $|I| \leq k$, at this point. The space of all such operators is denoted by $\mathcal{C}_k(P, P')$ and we also set $\mathcal{C}(P, P') = \cup_k \mathcal{C}_k(P, P')$.

A closely related notion to that of a $\mathcal{C}$-differential operator is *horizontal jets*. Let $P$ be as above. We say that two sections $p$, $p' \in P$ are *horizontally k-equivalent* (the

case $k = \infty$ is included) at a point $\theta \in J^\infty(\pi)$ if $D_I(p) = D_I(p')$ at $\theta$ for all $I$ such that $|I| \leq k$. Denote the equivalence class by $\{p\}^k_\theta$. The set

$$J^k_h(P) = \{ \, \{p\}^k_\theta \mid \theta \in J^\infty(\pi), p \in P \, \}$$

forms a smooth manifold that is fibred over $J^\infty(\pi)$,

$$\xi_k \colon J^k_h(P) \to J^\infty(\pi).$$

The section $j^h_k(p) \colon J^\infty(\pi) \to J^k_h(P)$, $\theta \mapsto \{p\}^k_\theta$, is called the *horizontal jet* of $p \in P$.

**Proposition 1.3.** *For any $\mathcal{C}$-differential operator $\Delta \in \mathcal{C}_k(P, P')$ there exists a unique morphism $\Phi_\Delta$ of vector bundles $\xi_k$ and $\xi'$ such that $\Delta(p) = \Phi_\Delta(j^h_k(p))$.*

Two natural identifications will be useful below.

**Proposition 1.4.** *For any vector bundle $\pi$ one has:*

  (i) *The module $\Lambda^1_c(\pi)$ is isomorphic to $\mathcal{C}(\varkappa(\pi), \mathcal{F}(\pi))$.*

 (ii) *The module $J^\infty_h(\varkappa(\pi))$ is isomorphic to $\mathcal{X}^v(\pi)$. The vector fields corresponding to sections of the form $j^h_\infty(p)$ are evolutionary fields.*

*Coordinates.* Choose an adapted coordinate system in the manifold $J^\infty(\pi)$ and let $r$, $r'$ be dimensions of the bundles $\xi$, $\xi'$, respectively. Then any operator $\Delta \in \mathcal{C}(P, P')$ is of the form

$$\Delta = \left\| \sum_I a^I_{\alpha\beta} D_I \right\|^{\beta=1,\dots,r}_{\alpha=1,\dots,r',} \qquad a^I_{\alpha\beta} \in \mathcal{F}(\pi). \tag{1.22}$$

If $v^1, \dots, v^r$ are fibre-wise coordinates in the bundle $\xi$ then adapted coordinates $v^l_K$ in $J^\infty_h(\xi)$, $K$ being a multi-index, $l = 1, \dots, r$, are determined by the equalities

$$j^h_\infty(s)^*(v^l_K) = D_K(s^l), \tag{1.23}$$

where $s = (s^1, \dots, s^r)$ is a local section of the bundle $\xi$.

*Remark* 2. The space of horizontal jets $J^\infty_h(P)$ is also endowed with an integrable distribution similar to the Cartan one: if $\theta \in J^\infty_h(P)$ then the corresponding plane $\mathcal{C}_\theta$ is tangent to the graphs of horizontal jets passing through this point. The differential of the map $\xi_\infty \colon J^\infty_h(P) \to J^\infty(\pi)$ isomorphically projects $\mathcal{C}_\theta$ to $\mathcal{C}_{\xi_\infty(\theta)}$.

Moreover, if $P$ is of the form $P = \Gamma(\pi^*_\infty(\xi))$, where $\xi$ is a vector bundle over $M$ and $\pi^*_\infty(\xi)$ is its pull-back, then one has a diffeomorphism

$$J^\infty_h(P) = J^\infty(\pi \times_M \xi),$$

where $\pi \times_M \xi$ is the Whitney product, and this isomorphism takes the Cartan distribution on $J^\infty_h(P)$ to the one on $J^\infty(\pi \times_M \xi)$.

*Remark* 3. In the case when the modules $P = \Gamma(\pi^*_\infty(\xi))$ and $P' = \Gamma(\pi^*_\infty(\xi'))$ are of the form considered in Remark 2 $\mathcal{C}$-differential operators from $P$ to $P'$ may be understood as non-linear differential operators that take sections of $\pi$ to linear differential operators acting from $\xi$ to $\xi'$.

### 1.6. Variational complex and Lagrangian formalism

Consider now decomposition (1.10) and note that it implies a more general splitting

$$\Lambda^k(\pi) = \bigoplus_{p+q=k} \Lambda_{\mathcal{C}}^p(\pi) \otimes \Lambda_h^q(\pi), \tag{1.24}$$

where

$$\Lambda_{\mathcal{C}}^p(\pi) = \underbrace{\Lambda_{\mathcal{C}}^1(\pi) \wedge \ldots \wedge \Lambda_{\mathcal{C}}^1(\pi)}_{p \text{ times}}, \qquad \Lambda_h^q(\pi) = \underbrace{\Lambda_h^1(\pi) \wedge \ldots \wedge \Lambda_h^1(\pi)}_{q \text{ times}}.$$

Introduce the notation $\Lambda_{\mathcal{C}}^p(\pi) \otimes \Lambda_h^q(\pi) = E_0^{p,q}(\pi)$. By Proposition 1.4 (i), this space is identified with the module $\mathcal{C}_p^{\mathrm{sk}}(\varkappa(\pi), \Lambda_h^q(\pi))$ of $p$-linear skew-symmetric $\mathcal{C}$-differential operators acting from $\varkappa(\pi)$ to $\Lambda_h^q(\pi)$.

The de Rham differential $\mathrm{d} \colon \Lambda^k(\pi) \to \Lambda^{k+1}(\pi)$, by (1.24), splits into two parts $\mathrm{d} = \mathrm{d}_{\mathcal{C}} + \mathrm{d}_h$, where

$$\mathrm{d}_{\mathcal{C}} = \mathrm{d}_{\mathcal{C}}^{p,q} \colon E_0^{p,q}(\pi) \to E_0^{p+1,q}(\pi), \qquad \mathrm{d}_h = \mathrm{d}_h^{p,q} \colon E_0^{p,q}(\pi) \to E_0^{p,q+1}(\pi) \tag{1.25}$$

are the *vertical* (or *Cartan*) and *horizontal* differentials, respectively These differentials anti-commute, i.e.,

$$\mathrm{d}_{\mathcal{C}} \circ \mathrm{d}_h + \mathrm{d}_h \circ \mathrm{d}_{\mathcal{C}} = 0. \tag{1.26}$$

*Coordinates.* For a function on $J^\infty(\pi)$ (i.e., a 0-form) the action of the Cartan differential is given by

$$\mathrm{d}_{\mathcal{C}} f = \sum_{I,j} \frac{\partial f}{\partial u_I^j} \mathrm{d}_{\mathcal{C}} u_I^j, \tag{1.27}$$

where $\mathrm{d}_{\mathcal{C}} u_I^j = \omega_I^j$ are the Cartan forms presented in (1.12), while the horizontal differential acts as follows

$$\mathrm{d}_h f = \sum_i D_i(f) \, \mathrm{d} x^i. \tag{1.28}$$

To compute the action on arbitrary forms it suffices to use (1.27) and (1.28) and the fact that $\mathrm{d}_{\mathcal{C}}$ and $\mathrm{d}_h$ differentiate the wedge product and anti-commute with the de Rham differential.

Let $E_1^{p,q}(\pi)$ be the cohomology of $\mathrm{d}_h$ at the term $E_0^{p,q}(\pi)$, i.e.,

$$E_1^{p,q}(\pi) = \ker \mathrm{d}_h^{p,q} / \operatorname{im} \mathrm{d}_h^{p,q-1}. \tag{1.29}$$

Due to (1.26), the vertical differential $\mathrm{d}_{\mathcal{C}}$ induces differentials $E_1^{p,q}(\pi) \to E_1^{p+1,q}(\pi)$ which will be denoted by $\delta^{p,q}$. The groups $E_1^{p,q}(\pi)$, together with the differentials $\delta^{p,q}$, play one of the most important rôles in the geometry of jets providing the background for Lagrangian formalism without constraints. To describe them we shall need new notions.

Let $\xi$ be a vector bundle over $J^\infty(\pi)$ and $P = \Gamma(\xi)$. Introduce the *adjoint* module $\hat{P} = \operatorname{Hom}_{\mathcal{F}(\pi)}(P, \Lambda_h^n(\pi))$. Consider another module of sections $Q$ and a $\mathcal{C}$-differential operator $\Delta \colon P \to Q$. Then the *adjoint* operator $\Delta^* \colon \hat{Q} \to \hat{P}$ is defined and it enjoys the *Green formula*

$$\langle \Delta(p), \hat{q} \rangle - \langle p, \Delta^*(\hat{q}) \rangle = \mathrm{d}_h \omega \tag{1.30}$$

for all $p \in P$, $\hat{q} \in \hat{Q}$ and some $\omega = \omega(p, \hat{q}) \in \Lambda_h^{n-1}(\pi)$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between the module and its adjoint. It is useful to keep in mind that the correspondence $(p, \hat{q}) \mapsto \omega(p, \hat{q})$ is a $\mathcal{C}$-differential operator with respect to both arguments.

An operator $\Delta$ is called *self-adjoint* if $\Delta^* = \Delta$ and *skew-adjoint* if $\Delta^* = -\Delta$.

*Coordinates.* If $\Delta = \sum_I a_I D_i$ is a scalar operator then

$$\Delta^* = \sum_I (-1)^{|I|} D_I \circ a_I. \tag{1.31}$$

For matrix $\mathcal{C}$-differential operators $\Delta = \|\Delta_{ij}\|$ one has

$$\Delta^* = \left\| \Delta_{ji}^* \right\|, \tag{1.32}$$

where $\Delta_{ji}^*$ is given by (1.31).

We can now describe the groups $E_1^{p,q}(\pi)$.

**Theorem 1.5** (One-line Theorem). *Let $\pi \colon E \to M$, $\dim M = n$, be a locally trivial vector bundle. Then:*

(i) *the groups $E_1^{0,q}(\pi)$, $q = 0, \ldots, n-1$, are isomorphic to the de Rham cohomology groups $H^q(M)$ of the manifold $M$;*

(ii) *the group $E_1^{0,n}(\pi)$ consists of the Lagrangians depending on the fields that are sections of $\pi$;*

(iii) *the groups $E_1^{p,n}(\pi)$, $p > 0$, are identified with the modules $\mathcal{C}_{p-1}^{\mathrm{sk}\,*}(\varkappa(\pi), \hat{\varkappa}(\pi))$ of $(p-1)$-linear skew-symmetric $\mathcal{C}$-differential operators that are skew-adjoint in each argument (in particular, $E_1^{1,n}(\pi) = \hat{\varkappa}(\pi)$);*

(iv) *all other terms are trivial.*

*Remark* 4. The group $E_1^{0,n}(\pi)$ is also called the $n$th horizontal cohomology group of $\pi$ and denoted by $H_h^n(\pi)$.

*Remark* 5. The construction above is a particular case of Vinogradov's $\mathcal{C}$-spectral sequence, [13, 14, 16–18].

In what follows, we shall assume the manifold $M$ to be cohomologically trivial, i.e., its de Rham cohomology is isomorphic to $\mathbb{R}$.

Define the operator $\delta \colon \Lambda_h^n(\pi) \to \hat{\varkappa}(\pi)$ as the composition of the projection $\Lambda_h^n(\pi) \to E_1^{0,n}$ and the differential $\delta_1^{0,n} \colon E_1^{0,n} \to E_1^{1,n}$.

*Remark* 6. In what follows, we shall also use the notation $\delta$ for the differential $\delta_1^{0,n}$ itself.

*Coordinates.* If $\omega \in \Lambda_h^n(\pi)$ is of the form $L \, dx^1 \wedge \ldots \wedge dx^n$ then

$$\delta(\omega) = \left( \frac{\delta L}{\delta u^1}, \ldots, \frac{\delta L}{\delta u^m} \right), \tag{1.33}$$

where

$$\frac{\delta L}{\delta u^j} = \sum_I (-1)^{|I|} D_I \frac{\partial L}{\partial u_I^j} \tag{1.34}$$

are *variational derivatives*. Thus, $\delta$ is the *Euler* operator and it takes a Lagrangian density $\omega$ to the corresponding *Euler-Lagrange operator*.

**Proposition 1.6.** *Let $\pi$ be locally trivial vector bundle over a cohomologically trivial manifold $M$. Then the complex*

$$\mathcal{F}(\pi) \xrightarrow{d_h^{0,0}} \Lambda_h^1(\pi) \xrightarrow{d_h^{0,1}} \cdots \to \Lambda_h^{n-1}(\pi) \xrightarrow{d_h^{0,n-1}} \Lambda_h^n(\pi)$$
$$\xrightarrow{\delta} \hat{\varkappa} \xrightarrow{\delta_1^{1,n}} \mathcal{C}_1^{\mathrm{sk}\,*}(\varkappa, \hat{\varkappa}) \xrightarrow{\delta_1^{2,n}} \mathcal{C}_2^{\mathrm{sk}\,*}(\varkappa, \hat{\varkappa}) \to \cdots$$

(1.35)

*is exact, i.e., the kernel of each differential coincides with the image of the preceding one.*

In the coordinate-free way, the Euler-Lagrange operator can be computed by

$$\delta(\omega) = \ell_\omega^*(1), \qquad \omega \in \Lambda_h^n(\pi),$$

(1.36)

*while the differentials $\delta_1^{p,n}$ enjoy the equality*

$$(\delta_1^{p,n}\Delta)(\varphi_1,\dots,\varphi_p) = \sum_{i=1}^p (-1)^{i+1}\ell_{\Delta,\varphi_1,\dots,\hat{\varphi}_i,\dots,\varphi_p}(\varphi_i) + (-1)^p \ell_{\Delta,\varphi_1,\dots,\varphi_{p-1}}^*(\varphi_p),$$

(1.37)

*where $\Delta \in \mathcal{C}_{p-1}^{\mathrm{sk}\,*}(\varkappa(\pi), \hat{\varkappa}(\pi))$ and $\varphi_1,\dots,\varphi_p \in \varkappa$. We use the notation $\ell_{\Delta,\varphi_1,\dots,\varphi_k}(\varphi) = \boldsymbol{E}_\varphi(\Delta)(\varphi_1,\dots,\varphi_k)$ In particular, if $\psi \in \hat{\varkappa}(\pi)$ then*

$$\delta_1^{1,n}(\psi) = \ell_\psi - \ell_\psi^*.$$

(1.38)

*Remark* 7. Complex (1.35) is exact starting from the term $\Lambda_h^n(\pi)$ independently of cohomological properties of the manifold $M$. This complex is called the *global variational complex* of the bundle $\pi$, see [89].

As a consequence of Proposition 1.6, we obtain the following result:

**Theorem 1.7.** *For a vector bundle $\pi$ one has:*

(i) *The action functional*

$$s \mapsto \int_M j_\infty(s)^*(\omega), \qquad s \in \Gamma(\pi), \quad \omega \in \Lambda_h^n(\pi),$$

*is stationary on a section $s$ if and only if $j_\infty(s)^*(\delta(\omega)) = 0$ (i.e., as we shall see below, $s$ is a solution of the Euler-Lagrange equation corresponding to $\omega$).*

(ii) *Variationally trivial Lagrangians are total divergences, which amounts to $\ker \delta = \operatorname{im} d_h^{0,n-1}$.*

(iii) *All null total divergences are total curls, i.e., $d_h^{0,n-1}\omega = 0$ if and only if $\omega = d_h^{0,n-2}\theta$ for some $\theta \in \Lambda_h^{n-2}(\pi)$.*

(iv) *If $\psi \in \hat{\varkappa}(\pi)$ then the nonlinear differential operator $\Delta_\psi \colon \varkappa(\pi) \to \hat{\varkappa}(\pi)$ is of the form $\delta\omega$ (i.e., is an Euler-Lagrange operator) if and only if $\ell_\psi = \ell_\psi^*$ (the Helmholtz condition).*

### 1.7. A parallel with finite-dimensional differential geometry. I

We want to indicate here a very useful and productive analogy between geometry of jet spaces (and, more generally, of differential equations) and classical differential geometry of finite-dimensional smooth manifolds. This parallel was exposed by A.M. Vinogradov

first within his philosophy of *Secondary Calculus* (cf. [14] and references therein) and we elaborate it further.

Two points of view on jet spaces as geometrical objects may exist. The first one is formal, traditional and straightforward. It was described on the previous pages and treats $J^\infty(\pi)$ as a particular case of general infinite-dimensional manifolds. Such an approach, being by all means necessary for a rigorous exposition of the theory, actually ignores essential intrinsic structures of jet spaces.

Another viewpoint is completely informal but incorporates these structures *ab ovo* and allows to reveal new and non-trivial relations and results just "translating" from the language of the classical differential geometry. To facilitate such translations, we shall compile a sort of a dictionary.

So, we consider the space $J^\infty(\pi)$ endowed with the Cartan distribution $\mathcal{C}$ and take for the points of the new "manifold" maximal integral submanifolds of $\mathcal{C}$. As it was indicated above, they are graphs of $j_\infty(s)$, $s \in \Gamma(\pi)$, and thus new points are sections of $\pi$ (i.e., fields).

Let $\omega \in \Lambda^n_h(\pi)$ be a horizontal $n$-form on $J^\infty(\pi)$ (or a Lagrangian density). Then to any "point" $s \in \Gamma(\pi)$ we can put into correspondence the number

$$\omega(s) = \int_M j_\infty(s)^*(\omega).$$

Thus, Lagrangians are understood as functions. Due to the identity

$$j_\infty(s)^*(\mathrm{d}_h\omega) = \mathrm{d}(j_\infty(s)^*(\omega))$$

and the Stokes' formula

$$\int_M \mathrm{d}\theta = \int_{\partial M} \theta, \qquad \theta \in \Lambda^{n-1}(M),$$

"functions" of the form $\mathrm{d}_h\omega$ vanish at all points. So, no-trivial functions are elements of the cohomology group $E_1^{0,n}(\pi) = H^h_h(\pi)$. Thus, the beginning of the dictionary is

| **Manifold** $M$ | | **Jet space** $J^\infty(\pi)$ |
|---|---|---|
| points | $\longleftrightarrow$ | sections of $\pi$ (fields) |
| functions | $\longleftrightarrow$ | Lagrangians $\omega = L\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ |
| value at a point, $f(x)$ | $\longleftrightarrow$ | integral $\omega(s) = \displaystyle\int_M j_\infty(s)^*\omega$ |

The next step is to define vector fields. They should be infinitesimal transformations of $J^\infty(\pi)$ that preserve the Cartan distribution (or, equivalently, move "points" to "points"). These are exactly the fields lying in $\mathcal{X}_c(\pi)$. But vector fields $X \in \mathcal{CX}(\pi)$ (i.e., lying in the Cartan distribution) are tangent to maximal integral manifolds of the latter and thus are trivial in the space of fields. Consequently, non-trivial vector fields are identified with elements of $\mathrm{sym}\,\pi$, i.e., with evolutionary vector fields on $J^\infty(\pi)$. On the other hand, as it was indicated above, they are integrable sections of $J^\infty_h(\varkappa)$. Hence, this bundle can be considered as the tangent bundle to $J^\infty(\pi)$. So, the dictionary can

be continued as follows:

|  **Manifold** $M$ |  | **Jet space** $J^\infty(\pi)$ |
| --- | --- | --- |
| vector fields, $\mathcal{X}(M)$ | $\longleftrightarrow$ | evolutionary vector fields, $\varkappa(\pi)$ |
| tangen bundle | $\longleftrightarrow$ | bundle of horizontal jets $J_h^\infty(\varkappa(\pi))$ |

Differential forms on a smooth manifold $M$ may be understood as multi-linear functions on the space of vector fields (or fibre-wise multi-linear functions on $T(M)$): we insert a vector field into a $p$-form and obtain a $(p-1)$-form. In our context, such objects are exactly elements of $\mathcal{C}_{p-1}^{\mathrm{sk}\,*}(\varkappa, \hat{\varkappa}) = E_1^{p,n}(\pi)$. We call them *variational forms* of degree $p$ and have the following parallel:

|  **Manifold** $M$ |  | **Jet space** $J^\infty(\pi)$ |
| --- | --- | --- |
| differential forms, $\Lambda^p(M)$ | $\longleftrightarrow$ | variational forms, $\mathcal{C}_{p-1}^{\mathrm{sk}\,*}(\varkappa, \hat{\varkappa})$ |
| de Rham complex | $\longleftrightarrow$ | variational complex |

*Remark* 8. It can also be shown that smooth maps $J^\infty(\pi) \to J^\infty(\pi')$ preserving the Cartan distributions are completely determined by non-linear differential operators from $\Gamma(\pi)$ to $\Gamma(\pi')$ while the differentials of these maps $J_h^\infty(\varkappa) \to J_h^\infty(\varkappa')$ correspond to linearizations. Unfortunately, a detailed exposition of this parallel is out of scope of our review.

We shall continue to compile our dictionary in Subsection 1.9.

