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Geometry of rotating disk and the Sagnac effect

In this paper we demonstrate that subsequent application of Lorentz transformations to the cylindrical coordinates on a rotating disk leaves the form $ds^2 = c^2dt^2 - dr^2 - r^2d\phi^2 - dz^2$ invariant. Therefore, the geometry on rotating disk is the Euclidean geometry, and any experiment which do not involve tidal forces or Coriolis forces cannot identify either the disk rotates or not. We also show that, from the point of view of external inertial observer, the difference in the transit times for the light running along a circle of radius R in the opposite directions (with respect to the rotation) is $\Delta t = 2\frac{\omega}{c^2}S$, where S is the area the circle.

We start with a historical review. In [2] H.A.Lorentz considered transformation from a resting coordinate system to another coordinate system moving with respect to the first one with velocity v as composition of the following transformations (see [1]):

1) The Galilean transformation

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t. \tag{1}$$

2) The transformation which gives the length shrinking with respect to the moving coordinate system and introduces the local time at any point of moving coordinate system

$$x'' = \frac{x'}{\sqrt{1 - \dfrac{v^2}{c^2}}}, \quad t'' = \sqrt{1 - \frac{v^2}{c^2}}\,t' - \frac{\dfrac{v}{c^2}x'}{\sqrt{1 - \dfrac{v^2}{c^2}}}, \tag{2}$$

where c is the light velocity.

In [3] A.Poincare gave the formulas for the resulting transformation

$$x'' = \frac{x - vt}{\sqrt{1 - \dfrac{v^2}{c^2}}}; \quad y'' = y, \quad z'' = z, \quad t'' = \frac{t - \dfrac{v}{c^2}x}{\sqrt{1 - \dfrac{v^2}{c^2}}} \tag{3}$$

In what follows we shall use the infinitesimal versions of transformations $(1 - 3)$:

$$dx' = dx - vdt, \quad dy' = dy, \quad dz' = dz, \quad dt' = dt;$$

$$dx'' = \frac{dx'}{\sqrt{1-\frac{v^2}{c^2}}} \quad dy'' = dy', \quad dz'' = dz'; \quad dt'' = \sqrt{1-\frac{v^2}{c^2}}dt' - \frac{\frac{v}{c^2}dx}{\sqrt{1-\frac{v^2}{c^2}}} \tag{1'}$$

(2')

$$dx'' = \frac{dx - vdt}{\sqrt{1-\frac{v^2}{c^2}}} \quad dy' = dy, \quad dz' = dz, \quad dt'' = \frac{dt - \frac{v}{c^2}dx}{\sqrt{1-\frac{v^2}{c^2}}} \tag{3'}$$

It is important to note that the transformation of local time (2') consists of two transformations

$$dt_1'' = \sqrt{1-\frac{v^2}{c^2}}dt' \qquad dt_2'' = \frac{-\frac{v}{c^2}dx}{\sqrt{1-\frac{v^2}{c^2}}} \tag{4}$$

Now let us consider coordinate transformations with respect to a rotating coordinate system. Following Möller [4], we intoduce an inertial reference system with cylindrical coordinates (r, θ, z,t), and a rotating coordinate system S with coordinates (r', θ', z',t') which are expressed through (r, θ, z,t) via the Galilean transformation (see (1)):

$$r' = r, \quad \theta' = \theta - \omega t, \quad z' = z, \quad t' = t. \tag{5}$$

where ω is the angle velocity of the disk. Then the infinitesimal transformation takes the form

$$dr' = dr, \quad d\theta' = d\theta - \omega dt, \quad dz' = dz, \quad dt' = dt, \tag{5'}$$

Here ω is the angle velocity of the disk measured by the inertial observer.

