Cambridge Studies in Advanced Mathematics 79

Soliton Equations and Their Algebro-Geometric Solutions

Volume 1: (1 + 1)- Dimensional Continuous Models

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Volume I: (1 + 1)-Dimensional Continuous Models

The focus of this book is on algebro-geometric solutions of completely integrable, nonlinear, partial differential equations in (1+1) dimensions, also known as soliton equations. Explicitly treated integrable models include the KdV, AKNS, sine–Gordon, and Camassa–Holm hierarchies as well as the classical massive Thirring system. An extensive treatment of the class of algebro-geometric solutions in the stationary as well as time-dependent contexts is provided. The formalism presented includes trace formulas, Dubrovin-type initial value problems, Baker–Akhiezer functions, and theta function representations of all relevant quantities involved. The book uses techniques from the theory of differential equations, spectral analysis, and elements of algebraic geometry (most notably, the theory of compact Riemann surfaces). The presentation is rigorous, detailed, and self-contained with ample background material in various appendices. Detailed notes for each chapter together with an extensive bibliography enhance the presentation offered in the main text.

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SOLITON EQUATIONS AND THEIR ALGEBRO-GEOMETRIC SOLUTIONS

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, United Kingdom

Published in the United States by Cambridge University Press, New York

www.cambridge.org

Information on this title: www.cambridge.org/9780521753074

First published in print format 2003

ISBN-13 978-0-511-06870-6 eBook (EBL)
ISBN-10 0-511-06870-0 eBook (EBL)

ISBN-13 978-0-521-75307-4 hardback
ISBN-10 0-521-75307-4 hardback

To our parents
Friederike and Franz Gesztesy
Kirsten Kiellerup (in memoriam) and Finn Holden

Contents

Acknowledgments

> . . . and the manuscript was becoming an albatross about my neck.
> There were two possibilities: to forget about it completely,
> or to publish it as it stood; and I preferred the second.
>
> *Robert P. Langlands*[1]

This monograph has grown out of work we have done over the past 15 years within the area of completely integrable nonlinear partial differential equations. Our starting point has been that of mathematical analysis with emphasis on spectral theoretic techniques. We have made every effort to be explicit and as detailed as possible in the presentation of results. Consequently, this volume has acquired a somewhat technical appearance. However, our experience in the extensive study of an area, especially one in which the notion of exactly solvable models dominates and hence explicit formulas abound, is that there can never be too many details — mathematics is not a spectator sport.

We are indebted to many co-workers in this endeavor for the joy of collaboration, among them, especially, Wolfgang Bulla, Ronnie Dickson, Victor Enol'skii, Ratnam Ratnaseelan, Walter Renger, Barry Simon, Wilhelm Sticka, Gerald Teschl, Karl Unterkofler, Rudi Weikard, and Zhongxin Zhao.

Special and heartfelt thanks are due to Harald Hanche-Olsen and Karl Unterkofler. Their around-the-clock unselfish assistance with all aspects, both mathematical and TeXnical, of an enterprise of this magnitude, has been of invaluable help to us. Thanks a lot, Harald and Karl!

We are grateful to all the individuals who read portions of the manuscript and supplied us with valuable comments and criticism. In particular, we would like to thank Wolfgang Bulla, Steve Clark, Harald Hanche-Olsen, Emma Previato, Gerald Teschl, Karl Unterkofler, and Rudi Weikard for their efforts to help us improve the presentation.

[1] *On the Functional Equations Satisfied by Eisenstein Series*, Lecture Notes in Mathematics, **544**, Springer, Berlin, 1976, p. III.

We have established a Web page with URL

`www.math.ntnu.no/~holden/solitons`

where we intend to keep an updated list of typographical errors for the benefit of the reader. We encourage the reader to send comments, corrections, etc., to the authors.

F.G. is indebted to Sergio Albeverio, Margarida de Faria, Gisèle Ruiz Goldstein, Jerry Goldstein, and Ludwig Streit for encouragement and support. Moreover, he thanks Gisèle and Jerry as well as Margarida and Ludwig for kind opportunities to lecture on various parts of the material in this volume at LSU, Baton Rouge, Louisiana, USA (April 1996) and CCM, Universidade da Madeira, Portugal (August 2001), respectively.

Research in this area has been funded in part by the University of Missouri Research Board (grant RB-97-086), the Norwegian Academy of Science and Letters, and last, but certainly not least, the Research Council of Norway.

July 2002

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Introduction

> It often happens that the understanding of the mathematical nature of an equation is impossible without a detailed understanding of its solutions.
>
> *Freeman J. Dyson*

Background: The discovery of solitary waves of translation goes back to Scott Russell in 1834, and during the remaining part of the 19th century the true nature of these waves remained controversial. It was only with the derivation by Korteweg and de Vries in 1895 of what is now called the Korteweg–de Vries (KdV) equation, that the one-soliton solution and hence the concept of solitary waves was put on a firm basis.[1] An extraordinary series of events took place around 1965 when Kruskal and Zabusky, while analyzing the numerical results of Fermi, Pasta, and Ulam on heat conductivity in solids, discovered that pulselike solitary wave solutions of the KdV equation, for which the name "solitons" was coined, interact elastically. This was followed by the 1967 discovery of Gardner, Greene, Kruskal, and Miura that the inverse scattering method allows one to solve initial value problems for the KdV equation with sufficiently fast-decaying initial data. Soon thereafter, in 1968, Lax found a new explanation of the isospectral nature of KdV solutions using the concept of Lax pairs and introduced a whole hierarchy of KdV equations. Subsequently, in the early 1970s, Zakharov and Shabat (ZS), and Ablowitz, Kaup, Newell, and Segur (AKNS) extended the inverse scattering method to a wide class of nonlinear partial differential equations of relevance in various scientific contexts ranging from nonlinear optics to condensed matter physics and elementary particle physics. In particular, solitons found numerous applications in classical and quantum field theory and in connection with optical communication devices.

Another decisive step forward in the development of completely integrable soliton equations was taken around 1974. Prior to that period, inverse spectral

[1] With hindsight, though, it is now clear that other researchers, such as Boussinesq, derived the KdV equation and its one-soliton solution prior to 1895, as described in the notes to Section 1.1.

methods in the context of nonlinear evolution equations had been restricted to spatially decaying solutions. In 1974–75, the arsenal of inverse spectral methods was extended considerably in scope to include periodic and certain classes of quasi-periodic and almost periodic KdV solutions. This new approach to constructing solutions of integrable nonlinear evolution equations, partly based on inverse spectral theory and partly relying on algebro-geometric methods developed by pioneers such as Dubrovin, Flaschka, Its, Krichever, Lax, Marchenko, Matveev, McKean, Novikov, van Moerbeke – to name just a few – was followed by very rapid development in the field. Within a few years of intense activity worldwide, the landscape of integrable systems was changed forever. By the early 1980s the theory was extended to a large class of nonlinear (including some multi-dimensional) evolution equations beyond the KdV equation, and the explicit theta function representations of quasi-periodic solutions of integrable equations (including, e.g., soliton solutions as special limiting cases) had introduced new algebro-geometric techniques into this area of nonlinear partial differential equations. Subsequently, this led to several new and deep results in nonlinear partial differential equations as well as in algebraic geometry (such as a solution of Schottky's problem).

Our series of monographs is devoted to this area of algebro-geometric solutions of hierarchies of soliton equations.

Scope: We aim for an elementary, yet self-contained and precise, presentation of hierarchies of integrable soliton equations and their algebro-geometric solutions. Our point of view is predominantly influenced by analytical methods, especially by spectral theoretic techniques. We hope this will make the presentation accessible and attractive to analysts working outside the traditional areas associated with soliton equations. Central to our approach is a simultaneous construction of all algebro-geometric solutions and their theta function representation of a given hierarchy. In this volume we focus on some of the key hierarchies in $(1+1)$-dimensions associated with continuous integrable models such as the Korteweg–de Vries hierarchy (KdV), the combined sine–Gordon modified Korteweg–de Vries hierarchy (sGmKdV), the Ablowitz–Kaup–Newell–Segur hierarchy[1] (AKNS), the classical massive Thirring system (Th), and the Camassa–Holm hierarchy (CH). The key equations defining the corresponding hierarchies read

$$\begin{aligned}
&\text{KdV:} && u_t + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x = 0,\\
&\text{sGmKdV:} && u_{xt} - \sin(u) = 0,\\
&\text{AKNS:} && \begin{pmatrix} p_t + \frac{i}{2}p_{xx} - ip^2q \\ q_t - \frac{i}{2}q_{xx} + ipq^2 \end{pmatrix} = 0,
\end{aligned} \tag{0.1}$$

[1] Using the gauge equivalence of the AKNS hierarchy and classical Boussinesq hierarchy, we also treat the latter.

$$\text{Th:} \qquad \begin{pmatrix} -iu_x + 2v + 2vv^*u \\ iu_x^* + 2v^* + 2vv^*u^* \\ -iv_t + 2u + 2uu^*v \\ iv_t^* + 2u^* + 2uu^*v^* \end{pmatrix} = 0,$$

$$\text{CH:} \qquad 4u_t - u_{xxt} - 2uu_{xxx} - 4u_xu_{xx} + 24uu_x = 0.$$

Our principal goal in this monograph is the construction of algebro-geometric solutions of the hierarchies associated with the equations listed in (0.1). Interest in the class of algebro-geometric solutions can be motivated in a variety of ways: It represents a natural extension of the classes of soliton and rational solutions, and similar to these, its elements can still be regarded as explicit solutions of the nonlinear integrable evolution equation in question (even though their complexity considerably increases compared with soliton solutions due to the underlying analysis on compact Riemann surfaces). Moreover, algebro-geometric solutions can be used to approximate more general solutions (such as almost periodic ones), although this is not a topic pursued in this monograph. Here we primarily focus on the construction of explicit solutions in terms of certain algebro-geometric data on a compact Riemann surface and their representation in terms of theta functions. For instance, in KdV-type contexts, solitons arise as the special case of solutions corresponding to an underlying singular hyperelliptic curve obtained by confluence of two or more branch points, and rational solutions correspond to a further singularization of the original curve. In either case, the theta function associated with the underlying algebraic curve degenerates into appropriate determinants with exponential, respectively, rational entries.

We use basic techniques from the theory of differential equations, some spectral analysis, and elements of algebraic geometry (most notably, the basic theory of compact Riemann surfaces). In particular, we do not employ more advanced tools such as loop groups, Grassmanians, Lie algebraic considerations, formal pseudo-differential expressions, etc. However, occasionally we bridge the gap to spectral theory and its vicinity and include some finer points of the basic formalism often omitted in this context. Thus, this volume strays off the mainstream, but we hope it appeals to spectral theorists and their kin and convinces them of the beauty of the subject. In particular, we hope a reader interested in quickly penetrating to the fundamentals of the algebro-geometric approach of constructing solutions of hierarchies of completely integrable evolution equations will not be disappointed.

Completely integrable systems, and especially nonlinear evolution equations of soliton-type, are an integral part of modern mathematical and theoretical physics with far-reaching implications from pure mathematics to the applied sciences. We intend to contribute to the dissemination of some of the beautiful techniques applied in this area.

Contents: In the present volume we provide an effective approach to the construction of algebro-geometric solutions of certain completely integrable nonlinear evolution equations by developing a technique that simultaneously applies to all equations of the hierarchy in question.

Starting with a specific integrable partial differential equation, one can build an infinite sequence of higher-order partial differential equations, the so-called hierarchy of the original soliton equation, by developing an explicit recursive formalism that reduces the construction of the entire hierarchy to elementary manipulations with polynomials and defines the associated Lax pairs or zero-curvature equations. Using this recursive polynomial formalism, we simultaneously construct algebro-geometric solutions for the entire hierarchy of soliton equations at hand. On a more technical level, our point of departure for the construction of algebro-geometric solutions is not directly based on Baker–Akhiezer functions and axiomatizations of algebro-geometric data but rather on Dubrovin-type equations, trace formulas, and a canonical meromorphic function ϕ on the underlying hyperelliptic Riemann surface $\mathcal{K}_n$ of genus $n \in \mathbb{N}$. More precisely, this fundamental meromorphic function ϕ carries the spectral information of the underlying Lax operator (such as the Schrödinger and Dirac operators in the KdV and AKNS contexts) and in many instances represents a direct generalization of the Weyl–Titchmarsh m-function, a fundamental device in the spectral theory of ordinary differential operators. Riccati-type differential equations satisfied by ϕ separately in the space and time variables then govern the time evolutions of all quantities of interest (such as that of the associated Baker–Akhiezer vector). The basic meromorphic function ϕ on $\mathcal{K}_n$ is then linked with solutions of equations of the underlying hierarchy via trace formulas and Dubrovin-type equations for (projections of) the pole divisor of ϕ. Subsequently, the Riemann theta function representation of ϕ is then obtained more or less simultaneously with those of the Baker–Akhiezer vector and the algebro-geometric solutions of the (stationary or time-dependent) equations of the hierarchy of evolution equations. This concisely summarizes our approach to all the $(1 + 1)$-dimensional, continuous integrable models discussed in this volume.

In the following we will detail this verbal description of our approach to algebro-geometric solutions of integrable hierarchies with the help of the KdV hierarchy. The latter consists of a sequence of nonlinear evolution equations for a function $u = u(x, t)$, the most prominent element of which, the KdV equation itself, is given by

$$u_t + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x = 0. \tag{0.2}$$

The KdV hierarchy is the simplest of all the hierarchies of nonlinear evolution equations studied in this volume, but the same strategy, with modifications to be discussed in the individual chapters, applies to all integrable systems treated in this monograph and is in fact typical for all $(1 + 1)$-dimensional integrable hierarchies of soliton equations.

A discussion of the KdV case then proceeds as follows.[1] In order to define the Lax pairs and zero-curvature pairs for the KdV hierarchy, one assumes u to be a smooth function on $\mathbb{R}$ (or meromorphic in $\mathbb{C}$) in the stationary context or a smooth function on $\mathbb{R}^2$ in the time-dependent case, and one introduces the recursion relation for some functions f_ℓ of u by

$$f_0 = 1, \quad f_{\ell,x} = -(1/4)f_{\ell-1,xxx} + uf_{\ell-1,x} + (1/2)u_x f_{\ell-1}, \quad \ell \in \mathbb{N}. \tag{0.3}$$

Given the recursively defined sequence $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ (whose elements turn out to be differential polynomials with respect to u defined up to certain integration constants) one defines the Lax pair of the KdV hierarchy by

$$L = -\frac{d^2}{dx^2} + u, \tag{0.4}$$

$$P_{2n+1} = \sum_{\ell=0}^{n} \Big(f_{n-\ell}\frac{d}{dx} - \frac{1}{2} f_{n-\ell,x}\Big) L^\ell. \tag{0.5}$$

The commutator of P_{2n+1} and L then reads[2]

$$[P_{2n+1}, L] = 2f_{n+1,x}, \tag{0.6}$$

using the recursion (0.3). Introducing a deformation (time) parameter[3] $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$ into u, the *KdV hierarchy* of nonlinear evolution equations is then defined by imposing the *Lax commutator relations*

$$\frac{d}{dt_n} L - [P_{2n+1}, L] = 0, \tag{0.7}$$

for each $n \in \mathbb{N}_0$. By (0.6), the latter are equivalent to the collection of evolution equations[4]

$$\mathrm{KdV}_n(u) = u_{t_n} - 2f_{n+1,x}(u) = 0, \quad n \in \mathbb{N}_0. \tag{0.8}$$

Explicitly,

$$\begin{aligned}
\mathrm{KdV}_0(u) &= u_{t_0} - u_x = 0,\\
\mathrm{KdV}_1(u) &= u_{t_1} + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x - c_1u_x = 0,\\
\mathrm{KdV}_2(u) &= u_{t_2} - \tfrac{1}{16}u_{xxxxx} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_xu_{xx} - \tfrac{15}{8}u^2u_x\\
&\qquad + c_1\big(\tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x\big) - c_2u_x = 0, \quad \text{etc.,}
\end{aligned}$$

[1] All details of the following construction are to be found in Chapter 1.

[2] The quantities P_{2n+1} and $\{f_\ell\}_{\ell=0,\dots,n}$ are constructed in such a manner that all differential operators in the commutator (0.6) vanish.

[3] Here we follow Hirota's notation and introduce a separate time variable t_n for the nth level in the KdV hierarchy.

[4] In a slight abuse of notation, we will occasionally stress the functional dependence of f_ℓ on u, writing $f_\ell(u)$.

represent the first few equations of the time-dependent KdV hierarchy. For $n = 1$ and $c_1 = 0$, we obtain *the* KdV equation (0.2). Introducing the polynomials ($z \in \mathbb{C}$),

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^{\ell}, \tag{0.9}$$

$$G_{n-1}(z) = -F_{n,x}(z)/2, \tag{0.10}$$

$$H_{n+1}(z) = (z-u)F_n(z) + (1/2)F_{n,xx}(z), \tag{0.11}$$

one can alternatively introduce the KdV hierarchy as follows. One defines a pair of 2×2 matrices $(U(z), V_{n+1}(z))$ depending polynomially on z by

$$U(z) = \begin{pmatrix} 0 & 1 \\ -z+u & 0 \end{pmatrix}, \tag{0.12}$$

$$V_{n+1}(z) = \begin{pmatrix} G_{n-1}(z) & F_n(z) \\ -H_{n+1}(z) & -G_{n-1}(z) \end{pmatrix}, \tag{0.13}$$

and then postulates the *zero-curvature equation*[1]

$$U_{t_n} - V_{n+1,x} + [U, V_{n+1}] = 0. \tag{0.14}$$

One easily verifies that both the Lax approach (0.8) as well as the zero-curvature approach (0.14) reduce to the basic equation

$$u_{t_n} + (1/2)F_{n,xxx} - 2(u-z)F_{n,x} - u_x F_n = 0. \tag{0.15}$$

Each one of (0.8), (0.14), and (0.15) defines the KdV hierarchy by varying $n \in \mathbb{N}_0$.

The strategy is as follows: We temporarily assume existence of a solution u and derive several of its properties. In particular, we show that u satisfies a trace formula (cf. (0.37) in the stationary case and (0.54) in the time-dependent case) expressed in terms of certain Dirichlet data that satisfy the so-called Dubrovin equations (cf. (0.38) in the stationary case and (0.55) in the time-dependent case), a first-order system of ordinary differential equations that can be shown at least locally to possess solutions. Furthermore, we deduce explicit formulas for the solution u, the so-called Its–Matveev formulas (cf. (0.40) in the stationary case and (0.57) in the time-dependent case).

The Lax and zero-curvature equations (0.7) and (0.14) imply a most remarkable isospectral deformation of L, as will be discussed later in this introduction. At this

[1] Equations $\Psi_x = U\Psi$, $\Psi_{t_n} = V_{n+1}\Psi$ and their compatibility condition (0.14), $U_{t_n} - V_{n+1,x} + [U, V_{n+1}] = 0$ permit a geometrical interpretation as follows: U and V_{n+1} may be considered local connection coefficients in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^2$ with space-time $\mathbb{R}^2$ the base and Ψ taking values in the fiber $\mathbb{C}^2$. The compatibility equation (0.14) then shows that the (U, V_{n+1})-connection has zero-curvature, and hence (0.14) is called a zero-curvature representation of a nonlinear evolution equation.

point, however, we interrupt our time-dependent KdV considerations for a while and take a closer look at the special stationary KdV equations defined by

$$u_{t_n} = 0, \quad n \in \mathbb{N}_0. \tag{0.16}$$

By (0.6)–(0.8) and (0.14), (0.15), the condition (0.16) is then equivalent to each one of the following collection of equations, with n ranging in $\mathbb{N}_0$, which then defines the *stationary KdV hierarchy*,

$$[P_{2n+1}, L] = 0, \tag{0.17}$$

$$f_{n+1,x} = 0, \tag{0.18}$$

$$-V_{n+1,x} + [U, V_{n+1}] = 0, \tag{0.19}$$

$$(1/2)F_{n,xxx} - 2(u-z)F_{n,x} - u_x F_n = 0. \tag{0.20}$$

To set the stationary KdV hierarchy apart from the general time-dependent one, we will denote it by

$$\text{s-KdV}_n(u) = -2f_{n+1,x}(u) = 0, \quad n \in \mathbb{N}_0.$$

Explicitly, the first few equations of the stationary KdV hierarchy then read as follows

$$\begin{aligned}
\text{s-KdV}_0(u) &= -u_x = 0, \\
\text{s-KdV}_1(u) &= \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x - c_1 u_x = 0, \\
\text{s-KdV}_2(u) &= -\tfrac{1}{16}u_{xxxxx} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_x u_{xx} - \tfrac{15}{8}u^2 u_x \\
&\qquad + c_1\left(\tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x\right) - c_2 u_x = 0, \quad \text{etc.}
\end{aligned}$$

The class of *algebro-geometric* KdV potentials, by definition, equals the set of solutions u of the stationary KdV hierarchy. In the following analysis we fix the value of n in (0.17)–(0.20), and hence we now turn to the investigation of algebro-geometric solutions u of the nth equation within the stationary KdV hierarchy. Equation (0.17) is of special interest because, by a 1923 result of Burchnall and Chaundy, commuting differential expressions (due to a common eigenfunction to be discussed below, cf. (0.33), (0.34)) give rise to an algebraic relationship between the two differential expressions. Similarly, (0.19) permits the important conclusion that

$$\partial_x \det(yI_2 - iV_{n+1}(z,x)) = 0 \tag{0.21}$$

and hence

$$\begin{aligned}
\det(yI_2 - iV_{n+1}(z,x)) &= y^2 - \det(V_{n+1}(z,x)) \\
&= y^2 + G_{n-1}(z,x)^2 - F_n(z,x)H_{n+1}(z,x) = y^2 - R_{2n+1}(z)
\end{aligned} \tag{0.22}$$

for some x-independent monic polynomial R_{2n+1}, which we write as

$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m) \text{ for some } \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}.$$

In particular, the combination

$$F_n(z,x)H_{n+1}(z,x) - G_{n-1}(z,x)^2 = R_{2n+1}(z) \tag{0.23}$$

is x-independent. Moreover, (0.20) can easily be integrated to yield

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2 = R_{2n+1} \tag{0.24}$$

with precisely the same integration constant $R_{2n+1}(z)$ as in (0.22). In fact, by (0.10) and (0.11), equations (0.23) and (0.24) are simply identical. Incidentally, the algebraic relationship between L and P_{2n+1} alluded to in connection with the vanishing of their commutator in (0.17) can be made precise as follows: Restricting P_{2n+1} to the (algebraic) kernel $\ker(L-z)$ of $L-z$, one computes, using (0.5) and (0.24),

$$\begin{aligned}\big(P_{2n+1}\big|_{\ker(L-z)}\big)^2 &= -\left(\frac{1}{2}F_{n,xx}F_n - \frac{1}{4}F_{n,x}^2 - (u-z)F_n^2\right)\bigg|_{\ker(L-z)}\\ &= -R_{2n+1}(L)\big|_{\ker(L-z)}.\end{aligned}$$

Thus, one concludes that P_{2n+1}^2 and $-R_{2n+1}(L)$ coincide on $\ker(L-z)$, and since $z \in \mathbb{C}$ is arbitrary, one infers that

$$P_{2n+1}^2 + R_{2n+1}(L) = 0 \tag{0.25}$$

holds once again with the same polynomial R_{2n+1}. The characteristic equation of iV_{n+1} (cf. (0.22)) and (0.25) naturally lead one to the introduction of the *hyperelliptic curve* $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) defined by

$$\mathcal{K}_n\colon \mathcal{F}_n(z,y) = y^2 - R_{2n+1}(z) = 0, \quad R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m). \tag{0.26}$$

We compactify the curve by adding the point P_∞ (still denoting it by $\mathcal{K}_n$ for simplicity) and note that points P on the curve are denoted by $P = (z,y) \in \mathcal{K}_n \setminus \{P_\infty\}$, where $y(\,\cdot\,)$ is a meromorphic function on $\mathcal{K}_n$ satisfying[1] $y^2 - R_{2n+1}(z) = 0$. For simplicity, we will assume in the following that the (affine part of the) curve $\mathcal{K}_n$ is nonsingular, that is, the zeros E_m of R_{2n+1} are all simple. Remaining within the stationary framework a bit longer, one can now introduce the fundamental meromorphic function ϕ on $\mathcal{K}_n$ alluded to earlier as follows,

$$\phi(P,x) = \frac{iy - G_{n-1,x}(z,x)}{F_n(z,x)} \tag{0.27}$$

$$= \frac{-H_{n+1}(z,x)}{iy + G_{n-1,x}(z,x)}, \quad P = (z,y) \in \mathcal{K}_n. \tag{0.28}$$

[1] For more details, refer to Appendix B and Chapter 1.

Equality of the two expressions (0.27) and (0.28) is an immediate consequence of the identity (0.23) and the fact $y^2 = R_{2n+1}(z)$. A comparison with (0.19) then readily reveals that ϕ satisfies the Riccati-type equation

$$\phi_x + \phi^2 = u - z. \tag{0.29}$$

The next step is crucial. It concerns the zeros and poles of ϕ and hence involves the zeros of $F_n(\,\cdot\,, x)$ and $H_{n+1}(\,\cdot\,, x)$. Isolating the latter by introducing the factorizations

$$F_n(z, x) = \prod_{j=1}^{n}(z - \mu_j(x)), \quad H_{n+1}(z, x) = \prod_{\ell=0}^{n}(z - \nu_\ell(x)),$$

one can use the zeros of F_n and H_{n+1} to define the following points $\hat{\mu}_j(x)$, $\hat{\nu}_\ell(x)$ on $\mathcal{K}_n$,

$$\hat{\mu}_j(x) = (\mu_j(x), iG_{n-1,x}(\mu_j(x), x)), \quad j = 1, \dots, n, \tag{0.30}$$

$$\hat{\nu}_\ell(x) = (\nu_\ell(x), -iG_{n-1,x}(\nu_\ell(x), x)), \quad \ell = 0, \dots, n. \tag{0.31}$$

The motivation for this choice stems from $y^2 = R_{2n+1}(z)$ by (0.22), the identity (0.23) (which combines to $F_n H_{n+1} - G_{n-1}^2 = y^2$), and a comparison of (0.27) and (0.28). Given (0.27)–(0.31), one obtains for the divisor $(\phi(\,\cdot\,, x))$ of the meromorphic function ϕ

$$(\phi(\,\cdot\,, x)) = \mathcal{D}_{\hat{\nu}_0(x)\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}. \tag{0.32}$$

Here we abbreviated $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \mathrm{Sym}^n(\mathcal{K}_n)$, with $\mathrm{Sym}^n(\mathcal{K}_n)$ the nth symmetric product of $\mathcal{K}_n$, and used our conventions[1] (A.43), (A.47), and (A.48) to denote positive divisors of degree n and $n+1$ on $\mathcal{K}_n$. Given $\phi(\,\cdot\,, x)$, one defines the *stationary Baker–Akhiezer vector* $\Psi(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_\infty\}$ by

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi_1(P, x, x_0) = \exp\left(\int_{x_0}^{x} dx'\, \phi(P, x')\right), \quad \psi_2 = \psi_{1,x}.$$

In particular, this implies

$$\phi = \psi_2/\psi_1$$

and the following normalization[2] of ψ_1, $\psi_1(P, x_0, x_0) = 1$, $P \in \mathcal{K}_n \setminus \{P_\infty\}$. The Riccati-type equation (0.29) satisfied by ϕ then shows that the Baker–Akhiezer

[1] $\mathcal{D}_{\underline{Q}}(P) = m$ if P occurs m times in $\{Q_1, \dots, Q_n\}$ and zero otherwise, $\underline{Q} = \{Q_1, \dots, Q_n\} \in \mathrm{Sym}^n(\mathcal{K}_n)$. Similarly, $\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}$, $\mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_n}$, $Q_0 \in \mathcal{K}_n$, and $\mathcal{D}_Q(P) = 1$ for $P = Q$ and zero otherwise.

[2] This normalization is less innocent than it might appear at first sight. It implies that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ are the divisors of zeros and poles of $\psi_1(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_\infty\}$.

function ψ_1 is the common formal eigenfunction of the commuting pair of Lax differential expressions L and P_{2n+1},

$$L\psi_1(P) = z\psi_1(P), \tag{0.33}$$

$$P_{n+1}\psi_1(P) = iy\psi_1(P), \quad P = (z, y), \tag{0.34}$$

and at the same time the Baker–Akhiezer vector Ψ satisfies the zero-curvature equations,

$$\Psi_x(P) = U(z)\Psi(P), \tag{0.35}$$

$$iy\Psi(P) = V_{n+1}(z)\Psi(P), \quad P = (z, y). \tag{0.36}$$

Moreover, one easily verifies that away from the (finite) branch points $(E_m, 0)$, $m = 0, \dots, 2n$, of the two-sheeted Riemann surface $\mathcal{K}_n$, the two branches of ψ_1 constitute a fundamental system of solutions of (0.33) and similarly, the two branches of Ψ yield a fundamental system of solutions of (0.35). Since $\psi_1(\,\cdot\,, x, x_0)$ vanishes at $\hat{\mu}_j(x)$, $j = 1, \dots, n$ and $\psi_2(\,\cdot\,, x, x_0) = \psi_{1,x}(\,\cdot\,, x, x_0)$ vanishes at $\hat{\nu}_\ell(x)$, $\ell = 0, \dots, n$, we may call $\{\hat{\mu}_j(x)\}_{j=1,\dots,n}$ and $\{\hat{\nu}_\ell(x)\}_{\ell=0,\dots,n}$ the *Dirichlet* and *Neumann data* of L at the point $x \in \mathbb{R}$, respectively.