*1.8. Hamiltonian formalism*

The objects dual to $\mathcal{C}_{p-1}^{\mathrm{sk}\,*}(\varkappa(\pi), \hat{\varkappa}(\pi))$ are the modules of *variational multivectors* $D_p(\pi) = \mathcal{C}_{p-1}^{\mathrm{sk}\,*}(\hat{\varkappa}(\pi), \varkappa(\pi))$. In particular, $D_1(\pi) = \varkappa(\pi)$. We also set $D_0(\pi) = \Lambda_h^n(\pi)/\operatorname{im} \mathrm{d}_h^{0,n-1} = E_1^{0,n}(\pi)$. To describe Hamiltonian formalism on $J^\infty(\pi)$, we first introduce the *variational Schouten bracket* [1]

$$[\![\cdot, \cdot]\!]\colon D_p(\pi) \times D_q(\pi) \to D_{p+q-1}(\pi)$$

in the following way (cf. with [90, 91], see also [77, 92]). If $B = [\omega] \in D_0(\pi)$ is a coset of a horizontal form $\omega \in \Lambda_h^n(\pi)$ and $A \in D_p(\pi)$, $p > 0$, then we set

$$[\![A, B]\!] = (-1)^p [\![B, A]\!] = A(\delta\omega),$$

while for any $B \in D_q(\pi)$, $q > 0$, and $\psi = \delta\omega \in \hat{\varkappa}(\pi)$, $\omega \in \Lambda_h^n(\pi)$,

$$[\![A, B]\!](\psi) = [\![A, B(\psi)]\!] + (-1)^{q-1}[\![A(\psi), B]\!],$$

and these two equalities define the bracket completely. In particular,

$$[\![\varphi, [\omega]]\!] = [\boldsymbol{E}_\varphi(\omega)], \qquad \varphi \in D_1(\pi) = \varkappa(\pi), \quad \omega \in \Lambda_h^n(\pi),$$

and

$$[\![\varphi, \varphi']\!] = \boldsymbol{E}_\varphi(\varphi') - \boldsymbol{E}_{\varphi'}(\varphi) = \{\varphi, \varphi'\}, \qquad \varphi, \varphi' \in D_1(\pi).$$

**Proposition 1.8.** *The variational Schouten bracket determines a super Lie algebra structure in the space* $D(\pi) = \sum_{p \geq 0} D_p(\pi)$ *in the following sense:*

$$[\![A, B]\!] = -(-1)^{(p-1)(q-1)}[\![B, A]\!], \tag{1.39}$$

$$(-1)^{(p+1)(r+1)}[\![[\![A, B]\!], C]\!] + (-1)^{(q+1)(p+1)}[\![[\![B, C]\!], A]\!] \tag{1.40}$$

$$+(-1)^{(r+1)(q+1)}[\![[\![C, A]\!], B]\!] = 0$$

*for all* $A \in D_p(\pi)$, $B \in D_q(\pi)$, $C \in D_r(\pi)$.

To compute the Schouten bracket explicitly, for any natural $n$ consider the set $S_n^i$ of all $(n-i)$-*unshuffles* consisting of all permutations $\sigma$ of the set $\{1, \ldots, n\}$ such that

$$\sigma(1) < \ldots < \sigma(i), \qquad \sigma(i+1) < \ldots < \sigma(n).$$

We formally set $S_n^i = \varnothing$ for $i < 0$ and $i > n$. We also use a short notation $\psi_{\sigma(k_1, k_2)}$ for $\psi_{\sigma(k_1)}, \ldots, \psi_{\sigma(k_2)}$.

Let now $A \in D_p(\pi)$ and $B \in D_q(\pi)$. Then for any $\psi_1, \ldots, \psi_{p+q-1} \in \hat{\varkappa}(\pi)$ we have

$$[\![A, B]\!](\psi_1, \ldots, \psi_{p+q-1}) = \sum_{\sigma \in S_{p+q-1}^{q-1}} (-1)^\sigma \ell_{B, \psi_{\sigma(1, q-1)}}(A(\psi_{\sigma(q, p+q-1)}))$$

$$-(-1)^{(p-1)(q-1)} \sum_{\sigma \in S_{p+q-1}^p} (-1)^\sigma B(\ell^*_{A, \psi_{\sigma(1, p-1)}}(\psi_{\sigma(p)}), \psi_{\sigma(p+1, p+q-1)})$$

$$-(-1)^{(p-1)(q-1)} \sum_{\sigma \in S_{p+q-1}^{p-1}} (-1)^\sigma \ell_{A, \psi_{\sigma(1, p-1)}}(B(\psi_{\sigma(p, p+q-1)}))$$

$$+ \sum_{\sigma \in S_{p+q-1}^q} (-1)^\sigma A(\ell^*_{B, \psi_{\sigma(1, q-1)}}(\psi_{\sigma(q)}), \psi_{\sigma(q+1, p+q-1)}), \tag{1.41}$$

where $(-1)^\sigma$ stands for the parity of $\sigma$ and, as before, $\ell_{\Delta, \psi_1, \ldots, \psi_k}(\varphi) = \boldsymbol{E}_\varphi(\Delta)(\psi_1, \ldots, \psi_k)$.

We say that a bivector $A \in D_2(\pi) = \mathcal{C}^{\mathrm{sk}*}(\hat{\varkappa}(\pi), \varkappa(\pi))$ is a *Hamiltonian structure* on $J^\infty(\pi)$ if

$$[\![A, A]\!] = 0. \tag{1.42}$$

*Remark* 9. A more appropriate name is *Poisson* structure but we follow here the tradition accepted in the theory of integrable systems.

Given a Hamiltonian structure, one can define a *Poisson bracket* on the set of Lagrangians:

$$\{\omega, \omega'\}_A = \langle A(\delta(\omega)), \delta(\omega') \rangle, \qquad \omega, \omega' \in D_0(\pi). \tag{1.43}$$

Two elements are *in involution* (with respect to the structure $A$) if

$$\{\omega, \omega'\}_A = 0.$$

**Proposition 1.9.** *For any* $A \in \mathcal{C}^{\mathrm{sk}*}(\hat{\varkappa}(\pi), \varkappa(\pi))$ *one has*

$$\{\omega, \omega'\}_A = -\{\omega', \omega\}_A \tag{1.44}$$

*If in addition* $A$ *satisfies (1.42) then*

$$\{\omega, \{\omega', \omega''\}_A\}_A + \{\omega', \{\omega'', \omega\}_A\}_A + \{\omega'', \{\omega, \omega'\}_A\}_A = 0. \tag{1.45}$$

One also has

$$A_{\{\omega,\omega'\}_A} = \{A_\omega, A_{\omega'}\}, \tag{1.46}$$

where $A_\omega = A(\delta(\omega)) \in \varkappa(\pi)$ and the curlies in the right-hand side denote the Jacobi bracket.

Chose a Hamiltonian structure $A$ and consider the sequence of operators

$$0 \to D_0(\pi) \xrightarrow{\partial_A} D_1(\pi) \to \cdots \to D_p(\pi) \xrightarrow{\partial_A} D_{p+1}(\pi) \to \cdots, \tag{1.47}$$

where $\partial_A(B) = [\![A, B]\!]$.

**Proposition 1.10.** *Sequence (1.47) is a complex, i.e., $\partial_A \circ \partial_A = 0$.*

We say that $\boldsymbol{E}_\varphi$ is a *Hamiltonian vector field* if $\varphi \in \ker \partial_A$. This is equivalent to

$$\boldsymbol{E}_\varphi(\{\omega, \omega'\}_A) = \{\boldsymbol{E}_\varphi(\omega), \omega'\}_A + \{\omega, \boldsymbol{E}_\varphi(\omega')\}_A, \tag{1.48}$$

i.e., $\boldsymbol{E}_\varphi$ preserves the Poisson bracket. Due to Proposition 1.10, a particular case of Hamiltonian fields are fields of the form $\boldsymbol{E}_{A(\delta(\omega))}$. In this case, $\omega \in D_0(\pi)$ is called the *Hamiltonian* of the field under consideration.

We say that $\omega \in D_0(\pi)$ is a *first integral* of a Hamiltonian field $\boldsymbol{E}_\varphi$ if $\boldsymbol{E}_\varphi(\omega) = 0$. A Hamiltonian field $\boldsymbol{E}_{\varphi'}$ is a *symmetry* for the field $\boldsymbol{E}_\varphi$ if $[\boldsymbol{E}_{\varphi'}, \boldsymbol{E}_\varphi] = 0$, or

$$(\boldsymbol{E}_\varphi - \ell_\varphi)(\varphi') = 0.$$

**Proposition 1.11.** *If $\boldsymbol{E}_\varphi$ is a Hamiltonian vector field with respect to Hamiltonian structure $A$ then the operator $A \circ \delta = \partial_A \colon D_0(\pi) \to D_1(\pi)$ takes first integrals of $\boldsymbol{E}_\varphi$ to its symmetries.*

A Hamiltonian structure $B \in D_2(\pi)$ is said to be *compatible* with the structure $A$ if $B \in \ker \partial_A$, or

$$[\![A, B]\!] = 0. \tag{1.49}$$

This is equivalent to the fact that all bivectors

$$\lambda A + \mu B, \qquad \lambda, \mu \in \mathbb{R}, \tag{1.50}$$

are Hamiltonian structures on $J^\infty(\pi)$. The family (1.50) is called a *Poisson pencil.* When two Hamiltonian structures are given, one also says that they form a *bi-Hamiltonian structure.*

*Coordinates.* Let us indicate how to verify conditions (1.42) and (1.49) in coordinates (the explanation will be given below in Subsection 1.9, see Remark 12). Take bivectors $A$, $B \in D_2(\pi) = C_1^{\mathrm{sk}*}(\hat{\varkappa}(\pi), \varkappa(\pi))$. Then $A$ and $B$, in adapted coordinates in $J^\infty(\pi)$, are represented as matrix $\mathcal{C}$-differential operators

$$A = \left\| \sum_\sigma a_\sigma^{ij} D_\sigma \right\|, \qquad B = \left\| \sum_\sigma b_\sigma^{ij} D_\sigma \right\|,$$

where $i, j = 1, \ldots, m = \dim \pi$. Let us put into correspondence to these operators the functions

$$W_A = \sum_{\sigma,i,j} a_\sigma^{ij} p_\sigma^i p^j, \qquad W_B = \sum_{\sigma,i,j} b_\sigma^{ij} p_\sigma^i p^j, \tag{1.51}$$

where $p_\sigma^i$ are *odd* variables. Then $A$ is a Hamiltonian structure if and only if

$$\delta \left( \sum_i \frac{\delta W_A}{\delta u^i} \frac{\delta W_A}{\delta p^i} \right) = 0, \tag{1.52}$$

while two Hamiltonian structures are compatible if and only if

$$\delta \left( \sum_i \left( \frac{\delta W_A}{\delta u^i} \frac{\delta W_B}{\delta p^i} + \frac{\delta W_B}{\delta u^i} \frac{\delta W_A}{\delta p^i} \right) \right) = 0. \tag{1.53}$$

**Theorem 1.12** (Magri Scheme, see [74, 90])**.** *Let $(A, B)$ be a bi-Hamiltonian structure on $J^\infty(\pi)$ and assume that the complex (1.47) is acyclic in the term $D_1(\pi)$, i.e., every Hamiltonian vector field with respect to $A$ possesses a Hamiltonian. Assume also that two densities $\omega_1$, $\omega_2 \in D_0(\pi)$ are given, such that $\partial_A(\omega_1) = \partial_B(\omega_2)$. Then:*

(i) *There exist elements $\omega_3, \dots, \omega_s, \dots \in D_0(\pi)$ satisfying*

$$\partial_A(\omega_s) = \partial_B(\omega_{s+1}). \tag{1.54}$$

(ii) *All elements $\omega_1, \dots, \omega_s, \dots$ are in involution with respect to both Hamiltonian structures, i.e.,*

$$\{\omega_\alpha, \omega_\beta\}_A = \{\omega_\alpha, \omega_\beta\}_B = 0$$

*for all $\alpha$, $\beta \geq 1$.*

**Example 1** (the KdV hierarchy)**.** Consider $J^\infty(\pi)$ for the trivial one-dimensional bundle $\pi \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Let $x$ be the independent variable and $u$ be the fibre coordinate (unknown function). Then the adapted coordinates $u = u_0, u_1, \dots, u_k, \dots$ in $J^\infty(\pi)$ arise, where $u_k$ corresponds to $\partial^k u / \partial x^k$. The operators

$$A = D_x = \frac{\partial}{\partial x} + \sum_{k=0}^\infty u_{k+1} \frac{\partial}{\partial u_k}$$

and

$$B = D_x^3 + 4u D_x + 2u_1,$$

as it can be easily checked using (1.52) and (1.53), constitute a bi-Hamiltonian structure on $J^\infty(\pi)$. Then obviously for the horizontal forms

$$\omega_1 = \frac{1}{2} u^2 \, \mathrm{d}x, \qquad \omega_2 = \frac{1}{2} u \, \mathrm{d}x$$

one has $\partial_A \omega_1 = A(u) = u_1$ and $\partial_B \omega_2 = B(1) = u_1$, i.e.,

$$\partial_A \omega_1 = \partial_B \omega_2.$$

The first cohomology group of $\partial_A$ is trivial (see [93]), and consequently we obtain an infinite series of first integrals and the corresponding symmetries. The second, after $u_1$, symmetry is $6uu_1 + u_3$:

$$6uu_1 + u_3 = \partial_A((u^3 - \frac{1}{2} u_1^2) \, \mathrm{d}x) = \partial_B(\frac{1}{2} u^2 \, \mathrm{d}x).$$

The corresponding flow on $J^\infty(\pi)$ is governed by the evolution equation

$$u_t = 6uu_x + u_{xxx};$$

thus, we obtain the Korteweg-de Vries equation and the corresponding hierarchy of commuting flows (higher KdV equations). The entire family of commuting flows can me obtained by applying the *Lenard recursion operator* (see [60])

$$R = B \circ A^{-1} = D_x^2 + 4u + 2u_1 D_x^{-1} \tag{1.55}$$

to the right-hand side of the first flow $u_t = u_x$ sufficiently many times.

**Example 2** (the Boussinesq hierarchy). Consider the adapted coordinates $x$, $u$, $v, \ldots, u_k, v_k, \ldots$ in the space $J^\infty(\pi)$, where $\pi \colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the trivial two-dimensional bundle over $\mathbb{R}$. Then the operators

$$A = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} \sigma D_x^3 + uD_x + \frac{1}{2}u_1 & \frac{1}{2}vD_x \\ \frac{1}{2}vD_x + \frac{1}{2}v_1 & D_x \end{pmatrix},$$

where $\sigma$ is a real constant, form a bi-Hamiltonian structure. For the differential forms $\omega_1 = 2(u + v)\,dx$ and $\omega_2 = uv\,dx$ one obviously has $\partial_A\omega_1 = \partial_B\omega_2$. The arising hierarchy of commuting flows corresponds to the evolution equation

$$\begin{aligned} u_t &= u_x v + uv_x + \sigma v_{xxx}, \\ v_t &= u_x + vv_x \end{aligned} \tag{1.56}$$

which is the *two-component Boussinesq system* which can be obtained from the Kaup equation, see [94]. Note that there exists another Hamiltonian operator

$$C = \begin{pmatrix} C^{uu} & C^{uv} \\ C^{vu} & C^{vv} \end{pmatrix},$$

where

$$C^{uu} = \sigma D_x^3 + \frac{3}{2}\sigma v_1 D_x^2 + (\sigma v_2 + uv)D_x + \frac{1}{2}(\sigma v_3 + uv_1 + u_1 v),$$

$$C^{uv} = \sigma D_x^3 + (u + \frac{1}{4}v^2)D_x + \frac{1}{2}u_1,$$

$$C^{vu} = \sigma D_x^3 + (u + \frac{1}{4}v^2)D_x + \frac{1}{2}(u_1 + vv_1),$$

$$C^{vv} = vD_x + \frac{1}{4}v_1,$$

which is compatible both with $A$ and $B$. In this sense, system (1.56) is *tri-Hamiltonian*.

**Example 3** (the KdV hierarchy, II). Let $\pi \colon \mathbb{R}^3 \times \mathbb{R}^1 \to \mathbb{R}^1$ be the trivial three-dimensional bundle with the coordinates $t$ in the base and $u$, $v$, $w$ in the fibre. Introduce the adapted coordinates $u_k$, $v_k$, $w_k$, where $k = 0, 1, 2, \ldots$, in $J^\infty(\pi)$ and consider the operators

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -6u \\ 0 & 6u & D_t \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -2u & -D_t - 2v \\ 2u & D_t & -12u^2 - 2w \\ -D_t + 2v & 12u^2 + 2w & 8uD_t + 4u_1 \end{pmatrix},$$

which form a bi-Hamiltonian structure on $J^\infty(\pi)$. It can be easily seen that $\partial_A \omega_1 = \partial_B \omega_2$, where

$$\omega_1 = (uw - \frac{1}{2}v^2 + 2u^3)\,\mathrm{d}t, \qquad \omega_2 = -(\frac{3}{2}u^2 + \frac{1}{2}w)\,\mathrm{d}t.$$

Thus, we obtain a hierarchy of commuting flows whose first term is

$$u_x = v, \qquad v_x = w, \qquad w_x = u_t - 6uv,$$

which is obviously the KdV equation rewritten in a different way (cf. with the paper [95]).

Note that the Lenard recursion operator (1.55) in the new representation of the KdV hierarchy acquires the form

$$R = \begin{pmatrix} 0 & -2u & -D_t - 2v \\ 2u & D_t & -12u^2 - 2w \\ -D_t + 2v & 12u^2 - 2w & 8uD_t + 4u_1 \end{pmatrix} \circ \begin{pmatrix} -36uD_t^{-1} \circ u & 1 & -6uD_t^{-1} \\ -1 & 0 & 0 \\ 6D_t^{-1} \circ u & 0 & D_t^{-1} \end{pmatrix}.$$

*Remark* 10. In a recently published paper [96], the authors formulate a much weaker than triviality of the first Poisson cohomology group criterion for feasibility of the Magri scheme. The criterion is given in the framework of Dirac structures [77, 97] (though it also admits a self-contained formulation) that unfortunately lie outside the scope of our review. Note that Dirac structures that merge the notions of symplectic and Hamiltonian operators constitute an interesting object for geometrical research in PDEs.

### 1.9. A parallel with finite-dimensional differential geometry. II

Let us now continue to construct the dictionary started in Section 1.7.

Of course, variational multivectors introduced in Section 1.8 are exact counterparts of classical multivector fields in differential geometry. These fields are naturally understood as smooth functions on $T^*M$ skew-symmetric and multi-linear with respect to fibre variables. Exactly the same interpretation is valid for variational vectors if one considers the bundle $\tau^*: J_h^\infty(\hat{\varkappa}) \to J^\infty(\pi)$¶. Thus, we have the following translations:

| **Manifold** $M$ | | **Jet space** $J^\infty(\pi)$ |
|---|---|---|
| multivector fields | $\longleftrightarrow$ | variational multivectors, $\mathcal{C}_{p-1}^{\mathrm{sk}\,*}(\hat{\varkappa}, \varkappa)$ |
| Schouten bracket | $\longleftrightarrow$ | variational Schouten bracket |
| Poisson structure | $\longleftrightarrow$ | Hamiltonian operator |
| cotangent bundle, | $\longleftrightarrow$ | variational cotangent bundle, |
| $T^*M \to M$ | $\longleftrightarrow$ | $J_h^\infty(\hat{\varkappa}) \to J^\infty(\pi)$ |

As it is well known, the tangent space $T^*M$ is endowed with the natural symplectic structure $\Omega = \mathrm{d}p \wedge \mathrm{d}q \in \Lambda^2(M)$ and, in addition, $\Omega = \mathrm{d}\rho$, where the form $\rho = p\,\mathrm{d}q$ is defined invariantly as well. Similar constructions exist on $J_h^\infty(\hat{\varkappa})$. Let us show this.

¶ Note that in such a way we independently arrived to Kupershmidt's notion of the cotangent bundle to a vector bundle, see [24].

To this end, recall (see Remark 2) that the manifold $J_h^\infty(\hat{\varkappa})$ is diffeomorphic to $J^\infty(\pi \times_M \hat{\pi})$, where $\hat{\pi} = \mathrm{Hom}(\pi, \bigwedge^n T^*M)$. Hence, the module of variational 1-forms on $J_h^\infty(\hat{\pi})$ is isomorphic to

$$\hat{\varkappa}(\pi \times_M \hat{\pi}) = \hat{\varkappa}(\pi) \times_{J^\infty(\pi)} \varkappa(\pi). \tag{1.57}$$

Then the 1-form $\rho_\pi$ (the analog of $p\,dq$) is uniquely defined by the condition

$$j_h^\infty(\psi)^*(\rho_\pi) = (\psi, 0), \tag{1.58}$$

where $\psi \in \hat{\varkappa}(\pi)$ is an arbitrary variational 1-form on $J^\infty(\pi)$.

Now, by the same reasons and dually to (1.57), the module of vector fields on $J_h^\infty(\hat{\varkappa})$ is

$$\varkappa(\pi \times_M \hat{\pi}) = \varkappa(\pi) \times_{J^\infty(\pi)} \varkappa(\hat{\pi}).$$

Consequently, the symplectic structure $\Omega_\pi$ must be an element of the module $\mathcal{C}^{\mathrm{sk}*}(\varkappa(\pi \times_M \hat{\pi}), \hat{\varkappa}(\pi \times_M \hat{\pi}))$. For any element $(\varphi, \psi) \in \varkappa(\pi \times_M \hat{\pi})$ we set

$$\Omega_\pi(\varphi, \psi) = (-\psi, \varphi); \tag{1.59}$$

this is a skew-adjoint $\mathcal{C}$-differential operator of order 0.