Now, following H.A.Lorentz [2] we introduce new local coordinates via the transformation

$$dr'' = dr', \quad r''d\theta'' = \frac{r'd\theta'}{\sqrt{1-\frac{v^2}{c^2}}}, \quad dt'' = \sqrt{1-\frac{v^2}{c^2}}dt' - \frac{\frac{v}{c^2}r'd\theta'}{\sqrt{1-\frac{v^2}{c^2}}} \tag{6'}$$

Here $V = \omega r$ is the linear speed of the rotating disk. The transformations (6') are written in view of the fact that the radial component of local coordinates is perpendicular to the velocity $\vec{V} = [\vec{\omega}, \vec{r}]$ of the disk, and, according to the postulates of special relativity, does not change. The infinitesimal coordinate $r'd\theta'$ shrinks by the factor $(1-\frac{v^2}{c^2})^{\frac{1}{2}}$ since this coordinate is parallel to the movement direction. The local time dt'' in (6') consists of two summands according to (2').

One can easily show that the "Lorentz transformation" for the infinitesimal coordinates on the rotating disk has the form

$$dr'' = dr, \quad r''d\theta'' = \frac{rd\theta - \omega r dt}{\sqrt{1 - \dfrac{\omega^2 r^2}{c^2}}}, \quad dt'' = \frac{dt - \dfrac{\omega r}{c^2} r d\theta}{\sqrt{1 - \dfrac{\omega^2 r^2}{c^2}}} \tag{7'}$$

and the transformation (7') preserves ds^2:

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\theta^2 = c^2 dt''^2 - dr''^2 - r''^2 d\theta''^2 \tag{8'}$$

The following important note concerning the transformation (7') need to be made. The transformation (3') can be obtained from the transformation (2) via differentiation. And (7') is an *infinitesimal* transformation connecting dr, $d\theta$, dt with dr'', $d\theta''$, dt''. The corresponding equations are not completely integrable, so it is impossible to express r'', θ'', t'' in terms of r, θ, t in a finite way! In fact, this is a peculiar feature of Lorentz transformation from an inertial reference system to an accelerated one. Transformations of this type are usually called local transformations, and are widely used (see e.g. [3], [5]). We can only note that, though there is no global connection between coordinates, we can consider r'', θ'', t'' as ordinary coordinates in the rotating reference system, and use them to solve mechanical and optical problems with respect to a rotating reference system (see e.g. discussion on the Sagnac effect in [3]). We suppose to construct mechanics with respect to rotating reference system in next paper.

According to the standard rules [4], from the space-time metric we separate out the space metric

$$dl''^2 = dr''^2 + r''^2 d\theta''^2, \tag{9'}$$

thus on the disk we get the Euclidean geometry, contrary to the conclusions of other researchers [4], [6] who obtained that

$$dl''^2 = dr''^2 + \frac{r''^2 d\theta''^2}{(1 - \frac{\omega^2 r''^2}{c^2})}, \tag{6}$$

The difference between (9') and (6) comes from the fact that they did not take into account the transformation of time in the rotating reference system.

The difference between our results on geometry of rotating disk and the results of the majority of scientists who believe that on the rotating disk the non-Euclidean geometry arises, leads to the difference in interpretations of effects in rotating reference system.

From our results it follows that, according to (8'), (9'), any experiment in a rotating reference system, which do not take into account the dynamical influence of rotation causing the centrifugal force and the Coriolis force, cannot detect the rotation of reference system. Therefore, the transit times for the light going in opposite directions around a circle $r'' = const$ coincide: $dt''_+ = dt''_-$, hence

$$(t''_+)_{r=r_0} = (t''_-)_{r=r_0}. \tag{7}$$

Substituting (7) into (7'), with $r = R$, we get

$$(dt_+)_{r=R} - (dt_-)_{r=R} = 2\frac{\omega R^2}{c^2} d\theta. \tag{8}$$

Then the total time difference with respect to the inertial reference system is

$$\Delta t = t_+ - t_- = \frac{4\omega\pi}{c^2} R^2 = 2\frac{\omega}{c^2} S, \tag{9}$$

where S is the area of circle of radius R.

The formula (9) gives the exact value of difference in the light transit times, and is called the Sagnac effect [6].

References

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