Now the stationary formalism is almost complete; we only need to relate the solution u of the nth stationary KdV equation and $\mathcal{K}_n$-associated data. This can be accomplished in several ways. We describe two of them next.

First we relate u and the zeros μ_j of F_n. This is easily done by comparing the coefficients of the power z^{2n} in (0.24) and results in the *trace formula*,

$$u = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j. \tag{0.37}$$

Next we will indicate how to reconstruct (at least locally) u from Dirichlet data at just one fixed point x_0. Combining the definition (0.30) of $\hat{\mu}_j$ and that of G_{n-1} in (0.10) yields, after a comparison with the x-derivative of $F_n(z, x) = \prod_{k=1}^n (z - \mu_k(x))$,

$$\begin{aligned} y(\hat{\mu}_j(x)) &= iG_{n-1}(\mu_j(x), x) = -(i/2)F_{n,x}(\mu_j(x), x) \\ &= (i/2)\mu_{j,x}(x) \prod_{\substack{k=1 \\ k\neq j}}^{n} (\mu_j(x) - \mu_k(x)), \quad j = 1, \dots, n. \end{aligned}$$

Hence, one arrives at the *Dubrovin equations* for $\hat{\mu}_j$, an autonomous first-order system of differential equations on $\mathcal{K}_n$,

$$\mu_{j,x} = -2iy(\hat{\mu}_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} \left(\mu_j - \mu_k\right)^{-1}, \quad j = 1, \dots, n. \tag{0.38}$$

Augmenting (0.38) with appropriate initial data

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{0.39}$$

for some $x_0 \in \mathbb{R}$, with $\mu_j(x_0)$, $j = 1, \dots, n$ assumed to be distinct, one can solve the Dubrovin system (0.38) at least locally[1] in a neighborhood of the point x_0 and then reconstruct u in that neighborhood using the trace formula (0.37). In other words, the Dirichlet data $\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n}$ in (0.39) at the point x_0 can be used to reconstruct u in a neighborhood of x_0. Since u can be shown to be meromorphic, this uniquely determines u (even though it is not necessarily clear from our discussion thus far how to reconstruct u globally). Furthermore, u satisfies $\text{s-KdV}_n(u) = 0$.

An alternative reconstruction of u, nicely complementing the one just discussed, can be given with the help of the *Riemann theta function*[2] associated with $\mathcal{K}_n$ and an appropriate homology basis of cycles on it. The known zeros and poles of ϕ (cf. (0.32)), and similarly, the set of zeros $\{\hat{\mu}_j(x)\}_{j=1,\dots,n}$ and poles $\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n}$ of the Baker–Akhiezer function $\psi_1(\,\cdot\,, x, x_0)$ together with the characteristic essential singularity of ψ_1 at P_∞, then permit one to find theta function representations for ϕ and ψ_1 by alluding to Riemann's vanishing theorem and the Riemann–Roch theorem.[3] The corresponding theta function representation of the algebro-geometric solution u of the nth stationary KdV equation then can be obtained from that of ψ_1 by an asymptotic expansion with respect to the spectral parameter near the point P_∞. Alternatively, one can use the trace formula (0.37) and apply the known theta function representations for symmetric functions of the projections $\mu_j(x)$ of the zeros $\hat{\mu}_j(x)$ of ψ_1 to the special case $\sum_{j=1}^n \mu_j(x)$ at hand. Either way, the resulting final expression for u, called the *Its–Matveev formula*, is of the type

$$u(x) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x)). \tag{0.40}$$

Here the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B} \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ (and its homology basis), and the constant $\underline{A} \in \mathbb{C}^n$ (related to the Abel map of the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$) is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \operatorname{Sym}^n(\mathcal{K}_n)$ at the point x_0 as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ is assumed to be nonspecial.[4] Moreover, the theta function representation (0.40) remains valid as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ stays nonspecial. We emphasize the remarkable fact that the argument of the theta function in (0.40) is linear with respect to x.

[1] In some situations, such as the case of periodic u, it is possible to elevate this procedure to a global reconstruction of u even in the presence of collisions of $\hat{\mu}_j$ on $\mathcal{K}_n$. But this requires an extensive analysis we mention in the notes to Appendix F.

[2] For details on the n-dimensional theta function $\theta(\underline{z})$, $\underline{z} \in \mathbb{C}^n$, we refer to Appendices A and B.

[3] We defer the analogous discussion of ψ_2 to Chapter 1 for simplicity.

[4] If $\mathcal{D} = n_1\mathcal{D}_{Q_1} + \cdots + n_k\mathcal{D}_{Q_k} \in \operatorname{Sym}^n(\mathcal{K}_n)$ for some $n_\ell \in \mathbb{N}$, $\ell = 1, \dots, k$, with $n_1 + \cdots + n_k = n$, then $\mathcal{D}$ is called nonspecial if there is no nonconstant meromorphic function on $\mathcal{K}_n$ that is holomorphic on $\mathcal{K}_n \setminus \{Q_1, \dots, Q_k\}$ with poles at most of order n_ℓ at Q_ℓ, $\ell = 1, \dots, k$.

The current discussion assumed that one started with a solution u of the nth stationary KdV equation and then either reconstructed it from the trace formula (0.37), or represented the given u in terms of the theta function associated with $\mathcal{K}_n$, as in (0.40). In addition to this procedure we also solve the following inverse problem: Given appropriate initial data (0.39) and solutions $\hat{\mu}_1(x), \dots, \hat{\mu}_n(x)$ of the first-order Dubrovin system (0.38) on an open interval $\Omega \subseteq \mathbb{R}$ containing the point x_0, we will define u on Ω in terms of the trace formula (0.37) and then prove that u so defined satisfies the nth stationary KdV equation on Ω.

This completes our somewhat lengthy excursion into the stationary KdV hierarchy. In the following we return to the time-dependent KdV hierarchy and describe the analogous steps involved to construct solutions $u = u(x, t_r)$ of the rth KdV equation with initial values being algebro-geometric solutions of the nth stationary KdV equation. More precisely, we are seeking a solution u of the following algebro-geometric initial value problem

$$\widetilde{\mathrm{KdV}}_r(u) = u_{t_r} - 2\tilde{f}_{r+1,x}(u) = 0, \quad u|_{t_r=t_{0,r}} = u^{(0)}, \tag{0.41}$$

$$\text{s-KdV}_n\big(u^{(0)}\big) = -2f_{n+1,x}\big(u^{(0)}\big) = 0 \tag{0.42}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$ and a fixed curve $\mathcal{K}_n$ associated with the stationary solution $u^{(0)}$ in (0.42).

We pause for a moment to reflect on the pair of equations (0.41), (0.42): As it turns out, they represent a dynamical system on the set of algebro-geometric solutions isospectral to the initial value $u^{(0)}$. The term *isospectral* here alludes to the fact that for any fixed t_r, the solution $u(\,\cdot\,, t_r)$ of (0.41), (0.42) is a stationary solution of (0.42),

$$\text{s-KdV}_n(u(\,\cdot\,, t_r)) = -2f_{n+1,x}(u(\,\cdot\,, t_r)) = 0$$

associated with the fixed underlying algebraic curve $\mathcal{K}_n$. Put differently, $u(\,\cdot\,, t_r)$ is an isospectral deformation of $u^{(0)}$ with t_r the corresponding deformation parameter. In particular, $u(\,\cdot\,, t_r)$ traces out a curve in the set of algebro-geometric solutions isospectral to $u^{(0)}$.

Since the integration constants in the functionals f_ℓ of u in the stationary and time-dependent contexts are independent of each other, we indicate this by placing a tilde over all the time-dependent quantities. Hence, we will employ the notation $\widetilde{P}_{2r+1}$, $\widetilde{V}_{r+1}$, $\widetilde{F}_r$, etc., to distinguish them from P_{2n+1}, V_{n+1}, F_n, etc. Thus, $\widetilde{P}_{2r+1}$, $\widetilde{V}_{r+1}$, $\widetilde{F}_r$, $\widetilde{H}_{r+1}$, $\tilde{f}_s$ are constructed in the same way as P_{2n+1}, V_{n+1}, F_n, H_n, f_ℓ using the recursion (0.3) with the only difference being that the set of integration constants $\tilde{c}_r$ in $\tilde{f}_s$ is independent of the set c_k used in computing f_ℓ.

Our strategy will be the same as in the stationary case: Assuming existence of a solution u, we will deduce many of its properties, which, in the end, will yield an explicit expression for the solution. In fact, we will go a step further, postulating the equations

$$u_{t_r} = -(1/2)\widetilde{F}_{r,xxx} + 2(u-z)\widetilde{F}_{r,x} + u_x\widetilde{F}_r, \tag{0.43}$$

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2 = R_{2n+1}, \tag{0.44}$$

where $u^{(0)} = u^{(0)}(x)$ in (0.42) has been replaced by $u = u(x, t_r)$ in (0.44). Here,

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell}z^\ell = \prod_{j=1}^{n}(z-\mu_j), \quad \widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s}z^s$$

for fixed $n, r \in \mathbb{N}_0$. Introducing G_{n-1}, H_{n+1}, U, V_{n+1} and $\widetilde{G}_{r-1}$, $\widetilde{H}_{r+1}$, $\widetilde{V}_{r+1}$ (replacing F_n by $\widetilde{F}_r$) as in (0.10)–(0.13), we observe that the basic equations (0.43), (0.44) are equivalent to the *Lax equations*

$$\begin{aligned} \frac{d}{dt_r}L - [\widetilde{P}_{2r+1}, L] = 0,\\ [P_{2n+1}, L] = 0, \end{aligned}$$

and to the *zero-curvature equations*

$$U_{t_r} - \widetilde{V}_{r+1,x} + [U, \widetilde{V}_{r+1}] = 0, \tag{0.45}$$

$$-V_{n+1,x} + [U, V_{n+1}] = 0. \tag{0.46}$$

Moreover, one computes in analogy to (0.21) and (0.22) that

$$\begin{aligned} \partial_x \det(yI_2 - iV_{n+1}(z,x,t_r)) = 0,\\ \partial_{t_r} \det(yI_2 - iV_{n+1}(z,x,t_r)) = 0, \end{aligned}$$

and hence

$$\begin{aligned} &\det(yI_2 - iV_{n+1}(z,x,t_r)) = y^2 - \det(V_{n+1}(z,x,t_r))\\ &\quad = y^2 + G_{n-1}(z,x,t_r)^2 - F_n(z,x,t_r)H_{n+1}(z,x,t_r) = y^2 - R_{2n+1}(z) \end{aligned} \tag{0.47}$$

is independent of (x, t_r). Thus,

$$F_nH_{n+1} - G_{n-1}^2 = R_{2n+1}, \tag{0.48}$$

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2 = R_{2n+1} \tag{0.49}$$

hold as in the stationary context. The independence of (0.47) of t_r can be interpreted as follows: The rth KdV flow represents an isospectral deformation of the curve $\mathcal{K}_n$ defined in (0.26); in particular,[1] the branch points of $\mathcal{K}_n$ remain invariant under

[1] Property (0.50) is weaker than the usually stated isospectral deformation of the Lax operator $L(t_r)$. However, the latter is a more delicate functional analytic problem marred by possible singularities of u and possible non-self-adjointness of $L(t_r)$.

these flows,

$$\partial_{t_r} E_m = 0, \quad m = 0, \dots, 2n. \tag{0.50}$$

As in the stationary case, one can now introduce the basic meromorphic function ϕ on $\mathcal{K}_n$ by

$$\begin{aligned}\phi(P, x, t_r) &= \frac{iy - G_{n-1}(z, x, t_r)}{F_n(z, x, t_r)} \\ &= \frac{-H_{n+1}(z, x, t_r)}{iy + G_{n-1,x}(z, x, t_r)}, \quad P = (z, y) \in \mathcal{K}_n,\end{aligned}$$

and a comparison with (0.45) and (0.46) then shows that ϕ satisfies the Riccati-type equations

$$\phi_x + \phi^2 = u - z, \tag{0.51}$$

$$\phi_{t_r} = \partial_x\big(\widetilde{F}_r \phi + \widetilde{G}_{r-1}\big) = -\widetilde{F}_r \phi^2 - 2\widetilde{G}_{r-1}\phi - \widetilde{H}_r. \tag{0.52}$$

Next, factorizing F_n and H_{n+1} as before,

$$F_n(z, x, t_r) = \prod_{j=1}^{n}(z - \mu_j(x, t_r)), \quad H_{n+1}(z, x, t_r) = \prod_{\ell=0}^{n}(z - \nu_\ell(x, t_r)),$$

one introduces points $\hat{\mu}_j(x, t_r)$, $\hat{\nu}_\ell(x, t_r)$ on $\mathcal{K}_n$ by

$$\begin{aligned}\hat{\mu}_j &= (\mu_j, iG_{n-1,x}(\mu_j)), \quad j = 1, \dots, n, \\ \hat{\nu}_\ell &= (\nu_\ell, -iG_{n-1,x}(\nu_\ell)), \quad \ell = 0, \dots, n\end{aligned}$$

and obtains for the divisor $(\phi(\,\cdot\,, x, t_r))$ of the meromorphic function ϕ

$$(\phi(\,\cdot\,, x, t_r)) = \mathcal{D}_{\hat{\nu}_0(x,t_r)\underline{\hat{\nu}}(x,t_r)} - \mathcal{D}_{P_\infty \underline{\hat{\mu}}(x,t_r)},$$

as in the stationary context. Given $\phi(\,\cdot\,, x, t_r)$, one then defines the *time-dependent Baker–Akhiezer vector* $\Psi(\,\cdot\,, x, x_0, t_r, t_{0,r})$ on $\mathcal{K}_n \setminus \{P_\infty\}$ by

$$\begin{aligned}&\Psi = \begin{pmatrix}\psi_1 \\ \psi_2\end{pmatrix}, \\ &\psi_1(P, x, x_0, t_r, t_{0,r}) = \exp\bigg(\int_{t_{0,r}}^{t_r} ds\,(\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \widetilde{G}_{r-1}(z, x_0, s)) \\ &\qquad\qquad + \int_{x_0}^{x} dx'\,\phi(P, x', t_r)\bigg), \\ &\psi_2 = \psi_{1,x}.\end{aligned}$$

The Riccati-type equations (0.51), (0.52) satisfied by ϕ then show that

$$-V_{n+1,t_r} + [\widetilde{V}_{r+1}, V_{n+1}] = 0 \tag{0.53}$$

in addition to (0.45), (0.46). Moreover, they yield again that the Baker–Akhiezer function ψ_1 is the common formal eigenfunction of the commuting pair of Lax differential expressions $L(t_r)$ and $P_{2n+1}(t_r)$,

$$\begin{aligned} L\psi_1(P) &= z\psi_1(P), \\ P_{n+1}\psi_1(P) &= iy\psi_1(P), \\ \psi_{t_r}(P) &= \widetilde{P}_{2r+1}\psi(P) \\ &= \widetilde{F}_r(z)\psi_x(P) + \widetilde{G}_{r-1}(z)\psi(P), \quad P=(z,y), \end{aligned}$$

and at the same time the Baker–Akhiezer vector Ψ satisfies the zero-curvature equations

$$\begin{aligned} \Psi_x(P) &= U(z)\Psi(P), \\ iy\Psi(P) &= V_{n+1}(z)\Psi(P), \\ \Psi_{t_r}(P) &= \widetilde{V}_{r+1}(z)\Psi(P), \quad P=(z,y). \end{aligned}$$

The remaining time-dependent constructions closely follow our stationary outline. First one notes again the *trace formula*

$$u(x,t_r) = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j(x,t_r) \tag{0.54}$$

as a consequence of (0.49). Next, to reconstruct u (locally) from *Dirichlet data* at just one fixed point $(x_0, t_{0,r})$, one derives the *Dubrovin equations*[1]

$$\begin{aligned} \mu_{j,x} &= -2iy(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j-\mu_k)^{-1}, \\ \mu_{j,t_r} &= -2i\widetilde{F}_r(\mu_j)y(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j-\mu_k)^{-1}, \end{aligned} \tag{0.55}$$

using (0.44), and (0.53) for F_{n,t_r}, as in the stationary case. Augmenting (0.55) with appropriate initial data

$$\{\hat{\mu}_j(x_0,t_{0,r})\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{0.56}$$

for some $(x_0, t_{0,r}) \in \mathbb{R}^2$, with $\mu_j(x_0,t_{0,r})$, $j=1,\dots,n$ assumed to be distinct, one can again solve the Dubrovin system (0.55), at least locally in a neighborhood of the point $(x_0,t_{0,r})$, and then reconstruct u in that neighborhood using the trace formula (0.54). In other words, the Dirichlet data $\{\hat{\mu}_j(x_0,t_{0,r})\}_{j=1,\dots,n}$ in (0.56) at the point $(x_0,t_{0,r})$ reconstruct u in a neighborhood of $(x_0,t_{0,r})$.

The corresponding representations of u, ϕ, and Ψ in terms of the *Riemann theta function* associated with $\mathcal{K}_n$ is then obtained in close analogy to the stationary case. Particularly, in the case of u, one obtains the *Its–Matveev formula*

$$u(x,t_r) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r)), \tag{0.57}$$

[1] To obtain a closed system of differential equations, one has to express $\widetilde{F}_r(\mu_j)$ solely in terms of $\mu_1,\dots,\mu_n$ and $E_0,\dots,E_{2n+1}$; see (1.222) and (1.223).

where the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B}, \underline{C}_r \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ and r, and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0, t_{0,r}) = (\hat{\mu}_1(x_0, t_{0,r}), \ldots, \hat{\mu}_n(x_0, t_{0,r})) \in \mathrm{Sym}^n(\mathcal{K}_n)$ at the point $(x_0, t_{0,r})$ as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0, t_{0,r})}$ is assumed to be nonspecial. Moreover, the theta function representation (0.57) remains valid as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x, t_r)}$ stays nonspecial. Again one notes the remarkable fact that the argument of the theta function in (0.57) is linear with respect to both x and t_r.

Again, the current discussion assumed one started with a solution u of the KdV initial value problem (0.41), (0.42) and then either reconstructed it from the trace formula (0.54) or represented the given u in terms of the theta function associated with $\mathcal{K}_n$, as in (0.57). In addition to this procedure we also solve the following inverse problem: Given appropriate initial data (0.56) and solutions $\hat{\mu}_1(x, t_r), \ldots, \hat{\mu}_n(x, t_r)$ of the first-order Dubrovin system (0.55) on a connected open set $\Omega \subseteq \mathbb{R}^2$ containing the point $(x_0, t_{0,r})$, we will define u on Ω in terms of the trace formula (0.54) and then prove that u so defined satisfies the KdV initial value problem (0.41), (0.42) on Ω.

The reader will have noticed that we used terms such as *integrability, soliton equations, isospectral deformations*, etc., without offering a precise definition for them. Arguably, an integrable system in connection with nonlinear evolution equations should possess several properties, including, for instance,

- infinitely many conservation laws
- isospectral deformations of a Lax operator
- action-angle variables, Hamiltonian formalism
- algebraic (spectral) curves
- infinitely many symmetries and transformation groups
- "explicit" solutions.

Although many of these properties apply to particular systems of interest, there is simply no generally accepted definition to date of what constitutes an integrable system.[1] That explicit but meromorphic (i.e., singular) solutions of systems such as the KdV hierarchy abound and local integrability of conserved densities as well as the functional analytic meaning of the Lax operator and its isospectral deformations in appropriate spaces are not obvious makes it plausible that no universally accepted notion of integrability can be achieved. Thus, different schools have necessarily introduced different shades of integrability (Liouville integrability, analytic integrability, algebraically complete integrability, etc.); in this monograph we found it useful to focus on the existence of underlying algebraic curves and explicit representations of solutions in terms of corresponding Riemann theta functions and limiting situations thereof.

[1] This has been eloquently discussed in Hitchin et al. (1999, p. 1ff). Most appropriate in this context seems Cherednik's statement, "All non-integrable equations are non-integrable the same way, all integrable ones are integrable in their own way," in the preface to Cherednik (1996).

Finally, a brief discussion of the content of each chapter is in order (additional details are collected in the list of contents at the beginning of each chapter). Chapter 1 is devoted to the KdV hierarchy and its algebro-geometric solutions. We have just given a fairly detailed outline of the KdV theory, and hence it suffices to mention that, in addition to that material, we will provide a general approach to trace formulas for Schrödinger operators L that is not restricted to the case of algebro-geometric potentials u. Throughout that chapter we often isolate the special case where u is real-valued and then describe the spectral theoretic properties of algebro-geometric Schrödinger operators. In Chapter 2 we turn to the sine–Gordon (sG) equation. In fact, we describe the algebro-geometric approach to a particular hierarchy of nonlinear evolution equations that link the sine–Gordon equation and the modified Korteweg–de Vries (mKdV) hierarchy, which we call the sGmKdV hierarchy. Next, in Chapter 3, we consider the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy (a complexified nonlinear Schrödinger (nS) hierarchy) of evolution equations and its algebro-geometric solutions. Employing the gauge equivalence of the AKNS and classical Boussinesq (cBsq) hierarchies, we also derive the algebro-geometric cBsq solutions. Chapter 4, devoted to the classical massive Thirring system (a complexified classical massive Thirring model), is somewhat of an exception, for we restrict ourselves to the basic equation itself and refrain from a discussion of the corresponding hierarchy. In our final chapter, Chapter 5, we discuss the algebro-geometric approach to the Camassa–Holm (CH) hierarchy – a hierarchy whose higher elements define nonlocal evolution equations with respect to u.

Presentation: Each chapter, together with appropriate appendices compiled in the second part of this volume, is intended to be essentially self-contained and hence can be read independently from the remaining chapters. Occasionally we provide more detail in the KdV chapter since it is the first and principal one in this volume and by far the simplest with respect to the complexity of the whole formalism involved. This attempt to organize chapters independently of one another comes at a price, of course: Similar arguments in the construction of algebro-geometric solutions for different hierarchies are repeated in different chapters. We believe this makes the results more easily accessible.

References are deferred to detailed notes for each section at the end of every chapter. In addition to comprehensive bibliographical documentation of the material dealt with in the main text, these notes also contain numerous additional comments and results.

Succinctly written appendices, many of which summarize subjects of interest on their own, such as compact (and particularly hyperelliptic) Riemann surfaces, Darboux transformations, elliptic functions, Weyl–Titchmarsh theory for second-order differential operators, and associated Herglotz functions, guarantee a fairly self-contained presentation accessible at the advanced graduate level.

An extensive bibliography is included at the end of this volume. Its size reflects the enormous interest this subject generated over the past three decades. It underscores the wide variety of techniques employed to study completely integrable systems. Even though we undertook every effort to provide an exhaustive list of references, the result in the end must necessarily be considered incomplete. We regret any omissions that have occurred. Publications with three or more authors are abbreviated with "First author et al. (year)" in the text. If more than one publication yield the same abbreviation, latin letters a,b,c, etc., are added after the year. In the bibliography, publications are alphabetically ordered using all authors' names and year of publication.

Future Projects: Volume II in our series will be devoted to $(1 + 1)$-dimensional lattice models associated with hyperelliptic curves and include the Toda, Kac–van Moerbeke, and variants of the Ablowitz–Ladik hierarchies. A subsequent project will treat completely integrable equations associated with non-hyperelliptic curves, including the Boussinesq and Gelfand–Dickey hierarchies in $(1 + 1)$-dimensions, the multi-dimensional Davey–Stewartson and Kadomtsev–Petviashvili equations, and certain systems associated with curves of infinite genus.

1
The KdV Hierarchy

> Im folgenden führe ich den Leser auf dem von mir selbst zurückgelegten, etwas indirekten und holperigen Wege, weil ich nur so hoffen kann, daß er dem Endergebnis Interesse entgegenbringe.
>
> *Albert Einstein*[1]

1.1 Contents

The Korteweg–de Vries (KdV) equation

$$u_t + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x = 0$$

for a function $u = u(x, t)$ with its origins in fluid dynamics has a long and interesting history,[2] but this chapter focuses on a relatively recent development since the mid-1970s, the construction of algebro-geometric solutions of the KdV hierarchy. Below we briefly summarize the principal content of each section. A more detailed discussion of the contents has been provided in the introduction to this volume.

Section 1.2.

- polynomial recursion formalism, Lax pairs (L, P_{2n+1})
- stationary and time-dependent KdV hierarchy
- Burchnall–Chaundy polynomial, hyperelliptic curve $\mathcal{K}_n$

Section 1.3. (stationary)

- properties of ϕ and the Baker–Akhiezer function ψ
- Dubrovin equations for Dirichlet, Neumann, and other auxiliary divisors

[1] Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie, *Königl. Preuß. Akad. Wissensch. (Berlin), Sitzungsber.* (1917), 142–152. ("In the present paragraph I shall conduct the reader over the road that I have myself traveled, rather a rough and winding road, because otherwise I cannot hope that he will take much interest in the result at the end of the journey.")

[2] A guide to the literature can be found in the detailed notes at the end of this chapter.

- trace formulas for u and higher-order KdV invariants
- theta function representations for ϕ, ψ, the Its–Matveev formula for u
- the algebro-geometric initial value problem
- examples

Section 1.4. (time-dependent)

- properties of ϕ and the Baker–Akhiezer function ψ
- Dubrovin equations for Dirichlet, Neumann, and other auxiliary divisors
- trace formulas for u and higher-order KdV invariants
- theta function representations for ϕ, ψ, the Its–Matveev formula for u
- the algebro-geometric initial value problem
- examples

Section 1.5. (trace formulas)

- general boundary conditions
- KdV invariants
- asymptotic spectral parameter expansions of Green's functions
- spectral shift function
- general trace formulas for KdV invariants

This chapter relies on terminology and notions developed in connection with compact Riemann surfaces. A brief summary of key results as well as definitions of some of the main quantities can be found in Appendices A, B, and F. Occasionally, we also draw from the spectral theory of Schrödinger operators, and some of the relevant material is summarized in Appendices G, I, and J.

1.2 The KdV Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section we provide the construction of the KdV hierarchy using a polynomial recursion formalism and derive the associated sequence of KdV Lax pairs. Moreover, we discuss the Burchnall–Chaundy polynomial in connection with the stationary KdV hierarchy and the underlying hyperelliptic curve.

Throughout this section we suppose the following hypothesis.

Hypothesis 1.1 *In the stationary case we assume that $u\colon \mathbb{R} \to \mathbb{C}$ is smooth,*[1] *that is,*

$$u \in C^\infty(\mathbb{R}). \tag{1.1}$$

In the time-dependent case we suppose that $u\colon \mathbb{R}^2 \to \mathbb{C}$ satisfies[2]

$$u(\,\cdot\,,t) \in C^\infty(\mathbb{R}),\ t \in \mathbb{R}, \quad u(x,\,\cdot\,) \in C^1(\mathbb{R}),\ x \in \mathbb{R}. \tag{1.2}$$

[1] Alternatively, one could suppose that $u\colon \mathbb{C} \to \mathbb{C}_\infty$ is meromorphic.
[2] Again one could assume that for fixed $t \in \mathbb{R}$, $u(\,\cdot\,,t)$ is meromorphic, etc.

Actually, up to (1.20) our analysis will be time-independent, and hence only the space variation of u will matter. Consider the one-dimensional second-order differential expression

$$L = -\frac{d^2}{dx^2} + u \tag{1.3}$$

of Schrödinger-type. To construct the KdV hierarchy we need a second differential expression of order $2n+1$, denoted by P_{2n+1}, $n \in \mathbb{N}_0$, defined recursively in the following. We take the quickest route to the construction of P_{2n+1} and hence to that of the KdV hierarchy by starting from the recursion relation (1.4) below. Subsequently, we will offer the motivation behind this approach (cf. Remark 1.4).