*Remark* 11. The form $\rho_\pi$ can be defined in a different way. Namely, let $X = (\varphi, \psi)$ be a vector field on $J_h^\infty(\hat{\varkappa})$. Then its 1st component is a vector field vertical with respect to the projection $J_h^\infty(\hat{\varkappa}) \to J^\infty(\pi)$ and may be understood as a function on $J_h^\infty(\hat{\varkappa})$. Then we set $\rho_\pi(X) = \varphi$. This definition is equivalent to (1.58).

Of course, the operator $\Omega_\pi$ is invertible and the inverse one $S_\pi = \Omega_\pi^{-1}$ is a bivector on the cotangent manifold $J_h^\infty(\hat{\varkappa})$. There exist two points of view at this manifold (as well as at $T^*M$). The first one treats it as a classical ("even") manifold. The second approach considers $J_h^\infty(\hat{\varkappa})$ as a super-manifold with odd coordinates along the projection $J_h^\infty(\hat{\varkappa}) \to J^\infty(\pi)$ and even ones in the base. If one takes the first approach the bivector $S_\pi$ will define the Poisson bracket for functions on $J_h^\infty(\hat{\varkappa})$. The second approach leads to functions multi-linear and skew-symmetric with respect to fibre variables. As it was stated above, these functions are identified with variational multivectors on $J^\infty(\pi)$. Then the super-bracket defined by $S_\pi$ coincides with the Schouten bracket. The bracket is given by the formula

$$S_\pi(\delta\omega_1)(\omega_2) = \langle S_\pi(\delta\omega_1), \delta\omega_2 \rangle = \begin{cases} \{\omega_1, \omega_2\}, & \text{the even case,} \\ [\![\omega_1, \omega_2]\!], & \text{the odd case,} \end{cases} \tag{1.60}$$

in both cases.

*Remark* 12. The above said clarifies the meaning of formulas (1.51)–(1.53). Namely, the correspondence $A \mapsto W_A$ given by (1.51) describes how to construct the function on $J_h^\infty(\hat{\varkappa})$ by a bivector $A$ (to be more precise, this function is the cohomology class of the form $W_A\,dx^1 \wedge \ldots \wedge dx^n$ in $H_h^n(\pi)$). The argument of $\delta$ in (1.55) is the coordinate expression of the bracket (1.60) in the odd case while (1.60) itself checks triviality of its horizontal cohomology class.

## 2. Differential Equations

With the concept of jet bundle at our disposal we give a geometric definition of differential equations.

*2.1. Definition of differential equations*

Suppose we have a system

$$F_s(x^i, u_I^j) = 0, \qquad s = 1, \dots, l, \tag{2.1}$$

of partial differential equations in $n$ independent variables $x^i$ and $m$ dependent variables $u^j$. Equations (2.1) determine a locus in the jet space $J^\infty(\pi)$ of a vector bundle $\pi \colon E \to M$, such that $\dim E = m + n$, $\dim M = n$.

The subset of $J^\infty(\pi)$ defined in this way is not an adequate geometric construction corresponding to the system at hand, because it does not take into account differential consequences of (2.1). So, we extend (2.1) to a larger system

$$D_I(F_s) = 0 \qquad \text{for all multi-indices } I \text{ and } s = 1, \dots, l, \tag{2.2}$$

and consider the locus $\mathcal{E} \subset J^\infty(\pi)$ defined by (2.2).

Thus, we get a correspondence

$$F_s(x^i, u_I^j) = 0 \quad (2.1) \qquad \mapsto \qquad \mathcal{E} \subset J^\infty(\pi).$$

This correspondence behaves nice with respect to solutions of (2.1): they are those sections of $\pi$ whose infinite jets lie in $\mathcal{E}$. To put this another way, the solutions of (2.1) are the maximal integral submanifolds of the Cartan distribution restricted to $\mathcal{E}$.

This shows that $\mathcal{E}$ endowed with the Cartan distribution can be taken as the geometric object corresponding to system (2.1), we call such a manifold $\mathcal{E}$ an *equation*.

An equation is generally of infinite dimension.

Without loss of generality we assume that (2.1) does not contain equations of zero order, in geometric language this means that the projection $\pi_{\infty,0}|_{\mathcal{E}} \colon \mathcal{E} \to J^0(\pi)$ is surjective. It is obvious that at every point $\theta \in \mathcal{E}$ the Cartan plane is tangent to the equation, $\mathcal{C}_\theta \subset T_\theta(\mathcal{E})$, so that the dimension of the Cartan distribution on an equation is equal to $n$, the same as on the jet space.

System (2.1) is a coordinate description of an equation $\mathcal{E}$. Every equation has many different coordinate descriptions.

**Example 4.** Take the KdV equation (cf. with Example 1 from Section 1)

$$u_t - 6uu_x - u_{xxx} = 0. \tag{2.3}$$

The bundle $\pi$ here is the projection $\pi \colon \mathbb{R}^3 \to \mathbb{R}^2$, $(x, t, u) \mapsto (x, t)$. The jet space $J^\infty(\pi)$ has coordinates $x$, $t$, $u$, $u_x$, $u_t$, $\dots$, $u_I$, $\dots$ The equation $\mathcal{E} \subset J^\infty(\pi)$ is given by the infinite series of equations

$$\begin{aligned}
u_t &= 6uu_x + u_{xxx}, \\
u_{tx} &= 6u_x^2 + 6uu_{xx} + u_{xxxx},
\end{aligned}$$

$$\cdots$$
$$u_{tI} = D_I(6uu_x + u_{xxx}),$$
$$\cdots$$

The Cartan distribution on $\mathcal{E}$ is two-dimensional and generated by

$$D_x = \frac{\partial}{\partial x} + \sum_s u_{s+1}\frac{\partial}{\partial u_s},$$

$$D_t = \frac{\partial}{\partial t} + \sum_s D_x^s(6uu_x + u_{xxx})\frac{\partial}{\partial u_s},$$

where $u_s = u_{x\ldots x}$ ($s$ times). The functions $x$, $t$, $u_s$ can be taken to be coordinates on $\mathcal{E}$.
   The system

$$u_x - v = 0, \quad v_x - w = 0, \quad w_x - u_t + 6uv = 0 \tag{2.4}$$

gives rise to the same equation $\mathcal{E} \subset J^\infty(\pi')$ with $\pi'\colon \mathbb{R}^5 \to \mathbb{R}^2$, $(x, t, u, v, w) \mapsto (x, t)$. To prove that (2.3) and (2.4) define the same equation consider the mappings

$$a\colon J^2(\pi) \to J^0(\pi'), \qquad a(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = (x, t, u, u_x, u_{xx}),$$
$$b\colon J^0(\pi') \to J^0(\pi), \qquad b(x, t, u, v, w) = (x, t, u).$$

Let $a^\infty\colon J^\infty(\pi) \to J^\infty(\pi')$ and $b^\infty\colon J^\infty(\pi') \to J^\infty(\pi)$ be the lifts of these maps. Then it is easy to see that $a^\infty|_{\mathcal{E}} \circ b^\infty|_{\mathcal{E}} = b^\infty|_{\mathcal{E}} \circ a^\infty|_{\mathcal{E}} = \mathrm{id}|_{\mathcal{E}}$. Lifts preserve the Cartan distributions, hence the maps $a^\infty|_{\mathcal{E}}$ and $b^\infty|_{\mathcal{E}}$ are isomorphisms of equations determined by (2.3) and (2.4).
   Thus, two different coordinate expressions (2.3) and (2.4) determine the same equation $\mathcal{E}$ included into two different jet spaces:

$$\begin{array}{c} J^\infty(\pi) \\ \mathcal{E} \nearrow \\ \phantom{\mathcal{E}} \searrow \\ J^\infty(\pi'). \end{array}$$

As usual, the choice of functions (2.1) should be restricted by some *regularity* assumptions. Namely, we require that there exists a subset $\Sigma \subset \{D_I(F_s)\}$ of functions on $J^\infty(\pi)$ such that

 (i) $F_s \in \Sigma$ for all $s = 1, \ldots, l$;

 (ii) the functions that belong to $\Sigma$ define the equation $\mathcal{E}$;

 (iii) the differentials $df$ are locally linearly independent on $\mathcal{E}$ for all $f \in \Sigma$.

We always require these conditions to be satisfied.
   In this review, we shall assume that an equation at hand $\mathcal{E} \subset J^\infty(\pi)$ is globally defined by a relation $F = 0$, where $F$ is a section of an appropriate $l$-dimensional vector bundle $\xi$ over the jet space $J^\infty(\pi)$. This is always possible. Equations (2.1) are local coordinate expressions for $F = 0$. Denote by $P$ the $\mathcal{F}(\pi)$-module of sections of $\xi$, so that $F \in P$.

The above regularity conditions imply the following very useful fact: a function $f \in J^\infty(\pi)$ vanishes on $\mathcal{E}$, $f|_\mathcal{E} = 0$, if and only if $f = \Delta(F)$ for some $\mathcal{C}$-differential operator $\Delta \colon P \to \mathcal{F}(\pi)$.

Let $\mathcal{E} = \{F = 0\}$ be the equation defined by a section $F \in P$. The section $F$ is called *normal* if for any $\mathcal{C}$-differential operator $\Delta \colon P \to \mathcal{F}(\pi)$ such that $\Delta(F) = 0$ we have $\Delta|_\mathcal{E} = 0$. The equation $\mathcal{E} \subset J^\infty(\pi)$ is called *normal* if it can be defined by a normal section.

**Example 5.** A simple example of an abnormal equation is the system

$$u_y - v_x = 0, \qquad u_z - w_x = 0, \qquad v_z - w_y = 0.$$

The gauge equations, including Maxwell, Yang-Mills, and Einstein equations, are not normal as well. Such equations are beyond the scope of our review. On the other hand, majority of equations of mathematical physics, in particular, all evolution equations, are normal.

The Cartan vector fields on an equation $\mathcal{E}$ form a Lie algebra $\mathcal{CX}(\mathcal{E})$. In the same manner as for jet spaces we define the Lie algebra of *symmetries* of $\mathcal{E}$ and spaces of *the Cartan* and *horizontal forms*:

$$\operatorname{sym} \mathcal{E} = \mathcal{X}_c(\mathcal{E})/\mathcal{CX}(\mathcal{E}),$$
$$\text{where } \mathcal{X}_c(\mathcal{E}) = \{\, X \in \mathcal{X}(E) \mid [X, \mathcal{CX}(\mathcal{E})] \subset \mathcal{CX}(\mathcal{E}) \,\},$$
$$\Lambda_c^p(\mathcal{E}) = \{\, \omega \in \Lambda^p(\mathcal{E}) \mid i_X(\omega) = 0 \quad \forall X \in \mathcal{CX}(\mathcal{E}) \,\},$$
$$\Lambda_h^1(\mathcal{E}) = \Lambda^1(\mathcal{E})/\Lambda_c^1(\mathcal{E}),$$
$$\Lambda_h^q(\mathcal{E}) = \Lambda_h^1(\mathcal{E}) \wedge \ldots \wedge \Lambda_h^1(\mathcal{E}).$$

We shall discuss them in more details below.

A morphism of equations $f \colon \mathcal{E} \to \mathcal{E}'$ is a smooth map that respects the Cartan distribution, i.e., for all points $\theta \in \mathcal{E}$ we have $f_*(\mathcal{C}_\theta) \subset \mathcal{C}_{f(\theta)}$, where $\mathcal{C}_\theta$ is the Cartan plane at a point $\theta \in \mathcal{E}$.

If a morphism $f \colon \mathcal{E} \to \mathcal{E}'$ is a fibre bundle and the map $f_* \colon \mathcal{C}_\theta \to \mathcal{C}_{f(\theta)}$ is an isomorphism of vector spaces for all points $\theta \in \mathcal{E}$ then $f$ is called a *covering*.

We shall discuss the theory of covering in Section 3.

**Example 6.** For an equation $\mathcal{E}$ we construct the quotient bundle

$$\tau \colon \mathcal{T}(\mathcal{E}) = T(\mathcal{E})/\mathcal{C} \to \mathcal{E},$$

where $T(\mathcal{E}) \to \mathcal{E}$ is the tangent bundle to $\mathcal{E}$, $\mathcal{C} \subset T(\mathcal{E})$ is the Cartan distribution thought of as a subbundle of $T(\mathcal{E})$. Given an inclusion $\mathcal{E} \subset J^\infty(\pi)$, $\pi \colon E \to M$, as above, the fibre bundle $\tau \colon \mathcal{T}(\mathcal{E}) \to \mathcal{E}$ can be identified with the vertical bundle with respect to the projection $\mathcal{E} \to M$.

Every Cartan vector field $X \in \mathcal{CX}(\mathcal{E})$ can be lifted to a vector field $\tilde{X} \in \mathcal{X}(\mathcal{T}(\mathcal{E}))$ as follows. It suffies to define action of $\tilde{X}$ on fibre-wise linear functions on $\mathcal{T}(\mathcal{E})$ that can be naturally identified with Cartan 1-forms $\omega \in \Lambda_c^1(\mathcal{E})$. We put

$$\tilde{X}(\omega) = L_X(\omega),$$

where $L_X$ denotes the Lie derivative. In coordinates, we have $\tilde{D}_i = D_i$, where $D_i$ in the right-hand side are the total derivatives on $\mathcal{T}(\mathcal{E})$.

Let us describe an inclusion of $\mathcal{T}(\mathcal{E})$ to a jet space. Assume that $\mathcal{E} \subset J^\infty(\pi)$ is given by $F = 0$. Let $\chi \colon J^\infty(\pi \times_M \pi) \to J^\infty(\pi)$ be defined by the projection to the first factor. Then $\mathcal{T}(\mathcal{E}) \subset J^\infty(\pi \times_M \pi)$ is defined by equations

$$\tilde{F} = 0, \qquad \tilde{\ell}_F(\boldsymbol{v}) = 0,$$

where the tilde denotes the pullback by $\chi$ (with the tilde for the Cartan vector fields defined above), $\boldsymbol{v} \in \tilde{\varkappa}(\pi) = \mathcal{F}(\pi \times_M \pi, \pi)$ is the projection to the second factor $\Gamma(\pi \times_M \pi) \to \Gamma(\pi)$.

So, $\mathcal{T}(\mathcal{E})$ is an equation and the vector fields of the form $\tilde{X}$ generate the Cartan distribution on it. Thus, the bundle $\tau \colon \mathcal{T}(\mathcal{E}) \to \mathcal{E}$ is a covering called the *tangent covering* to $\mathcal{E}$.

In coordinates, we have $\boldsymbol{v} = (v^1, \dots, v^m)$ if the coordinates on $J^\infty(\pi \times_M \pi)$ are $x^i$, $u_I^j$, $v_I^j$, with $u_I^j$ and $v_I^j$ corresponding to the first and second factors, respectively. Thus, coordinate description of $\mathcal{T}(\mathcal{E})$ has the form

$$F_s(x^i, u_I^j) = 0, \qquad \sum_{\alpha, I} \frac{\partial F_j}{\partial u_I^\alpha} v_I^\alpha = 0.$$

Note that if $\mathcal{E}$ is normal equation then $\mathcal{T}(\mathcal{E})$ is normal as well.

The reader will find more examples and a detailed discussion of coverings in Section 3 below.

*2.2. Linearization*

Let $\mathcal{E} \subset J^\infty(\pi)$ be an equation defined by a section $F \in P$. Denote by $\varkappa$ the restriction of the module $\varkappa(\pi) = \mathcal{F}(\pi, \pi)$ to $\mathcal{E}$. The *linearization* of the equation $\mathcal{E}$ is the restriction to $\mathcal{E}$ of the linearization of $F$:

$$\bar{\ell}_F = \ell_F|_{\mathcal{E}} \colon \varkappa \to P.$$

We denote by bar the restriction of a $\mathcal{C}$-differential operator to $\mathcal{E}$ and preserve the notation of modules for their restrictions.

*Remark* 13. The operator $\ell_F$ is well-defined globally only if the module $P$ has the form $P = \mathcal{F}(\pi, \pi')$. For an arbitrary module $P$ the operator $\ell_F$ is defined only locally. But its restriction $\bar{\ell}_F$ is well-defined globally on the whole $\mathcal{E}$.

*Remark* 14. A $\mathcal{C}$-differential operator $\Delta$ on $\mathcal{E}$ is called *normal* if for any $\mathcal{C}$-differential operator $\square$ the condition $\square \circ \Delta = 0$ implies $\square = 0$.

The linearization operator $\bar{\ell}_F$ of an equation $\mathcal{E}$ is normal if and only if the section $F$ is normal.

Recall that in coordinates the linearization $\bar{\ell}_F$ has the form (cf. with (1.20)):

$$\bar{\ell}_F = \left\| \sum_I \frac{\partial F_j}{\partial u_I^\alpha} D_I \right\|_{\alpha=1,\dots,m.}^{j=1,\dots,l}$$

If an equation $\mathcal{E}$ is defined by two different sections $F_1 \in P_1$ an $F_2 \in P_2$ in different, generally speaking, jet spaces

$$
\mathcal{E} \begin{array}{c} \nearrow \quad J^\infty(\pi_1) \\ \\ \searrow \quad J^\infty(\pi_2) \end{array}
$$
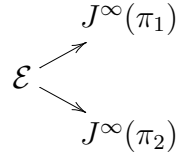
then the corresponding linearizations $\bar{\ell}_{F_1} \colon \varkappa_1 \to P_1$ and $\bar{\ell}_{F_2} \colon \varkappa_2 \to P_2$ are *equivalent* [8, 98] in the sense that there exist $\mathcal{C}$-differential operators $\alpha$, $\beta$, $\alpha'$, $\beta'$, $s_1$, and $s_2$ on $\mathcal{E}$

$$
\begin{array}{ccc}
& \xleftarrow{\;\;s_1\;\;} & \\
\varkappa_1 & \xrightarrow{\;\;\bar{\ell}_{F_1}\;\;} & P_1 \\
\beta \Big\updownarrow \alpha & & \beta' \Big\updownarrow \alpha' \\
\varkappa_2 & \xrightarrow{\;\;\bar{\ell}_{F_2}\;\;} & P_2 \\
& \xleftarrow{\;\;s_2\;\;} &
\end{array}
\tag{2.5}
$$

such that

$$
\bar{\ell}_{F_1}\beta = \beta'\bar{\ell}_{F_2}, \quad \bar{\ell}_{F_2}\alpha = \alpha'\bar{\ell}_{F_1}, \quad \beta\alpha = \mathrm{id} + s_1\bar{\ell}_{F_1}, \quad \alpha\beta = \mathrm{id} + s_2\bar{\ell}_{F_2}.
$$

**Example 7.** Consider two presentations (2.3) and (2.4) of the KdV equation from Example 4. The operators of diagram (2.5) are:

$$
\bar{\ell}_{F_1} = D_t - D_{xxx} - 6uD_x - 6u_x,
$$

$$
\bar{\ell}_{F_2} = \begin{pmatrix} D_x & -1 & 0 \\ 0 & D_x & -1 \\ -D_t + 6v & 6u & D_x \end{pmatrix}
$$

and

$$
\alpha = \begin{pmatrix} 1 \\ D_x \\ D_{xx} \end{pmatrix},
$$

$$
\beta = (\,1 \quad 0 \quad 0\,),
$$

$$
\alpha' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},
$$

$$
\beta' = (\,-D_{xx} - 6u \quad -D_x \quad -1\,),
$$

$$
s_1 = 0,
$$

$$
s_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ D_x & 1 & 0 \end{pmatrix}.
$$

The form of operators $\alpha$ and $\beta$ is obvious from the form of operators $a$ and $b$ in Example 4. The operators $\alpha'$ and $\beta'$ show how equations (2.3) and (2.4) are obtained one from the other.

**Example 8.** Let the number of dependent variables $m$ be equal to 1. Consider the lift $L\colon J^\infty(\pi) \to J^\infty(\pi)$ of the Legendre transformation

$$L|_{J^1(\pi)}\,(x^i, u, u_{x^i}) = (u_{x^i}, \textstyle\sum_\alpha x^\alpha u_{x^\alpha} - u, x^i).$$

It is defined wherever $\det \|u_{x^i x^j}\| \neq 0$ and preserves the Cartan distribution. Of course, $L$ is not a map of fibre bundles.

Consider an equation $\mathcal{E}$ defined by $F = 0$, then $L^*(F) = 0$ defines the same equation $\mathcal{E}$. The operators in diagram (2.5) are as follows:

$$F_1 = F,$$
$$F_2 = L^*(F)$$

and

$$\begin{array}{lll} \alpha = -1, & \alpha' = 1, & s_1 = 0, \\ \beta = -1, & \beta' = 1, & s_2 = 0. \end{array}$$

Let us explain how to compute the maps $\alpha$ and $\beta$. To this end, we have to take a symmetry of the Cartan distribution $X \in \mathcal{X}_c(\pi)/\mathcal{CX}(\pi)$ and see how the generating section of $X$ transforms under the Legendre map. The generating section can be computed by the formula $\varphi = \omega(X)$, where $\omega = du - \sum_i u_{x^i}\, dx^i$ is a Cartan form. So, $\alpha(\varphi) = \omega(L(X)) = L^*(\omega)(X) = -\omega(X) = -\varphi$.