Define $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$f_0 = 1, \quad f_{\ell,x} = -(1/4)f_{\ell-1,xxx} + uf_{\ell-1,x} + (1/2)u_x f_{\ell-1}, \quad \ell \in \mathbb{N}. \tag{1.4}$$

Explicitly, one finds

$$\begin{aligned} f_0 &= 1, \\ f_1 &= \tfrac{1}{2}u + c_1, \\ f_2 &= -\tfrac{1}{8}u_{xx} + \tfrac{3}{8}u^2 + c_1\tfrac{1}{2}u + c_2, \\ f_3 &= \tfrac{1}{32}u_{xxxx} - \tfrac{5}{16}uu_{xx} - \tfrac{5}{32}u_x^2 + \tfrac{5}{16}u^3 \\ &\quad + c_1\big(-\tfrac{1}{8}u_{xx} + \tfrac{3}{8}u^2\big) + c_2\tfrac{1}{2}u + c_3, \quad \text{etc.} \end{aligned} \tag{1.5}$$

Here $\{c_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}$ denote integration constants that naturally arise when solving (1.4). Subsequently, it will be convenient also to introduce the corresponding homogeneous coefficients $\hat{f}_\ell$ defined by the vanishing of the integration constants c_k for $k = 1, \dots, \ell$,

$$\hat{f}_0 = f_0 = 1, \quad \hat{f}_\ell = f_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{1.6}$$

Hence,

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{f}_k, \quad \ell \in \mathbb{N}_0,$$

introducing

$$c_0 = 1.$$

Remark 1.2 Using the nonlinear recursion (D.8) in Theorem D.1, one infers inductively that all homogeneous elements $\hat{f}_\ell$ (and hence all f_ℓ), $\ell \in \mathbb{N}_0$, are differential polynomials in u, that is, polynomials with respect to u and (some of) its x-derivatives. (Alternatively, one can prove directly by induction that the nonlinear recursion (D.8) is equivalent to that in (1.4) with all integration constants put to zero, $c_\ell = 0$, $\ell \in \mathbb{N}$.)

Next we define differential expressions P_{2n+1} of order $2n+1$ by

$$P_{2n+1} = \sum_{\ell=0}^{n} \Big(f_{n-\ell} \frac{d}{dx} - \frac{1}{2} f_{n-\ell,x} \Big) L^{\ell}, \quad n \in \mathbb{N}_0. \tag{1.7}$$

We record the first few P_{2n+1},

$$\begin{aligned}
P_1 &= \tfrac{d}{dx}, \\
P_3 &= -\tfrac{d^3}{dx^3} + \tfrac{3}{2} u \tfrac{d}{dx} + \tfrac{3}{4} u_x + c_1 \tfrac{d}{dx}, \\
P_5 &= \tfrac{d^5}{dx^5} - \tfrac{5}{4} u \tfrac{d^3}{dx^3} + \tfrac{7}{2} u_x \tfrac{d^2}{dx^2} + \big(\tfrac{3}{2} u^2 - 3u_{xx}\big) \tfrac{d}{dx} + 2uu_x - \tfrac{15}{16} u_{xxx} \\
&\quad + \tfrac{3}{8} u^2 - \tfrac{1}{8} u_{xx} + c_1 \big(- \tfrac{d^3}{dx^3} + \tfrac{3}{2} u \tfrac{d}{dx} + \tfrac{3}{4} u_x \big) + c_2 \tfrac{d}{dx}, \text{ etc.}
\end{aligned}$$

Introducing the corresponding homogeneous differential expressions $\widehat{P}_{2\ell+1}$ defined by

$$\widehat{P}_{2\ell+1} = P_{2\ell+1}\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}_0,$$

one finds

$$P_{2n+1} = \sum_{\ell=0}^{n} c_{n-\ell} \widehat{P}_{2\ell+1}. \tag{1.8}$$

Using the recursion (1.4), the commutator of P_{2n+1} and L can be explicitly computed and yields[1]

$$[P_{2n+1}, L] = 2 f_{n+1,x}, \quad n \in \mathbb{N}_0. \tag{1.9}$$

In particular, (L, P_{2n+1}) represents the celebrated *Lax pair* of the KdV hierarchy. Varying $n \in \mathbb{N}_0$, the stationary KdV hierarchy is then defined in terms of the vanishing of the commutator of P_{2n+1} and L in (1.9) by,[2]

$$-[P_{2n+1}, L] = -2 f_{n+1,x}(u) = \text{s-KdV}_n(u) = 0, \quad n \in \mathbb{N}_0. \tag{1.10}$$

Explicitly,

$$\begin{aligned}
\text{s-KdV}_0(u) &= -u_x = 0, \\
\text{s-KdV}_1(u) &= \tfrac{1}{4} u_{xxx} - \tfrac{3}{2} uu_x + c_1(-u_x) = 0, \\
\text{s-KdV}_2(u) &= -\tfrac{1}{16} u_{xxxxx} + \tfrac{5}{8} uu_{xxx} + \tfrac{5}{4} u_x u_{xx} - \tfrac{15}{8} u^2 u_x \\
&\qquad + c_1 \big(\tfrac{1}{4} u_{xxx} - \tfrac{3}{2} uu_x \big) + c_2(-u_x) = 0, \quad \text{etc.,}
\end{aligned}$$

represent the first few equations of the stationary KdV hierarchy. By definition, the set of solutions of (1.10), with n ranging in $\mathbb{N}_0$ and c_ℓ in $\mathbb{C}$, $\ell \in \mathbb{N}$, represents

[1] The recursion (1.4) is constructed in such a manner that the commutator of P_{2n+1} and L ceases to be a higher-order differential expression but results in multiplication by $2 f_{n+1,x}$ only.

[2] In a slight abuse of notation we will occasionally stress the functional dependence of f_ℓ on u, writing $f_\ell(u)$.

the class of algebro-geometric KdV solutions. At times it will be convenient to abbreviate algebro-geometric stationary KdV solutions u simply as KdV *potentials*.

In the following we will frequently assume that u satisfies the nth stationary KdV equation. By this we mean it satisfies one of the nth stationary KdV equations after a particular choice of integration constants $c_\ell \in \mathbb{C}$, $\ell = 1, \dots, n$, $n \in \mathbb{N}$, has been made.

In accordance with our notation introduced in (1.6) and (1.8), the corresponding homogeneous stationary KdV equations are defined by

$$\text{s-}\widehat{\mathrm{KdV}}_n(u) = \text{s-KdV}_n(u)\big|_{c_\ell=0,\, \ell=1,\dots,n} = 0, \quad n \in \mathbb{N}_0.$$

Next, we introduce a polynomial F_n of degree n with respect to the spectral parameter $z \in \mathbb{C}$ by

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell = \sum_{\ell=0}^{n} c_{n-\ell} \widehat{F}_\ell(z), \tag{1.11}$$

where $\widehat{F}_\ell$ denotes the corresponding homogeneous polynomials defined by

$$\widehat{F}_0(z) = F_0(z) = 1, \quad \widehat{F}_\ell(z) = F_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{f}_{\ell-k} z^k, \quad \ell \in \mathbb{N}_0.$$

Explicitly, one obtains

$$\begin{aligned}
F_0 &= 1,\\
F_1 &= z + \tfrac{1}{2}u + c_1,\\
F_2 &= z^2 + \tfrac{1}{2}uz - \tfrac{1}{8}u_{xx} + \tfrac{3}{8}u^2 + c_1\big(\tfrac{1}{2}u + z\big) + c_2,\\
F_3 &= z^3 + \tfrac{1}{2}uz^2 + \big(-\tfrac{1}{8}u_{xx} + \tfrac{3}{8}u^2\big)z + \tfrac{1}{32}u_{xxxx} - \tfrac{5}{16}uu_{xx} - \tfrac{5}{32}u_x^2\\
&\quad + \tfrac{5}{16}u^3 + c_1\big(z^2 + \tfrac{1}{2}uz - \tfrac{1}{8}u_{xx} + \tfrac{3}{8}u^2\big) + c_2\big(z + \tfrac{1}{2}u\big) + c_3, \quad \text{etc.}
\end{aligned}$$

The recursion relation (1.4) together with (1.10) implies that

$$F_{n,xxx} - 4(u-z)F_{n,x} - 2u_x F_n = 0. \tag{1.12}$$

Multiplying (1.12) by F_n a subsequent integration with respect to x results in

$$(1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2 = R_{2n+1}, \tag{1.13}$$

where R_{2n+1} is a monic polynomial of degree $2n+1$. We denote its roots[1] by $\{E_m\}_{m=0,\dots,2n}$ and hence write

$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}. \tag{1.14}$$

[1] The roots of R_{2n+1} are related to the spectrum of a closed realization of L in $L^2(\mathbb{R})$ (see, e.g., (J.38) in the special self-adjoint case).

Equation (1.13) can be used to derive nonlinear recursion relations for the homogeneous coefficients $\hat{f}_\ell$ (i.e., the ones satisfying (1.6) in the case of vanishing integration constants), as proved in Theorem D.1 in Appendix D. This has interesting applications to the asymptotic expansion of the Green's function of L with respect to the spectral parameter, as briefly discussed in Remark D.2, and also yields a proof that f_ℓ are differential polynomials in u (cf. Remark 1.2). In addition, as proven in Theorem D.1, (1.13) leads to an explicit determination of the integration constants $c_1, \dots, c_n$ in

$$\text{s-KdV}_n(u) = -2 f_{n+1,x}(u) = 0$$

in terms of the zeros $E_0, \dots, E_{2n}$ of the associated polynomial R_{2n+1} in (1.14). In fact, one can prove (cf. (D.9))

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n, \tag{1.15}$$

where

$$\begin{aligned} &c_0(\underline{E}) = 1, \\ &c_k(\underline{E}) = -\sum_{\substack{j_0,\dots,j_{2n}=0 \\ j_0+\cdots+j_{2n}=k}}^{k} \frac{(2j_0)!\cdots(2j_{2n})!}{2^{2k}(j_0!)^2\cdots(j_{2n}!)^2(2j_0-1)\cdots(2j_{2n}-1)} E_0^{j_0}\cdots E_{2n}^{j_{2n}}, \\ &\hspace{20em} k = 1, \dots, n. \end{aligned} \tag{1.16}$$

Next, we study the restriction of the differential expression P_{2n+1} to the two-dimensional kernel (i.e., the formal null space in an algebraic sense as opposed to the functional analytic one) of $(L - z)$. More precisely, let[1]

$$\ker(L - z) = \{\psi \colon \mathbb{R} \to \mathbb{C}_\infty \text{ meromorphic} \mid (L - z)\psi = 0\}, \quad z \in \mathbb{C}, \tag{1.17}$$

then, (1.7) implies

$$P_{2n+1}\big|_{\ker(L-z)} = \Big(F_n(z)\frac{d}{dx} - \frac{1}{2}F_{n,x}(z)\Big)\Big|_{\ker(L-z)}. \tag{1.18}$$

We emphasize that the result (1.18) is valid independently of whether P_{2n+1} and L commute. However, if one makes the additional assumption that P_{2n+1} and L commute, we will now prove that this implies an algebraic relationship between P_{2n+1} and L.

Theorem 1.3 (Burchnall–Chaundy) *Assume that P_{2n+1} and L commute, $[P_{2n+1}, L] = 0$, or equivalently, suppose* s-KdV$_n(u) = -2f_{n+1,x}(u) = 0$ *for some*

[1] If u is considered on $\mathbb{C}$, then ψ in (1.17) should be considered on $\mathbb{C}$ too.

$n \in \mathbb{N}_0$. *Then L and P_{2n+1} satisfy an algebraic relationship of the type (cf.* (1.14)*)*

$$\mathcal{F}_n(L, -iP_{2n+1}) = -P_{2n+1}^2 - R_{2n+1}(L) = 0,$$
$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad z \in \mathbb{C}. \tag{1.19}$$

Proof Using relations (1.18) and (1.13) one finds that

$$\big(P_{2n+1}\big|_{\ker(L-z)}\big)^2 = -\big((1/2)F_{n,xx}F_n - (1/4)F_{n,x}^2 - (u-z)F_n^2\big)\big|_{\ker(L-z)}$$
$$= -R_{2n+1}(L)\big|_{\ker(L-z)}.$$

Thus, one concludes that P_{2n+1}^2 and $-R_{2n+1}(L)$ coincide on $\ker(L-z)$, and since $z \in \mathbb{C}$ is arbitrary, one infers that (1.19) holds. □

The expression $\mathcal{F}_n(L, -iP_{2n+1})$ is called the Burchnall–Chaundy polynomial of the pair (L, P_{2n+1}). Equation (1.19) naturally leads to the hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part), where

$$\mathcal{K}_n\colon \mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0,$$
$$R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}. \tag{1.20}$$

Remark 1.4 At this point it is easy to motivate the recursion relation (1.4) used as our starting point for constructing the KdV hierarchy. If one is interested in determining differential expressions P commuting with L (other than simply polynomials of L or the case where P and L are polynomials of a third differential expression), one can proceed as follows. Restricting P to the two-dimensional null space, $\ker(L-z)$, of $(L-z)$, one can systematically replace d^2/dx^2 by $(u-z)$ and hence effectively reduce P on $\ker(L-z)$ to a first-order differential expression of the type $P\big|_{\ker(L-z)} = (F(z)d/dx + G(z))\big|_{\ker(L-z)}$, where F and G are polynomials. Imposing commutativity of P and L on $\ker(L-z)$ then yields the relation $G = -F_x/2$ between F and G and as a consequence of this and $[P, L]\big|_{\ker(L-z)} = 0$ also yields the equation

$$F_{xxx} - 4(u-z)F_x - 2u_xF = 0. \tag{1.21}$$

Moreover, we reproduced identity (1.18). Making the polynomial ansatz $F(z) = \sum_{\ell=0}^{n} f_{n-\ell}z^\ell$ and inserting it into (1.21) then readily yields the recursion relation (1.4) for $f_0, \dots, f_n$ together with $-\frac{1}{4}f_{n,xxx} + uf_{n,x} + \frac{1}{2}u_xf_n = 0$. In other words, one obtains the beginning of the recursion relation (1.4) as well as relation (1.10) defining the nth stationary KdV equation.

Remark 1.5 If u satisfies one of the stationary KdV equations in (1.10) for a particular value of n, s-KdV$_n(u) = 0$, then it satisfies infinitely many such equations of order higher than n for certain choices of integration constants c_ℓ. In fact, it satisfies a certain stationary KdV equation s-KdV$_p(u) = 0$ for every $p \geq n+1$. This is seen as follows. Assuming $f_{n+1,x} = 0$ for some $n \in \mathbb{N}$ and some set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$, one infers

$$f_{n+1} = d_{n+1}$$

for some constant $d_{n+1} \in \mathbb{C}$. Subtracting the constant d_{n+1} (i.e., writing $f_{n+1} = \sum_{\ell=0}^{n+1} \check{c}_{n+1-\ell} \hat{f}_\ell$ for some set of constants $\{\check{c}_\ell\}_{\ell=1,\dots,n+1} \subset \mathbb{C}$ and absorbing d_{n+1} into $\check{c}_{n+1}$), we may without loss of generality assume that $f_{n+1} = 0$, and hence the recursion (1.4) implies

$$f_{n+2} = d_{n+2}$$

for some constant $d_{n+2} \in \mathbb{C}$. Iterating this procedure yields

$$f_{n+q,x} = 0, \quad q \geq 2.$$

Hence, s-KdV$_p(u) = 0$ for all $p \geq n+1$ (corresponding to some p-dependent choice of integration constants $\{\check{c}_\ell\}_{\ell=1,\dots,p} \subset \mathbb{C}$).

We illustrate this remark by the following example. In it we denote by $\wp(\,\cdot\,) = \wp(\,\cdot \mid \omega_1, \omega_3) = \wp(\,\cdot\,; g_2, g_3)$ the Weierstrass $\wp$-function with periods $2\omega_j$, $j = 1, 3$, $\mathrm{Im}(\omega_3/\omega_1) \neq 0$, $\omega_2 = \omega_1 + \omega_3$, and invariants g_2 and g_3 (cf. Appendix H).

Example 1.6 Consider the genus $n = 1$ elliptic KdV potential

$$u(x) = 2\wp(x) + c, \quad c \in \mathbb{C}.$$

Then one infers

$$\begin{aligned}
\hat{f}_1 &= \tfrac{1}{2}u = \wp + \tfrac{1}{2}c, \\
\hat{f}_2 &= -\tfrac{1}{8}u_{xx} + \tfrac{3}{8}u^2 = \tfrac{3}{2}c\wp + \tfrac{1}{8}g_2 + \tfrac{3}{8}c^2, \\
\hat{f}_3 &= \tfrac{1}{32}u_{xxxx} - \tfrac{5}{16}uu_{xx} - \tfrac{5}{32}u_x^2 + \tfrac{5}{16}u^3 \\
&= \big(\tfrac{15}{8}c^2 + \tfrac{1}{8}g_2\big)\wp + \tfrac{5}{16}cg_2 + \tfrac{5}{16}c^3 - \tfrac{1}{8}g_3, \quad \text{etc.}, \\
F_1(z) &= z + \wp - c, \\
F_2(z) &= (z - c)(z + \wp - c), \quad \text{etc.}
\end{aligned}$$

Thus, u satisfies an s-KdV$_1$ equation of the form

$$\text{s-KdV}_1(u) = \text{s-}\widehat{\text{KdV}}_1(u) - \frac{3}{2}c\,\text{s-}\widehat{\text{KdV}}_0(u) = 0,$$

with the associated genus $n = 1$ curve given by

$$\mathcal{K}_1\colon\ y^2 - R_3(z) = 0,$$
$$R_3(z) = \prod_{m=0}^{2}(z - c + e_m), \quad e_m = \wp(\omega_{m+1}),\ m = 0, 1, 2,$$
$$c_1(\underline{E}) = -3c/2.$$

Moreover, u satisfies the s-KdV$_2$ equation

$$\text{s-KdV}_2(u) = \text{s-}\widehat{\text{KdV}}_2(u) - \Big(\frac{g_2}{8} + \frac{15}{8}c^2\Big)\text{s-}\widehat{\text{KdV}}_0(u) = 0,$$

with the associated singular (arithmetic) genus $n = 2$ curve given by

$$\mathcal{K}_2\colon\ y^2 - R_5(z) = 0,$$
$$R_5(z) = (z - c)^2\prod_{m=0}^{2}(z - c + e_m),$$
$$c_1(\underline{E}) = -5c/2, \quad c_2(\underline{E}) = (15/8)c^2 - (g_2/8).$$

Analogous formulas can be derived for all higher-order s-KdV$_n$ equations for $n \geq 3$.

Next we turn to the time-dependent KdV hierarchy. This means that u is now considered as a function of both space and time. For each equation in the hierarchy, that is, for each n, we introduce a deformation (time) parameter $t_n \in \mathbb{R}$ in u, replacing $u(x)$ by $u(x, t_n)$. The second-order differential expression L (cf. (1.3)) now reads

$$L(t_n) = -\frac{d^2}{dx^2} + u(\,\cdot\,, t_n). \tag{1.22}$$

The quantities $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ and P_{2n+1}, $n \in \mathbb{N}_0$, are still defined by (1.4) and (1.7), respectively. The time-dependent KdV hierarchy is then obtained by imposing the Lax commutator equations

$$\frac{d}{dt_n}L(t_n) - [P_{2n+1}(t_n), L(t_n)] = 0, \quad t_n \in \mathbb{R}, \tag{1.23}$$

varying $n \in \mathbb{N}_0$. The latter are equivalent to the collection of evolution equations

$$\text{KdV}_n(u) = u_{t_n} - 2f_{n+1,x}(u) = 0, \quad (x, t_n) \in \mathbb{R}^2,\ \ n \in \mathbb{N}_0. \tag{1.24}$$

Explicitly,

$$\text{KdV}_0(u) = u_{t_0} - u_x = 0,$$
$$\text{KdV}_1(u) = u_{t_1} + \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x - c_1u_x = 0,$$
$$\text{KdV}_2(u) = u_{t_2} - \tfrac{1}{16}u_{xxxxx} + \tfrac{5}{8}uu_{xxx} + \tfrac{5}{4}u_xu_{xx} - \tfrac{15}{8}u^2u_x$$
$$+ c_1\big(\tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x\big) - c_2u_x = 0, \quad \text{etc.},$$

represent the first few equations of the time-dependent KdV hierarchy. The equation $\mathrm{KdV}_1(u) = 0$ (with $c_1 = 0$) is of course *the* Korteweg–de Vries equation. Similarly, the corresponding homogeneous KdV equations are defined by

$$\widehat{\mathrm{KdV}}_n(u) = \mathrm{KdV}_n(u)\big|_{c_\ell=0,\, \ell=1,\dots,n} = 0, \quad n \in \mathbb{N}_0.$$

Later on we also use the following alternative formulation of the KdV hierarchy. Consider once more P_{2n+1} restricted to $\ker(L - z)$. On this null space the Lax equation (1.23) reads

$$\big(L_{t_n} - [P_{2n+1}, L]\big)\big|_{\ker(L-z)} = \big(u_{t_n} - (L - z)P_{2n+1}\big)\big|_{\ker(L-z)} = 0,$$

which simplifies to

$$u_{t_n} + (1/2)F_{n,xxx} - 2(u - z)F_{n,x} - u_x F_n = 0. \tag{1.25}$$

Equation (1.25) is just another way of writing the nth KdV equation (1.24).

We conclude this section by pointing out an alternative construction of the KdV hierarchy using a zero-curvature approach instead of Lax pairs (L, P_{2n+1}).

Remark 1.7 The zero-curvature formalism for the KdV hierarchy can be set up as follows. One defines the 2×2 matrices

$$U(z) = \begin{pmatrix} 0 & 1 \\ -z + u & 0 \end{pmatrix},$$
$$V_{n+1}(z) = \begin{pmatrix} G_{n-1}(z) & F_n(z) \\ -H_{n+1}(z) & -G_{n-1}(z) \end{pmatrix}, \quad n \in \mathbb{N}_0.$$

Then the stationary part of this section can equivalently be based on the zero-curvature equation

$$\begin{aligned} 0 &= -V_{n+1,x} + [U, V_{n+1}] \\ &= \begin{pmatrix} -G_{n-1,x} + (z - u)F_n - H_{n+1} & -F_{n,x} - 2G_{n-1} \\ H_{n+1,x} - 2(z - u)G_{n-1} & G_{n-1,x} - (z - u)F_n + H_{n+1} \end{pmatrix}. \end{aligned}$$

Thus, one obtains,

$$G_{n-1} = -F_{n,x}/2, \tag{1.26}$$
$$G_{n-1,x} = -F_{n,xx}/2 = (z - u)F_n - H_{n+1}, \tag{1.27}$$
$$H_{n+1,x} = (u - z)F_{n,x}, \tag{1.28}$$

implying the basic stationary equation (1.12). The hyperelliptic curve $\mathcal{K}_n$ in (1.20) is then obtained from the characteristic equation of $i\,V_{n+1}(z)$ by[1]

$$\begin{aligned} \det(yI_2 - i\,V_{n+1}(z, x)) &= y^2 - \det(V_{n+1}(z, x)) \\ &= y^2 + G_{n-1}(z, x)^2 - F_n(z, x)H_{n+1}(z, x) = y^2 - R_{2n+1}(z) = 0 \end{aligned}$$

[1] I_2 denotes the identity matrix in $\mathbb{C}^2$.

using (1.26) and (1.27). Similarly, using (1.26)–(1.28), one can equivalently develop the time-dependent part (1.22)–(1.25) from the zero-curvature equation

$$\begin{aligned} 0 &= U_{t_n} - V_{n+1,x} + [U, V_{n+1}] \\ &= \begin{pmatrix} -G_{n-1,x} + (z-u)F_n - H_{n+1} & -F_{n,x} - 2G_{n-1} \\ u_t + H_{n+1,x} - 2(z-u)G_{n-1} & G_{n-1,x} - (z-u)F_n + H_{n+1} \end{pmatrix}, \end{aligned}$$

implying

$$\begin{aligned} G_{n-1} &= -F_{n,x}/2, \\ G_{n-1,x} &= -F_{n,xx}/2 = (z-u)F_n - H_{n+1}, \\ u_t &= -H_{n+1,x} + (u-z)F_{n,x} \\ &= -(1/2)F_{n,xxx} - 2(z-u)F_{n,x} + u_x F_n \end{aligned}$$

in agreement with (1.25).

1.3 The Stationary KdV Formalism

As shown in Section 1.2, the stationary KdV hierarchy is intimately connected with pairs of commuting differential expressions P_{2n+1} and L of orders $2n+1$ and 2, respectively, and a hyperelliptic curve $\mathcal{K}_n$. In this section we study this relationship more closely and present a detailed study of the stationary KdV hierarchy and its algebro-geometric solutions u. Our principal tools are derived from combining the polynomial recursion formalism introduced in Section 1.2 and a fundamental meromorphic function ϕ on $\mathcal{K}_n$, the analog of the Weyl–Titchmarsh function of L. With the help of ϕ we study the Baker–Akhiezer function ψ, the common eigenfunction of P_{2n+1} and L, Dubrovin equations governing the motion of auxiliary divisors on $\mathcal{K}_n$, trace formulas, and theta function representations of ϕ, ψ, and u. We also discuss the algebro-geometric intitial value problem of constructing u from the Dubrovin equations and auxiliary divisors as initial data.

For major parts of this section we suppose that

$$u \in C^\infty(\mathbb{R}) \tag{1.29}$$

(which could be replaced by $u\colon \mathbb{C} \to \mathbb{C}_\infty$ meromorphic) and assume (1.10) (respectively (1.12)) and (1.11); we then freely employ the formalism developed in (1.4)–(1.19), keeping $n \in \mathbb{N}_0$ fixed.

We recall the Burchnall–Chaundy curve

$$\begin{aligned} &\mathcal{K}_n\colon \mathcal{F}_n(z,y) = y^2 - R_{2n+1}(z) = 0, \\ &R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C} \end{aligned} \tag{1.30}$$

as introduced in (1.20). The curve $\mathcal{K}_n$ is compactified by joining the point P_∞, but for notational simplicity the compactification is also denoted by $\mathcal{K}_n$.

Points P on $\mathcal{K}_n \setminus \{P_\infty\}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = 0$. The complex structure on $\mathcal{K}_n$ is then defined in the usual way (see Appendix B). Hence, $\mathcal{K}_n$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_n$ (i.e., by fixing $E_0, \dots, E_{2n}$), the integration constants $c_1, \dots, c_n$ in $f_{n+1,x}$ (and hence in the corresponding stationary KdV_n equation) are uniquely determined, as is clear from (1.15), (1.16), which establish the integration constants c_ℓ as symmetric functions of $E_0, \dots, E_{2n}$.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$. (The trivial case $n = 0$ is explicitly treated in Example 1.25.)

In the following, the zeros[1] of the polynomial $F_n(\,\cdot\,, x)$ (cf. (1.11)) will play a special role. We denote them by $\{\mu_j(x)\}_{j=1,\dots,n}$ and hence write

$$F_n(z) = \prod_{j=1}^{n} (z - \mu_j). \tag{1.31}$$

From (1.13) we see that

$$R_{2n+1} + (1/4)F_{n,x}^2 = F_n H_{n+1}, \tag{1.32}$$

where

$$H_{n+1}(z) = (1/2)F_{n,xx}(z) + (z - u)F_n(z) \tag{1.33}$$

is a monic polynomial of degree $n + 1$. We introduce the corresponding roots[2] $\{\nu_\ell(x)\}_{\ell=0,\dots,n}$ of $H_{n+1}(\,\cdot\,, x)$ and its associated homogeneous polynomials $\widehat{H}_{\ell+1}$, which are defined by the vanishing of the integration constants c_k for $k = 1, \dots, \ell$, by

$$H_{n+1}(z) = \prod_{\ell=0}^{n} (z - \nu_\ell) = \sum_{\ell=0}^{n} c_{n-\ell} \widehat{H}_{\ell+1}(z), \tag{1.34}$$

where

$$\widehat{H}_1(z) = H_1(z) = z - u, \quad \widehat{H}_{\ell+1}(z) = H_{\ell+1}(z)\big|_{c_k=0,\ k=1,\dots,\ell}, \quad \ell = 0, \dots, n.$$

Explicitly, one computes from (1.5) and (1.11)

$$\begin{aligned}
H_1 &= z - u,\\
H_2 &= z^2 - \tfrac{1}{2}uz + \tfrac{1}{4}u_{xx} - \tfrac{1}{2}u^2 + c_1(z - u),\\
H_3 &= z^3 - \tfrac{1}{2}uz^2 + \tfrac{1}{8}\big(u_{xx} - u^2\big)z - \tfrac{1}{16}u_{xxxx} + \tfrac{3}{8}u_x^2 + \tfrac{1}{2}uu_{xx}\\
&\quad - \tfrac{3}{8}u^3 + c_1\big(z^2 - \tfrac{1}{2}uz + \tfrac{1}{4}u_{xx} - \tfrac{1}{2}u^2\big) + c_2(z - u), \quad \text{etc.}
\end{aligned}$$

[1] If $u \in L^\infty(\mathbb{R})$, these zeros are the Dirichlet eigenvalues of a closed operator in $L^2(\mathbb{R})$ associated with the differential expression L and a Dirichlet boundary condition at $x \in \mathbb{R}$ (cf. Appendix J).

[2] If $u \in L^\infty(\mathbb{R})$, these roots are the Neumann eigenvalues of a closed operator in $L^2(\mathbb{R})$ associated with L and a Neumann boundary condition at $x \in \mathbb{R}$ (cf. Appendix J).

The next step is crucial; it permits us to "lift" the zeros μ_j and ν_ℓ of F_n and H_{n+1} from $\mathbb{C}$ to the curve $\mathcal{K}_n$. From (1.32) one infers

$$R_{2n+1}(z) + (1/4)F_{n,x}(z)^2 = 0, \quad z \in \{\mu_j, \nu_\ell\}_{j=1,\dots,n,\ell=0,\dots,n}.$$

We now introduce $\{\hat{\mu}_j(x)\}_{j=1,\dots,n} \subset \mathcal{K}_n$ and $\{\hat{\nu}_\ell(x)\}_{\ell=0,\dots,n} \subset \mathcal{K}_n$ by

$$\hat{\mu}_j(x) = (\mu_j(x), -(i/2)F_{n,x}(\mu_j(x), x)), \quad j = 1, \dots, n, \ x \in \mathbb{R} \tag{1.35}$$

and

$$\hat{\nu}_\ell(x) = (\nu_\ell(x), (i/2)F_{n,x}(\nu_\ell(x), x)), \quad \ell = 0, \dots, n, \ x \in \mathbb{R}. \tag{1.36}$$

Due to the $C^\infty(\mathbb{R})$ assumption (1.29) on u, $F_n(z, \cdot\,) \in C^\infty(\mathbb{R})$ by (1.4) and (1.11), and hence also $H_{n+1}(z, \cdot\,) \in C^\infty(\mathbb{R})$ by (1.33). Thus, one concludes

$$\mu_j, \nu_\ell \in C(\mathbb{R}), \ j = 1, \dots, n, \ \ell = 0, \dots, n, \tag{1.37}$$

taking multiplicities (and appropriate renumbering) of the zeros of F_n and H_{n+1} into account. (Away from collisions of zeros, μ_j and ν_ℓ are of course C^∞.)

Next, we define the fundamental meromorphic function $\phi(\,\cdot\,, x)$ on $\mathcal{K}_n$,

$$\phi(P, x) = \frac{iy + (1/2)F_{n,x}(z, x)}{F_n(z, x)} \tag{1.38}$$

$$= \frac{-H_{n+1}(z, x)}{iy - (1/2)F_{n,x}(z, x)}, \tag{1.39}$$

$$P = (z, y) \in \mathcal{K}_n, \ x \in \mathbb{R}$$

with divisor $(\phi(\,\cdot\,, x))$ of $\phi(\,\cdot\,, x)$ given by

$$(\phi(\,\cdot\,, x)) = \mathcal{D}_{\hat{\nu}_0(x)\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}, \tag{1.40}$$

using (1.31), (1.34), and (1.37). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n).$$

Given $\phi(\,\cdot\,, x)$, we define the stationary Baker–Akhiezer function $\psi(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_\infty\}$ by

$$\psi(P, x, x_0) = \exp\left(\int_{x_0}^{x} dx'\, \phi(P, x')\right), \quad P \in \mathcal{K}_n \setminus \{P_\infty\}, \ (x, x_0) \in \mathbb{R}^2. \tag{1.41}$$

Basic properties of ϕ and ψ are summarized in the following result ($W(f, g) = fg' - f'g$ denotes the Wronskian of f and g).

Lemma 1.8 *Suppose $u \in C^\infty(\mathbb{R})$ satisfies the nth stationary KdV equation* (1.10)*. Moreover, let $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}$, $(x, x_0) \in \mathbb{R}^2$. Then ϕ satisfies the Riccati-type equation*

$$\phi_x(P) + \phi(P)^2 = u - z \tag{1.42}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_{n+1}(z)}{F_n(z)}, \tag{1.43}$$

$$\phi(P) + \phi(P^*) = \frac{F_{n,x}(z)}{F_n(z)}, \tag{1.44}$$

$$\phi(P) - \phi(P^*) = \frac{2iy}{F_n(z)}. \tag{1.45}$$

Moreover, ψ satisfies

$$(L - z(P))\psi(P) = 0, \quad (P_{n+1} - iy(P))\psi(P) = 0, \tag{1.46}$$

$$\psi(P, x, x_0) = \left(\frac{F_n(z, x)}{F_n(z, x_0)}\right)^{1/2} \exp\left(iy \int_{x_0}^{x} dx'\, F_n(z, x')^{-1}\right), \tag{1.47}$$

$$\psi(P, x, x_0)\psi(P^*, x, x_0) = \frac{F_n(z, x)}{F_n(z, x_0)}, \tag{1.48}$$

$$\psi_x(P, x, x_0)\psi_x(P^*, x, x_0) = \frac{H_{n+1}(z, x)}{F_n(z, x_0)}, \tag{1.49}$$

$$\psi(P, x, x_0)\psi_x(P^*, x, x_0) + \psi(P^*, x, x_0)\psi_x(P, x, x_0) = \frac{F_{n,x}(z, x)}{F_n(z, x_0)}, \tag{1.50}$$

$$W(\psi(P, \,\cdot\,, x_0), \psi(P^*, \,\cdot\,, x_0)) = -\frac{2iy}{F_n(z, x_0)}. \tag{1.51}$$

In addition, as long as the zeros of $F_n(\,\cdot\,, x)$ are all simple for $x \in \Omega$, $\Omega \subseteq \mathbb{R}$ an open interval, $\psi(\,\cdot\,, x, x_0)$ is meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ for $x, x_0 \in \Omega$.

Proof Relation (1.42) follows by combining (1.13) and (1.38). Equation (1.43) follows by multiplying (1.38) and (1.39), replacing P by P^* in one of the two factors. Equations (1.44) and (1.45) are clear from (1.38) and (1.39). By (1.41), $\psi(\,\cdot\,, x, x_0)$ is meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ away from the poles $\hat{\mu}_j(x')$ of $\phi(\,\cdot\,, x')$. By (1.35) and (1.38),

$$\phi(P, x') \underset{P \to \hat{\mu}_j(x')}{=} \partial_{x'} \ln\big(F(z, x')\big) + O(1) \text{ as } z \to \mu_j(x'), \tag{1.52}$$

and hence $\psi(\,\cdot\,, x, x_0)$ is meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ as long as the zeros of $F_n(\,\cdot\,, x)$ are all simple. This follows from (1.41) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j$ of $\{\hat{\mu}_j(x') \in \mathcal{K}_n \mid x' \in \Omega,\ x' \in [x_0, x]\}$ such that $\hat{\mu}_k(x') \notin \mathcal{U}_j$ for all $x' \in [x_0, x]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$. To prove (1.46), one employs $\psi_{xx}/\psi = \phi_x + \phi^2 = u - z$ to arrive at $L\psi = z\psi$. Next, one uses (1.18) and (1.38) to compute $P_{n+1}\psi = F_n\phi\psi - \frac{1}{2}F_{n,x}\psi = iy\psi$. Equation (1.47) follows by invoking (1.38) and (1.41). Equation (1.48) follows by combining (1.41) and (1.43). Equation (1.49) is a consequence of (1.43) and (1.48), and the fact that $\psi_x = \phi\psi$. Equations (1.50) and (1.51) follow from (1.41), (1.44), and (1.45). $\square$

The normalization chosen for the Baker–Akhiezer function ψ in (1.41) (basically, $\psi(P, x, x_0)$ equals $\tilde{\psi}(P, x)/\tilde{\psi}(P, x_0)$ for a certain (not necessarily normalized) solution $\tilde{\psi}$ of $(L - z)\psi = 0$) has some interesting consequences and is not quite as innocent as it may appear at first glance. In fact, by (1.48), one infers that its divisor of zeros and poles on $\mathcal{K}_n \setminus \{P_\infty\}$ is precisely given by $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$, respectively.

Equations (1.48)–(1.51) show that the basic identity (1.13), rewritten in the form $-G_{n-1}^2 + F_n H_{n+1} = R_{2n+1}$, where $G_{n-1} = -F_{n,x}/2$, is equivalent to the elementary fact

$$(\psi_{1,+}\psi_{2,-} + \psi_{1,-}\psi_{2,+})^2 - 4\psi_{1,+}\psi_{1,-}\psi_{2,+}\psi_{2,-} = (\psi_{1,+}\psi_{2,-} - \psi_{1,-}\psi_{2,+})^2, \tag{1.53}$$

identifying $\psi(P) = \psi_{1,+}$, $\psi(P^*) = \psi_{1,-}$, $\psi_x(P) = \psi_{2,+}$, $\psi_x(P^*) = \psi_{2,-}$. This provides the intimate link between our approach and the squared function systems also employed in the literature in connection with algebro-geometric solutions of the KdV hierarchy.

If $u \in L^\infty(\mathbb{R})$, the zeros of $\mu_j(x)$ of $F_n(\,\cdot\,, x)$, respectively, the zeros $\nu_\ell(x)$ of $H_{n+1}(\,\cdot\,, x)$, are naturally associated with Dirichlet, respectively, Neumann boundary conditions of L at the point $x \in \mathbb{R}$. In other words, $\mu_j(x)$ are associated with the boundary condition $g(x) = 0$ for an element g in the domain of an $L^2(\mathbb{R})$ operator realization of L, whereas $\nu_\ell(x)$ corresponds to $g'(x) = 0$. Next, we "interpolate" between these two boundary conditions and consider the general case

$$g'(x) + \beta g(x) = 0, \quad \beta \in \mathbb{R} \tag{1.54}$$

(cf. Appendix J for more details in the special case in which u is real-valued). The values $\beta = \infty$ and $\beta = 0$ then represent the Dirichlet and Neumann cases, respectively.

To this end we introduce the additional polynomial $K_{n+1}^\beta(z)$, $\beta \in \mathbb{R}$ of degree $n + 1$ by

$$\begin{aligned} K_{n+1}^\beta(z) &= H_{n+1}(z) + \beta F_{n,x}(z) + \beta^2 F_n(z) = \prod_{\ell=0}^n \left(z - \lambda_\ell^\beta\right) \\ &= \sum_{\ell=0}^n c_{n-\ell} \widehat{K}_{\ell+1}^\beta(z), \quad \beta \in \mathbb{R}. \end{aligned} \tag{1.55}$$

Here $\widehat{K}_{\ell+1}^\beta$ denote the corresponding homogeneous polynomials defined by the vanishing of the integration constants c_k for $k = 1, \dots, \ell$,

$$\widehat{K}_1^\beta(z) = K_1^\beta(z) = z + \beta^2 - u, \quad \widehat{K}_{\ell+1}^\beta(z) = K_{\ell+1}^\beta(z)\big|_{c_k=0,\ k=1,\dots,\ell}, \quad \ell = 0, \dots, n.$$

In particular,

$$K^0_{n+1}(z) = H_{n+1}(z), \quad \lambda^0_\ell = \nu_\ell, \quad \ell = 0, \dots, n.$$

Explicitly, one computes

$$\begin{aligned}
K^\beta_1 &= z + \beta^2 - u, \\
K^\beta_2 &= z^2 + \big(\beta^2 - \tfrac{1}{2}u\big)z + \tfrac{1}{4}u_{xx} - \tfrac{1}{2}u^2 + \tfrac{1}{2}\beta u_x + \tfrac{1}{2}\beta^2 u + c_1(z + \beta^2 - u), \\
K^\beta_3 &= z^3 + \big(\beta^2 - \tfrac{1}{2}u\big)z^2 + \big(\tfrac{1}{2}\beta u_x + \tfrac{1}{2}\beta^2 u + \tfrac{1}{8}u_{xx} - \tfrac{1}{8}u^2\big)z - \tfrac{1}{8}\beta u_{xxx} \\
&\quad + \tfrac{3}{4}\beta u u_x - \tfrac{1}{8}\beta^2 u_{xx} + \tfrac{3}{8}\beta^2 u^2 - \tfrac{1}{16}u_{xxxx} + \tfrac{3}{8}u_x^2 + \tfrac{1}{2}u u_{xx} - \tfrac{3}{8}u^3 \\
&\quad + c_1\big(z^2 + \big(\beta^2 - \tfrac{1}{2}u\big)z + \tfrac{1}{4}u_{xx} - \tfrac{1}{2}u^2 + \tfrac{1}{2}\beta u_x + \tfrac{1}{2}\beta^2 u\big) \\
&\quad + c_2(z + \beta^2 - u), \quad \text{etc.}
\end{aligned} \tag{1.56}$$

Strictly speaking, the Dirichlet eigenvalues $\mu_j(x)$ of $H^D_x = H^\infty_x$, the Neumann eigenvalues $\nu_\ell(x)$ of $H^N_x = H^0_x$, and the eigenvalues $\lambda^\beta_\ell(x)$ of H^β_x for general $\beta \in \mathbb{R}$ are introduced in Appendix J only in the special case in which $u \in L^1_{\text{loc}}(\mathbb{R})$ is real-valued and the differential expression $L = -\frac{d^2}{dx^2} + u$ is in the limit point case at $\pm\infty$ (and hence H^β_x is self-adjoint in $L^2(\mathbb{R})$ for $x \in \mathbb{R}$ and $\beta \in \mathbb{R} \cup \{\infty\}$). However, this spectral interpretation immediately extends to the case in which $u \in L^\infty(\mathbb{R})$ is complex-valued; hence, we generally call $\mu_j(x)$ and $\nu_\ell(x)$ the Dirichlet and Neumann eigenvalues associated with the closed $L^2(\mathbb{R})$-realization of L, respectively.

Next, combining (1.38), (1.39), and (1.55) yields

$$\phi(P) + \beta = \frac{iy + \frac{1}{2}F_{n,x}(z) + \beta F_n(z)}{F_n(z)} \tag{1.57}$$

$$= \frac{-K^\beta_{n+1}(z)}{iy - \frac{1}{2}F_{n,x}(z) - \beta F_n(z)}. \tag{1.58}$$

One verifies, as before (cf. Lemma 1.8), that

$$\begin{aligned}
&R_{2n+1}(z) + \big((1/2)F_{n,x}(z) + \beta F_n(z)\big)^2 = F_n(z)K^\beta_{n+1}(z), \\
&(\phi(P) + \beta)(\phi(P^*) + \beta) = \frac{K^\beta_{n+1}(z)}{F_n(z)}, \\
&(\psi_x(P, x, x_0) + \beta\psi(P, x, x_0))(\psi_x(P^*, x, x_0) + \beta\psi(P^*, x, x_0)) = \frac{K^\beta_{n+1}(z, x)}{F_n(z, x_0)},
\end{aligned} \tag{1.59}$$

where the Baker–Akhiezer function $\psi(\,\cdot\,, x, x_0)$ is defined in (1.41). The divisor $(\phi(\,\cdot\,, x) + \beta)$ of $\phi(\,\cdot\,, x) + \beta$, $\beta \in \mathbb{R}$, is then given by

$$(\phi(\,\cdot\,, x) + \beta) = \mathcal{D}_{\hat{\lambda}^\beta_0(x)\underline{\hat{\lambda}}^\beta(x)} - \mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}, \tag{1.60}$$

with

$$\hat{\lambda}^\beta_\ell(x) = \big(\lambda^\beta_\ell(x), (i/2)F_{n,x}\big(\lambda^\beta_\ell(x), x\big) + i\beta F_n\big(\lambda^\beta_\ell(x), x\big)\big), \quad \ell = 0, \dots, n, \ \beta \in \mathbb{R}. \tag{1.61}$$

Remark 1.9 Our notation $\mathcal{D}_{\hat{\lambda}_0^\beta \underline{\hat{\lambda}}^\beta}$, $\underline{\hat{\lambda}}^\beta = \{\hat{\lambda}_1^\beta, \dots, \hat{\lambda}_n^\beta\}$ in the general case, where u is complex-valued, is somewhat misleading since

$$\mathcal{D}_{\hat{\lambda}_0^\beta \underline{\hat{\lambda}}^\beta} = \sum_{\ell=0}^{n} \mathcal{D}_{\hat{\lambda}_\ell^\beta} \in \operatorname{Sym}^{n+1}(\mathcal{K}_n)$$

is symmetric in $\hat{\lambda}_0^\beta, \dots, \hat{\lambda}_n^\beta$ and there is no natural way to distinguish $\hat{\lambda}_0^\beta$ from $\hat{\lambda}_\ell^\beta$, $\ell = 1, \dots, n$. In particular,

$$\mathcal{D}_{\hat{\lambda}_0^\beta \underline{\hat{\lambda}}^\beta} = \mathcal{D}_{\hat{\lambda}_\ell^\beta \underline{\hat{\lambda}}^{\beta,\ell}},$$

where

$$\begin{aligned}
\underline{\hat{\lambda}}^{\beta,1} &= \{\hat{\lambda}_0^\beta, \hat{\lambda}_2^\beta, \dots, \hat{\lambda}_n^\beta\}, \\
\underline{\hat{\lambda}}^{\beta,\ell} &= \{\hat{\lambda}_0^\beta, \hat{\lambda}_1^\beta, \dots, \hat{\lambda}_{\ell-1}^\beta, \hat{\lambda}_{\ell+1}^\beta, \hat{\lambda}_n^\beta\}, \quad \ell = 2, \dots, n-1, \\
\underline{\hat{\lambda}}^{\beta,n} &= \{\hat{\lambda}_0^\beta, \hat{\lambda}_1^\beta, \dots, \hat{\lambda}_{n-1}^\beta\}.
\end{aligned}$$

In the special case, where u is real-valued and nonsingular, a distinction between $\hat{\lambda}_0^\beta$ and $\hat{\lambda}_\ell^\beta$, $\ell = 1, \dots, n$ can be made naturally by supposing

$$\lambda_0^\beta \le E_0, \quad \lambda_\ell^\beta \in [E_{2\ell-1}, E_{2\ell}], \quad \ell = 1, \dots, n.$$

For notational convenience in connection with positive divisors of degree n on $\mathcal{K}_n$ and their subsequent use in the associated n-dimensional theta function, we will keep the abbreviation $\mathcal{D}_{\hat{\lambda}_0^\beta \underline{\hat{\lambda}}^\beta}$ for general complex-valued u but occasionally will caution the reader about this convention.

Next, we recall that the affine part of $\mathcal{K}_n$ is nonsingular if

$$E_m \neq E_{m'} \text{ for } m \neq m', \quad m, m' = 0, 1, \dots, 2n. \tag{1.62}$$

In the special case in which $\{E_m\}_{m=0,\dots,2n} \subset \mathbb{R}$, we will from now on always assume the ordering

$$E_m < E_{m+1} \text{ for } m = 0, 1, \dots, 2n-1. \tag{1.63}$$

In particular, if $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is assumed to be real-valued, then necessarily $\{\mu_j(x)\}_{j=1,\dots,n} \subset \mathbb{R}$ and $\{\lambda_\ell^\beta(x)\}_{\ell=0,\dots,n} \subset \mathbb{R}$ for all $x \in \mathbb{R}$ since one is then dealing with self-adjoint boundary value problems in $L^2(\mathbb{R})$ (cf. also the explicit argument presented for the Dirichlet case in the proof of Lemma 1.10 (ii) below); hence, we will also always assume the ordering

$$\mu_j(x) < \mu_{j+1}(x) \text{ for } j = 1, \dots, n-1, \ x \in \mathbb{R}, \tag{1.64}$$

$$\lambda_\ell^\beta(x) < \lambda_{\ell+1}^\beta(x) \text{ for } \ell = 0, \dots, n-1, \ x \in \mathbb{R} \tag{1.65}$$

in this case.

The dynamics of λ_ℓ^β, $\beta \in \mathbb{R} \cup \{\infty\}$ with respect to variations of x can be described by a first-order system of nonlinear differential equations traditionally called Dubrovin equations in the Dirichlet case $\beta = \infty$. We first treat the Dirichlet case $\beta = \infty$ and then turn to the case $\beta \in \mathbb{R}$.

Lemma 1.10 (The Dubrovin Equations)
(i) Suppose that $u \in C^\infty(\widetilde{\Omega}_\mu)$ satisfies the nth stationary KdV equation (1.10) *on an open interval $\widetilde{\Omega}_\mu \subseteq \mathbb{R}$. Moreover, assume that the zeros μ_j, $j = 1, \dots, n$, of $F_n(\,\cdot\,)$ remain distinct on $\widetilde{\Omega}_\mu$. Then $\{\hat{\mu}_j\}_{j=1,\dots,n}$, defined by* (1.35)*, satisfies the following first-order system of differential equations on $\widetilde{\Omega}_\mu$*

$$\mu_{j,x} = -2iy(\hat{\mu}_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} \left(\mu_j - \mu_k\right)^{-1}, \quad j = 1, \dots, n. \tag{1.66}$$

Next, assume the affine part of $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{1.67}$$

for some $x_0 \in \mathbb{R}$, where $\mu_j(x_0)$, $j = 1, \dots, n$, are assumed to be distinct. Then there exists an open interval $\Omega_\mu \subseteq \mathbb{R}$, with $x_0 \in \Omega_\mu$, such that the initial value problem (1.66), (1.67) *has a unique solution $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{1.68}$$

and μ_j, $j = 1, \dots, n$, remain distinct on Ω_μ.
(ii) Suppose in addition to (1.10) *that $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is real-valued and the affine part of $\mathcal{K}_n$ is nonsingular. Moreover, assume the eigenvalue orderings* (1.63), (1.64)*. Then $\{\hat{\mu}_j\}_{j=1,\dots,n}$, with the projections $\mu_j(x)$, $j = 1, \dots, n$, the Dirichlet eigenvalues of $-d^2/dx^2 + u$ corresponding to a Dirichlet boundary condition at $x \in \mathbb{R}$ (i.e., the eigenvalues of H_x^D), satisfies the differential equation* (1.66) *for $x \in \mathbb{R}$. Furthermore, given initial data satisfying $\mu_j(x_0) \in [E_{2j-1}, E_{2j}]$, $j = 1, \dots, n$, then*

$$\mu_j(x) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n, \; x \in \mathbb{R}. \tag{1.69}$$

In particular, $\hat{\mu}_j(x)$ changes sheets whenever it hits E_{2j-1} or E_{2j} and its projection $\mu_j(x)$ remains trapped in $[E_{2j-1}, E_{2j}]$ for all $j = 1, \dots, n$ and $x \in \mathbb{R}$.

Proof Equations (1.31) and (1.35) imply

$$F_{n,x}(\mu_j) = -\mu_{j,x} \prod_{\substack{k=1 \\ k \neq j}}^{n} (\mu_j - \mu_k) = 2iy(\hat{\mu}_j),$$

proving (1.66). To verify property (1.68) of the solutions $\hat{\mu}_j$, one invokes the charts (B.3)–(B.6) and (B.12)–(B.15). In particular, the only nontrivial issue to check is the case in which $\hat{\mu}_j$ hits one of the branch points $(E_m, 0) \in \mathcal{B}(\mathcal{K}_n)$ and hence the right-hand side of (1.66) vanishes. Therefore, we suppose

$$\mu_{j_0}(x) \to E_{m_0} \text{ as } x \to x_0 \in \Omega_\mu$$

for some $j_0 \in \{1, \dots, n\}$, $m_0 \in \{0, \dots, 2n\}$. Introducing

$$\zeta_{j_0}(x) = \sigma(\mu_{j_0}(x) - E_{m_0})^{1/2},\ \sigma = \pm 1, \quad \mu_{j_0}(x) = E_{m_0} + \zeta_{j_0}(x)^2$$

for x in an open interval centered around x_0, one finds that the Dubrovin equation (1.66) for μ_{j_0} becomes

$$\begin{aligned}\zeta_{j_0,x}(x) \underset{x\to x_0}{=} c(\sigma) &\Bigg(\prod_{\substack{m=0\\ m\neq m_0}}^{2n} (E_{m_0} - E_m)\Bigg)^{1/2} \\ &\times \Bigg(\prod_{\substack{k=1\\ k\neq j_0}}^{n} \big(E_{m_0} - \mu_k(x)\big)^{-1}\Bigg) \big(1 + O(\zeta_{j_0}(x)^2)\big)\end{aligned}$$

for some $|c(\sigma)| = 1$ and concludes (1.68). A simple strategy of proof of part (ii) in our context proceeds as follows. First one invokes the fact that the diagonal Green's function $g(z, x_0)$ associated with the $L^2(\mathbb{R})$-realization H of the differential expression $L = -d^2/dx^2 + u$ on all of $\mathbb{R}$ is given by

$$g(z, x_0) = \frac{i F_n(z, x_0)}{2 R_{2n+1}(z)^{1/2}}, \quad z \in \mathbb{C}_+.$$

This is discussed in detail in Appendix J (cf. (J.46)). The Herglotz property of $g(\,\cdot\,, x_0)$ (cf. (J.15)) together with Theorem I.3 then yields the interlacing property of $\{\mu_j(x_0)\}_{j=1,\dots,n}$ and $\{E_m\}_{m=0,\dots,2n}$, as described in (1.69) for $x = x_0$. Since $x_0 \in \mathbb{R}$ was arbitrary, this proves (1.69). $\square$

The analogous result for the general β-boundary conditions (1.54) reads as follows.

Lemma 1.11 *Let* $\beta \in \mathbb{R}$.
(i) Suppose that $u \in C^\infty(\widetilde{\Omega}_\lambda)$ *satisfies the nth stationary KdV equation* (1.10) *on an open interval* $\widetilde{\Omega}_\lambda \subseteq \mathbb{R}$. *Moreover, assume that the zeros* λ_ℓ^β, $\ell = 0, \dots, n$, *of* $K_{n+1}^\beta(\,\cdot\,)$ *remain distinct on* $\widetilde{\Omega}_\lambda$. *Then* $\{\hat{\lambda}_\ell^\beta\}_{\ell=0,\dots,n}$, *defined by* (1.61), *satisfies the following first-order system of differential equations on* $\widetilde{\Omega}_\lambda$

$$\lambda_{\ell,x}^\beta = -2i\big(\beta^2 - u + \lambda_\ell^\beta\big) y\big(\hat{\lambda}_\ell^\beta\big) \prod_{\substack{m=0\\ m\neq \ell}}^{n} \big(\lambda_\ell^\beta - \lambda_m^\beta\big)^{-1}, \quad \ell = 0, \dots, n. \quad (1.70)$$

Next, assume the affine part of $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\big\{\hat{\lambda}_\ell^\beta(x_0)\big\}_{\ell=0,\dots,n} \subset \mathcal{K}_n \tag{1.71}$$

for some $x_0 \in \mathbb{R}$, where $\lambda_\ell^\beta(x_0)$, $\ell = 0, \dots, n$, are assumed to be distinct. Then there exists an open interval $\Omega_\lambda \subseteq \mathbb{R}$, with $x_0 \in \Omega_\lambda$, such that the initial value problem (1.70), (1.71) *has a unique solution $\{\hat{\lambda}_\ell^\beta\}_{\ell=0,\dots,n} \subset \mathcal{K}_n$ satisfying*

$$\hat{\lambda}_\ell^\beta \in C^\infty(\Omega_\lambda, \mathcal{K}_n), \quad \ell = 0, \dots, n, \tag{1.72}$$

and λ_ℓ^β, $\ell = 0, \dots, n$, remain distinct on Ω_λ.
(ii) Suppose in addition to (1.10) *that $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is real-valued and the affine part of $\mathcal{K}_n$ is nonsingular. Moreover, assume the eigenvalue orderings* (1.63), (1.65)*. Then $\{\hat{\lambda}_\ell^\beta\}_{\ell=0,\dots,n}$, with $\lambda_\ell^\beta(x)$, $\ell = 0, \dots, n$, the eigenvalues of H_x^β, satisfies the differential equation* (1.70) *for $x \in \mathbb{R}$. Furthermore, given initial data $\lambda_0^\beta(x_0) \leq E_0,\ \lambda_\ell^\beta(x_0) \in [E_{2\ell-1}, E_{2\ell}]$, $\ell = 1, \dots, n$, then*

$$\lambda_0^\beta(x) \leq E_0,\ \lambda_\ell^\beta(x) \in [E_{2\ell-1}, E_{2\ell}], \quad \ell = 1, \dots, n,\ x \in \mathbb{R}. \tag{1.73}$$

In particular, $\hat{\lambda}_\ell^\beta(x)$ changes sheets whenever it hits $E_{2\ell-1}$ or $E_{2\ell}$ and its projection $\lambda_\ell^\beta(x)$ remains trapped in $[E_{2\ell-1}, E_{2\ell}]$ for all $\ell = 1, \dots, n$ and $x \in \mathbb{R}$ (and similarly for $\hat{\lambda}_0^\beta(x)$).