### 2.3. Symmetries and recursions

We have defined symmetries of differential equation $\mathcal{E}$ as elements of quotient space

$$\operatorname{sym}\mathcal{E} = \mathcal{X}_c(\mathcal{E})/\mathcal{CX}(\mathcal{E}),$$

where $\mathcal{X}_c(\mathcal{E}) = \{\, X \in \mathcal{X}(\mathcal{E}) \mid [X, \mathcal{CX}(\mathcal{E})] \subset \mathcal{CX}(\mathcal{E}) \,\}$, that is, symmetries of equations are symmetries of the Cartan distribution on it modulo trivial symmetries (the ones belonging to the Cartan distribution).

Obviously, symmetries of an equation form a Lie algebra with respect to the commutator.

Given an inclusion $\mathcal{E} \subset J^\infty(\pi)$, each symmetry $X \in \operatorname{sym}\mathcal{E}$ contains exactly one vector filed $X^v \in \mathcal{X}_c(\mathcal{E})$ vertical with respect to the projection $\pi_\infty\colon \mathcal{E} \to M$, where $M$ is the base manifold of the bundle $\pi$. Next, for every such a field $X^v$ there exists a unique element $\varphi \in \varkappa = \varkappa(\pi)|_{\mathcal{E}}$ such that

$$X^v = \boldsymbol{E}_{\varphi'}|_{\mathcal{E}},$$

where $\varphi' \in \varkappa(\pi)$ is an arbitrary extension of $\varphi$ to $J^\infty(\pi)$. Hence, we can denote the field $X^v$ by $\boldsymbol{E}_\varphi$.

Element $\varphi \in \varkappa$ such that there exists symmetry $\boldsymbol{E}_\varphi$ is called the *generating section* of this symmetry.

If the equation at hand $\mathcal{E}$ is defined by an equality $F = 0$, then the existence of a symmetry $\boldsymbol{E}_\varphi$ boils down to the condition

$$\boldsymbol{E}_{\varphi'}(F)|_{\mathcal{E}} = 0,$$

where $\varphi' \in \varkappa(\pi)$ is an arbitrary extension of $\varphi$, which is equivalent to the condition

$$\bar{\ell}_F(\varphi) = 0 \quad \text{on } \mathcal{E}. \tag{2.6}$$

This is the determining equation for the symmetries of the equation $\mathcal{E} = \{F = 0\}$.

The Jacobi bracket (1.15) yields the Lie algebra structure on $\text{sym}\,\mathcal{E}$ in terms of generating sections:

$$\{\varphi, \psi\} = \bar{\ell}_\psi(\varphi) - \bar{\ell}_\varphi(\psi), \qquad \varphi, \psi \in \ker \bar{\ell}_F \subset \varkappa. \tag{2.7}$$

Searching symmetries, that is solving (2.6), one usually begins with choosing *internal coordinates* on $\mathcal{E}$.

**Example 9.** We have seen in Example 4 that on the KdV equation

$$u_t - 6uu_x - u_{xxx} = 0$$

the functions $x$, $t$, $u_s = u_{x\ldots x}$ ($s$ times) can be taken to be coordinates on $\mathcal{E}$. These are internal coordinates for the KdV equation.

Of course, the choice of internal coordinates is not unique.

Next, one can start with finding symmetries whose generating sections depend on derivatives of order less than some number $k$, $\varphi = \varphi(x, t, u, u_1, \ldots, u_k)$.

**Example 10.** Let us find symmetries of the KdV equation such that $\varphi = \varphi(x, t, u, u_1)$.

The determining equation for symmetries has the form:

$$(D_t - D_x^3 - 6uD_x - 6u_x)\varphi(x, t, u, u_1) = 0. \tag{2.8}$$

The left-hand side of this equation is a polynomial in $u_3$ and $u_2$. The coefficient of the product $u_2 u_3$ is equal to $-3\varphi_{u_1 u_1}$, so $\varphi_{u_1 u_1} = 0$, and we have

$$\varphi = \varphi^0(x, t, u) + \varphi^1(x, t, u)u_1.$$

Now, the left-hand side of (2.8) is a polynomial in $u_3$, $u_2$ and $u_1$, with coefficient of $u_3$ equal to $-3D_x(\varphi^1)$. Hence,

$$\varphi = \varphi^0(x, t, u) + \varphi^1(t)u_1.$$

With such $\varphi$, the left-hand side of (2.8) does not depend on $u_3$ any more. The coefficient of the product $u_1 u_2$ is $-3\varphi_{uu}^0$, hence

$$\varphi = \varphi^{00}(x, t) + u\varphi^{01}(t, x) + \varphi^1(t)u_1.$$

The coefficient of $u_2$ is $-3\varphi_x^{01}$, so that

$$\varphi = \varphi^{00}(x, t) + u\varphi^{01}(t) + \varphi^1(t)u_1$$

and the left-hand side of (2.8) is a polynomial in $u_1$ and $u$. The coefficient of the product $uu_1$ shows that $\varphi^{01} = 0$. The remaining coefficients of $u_1$ and $u$, and the free term reveal that

$$\varphi^{00} = c_0, \qquad \varphi^1 = 6\varphi^{00}t + c_1,$$

where $c_0$ and $c_1$ are arbitrary constants. Therefore we have found two independent symmetries of the KdV equation

$$\varphi_1 = u_x \quad \text{and} \quad \varphi_2 = 6tu_x + 1. \tag{2.9}$$

Let $X \in \operatorname{sym} \mathcal{E} = \mathcal{X}_c(\mathcal{E})/\mathcal{C}\mathcal{X}(\mathcal{E})$ be a symmetry. Vector fields $Y \in \mathcal{X}_c(\mathcal{E})$ belonging the equivalent class $X$ we shall call *representatives* of $X$. As we explained above, any symmetry has a unique representative of the form $\boldsymbol{E}_\varphi$. But this representative can be not the simplest one.

**Example 11.** Symmetries (2.9) of the KdV equation can be represented, respectively, by the fields $Y_1 = -\partial/\partial x$ and $Y_2 = \partial/\partial u - 6t\partial/\partial x$, because

$$\boldsymbol{E}_{u_x} - Y_1 = D_x, \qquad \boldsymbol{E}_{6tu_x+1} - Y_2 = 6tD_x.$$

The fields $Y_1$ and $Y_2$ are the lifts from the zero order jet space, hence they correspond to one-parameter groups of transformations:

$$\begin{aligned} Y_1\colon & \quad x' = x - \varepsilon & \text{(translation along } x\text{)}, \\ Y_2\colon & \quad x' = x - 6\varepsilon t \quad u' = u + \varepsilon & \text{(Galilean symmetry)}. \end{aligned}$$

A symmetry $X$ is called *classical* if it can be represented by a field $Y|_{\mathcal{E}}$, with $Y \in \mathcal{X}(\pi)$ being the lift from a finite order jet space. Classical symmetries form a subalgebra of the Lie algebra $\operatorname{sym} \mathcal{E}$. By the Lie-Bäcklund theorem (see [12]), $Y$ is a lift from the zero order jet space (if the number of dependent variables $m > 1$) or from the first order jet space (if the number of dependent variables $m = 1$). Thus, in coordinate language, the generating section of a classical symmetry has the form

$$\varphi = \begin{cases} (\varphi_1, \ldots, \varphi_m) & \text{for } m > 1, \\ \varphi(x^i, u, u_i) & \text{for } m = 1, \end{cases} \tag{2.10}$$

where $\varphi_\alpha = b_\alpha(x^i, u^j) + \sum_{k=1}^n a_k(x^i, u^j)u_k^\alpha$, while $\varphi$ is an arbitrary smooth function. The vector field $Y$ on $J^0(\pi)$ or $J^1(\pi)$ that represents the symmetry with generating section (2.10) has the form

$$Y = \begin{cases} \sum_{j=1}^m b_j \frac{\partial}{\partial u^j} - \sum_{k=1}^n a_k \frac{\partial}{\partial x^k} & \text{for } m > 1, \\ -\sum_{i=1}^n \frac{\partial \varphi}{\partial u_i}\frac{\partial}{\partial x^i} + (\varphi - \sum_{i=1}^n u_i \frac{\partial \varphi}{\partial u_i})\frac{\partial}{\partial u} + \sum_{i=1}^n (\frac{\partial \varphi}{\partial x^i} + u_i \frac{\partial \varphi}{\partial u})\frac{\partial}{\partial u_i} & \text{for } m = 1. \end{cases}$$

**Example 12.** In Example 10 above we computed symmetries of the KdV equation with generating sections depending on $x$, $t$, $u$, and $u_x$. To find all classical symmetries, we allow also dependence on $u_t$ and compute symmetries in the same manner as in Example 10. We get two additional classical symmetries:

$$\varphi_3 = u_t \quad \text{and} \quad \varphi_4 = xu_x + 3tu_t + 2u,$$

with the corresponding one-parameter groups of transformations being

$$\begin{aligned} \varphi_3\colon & \quad t' = t - \varepsilon & \text{(translation along } t\text{)}, \\ \varphi_4\colon & \quad x' = \mathrm{e}^{-\varepsilon}x \quad t' = \mathrm{e}^{-3\varepsilon}t \quad u' = \mathrm{e}^{2\varepsilon}u & \text{(scale symmetry)}. \end{aligned}$$

Solving (2.6) for sections $\varphi$ that depend on at most $k$th order derivatives will not give us a complete description of all symmetries. We can find all classical symmetries or a slightly large subspace of symmetries, which can be considered to be a *lower estimate* of the full symmetry algebra. Letting the maximal order $k$ be arbitrary and by solving (2.6) describe the dependence of $\varphi$ on derivatives of order $k$, $k - 1$, etc. This will be an

*upper estimate* of the symmetry algebra. Sometimes these estimates allow to find more symmetries and, in some cases, obtain a complete description of the symmetry algebra. We refer the reader to [12] for examples of such calculations.

A different approach is to look for a *recursion operator* that is $\mathcal{C}$-differential operator $R\colon \varkappa \to \varkappa$ such that there exists another $\mathcal{C}$-differential operator $R'$ satisfying the condition

$$\bar{\ell}_F R = R' \bar{\ell}_F. \tag{2.11}$$

Operators of the form $R = \Box \bar{\ell}_F$, with $\Box$ being arbitrary, enjoy (2.11) for all equations, so we consider recursion operators modulo such trivial ones. Obviously, $R(\operatorname{sym}\mathcal{E}) \subset \operatorname{sym}\mathcal{E}$, so that having a recursion operator we can produce an infinite number of symmetries from a given one.

**Example 13.** The heat equation $u_t = u_{xx}$ has two recursion operators of the first order

$$R_1 = D_x \quad \text{and} \quad R_2 = 2tD_x + x$$

(and, of course, the identical operator, which is of no interest).

For nonlinear equations we often are able to find a nontrivial recursion operator only if we allow it to contain the "integration" operator $D_x^{-1}$. In Section 3.3 we explain how to define such recursion operators in a rigorous way.

**Example 14.** As it was already mentioned, the KdV equation has the Lenard recursion operator

$$R = D_x^2 + 4u + 2u_x D_x^{-1}. \tag{2.12}$$

It is obvious that $R(u_x) = u_t$ and $R^2(u_x) = R(u_t) = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x$. Thus, we have a new *higher* (that is non-classical) symmetry of KdV. We can proceed in the same way and compute $R^3(u_x)$, $R^4(u_x)$ and so on. As it was already mentioned (see Section 1.8), all $R^k(u_x)$ exist, that is the computation of $D_x^{-1}$ will be possible for all $k$. So, we have constructed an infinite set of symmetries of the KdV equation.

If we try to apply the Lenard recursion operator to the other two classical symmetries, $\varphi_2$ (Galilean) and $\varphi_4$ (scale), we get $R(\varphi_2) = 2\varphi_4$, but $R(\varphi_4)$ does not exist. In fact, $R(\varphi_4)$ is a *nonlocal* symmetry, as explained below in Section 3.

Now, let us explain how to compute recursion operators. To this end, note that $\mathcal{C}$-differential operators $\varkappa \to \varkappa$ modulo operators of the form $\Box \bar{\ell}_F$ can be naturally identified with elements of $\Lambda^1_{\mathcal{C}}(\mathcal{E}) \otimes_{\mathcal{F}(\mathcal{E})} \varkappa$, that is with $\varkappa$-valued Cartan 1-froms. This identification takes an operator $R\colon \varkappa \to \varkappa$ to the form $\omega_R \in \Lambda^1_{\mathcal{C}}(\mathcal{E}) \otimes_{\mathcal{F}(\mathcal{E})} \varkappa$, such that $\omega_R(\boldsymbol{E}_\varphi) = R(\varphi)$.

Next, recall that the Cartan 1-forms $\omega \in \Lambda^1_{\mathcal{C}}(\mathcal{E})$ are functions on the tangent covering $\mathcal{T}(\mathcal{E})$ (see Example 6) that are linear along the fibres of the projection $\tau\colon \mathcal{T}(\mathcal{E}) \to \mathcal{E}$. Hence the forms $\omega \in \Lambda^1_{\mathcal{C}}(\mathcal{E}) \otimes_{\mathcal{F}(\mathcal{E})} \varkappa$ are elements of the pullback $\tilde{\varkappa}$ of the module $\varkappa$ by $\tau$. Thus, to a recursion operator $R\colon \varkappa \to \varkappa$ we assign a fibre-wise linear element $\omega_R \in \tilde{\varkappa}$.

In coordinates, to the operator $R = \left\| \sum_I f_I^{\alpha\beta} D_I \right\|$ there corresponds the element

$$\omega_R = \left( \sum_{\beta,I} f_I^{1\beta} v_I^\beta, \ldots, \sum_{\beta,I} f_I^{m\beta} v_I^\beta \right)$$

with $v_I^\beta$ being coordinates along the fibres of $\tau$.

Condition (2.11) on $R$ is equivalent to the following condition on $\omega_R$:

$$\tilde{\ell}_F(\omega_R)\Big|_{\mathcal{T}(\mathcal{E})} = 0. \tag{2.13}$$

**Example 15.** For the heat equation $u_t = u_{xx}$ the tangent covering is given by

$$u_t = u_{xx}, \qquad v_t = v_{xx}.$$

Let $x$, $t$, $u_k = u_{x\ldots x}$, and $v_k = v_{x\ldots x}$ be internal coordinates on it. To compute recursion operators of order 1 in $D_x$ we are to solve the equation

$$(D_t - D_x^2)(f(x, t, u_k)v_x + g(x, t, u_k)v) = 0.$$

It is easy to show that the solutions are: $v$, $v_x$, and $2tv_x + xv$, with the corresponding operators id, $D_x$, and $2tD_x + x$.

**Example 16.** The tangent covering over the KdV equation has the form

$$u_t - u_{xxx} - 6uu_x = 0, \qquad v_t - v_{xxx} - 6uv_x - 6u_x v = 0.$$

Computation of recursion operators amounts to solving equations like the following:

$$(D_t - D_x^3 - 6uD_x - 6u_x)(f_2 v_{xx} + f_1 v_x + f_0 v + f_{-1} v_{-1}), \tag{2.14}$$

where $f_i = f_i(x, t, u_k)$ and $v_{-1}$ is s new variable such that

$$D_x(v_{-1}) = v, \tag{2.15}$$

$$D_t(v_{-1}) = v_{xx} + 6uv. \tag{2.16}$$

As we saw above, the variables $v_I$ correspond to the total derivatives $D_I$, so that the variable $v_{-1}$ is defined in (2.15) to correspond to $D_x^{-1}$, while (2.16) provides the equality $D_t(D_x(v_{-1})) = D_x(D_t(v_{-1}))$. We shall defer a rigorous explanation of this computation until Section 3.

One can check that $v_{xx} + 4uv + 2u_x v_{-1}$ is a solution of (2.14), it yields the Lenard recursion operator (2.12).

The Lie algebra structure on $\ker \bar{\ell}_F \subset \varkappa$ (the Jacobi bracket on symmetries) has a natural extension to a Lie superalgebra on $\ker \tilde{\ell}_F\Big|_{\Lambda_{\mathcal{C}}^*(\mathcal{E}) \otimes_{\mathcal{F}(\mathcal{E})} \varkappa}$, called the *Nijenhuis* (also *Frölicher-Nijenhuis*) *bracket* and denoted by $[\![\cdot,\cdot]\!]$. For detailed definition we refer the reader to [15]. Since we identified recursion operators with elements of $\ker \tilde{\ell}_F\Big|_{\Lambda_{\mathcal{C}}^1(\mathcal{E}) \otimes_{\mathcal{F}(\mathcal{E})} \varkappa}$, we can compute the Nijenhuis bracket of them. Two particular cases are of importance for us here: the bracket of a recursion operator with itself (the Nijenhuis torsion):

$$\frac{1}{2}[\![R, R,]\!](\varphi_1, \varphi_2) = \{R(\varphi_1), R(\varphi_2)\} - R\{R(\varphi_1), \varphi_2\} - R\{\varphi_1, R(\varphi_2)\} + R^2\{\varphi_1, \varphi_2\}$$

$$= \ell_{R,\varphi_2}(R(\varphi_1)) - \ell_{R,\varphi_1}(R(\varphi_2)) + R\ell_{R,\varphi_1}(\varphi_2) - R\ell_{R,\varphi_2}(\varphi_2),$$

where $\varphi_1$, $\varphi_2 \in \varkappa$, and the bracket of a recursion operator and a symmetry with generating section $\varphi \in \ker \bar{\ell}_F \subset \varkappa$ (the Lie derivative):

$$L_\varphi(R)(\varphi') = [\![\varphi, R]\!](\varphi') = \{\varphi, R(\varphi')\} - R\{\varphi, \varphi'\}, \qquad \varphi' \in \varkappa,$$
$$\text{or} \quad L_\varphi(R) = \boldsymbol{E}_\varphi(R) - [\ell_\varphi, R].$$

A recursion operator $R$ is called *Nijenhuis* (or *hereditary*) if its Nijenhuis torsion vanishes, i.e., $[\![R, R]\!] = 0$. Almost all known recursion operators are Nijenhuis, including the operators encountered above. The main property of Nijenhuis operators is the following: for every two symmetries with generating sections $\varphi_1$, $\varphi_2 \in \varkappa$ such that $\{\varphi_1, \varphi_2\} = 0$, $L_{\varphi_1}(R) = 0$, $L_{\varphi_2}(R) = 0$ and for arbitrary $k_1$ and $k_2$ we have $\{R^{k_1}(\varphi_1), R^{k_2}(\varphi_2)\} = 0$.

**Example 17.** For the KdV equation take $\varphi_1 = \varphi_2 = u_x$. Obviously, we have $L_{u_x}(R) = 0$, where $R$ is the Lenard recursion operator (2.12), so that all symmetries $R^k(u_x)$ commute.

*2.4. Conservation laws*

A *conserved current* $\omega$ on an equation $\mathcal{E}$ with $n$ independent variables is a closed horizontal $(n-1)$-form on $\mathcal{E}$, i.e., $\omega \in \Lambda_h^{n-1}(\mathcal{E})$, $\mathrm{d}_h\omega = 0$.

**Example 18.** Consider the equation of continuity in fluid dynamics

$$\rho_t + (\rho v^1)_{x_1} + (\rho v^2)_{x_2} + (\rho v^3)_{x_3} = 0.$$

The form $\omega = \rho\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - \rho v^1\,\mathrm{d}t \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 + \rho v^2\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_3 - \rho v^3\,\mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2$ is a conserved current.

**Example 19.** For the KdV equation $u_t - u_{xxx} - 6uu_x = 0$ the forms

$$u\,\mathrm{d}x + (u_{xx} + 3u^2)\,\mathrm{d}t,$$
$$u^2\,\mathrm{d}x + (2uu_{xx} - u_x^2 + 4u^3)\,\mathrm{d}t,$$
$$(u_x^2/2 - u^3)\,\mathrm{d}x + (u_x u_{xxx} - u_{xx}^2/2 - 3u^2 u_{xx} + 6uu_x^2 - 9u^4/2)\,\mathrm{d}t$$

are conserved currents.

For any $\eta \in \Lambda_h^{n-2}(\mathcal{E})$ the form $\omega = \mathrm{d}_h\eta$ is always a conserved current. Such currents are called *trivial* because they are not related to the equation properties and hence are of no interest. Consider the quotient space

$$H_h^{n-1}(\mathcal{E}) = \{\, \omega \in \Lambda_h^{n-1}(\mathcal{E}) \mid \mathrm{d}_h\omega = 0 \,\}/\{\, \omega \in \Lambda_h^{n-1}(\mathcal{E}) \mid \omega = \mathrm{d}_h\eta, \quad \eta \in \Lambda_h^{n-2}(\mathcal{E}) \,\}$$

of *horizontal cohomology* of $\mathcal{E}$. We also want to quotient out the *topological* conserved currents that lie in the image of the map $\zeta\colon H^{n-1}(\mathcal{E}) \to H_h^{n-1}(\mathcal{E})$ induced by the natural projection $\Lambda^{n-1}(\mathcal{E}) \to \Lambda_h^{n-1}(\mathcal{E})$; here $H^{n-1}(\mathcal{E})$ is the $(n-1)$st group of the de Rham cohomology of the space $\mathcal{E}$. Such currents are related to the topology of the equation $\mathcal{E}$ only. Thus, we define $\mathrm{cl}(\mathcal{E}) = H_h^{n-1}(\mathcal{E})/\operatorname{im}\zeta$ to be the set of *conservation laws* of equation $\mathcal{E}$.