Proof The derivative with respect to x evaluated at $z = \lambda_\ell^\beta(x)$ of (1.55) and (1.59) reads

$$K_{n+1,x}^\beta\big(\lambda_\ell^\beta\big) = -\lambda_{\ell,x}^\beta \prod_{\substack{m=0\\ m\neq\ell}}^{n} \big(\lambda_\ell^\beta - \lambda_m^\beta\big). \tag{1.74}$$

and

$$\begin{aligned} &2\big((1/2)F_{n,x}\big(\lambda_\ell^\beta\big) + \beta F_n\big(\lambda_\ell^\beta\big)\big)\big((1/2)F_{n,xx}\big(\lambda_\ell^\beta\big) + \beta F_{n,x}\big(\lambda_\ell^\beta\big)\big) \\ &\quad = F_n\big(\lambda_\ell^\beta\big)K_{n+1,x}^\beta\big(\lambda_\ell^\beta\big), \end{aligned} \tag{1.75}$$

respectively. We will use (1.75) to evaluate the left-hand side of (1.74). From (1.61) we see that

$$(1/2)F_{n,x}\big(\lambda_\ell^\beta\big) + \beta F_n\big(\lambda_\ell^\beta\big) = -iy\big(\hat{\lambda}_\ell^\beta\big). \tag{1.76}$$

Eliminating $R_{2n+1}(z)$ using (1.13) and (1.59), taking $z = \lambda_\ell^\beta(x)$, implies

$$(1/2)F_{n,xx}\big(\lambda_\ell^\beta\big) + \beta F_{n,x}\big(\lambda_\ell^\beta\big) = (u - \lambda_\ell^\beta - \beta^2)F_n\big(\lambda_\ell^\beta\big). \tag{1.77}$$

Inserting (1.76) and (1.77) into (1.75) and subsequently into (1.74) proves (1.70). Equation (1.72) and the remainder of the proof follows that of Lemma 1.10 step by step, using (J.23) and (J.48) in connection with the β-boundary conditions (1.54) at the point $x_0 \in \mathbb{R}$. $\square$

In the following remark, the Neumann case, that is, the special case $\beta = 0$, is stated as a separate result.

Remark 1.12 In the Neumann case $\beta = 0$, equation (1.70) simplifies to

$$\nu_{\ell,x} = -2i\big(-u + \nu_\ell\big)y(\hat{\nu}_\ell)\prod_{\substack{m=0\\ m\neq\ell}}^{n}\big(\nu_\ell - \nu_m\big)^{-1}, \quad \ell = 0, \dots, n. \tag{1.78}$$

We remark that u in (1.70) and (1.78) has been used for reasons of brevity only. In order to obtain a coupled system of differential equations for λ_ℓ^β and ν_ℓ, one needs to replace u by the corresponding trace formula for u in Lemma 1.17.

Remark 1.13 Due to our convention (B.19) for $y(\,\cdot\,)$, the differential equations (1.66) and (1.70) exhibit the well-known (piecewise) monotonicity properties of $\mu_j(x)$ and $\lambda_\ell^\beta(x)$, $\beta \in \mathbb{R}$, $j \in \mathbb{N}$, $\ell \in \mathbb{N}_0$, with respect to $x \in \mathbb{R}$. For instance, Dirichlet eigenvalues corresponding to the right (left) half-axis (x, ∞) $((-\infty, x))$ associated with the decomposition (J.19) are always increasing (decreasing) with respect to $x \in \mathbb{R}$, etc.

We also mention the following well-known result connecting Dirichlet and Neumann eigenvalues in the self-adjoint case.

Lemma 1.14 *Suppose that $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is real-valued and satisfies the nth stationary KdV equation* (1.10). *Moreover, assume the affine part of $\mathcal{K}_n$ to be nonsingular. Suppose that $\mu_j(x_0) \in \{E_{2j-1}, E_{2j}\}$, $j = 1, \dots, n$. Then $\nu_0(x_0) = E_0$, $\nu_j(x_0) \in \{E_{2j-1}, E_{2j}\} \setminus \{\mu_j(x_0)\}$, $j = 1, \dots, n$. Conversely, suppose $\nu_j(x_0) \in \{E_{2j-1}, E_{2j}\}$, $j = 1, \dots, n$. Then $\nu_0(x_0) = E_0$, $\mu_j(x_0) \in \{E_{2j-1}, E_{2j}\} \setminus \{\nu_j(x_0)\}$, $j = 1, \dots, n$.*

Proof The derivative of (1.31), inserting the Dubrovin equations (1.66), yields

$$F_{n,x}(z) = 2i\sum_{j=1}^{n} iy(\hat{\mu}_j)\prod_{\substack{k=1\\ k\neq j}}^{n}(z - \mu_k)(\mu_j - \mu_k)^{-1}, \tag{1.79}$$

from which we infer that $F_{n,x}(z, x_0) = 0$ for all $z \in \mathbb{C}$. Equation (1.32) then implies that $R_{2n+1}(z) = F_n(z, x_0)H_{n+1}(z, x_0)$ for all z and fixed x_0. This proves the first claim.

Conversely, assuming $\nu_j(x_0) \in \{E_{2j-1}, E_{2j}\}$ for all $j = 1, \dots, n$, one infers from (1.36) that $F_{n,x}(\nu_j(x_0), x_0) = -2iy(\hat{\nu}_j(x_0)) = 0$ for all $j = 1, \dots, n$, that is, the polynomial $F_{n,x}$, which is of degree $n-1$, has at least n zeros. Hence, $F_{n,x}(z,x_0)=0$ for all z. We conclude that $R_{2n+1}(z)=F_n(z,x_0)H_{n+1}(z,x_0)$, which proves the second claim. $\square$

In an analogous fashion one can analyze the behavior of $\lambda_\ell^\beta(x)$ as a function of the boundary condition parameter $\beta\in\mathbb{R}$. In fact, (1.55) yields

$$\partial_\beta K_{n+1}^\beta(z) = -F_{n,x}(z) + 2\beta F_n(z), \tag{1.80}$$

and hence

$$\begin{aligned}\partial_\beta K_{n+1}^\beta(z)\big|_{z=\lambda_\ell^\beta} &= -\big(\partial_\beta\lambda_\ell^\beta\big)\prod_{\substack{m=0\\ m\neq\ell}}^n\big(\lambda_\ell^\beta-\lambda_m^\beta\big) = -F_{n,x}\big(\lambda_\ell^\beta\big)+2\beta F_n\big(\lambda_\ell^\beta\big)\\ &= -2iy\big(\hat\lambda_\ell^\beta\big)\end{aligned} \tag{1.81}$$

by (1.61). This implies the following result for the β-variation of the eigenvalues $\lambda_\ell^\beta(x)$.

Lemma 1.15 *Let $(x,\beta)\in\Omega\times\mathcal{U}$, where $\Omega,\mathcal{U}\subseteq\mathbb{R}$ are open intervals. Suppose that $u\in C^\infty(\Omega)$, satisfies the nth stationary KdV equation* (1.10) *on Ω, and assume that the zeros $\lambda_\ell^\beta(x)$, $\ell=0,\dots,n$, of $K_{n+1}^\beta(\,\cdot\,,x)$ remain distinct for $(x,\beta)\in\Omega\times\mathcal{U}$. Then $\{\hat\lambda_\ell^\beta\}_{\ell=0,\dots,n}$, defined by* (1.61)*, satisfies the following first-order system of differential equations on Ω*

$$\partial_\beta\lambda_\ell^\beta = 2iy\big(\hat\lambda_\ell^\beta\big)\prod_{\substack{m=0\\ m\neq\ell}}^n\big(\lambda_\ell^\beta-\lambda_m^\beta\big)^{-1},\ \ell=0,\dots,n. \tag{1.82}$$

Proof This follows from (1.81). $\square$

Combining the polynomial approach of Section 1.2 with (1.31) readily yields trace formulas for the KdV invariants, that is, expressions of f_ℓ in terms of symmetric functions of the zeros μ_j of F_n.

Lemma 1.16 *Suppose $u\in C^\infty(\mathbb{R})$ satisfies the nth stationary KdV equation* (1.10)*. Then,*

$$u = \sum_{m=0}^{2n}E_m - 2\sum_{j=1}^n\mu_j, \tag{1.83}$$

$$u^2-(1/2)u_{xx} = \sum_{m=0}^{2n}E_m^2 - 2\sum_{j=1}^n\mu_j^2,\ \textit{etc.} \tag{1.84}$$

Proof Relations (1.83) and (1.84) follow by comparison of powers of z^{n-1} and z^{n-2} in (1.31) for F_n with (1.5) taken into account. □

The analogous result for the general β-boundary conditions (1.54) then reads as follows.

Lemma 1.17 *Let $\beta \in \mathbb{R}$. Suppose $u \in C^\infty(\mathbb{R})$ satisfies the nth stationary KdV equation* (1.10). *Then,*

$$2\beta^2 - u = \sum_{m=0}^{2n} E_m - 2\sum_{\ell=0}^{n} \lambda_\ell^\beta, \tag{1.85}$$

$$(1/2)u_{xx} - u^2 + 2\beta u_x + 4\beta^2 u - 2\beta^4 = \sum_{m=0}^{2n} E_m^2 - 2\sum_{\ell=0}^{n} \big(\lambda_\ell^\beta\big)^2, \ \textit{etc.} \tag{1.86}$$

Proof Relations (1.83) and (1.84) are proved by comparing powers of z^n and z^{n-1} in (1.55) for K_{n+1} with (1.56) taken into account. □

Equations (1.83) and (1.85) represent trace formulas for the algebro-geometric potential u. Equations (1.83)–(1.86) (as well as the method of proof) indicate that higher-order trace formulas associated with the KdV hierarchy can be obtained from (1.31) and (1.55) when comparing powers of z. Since we will systematically derive trace formulas for general potentials in Section 1.5, we postpone the special case of algebro-geometric potentials at this point and refer to Example 1.60.

Since nonspecial divisors play a fundamental role in this section and the next, we now take a closer look at them.

Lemma 1.18 *Assume that $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfies the nth stationary KdV equation* (1.10). *Let $\mathcal{D}_{\underline{\hat{\mu}}}$, $\underline{\hat{\mu}} = (\hat{\mu}_1, \dots, \hat{\mu}_n)$ be the Dirichlet divisor of degree n associated with u defined according to* (1.35), *that is,*

$$\hat{\mu}_j(x) = (\mu_j(x), -(i/2)F_{n,x}(\mu_j(x), x)), \quad j = 1, \dots, n, \ x \in \mathbb{R}.$$

Then, $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ is nonspecial for all $x \in \mathbb{R}$.

Proof Since $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and μ_j vary continuously on $\mathbb{R}$ (with multiplicities and appropriate renumbering of the zeros of F_n taken into account), one infers the existence of constants $C_j > 0$, $j = 1, 2$, such that

$$\operatorname{Re}(\mu_j(x)) \geq -C_1, \quad |\operatorname{Im}(\mu_j(x))| \leq C_2, \quad j = 1, \dots, n, \ x \in \mathbb{R}.$$

In particular, since $u(x) = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j(x)$ according to the trace formula (1.83), this yields the existence of a constant $C > 0$ such that

$$|\mu_j(x)| \le C, \quad j = 1, \dots, n, \ x \in \mathbb{R}. \tag{1.87}$$

By Theorem A.30, $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ is special if and only if $\{\hat{\mu}_1(x), \dots, \hat{\mu}_n(x)\}$ contains at least one pair of the type $\{\hat{\mu}(x), \hat{\mu}^*(x)\}$. Hence, $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ is certainly nonspecial as long as the projections $\mu_j(x)$ of $\hat{\mu}_j(x)$ are mutually distinct, $\mu_j(x) \neq \mu_k(x)$ for $j \neq k$. On the other hand, if two or more projections collide for some $x_0 \in \mathbb{R}$, for instance,

$$\lim_{x \to x_0} \mu_{j_p}(x) = \mu_0, \quad p = 1, 2, \dots, N, \ N \in \{2, \dots, n\},$$

then $F_{n,x}(\mu_0, x_0) \neq 0$ as long as $\mu_0 \notin \{E_0, \dots, E_{2n}\}$. This fact immediately follows from (1.13), since $F_n(\mu_0, x_0) = 0$ but $R_{2n+1}(\mu_0) \neq 0$ by hypothesis. In particular, $\hat{\mu}_{j_1}(x), \dots, \hat{\mu}_{j_N}(x)$ all meet on the same sheet since

$$\lim_{x \to x_0} \hat{\mu}_{j_p}(x) = (\mu_0, F_{n,x}(\mu_0, x_0)), \quad p = 1, \dots, N;$$

hence, no special divisor can arise in this manner. It remains to study the case in which two or more projections collide at a branch point, say at $(E_{m_0}, 0)$ for some $x_0 \in \mathbb{R}$. In this case, one concludes

$$F_n(z, x_0) \underset{z \to E_{m_0}}{=} O\big((z - E_{m_0})^2\big)$$

and

$$F_{n,x}(E_{m_0}, x_0) = 0, \tag{1.88}$$

using again (1.13) and $F_n(E_{m_0}, x_0) = R_{2n+1}(E_{m_0}) = 0$. Since $F_{n,x}(\,\cdot\,, x_0)$ is a polynomial (of degree $n-1$), (1.88) implies

$$F_{n,x}(z, x_0) \underset{z \to E_{m_0}}{=} O((z - E_{m_0})).$$

Thus, using (1.13) once more, one obtains the contradiction,

$$\begin{aligned} O\big((z - E_{m_0})^2\big) &\underset{z \to E_{m_0}}{=} R_{2n+1}(z) \\ &\underset{z \to E_{m_0}}{=} (z - E_{m_0}) \left(\prod_{\substack{m=1 \\ m \neq m_0}}^{2n} \big(E_{m_0} - E_m\big) + O(z - E_{m_0}) \right). \end{aligned}$$

Consequently, at most one $\hat{\mu}_j(x)$ can hit a branch point at a time and again no special divisor arises. Finally, by (1.87), $\hat{\mu}_j(x)$ never reaches the branch point P_∞, completing the proof. $\square$

Next we turn to asymptotic properties of ϕ and ψ.

Lemma 1.19 *Suppose $u \in C^\infty(\mathbb{R})$ satisfies the nth stationary KdV equation* (1.10). *Moreover, let $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}$, $(x, x_0) \in \mathbb{R}^2$. Then,*

$$\phi(P) \underset{\zeta\to 0}{=} i\zeta^{-1} - (i/2)u\zeta + O(\zeta^2) \text{ as } P \to P_\infty, \tag{1.89}$$

$$\psi(P, x, x_0) \underset{\zeta\to 0}{=} \exp(i\zeta^{-1}(x - x_0))\Big(1 - (i/2)\int_{x_0}^x dx'\, u(x')\zeta + O(\zeta^2)\Big)$$
$$\text{as } P \to P_\infty, \quad \zeta = \sigma/z^{1/2},\ \sigma = \pm 1. \tag{1.90}$$

Proof The existence of the asymptotic expansion of ϕ in terms of the local coordinate $\zeta = \sigma/z^{1/2}$, $\sigma = \pm 1$ near P_∞ (cf. (B.7)–(B.11)) is clear from the explicit form of ϕ in (1.38). Insertion of the polynomial F_n into (1.38) then yields the explicit expansion coefficients in (1.89). Alternatively, one can insert the ansatz

$$\phi \underset{z\to\infty}{=} \phi_{-1}z^{1/2} + \phi_0 + \phi_1 z^{-1/2} + O(z^{-1}) \tag{1.91}$$

into the Riccati-type equation (1.42). A comparison of powers of $z^{-1/2}$ then proves (1.89). Equation (1.90) then follows from inserting (1.89) into (1.41). □

For subsequent use we note the following asymptotic spectral parameter expansion of F_n/y as $P \to P_\infty$,

$$\frac{F_n(z)}{y} \underset{\zeta\to 0}{=} \zeta \sum_{\ell=0}^\infty \hat{f}_\ell \zeta^{2\ell}, \quad \zeta = \sigma/z^{1/2},\ \sigma = \pm 1. \tag{1.92}$$

Here, $\hat{f}_\ell$ denote the homogeneous coefficients in (1.6) (i.e., the ones satisfying (1.4) with vanishing integration constants). In particular, $\hat{f}_\ell$ can be computed from a nonlinear recursion relation as proven in Theorem D.1 in Appendix D. The analogous expansion can be derived for H_{n+1}/y, K_{n+1}^β/y, $\beta \in \mathbb{R}$, etc. The spectral theoretic content of the polynomials F_n, H_{n+1}, and K_{n+1}^β is clearly displayed in (J.32)–(J.48) (particularly in the Green's function formulas (J.45) and (J.46) in connection with F_n).

We continue with the theta function representation for ϕ, ψ, and u. For the general background and fundamental notation we refer to Appendices A and B. To avoid the trivial case $n = 0$ (considered in Example 1.25), we assume $n \in \mathbb{N}$ for the remainder of this argument.

Let θ denote the Riemann theta function associated with $\mathcal{K}_n$ (whose affine part is assumed to be nonsingular) and $\{a_j, b_j\}_{j=1,\dots,n}$ be a fixed homology basis on $\mathcal{K}_n$. Next, choosing as a convenient base point Q_0 one of the branch points $(E_m, 0)$, $m \in \{0, \dots, 2n\}$, the Abel maps $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ are defined by (A.34) and (A.35), and the Riemann vector $\underline{\Xi}_{Q_0}$ is given by (A.45). Let $\omega^{(3)}_{P_\infty,\hat{\lambda}_0^\beta(x)}$ be the normal differential of the third kind holomorphic on $\mathcal{K}_n \setminus \{P_\infty, \hat{\lambda}_0^\beta(x)\}$ with simple poles at P_∞ and $\hat{\lambda}_0^\beta(x)$ and residues $+1$ and -1, respectively (cf. (A.23)–(A.26), (B.40)),

$$\omega^{(3)}_{P_\infty,\hat{\lambda}_0^\beta(x)} \underset{\zeta\to 0}{=} (\zeta^{-1} + O(\zeta))d\zeta \text{ as } P \to P_\infty, \tag{1.93}$$

$$\omega^{(3)}_{P_\infty,\hat{\lambda}_0^\beta(x)} \underset{\zeta\to 0}{=} (-\zeta^{-1} + O(1))d\zeta \text{ as } P \to \hat{\lambda}_0^\beta(x), \tag{1.94}$$

where ζ in (1.93) denotes the local coordinate

$$\zeta = \sigma/z^{1/2} \text{ for } P \text{ near } P_\infty, \quad \sigma = \pm 1$$

near P_∞, and analogously, ζ in (1.94) that near $\hat{\lambda}_0^\beta(x)$. In particular,

$$\int_{a_j} \omega^{(3)}_{P_\infty,\hat{\lambda}_0^\beta(x)} = 0, \quad j = 1, \dots, n, \tag{1.95}$$

and with $Q_0 = (E_{m_0}, 0)$,

$$\int_{Q_0}^P \omega^{(3)}_{P_\infty,\hat{\lambda}_0^\beta(x)} \underset{\zeta\to 0}{=} \ln(\zeta) + (1/2)\ln\big(E_{m_0} - \lambda_0^\beta(x)\big) + O(\zeta) \text{ as } P \to P_\infty, \tag{1.96}$$

$$\int_{Q_0}^P \omega^{(3)}_{P_\infty,\hat{\lambda}_0^\beta(x)} \underset{\zeta\to 0}{=} -\ln(\zeta) + (1/2)\ln(E_{m_0} - \lambda_0^\beta(x)) + O(\zeta) \text{ as } P \to \hat{\lambda}_0^\beta(x). \tag{1.97}$$

Equations (1.96) and (1.97) follow from (B.40) by computing $\int_{Q_0}^P \omega^{(3)}_{P_\infty,\hat{\lambda}_0^\beta(x)} + \int_{Q_0}^{P^*} \omega^{(3)}_{P_\infty,\hat{\lambda}_0^\beta(x)}$, choosing the same path of integration on both sheets $\Pi_\pm$.

Next, let $\omega^{(2)}_{P_\infty,0}$ denote the normalized differential of the second kind defined by

$$\omega^{(2)}_{P_\infty,0} = -\frac{1}{2y}\prod_{j=1}^n (z - \lambda_j)dz \underset{\zeta\to 0}{=} (\zeta^{-2} + O(1))d\zeta \text{ as } P \to P_\infty, \tag{1.98}$$

where the constants $\lambda_j \in \mathbb{C}$, $j = 1, \dots, n$ are determined by employing the normalization

$$\int_{a_j} \omega^{(2)}_{P_\infty,0} = 0, \quad j = 1, \dots, n. \tag{1.99}$$

One infers

$$\int_{Q_0}^P \omega^{(2)}_{P_\infty,0} \underset{\zeta\to 0}{=} -\zeta^{-1} + O(\zeta) \text{ as } P \to P_\infty, \tag{1.100}$$

since by (1.98), $\int_{Q_0}^P \omega^{(2)}_{P_\infty,0} + \int_{Q_0}^{P^*} \omega^{(2)}_{P_\infty,0} = 0$, choosing the same path of integration on both sheets $\Pi_\pm$. Thus, the right-hand side of (1.100) is odd with respect to ζ and hence contains no constant term. The vector of b-periods of $\omega^{(2)}_{P_\infty,0}/(2\pi i)$ is denoted by

$$\underline{U}_0^{(2)} = \big(U_{0,1}^{(2)}, \dots, U_{0,n}^{(2)}\big), \quad U_{0,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \omega_{P_\infty,0}^{(2)}, \ j = 1, \dots, n. \quad (1.101)$$

By (B.33) one concludes

$$U_{0,j}^{(2)} = -2c_j(n), \quad j = 1, \dots, n. \quad (1.102)$$

In the following it will be convenient to introduce the abbreviation

$$\begin{aligned} &\underline{z}(P, \underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ &P \in \mathcal{K}_n, \ \underline{Q} = \{Q_1, \dots, Q_n\} \in \operatorname{Sym}^n(\mathcal{K}_n). \end{aligned} \quad (1.103)$$

We note that by (A.52) and (A.53), $\underline{z}(\,\cdot\,, \underline{Q})$ is independent of the choice of base point Q_0.

Theorem 1.20 *Suppose that $u \in C^\infty(\Omega)$ satisfies the nth stationary KdV equation* (1.10) *on an open interval $\Omega \subseteq \mathbb{R}$. In addition, assume the affine part of $\mathcal{K}_n$ to be nonsingular and let $P \in \mathcal{K}_n \setminus \{P_\infty\}$, $\beta \in \mathbb{R}$, and $x, x_0 \in \Omega$. Moreover, suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\lambda}}^\beta(x)}$ is nonspecial for $x \in \Omega$. Then,*[1]

$$\begin{aligned} \phi(P, x) = -\beta + i\,&\frac{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x)))\theta(\underline{z}(P, \underline{\hat{\lambda}}^\beta(x)))}{\theta(\underline{z}(P_\infty, \underline{\hat{\lambda}}^\beta(x)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))} \\ &\times \exp\bigg(-\int_{Q_0}^P \omega_{P_\infty, \hat{\lambda}_0^\beta(x)}^{(3)} + (1/2)\ln(E_{m_0} - \lambda_0^\beta(x))\bigg), \end{aligned} \quad (1.104)$$

and[2]

$$\psi(P, x, x_0) = \frac{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0)))} \exp\bigg(-i(x - x_0)\int_{Q_0}^P \omega_{P_\infty,0}^{(2)}\bigg), \quad (1.105)$$

with the linearizing property of the Abel map,

$$\underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat{\mu}}(x)}\big) = \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}\big) + i\underline{U}_0^{(2)}(x - x_0), \quad (1.106)$$

$$\underline{\alpha}_{Q_0}\big(\mathcal{D}_{\hat{\lambda}_0^\beta(x)\underline{\hat{\lambda}}^\beta(x)}\big) = \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\hat{\lambda}_0^\beta(x_0)\underline{\hat{\lambda}}^\beta(x_0)}\big) + i\underline{U}_0^{(2)}(x - x_0). \quad (1.107)$$

[1] According to Remark 1.9, the right-hand side of (1.104) is symmetric with respect to $\hat{\lambda}_\ell^\beta$, $\ell = 0, \dots, n$; hence, the pair $\big(\hat{\lambda}_0^\beta, \underline{\hat{\lambda}}^\beta\big)$ can be replaced by any of the pairs $\big(\hat{\lambda}_\ell^\beta, \underline{\hat{\lambda}}^{\beta,\ell}\big)$, $\ell = 1, \dots, n$.

[2] To avoid multi-valued expressions in formulas such as (1.104), (1.105), etc., we agree always to choose the same path of integration connecting Q_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

The Its–Matveev formula for u reads

$$
\begin{aligned}
u(x) &= E_0 + \sum_{j=1}^{n}(E_{2j-1} + E_{2j} - 2\lambda_j) \\
&\quad - 2\partial_x^2 \ln\big(\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}))\big) \qquad (1.108)\\
&= E_0 + \sum_{j=1}^{n}(E_{2j-1} + E_{2j} - 2\lambda_j) \\
&\quad - 2\partial_x^2 \ln\big(\theta\big(\underline{\Xi}_{Q_0} + \underline{A}_{Q_0}\big(\hat{\lambda}_0^\beta(x)\big) + \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat{\lambda}}^\beta(x)}\big)\big)\big). \qquad (1.109)
\end{aligned}
$$

Proof First we temporarily assume that

$$
\mu_j(x) \neq \mu_{j'}(x),\ \lambda_k^\beta(x) \neq \lambda_{k'}^\beta(x) \text{ for } j \neq j', k \neq k' \text{ and } x \in \widetilde{\Omega} \tag{1.110}
$$

for appropriate $\widetilde{\Omega} \subseteq \Omega$. Since by (1.40), $\mathcal{D}_{\hat{\nu}_0\underline{\hat{\nu}}} \sim \mathcal{D}_{P_\infty\underline{\hat{\mu}}}$, and $P_\infty = (P_\infty)^* \notin \{\hat{\mu}_1, \ldots, \hat{\mu}_n\}$ by hypothesis, one can apply Theorem A.31 to conclude that $\mathcal{D}_{\underline{\hat{\nu}}} \in \mathrm{Sym}^n(\mathcal{K}_n)$ is nonspecial. This argument is of course symmetric with respect to $\underline{\hat{\mu}}$ and $\underline{\hat{\nu}}$. Thus, $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial if and only if $\mathcal{D}_{\underline{\hat{\nu}}}$ is. Next, let $\tilde{\phi}$ denote the right-hand side of (1.104) with the aim of proving $\phi = \tilde{\phi}$, with ϕ given by (1.38) (or (1.57)). By (1.60) one infers that $\phi(\,\cdot\,, x)$ has simple poles at $\underline{\hat{\mu}}(x)$ and P_∞ and simple zeros at $\hat{\lambda}_0^\beta(x)$ and $\underline{\hat{\lambda}}^\beta(x)$. By inspection, the function $\tilde{\phi}$ shares these properties using (1.96), (1.97), the expression (1.104) for $\tilde{\phi}$, and a special case of Riemann's vanishing theorem (cf. Theorem A.26). By the Riemann–Roch theorem (Theorem A.13) and since ϕ and $\tilde{\phi}$ share common zeros, one infers that $\tilde{\phi}/\phi = c$ for some constant $c \in \mathbb{C}$. (Actually, the Riemann–Roch theorem implies $\tilde{\phi} = c\phi + d$ for some $c, d \in \mathbb{C}$ since $\deg(\mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}) = n + 1$ and $i(\mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}) = 0$. However, since $\tilde{\phi}$ and ϕ have common zeros, one concludes that $d = 0$.) Using (1.89) and (1.96), one computes

$$
\frac{\tilde{\phi}}{\phi} \underset{\zeta\to 0}{=} \frac{-\beta + i(1 + O(\zeta))(\zeta^{-1} + O(1))}{i\zeta^{-1} + O(\zeta)} \underset{\zeta\to 0}{=} 1 + O(\zeta);
$$

hence, $c = 1$. This proves (1.104) subject to (1.110).

For the Baker–Akhiezer function ψ we will use the same strategy. However, the situation is slightly more involved in that ψ has an essential singularity at P_∞. Denote by $\tilde{\psi}$ the right-hand side of (1.105). To prove that $\psi = \tilde{\psi}$, with ψ given by (1.41), one first observes, using (1.35), the definition (1.38) of ϕ and the Dubrovin equations (1.66), that

$$
\phi(P, x') \underset{P\to\hat{\mu}_j(x')}{=} \partial_{x'} \ln(z - \mu_j(x')) + O(1). \tag{1.111}
$$

Together with (1.41), this implies

$$\psi(P,x,x_0) = \begin{cases} (z-\mu_j(x))O(1) & \text{as } P \to \hat{\mu}_j(x) \neq \hat{\mu}_j(x_0), \\ O(1) & \text{as } P \to \hat{\mu}_j(x) = \hat{\mu}_j(x_0), \\ (z-\mu_j(x_0))^{-1}O(1) & \text{as } P \to \hat{\mu}_j(x_0) \neq \hat{\mu}_j(x), \end{cases} \tag{1.112}$$
$$P = (z,y) \in \mathcal{K}_n,\ x, x_0 \in \widetilde{\Omega},$$

where $O(1) \neq 0$ in (1.112). Consequently, all zeros and poles of ψ and $\tilde{\psi}$ on $\mathcal{K}_n \setminus \{P_\infty\}$ are simple and coincide. Hence, one concludes by Theorem A.26 that ψ contains a factor $\theta(\underline{z}(P,\underline{\hat{\mu}}(x)))/\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0)))$. It remains to identify the essential singularity of ψ and $\tilde{\psi}$ at P_∞. The asymptotic spectral parameter expansion (1.89) of ϕ yields

$$\int_{x_0}^{x} dx'\, \phi(P,x') \underset{\zeta\to 0}{=} i(\zeta^{-1} + O(\zeta))(x-x_0) \text{ as } P \to P_\infty. \tag{1.113}$$

Thus, comparing (1.41), (1.100), the expression (1.105) for $\tilde{\psi}$, and (1.113) then shows that ψ and $\tilde{\psi}$ have identical exponential behavior up to order $O(\zeta)$ near P_∞. Consequently, ψ and $\tilde{\psi}$ share the same singularities and zeros, and the Riemann–Roch-type uniqueness result in Lemma B.2 (taking $t_r = t_{0,r}$) then proves that ψ and $\tilde{\psi}$ coincide up to normalization. The latter is determined by (1.48) (or (1.90)), implying

$$\psi(P,x,x_0)\psi(P^*,x,x_0) \underset{P\to P_\infty}{=} 1. \tag{1.114}$$

Hence, (1.105) holds subject to (1.110).

The Its–Matveev formula requires a more detailed analysis of the behavior of ψ near P_∞. For this, one needs to expand $\omega^{(2)}_{P_\infty,0}$ up to second-order with respect to ζ. By (1.98), one finds

$$\omega^{(2)}_{P_\infty,0} \underset{\zeta\to 0}{=} (\zeta^{-2} + \Lambda_0 + O(\zeta^2))\, d\zeta \text{ near } P_\infty, \tag{1.115}$$

abbreviating

$$\Lambda_0 = \frac{1}{2}\biggl(E_0 + \sum_{j=1}^{n}(E_{2j-1} + E_{2j} - 2\lambda_j)\biggr).$$

Combining (1.105) and (1.115), one computes

$$\psi(P,x,x_0) \underset{\zeta\to 0}{=} \exp\bigl(i(x-x_0)(\zeta^{-1} - \Lambda_0\zeta + O(\zeta^3))\bigr)\bigl(1 + c_1(x)\zeta + O(\zeta^2)\bigr), \tag{1.116}$$

where c_1 is yet to be determined. This implies that

$$\psi_{xx}(P,x,x_0) \underset{\zeta\to 0}{=} \bigl(-\zeta^{-2} + 2\Lambda_0 + 2ic_{1,x}(x) + O(\zeta)\bigr)\psi(P,x,x_0),$$

which results in

$$-\psi_{xx} + \big(2\Lambda_0 + 2ic_{1,x} - \zeta^{-2}\big)\psi \underset{\zeta\to 0}{=} O(\zeta)\psi.$$

The right-hand side is yet another Baker–Akhiezer function with the same divisors and same essential singularity at P_∞ as ψ. By the uniqueness theorem for functions of that type (see Lemma B.2) one infers

$$-\psi_{xx} + \big(2\Lambda_0 + 2ic_{1,x} - \zeta^{-2}\big)\psi = 0,$$

and hence that

$$u = 2\Lambda_0 + 2ic_{1,x}, \tag{1.117}$$

in agreement with (1.90). It remains to determine $c_{1,x}$. Since only the x-derivative of c_1 enters the expression for u, we merely have to analyze the first ratio of theta functions as the other ratio in the expression (1.105) for ψ is x-independent. From (B.31), one infers

$$\underline{\omega} \underset{\zeta\to 0}{=} (-2\underline{c}(n) + O(\zeta^2))d\zeta \text{ as } P \to P_\infty,$$

and hence

$$\begin{aligned}\underline{A}_{Q_0}(P) = \int_{Q_0}^{P} \underline{\omega} \pmod{L_n} &\underset{\zeta\to 0}{=} \underline{A}_{Q_0}(P_\infty) - 2\underline{c}(n)\zeta + O(\zeta^3)\\ &= \underline{A}_{Q_0}(P_\infty) + \underline{U}_0^{(2)}\zeta + O(\zeta^2) \text{ as } P \to P_\infty,\end{aligned}$$

using (1.102). Thus,

$$\begin{aligned}&\frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}))}\\ &\underset{\zeta\to 0}{=} 1 - \frac{\sum_{j=1}^{n} U_{0,j}^{(2)}\partial_{w_j}\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{w} + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}))\big|_{\underline{w}=0}}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}))}\zeta + O(\zeta^3),\end{aligned} \tag{1.118}$$

where $\sum_{j=1}^{n} U_{0,j}^{(2)}\partial_{w_j}$ denotes the directional derivative in the $\underline{U}_0^{(2)}$-direction.

Next we prove the linearity of the Abel map with respect to x in (1.106) subject to (1.110). From

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = \bigg(\sum_{j=1}^{n}\int_{Q_0}^{\hat{\mu}_j} \underline{\omega}\bigg) \pmod{L_n},$$

(B.30), and the Dubrovin equations (1.66) one infers

$$\partial_x \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = \sum_{j=1}^{n} \mu_{j,x} \sum_{k=1}^{n} \underline{c}(k)\frac{\mu_j^{k-1}}{y(\hat{\mu}_j)} = -\sum_{j,k=1}^{n} \underline{c}(k)\frac{2i\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j - \mu_\ell)}.$$

The following special case of Lagrange's interpolation formula (cf. Appendix E)

$$\sum_{j=1}^{n} \mu_j^{k-1} \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} (\mu_j - \mu_\ell)^{-1} = \delta_{k,n}, \quad \mu_j \in \mathbb{C}, \ j,k = 1,\dots,n$$

then yields (cf. also Theorem F.10)

$$\partial_x \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = -2i\underline{c}(n) = i\underline{U}_0^{(2)}, \quad x \in \widetilde{\Omega}. \tag{1.119}$$

Given (1.119) we may write

$$\begin{aligned}
&\sum_{j=1}^{n} iU_{0,j}^{(2)} \partial_{w_j} \theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{w} + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) + i\underline{U}_0^{(2)}(x - x_0)\big)\big|_{\underline{w}=0} \\
&\quad = \frac{d}{dx}\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) + i\underline{U}_0^{(2)}(x - x_0)\big)
\end{aligned}$$

and hence obtain from (1.118)

$$\frac{\theta(\underline{z}(P, \underline{\hat{\mu}}))}{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}))} \underset{\zeta \to 0}{=} 1 + i\partial_x \ln\big(\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}))\big)\zeta + O(\zeta^3).$$

Using (1.105) and (1.116) we may identify

$$c_{1,x} = i\partial_x^2 \ln\big(\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}))\big) \text{ on } \widetilde{\Omega},$$

and hence obtain the Its–Matveev formula (1.108) as a consequence of (1.117), assuming (1.110). The second equality (1.109) then follows from the linear equivalence $\mathcal{D}_{P_\infty \underline{\hat{\mu}}} \sim \mathcal{D}_{\hat{\lambda}_0^\beta \underline{\hat{\lambda}}^\beta}$, that is,

$$\underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = \underline{A}_{Q_0}\big(\hat{\lambda}_0^\beta\big) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^\beta}), \tag{1.120}$$

and because $\underline{A}_{Q_0}(P_\infty)$ is a half-period,

$$2\underline{A}_{Q_0}(P_\infty) = 0 \pmod{L_g}.$$

The extension of all these results from $\widetilde{\Omega}$ to Ω then simply follows from the continuity of $\underline{\alpha}_{P_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}}$ being nonspecial on Ω. Equation (1.107) then follows from (1.106) and (1.120). $\square$

Combining (1.106) and (1.108) shows the remarkable linearity of the theta function with respect to x in the Its–Matveev formula for u. In fact, one can rewrite (1.108) as

$$u(x) = \Lambda_0 - 2\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x)), \tag{1.121}$$

where

$$\underline{A} = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) - i\underline{U}_0^{(2)} x_0 + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}), \tag{1.122}$$

$$\underline{B} = i\underline{U}_0^{(2)}, \tag{1.123}$$

$$\Lambda_0 = E_0 + \sum_{j=1}^{n}(E_{2j-1} + E_{2j} - 2\lambda_j), \tag{1.124}$$

and hence the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B} \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ (and its homology basis), and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \mathrm{Sym}^n(\mathcal{K}_n)$ at the point x_0 as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ is assumed to be nonspecial.

Remark 1.21 If $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial and $P_\infty \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$, then $\mathcal{D}_{\underline{\hat{\lambda}}^\beta}$ is nonspecial by Theorem A.31 (cf. also Lemma 1.18).

Remark 1.22 The explicit representation (1.105) for ψ complements Lemma 1.8 and shows that ψ stays meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial (assuming the affine part of $\mathcal{K}_n$ to be nonsingular).

The algebro-geometric KdV potential u in the Its–Matveev formula (1.108) is complex-valued in general. To obtain real-valued potentials, one needs to impose certain symmetry constraints on $\mathcal{K}_n$ and additional constraints on $\underline{A}$ in (1.121), (1.122), which we will discuss next. In particular, the formal self-adjointness of the Lax differential expression $L = -d^2/dx^2 + u$ with a real-valued potential u leads to the reality constraints

$$E_0 < E_1 < \cdots < E_{2n} \tag{1.125}$$

on the zeros of R_{2n+1}, that is, all branch points of $\mathcal{K}_n$ different from P_∞ are assumed to be in real position.

Lemma 1.23 *Assume* (1.125)*, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ *is nonspecial for some* $x_0 \in \mathbb{R}$*, and choose the homology basis* $\{a_j, b_j\}_{j=1}^n$ *according to Theorem* A.36 *(i) (cf. the discussion in the paragraph following* (B.20) *and compare with Figure* B.2*, implementing the additional constraint* (1.125)*). Then the meromorphic solution u in the Its–Matveev formula* (1.108) *is real-valued if and only if* $\underline{A}$ *in* (1.122) *satisfies the constraint*

$$\mathrm{Re}(\underline{A}) = (1/2)\underline{\chi} \pmod{\mathbb{Z}^n}, \quad \underline{\chi} = (\chi_1, \dots, \chi_n), \quad \chi_j \in \{0, 1\}, \ j = 1, \dots, n. \tag{1.126}$$

In particular, under the present hypotheses, the set of real-valued stationary algebro-geometric KdV potentials V in (1.121) *consists of* 2^n *connected components* *indexed by* $\underline{\chi} = (\chi_1, \dots, \chi_n)$*,* $\chi_j \in \{0, 1\}$*,* $j = 1, \dots, n$*, and the component associated with* $\underline{\chi} = 0$ *comprises all real-valued smooth potentials* $u \in C^\infty(\mathbb{R})$*.*

Proof Define the antiholomorphic involution $\rho_+ \colon (z, y) \mapsto (\overline{z}, \overline{y})$, as in Example A.35 (iii). By Example A.35 (iii), Theorem A.36 (cf. (A.65), (A.69)–(A.71)), (B.27)–(B.29), (B.33), and (B.37)–(B.39) one infers that $(\mathcal{K}_n, \rho_+)$ is of dividing type and hence

$$\begin{aligned}
&r = n+1, \quad \overline{\tau} = -\tau, \quad R = 0, \quad \overline{\theta(\underline{z})} = \theta(\underline{\overline{z}}), \ \underline{z} \in \mathbb{C}^n, \\
&\rho_+(a_j) = a_j, \quad \rho_+(b_j) = -b_j, \ j = 1, \dots, n, \\
&\underline{c}(k) \in \mathbb{R}^n, \ k = 1, \dots, n, \quad \underline{U}_0^{(2)} \in \mathbb{R}^n, \\
&\lambda_j \in \mathbb{R}, \ j = 1, \dots, n.
\end{aligned}$$

Thus,

$$\overline{\underline{B}} = -\underline{B}$$

by (1.123); hence, real-valuedness of u in (1.121) is equivalent to

$$\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x)) = \overline{\partial_x^2 \ln(\theta(\underline{A} + \underline{B}x))} = \partial_x^2 \ln(\theta(-\overline{\underline{A}} + \underline{B}x)).$$

This, in turn, is equivalent to

$$\underline{A} = -\overline{\underline{A}} + \underline{m}_1 + \underline{n}_1 \tau, \quad \underline{m}_1, \underline{n}_1 \in \mathbb{Z}^n$$

and hence to

$$\mathrm{Re}(\underline{A}) = (1/2)\underline{m}_1, \quad \underline{m}_1 \in \mathbb{Z}^n.$$

Replacing $\underline{A}$ by $\underline{A} + \underline{m} + \underline{n}\tau$ with $\underline{m}, \underline{n} \in \mathbb{Z}^n$ then yields

$$\mathrm{Re}(\underline{A}) = (1/2)\underline{m}_1 - \underline{m}, \quad \underline{m}_1, \underline{m} \in \mathbb{Z}^n$$

and hence (1.126). Finally, since u is of the type

$$\begin{aligned}
&u(x) = \Lambda_0 - 2\partial_x^2 \ln(\theta(i\mathrm{Im}(\underline{A}) + i\mathrm{Im}(\underline{B})x + (1/2)\underline{\chi})), \\
&\underline{\chi} = (\chi_1, \dots, \chi_n), \ \ \chi_j \in \{0, 1\}, \ j = 1, \dots, n,
\end{aligned}$$

$u \in C^\infty(\mathbb{R})$ if and only if $\underline{\chi} = 0$ by (A.73) (with $\ell = 0$). □

Remark 1.24 The connected component of nonsingular KdV potentials (1.121) associated with $\underline{\chi} = 0$ in Lemma 1.23 can be described as follows: The initial position of $\hat{\mu}_j(x_0) \in \mathcal{K}_n$ must be chosen in real position with its projections lying in the spectral gaps of H, that is,

$$\mu_j(x_0) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n, \tag{1.127}$$

in order to render $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)})$ purely imaginary $\mod \mathbb{Z}^n$ and $u \in L^\infty(\mathbb{R})$. One can show that all real-valued and bounded algebro-geometric KdV potentials arise in this manner. In particular, as x varies, the motion of the projection $\mu_j(x)$ of $\hat{\mu}_j(x) \in \mathcal{K}_n$ remains confined to the spectral gap interval $[E_{2j-1}, E_{2j}]$, and $\hat{\mu}_j$ changes sheets whenever it hits a branch point. Topologically, this motion corresponds to one on a circle. Moreover, the initial data $\hat{\mu}_j(x_0)$, with the projections $\mu_j(x_0)$ constrained by (1.127), are independent of each other. Thus, the corresponding isospectral set of all smooth (algebro-geometric) KdV potentials $u \in C^\infty(\mathbb{R})$ corresponding to a fixed curve $\mathcal{K}_n$ constrained by (1.125), that is, the connected component associated with $\underline{\chi} = 0$ in Lemma 1.23, can then be identified with the n-dimensional real torus $\mathbb{T}^n$. Effective coordinates on this torus uniquely characterizing u are then the Dirichlet data $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0))$ (cf. also the notes to Section 1.3), or equivalently, Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ in real position constrained by (1.127). Solving the Dubrovin equations (1.66) with these initial data then recovers u for all $x \in \mathbb{R}$ from the trace formula (1.83). The Its–Matveev formula (1.108) for u then provides a concrete representation of the elements of this isospectral torus $\mathbb{T}^n$. The potentials u in (1.108), in general, will be quasi-periodic[1] with respect to $x \in \mathbb{R}$.

Real-valued KdV potentials u associated with $\mathcal{K}_n$ constrained by (1.125) can also be constructed by "misplacing" one or several initial values in the "wrong" spectral gap $(-\infty, E_0]$. This then results in the remaining $2^n - 1$ connected but noncompact components of isospectral and singular KdV potentials (the singularities being certain poles in x) in Lemma 1.23.

If in addition one is interested in periodic KdV potentials u with a real period $\Omega > 0$, the additional periodicity constraints

$$i\Omega \underline{U}_0^{(2)} \in \mathbb{Z}^n \setminus \{0\} \tag{1.128}$$

must be imposed. (By (B.45) this is equivalent to $2i\Omega\underline{c}(n) \in \mathbb{Z}^n \setminus \{0\}$.) In fact, the integers $m_j \in \mathbb{Z} \setminus \{0\}$ arising in (1.128), $i\Omega U_{0,j}^{(2)} = m_j$, have a topological interpretation as winding numbers since by oscillation theoretic arguments they describe the number of full revolutions of $\mu_j(x)$ in the jth spectral gap $[E_{2j-1}, E_{2j}]$, $j = 1, \dots, n$, as x traverses a periodicity interval of length Ω.

An alternative strategy of proof of the Its–Matveev formula (1.108) based on the trace formula (1.83) and theta function representations for symmetric functions of μ_j is outlined in (B.44)–(B.47).

Next, we briefly consider the trivial case $n = 0$ excluded in Theorem 1.20.

[1] A function $f \in C(\mathbb{R})$ is called quasi-periodic with fundamental periods $(\omega_1, \dots, \omega_n) \in (0, \infty)^n$ if $\omega_1, \dots, \omega_n$ are linearly independent over $\mathbb{Z}$ and there exists an $F \in C(\mathbb{R}^n)$ with periods $\omega_1, \dots, \omega_n$, $F(x_1, \dots, x_{j-1}, x_j + \omega_j, x_{j+1}, \dots, x_n) = F(x_1, \dots, x_n)$ such that $f(x) = F(x, \dots, x)$. f becomes periodic with period $\omega > 0$ if and only if $\omega_j = m_j\omega$ for $m_j \in \mathbb{N}$, $j = 1, \dots, n$.

Example 1.25 Assume $n = 0$, $P = (z, y) \in \mathcal{K}_0 \setminus \{P_\infty\}$, and let $(x, x_0) \in \mathbb{R}^2$. Then

$$\begin{aligned}
&\mathcal{K}_0 \colon \mathcal{F}_0(z, y) = y^2 - R_1(z) = y^2 - (z - E_0) = 0, \quad E_0 \in \mathbb{C},\\
&u(x) = E_0,\\
&\text{s-}\widehat{\mathrm{KdV}}_m(u) = 0, \ m \geq 0,\\
&L = -\frac{d^2}{dx^2} + E_0, \quad P_1 = \frac{d}{dx},\\
&F_0(z, x) = 1, \quad H_1(z, x) = z - E_0, \quad \nu_0(x) = E_0,\\
&\phi(P, x) = iy,\\
&\psi(P, x, x_0) = \exp(iy(x - x_0)).
\end{aligned}$$

Up to this point we assumed $u \in C^\infty(\mathbb{R})$ satisfies the stationary KdV equation (1.10) for some fixed $n \in \mathbb{N}_0$. Next we will show that solvability of the Dubrovin equations (1.66) on $\Omega_\mu \subseteq \mathbb{R}$ in fact implies equation (1.10) on Ω_μ. As pointed out in Remark 1.29, this amounts to solving the algebro-geometric initial value problem in the stationary case.

Theorem 1.26 *Fix $n \in \mathbb{N}$, assume the affine part of $\mathcal{K}_n$ to be nonsingular, and suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the stationary Dubrovin equations* (1.66) *on an open interval $\Omega_\mu \subseteq \mathbb{R}$ such that μ_j, $j = 1, \dots, n$, remain distinct on Ω_μ. Then $u \in C^\infty(\Omega_\mu)$, defined by*

$$u = E_0 + \sum_{j=1}^{n} \big(E_{2j-1} + E_{2j} - 2\mu_j\big), \tag{1.129}$$

satisfies the nth stationary KdV equation (1.10)*, that is,*

$$\text{s-KdV}_n(u) = 0 \text{ on } \Omega_\mu. \tag{1.130}$$

Proof Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (1.66), we introduce

$$F_n(z) = \prod_{j=1}^{n} (z - \mu_j) \text{ on } \mathbb{C} \times \Omega_\mu \tag{1.131}$$

and note that on Ω_μ,

$$\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) = (\mu_j, -(i/2)F_{n,x}(\mu_j)), \quad j = 1, \dots, n, \tag{1.132}$$

by (1.66) and $F_{n,x}(\mu_j) = -\mu_{j,x} \prod_{\substack{k=1\\k\neq j}}^{n} (\mu_j - \mu_k)$. Next we define a monic polynomial H_{n+1} of degree $n + 1$ on $\mathbb{C} \times \Omega_\mu$ such that (1.13) holds, that is,

$$R_{2n+1}(z) + (1/4)F_{n,x}(z)^2 = F_n(z)H_{n+1}(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{1.133}$$

The polynomial H_{n+1} exists since the left-hand side of (1.133) has zeros at $z = \mu_j$, $j = 1, \dots, n$ by (1.132). Hence, we may factor H_{n+1} as

$$H_{n+1}(z) = \prod_{\ell=0}^{n} (z - \nu_\ell).$$

A comparison of the coefficients of z^{2n} in (1.133) then yields

$$\sum_{m=0}^{2n} E_m = \sum_{j=1}^{n} \mu_j + \sum_{\ell=0}^{n} \nu_\ell \text{ on } \Omega_\mu. \tag{1.134}$$

Introducing u as in (1.129) and noticing $u \in C^\infty(\Omega_\mu)$ since $\mu_j \in C^\infty(\Omega_\mu)$, $j = 1, \dots, n$, we next define the polynomial P_{n-1} by

$$P_{n-1}(z) = H_{n+1}(z) - (1/2)F_{n,xx}(z) + (u - z)F_n(z) \text{ on } \mathbb{C} \times \Omega_\mu.$$

Since by (1.129) and (1.134)

$$\begin{aligned} P_{n-1}(z) &= z^{n+1} - z^n \sum_{\ell=0}^{n} \nu_\ell + z^n u - z^{n+1} + z^n \sum_{j=1}^{n} \mu_j + O(z^{n-1}) \\ &= O(z^{n-1}) \text{ as } |z| \to \infty, \end{aligned}$$

P_{n-1} is a polynomial of degree $n - 1$. Differentiating (1.133) with respect to x yields

$$F_{n,x}(z)H_{n+1}(z) + F_n(z)H_{n+1,x}(z) - (1/2)F_{n,x}(z)F_{n,xx}(z) = 0,$$

and hence

$$F_{n,x}(\mu_j)P_{n-1}(\mu_j) = F_n(\mu_j)(-H_{n+1,x}(\mu_j) + (u - \mu_j)F_{n,x}(\mu_j)) = 0$$

on Ω_μ. Restricting $x \in \Omega_\mu$ temporarily to $x \in \widetilde{\Omega}_\mu$, where $\widetilde{\Omega}_\mu \subseteq \Omega_\mu$ is defined by

$$\begin{aligned} \widetilde{\Omega}_\mu &= \{x \in \Omega_\mu \mid F_{n,x}(\mu_j(x), x) = 2iy(\hat{\mu}_j(x)) \neq 0, \ j = 1, \dots, n\} \\ &= \{x \in \Omega_\mu \mid \mu_j(x) \notin \{E_m\}_{m=0,\dots,2n}, \ j = 1, \dots, n\}, \end{aligned}$$

one infers

$$P_{n-1}(\mu_j(x), x) = 0, \quad j = 1, \dots, n, \ x \in \widetilde{\Omega}_\mu. \tag{1.135}$$

Since P_{n-1} is a polynomial of degree $n - 1$, (1.135) then yields

$$P_{n-1} = 0 \text{ on } \mathbb{C} \times \widetilde{\Omega}_\mu, \tag{1.136}$$

and hence (1.33),

$$H_{n+1}(z) = (1/2)F_{n,xx}(z) - (u - z)F_n(z) \text{ on } \mathbb{C} \times \widetilde{\Omega}_\mu, \tag{1.137}$$

and (1.13),

$$F_{n,xx}(z)F_n(z) - (1/2)F_{n,x}(z)^2 - 2(u-z)F_n(z)^2 = 2R_{2n+1}(z) \text{ on } \mathbb{C} \times \widetilde{\Omega}_\mu. \tag{1.138}$$

Differentiating (1.138) with respect to x then yields

$$F_{n,xxx}(z) - 4(u-z)F_{n,x}(z) - 2u_x F_n(z) = 0 \text{ on } \mathbb{C} \times \widetilde{\Omega}_\mu, \tag{1.139}$$

the fundamental equation (1.12). Introducing f_j, $j = 1, \dots, n$, on $\widetilde{\Omega}_\mu$ by

$$F_n(z,x) = \sum_{\ell=0}^{n} f_{n-\ell}(x) z^\ell, \quad (z,x) \in \mathbb{C} \times \widetilde{\Omega}_\mu,$$

(1.139) then yields the beginning of the basic recursion relation (1.4)

$$f_0 = 1, \quad f_{j,x} = -\tfrac{1}{4} f_{j-1,xxx} + u f_{j-1,x} + \tfrac{1}{2} u_x f_{j-1}, \quad j = 1, \dots, n, \ x \in \widetilde{\Omega}_\mu. \tag{1.140}$$

Define $f_{n+1,x}$ by (1.140) with $j = n+1$. Then

$$\begin{aligned} f_{n+1,x} &= -(1/4) f_{n,xxx} + u f_{n,x} + (1/2) u_x f_n \\ &= -(1/4) F_{n,xxx}(0) + u F_{n,x}(0) + (1/2) u_x F_n(0) = 0 \text{ on } \widetilde{\Omega}_\mu. \end{aligned}$$

Thus,

$$0 = -2 f_{n+1,x}(u) = \text{s-KdV}_n(u) \text{ on } \widetilde{\Omega}_\mu. \tag{1.141}$$

To extend (1.141) to all $x \in \Omega_\mu$, we next consider the case in which $\hat{\mu}_j$ hits one of the branch points $(E_m, 0)$. Hence, we suppose

$$\mu_{j_0}(x) \to E_{m_0} \text{ as } x \to x_0 \in \Omega_\mu$$

for some $j_0 \in \{1, \dots, n\}$, $m_0 \in \{0, \dots, 2n\}$. If one introduces

$$\zeta_{j_0}(x) = \sigma(\mu_{j_0}(x) - E_{m_0})^{1/2}, \ \sigma = \pm 1, \quad \mu_{j_0}(x) = E_{m_0} + \zeta_{j_0}(x)^2$$

for x in an open interval centered around x_0, the Dubrovin equation (1.66) for μ_{j_0} becomes

$$\begin{aligned} \zeta_{j_0,x}(x) \underset{x \to x_0}{=} c(\sigma) &\Bigg(\prod_{\substack{m=0 \\ m \neq m_0}}^{2n} (E_{m_0} - E_m) \Bigg)^{1/2} \\ &\times \prod_{\substack{k=1 \\ k \neq j_0}}^{n} (E_{m_0} - \mu_k(x))^{-1} \big(1 + O(\zeta_{j_0}(x)^2)\big) \end{aligned}$$

for some $|c(\sigma)| = 1$, and hence (1.135) extends to Ω_μ by continuity. Consequently, (1.136)–(1.141) extend to Ω_μ by continuity. $\square$

Remark 1.27 The explicit theta function representation (1.108) of u on Ω_μ in (1.129) then permits one to extend u beyond Ω_μ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ remains nonspecial (cf. Remark 1.22). This observation extends to all elementary symmetric functions of the Dirichlet eigenvalues and hence to higher-order KdV invariants.

Remark 1.28 Although we formulated Theorem 1.26 in terms of Dirichlet eigenvalues μ_j, $j = 1, \dots, n$ only, the analogous result (and strategy of proof) works for all β-boundary conditions (1.54) in terms of λ_ℓ^β, $\ell = 0, \dots, n$ for each $\beta \in \mathbb{R}$.

Remark 1.29 A closer look at Theorem 1.26 reveals that u is uniquely determined in an open neighborhood Ω of x_0 by $\mathcal{K}_n$ and the initial condition $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \operatorname{Sym}^n(\mathcal{K}_n)$, or equivalently, by the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)} \in \operatorname{Sym}^n(\mathcal{K}_n)$ at $x = x_0$. Since u can be extended meromorphically to $\mathbb{C}$ with singularities given by second-order poles, u is actually globally uniquely determined by $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$. Conversely, given $\mathcal{K}_n$ and u in an open neighborhood Ω of x_0, one can construct the corresponding polynomial $F_n(\,\cdot\,, x)$ for $x \in \Omega$ (using the recursion relation (1.4) to determine the homogeneous elements $\hat{f}_\ell$ and (D.9) to determine $c_\ell = c_\ell(\underline{E})$, $\ell = 0, \dots, n$) and then recover the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ for $x \in \Omega$ from the zeros of $F_n(\,\cdot\,, x)$ and from (1.35). This remark is of relevance in connection with determining the isospectral set of KdV potentials u in the sense that once the curve $\mathcal{K}_n$ is fixed, elements of the isospectral class of potentials are parametrized by (nonspecial) Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ (cf. Remark 1.24).