We shall now discuss how to compute conservation laws for *normal* equations. To this end, let us consider the following complex:

$$0 \to \Omega^0(\mathcal{E}) \xrightarrow{\delta} \Omega^1(\mathcal{E}) \xrightarrow{\delta} \Omega^2(\mathcal{E}) \xrightarrow{\delta} \dots, \tag{2.17}$$

where

$$\begin{aligned} \Omega^0(\mathcal{E}) &= \mathrm{cl}(\mathcal{E}), \\ \Omega^p(\mathcal{E}) &= Z^p / \mathrm{d}_h(\Lambda_c^p(\mathcal{E}) \otimes_{\mathcal{F}(\mathcal{E})} \Lambda_h^{n-2}(\mathcal{E})), \end{aligned} \tag{2.18}$$

and $Z^p = \ker \mathrm{d}_h \subset \Lambda_c^p(\mathcal{E}) \otimes_{\mathcal{F}(\mathcal{E})} \Lambda_h^{n-1}(\mathcal{E})$, for $p > 0$. The differential $\delta$ is induced by the Cartan differential $\mathrm{d}_c$, so that $\delta^2 = 0$.

*Remark* 15. Complex (2.17) is a part of Vinogradov's $\mathcal{C}$-spectral sequence (see [14, 16–18]), namely $\Omega^p(\mathcal{E}) = E_1^{p,n-1}(\mathcal{E})$ for $p > 0$, $\Omega^0(\mathcal{E}) = E_1^{0,n-1}(\mathcal{E})/H^{n-1}(\mathcal{E})$, and $\delta = d_1^{*,n-1}$.

Assume that $\mathcal{E}$ is a normal equation defined by a normal section $F \in P$. In this case complex (2.17) can be described in the following way:

$$\Omega^p(\mathcal{E}) = \Theta^p / \Theta_\ell^p, \tag{2.19}$$

where $\Theta^p$ is a subset of $\mathcal{C}_{p-1}^{\mathrm{sk}}(\varkappa, \hat{P})$ that consists of operators $\Delta \in \mathcal{C}_{p-1}^{\mathrm{sk}}(\varkappa, \hat{P})$ such that

$$\bar{\ell}_F^* \Delta(\varphi_1, \dots, \varphi_{p-1}) - \sum_{\alpha=1}^{p-1} \Delta^{*\alpha}(\varphi_1, \dots, \bar{\ell}_F(\varphi_\alpha), \dots, \varphi_{p-1}) = 0$$

for all $\varphi_1, \dots, \varphi_{p-1} \in \varkappa$, where $*\alpha$ denotes the operation of taking adjoint with respect to the $\alpha$th argument. The subset $\Theta_\ell^p \subset \Theta^p$ consists of operators $\Delta \in \Theta^p$ of the form

$$\Delta(\varphi_1, \dots, \varphi_{p-1}) = \sum_{\alpha=1}^{p-1} (-1)^{\alpha+1} \Delta'(\bar{\ell}_F(\varphi_\alpha), \varphi_1, \dots, \hat{\varphi}_\alpha, \dots, \varphi_{p-1}) \tag{2.20}$$

for some $\mathcal{C}$-differential operators $\Delta' \colon P \times \varkappa \times \dots \times \varkappa \to \hat{P}$.

In particular, for $p = 1$, we have

$$\Omega^1(\mathcal{E}) = \ker \bar{\ell}_F^* \subset \hat{P}. \tag{2.21}$$

If $p = 2$ then

$$\Omega^2(\mathcal{E}) = \{ \Delta \in \mathcal{C}(\varkappa, \hat{P}) \mid \bar{\ell}_F^* \Delta = \Delta^* \bar{\ell}_F \} / \{ \Delta' \bar{\ell}_F \mid \Delta' \in \mathcal{C}(P, \hat{P}), \ \Delta'^* = \Delta' \}.$$

The differential $\delta \colon \Omega^0(\mathcal{E}) \to \Omega^1(\mathcal{E})$ is given by the formula $\delta(\omega) = \nabla^*(1)|_{\mathcal{E}}$, with $\nabla$ being a $\mathcal{C}$-differential operator from $P$ to $\Lambda_h^n(\pi)$ such that $\mathrm{d}_h \omega = \nabla(F)$ on $J^\infty(\pi)$.

The differential $\delta \colon \Omega^p(\mathcal{E}) \to \Omega^{p+1}(\mathcal{E})$, $p \geq 1$, has the form:

$$\delta(\Delta)(\varphi_1, \dots, \varphi_p) = \sum_{\alpha=1}^{p} (-1)^{\alpha+1} \ell_{\Delta, \varphi_1, \dots, \hat{\varphi}_\alpha, \dots, \varphi_p}(\varphi_\alpha) + \nabla|_{\mathcal{E}}^{*1}(\varphi_1, \dots, \varphi_p),$$

where $\nabla$ is a $\mathcal{C}$-differential operator $\nabla \colon P \times \varkappa(\pi) \times \dots \times \varkappa(\pi) \to \hat{\varkappa}(\pi)$ that satisfies the relation

$$\ell_F^* \Delta(\varphi_1, \dots, \varphi_{p-1}) - \sum_{\alpha=1}^{p-1} \Delta^{*\alpha}(\varphi_1, \dots, \ell_F(\varphi_\alpha), \dots, \varphi_{p-1}) = \nabla(F, \varphi_1, \dots, \varphi_{p-1}) \tag{2.22}$$

on $J^\infty(\pi)$.

In the case of evolution equation $F = u_t - f$ we have $\Delta \in \mathcal{C}^{\mathrm{sk}\,*}(\varkappa, \hat{\varkappa})$ and the operator $\nabla$ can be chosen in the form:

$$\nabla(h, \varphi_1, \ldots, \varphi_{p-1}) = -\ell_{\Delta, \varphi_1, \ldots, \varphi_{p-1}}(h),$$

where $h \in P$.

If $p = 1$ and $\psi \in \operatorname{im}\delta \subset \Omega^1(\mathcal{E})$, then we can put $\nabla = -\ell_\psi^*$.

The above description of elements in $\Omega^p(\mathcal{E})$ in terms of $\mathcal{C}$-differential operators makes sense only for a given inclusion of equation $\mathcal{E}$ to a jet space. If we consider two inclusions

$$\mathcal{E} \begin{array}{c} \nearrow J^\infty(\pi_1) \\ \searrow J^\infty(\pi_2), \end{array}$$

so that the corresponding linearizations are equivalent and we have diagram (2.5), then operators $\Delta_1$ and $\Delta_2$ that define the same element of $\Omega^p(\mathcal{E})$ with respect to two inclusions are related as follows:

$$\Delta_1 = \alpha'^* \Delta_2(\alpha(\,\cdot\,), \ldots, \alpha(\,\cdot\,)), \qquad \Delta_2 = \beta'^* \Delta_1(\beta(\,\cdot\,), \ldots, \beta(\,\cdot\,)).$$

Elements of $\Omega^1(\mathcal{E})$ are said to be *cosymmetries* of the equation $\mathcal{E}$. We say that isomorphism (2.21) takes a cosymmery to its *generating section* belonging to $\ker \bar{\ell}_F^* \subset \hat{P}$.

The "Two-line Theorem" by Vinogradov (see, e.g., [12]) implies that complex (2.17) is exact at the term $\Omega^0(\mathcal{E})$, hence the conservation laws of $\mathcal{E}$ form a subset of the space of cosymmetries, $\mathrm{cl}(\mathcal{E}) \subset \Omega^1(\mathcal{E})$. The generating section of a cosymmery that belongs to $\delta(\mathrm{cl}(\mathcal{E}))$ is called the *generating section of the conservation law.*

As noted above, to compute the generating section of a conservation law we extend it arbitrarily to a form $\omega \in \Lambda_h^{n-1}(\pi)$ on the jet space $J^\infty(\pi)$, so that $\mathrm{d}_h\omega|_{\mathcal{E}} = 0$. Hence there exists a $\mathcal{C}$-differential operator $\Delta\colon P \to \Lambda_h^n(\pi)$ such that $\mathrm{d}_h\omega = \Delta(F)$; the element $\psi = \Delta^*|_{\mathcal{E}}(1) \in \hat{P}$ is the generating section of the conservation law under consideration.

The generating section $\psi$ of a conservation law can always be extended to the jet space $J^\infty(\pi)$ in such a way that $\langle \psi, F \rangle = \mathrm{d}_h\omega$, with $\omega$ being a conserved current for the same conservation law.

**Example 20.** The generating section of the conservation law from Example 18 is equal to 1.

The generating sections of the conservation laws from Example 19 are 1, $2u$, and $u_{xx} + 3u^2$.

The determining equation for the conservation laws of equation $\mathcal{E} = \{F = 0\}$ is

$$\bar{\ell}_F^*(\psi) = 0. \tag{2.23}$$

This equation is dual to equation (2.6) for symmetries. The computations involved in solving (2.23) are very similar to those used to compute symmetries. However, unlike

the case of symmetries, not all solutions of (2.23) give conservation laws. To check if a generating section of a cosymmery corresponds to a conservation law we can use the following corollary of the "Two-line Theorem" by Vinogradov (see [12]): if the de Rham cohomology $H^n(\mathcal{E}) = 0$, complex (2.17) is exact at the term $\Omega^1(\mathcal{E})$. So, in this case the generating section $\psi \in \ker \bar{\ell}_F^*$ is a generating section of a conservation law if and only if $\delta(\psi) = 0$, that is, there exists a self-adjoint operator $\Delta' = \Delta'^* \in \mathcal{C}(P, \hat{P})$ such that

$$\ell_\psi + \nabla|_{\mathcal{E}}{}^* = \Delta' \bar{\ell}_F,$$

where $\nabla \in \mathcal{C}(\varkappa(\pi), \hat{P})$ satisfies the equality

$$\ell_F^*(\psi) = \nabla(F) \quad \text{on } J^\infty(\pi).$$

Note that the generating section $\psi$ of a conservation law can always be extended to the jet space $J^\infty(\pi)$ in such a way that the horizontal $n$-form $\langle \psi, F \rangle$ will be exact: $\langle \psi, F \rangle = \mathrm{d}_h \omega$, with $\omega|_{\mathcal{E}}$ being a conserved current that corresponds to $\psi$.

### 2.5. A parallel with finite-dimensional differential geometry. III

In this section we begin compilation of a dictionary between the geometry of normal differential equations and finite-dimensional differential geometry, similar to the one for jet spaces from Sections 1.7 and 1.9.

We start just as we did for jet spaces: we consider the space $\mathcal{E}$ endowed with the Cartan distribution $\mathcal{C}$ and take for the points of that "manifold" the maximal integral submanifolds of $\mathcal{C}$, i.e., the solutions of $\mathcal{E}$. Further, the dictionary reads:

| **Manifold $M$** | | **Normal equation $\mathcal{E}$** |
|---|---|---|
| points | $\longleftrightarrow$ | solutions |
| functions $C^\infty(M)$ | $\longleftrightarrow$ | conservation laws $\mathrm{cl}(\mathcal{E})$ |
| the de Rham complex of differential forms | $\longleftrightarrow$ | complex (2.17) $\ldots \to \Omega^{p-1}(\mathcal{E}) \xrightarrow{\delta} \Omega^p(\mathcal{E}) \to \ldots$ |
| vector fields | $\longleftrightarrow$ | symmetries |
| the tangent bundle | $\longleftrightarrow$ | the tangent covering $\tau \colon \mathcal{T}(\mathcal{E}) \to \mathcal{E}$ |

In addition to considerations from Section 1.7 on jet spaces, this dictionary is justified by the following facts.

First, on a finite-dimensional manifold the differential forms are functions on the tangent bundle with odd fibres. Correspondingly, definition (2.18) shows that

$$\Omega^*(\mathcal{E}) = \mathrm{cl}(\mathcal{T}(\mathcal{E})),$$

with elements of $\Omega^p(\mathcal{E})$ given by fibre-wise $p$-linear conserved currents. Fibres of the tangent covering $\tau \colon \mathcal{T}(\mathcal{E}) \to \mathcal{E}$ are assumed to be odd.

Since elements of $\Omega^p(\mathcal{E})$ can be understood as conservation laws on $\mathcal{T}(\mathcal{E})$, we can ask what are the generating section of these conservation laws? For an element of $\Omega^p(\mathcal{E})$

that corresponds to an operator $\Delta \in \mathcal{C}^{\mathrm{sk}}_{p-1}(\varkappa, \hat{P})$ the generating section is $(-\nabla^{*_1}, \Delta)$, where the operator $\nabla$ is given by (2.22). Here we interpret skew-symmetric $\mathcal{C}$-differential operators $\varkappa \times \ldots \times \varkappa \to Q$, modulo operators of form (2.20), as elements of the module $Q$ pulled back on $\mathcal{T}(\mathcal{E})$. This describes the isomorphism (2.19).

Second, on a finite-dimensional manifold vector fields define the interior product and the Lie derivative on differential forms. Correspondingly, on an equation $\mathcal{E}$ the evolution field $\boldsymbol{E}_\varphi$, $\varphi \in \mathrm{sym}\,\mathcal{E}$, induces the interior product and the Lie derivative on $\Omega^*(\mathcal{E})$ defined by (2.18):

$$i_\varphi \colon \Omega^p(\mathcal{E}) \to \Omega^{p-1}(\mathcal{E}), \qquad L_\varphi \colon \Omega^p(\mathcal{E}) \to \Omega^{p+1}(\mathcal{E}).$$

These operations are related to the differential $\delta$ by the usual identity

$$L_\varphi = \delta i_\varphi + i_\varphi \delta.$$

In terms of $\mathcal{C}$-differential operators, the interior product $i_\varphi \colon \Omega^p(\mathcal{E}) \to \Omega^{p-1}(\mathcal{E})$ for $p > 1$ is the contraction of the operator with $\varphi$. For $p = 1$, the interior product $i_\varphi(\psi)$, $\psi \in \ker \bar{\ell}_F^*$, is the conservation law defined by the conserved current $\omega|_{\mathcal{E}} \in \Lambda_h^{n-1}(\mathcal{E})$ such that

$$\langle \ell_F(\varphi), \psi \rangle - \langle \varphi, \ell_F^*(\psi) \rangle = \mathrm{d}_h \omega \quad \text{on } J^\infty(\pi).$$

If $\psi \in \hat{P}$ is a generating section of a conservation law, then a symmetry $\varphi \in \mathrm{sym}\,\mathcal{E}$ acts on it by the formula

$$L_\varphi(\psi) = \boldsymbol{E}_\varphi(\psi) + \Box^*(\psi),$$

with an operator $\Box \in \mathcal{C}(P, P)$ satisfying the equality $\ell_F(\varphi) = \Box(F)$ on $J^\infty(\pi)$.

Third, on a finite-dimensional manifold a symplectic form gives rise to a Poisson bracket. The corresponding construction for an equation $\mathcal{E}$ relies on the notion of a *symplectic structure* that is a closed element of $\Omega^2(\mathcal{E})$. We do not assume that the symplectic form is nondegenerate, so the Poisson bracket will be defined on a subset of $\mathrm{cl}\,\mathcal{E}$ (recall that conservation laws are analogues of functions on $\mathcal{E}$.)

In terms of $\mathcal{C}$-differential operators, a symplectic structure is the equivalence class of operators $\Delta \in \mathcal{C}(\varkappa, \hat{P})$ such that

$$\begin{aligned}
&\bar{\ell}_F^* \Delta = \Delta^* \bar{\ell}_F, \\
&\ell_{\Delta,\varphi_2}(\varphi_1) - \ell_{\Delta,\varphi_1}(\varphi_2) + \nabla|_{\mathcal{E}}^{*_1}(\varphi_1, \varphi_2) = 0,
\end{aligned} \tag{2.24}$$

where $\varphi_1, \varphi_2 \in \varkappa$, $\nabla \colon P \times \varkappa \to \varkappa$ is a $\mathcal{C}$-differential operator such that

$$\ell_F^* \Delta - \Delta^* \ell_F = \nabla(F, \cdot) \quad \text{on } J^\infty(\pi),$$

modulo operators of the form $\Delta' \bar{\ell}_F$, $\Delta' \in \mathcal{C}(P, \hat{P})$, $\Delta'^* = \Delta'$.

**Example 21.** For the simplest WDVV equation

$$u_{yyy} - u_{xxy}^2 + u_{xxx} u_{xyy} = 0$$

the operator $D_x$ is a symplectic structure.

For evolution equations conditions (2.24) amount to

$$\Delta^* = -\Delta,$$
$$\ell_{\Delta,\varphi_1}(\varphi_2) - \ell_{\Delta,\varphi_2}(\varphi_1) = \ell^*_{\Delta,\varphi_1}(\varphi_2).$$

The construction of the Poisson bracket is similar to the one on a finite-dimensional manifold. Let $\Omega \in \Omega^2(\mathcal{E})$ be a symplectic structure, i.e., $\delta(\Omega) = 0$. A conservation law with the generating section $\psi$ is called *admissible* if there exist a symmetry $\varphi \in \operatorname{sym}\mathcal{E}$ such that

$$\psi = i_\varphi(\Omega). \tag{2.25}$$

Symmetries that correspond to admissible conservation laws in the sense of (2.25) are called *Hamiltonian symmetries.*

By definition, *the Poisson bracket of two admissible conservation laws $\omega$ and $\omega'$* with the generating sections $\psi$ and $\psi'$, respectively, has the generating section

$$\{\omega, \omega'\}_\Omega = L_\varphi(\psi') = i_\varphi i_{\varphi'}\Omega,$$

where $\varphi$ and $\varphi'$ are Hamiltonian symmetries corresponding to conservation laws $\omega$ and $\omega'$. To the conservation law $\{\omega, \omega'\}_\Omega$ there corresponds the Jacobi bracket $\{\varphi, \varphi'\}$, so that Hamiltonian symmetries form a Lie algebra: $[L_\varphi, L_{\varphi'}] = L_{\{\varphi,\varphi'\}}$.

As we see from (2.25), a symplectic structure takes a symmetry to a conservation law, in terms of operators (2.24), this map has the form $\psi = \Delta(\varphi)$.

### 2.6. Cotangent covering to a normal equation

In the previous section we discussed geometry related to the tangent covering and functions on it (forms). But what about the cotangent covering?

We have defined the tangent covering for an equation $\mathcal{E}$ without fixing an inclusion of $\mathcal{E}$ to a jet space. Dualizing such an invariant definition requires use of homological algebra, so we will define the cotangent covering for an equation $\mathcal{E}$ embedded to a jet space, $\mathcal{E} \subset J^\infty(\pi)$, and then check that the construction does not depend on the choice of the embedding.

For a normal equation $\mathcal{E}$ given by a normal section $F = 0$, $F \in P$, with $P$ being a module over $J^\infty(\pi)$, we define an equation $\mathcal{T}^*(\mathcal{E}) \subset J^\infty_h(\hat{P})$ by the equations

$$\tilde{F} = 0, \qquad \tilde{\ell}^*_F(\boldsymbol{p}) = 0,$$

where the tilde denotes the pullback to $J^\infty_h(\hat{P})$ and $\boldsymbol{p} \in \tilde{\hat{P}}$ corresponds to the identity operator $\hat{P} \to \hat{P}$ under the identification $\tilde{\hat{P}} = \mathcal{C}(\hat{P}, \hat{P})$. The natural projection $\tau^*\colon \mathcal{T}^*(\mathcal{E}) \to \mathcal{E}$ is called the *cotangent covering* to $\mathcal{E}$.

In coordinates, we have $\boldsymbol{p} = (p^1, \ldots, p^l)$ if the coordinates on $J^\infty_h(\hat{P})$ are $x^i$, $u^j_I$, $p^j_I$, with $u^j_I$ and $p^j_I$ being fibre coordinates respectively along projections $\mathcal{E} \to M$ and $J^\infty_h(\hat{P}) \to \mathcal{E}$.

Below we assume that not only equation $\mathcal{E}$ is normal but the operator $\bar{\ell}^*_F$ is normal (see Remark 14) as well. Then $\mathcal{T}^*(\mathcal{E})$ will also be a normal equation.

*Remark* 16. Obviously, for every $\mathcal{E}$ the cotangent equation $\mathcal{T}^*(\mathcal{E})$ is an Euler-Lagrange equation with the Lagrangian density $L = \langle F, \boldsymbol{p} \rangle$. In applications, considering $\mathcal{T}^*(\mathcal{E})$ instead of $\mathcal{E}$ is occasionally useful to handle the equation as though it were Lagrangian (see, e.g., [99, Volume 1, Sections 3.2, 3.3]). We refer to [100, Section 4.5.1] and [101, Section 5] for more details and references.

If we have two inclusions of equation $\mathcal{E}$ to jet spaces

$$
\mathcal{E} \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{c} J^\infty(\pi_1) \\ \\ J^\infty(\pi_2), \end{array}
$$

then the adjoint linearizations $\bar{\ell}^*_{F_1}$ and $\bar{\ell}^*_{F_2}$ are equivalent:



$$(2.26)$$

such that

$$
\bar{\ell}^*_{F_1} \alpha'^* = \alpha^* \bar{\ell}^*_{F_2}, \quad \bar{\ell}^*_{F_2} \beta'^* = \beta^* \bar{\ell}^*_{F_1}, \quad \alpha'^* \beta'^* = \mathrm{id} + s_1^* \bar{\ell}^*_{F_1}, \quad \beta'^* \alpha'^* = \mathrm{id} + s_2^* \bar{\ell}^*_{F_2},
$$

where the operators $\alpha$, $\beta$, $\alpha'$, $\beta'$, $s_1$, and $s_2$ are defined in (2.5). Therefore, the cotangent coverings constructed using $\ell^*_{F_1}$ and $\ell^*_{F_2}$ are isomorphic, thus the cotangent coverings do not depend on the choice of inclusion $\mathcal{E} \subset J^\infty(\pi)$.