We will end this section by providing some examples, we hope will aid in illustrating the general results of this section. We consider also some examples involving singular curves and/or singular (i.e., meromorphic) algebro-geometric stationary KdV solutions u, even though the principal results of this section were predominantly formulated for curves with nonsingular affine parts. We recall our convention abbreviating algebro-geometric stationary solutions of some (and hence infinitely many such) stationary KdV equations as KdV potentials.

The case of *rational* KdV potentials is treated first.

Example 1.30 The case of rational KdV potentials.
(i) The simplest nontrivial rational potential arises in connection with the arithmetic genus one case, $n = 1$, where

$$\begin{aligned} u_1(x) &= \frac{2}{x^2}, \quad x \in \mathbb{R} \setminus \{0\}, \\ \text{s-}\widehat{\mathrm{KdV}}_m(u_1) &= 0, \;\; m \in \mathbb{N}. \end{aligned}$$

Then,

$$L = -\frac{d^2}{dx^2} + \frac{2}{x^2}, \qquad P_3 = -\frac{d^3}{dx^3} + \frac{3}{x^2}\frac{d}{dx} - \frac{3}{x^3},$$

and the corresponding rational curve reads

$$\mathcal{F}_1(z,y) = y^2 + z^3 = 0, \qquad E_0 = E_1 = E_2 = 0.$$

Furthermore,

$$F_1(z,x) = z + \frac{1}{x^2}, \qquad \mu_1(x) = -\frac{1}{x^2},$$
$$H_2(z,x) = z^2 - \frac{1}{x^2}z + \frac{1}{x^4}, \quad \nu_\ell(x) = \frac{1}{2x^2}\big(1 + (-1)^\ell\sqrt{3}\,i\big), \quad \ell = 0, 1,$$

and hence one obtains for the two branches ϕ_j, $j = 1, 2$, of ϕ

$$\phi_j(z,x) = \frac{y_j - \frac{1}{x^3}}{z + \frac{1}{x^2}} = -\frac{z^2 - \frac{1}{x^2}z + \frac{1}{x^4}}{y_j + \frac{1}{x^3}}, \quad j = 1, 2,$$
$$y_1 = iz^{3/2}, \; y_2 = -iz^{3/2}.$$

(ii) The next example is the arithmetic genus two case, $n = 2$. Here

$$u_2(x) = \frac{6}{x^2}, \quad x \in \mathbb{R}\setminus\{0\},$$
$$\text{s-}\widehat{\mathrm{KdV}}_m(u_2) = 0, \; m \geq 2,$$

and

$$L = -\frac{d^2}{dx^2} + \frac{6}{x^2}, \qquad P_5 = \frac{d^5}{dx^5} - \frac{15}{x^2}\frac{d^3}{dx^3} + \frac{45}{x^3}\frac{d^2}{dx^2} - \frac{45}{x^4}\frac{d}{dx}$$

with the rational curve

$$\mathcal{F}_2(z,y) = y^2 + z^5 = 0, \qquad E_m = 0, \; m = 0, \dots, 4. \tag{1.142}$$

Moreover,

$$F_2(z,x) = z^2 + \frac{3}{x^2}z + \frac{9}{x^4}, \quad \mu_j(x) = \frac{3}{2x^2}\big(-1 + (-1)^j\sqrt{3}\,i\big), \quad j = 1, 2,$$
$$H_3(z,x) = z^3 - \frac{3}{x^2}z^2 + \frac{36}{x^6},$$
$$\nu_\ell(x) = \frac{n_\ell}{x^2}, \quad \ell = 0, 1, 2,$$
$$n_0 = \big(1 - (17 - 12\sqrt{2})^{-1/3} - (17 - 12\sqrt{2})^{1/3}\big),$$
$$n_1 = \big(1 + (17 - 12\sqrt{2})^{-1/3}(1 + i\sqrt{3})/2 + (17 - 12\sqrt{2})^{1/3}(1 - i\sqrt{3})/2\big),$$
$$n_2 = \big(1 + (17 - 12\sqrt{2})^{-1/3}(1 - i\sqrt{3})/2 + (17 - 12\sqrt{2})^{1/3}(1 + i\sqrt{3})/2\big).$$

Thus,

$$\phi_j(z,x) = \frac{y_j - \frac{3}{x^3}z - \frac{18}{x^5}}{z^2 + \frac{3}{x^2}z + \frac{9}{x^4}} = -\frac{z^3 - \frac{3}{x^2}z^2 + \frac{36}{x^6}}{y_j + \frac{3}{x^3}z + \frac{18}{x^5}}, \quad j = 1, 2,$$
$$y_1 = -iz^{5/2}, \ y_2 = iz^{5/2}.$$

(iii) Next, we treat the arithmetic genus three case, $n = 3$. Here

$$u_3(x) = \frac{12}{x^2}, \quad x \in \mathbb{R} \setminus \{0\},$$
$$\text{s-}\widehat{\text{KdV}}_m(u_3) = 0, \ m \geq 3,$$

and thus

$$L = -\frac{d^2}{dx^2} + \frac{12}{x^2},$$
$$P_7 = -\frac{d^7}{dx^7} + \frac{42}{x^2}\frac{d^5}{dx^5} - \frac{210}{x^3}\frac{d^4}{dx^4} + \frac{315}{x^4}\frac{d^3}{dx^3} + \frac{630}{x^5}\frac{d^2}{dx^2} - \frac{2835}{x^6}\frac{d}{dx} + \frac{2835}{x^7}$$

with rational curve

$$\mathcal{F}_3(z,y) = y^2 + z^7 = 0, \qquad E_m = 0, \ m = 0, \dots, 6.$$

Furthermore,

$$F_3(z,x) = z^3 + \frac{6}{x^2}z^2 + \frac{45}{x^4}z + \frac{225}{x^6},$$
$$\mu_j(x) = \frac{m_j}{x^2}, \quad j = 1, 2, 3,$$
$$m_1 = \Big(-2 - 2^{1/3}11\big(-151 + 75\sqrt{5}\big)^{-1/3} + 2^{-1/3}\big(-151 + 75\sqrt{5}\big)^{1/3}\Big),$$
$$m_2 = \Big(-2 + 2^{-2/3}11(1 + i\sqrt{3})\big(-151 + 75\sqrt{5}\big)^{-1/3} - 2^{-4/3}(1 - i\sqrt{3})\big(-151 + 75\sqrt{5}\big)^{1/3}\Big),$$
$$m_3 = \Big(-2 + 2^{-2/3}11(1 - i\sqrt{3})\big(-151 + 75\sqrt{5}\big)^{-1/3} - 2^{-4/3}(1 + i\sqrt{3})\big(-151 + 75\sqrt{5}\big)^{1/3}\Big),$$
$$H_4(z,x) = z^4 - \frac{6}{x^2}z^3 - \frac{9}{x^4}z^2 + \frac{135}{x^6}z + \frac{2025}{x^8},$$
$$\nu_\ell(x) = \frac{n_\ell}{x^2}, \quad \ell = 0, \dots, 3,$$

$$\begin{aligned}
n_0 = \frac{3}{2} + \frac{1}{2}\bigg(& 15 + 993\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& + 3\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3}\bigg)^{1/2} \\
& - \frac{1}{2}\bigg(30 - 993\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& - 3\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3} \\
& - 54\bigg(\frac{5}{3} + \frac{331}{3}\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& + \frac{1}{3}\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3}\bigg)^{-1/2}\bigg)^{1/2}, \\
n_1 = \frac{3}{2} + \frac{1}{2}\bigg(& 15 + 993\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& + 3(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,))^{1/3}\bigg)^{1/2} \\
& + \frac{1}{2}\bigg(30 - 993\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& - 3\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3} \\
& - 54\bigg(\frac{5}{3} + \frac{331}{3}\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& + \frac{1}{3}\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3}\bigg)^{-1/2}\bigg)^{1/2}, \\
n_2 = \frac{3}{2} - \frac{1}{2}\bigg(& 15 + 993\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& + 3\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3}\bigg)^{1/2} \\
& - \frac{1}{2}\bigg(30 - 993\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& - 3\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3} \\
& + 54\bigg(\frac{5}{3} + \frac{331}{3}\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& + \frac{1}{3}\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3}\bigg)^{-1/2}\bigg)^{1/2}, \\
n_3 = \frac{3}{2} - \frac{1}{2}\bigg(& 15 + 993\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{-1/3} \\
& + 3\big(\tfrac{1}{2}(5083 + 225i\sqrt{2355}\,)\big)^{1/3}\bigg)^{1/2}
\end{aligned}$$

$$+\frac{1}{2}\Bigg(30-993\big(\tfrac{1}{2}(5083+225i\sqrt{2355}\,)\big)^{-1/3}$$
$$-3\big(\tfrac{1}{2}(5083+225i\sqrt{2355}\,)\big)^{1/3}$$
$$+54\Bigg(\frac{5}{3}+\frac{331}{3}\big(\tfrac{1}{2}(5083+225i\sqrt{2355}\,)\big)^{-1/3}$$
$$+\frac{1}{3}\big(\tfrac{1}{2}(5083+225i\sqrt{2355}\,)\big)^{1/3}\Bigg)^{-1/2}\Bigg)^{1/2}.$$

Thus,

$$\begin{aligned}\phi_j(z,x)&=\frac{y_j-\frac{6}{x^3}z^2-\frac{90}{x^5}z-\frac{675}{x^7}}{z^3+\frac{6}{x^2}z^2+\frac{45}{x^4}z+\frac{225}{x^6}}\\&=-\frac{z^4-\frac{6}{x^2}z^3-\frac{9}{x^4}z^2+\frac{135}{x^6}z+\frac{2025}{x^8}}{y_j+\frac{6}{x^3}z^2+\frac{90}{x^5}z+\frac{675}{x^7}},\quad j=1,2,\\ y_1&=-iz^{7/2},\ y_2=iz^{7/2}.\end{aligned}$$

(iv) In the general arithmetic genus $n\in\mathbb{N}$ case, the corresponding rational potential is given by

$$u_n(x)=\frac{n(n+1)}{x^2},\quad x\in\mathbb{R}\setminus\{0\},$$
$$\text{s-}\widehat{\mathrm{KdV}}_m(u_n)=0,\ m\geq n,$$

with associated rational curve

$$\mathcal{F}_n(z,y)=y^2+z^{2n+1}=0,\qquad E_m=0,\ m=0,\dots,2n. \tag{1.143}$$

(v) More generally, all rational (nonconstant) KdV potentials u_n in the arithmetic genus $n\in\mathbb{N}$ case, vanishing at infinity, are obtained as follows. Let $M\in\mathbb{N}$, $s_k\in\mathbb{N}$, and $x_k\in\mathbb{C}$ be pairwise distinct, $k=1,\dots,M$. Consider

$$u_n(x)=\sum_{k=1}^{M}s_k(s_k+1)(x-x_k)^{-2},\quad x\in\mathbb{R}\setminus\{x_k\}_{k=1,\dots,N} \tag{1.144}$$

subject to the constraints

$$n(n+1)=\sum_{k=1}^{M}s_k(s_k+1), \tag{1.145}$$

$$\sum_{\substack{k'=1\\k'\neq k}}^{M}\frac{s_{k'}(s_{k'}+1)}{(x_k-x_{k'})^{2\ell+1}}=0\quad\text{for }\ell=1,\dots,s_k\ \text{ and }\ k=1,\dots,M. \tag{1.146}$$

Then u is a rational KdV potential vanishing at infinity if and only if u is of the type (1.144) and the constraints (1.145), (1.146) hold. In particular, for fixed n, the constraints (1.145), (1.146) characterize the isospectral class of all rational KdV potentials associated with the rational curve (1.143), and all KdV potentials of the type (1.144)–(1.146) satisfy

$$\text{s-}\widehat{\text{KdV}}_m(u_n) = 0, \;\; m \geq n.$$

Our second example describes the *n-soliton* potentials of the stationary KdV hierarchy.

Example 1.31 The case of n-soliton KdV potentials.
Let $n \in \mathbb{N}$. Then

$$\begin{aligned}
&u_n(x) = -2\frac{d^2}{dx^2}\ln(\tau_n(x)), \quad x \in \mathbb{R} \setminus \{y \in \mathbb{R} \,|\, \tau_n(y) = 0\},\\
&\tau_n(x) = \det(I_n + C_n(x)),\\
&C_n(x) = \big(c_j c_k (\kappa_j + \kappa_k)^{-1} \exp(-(\kappa_j + \kappa_k)x)\big)_{j,k=1,\dots,n},\\
&\qquad\qquad\qquad\qquad\qquad c_j, \kappa_j \in \mathbb{C}, \;\; j = 1, \dots, n,\\
&\text{s-}\widehat{\text{KdV}}_n(u_n) = 0,
\end{aligned}$$

with associated singular curve

$$\begin{aligned}
&\mathcal{F}_n(z, y) = y^2 + z\prod_{j=1}^{n}\big(z + \kappa_j^2\big)^2 = 0,\\
&E_{2j-2} = E_{2j-1} = -\kappa_j^2, \;\; j = 1, \dots, n, \;\; E_{2n} = 0.
\end{aligned}$$

Nonsingular soliton potentials are obtained upon imposing the restrictions $c_j > 0$, $\kappa_j > 0$, $j = 1, \dots, n$.

Finally we consider *elliptic* KdV potentials. By $\wp(\,\cdot\,) = \wp(\,\cdot \,|\, \omega_1, \omega_3) = \wp(\,\cdot\,; g_2, g_3)$, we denote the Weierstrass $\wp$-function with periods $2\omega_j$, $j = 1, 3$, $\mathrm{Im}(\omega_3/\omega_1) \neq 0$, $\omega_2 = \omega_1 + \omega_3$, invariants g_2 and g_3, and associated fundamental period parallelogram Δ (cf. Appendix H).

Example 1.32 The case of Lamé potentials.
(i) In the genus one case, $n = 1$, one has

$$\begin{aligned}
&u_1(x) = 2\wp(x), \quad x \in \mathbb{R}, \; x \neq 0 \pmod{\Delta},\\
&\text{s-}\widehat{\text{KdV}}_1(u_1) = 0,\\
&\text{s-}\widehat{\text{KdV}}_2(u_1) - \frac{1}{8}g_2\text{s-}\widehat{\text{KdV}}_0(u_1) = 0, \;\text{etc.},
\end{aligned}$$

and

$$L = -\frac{d^2}{dx^2} + 2\wp(x), \quad P_3 = -\frac{d^3}{dx^3} + 3\wp(x)\frac{d}{dx} + \frac{3}{2}\wp'(x),$$

with elliptic curve

$$\mathcal{F}_1(z, y) = y^2 + \left(z^3 - \frac{g_2}{4}z + \frac{g_3}{4}\right) = 0,$$
$$E_0 = -\wp(\omega_1),\ E_1 = -\wp(\omega_2),\ E_2 = -\wp(\omega_3).$$

Moreover,

$$F_1(z, x) = z + \wp(x), \quad \mu_1(x) = -\wp(x),$$
$$H_2(z, x) = z^2 - \wp(x)z + \wp(x)^2 - \frac{g_2}{4},$$
$$\nu_\ell(x) = \frac{1}{2}\big(\wp(x) - (-1)^\ell (g_2 - 3\wp(x)^2)^{1/2}\big), \quad \ell = 0, 1.$$

Thus,

$$\phi_j(z, x) = \frac{y_j + \frac{1}{2}\wp'(x)}{z + \wp(x)} = -\frac{z^2 - \wp(x)z + \wp(x)^2 - \frac{g_2}{4}}{y_j - \frac{1}{2}\wp'(x)}, \quad j = 1, 2,$$
$$y_j = (-1)^{j+1} i\left(z^3 - \frac{g_2}{4}z + \frac{g_3}{4}\right)^{1/2}, \quad j = 1, 2.$$

(ii) In the genus two case, $n = 2$, one obtains

$$u_2(x) = 6\wp(x), \quad x \in \mathbb{R},\ x \neq 0 \pmod{\Delta},$$
$$\text{s-}\widehat{\text{KdV}}_2(u_2) - \frac{21}{8}g_2\text{s-}\widehat{\text{KdV}}_0(u_2) = 0,$$
$$\text{s-}\widehat{\text{KdV}}_3(u_2) - \frac{21}{8}g_2\text{s-}\widehat{\text{KdV}}_1(u_2) - \frac{27}{8}g_3\text{s-}\widehat{\text{KdV}}_0(u_2) = 0, \text{ etc.},$$

and

$$L = -\frac{d^2}{dx^2} + 6\wp(x),$$
$$P_5 = \frac{d^5}{dx^5} - 15\wp(x)\frac{d^3}{dx^3} - \frac{45}{2}\wp'(x)\frac{d^2}{dx^2} + \left(\frac{27}{4}g_2 - 45\wp(x)^2\right)\frac{d}{dx},$$

with hyperelliptic curve

$$\mathcal{F}_2(z, y) = y^2 + \left(z^5 - \frac{21}{4}g_2z^3 - \frac{27}{4}g_3z^2 + \frac{27}{4}g_2^2z + \frac{81}{4}g_2g_3\right)$$
$$= y^2 + (z^2 - 3g_2)(z^3 - \frac{9}{4}g_2z - \frac{27}{4}g_3) = 0,$$
$$E_0 = -(3g_2)^{1/2},\ E_1 = 3\wp(\omega_3),\ E_2 = 3\wp(\omega_2),$$
$$E_3 = 3\wp(\omega_1),\ E_4 = (3g_2)^{1/2}.$$

Furthermore,

$$\begin{aligned}
F_2(z,x) &= z^2 + 3\wp(x)z + 9\wp(x)^2 - \frac{9}{4}g_2,\\
\mu_j(x) &= \frac{3}{2}\big(-\wp(x) + (-1)^j(g_2 - 3\wp(x)^2)^{1/2}\big), \quad j=1,2,\\
H_3(z,x) &= z^3 - 3\wp(x)z^2 - 3g_2 z - 9g_3 + 36\wp(x)^3,\\
\nu_0(x) &= \wp(x) + 2^{1/3}\left(g_2 + \wp(x)^2\right)A(x)^{-1/3} + 2^{-\frac{1}{3}}A(x)^{1/3},\\
\nu_1(x) &= \wp(x) - 2^{-2/3}\,(1+\sqrt{3}i)\,\left(g_2 + \wp(x)^2\right)A(x)^{-1/3}\\
&\quad - 2^{-4/3}(1-\sqrt{3}i)\,A(x)^{1/3},\\
\nu_2(x) &= \wp(x) - 2^{-2/3}\,(1-\sqrt{3}i)\,\left(g_2 + \wp(x)^2\right)A(x)^{-1/3}\\
&\quad - 2^{-4/3}(1+\sqrt{3}i)\,A(x)^{1/3},
\end{aligned}$$

where we used the abbreviation

$$\begin{aligned}
A(x) &= 9\,g_3 + 3\,g_2\,\wp(x) - 34\,\wp(x)^3\\
&\quad + \left(\left(9\,g_3 + 3\,g_2\,\wp(x) - 34\,\wp(x)^3\right)^2 - 4\left(g_2 + \wp(x)^2\right)^3\right)^{1/2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\phi_j(z,x) &= \frac{y_j + \frac{3}{2}\wp'(x)z + 9\wp(x)\wp'(x)}{z^2 + 3\wp(x)z + 9\wp(x)^2 - \frac{9}{4}g_2}\\
&= -\frac{z^3 - 3\wp(x)z^2 - 3g_2 z - 9g_3 + 36\wp(x)^3}{y_j - \frac{3}{2}\wp'(x)z - 9\wp(x)\wp'(x)}, \quad j=1,2,
\end{aligned}$$

$$y_j = (-1)^j i\Big(z^5 - \frac{21}{4}g_2 z^3 - \frac{27}{4}g_3 z^2 + \frac{27}{4}g_2^2 z + \frac{81}{4}g_2 g_3\Big)^{1/2}, \quad j=1,2.$$

(iii) In the general genus $n \in \mathbb{N}$ case, the corresponding Lamé potential is given by

$$\begin{aligned}
&u_n(x) = n(n+1)\wp(x), \quad x\in\mathbb{R},\ x\neq 0 \pmod{\Delta},\\
&\text{s-KdV}_n(u_n) = 0
\end{aligned} \tag{1.147}$$

for a particular set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$ in (1.147).
(iv) More generally, all elliptic (nonconstant) KdV potentials u_n in the genus $n \in \mathbb{N}$ case are obtained as follows. Let $M \in \mathbb{N}$, $s_k \in \mathbb{N}$, $u_0 \in \mathbb{C}$, and $x_k \in \mathbb{C}$ be pairwise distinct $\pmod{\Delta}$, $k = 1,\dots,M$. Consider

$$\begin{aligned}
&u_n(x) = u_0 + \sum_{k=1}^{M} s_k(s_k+1)\wp(x - x_k),\\
&x\in\mathbb{R},\ x\neq x_k \pmod{\Delta},\ k=1,\dots,N
\end{aligned} \tag{1.148}$$

subject to the constraints

$$n(n+1) = \sum_{k=1}^{M} s_k(s_k+1), \tag{1.149}$$

$$\sum_{k'=1 k'\neq k}^{M} s_{k'}(s_{k'}+1)\wp^{(\ell)}(x_k - x_{k'}) = 0 \quad \text{for } \ell = 1,\dots,s_k \ \text{ and } \ k = 1,\dots,M. \tag{1.150}$$

Then u is an elliptic KdV potential if and only if u is of the type (1.148) and the constraints (1.149), (1.150) hold. In particular, for fixed n, the constraints (1.149), (1.150) characterize the isospectral class of all elliptic KdV potentials associated with a curve of the form (1.30), (1.62). All KdV potentials of the type (1.148)–(1.150) satisfy

$$\text{s-KdV}_n(u_n) = 0 \tag{1.151}$$

for a particular set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$ in (1.151).

The rational case studied in Example 1.30 is obtained by letting ω_1 and ω_3 tend to infinity in (1.147) assuming that ω_3/ω_1 and ω_1/ω_3 do not converge to a real number. Under these circumstances,

$$\lim_{\omega_1,\omega_3\to\infty} \wp(x\,|\,\omega_1,\omega_3) = x^{-2}.$$

Similarly, the soliton case described in Example 1.31 is obtained by letting ω_1 tend to infinity in (1.147), keeping ω_3 fixed.

1.4 The Time-Dependent KdV Formalism

In this section we extend the algebro-geometric analysis of Section 1.3 to the time-dependent KdV hierarchy.

For most of this section we assume the following hypothesis.

Hypothesis 1.33 *Suppose that* $u\colon \mathbb{R}^2 \to \mathbb{C}$ *satisfies*

$$u(\,\cdot\,,t) \in C^\infty(\mathbb{R}),\ t\in\mathbb{R}, \quad u(x,\,\cdot\,) \in C^1(\mathbb{R}),\ x\in\mathbb{R}. \tag{1.152}$$

The basic problem in the analysis of algebro-geometric solutions of the KdV hierarchy consists in solving the time-dependent rth KdV flow with initial data a stationary solution of the nth equation in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, consider a solution $u^{(0)}$ of the nth stationary KdV equation $\text{s-KdV}_n(u^{(0)}) = 0$ associated with $\mathcal{K}_n$ and a given set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to construct a solution u of the rth KdV flow $\text{KdV}_r(u) = 0$ with $u(t_{0,r}) = u^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the integration constants in the definitions of the stationary and the time-dependent KdV equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence, we employ the notation $\widetilde{P}_{2r+1}$, $\widetilde{F}_r$, $\widetilde{H}_{r+1}$, $\widetilde{K}^{\beta}_{r+1}$, $\tilde{f}_s$, $\tilde{c}_s$ to distinguish them from P_{2n+1}, F_n, H_{n+1}, K^{β}_{n+1}, f_ℓ, c_ℓ in the following. In addition, we will follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth KdV flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution u of the time-dependent algebro-geometric initial value problem

$$\widetilde{\mathrm{KdV}}_r(u) = u_{t_r} - 2\tilde{f}_{r+1,x}(u) = 0, \quad u\big|_{t_r=t_{0,r}} = u^{(0)}, \tag{1.153}$$

$$\text{s-KdV}_n(u^{(0)}) = -2f_{n+1,x}(u^{(0)}) = 0 \tag{1.154}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$, where $u = u(x, t_r)$ satisfies (1.152) and a fixed curve $\mathcal{K}_n$ is associated with the stationary KdV solution $u^{(0)}$ in (1.154). In terms of Lax pairs this amounts to solving

$$\frac{d}{dt_r}L(t_r) - [\widetilde{P}_{2r+1}(t_r), L(t_r)] = 0, \quad t_r \in \mathbb{R}, \tag{1.155}$$

$$[P_{2n+1}(t_{0,r}), L(t_{0,r})] = 0. \tag{1.156}$$

In anticipating that the KdV_r flows are isospectral deformations of $L(t_{0,r})$, we are going a step further, replacing (1.156) by

$$[P_{2n+1}(t_r), L(t_r)] = 0, \quad t_r \in \mathbb{R}. \tag{1.157}$$

This then implies

$$-P^2_{2n+1}(t_r) = R_{2n+1}(L(t_r)) = \prod_{j=0}^{2n}(L(t_r) - E_j), \quad t_r \in \mathbb{R}. \tag{1.158}$$

Here we base the explicit solution of (1.153) not directly on (1.155), (1.157), and (1.158), but instead take the following equations as our point of departure,

$$u_{t_r} = -(1/2)\widetilde{F}_{r,xxx}(z) + 2(u-z)\widetilde{F}_{r,x}(z) + u_x\widetilde{F}_r(z), \tag{1.159}$$

$$(1/2)F_{n,xx}(z)F_n(z) - (1/4)F_{n,x}(z)^2 - (u-z)F_n(z)^2 = R_{2n+1}(z), \tag{1.160}$$

where

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell}z^\ell = \prod_{j=1}^{n}(z-\mu_j), \tag{1.161}$$

$$\widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s}z^s \tag{1.162}$$

for fixed $n, r \in \mathbb{N}_0$. Here f_ℓ, $\ell = 0, \dots, n$, and $\tilde{f}_s$, $s = 0, \dots, r$, are defined as in (1.5) with appropriate sets of integration constants c_ℓ, $\tilde{c}_s$, etc.

First we will assume the existence of a solution u of equations (1.159), (1.160) and derive an explicit formula for u in terms of Riemann theta functions. In addition, we will show in Theorem 1.48 that (1.159), (1.160), and hence the algebro-geometric initial value problem (1.153), (1.154) has a solution at least locally, that is, for $(x,t_r)\in\Omega$ for some open and connected set $\Omega\subseteq\mathbb{R}^2$.

We recall from (1.79), (1.32)–(1.34), (1.35), and (1.36) that

$$-(1/2)F_{n,x}(z)=-i\sum_{j=1}^{n}y(\hat\mu_j)\prod_{\substack{k=1\\k\neq j}}^{n}(z-\mu_k)(\mu_j-\mu_k)^{-1},$$

$$R_{2n+1}(z)+(1/4)F_{n,x}(z)^2=F_n(z)H_{n+1}(z),$$

$$H_{n+1}(z)=\prod_{\ell=0}^{n}(z-\nu_\ell)=(1/2)F_{n,xx}(z)-(u-z)F_n(z),$$

$$\hat\mu_j(x,t_r)=\big(\mu_j(x,t_r),-(i/2)F_{n,x}(\mu_j(x,t_r),x,t_r)\big)\in\mathcal{K}_n, \quad j=1,\dots,n,\ (x,t_r)\in\mathbb{R}^2, \tag{1.163}$$

$$\hat\nu_\ell(x,t_r)=\big(\nu_\ell(x,t_r),(i/2)F_{n,x}(\nu_j(x,t_r),x,t_r)\big)\in\mathcal{K}_n, \quad \ell=0,\dots,n,\ (x,t_r)\in\mathbb{R}^2. \tag{1.164}$$

As in Section 1.3, the regularity assumptions (1.152) on u imply analogous regularity assumptions on F_n, H_{n+1}, μ_j, and ν_ℓ.