Now we describe an isomorphism between $\mathrm{sym}\,\mathcal{E}$ and the subspace of $\mathrm{cl}\,\mathcal{T}^*(\mathcal{E})$ that consists of conservation laws with fibre-wise linear conserved currents. Thus we will justify the first parallel in the prolongation of our dictionary:

| **Manifold** $M$ | | **Normal equation** $\mathcal{E}$ |
|---|---|---|
| the cotangent bundle | $\longleftrightarrow$ | the cotangent covering $\tau^*: \mathcal{T}^*(\mathcal{E}) \to \mathcal{E}$ |
| multivector fields | $\longleftrightarrow$ | conservation laws $\mathrm{cl}(\mathcal{T}^*(\mathcal{E}))$ $\qquad$ (2.27) |

Let $\varphi \in \varkappa$ be the generating section of a symmetry of $\mathcal{E}$. Extend it to an element $\varphi \in \varkappa(\pi)$ and consider the Green formula

$$
\langle \ell_F(\varphi), \psi \rangle - \langle \varphi, \ell_F^*(\psi) \rangle = \mathrm{d}_h \omega(\varphi, \psi), \tag{2.28}
$$

where $\psi \in \hat{P}$, $\omega(\varphi, \psi) \in \Lambda_h^{n-1}(\pi)$. The mapping $\psi \mapsto \omega(\varphi, \psi)$ is a $\mathcal{C}$-differential operator $\hat{P} \to \Lambda_h^{n-1}(\pi)$, so that it gives rise to a closed form $\omega_\varphi \in \Lambda_h^{n-1}(\mathcal{T}^*(\mathcal{E}))$. The induced map $\mathrm{sym}\,\mathcal{E} \to \mathrm{cl}\,\mathcal{T}^*(\mathcal{E})$, which takes the symmetry with generating functions $\varphi$ to the conservation law with the current $\omega_\varphi$, gives the desired isomorphism.

In fact, formula (2.28) gives more. It holds not only for generating sections of symmetries of $\mathcal{E}$, but also for arbitrary $\varphi \in \varkappa$, so that we obtain a map $\varkappa \to \Lambda_h^{n-1}(\mathcal{T}^*(\mathcal{E}))$. Since $\tilde{\varkappa}$ is a direct summand in $\varkappa(\mathcal{T}^*(\mathcal{E}))$, the element $\omega(\varphi, \psi)$ yields an element $\rho \in \Omega^1(\mathcal{T}^*(\mathcal{E}))$. One can prove that $\rho$ does not depend on the choice of inclusion $\mathcal{E} \to J^\infty(\pi)$ used in its construction.

In terms of isomorphism (2.19), we have $\rho = (\boldsymbol{p}, 0)$.

This is a very important element since it plays the rôle of the canonical 1-form $p \, \mathrm{d}q$ on a finite-dimensional cotangent space:

| **Manifold** $M$ | | **Normal equation** $\mathcal{E}$ |
|---|---|---|
| the canonical 1-form $p \, \mathrm{d}q$ | $\longleftrightarrow$ | $\rho \in \Omega^1(\mathcal{T}^*(\mathcal{E}))$ |
| the canonical symplectic form $\mathrm{d}p \wedge \mathrm{d}q$ | $\longleftrightarrow$ $\longleftrightarrow$ | canonical symplectic structure $\Omega = \delta(\rho) \in \Omega^2(\mathcal{T}^*(\mathcal{E}))$ |

In terms of operators (2.19), the canonical symplectic structure on $\mathcal{T}^*(\mathcal{E})$ has the form

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As to entry (2.27) of our dictionary, we take it for the definition of multivectors. Since we are interested in skew-symmetric multivectors, we assume the fibres of the cotangent covering $\mathcal{T}^*(\mathcal{E}) \to \mathcal{E}$ to be *odd*.

Conservation laws of $\mathcal{T}^*(\mathcal{E})$ whose currents are fibre-wise $p$-linear we call *variational $p$-vectors* on $\mathcal{E}$ and denote by $D_p(\mathcal{E})$. Thus, $D_0(\mathcal{E}) = \mathrm{cl}\,\mathcal{E}$ and $D_1(\mathcal{E}) = \mathrm{sym}\,\mathcal{E}$.

For $D_p(\mathcal{E})$ we have a description in terms of $\mathcal{C}$-differential operators similar to (2.19). Namely,

$$D_p(\mathcal{E}) = \Xi_p / \Xi_p^\ell,$$

where $\Xi_p$ is a subset of $\mathcal{C}_{p-1}^{\mathrm{sk}}(\hat{P}, \varkappa)$ that consists of operators $\Delta \in \mathcal{C}_{p-1}^{\mathrm{sk}}(\hat{P}, \varkappa)$ such that

$$\bar{\ell}_F \Delta(\psi_1, \ldots, \psi_{p-1}) - \sum_{\alpha=1}^{p-1} \Delta^{*\alpha}(\psi_1, \ldots, \bar{\ell}_F^*(\psi_\alpha), \ldots, \psi_{p-1}) = 0$$

for all $\psi_1, \ldots, \psi_{p-1} \in \hat{P}$, where $^{*\alpha}$ denotes, as before, the operation of taking adjoint with respect to the $\alpha$th argument. The subset $\Xi_p^\ell \subset \Xi_p$ consists of operators $\Delta \in \Xi_p$ of the form

$$\Delta(\psi_1, \ldots, \psi_{p-1}) = \sum_{\alpha=1}^{p-1} (-1)^{\alpha+1} \Delta'(\bar{\ell}_F^*(\psi_\alpha), \psi_1, \ldots, \hat{\psi}_\alpha, \ldots, \psi_{p-1})$$

for some $\mathcal{C}$-differential operators $\Delta' \colon \hat{\varkappa} \times \hat{P} \times \ldots \times \hat{P} \to \varkappa$.

In particular, for $p = 2$, we have

$$D_2(\mathcal{E}) = \{\, \Delta \in \mathcal{C}(\hat{P}, \varkappa) \mid \bar{\ell}_F \Delta = \Delta^* \bar{\ell}_F^* \,\} / \{\, \Delta' \bar{\ell}_F^* \mid \Delta' \in \mathcal{C}(\hat{\varkappa}, \varkappa),\ \Delta'^* = \Delta' \,\}.$$

For the element of $D_p(\mathcal{E})$ that corresponds to an operator $\Delta \in \mathcal{C}^{\mathrm{sk}}_{p-1}(\hat{P}, \varkappa)$ the generating section is $(-\nabla^{*_1}, \Delta)$, where the operator $\nabla \colon P \times \hat{P} \times \ldots \times \hat{P} \to P$ is given by

$$\ell_F \Delta(\psi_1, \ldots, \psi_{p-1}) - \sum_{\alpha=1}^{p-1} \Delta^{*\alpha}(\psi_1, \ldots, \ell_F^*(\psi_\alpha), \ldots, \psi_{p-1}) = \nabla(F, \psi_1, \ldots, \psi_{p-1}). \qquad (2.29)$$

In the case of evolution equation $F = u_t - f$ we have $\Delta \in \mathcal{C}^{\mathrm{sk}\,*}(\hat{\varkappa}, \varkappa)$ and the operator $\nabla$ can be chosen in the form:

$$\nabla(h, \psi_1, \ldots, \psi_{p-1}) = \ell_{\Delta, \psi_1, \ldots, \psi_{p-1}}(h),$$

where $h \in P$.

Since we assume the fibres of cotangent covering to be odd, the bracket defined on $\mathrm{cl}(\mathcal{T}^*(\mathcal{E}))$ by the canonical symplectic structure will be the *variational Schouten bracket*

$$[\![\,\cdot\,,\cdot\,]\!] \colon D_k \times D_l \to D_{k+l-1}.$$

In terms of $\mathcal{C}$-differential operators, this bracket has the form:

$$[\![\Delta_1, \Delta_2]\!](\psi_1, \ldots, \psi_{k+l-2}) = \sum_{\sigma \in S^{l-1}_{k+l-2}} (-1)^\sigma \ell_{\Delta_2, \psi_{\sigma(1,l-1)}}(\Delta_1(\psi_{\sigma(l,k+l-2)}))$$

$$- (-1)^{(k-1)l} \sum_{\sigma \in S^k_{k+l-2}} (-1)^\sigma \Delta_2(\nabla_1^{*_1}(\psi_{\sigma(1,k)}), \psi_{\sigma(k+1,k+l-2)})$$

$$- (-1)^{(k-1)(l-1)} \sum_{\sigma \in S^{k-1}_{k+l-2}} (-1)^\sigma \ell_{\Delta_1, \psi_{\sigma(1,k-1)}}(\Delta_2(\psi_{\sigma(k,k+l-2)}))$$

$$+ (-1)^{l-1} \sum_{\sigma \in S^l_{k+l-2}} (-1)^\sigma \Delta_1(\nabla_2^{*_1}(\psi_{\sigma(1,l)}), \psi_{\sigma(l+1,k+l-2)}),$$

where $\Delta_1 \in D_k(P)$, $\Delta_2 \in D_l(P)$, $\nabla_1$ and $\nabla_2$ are defined by (2.29), $\psi_1$, $\ldots$, $\psi_{k+l-2} \in \hat{P}$, cf. with with (1.41).

The above description of variational multivectors in terms of $\mathcal{C}$-differential operators makes sense only for a given inclusion of equation $\mathcal{E}$ to a jet space. If we consider two inclusions

$$
\mathcal{E} 
\begin{array}{c}
\nearrow J^\infty(\pi_1) \\
\\
\searrow J^\infty(\pi_2),
\end{array}
$$

so that the corresponding linearizations and adjoint linearizations are equivalent and we have diagrams (2.5) and (2.26), then operators $\Delta_1$ and $\Delta_2$ that define the same element of $D_p(\mathcal{E})$ with respect to two the inclusions are related as follows:

$$\Delta_2 = \alpha \Delta_1(\alpha'^*(\,\cdot\,), \ldots, \alpha'^*(\,\cdot\,)), \qquad \Delta_1 = \beta \Delta_2(\beta'^*(\,\cdot\,), \ldots, \beta'^*(\,\cdot\,)).$$

An element $A \in D_2(\mathcal{E})$ is called the *Hamiltonian structure* on $\mathcal{E}$ if $[\![A, A]\!] = 0$. Two Hamiltonian operators $A_1$ and $A_2$ are said to be *compatible* if $[\![A_1, A_2]\!] = 0$ (cf. with Section 1.8).

*Remark* 17. Using this definition of the Hamiltonian structure, the Hamiltonian formalisms on jet spaces, including the Magri scheme, explained in Section 1.8, can be extended straightforwardly to the case of equations described here.

**Example 22.** The Camassa-Holm equation [102]

$$u_t - u_{txx} - uu_{xxx} - 2u_x u_{xx} + 3uu_x = 0$$

has a bi-Hamiltonian structure:

$$A_1 = D_x, \qquad A_2 = -D_t - uD_x + u_x.$$

This equation is often written in the form

$$m_t + um_x + 2u_x m = 0,$$
$$m - u + u_{xx} = 0.$$

Then the bi-Hamiltonian structure takes the form

$$A_1' = \begin{pmatrix} D_x & 0 \\ D_x - D_x^3 & 0 \end{pmatrix}, \qquad A_2' = \begin{pmatrix} 0 & -1 \\ 2mD_x + m_x & 0 \end{pmatrix}.$$

**Example 23.** Let $\mathcal{E}$ be a bi-Hamiltonian equation given by $F = 0$ and $A_1$ and $A_2$ be the corresponding Hamiltonian operators. The Kupershmidt deformation [103, 104] $\tilde{\mathcal{E}}$ of $\mathcal{E}$ has the form

$$F + A_1^*(w) = 0, \qquad A_2^*(w) = 0,$$

where $w = (w^1, \ldots, w^l)$ are new dependent variables. For example, the KdV6 equation [105]

$$v_t + v_{xxx} + 12vv_x - w_x = 0, \qquad w_{xxx} + 8vw_x + 4wv_x = 0,$$

is a Kupershmidt deformation of the KdV.

The following two variational bivectors define a bi-Hamiltonian structures on $\tilde{\mathcal{E}}$:

$$\tilde{A}_1 = \begin{pmatrix} A_1 & -A_1 \\ 0 & \ell_{F+A_1^*(w)+A_2^*(w)} \end{pmatrix}, \qquad \tilde{A}_2 = \begin{pmatrix} A_2 & -A_2 \\ -\ell_{F+A_1^*(w)+A_2^*(w)} & 0 \end{pmatrix}.$$

**Example 24.** The equation

$$z_{yy} + (1/z)_{xx} + 2 = 0$$

associated with an integrable class of Weingarten surfaces [106] is bi-Hamiltonian with operators $D_x^2$ and $2zD_{xy} - z_y D_x + z_x D_y$.

## 3. Nonlocal theory

Nonlocal phenomena in the theory of integrable systems is quite common. Here by *nonlocality* we mean the extension of the initial system by new variables (fields) that are related to the old ones by differential relations. Perhaps, the simplest way to observe how nonlocal objects originate is to analyse the action of recursion operators on symmetries.

**Example 25.** Consider recursion operator (1.55) $R = D_x^2 + 4u + 2u_1 D_x^{-1}$ that generates higher KdV equations (it can be shown that successive application of $R$ to the first symmetry $\varphi_1 = u_1$ results in polynomial expressions in $u$, $u_1, \ldots, u_k, \ldots$, see, e.g., [107, 108]). When one applies the operator $R$ to the first $(x,t)$-dependent symmetry

$$\bar{\varphi}_1 = tu_1 + \frac{1}{6}$$

(the Galilean boost), this will result in the scaling symmetry

$$\bar{\varphi}_3 = tu_3 + (6tu + \frac{1}{3}x)u_1 + \frac{2}{3}u,$$

but application of the recursion operator to $\bar{\varphi}_3$ leads to an expression that contains the nonlocal term $D_x^{-1}(u)$ which can not be expressed in the geometrical terms introduced above[+].

An apparent way to incorporate this nonlocal object into the initial geometric setting is to introduce a new variable, say $w$, that is related with the old one by $w_x = u$. This relation, due to the KdV equation, implies another one: $w_t = 3u^2 + u_{xx}$ and thus we shall result in the system

$$u_t = 6uu_x + u_{xxx}, \qquad w_x = u, \qquad w_t = 3u^2 + u_{xx}.$$

A general geometric formulation of this construction was first introduced in [71, 72] and below we shall give a concise exposition of the theory together with a number of applications.

### 3.1. Differential coverings

The notion of a covering was already used in Section 2 in the context of the tangent and cotangent coverings. Here we discuss it in more detail.

Let $\mathcal{E} \subset J^\infty(\pi)$, where $\pi \colon E \to M$, $\dim M = n$, be an equation. Consider a locally trivial bundle $\tau \colon \tilde{\mathcal{E}} \to \mathcal{E}$ and endow the manifold $\tilde{\mathcal{E}}$ with an $n$-dimensional distribution $\tilde{\mathcal{C}}$ in such a way that

(i) $\tilde{\mathcal{C}}$ is integrable and

(ii) for any point $\tilde{\theta} \in \tilde{\mathcal{E}}$ the restriction $d\tau|_{\tilde{\mathcal{C}}_{\tilde{\theta}}}$ is a one-to-one correspondence between $\tilde{\mathcal{C}}_{\tilde{\theta}}$ and the Cartan plane $\mathcal{C}_{\tau(\tilde{\theta})}$.

Then we say that $\psi$ is endowed with the structure of a *differential covering* (or simply a *covering*, to be short) over $\mathcal{E}$.

*Coordinates.* Consider a trivialization of the bundle $\tau$ and let $w^1, \ldots, w^j, \ldots$ be fibrewise coordinates (the so-called *nonlocal variables*). The number $r$ of these coordinates is called the *dimension* of $\tau$.

Let $D_1, \ldots, D_n$ be the total derivatives on $\mathcal{E}$. By Property (ii) in the definition of the distribution $\tilde{\mathcal{C}}$ there exist $\tau$-vertical vector fields $X_1, \ldots, X_n$ on $\tilde{\mathcal{E}}$ such that the fields

$$\tilde{D}_i = D_i + X_i, \qquad i = 1, \ldots, n,$$

[+] Of course, the Lenard operator itself contains a nonlocal summand, but we can consider it just as a convenient reformulation of the Magri relation (1.54).

lie in $\tilde{\mathcal{C}}$. Then Property (i) is equivalent to the system of equations

$$D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \qquad 1 \le i < j \le n, \tag{3.1}$$

where $X_1, \ldots, X_n$ are $\tau$-vertical fields and $D_i(X_j)$ denotes the component-wise action. Since the vector fields $X_i$ are $\tau$-vertical fields, they can be presented in the form

$$X_i = \sum_{j=1}^{r} X_i^j \frac{\partial}{\partial w^j},$$

where $X_i^j$ are smooth functions on $\tilde{\mathcal{E}}$, while $\mathcal{E}$, as a manifold with distribution, is isomorphic to the infinite prolongation of the system of PDEs

$$\frac{\partial w^j}{\partial x^i} = X_i^j, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, r,$$

which extends the initial equation $\mathcal{E}$ and is compatible over it due to (3.1). This system is called the *covering equation.*

**Example 26.** Consider the one-dimensional covering over the KdV equation determined by

$$\tilde{D}_x = D_x + u\frac{\partial}{\partial w}, \qquad \tilde{D}_t = D_t + (3u^2 + u_2)\frac{\partial}{\partial w}.$$

The covering equation in this case is

$$\frac{\partial w}{\partial x} = u, \qquad \frac{\partial w}{\partial t} = 3u^2 + u_{xx}$$

and is isomorphic to the potential KdV equation $w_t = 3w_x^2 + w_{xxx}$.

Note the the relation between $w$ and $u$ may be expressed in the form $w = \int u\,dx$, or $w = D_x^{-1}u$ and thus this is exactly the nonlocality that arose in Example 25.

**Example 27.** Let again the base equation be the KdV and the covering be described by the system

$$X = u + w^2 + \lambda, \qquad T = u_2 + 2wu_1 + 2u^2 + 2(w^2 - \lambda)u - 4\lambda(w^2 + 1), \tag{3.2}$$

where $\lambda \in \mathbb{R}$. Actually, (3.2) determines a one-parameter family of covering structures in the trivial bundle $\mathcal{E} \times \mathbb{R} \to \mathcal{E}$, but the covering equation is isomorphic to the modified KdV equation $w_t = 6w^2 w_x + w_{xxx}$. Of course, the covering under consideration is a geometric realization of the Miura transformation [55].

*Remark* 18. Note that any differential substitution $u = \varphi(x, w, \ldots, w_I, \ldots)$ is associated with a covering over the initial equation, though this covering may be infinite-dimensional. For example, such is the covering over the KdV equation $u_t - 6uu_x + u_{xxx} = 0$ determined with the Hirota substitution

$$u = -2\frac{\partial^2}{\partial x^2} \ln w$$

(see [109]). Nevertheless, in spite of the infinite dimension of this covering, the covering space is isomorphic to the fourth-order scalar equation

$$ww_{xt} - w_t w_x + w_{xxxx}w - 4w_{xxx}w_x + 3w_{xx}^2 = 0$$

in one unknown function.

**Example 28.** Let $P$ be the module of sections for some vector bundle $\xi$ over $\mathcal{E}$. Then the bundle $j_\infty^h \colon J_h^\infty(P) \to \mathcal{E}$ of horizontal jets is an infinite-dimensional covering over $\mathcal{E}$. If $v_K^l$ are adapted coordinates in $J_h^\infty(P)$ then the total derivatives lifted to $J_h^\infty(P)$ are of the form

$$\tilde{D}_i = D_i + \sum_{l,K} v_{Ki}^l \frac{\partial}{\partial v_K^l}. \tag{3.3}$$

This construction is generalized in the next example.

**Example 29** ($\Delta$-coverings). Let $\mathcal{E}$ be an equation and consider a $\mathcal{C}$-differential operator $\Delta \colon P \to Q$, where $P$ and $Q$ are modules of sections for some vector bundles over $\mathcal{E}$. Let $\Phi_\Delta \colon J_h^\infty(P) \to Q$ be the corresponding morphism of vector bundles (see Proposition 1.3). Then, under natural conditions of non-degeneracy, $\tilde{\mathcal{E}}_\Delta = \ker \Phi_\Delta$ is a sub-bundle in $j_\infty^h \colon J_h^\infty(P) \to \mathcal{E}$ that carries a natural structure of a covering: the total derivatives in this covering are obtained by restriction of the operators (3.3) to $\tilde{\mathcal{E}}_\Delta$. We call this covering the $\Delta$-*covering* over $\mathcal{E}$.

If the operator $\Delta$ is locally given in the matrix form $\Delta = \| \sum_K d_{\alpha\beta}^K D_K \|$ then the subspace $\tilde{\mathcal{E}}_\Delta \subset J_h^\infty(\xi)$ is described by the relations

$$\sum_{\alpha,K} d_{\alpha\beta}^K v_K^\alpha = 0$$

and their prolongations. Obviously, the tangent and cotangent coverings are particular cases of this construction.

$\Delta$-coverings play the key rôle in solving the following factorization problem: let $\Delta' \colon P' \to Q'$ be another $\mathcal{C}$-differential operator; how to find all operators $A \colon P \to P'$ such that

$$\Delta' \circ A = B \circ \Delta, \tag{3.4}$$

i.e., such that the diagram

$$
\begin{array}{ccc}
P & \xrightarrow{\ \Delta\ } & Q \\
{\scriptstyle A}\downarrow & & \downarrow{\scriptstyle B} \\
P' & \xrightarrow[\ \Delta'\ ]{} & Q'
\end{array}
$$

is commutative? Note that any operator $A$ of the form $A = B' \circ \Delta$, where $B' \colon Q \to P'$ is an arbitrary $\mathcal{C}$-differential operator, is a solution to (3.4). Such solutions will be called *trivial*.

To find nontrivial solutions, first note that since $\Delta'$ is a $\mathcal{C}$-differential operator it can be lifted to the covering just by changing the total derivatives $D_i$ to the lifted ones $\tilde{D}_i$. Denote this lift by $\tilde{\Delta}'$. Second, let us put into correspondence to any operator $A = \| \sum_K a_{\alpha\beta}^K \|$ the vector-function

$$\tilde{\Phi}_A = \left. \left( \sum_{\alpha,K} a_{\alpha,1}^K v_K^\alpha, \dots, \sum_{\alpha,K} a_{\alpha,r'}^K v_K^\alpha \right) \right|_{\tilde{\mathcal{E}}_\Delta}, \qquad r' = \dim P',$$

Then one has:

**Proposition 3.1.** *Classes of solutions of Equation (3.4) modulo trivial ones are in one-to-one correspondence with solutions of*

$$\tilde{\Delta}'(\tilde{\Phi}_A) = 0.$$

*Operators satisfying* (3.4) *take elements of* $\ker \Delta$ *to those of* $\ker \Delta'$.

Consider system (3.1) that determines a covering structure in the space $\tilde{\mathcal{E}}$ and assume that the coefficients $X_i^j$ of the vertical vector fields $X_i$ are independent of nonlocal variables $w^\alpha$. In this case, (3.1) reduces to

$$D_i(X_j) = D_j(X_i), \qquad 1 \le i < j \le n; \tag{3.5}$$

the corresponding covering is called *Abelian*. The covering in Example 26 is an Abelian one, while the covering associated with the Miura transformation (Example 27) is not.

Let $\dim \tau = 1$ and define a differential horizontal 1-form on $\mathcal{E}$ by setting

$$\omega_\tau = \sum_{i=1}^{n} X_i \, \mathrm{d}x^i. \tag{3.6}$$

Then (3.5) amounts to the equation

$$\mathrm{d}_h \omega_\tau = 0, \tag{3.7}$$

where $\mathrm{d}_h$ is the horizontal differential on $\mathcal{E}$. Thus, one-dimensional Abelian coverings over $\mathcal{E}$ are in one-to-one correspondence with closed horizontal $(n-1)$-forms.

In Example 27 we presented a one-parameter family of coverings over the KdV equation. Are these coverings different for different values of the parameter $\lambda$ or not and what the word "different" means in this context? The answer is the following.

Consider an equation $\mathcal{E}$ and two coverings $\tau_i \colon \tilde{\mathcal{E}}_i \to \mathcal{E}$, $i = 1, 2$, over $\mathcal{E}$. We say that these coverings are *gauge equivalent* (or simply *equivalent*) if there exists an isomorphism $\varphi \colon \tilde{\mathcal{E}}_1 \to \tilde{\mathcal{E}}_2$ of the equations $\tilde{\mathcal{E}}_1$ and $\tilde{\mathcal{E}}_2$ such that the diagram

$$
\begin{array}{ccc}
\tilde{\mathcal{E}}_1 & \xrightarrow{\varphi} & \tilde{\mathcal{E}}_2 \\
& {\scriptstyle \tau_1} \searrow \quad \swarrow {\scriptstyle \tau_2} & \\
& \mathcal{E} &
\end{array}
$$

is commutative, i.e., $\tau_2 \circ \varphi = \tau_1$. In this sense, all coverings (3.2) are different, i.e., pair-wise non-equivalent for different values of $\lambda$. General cohomological technique to check whether a parameter is "fake" or not was suggested in [110, 111].

We say that a covering $\tau \colon \tilde{\mathcal{E}} \to \mathcal{E}$ is *trivial* if for any point $\theta \in \mathcal{E}$ there exists a neighbourhood $\mathcal{U} \ni \theta$ such that

(i) $\tau|_{\mathcal{U}}$ is a trivial bundle;

(ii) there exist an adapted coordinate system in $\mathcal{U}$ for which the fields $\tilde{D}_i$ are of the form $\tilde{D}_i = D_i$, $i = 1, \dots, \dim M$.

**Theorem 3.2.** *There exists a one-to-one correspondence between equivalence classes of one-dimensional Abelian coverings over $\mathcal{E}$ and elements of the horizontal cohomology group $H_h^1(\mathcal{E})$ given by (3.6). In particular, a covering $\tau$ is trivial if and only if the form $\omega_\tau$ is a co-boundary, i.e., $\omega_\tau = \mathrm{d}_h(f)$.*

Since $H_h^1(\mathcal{E})$ coincides with the group of conservation laws when $\dim M = 2$, Theorem 3.2 allows one to construct special type of coverings by conservation laws of the equation at hand.

**Example 30.** The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_xu_{xx} + uu_{xxx}$$

admits the conservation law

$$\omega = (u - u_{xx})\,\mathrm{d}x + \frac{1}{2}(u_x^2 - 3u^2 + 2uu_{xx})\,\mathrm{d}t;$$

consequently, the corresponding covering is given by

$$w_x = u - u_{xx}, \qquad w_t = \frac{1}{2}(u_x^2 - 3u^2 + 2uu_{xx}).$$

*Remark* 19. Since the group $H_h^1(\mathcal{E})$ is trivial for normal equations in the case $\dim M = 2$, this implies that such equations possess no nontrivial one-dimensional Abelian covering. Actually, there exist very strong indications that these equations do not have finite-dimensional coverings at all (see [112]).

**Example 31** (the KP equation)**.** Consider the dispersionless Kadomtsev-Petviashvili equation

$$(u_t - 6uu_x + u_{xxx})_x = u_{yy}.$$

It admits an obvious covering

$$w_x = u_y, \qquad w_y = u_t - 6uu_x + u_{xxx}, \tag{3.8}$$

which at first glance seems to be one-dimensional. But this is not the case because Equations (3.8) do not contain information on the derivative $w_t$. To incorporate these data, we must introduce infinite number of nonlocal variables $w^0$, $w^1, \ldots$ such that

$$w^0 = w,\ w_t^0 = w^1,\ \ldots,\ w_t^r = w^{r+1},\ \ldots$$

and express their $x$- and $y$-derivatives using (3.8). Thus, the covering is infinite-dimensional actually.

It was shown above that one-dimensional Abelian coverings can be constructed using conservation laws of the equation. Another type of coverings is related to Wahlquist-Estabrook prolongation structures [113–115] and their description is based on the following *ansatz*: Let $u_t = f(u, u_1, \ldots, u_k)$ be a system of evolution equations, $u = (u^1, \ldots, u^m)$, $f = (f^1, \ldots, f^m)$ being vectors and $u_i$ denoting the $i$th derivative with respect to $x$. Let us look for coverings such that the coefficients of the fields $X$ and $T$ in

$$\tilde{D}_x = D_x + X, \qquad \tilde{D}_t = D_t + T$$

depend on $u, u_1, \ldots, u_{k-1}$ and nonlocal variables only. Then description of such coverings locally reduces to representations of a certain free Lie algebra (the so-called *Wahlquist-Estabrook algebra*) in vector fields on the fibre $W$ of the trivial bundle $\tau\colon \mathcal{E} \times W \to \mathcal{E}$.

**Example 32.** Consider the potential KdV equation

$$u_t = u_x^2 + u_{xxx} \tag{3.9}$$

and let us describe coverings $\tilde{D}_x = D_x + X$, $\tilde{D}_t = D_t + T$ over $\mathcal{E}$ such that the fields $X$ and $T$ depend on $u$, $u_1$ and $u_2$ only. Straightforward computations show that all these coverings are of the form

$$X = u^2 \mathbf{a} + u\mathbf{b} + \mathbf{c},$$

$$T = (2uu_2 - u_1^2 + 2u^2 u_1)\mathbf{a} + (u_2 + 2uu_1)\mathbf{b} + u_1[\mathbf{c}, \mathbf{b}] + \frac{1}{2}u^2[\mathbf{b}, \mathbf{d}] + u[\mathbf{c}, \mathbf{d}] + \mathbf{e}, \tag{3.10}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ are vector field on the fibre $W$ of the covering (i.e., such that they do not depend on the equation coordinates) which enjoy the commutator relations

$$2\mathbf{a} = [\mathbf{a}, \mathbf{b}], \quad \mathbf{b} = [\mathbf{a}, \mathbf{c}], \quad \mathbf{d} = 2\mathbf{c} + [\mathbf{c}, \mathbf{b}],$$

$$[\mathbf{a}, \mathbf{d}] = [\mathbf{c}, \mathbf{e}] = 0,$$

$$[\mathbf{b}, \mathbf{d}] + \frac{1}{2}[\mathbf{b}, [\mathbf{b}, \mathbf{d}]] = 0, \quad [\mathbf{b}, \mathbf{e}] + [\mathbf{c}, [\mathbf{c}, \mathbf{d}]] = 0,$$

$$[\mathbf{a}, \mathbf{e}] + [\mathbf{b}, [\mathbf{c}, \mathbf{d}]] + \frac{1}{2}[\mathbf{c}, [\mathbf{b}, \mathbf{d}]] = 0.$$

Now, to find all Wahlquist-Estabrook coverings for (3.9) amounts to describing representations as vector fields on $W$ of the free Lie algebra generated by the elements $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and $\mathbf{e}$ with the above indicated relations.

If $W = \mathbb{R}$ then all such representations, up to an isomorphism, are

$$\mathbf{a} \mapsto \frac{\partial}{\partial w}, \quad \mathbf{b} \mapsto (2w + \beta)\frac{\partial}{\partial w}, \quad \mathbf{c} \mapsto (w^2 + \beta w + \gamma)\frac{\partial}{\partial w},$$

$$\mathbf{d} \mapsto -\Delta \frac{\partial}{\partial w}, \quad \mathbf{e} \mapsto \Delta(w^2 + \beta w + \gamma)\frac{\partial}{\partial w},$$

where $\beta, \gamma \in \mathbb{R}$ and $\Delta = \beta^2 - 4\gamma$. The corresponding one-dimensional coverings, up to gauge equivalence, are of the form

$$X = (u^2 + 2wu + w^2 + \gamma)\frac{\partial}{\partial w}$$

(the parameter $\beta$ can be removed by a gauge transformation) and with $T$ given by (3.10); they are pair-wise inequivalent for different values of $\gamma$.

*Remark* 20. The term *covering* also refers to the parallel between classical differential geometry and geometry of PDEs. Namely, if we define dimension of an equation $\mathcal{E}$ (or of a jet space $J^\infty(\pi)$) as that of the corresponding Cartan distribution (i.e., the number of independent variables) then fibres of a differential covering become zero-dimensional, and this complies with the definition of a topological covering. That was the initial reason to name the object in [71].

But the parallel goes far beyond this trivial observation. In [116] a new powerful invariant (the *fundamental Lie algebra*) of differential equations was proposed whose rôle in the theory of differential coverings is quite similar to the one that the fundamental group plays in topology. In particular, the fundamental Lie algebra allows one to

enumerate all coverings over a given equation in the same way as conjugacy classes of subgroups of the fundamental group enumerate topological coverings. So, the dictionary evolved in the previous sections can be continued:

| **Manifold $M$** | | **Differential equation $\mathcal{E}$** |
|---|---|---|
| topological dimension | $\longleftrightarrow$ | differential dimension |
| topological coverings | $\longleftrightarrow$ | differential coverings |
| fundamental group | $\longleftrightarrow$ | fundamental Lie algebra |

Note also that using the fundamental Lie algebra technique the author of [116] proved *inexistence* of Bäcklund transformations for some pairs of differential equations. It seems that it is impossible to achieve such a result by other methods.

### 3.2. Nonlocal symmetries

The concept of a symmetry discussed in Section 2 can be generalized to the nonlocal situation. Consider an example.

**Example 33.** Let

$$u_t = uu_x + u_{xx} \tag{3.11}$$

be the Burgers equation (its Lie algebra of symmetries was fully described in [71]). Direct computations show that (3.11) does not possess symmetries of the form $\varphi = \varphi(x, t, u)$, but if one extends the setting by a new (nonlocal) variable $w$ such that

$$w_x = u, \qquad w_t = \frac{1}{2}u^2 + u_x \tag{3.12}$$

then the equation $\ell_{\mathcal{E}}(\varphi) = 0$ will acquire a new family of solutions of the form

$$\varphi = (au - 2a_x)\mathrm{e}^{-\frac{1}{2}w}, \tag{3.13}$$

where $a = a(x, t)$ is an arbitrary solution of the heat equation $a_t = a_{xx}$.

The question is: can functions (3.13) be considered as symmetries of Equation (3.11) in some natural sense? To answer this question, consider an arbitrary equation $\mathcal{E} \subset J^\infty(\pi)$ and a covering $\tau \colon \tilde{\mathcal{E}} \to \mathcal{E}$. We say that $\varphi$ is a *nonlocal symmetry* (or $\tau$-symmetry) of $\mathcal{E}$ if it is a symmetry of $\tilde{\mathcal{E}}$.

*Coordinates.* Let $\mathcal{E} \subset J^\infty(\pi)$ be an equation and $\tau \colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering locally given by the total derivatives

$$\tilde{D}_i = D_i + \sum_j X_i^j \frac{\partial}{\partial w^j}, \qquad i = 1, \ldots, \dim M.$$

Then any nonlocal $\tau$-symmetry is of the form

$$\tilde{\boldsymbol{E}}_\varphi + \sum_j \psi^j \frac{\partial}{\partial w^j}. \tag{3.14}$$

Here $\varphi$ is an $m$-component vector-function on $\tilde{\mathcal{E}}$ that satisfies the equation

$$\tilde{\ell}_{\mathcal{E}}(\varphi) = 0, \tag{3.15}$$

$\psi^j$ are functions on $\tilde{\mathcal{E}}$ such that

$$\tilde{D}_i(\psi^j) = \tilde{\ell}_{X_i^j}(\varphi) + \sum_\alpha \frac{\partial X_i^j}{\partial w^\alpha} \psi^\alpha \tag{3.16}$$

and

$$\tilde{\boldsymbol{E}}_\varphi = \sum_{\substack{\text{over internal} \\ \text{coordinates}}} \tilde{D}_I(\varphi^j) \frac{\partial}{\partial u_I^j} \tag{3.17}$$

(recall that "tilde" over a $\mathcal{C}$-differential operator denotes its natural lifting to the covering).

**Example 34.** Let us consider Example 33 again. In the case of covering (3.12) Equations (3.16) take the form

$$\tilde{D}_x(\psi) = \varphi, \qquad \tilde{D}_t(\psi) = u\varphi + \tilde{D}_x(\varphi) \tag{3.18}$$

and for $\varphi$ of the form (3.13) we see that

$$\psi = -2ae^{-\frac{1}{2}w}$$

satisfies (3.18). Thus, the pair of functions $\varphi$ and $\psi$ determine a nonlocal symmetry of the Burgers equation in the sense of the above definition.

However, the situation of the previous example is not general.

**Example 35.** Consider the covering

$$w_x = u, \qquad w_t = 3u^2 + u_{xx}$$

over the KdV equation $u_t = 6uu_x + u_{xxx}$ and let us try to find nonlocal symmetries in this covering. In the case under consideration, Equations (3.16) acquire the form

$$\tilde{D}_x(\psi) = \varphi, \qquad \tilde{D}_t(\psi) = 6u\varphi + \tilde{D}_x^2(\varphi), \tag{3.19}$$

while (3.15) is

$$\tilde{D}_t(\varphi) = 6u_1\varphi + 6u\tilde{D}_x(\varphi) + \tilde{D}_x^3(\varphi).$$

The simplest solution of the last equation that depends on $w$ is

$$\varphi = tu_5 + \left(10tu + \frac{1}{3}x\right)u_3 + 4\left(5tu_1 + \frac{1}{3}\right)u_2 + 2\left(15tu^2 + xu + \frac{1}{3}w\right)u_1 + \frac{8}{3}u^2.$$

But solving (3.19) with $\varphi$ of the above form leads to contradiction: no function $\psi$ exist on $\mathcal{E}$ such that (3.19) is valid for our $\varphi$.

Nevertheless, if we introduce another nonlocal variable $w'$ satisfying

$$w'_x = u^2, \qquad w'_t = 4u^3 - u_x^2 + 2uu_{xx}$$

then (3.19) will be resolved in the new setting.

But a similar problem arises at this step: now we need to reconstruct the coefficient $\psi'$ at $\partial/\partial w'$.

The procedure we encountered in Example 35 is typical and we shall describe it in general terms now. Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering and denote by $\mathcal{F}$ and $\tilde{\mathcal{F}}$ the function algebras on $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively. An $\mathbb{R}$-linear map $X\colon \mathcal{F} \to \tilde{\mathcal{F}}$ is called a $\tau$-*shadow* if

(i) $X$ is a derivation, i.e.,

$$X(fg) = fX(g) + gX(f)$$

for all $f$, $g \in \mathcal{F}$;

(ii) the action of $X$ preserves the Cartan distribution, i.e., $\mathrm{L}_X(\omega) \in \Lambda_c(\tilde{\mathcal{E}})$ as soon as $\omega \in \Lambda_c(\mathcal{E})$ (or, equivalently, for any Cartan field $\tilde{y}$ on $\tilde{\mathcal{E}}$ and its projection $Y$ to $\mathcal{E}$ the commutator $[X, \tilde{Y}] = XY - \tilde{Y}X$ is a Cartan field again).

In particular, any symmetry of $\tilde{\mathcal{E}}$ can be considered as a shadow in an arbitrary covering $\tau$.

*Coordinates.* Let $\mathcal{U}$ be the set of internal coordinates on $\mathcal{E}$. Then any $\tau$-shadow is given by the formula

$$\boldsymbol{\tilde{E}}_\varphi = \sum_{u_I^j \in \mathcal{U}} \tilde{D}_I(\varphi^j) \frac{\partial}{\partial u_I^j},$$

where $\varphi^1, \ldots, \varphi^m$ are functions on $\tilde{\mathcal{E}}$ and $\tilde{D}_1, \ldots, \tilde{D}_n$ are total derivatives on $\tilde{\mathcal{E}}$ (cf. with (3.17)).

We say that a $\tau$-shadow $X$ is *reconstructed* in $\tau$ if there exists a nonlocal $\tau$-symmetry $\tilde{X}$ such that $\tilde{X}\big|_{\mathcal{F}} = X$. As Examples 34 and 35 show, not all shadows can be reconstructed in a straightforward way. A general result that describes the reconstruction procedure was proved in [117] (see also [72]):

**Proposition 3.3.** *Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering and $X$ be a $\tau$-shadow. Then there exists another covering $\bar{\tau}\colon \bar{\mathcal{E}} \to \tilde{\mathcal{E}}$ and a $\bar{\tau}$-shadow $\bar{X}$ such that $\bar{X}\big|_{\mathcal{F}} = X$.*

Thus, putting $\tau = \tau_0$ and $\tau_{i+1} = \tilde{\tau}_i$ and applying Proposition 3.3 sufficiently (maybe, infinitely) many times we shall arrive to a covering in which the given shadow is reconstructed.

*Coordinates.* Actually, the results of [117] not just state an existence of the needed covering but provide a canonical way to construct the one. The construction is in a sense tautological and copies relations (3.16).

**Proposition 3.4.** *Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering over $\mathcal{E}$ with nonlocal coordinates $w^1, \ldots, w^j, \ldots$ and $\tilde{D}_1, \ldots, \tilde{D}_n$ be total derivatives in this covering. Let also $\varphi$ be a $\tau$-shadow. Then:*

(i) *the relations*

$$\frac{\partial \tilde{w}^j}{\partial x^i} = \tilde{\ell}_{X_i^j}(\varphi) + \sum_\alpha \frac{\partial X_i^j}{\partial w^\alpha} \tilde{w}^\alpha, \qquad i = 1, \ldots, n, \quad j = 1, \ldots, \dim \tau, \qquad (3.20)$$

*define a covering over $\tilde{\mathcal{E}}$ whose dimension equals that of $\tau$;*

*(ii) equations (3.16) are solvable in this covering.*

*Remark* 21. From (3.20) it follows that for an Abelian covering $\tau$ the covering $\tilde{\tau}$ is Abelian as well. Hence, at every step of reconstruction obstructions to solving (3.16) lie in the horizontal cohomology group of the corresponding equation. Consequently, if $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ is an Abelian covering and $H^1_h(\tilde{\mathcal{E}}) = 0$ then any $\tau$-shadow can be reconstructed to a nonlocal $\tau$-symmetry. In particular, any local symmetry of $\mathcal{E}$ can be lifted to $\tilde{\mathcal{E}}$.

Let $\mathcal{E}$ be an equation and $\{[\omega^\alpha]\}$, $\omega^\alpha \in \Lambda^1_h(\mathcal{E})$, be an $\mathbb{R}$-basis of the group $H^1_h(\mathcal{E})$. Assume that

$$\omega^\alpha = X^\alpha_1 \, dx^1 + \ldots + X^\alpha_n \, dx^n$$

and consider the covering $\tau_1\colon \mathcal{E}_1 \to \mathcal{E}$ determined by

$$\frac{\partial w^\alpha}{\partial x^i} = X^\alpha_i$$

for all $\alpha$ and $i = 1, \ldots, n$. For $\mathcal{E}_1$ let us construct the covering $\tau_2\colon \mathcal{E}_2 \to \mathcal{E}_1$ in a similar way, etc. The covering $\tau_*\colon \mathcal{E}_* \to \mathcal{E}$ obtained as the inverse limit of the sequence

$$\ldots \xrightarrow{\tau_{i+1}} \mathcal{E}_i \xrightarrow{\tau_i} \mathcal{E}_{i-1} \xrightarrow{\tau_{i-1}} \ldots \xrightarrow{\tau_2} \mathcal{E}_1 \xrightarrow{\tau_1} \mathcal{E}$$

is called the *universal Abelian covering* over $\mathcal{E}$. By construction, $H^1_h(\mathcal{E}_*) = 0$.

**Proposition 3.5.** *For an arbitrary Abelian covering $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ there exists a uniquely (up to a gauge equivalence) defined morphism*



*and any $\tau$-shadow can be reconstructed in $\tau_*$. In particular, any symmetry of $\mathcal{E}$ can be lifted to $\mathcal{E}_*$.*

*Remark* 22. Though the covering $\tilde{\tau}$ whose existence is stated in Proposition 3.3 is determined canonically by the shadow $X$, the new shadow $\tilde{X}$ is not unique, but is defined up to an infinitesimal gauge symmetries of $\tau$, i.e., up to $Y \in \mathrm{sym}(\tilde{\mathcal{E}})$ such that $Y|_{\mathcal{F}} = 0$. Due to (3.16), these symmetries are given by the equations

$$\tilde{D}_i(\psi^j) = \sum_\alpha \frac{\partial X^j_i}{\partial w^\alpha} \psi^\alpha \tag{3.21}$$

and are of the form $Y = \sum_\alpha \psi^\alpha \partial/\partial w^\alpha$.

**Example 36.** For Example 26, Equations (3.21) take the form

$$\tilde{D}_x(\psi) = 0, \qquad \tilde{D}_x(\psi) = 0$$

and consequently infinitesimal gauge symmetries are $\gamma \partial/\partial w$ in this case, $\gamma \in \mathbb{R}$.

On the other hand, if we consider Example 27 then Equations (3.21) are written as

$$\tilde{D}_x(\psi) = 2w\psi, \qquad \tilde{D}_t(\psi) = 2(u_1 + 2uw - 4\lambda w)\psi.$$

The only solution of this system is $\psi = 0$ and thus there is no ambiguity in shadow reconstruction in this case.

Non-uniqueness of solution to the problem of reconstruction leads, in turn, to the problem of commutation for shadows: no well defined way to compute the Lie bracket of shadows is known. This problem was first indicated in [118]. A way to solve it was suggested in [119], but a practical realization of the approach is somewhat cumbersome.

To conclude this subsection, let us make a remark also related to the problem of reconstruction. Let $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a finite-dimensional covering and $X$ be a symmetry of $\mathcal{E}$. One can (at least, locally) lift $X$ to $\tilde{\mathcal{E}}$ in an arbitrary way. Then, if $X$ is an integrable vector field (i.e., if it possesses the corresponding one-parameter group of transformations) then the lifted field $\tilde{X}$ is integrable as well. If $\tilde{X}$ is a symmetry of $\tilde{\mathcal{E}}$ then this means that we managed to reconstruct $X$ up to a nonlocal symmetry in the covering $\tau$.

Otherwise, consider the one-parameter group of transformations $\{\tilde{A}_\lambda\}$ corresponding to $\tilde{X}$ and for any $\lambda \in \mathbb{R}$ define an $n$-dimensional distribution $\tilde{\mathcal{C}}^\lambda$ on $\tilde{\mathcal{E}}$ by

$$\tilde{\mathcal{C}}^\lambda \colon \theta \mapsto \tilde{\mathcal{C}}_\theta^\lambda = \tilde{A}_{\lambda,*}\left(\tilde{\mathcal{C}}_{\tilde{A}_\lambda^{-1}(\theta)}\right) \tag{3.22}$$

where $\theta \in \tilde{\mathcal{E}}$, $\tilde{\mathcal{C}}_\theta$ is the Cartan plane at the point $\theta$ and $F_*$ denotes the differential of the map $F$.

**Proposition 3.6.** *Correspondence (3.22) determines a one-parameter family $\tau_\lambda$ of pairwise inequivalent coverings over $\tilde{\mathcal{E}}$ such that $\tau_0 = \tau$.*

**Example 37.** Take the covering

$$X = u + w^2, \qquad T = u_2 + 2wu_1 + 2u^2 + 2w^2 u$$

over the KdV equation and apply Proposition 3.4 using the Galilean boost $tu_1 + 1/6$ for the symmetry $X$. This will result in the Miura covering described in Example 27.

Not all one-parameter families of coverings can be obtained by this procedure (a counter-example can be found in [120, 121]). But a weaker result was proved in [122]:

**Theorem 3.7.** *Let $\tau_\lambda\colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a one-parameter family of coverings regarded as a deformation of the covering $\tau = \tau_0$. Then the corresponding infinitesimal deformation is a $\tau$-shadow.*

### 3.3. Bäcklund transformations and zero-curvature representations

In conclusion, let us briefly discuss how the constructions of Bäcklund transformations [123, 124] and zero-curvature representations [125] are translated to geometrical language.

A *Bäcklund transformation* between two equations $\mathcal{E}'$ and $\mathcal{E}''$ with the unknown functions $u'$ and $u''$, respectively, is another equation $\mathcal{E}$ in unknown functions both $u'$ and $u''$ such that for any solution $u'$ of $\mathcal{E}'$ a solution $u''$ of $\mathcal{E}$ is a solution of $\mathcal{E}''$ as well and vice versa. If $\mathcal{E}'$ coincides with $\mathcal{E}''$ then one speaks about *auto-Bäcklund transformation*.

**Example 38.** Consider the sine-Gordon equation

$$u_{xy} = \sin u. \tag{3.23}$$

Then the system

$$v_y - u_y = 2\lambda \sin \frac{v+u}{2}, \qquad v_x + u_x = \frac{2}{\lambda} \sin \frac{v-u}{2}, \tag{3.24}$$

where $\lambda \neq 0$ is a real parameter, determines the classical auto-Bäcklund transformation (a one-parameter family, actually) for (3.23), see [126].

**Example 39.** The second example (which now can also be considered as a classical one) was found in [113]. It is of the form

$$\left(\frac{v+w}{2}\right)_x + \left(\frac{v-w}{2}\right)^2 + \lambda^2 = 0, \qquad \lambda \in \mathbb{R},$$

$$\left(\frac{v-w}{2}\right)_t + 6\left(\frac{v+w}{2}\right)_x \left(\frac{v-w}{2}\right)_x + \left(\frac{v-w}{2}\right)_{xxx} = 0 \tag{3.25}$$

and relates solutions of the KdV equation to each other (or, to be more precise, system (3.25) is a Bäcklund transformation for the potential KdV equation, while solutions of the KdV itself are obtained by $u = v_x$).

Analysis of these two examples (as well as other ones) shows that a Bäcklund transformation between equations $\mathcal{E}'$ and $\mathcal{E}''$ is a diagram



where $\tau'$ and $\tau''$ are coverings. The correspondence between solutions of $\mathcal{E}'$ and $\mathcal{E}''$ is achieved in the following way. Let $u' = u'(x')$ be a solution of $\mathcal{E}'$ and assume that $\tau'$ is a finite-dimensional covering. Then the Cartan distribution of the equation $\mathcal{E}$ induces on the finite-dimensional manifold $\mathcal{E}_{u'} = (\tau')^{-1}(u') \subset \mathcal{E}$ an $n$-dimensional integrable distribution. In the vicinity of a generic point the latter possesses a $(\dim \tau')$-parameter family of maximal integral manifolds that are projected to $u'$ by $\tau'$ and to the corresponding family of solutions of $\mathcal{E}''$ by $\tau''$. Generically, such a correspondence is non-trivial provided $\tau'$ and $\tau''$ are not gauge equivalent.

**Example 40.** A common way to construct non-trivial Bäcklund transformations is the following. Let $\tau \colon \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering and $f \colon \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ be a finite symmetry of $\tilde{\mathcal{E}}$, i.e., a diffeomorphism preserving the Cartan distribution. Then the composition $\tau' = \tau \circ f$ is a covering as well and the pair $(\tau, \tau')$ is an auto-Bäcklund transformation for $\mathcal{E}$. If $f$ is not a gauge equivalence then this transformation is non-trivial.

Consider covering (3.2) from Example 27 and note that the change of the nonlocal variable $w \leftrightarrow -w$ is a symmetry of the covering equation (the mKdV one), but is not a gauge symmetry of the covering itself. Thus, for any value of the parameter $\lambda$ we get an auto-Bäcklund transformation of the KdV equation, i.e., a one-parameter family of Bäcklund transformations.

Note that the Wahlquist-Estabrook construction (Example 39) is a consequence of the latter one.

*Remark* 23. Families of Bäcklund transformations, like the ones from Examples 38 and 39, provide a base to construct special exact solutions of integrable equations (such as multi-kink solutions for the sine-Gordon equation, multi-soliton solutions for the KdV, etc.). The construction uses the *nonlinear superposition principle* which, in turn, is based on the following informal statement:

**Theorem 3.8** (Bianchi Permutability Theorem). *Assume that an equation $\mathcal{E}$ possesses a one-parameter family of auto-Bäcklund transformations $\mathcal{B}_\lambda$ and let $\lambda \in \mathbb{R}$ be the parameter. For any solution $u = u(x)$ of $\mathcal{E}$ denote by $\mathcal{B}_\lambda(u)$ the set of solutions obtained from $u$ by means of $\mathcal{B}_\lambda$. Then for any $\lambda_1 \neq \lambda_2$ there exists a solution $u_{\lambda_1,\lambda_2} \in \mathcal{B}_{\lambda_1}(\mathcal{B}_{\lambda_2}(u)) \cap \mathcal{B}_{\lambda_2}(\mathcal{B}_{\lambda_1}(u))$ that is expressed as a bi-differential operator applied to some solutions $u_1 \in \mathcal{B}_{\lambda_1}(u)$ and $u_2 \in \mathcal{B}_{\lambda_2}(u)$.*

This "theorem" was first observed by Bianchi in [36] (see also [123]) in application to the sine-Gordon equation (Example 38) and since then dozens of examples were computed, but nevertheless a general formulation of this statement is unknown to us. Some hints to a rigorous approach to the problem can be found in [127].

Geometrical theory of Bäcklund transformations is also related to an unorthodox approach to recursion operators [128]. Consider an equation $\mathcal{E}$ and its tangent covering $\tau \colon \mathcal{T}(\mathcal{E}) \to \mathcal{E}$ (see Section 2). Recall that symmetries of $\mathcal{E}$ are identified with sections of $\tau$ that take the Cartan distribution on $\mathcal{E}$ to that on $\mathcal{T}(\mathcal{E})$. Hence, if we consider a diagram of the form
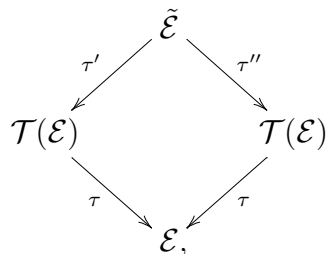


where $\tau'$ and $\tau''$ are coverings, then this Bäcklund transformation will relate symmetries of $\mathcal{E}$ to each other. Thus, this Bäcklund transformation plays the rôle of a recursion operator for symmetries of $\mathcal{E}$.

**Example 41.** Consider the KdV equation $u_t = 6uu_x + u_{xxx}$ and two copies of its tangent covering with the new dependent variable $v$ that enjoys the the additional equation

$$v_t = 6u_x v + 6u v_x + v_{xxx}.$$

Introduce a nonlocal variable $\tilde{v}$ by setting

$$\tilde{v}_x = v, \qquad \tilde{v}_t = 6uv + v_{xx}.$$

Thus, internal coordinates in $\tilde{\mathcal{E}}$ are

$$x, \ t, \ u = u_0, \ u_x = u_1, \ \dots, \ v = v_0, \ v_x = v_1, \ \dots, \ \tilde{v}.$$

Define the covering $\tau'$ by

$$\tau': (x, t, u_k, v_k, \tilde{v}) \mapsto (x, t, u_k, v_k)$$

and the covering $\tau''$ by

$$\tau': (x, t, u_k, v_k, \tilde{v}) \mapsto (x, t, u_k, D_x^k(v_2 + 4uv + 2u_1\tilde{v})).$$

The Bäcklund transformation obtained in such a way is the geometrical realization of the Lenard recursion operator $R = D_x^2 + 4u + 2u_1 D_x^{-1}$ (1.55).

Another, less trivial example will be considered later (see Example 43 below), after discussing the concept of *zero-curvature representations* (ZCR).

Let $\tau: \tilde{\mathcal{E}} \to \mathcal{E}$ be a covering. We say that it is *linear* if

(i) $\tau$ is a vector bundle;

(ii) the action of vector fields $\tilde{D}_1, \ldots, \tilde{D}_n$ on $\mathcal{F}(\tilde{\mathcal{E}})$ preserves the subspace of fibre-wise linear functions.

*Coordinates.* Let $v^1, \ldots, v^r, \ldots$ be local coordinates along the fibre of $\tau$ and the covering be given by the total derivatives

$$\tilde{D}_i = D_i + \sum_r X_i^r \frac{\partial}{\partial v^r}, \qquad i = 1, \ldots, n. \tag{3.26}$$

Then the covering is linear if and only if the coefficients $X_i^r$ in (3.26) are of the form

$$X_i^r = \sum_\alpha X_{i\alpha}^r v^\alpha,$$

where $X_{i\alpha}^r$ are smooth functions on $\mathcal{E}$. If we now identify the vertical terms $X_i = \sum_r X_i^r \partial/\partial v^r$ in (3.26) with the function-valued matrices

$$X_i = \begin{pmatrix} X_{i1}^1 & \ldots & X_{i1}^n \\ \ldots & \ldots & \ldots \\ X_{in}^1 & \ldots & X_{in}^n \end{pmatrix}$$

then (3.26) will be rewritten as

$$\tilde{D}_i = D_i + X_i, \qquad i = 1, \ldots, n,$$

while the conditions $[\tilde{D}_i, \tilde{D}_j] = 0$ will acquire the form

$$D_i(X_j) - D_j(X_i) + [X_i, X_j] = 0, \qquad 1 \le i < j \le n.$$

In other words, we arrive to the classical definition of a ZCR (cf. with [125]).

**Example 42.** The well known two-dimensional ZCR for the KdV equation (see [125]) is given by

$$\tilde{D}_x = D_x + A, \qquad \tilde{D}_t = D_t + B,$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -u + \lambda & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -u_x & 2u - 4\lambda \\ -u_{xx} - 2u^2 + 2\lambda u + 4\lambda^2 & u_x \end{pmatrix}.$$

**Example 43** (vacuum Einstein equations)**.** Consider the Lewis metric $\mathrm{d}s^2 = 2f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \sum_{j\leq j} g_{ij}\,\mathrm{d}z^i\,\mathrm{d}z^j$ in $\mathbb{R}^4$ with coordinates $x$, $y$, $z^1$, and $z^2$ (see [129]). The the vacuum Einstein equations read

$$(\sqrt{\det g}\,g_x g^{-1})_y + (\sqrt{\det g}\,g_y g^{-1})_x = 0. \tag{3.27}$$

After a re-parameterization, (3.27) acquires the form

$$u_{xy} = \frac{u_x u_y - v_x v_y}{u} - \frac{1}{2}\frac{u_x + u_y}{x+y}, \qquad v_{xy} = \frac{v_x u + y + u_x v_y}{u} - \frac{1}{2}\frac{v_x + v_y}{x+y}. \tag{3.28}$$

Bäcklund transformations and ZCR for (3.28) were constructed in many papers (see, e.g., [130–133]). The latter is of the form

$$\tilde{D}_x = D_x + A, \qquad \tilde{D}_y = D_y + B,$$

where

$$A = \frac{1}{2}\begin{pmatrix} -\frac{(\theta+1)u_x}{u} & \frac{(\theta+1)v_x}{u^2} \\ (\theta-1)v_x, & \frac{(\theta+1)u_x}{u} \end{pmatrix}, \qquad B = \frac{1}{2\theta}\begin{pmatrix} -\frac{(\theta+1)u_y}{u} & \frac{(\theta+1)v_y}{u^2} \\ (1-\theta)v_y & \frac{(\theta+1)u_y}{u} \end{pmatrix}; \tag{3.29}$$

here $\theta = \sqrt{(\lambda+y)(\lambda-x)}$ and $\lambda$ is the spectral parameter.

Using ZCR (3.29), a three-dimensional covering over $\mathcal{T}(\mathcal{E})$ can be constructed (see [134]). Let $U$ and $V$ be the variables in $\mathcal{T}(\mathcal{E})$ corresponding to $u$ and $v$, respectively, and $w^1$, $w^2$, $w^3$ be the nonlocal variables. Then the covering is given by the relations

$$w_x^1 = \frac{1-\theta}{2}v_x w^2 + \frac{1+\theta}{2u^2}v_x w^3 - \frac{1+\theta}{2u}U_x + \frac{1+\theta}{2u^2}u_x U,$$

$$w_x^2 = -\frac{1+\theta}{u^2}v_x w^1 - \frac{1+\theta}{u}u_x w^2 - \frac{1+\theta}{u^3}v_x U + \frac{1+\theta}{2u^2}V_x,$$

$$w_x^3 = (\theta-1)v_x w^1 + \frac{1+\theta}{u}u_x w^3 + \frac{\theta-1}{2}V_x$$

and

$$w_y^1 = \frac{\theta-1}{2\theta}v_y w^2 + \frac{1+\theta}{2\theta u^2}v_y w^3 + \frac{1+\theta}{2\theta u^2}u_y U - \frac{1+\theta}{2\theta u}U_y,$$

$$w_y^2 = -\frac{1+\theta}{\theta u^2}v_y w^1 - \frac{1+\theta}{\theta u}u_y w^2 - \frac{1+\theta}{\theta u^3}v_y U + \frac{1+\theta}{2\theta u^2}V_y,$$

$$w_y^3 = \frac{1-\theta}{\theta}v_y w^1 + \frac{1+\theta}{\theta u}u_y w^3 + \frac{1-\theta}{2\theta}V_y.$$

This covering gives rise to a Bäcklund transformation of the form

$$\theta U' = 2uw^1 + U, \qquad \theta V' = -u^2 w^2 - w^3,$$

i.e., to a recursion operator for symmetries.

*Remark* 24. Note that with an arbitrary covering $\tau\colon \tilde{\mathcal{E}} \to \mathcal{E}$ one can naturally associate a linear covering $\tau^v\colon \mathcal{T}^v\tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$. The space $\mathcal{T}^v\tilde{\mathcal{E}}$ is a submanifold in $\mathcal{T}\tilde{\mathcal{E}}$ and consists of tangent vectors that vanish under the action of the differential $\tau_*$.

Note also the existence of the exact sequence of coverings

$$0 \longrightarrow \mathcal{T}^v\tilde{\mathcal{E}} \longrightarrow \mathcal{T}\tilde{\mathcal{E}} \longrightarrow \mathcal{T}^s\tilde{\mathcal{E}} \longrightarrow 0$$

$$\tau^v \searrow \quad \downarrow \tilde{\tau} \quad \swarrow \tau^s$$

$$\tilde{\mathcal{E}},$$

where $\tau^s \colon \mathcal{T}^s \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}$ is the quotient. The terms of this sequence possess the following characteristic property: integrable sections* of $\tau^v$ are infinitesimal gauge symmetries of $\tau$, integrable sections of $\tilde{\tau}$ are, as it was mentioned above, nonlocal $\tau$-symmetries of $\mathcal{E}$, and integrable sections of $\tau^s$ are $\tau$-shadows.

## Concluding remarks

We described a geometrical approach to partial differential equations which proved to be efficient both from the theoretical point of view and in particular applications. Based on this approach, in particular, Hamiltonian formalism for arbitrary normal (2-line) equations is constructed. On the other hand, a number of interesting and important problems are waiting for their solution. We plan to continue the research along the following lines:

- Generalization of the Hamiltonian formalism from normal equations to arbitrary $p$-line ones that, in particular, include gauge-invariant systems.
- Incorporation of Dirac structures into the above described scheme and elaboration of their computation and use.
- Further development of the nonlocal theory and, in particular, analysis of differential coverings over the systems with the number of independent variables greater than two and generalization the theory of variational brackets to nonlocal structures.

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* We say that a section is integrable if it preserves the Cartan distributions.

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