In analogy to (1.38), (1.39), and (1.41) one then considers the fundamental meromorphic function $\phi(\,\cdot\,,x,t_r)$ on $\mathcal{K}_n$,

$$\phi(P,x,t_r)=\frac{iy+(1/2)F_{n,x}(z,x,t_r)}{F_n(z,x,t_r)} \tag{1.165}$$

$$=\frac{-H_{n+1}(z,x,t_r)}{iy-(1/2)F_{n,x}(z,x,t_r)}, \tag{1.166}$$

$$P=(z,y)\in\mathcal{K}_n,\ (x,t_r)\in\mathbb{R}^2$$

with divisor $(\phi(\,\cdot\,,x,t_r))$ of $\phi(\,\cdot\,,x,t_r)$ given by[1]

$$(\phi(\,\cdot\,,x,t_r))=\mathcal{D}_{\hat\nu_0(x,t_r)\underline{\hat\nu}(x,t_r)}-\mathcal{D}_{P_\infty\underline{\hat\mu}(x,t_r)}, \tag{1.167}$$

where

$$\underline{\hat\mu}=\{\hat\mu_1,\dots,\hat\mu_n\},\ \underline{\hat\nu}=\{\hat\nu_1,\dots,\hat\nu_n\}\in\operatorname{Sym}^n(\mathcal{K}_n),$$

and the time-dependent Baker–Akhiezer function $\psi(\,\cdot\,,x,x_0,t_r,t_{0,r})$ on $\mathcal{K}_n\setminus\{P_\infty\}$,

$$\psi(P,x,x_0,t_r,t_{0,r})=\exp\bigg(\int_{t_{0,r}}^{t_r}ds\,\big(\widetilde F_r(z,x_0,s)\phi(P,x_0,s)-\tfrac12\widetilde F_{r,x}(z,x_0,s)\big) + \int_{x_0}^{x}dx'\,\phi(P,x',t_r)\bigg),\quad (x,x_0,t_r,t_{0,r})\in\mathbb{R}^4. \tag{1.168}$$

[1] According to Remark 1.9, the right-hand side of (1.167) is symmetric with respect to $\hat\nu_\ell^\beta$, $\ell=0,\dots,n$, and hence the pair $\big(\hat\nu_0^\beta,\underline{\hat\nu}^\beta\big)$ can be replaced by any of the pairs $\big(\hat\nu_\ell^\beta,\underline{\hat\nu}^{\beta,\ell}\big)$, $\ell=1,\dots,n$.

Moreover, in analogy to (1.33) we also introduce the polynomial

$$\widetilde{H}_{r+1}(z) = (1/2)\widetilde{F}_{r,xx}(z) - (u-z)\widetilde{F}_r(z). \tag{1.169}$$

From (1.159) and (1.160) one then computes

$$\widetilde{H}_{r+1,x} = (1/2)\widetilde{F}_{r,xxx} - (u-z)\widetilde{F}_{r,x} - u_x\widetilde{F}_r = -u_{t_r} + (u-z)\widetilde{F}_{r,x}. \tag{1.170}$$

The following lemma records basic properties of ϕ and Ψ in analogy to the stationary case discussed in Lemma 1.8.

Lemma 1.34 *Assume Hypothesis* 1.33 *and suppose that* (1.159), (1.160) *hold. Moreover, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}$, $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$. *Then* ϕ *satisfies*

$$\phi_x(P) + \phi(P)^2 = u - z, \tag{1.171}$$

$$\phi_{t_r}(P) = \partial_x\big(\widetilde{F}_r(z)\phi(P) - (1/2)\widetilde{F}_{r,x}(z)\big) \tag{1.172}$$

$$= -\widetilde{F}_r(z)\phi(P)^2 + \widetilde{F}_{r,x}(z)\phi(P) - \widetilde{H}_r(z), \tag{1.173}$$

$$\phi(P)\phi(P^*) = \frac{H_{n+1}(z)}{F_n(z)}, \tag{1.174}$$

$$\phi(P) + \phi(P^*) = \frac{F_{n,x}(z)}{F_n(z)}, \tag{1.175}$$

$$\phi(P) - \phi(P^*) = \frac{2iy}{F_n(z)}. \tag{1.176}$$

Moreover, ψ *satisfies*

$$(L(t_r) - z(P))\psi(P) = 0, \tag{1.177}$$

$$(P_{2n+1}(t_r) - iy(P))\psi(P) = 0, \tag{1.178}$$

$$\psi_{t_r}(P) = \widetilde{P}_{2r+1}(t_r)\psi(P) \tag{1.179}$$

$$= \widetilde{F}_r(z)\psi_x(P) - (1/2)\widetilde{F}_{r,x}(z)\psi(P). \tag{1.180}$$

In addition, as long as the zeros of $F_n(\,\cdot\,, x, t_r)$ *are all simple for* $(x, t_r) \in \Omega$, $\Omega \subseteq \mathbb{R}^2$ *open and connected,* $\psi(\,\cdot\,, x, x_0, t_r, t_{0,r})$ *is meromorphic on* $\mathcal{K}_n \setminus \{P_\infty\}$ *for* $(x, t_r), (x_0, t_{0,r}) \in \Omega$.

Proof The proofs of (1.171), (1.177), (1.178), and (1.174)–(1.176) are analogous to those in Lemma 1.8. To prove (1.172) one can argue as follows. By (1.159) and (1.171),

$$\begin{aligned}\partial_{t_r}(\phi_x + \phi^2) &= \phi_{t_r,x} + 2\phi\phi_{t_r} \\ &= u_{t_r} = \big(\widetilde{F}_r\phi - (1/2)\widetilde{F}_{r,x}\big)_{xx} + 2\phi\big(\widetilde{F}_r\phi - (1/2)\widetilde{F}_{r,x}\big)_x,\end{aligned}$$

which implies

$$(\partial_x + 2\phi)\big(\phi_{t_r} - \big(\widetilde{F}_r\phi - (1/2)\widetilde{F}_{r,x}\big)_x\big) = 0,$$

and hence

$$\phi_{t_r} = \big(\widetilde{F}_r\phi - (1/2)\widetilde{F}_{r,x}\big)_x + C\exp\bigg(-2\int^x dx'\,\phi\bigg),$$

where C is independent of x (but may depend on P and t_r). The behavior of $\phi(P, x, t_r)$ as the spectral parameter tends to infinity, derived from (1.167) for fixed $(x, t_r) \in \mathbb{R}^2$, yields $\phi(P, x, t_r) = \pm i z^{1/2} + O(1)$ as $z \to \infty$, $P \in \Pi_\pm$; hence, $C = 0$ since $\phi_{t_r} - (\widetilde{F}_r\phi - 2^{-1}\widetilde{F}_{r,x})_x$ is meromorphic on $\mathcal{K}_n$, and hence especially near P_∞, while $C\exp\big(-2\int^x dx'\phi\big)$ is meromorphic near P_∞ only if $C = 0$. This proves (1.172). Equation (1.173) is then clear from (1.169) and (1.172). By (1.168), $\psi(\,\cdot\,, x, x_0, t_r, t_{0,r})$ is meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ away from the poles $\hat{\mu}_j(x_0, s)$ of $\phi(\,\cdot\,, x_0, s)$ and $\hat{\mu}_k(x', t_r)$ of $\phi(\,\cdot\,, x', t_r)$. That $\psi(\,\cdot\,, x, x_0, t_r, t_{0,r})$ is meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ if $F_n(\,\cdot\,, x, t_r)$ has only simple zeros is a consequence of

$$\phi(P, x', t_r) \underset{P\to\hat{\mu}_j(x',t_r)}{=} \partial_{x'} \ln\big(F_n(z, x', t_r)\big) + O(1) \text{ as } z \to \mu_j(x', t_r)$$

(cf. (1.52)) and from

$$\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) \underset{P\to\hat{\mu}_j(x_0,s)}{=} \partial_s \ln\big(F_n(z, x_0, s)\big) + O(1) \text{ as } z \to \mu_j(x_0, s),$$

using (1.163), (1.167), and (1.181) ((1.181) in Lemma 1.35 follows from (1.172) and (1.176), which have already been proven). This follows from (1.168) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j(x_0)$ of $\{\hat{\mu}_j(x_0, s) \in \mathcal{K}_n \mid (x_0, s) \in \Omega,\ s \in [t_{0,r}, t_r]\}$ such that $\hat{\mu}_k(x_0, s) \notin \mathcal{U}_j(x_0)$ for all $s \in [t_{0,r}, t_r]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$ and by simultaneously restricting P to a sufficiently small neighborhood $\mathcal{U}_j(t_r)$ of $\{\hat{\mu}_j(x', t_r) \in \mathcal{K}_n \mid (x', t_r) \in \Omega,\ x' \in [x_0, x]\}$ such that $\hat{\mu}_k(x', t_r) \notin \mathcal{U}_j(t_r)$ for all $x' \in [x_0, x]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$. Finally, (1.180) immediately follows from (1.168) and (1.172). $\square$

The t_r-dependence of F_n and H_{n+1} is governed by the following result.

Lemma 1.35 *Assume Hypothesis* 1.33 *and suppose that* (1.159), (1.160) *hold. Then,*

$$F_{n,t_r} = F_{n,x}\widetilde{F}_r - F_n\widetilde{F}_{r,x}, \tag{1.181}$$

$$F_{n,xt_r} = 2(H_{n+1}\widetilde{F}_r - F_n\widetilde{H}_{r+1}), \tag{1.182}$$

$$H_{n+1,t_r} = H_{n+1}\widetilde{F}_{r,x} - F_{n,x}\widetilde{H}_{r+1}. \tag{1.183}$$

Proof By (1.172) and (1.176),

$$\begin{aligned}\phi_{t_r}(P) - \phi_{t_r}(P^*) &= -2iyF_n^{-2}F_{n,t_r} = \partial_x\big(\widetilde{F}_r(\phi(P) - \phi(P^*))\big)\\ &= 2iy(F_n\widetilde{F}_{r,x} - F_{n,x}\widetilde{F}_r)F_n^{-2},\end{aligned}$$

implying (1.181). Similarly, by (1.169),

$$-(1/2)F_{n,t_rx} = (1/2)(F_n\widetilde{F}_{r,xx} - \widetilde{F}_r F_{n,xx}) = F_n\widetilde{H}_{r+1} - \widetilde{F}_r H_{n+1}.$$

Finally, (1.12), (1.33), (1.170), and (1.181) yield

$$\begin{aligned} H_{n+1,t_r} &= (1/2)F_{n,t_rxx} - (u-z)F_{n,t_r} - u_{t_r}F_n \\ &= -F_{n,x}\widetilde{H}_{r+1} - F_n\widetilde{H}_{r+1,x} + \widetilde{F}_{r,x}H_{n+1} + (u-z)F_n\widetilde{F}_{r,x} - u_{t_r}F_n \\ &= -F_{n,x}\widetilde{H}_{r+1} + \widetilde{F}_{r,x}H_{n+1} \end{aligned}$$

and hence (1.183). □

Next we record the remaining t_r-dependent analogs of Lemma 1.8.

Lemma 1.36 *Assume Hypothesis* 1.33 *and suppose that* (1.159), (1.160) *hold. Moreover, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}$ *and* $(x, x_0, t, t_0) \in \mathbb{R}^4$. *Then,*

$$\psi(P, x, x_0, t_r, t_{0,r}) = \left(\frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})}\right)^{1/2} \tag{1.184}$$

$$\times \exp\left(iy\int_{t_0}^{t_r} ds\, \widetilde{F}_r(z, x_0, s)F_n(z, x_0, s)^{-1} + iy\int_{x_0}^{x} dx'\, F_n(z, x', t_r)^{-1}\right),$$

$$\psi(P, x, x_0, t_r, t_{0,r})\psi(P^*, x, x_0, t_r, t_{0,r}) = \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})}, \tag{1.185}$$

$$\psi_x(P, x, x_0, t_r, t_{0,r})\psi_x(P^*, x, x_0, t_r, t_{0,r}) = \frac{H_{n+1}(z, x, t_r)}{F_n(z, x_0, t_{0,r})}, \tag{1.186}$$

$$\begin{aligned} &\psi(P, x, x_0, t_r, t_{0,r})\psi_x(P^*, x, x_0, t_r, t_{0,r}) \\ &\quad + \psi(P^*, x, x_0, t_r, t_{0,r})\psi_x(P, x, x_0, t_r, t_{0,r}) = \frac{F_{n,x}(z, x, t_r)}{F_{n,x}(z, x_0.t_{0,r})}, \end{aligned} \tag{1.187}$$

$$W(\psi(P,\,\cdot\,, x_0, t_r, t_{0,r}), \psi(P^*,\,\cdot\,, x_0, t_r, t_{0,r})) = -\frac{2iy}{F_n(z, x_0.t_{0,r})}. \tag{1.188}$$

Proof Equation (1.184) follows from (1.165), (1.168), and (1.181). Combining (1.168), (1.175), and (1.181) yields

$$\begin{aligned} &\psi(P, x, x_0, t_r, t_{0,r})\psi(P^*, x, x_0, t_r, t_{0,r}) \\ &\quad = \exp\left(\int_{t_{0,r}}^{t_r} ds\, F_{n,s}(z, x_0, s)F_n(z, x_0, s)^{-1}\right. \\ &\qquad\qquad \left. + \int_{x_0}^{x} dx'\, F_{n,x'}(z, x', t_r)F_n(z, x', t_r)^{-1}\right) \\ &\quad = (F_n(z, x_0, t_r)/F_n(z, x_0, t_{0,r}))(F_n(z, x, t_r)/F_n(z, x_0, t_r)) \\ &\quad = F_n(z, x, t_r)/F_n(z, x_0, t_{0,r}), \end{aligned}$$

proving (1.185). Equations (1.174), (1.185) and $\psi_x = \phi\psi$ imply

$$\begin{aligned}&\psi_x(P, x, x_0, t_r, t_{0,r})\psi_x(P^*, x, x_0, t_r, t_{0,r})\\ &\quad = (H_{n+1}(z, x, t_r)/F_n(z, x, t_r))(F_n(z, x, t_r)/F_n(z, x_0, t_{0,r}))\\ &\quad = H_{n+1}(z, x, t_r)/F_n(z, x_0, t_{0,r}).\end{aligned}$$

This proves (1.186). Equations (1.187) and (1.188) follow from (1.168), (1.175), and (1.176). $\square$

Turning to the t_r-dependent analog of (1.55)–(1.61) we start by introducing

$$K^{\beta}_{n+1}(z) = H_{n+1}(z) + \beta F_{n,x}(z) + \beta^2 F_n(z) = \prod_{\ell=0}^{n} \big(z - \lambda^{\beta}_{\ell}\big), \quad \beta \in \mathbb{R}, \quad (1.189)$$

with

$$H_{n+1}(z) = K^0_{n+1}(z), \quad \nu_\ell = \lambda^0_\ell, \quad \ell = 0, \dots, n.$$

One then verifies in analogy to (1.57)–(1.61) that

$$\begin{aligned}&\phi(P) + \beta = \frac{iy + \frac{1}{2}F_{n,x}(z) + \beta F_n(z)}{F_n(z)}\\ &\qquad\qquad = \frac{-K^{\beta}_{n+1}(z)}{iy - (1/2)F_{n,x}(z) - \beta F_n(z)},\\ &R_{2n+1}(z) + \big((1/2)F_{n,x}(z) + \beta F_n(z)\big)^2 = F_n(z)K^{\beta}_{n+1}(z),\\ &(\phi(P) + \beta)(\phi(P^*) + \beta) = \frac{K^{\beta}_{n+1}(z)}{F_n(z)},\\ &(\psi_x(P, x, x_0, t_r, t_{0,r}) + \beta\psi(P, x, x_0, t_r, t_{0,r}))\\ &\quad \times (\psi_x(P^*, x, x_0, t_r, t_{0,r}) + \beta\psi(P^*, x, x_0, t_r, t_{0,r})) = \frac{K^{\beta}_{n+1}(z, x, t_r)}{F_n(z, x_0, t_{0,r})},\\ &(\phi + \beta) = \mathcal{D}_{\hat{\lambda}^{\beta}_0 \underline{\hat{\lambda}}^{\beta}} - \mathcal{D}_{P_\infty \underline{\hat{\mu}}}\end{aligned}$$

with

$$\hat{\lambda}^{\beta}_{\ell}(x, t_r) = \big(\lambda^{\beta}_{\ell}(x, t_r), (i/2)F_{n,x}\big(\lambda^{\beta}_{\ell}(x, t_r), x, t_r\big) + i\beta F_n\big(\lambda^{\beta}_{\ell}(x, t_r), x, t_r\big)\big),$$
$$\ell = 0, \dots, n. \quad (1.190)$$

Equation (1.189) and Lemma 1.34 then yield

$$K^{\beta}_{n+1,t_r} = K^{\beta}_{n+1}\big(\widetilde{F}_{r,x} + 2\beta \widetilde{F}_r\big) - (F_{n,x} + 2\beta F_n)\widetilde{K}^{\beta}_{r+1}. \quad (1.191)$$

Before turning to the analog of Dubrovin equations in the time-dependent setting, we recall that the affine part of $\mathcal{K}_n$ is nonsingular if (1.62) holds. Moreover, as in Section 1.3, we will always assume the eigenvalue ordering (1.63) in the special case in which $\{E_m\}_{m=0,\dots,2n} \subset \mathbb{R}$, that is,

$$E_m < E_{m+1} \text{ for } m = 0, 1, \dots, 2n-1. \tag{1.192}$$

In particular, if $u(\,\cdot\,, t_r) \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is real-valued, $t_r \in \mathbb{R}$, then again $\{\mu_j\}_{j=1,\dots,n} \subset \mathbb{R}$ and $\{\lambda_\ell^\beta\}_{\ell=0,\dots,n} \subset \mathbb{R}$ and hence we will also always assume the ordering

$$\mu_j(x, t_r) < \mu_{j+1}(x, t_r) \text{ for } j = 1, \dots, n-1, (x, t_r) \in \mathbb{R}^2, \tag{1.193}$$

$$\lambda_\ell^\beta(x, t_r) < \lambda_{\ell+1}^\beta(x, t_r) \text{ for } \ell = 0, \dots, n-1, (x, t_r) \in \mathbb{R}^2 \tag{1.194}$$

in this case.

The stationary Dubrovin equations in Lemmas 1.10 and 1.11 have analogs for each KdV_r flow (indexed by the parameter t_r), which govern the dynamics of μ_j and λ_ℓ^β with respect to variations of x and t_r. In this context the stationary case simply corresponds to the special case $r = 0$. We first provide the result in connection with Dirichlet boundary conditions (where $\beta = \infty$) and then turn to the general boundary conditions (1.54) parametrized by $\beta \in \mathbb{R}$.

Lemma 1.37 (The Dubrovin Equations)
(i) *Assume Hypothesis* 1.33 *and* (1.159), (1.160) *hold on an open and connected set* $\widetilde{\Omega}_\mu \subseteq \mathbb{R}^2$. *Moreover, suppose that the zeros* μ_j, $j = 1, \dots, n$, *of* $F_n(\,\cdot\,)$ *remain distinct on* $\widetilde{\Omega}_\mu$. *Then* $\{\hat{\mu}_j\}_{j=1,\dots,n}$, *defined by* (1.163), *satisfies the following first-order system of differential equations on* $\widetilde{\Omega}_\mu$

$$\mu_{j,x} = -2iy(\hat{\mu}_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} (\mu_j - \mu_k)^{-1}, \tag{1.195}$$

$$\mu_{j,t_r} = -2i\widetilde{F}_r(\mu_j) y(\hat{\mu}_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, n. \tag{1.196}$$

Next, assume the affine part of $\mathcal{K}_n$ *to be nonsingular and introduce the initial condition*

$$\{\hat{\mu}_j(x_0, t_{0,r})\}_{j=1,\dots,n} \subset \mathcal{K}_n, \tag{1.197}$$

for some $(x_0, t_{0,r}) \in \mathbb{R}^2$, *where* $\mu_j(x_0, t_{0,r})$, $j = 1, \dots, n$, *are assumed to be distinct. Then there exists an open and connected set* $\Omega_\mu \subseteq \mathbb{R}^2$, *with* $(x_0, t_{0,r}) \in \Omega_\mu$, *such that the initial value problem* (1.195)–(1.197) *has a unique solution* $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ *satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{1.198}$$

and μ_j, $j = 1, \dots, n$, *remain distinct on* Ω_μ.
(ii) *Suppose in addition to Hypothesis* 1.33 *and* (1.159), (1.160) *that* u *is real-valued and bounded and that the affine part of* $\mathcal{K}_n$ *is nonsingular. Moreover, assume the eigenvalue orderings* (1.192), (1.193). *Then* $\{\hat{\mu}_j\}_{j=1,\dots,n}$, *with* $\mu_j(x, t_r)$, $j = 1, \dots, n$, *the Dirichlet eigenvalues of* $-\frac{d^2}{dx^2} + u(\,\cdot\,, t_r)$ *corresponding to a Dirichlet boundary condition at* $x \in \mathbb{R}$ *(i.e., the eigenvalues of* $H_x^D(t_r)$*), satisfies* (1.196) *on* $\mathbb{R}^2$. *Furthermore, given initial data satisfying* $\mu_j(x_0, t_{0,r}) = \mu_j^{(0)}(x_0) \in [E_{2j-1}, E_{2j}]$, $j = 1, \dots n$, *then*

$$\mu_j(x, t_r) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots n, \ (x, t_r) \in \mathbb{R}^2.$$

In particular, $\hat{\mu}_j(x, t_r)$ *changes sheets whenever it hits* E_{2j-1} *or* E_{2j}, *and its projection* $\mu_j(x, t_r)$ *remains trapped in* $[E_{2j-1}, E_{2j}]$ *for all* $j = 1, \dots, n$ *and* $(x, t_r) \in \mathbb{R}^2$.

Proof Since $\widetilde{F}_0 = 1$, the proof of (1.195) is identical to that in Lemma 1.10 in the stationary case. Taking $z = \mu_j$ in (1.181) and observing (1.161) and (1.163) immediately yield

$$F_{n,t_r}(\mu_j) = -\mu_{j,t_r} \prod_{\substack{k=1 \\ k \neq j}}^{n} (\mu_j - \mu_k) = F_{n,x}(\mu_j)\widetilde{F}_r(\mu_j) = 2iy(\hat{\mu}_j)\widetilde{F}_r(\mu_j)$$

and hence (1.196). Similarly, the argument proving Lemma 1.10 (ii) applies to the present time-dependent context line by line. For the proof of (1.198), one invokes again the charts (B.3)–(B.6) and (B.12)–(B.15). As in the stationary case, the only nontrivial issue to check is the case in which $\hat{\mu}_j$ hits one of the branch points $(E_m, 0) \in \mathcal{B}(\mathcal{K}_n)$, and hence the right-hand sides of (1.195) and (1.196) vanish. We suppose therefore that

$$\mu_{j_0}(x, t_r) \to E_{m_0} \text{ as } (x, t_r) \to (x_0, t_{0,r}) \in \Omega_\mu$$

for some $j_0 \in \{1, \dots, n\}$, $m_0 \in \{0, \dots, 2n\}$. Introducing

$$\zeta_{j_0}(x, t_r) = \sigma(\mu_{j_0}(x, t_r) - E_{m_0})^{1/2}, \ \sigma = \pm 1,$$
$$\mu_{j_0}(x, t_r) = E_{m_0} + \zeta_{j_0}(x, t_r)^2$$

for (x, t_r) in a sufficiently small neighborhood of $(x_0, t_{0,r})$, the Dubrovin equations (1.195), (1.196) for μ_{j_0} become

$$\zeta_{j_0,x}(x,t_r) \underset{(x,t_r)\to(x_0,t_{0,r})}{=} c(\sigma)\Bigg(\prod_{\substack{m=0\\ m\neq m_0}}^{2n}(E_{m_0}-E_m)\Bigg)^{1/2} \times\Bigg(\prod_{\substack{k=1\\ k\neq j_0}}^{n}\big(E_{m_0}-\mu_k(x,t_r)\big)^{-1}\Bigg)\big(1+O(\zeta_{j_0}(x,t_r)^2)\big),$$

$$\zeta_{j_0,t_r}(x,t_r) \underset{(x,t_r)\to(x_0,t_{0,r})}{=} c(\sigma)\widetilde F_r(E_{m_0},x_0,t_{0,r})\Bigg(\prod_{\substack{m=0\\ m\neq m_0}}^{2n}(E_{m_0}-E_m)\Bigg)^{1/2} \times\Bigg(\prod_{\substack{k=1\\ k\neq j_0}}^{n}\big(E_{m_0}-\mu_k(x,t_r)\big)^{-1}\Bigg)\big(1+O(\zeta_{j_0}(x,t_r)^2)\big)$$

for some $|c(\sigma)|=1$, and one concludes (1.198). $\square$

For the general β boundary conditions (cf. (1.54)), we record the following result.

Lemma 1.38 *Let $\beta\in\mathbb{R}$.*
(i) Assume Hypothesis 1.33 *and* (1.159), (1.160) *hold on an open and connected set $\widetilde\Omega_\lambda\subseteq\mathbb{R}^2$. Moreover, suppose that the zeros λ_ℓ^β, $\ell=0,\dots,n$, of $K_{n+1}^\beta(\,\cdot\,)$ remain distinct on $\widetilde\Omega_\lambda$. Then $\{\hat\lambda_\ell^\beta\}_{\ell=0,\dots,n}$, defined by* (1.190)*, satisfies the following first-order system of differential equations on $\widetilde\Omega_\lambda$*

$$\lambda_{\ell,x}^\beta = -2i\big(\beta^2-u+\lambda_\ell^\beta\big)y\big(\hat\lambda_\ell^\beta\big)\prod_{\substack{m=0\\ m\neq\ell}}^{n}\big(\lambda_\ell^\beta-\lambda_m^\beta\big)^{-1}, \tag{1.199}$$

$$\lambda_{\ell,t_r}^\beta = -2i\widetilde K_{r+1}^\beta\big(\lambda_\ell^\beta\big)y\big(\hat\lambda_\ell^\beta\big)\prod_{\substack{m=0\\ m\neq\ell}}^{n}\big(\lambda_\ell^\beta-\lambda_m^\beta\big)^{-1}, \quad \ell=0,\dots,n. \tag{1.200}$$

Next, assume the affine part of $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\big\{\hat\lambda_\ell^\beta(x_0,t_{0,r})\big\}_{\ell=0,\dots,n}\subset\mathcal{K}_n \tag{1.201}$$

for some $(x_0,t_{0,r})\in\mathbb{R}^2$, where $\lambda_\ell^\beta(x_0,t_{0,r})$, $\ell=0,\dots,n$, are assumed to be distinct. Then there exists an open and connected set $\Omega_\lambda\subseteq\mathbb{R}^2$, with $(x_0,t_{0,r})\in\Omega_\lambda$, such that the initial value problem (1.199)–(1.201) *has a unique solution*

$\{\hat{\lambda}_\ell^\beta\}_{\ell=0,\dots,n} \subset \mathcal{K}_n$ satisfying

$$\hat{\lambda}_\ell^\beta \in C^\infty(\Omega_\lambda, \mathcal{K}_n), \quad \ell = 0, \dots, n, \tag{1.202}$$

and λ_ℓ^β, $\ell = 0, \dots, n$ remain distinct on Ω_λ.
(ii) Suppose in addition to Hypothesis 1.33 *and* (1.159), (1.160) *that u is real-valued and bounded and that the affine part of $\mathcal{K}_n$ is nonsingular. Moreover, assume the eigenvalue orderings* (1.192), (1.194). *Then $\{\hat{\lambda}_\ell^\beta\}_{\ell=0,\dots,n}$, with $\lambda_\ell^\beta(x,t_r)$, $\ell = 0, \dots, n$, the eigenvalues of $H_x^\beta(t_r)$, satisfies* (1.200) *on $\mathbb{R}^2$. Furthermore, given initial data satisfying $\lambda_0(x_0, t_{0,r}) \le E_0$ and $\lambda_\ell(x_0, t_{0,r}) \in [E_{2\ell-1}, E_{2\ell}]$, $\ell = 1, \dots n$, then*

$$\lambda_0^\beta(x,t_r) \le E_0, \quad \lambda_\ell^\beta(x,t_r) \in [E_{2\ell-1}, E_{2\ell}], \quad \ell = 1, \dots n, \ (x,t_r) \in \mathbb{R}^2.$$

In particular, $\hat{\lambda}_\ell^\beta(x,t_r)$ changes sheets whenever it hits $E_{2\ell-1}$ or $E_{2\ell}$, and its projection $\lambda_\ell^\beta(x,t_r)$ remains trapped in $[E_{2\ell-1}, E_{2\ell}]$ for all $\ell = 1, \dots, n$ and $(x,t_r) \in \mathbb{R}^2$ (and similarly for $\hat{\lambda}_0^\beta(x,t_r)$).

Proof Again the proofs of (1.199), (1.202) and part (ii) parallel those of Lemmas 1.10 and 1.11 line by line. Inserting (1.189) into (1.191), taking $z = \lambda_\ell^\beta(x,t_r)$, and applying (1.190) yield (1.200). □

In a fashion analogous to (1.80)–(1.82), one can analyze the behavior of $\lambda_\ell^\beta(x,t_r)$ as a function of the boundary condition parameter $\beta \in \mathbb{R}$. In fact, (1.189) yields

$$\partial_\beta K_{n+1}^\beta = -F_{n,x} + 2\beta F_n, \tag{1.203}$$

and hence

$$\begin{aligned} \partial_\beta K_{n+1}^\beta(z)\big|_{z=\lambda_\ell^\beta} &= -\big(\partial_\beta \lambda_\ell^\beta\big) \prod_{\substack{m=0\\ m\neq\ell}}^n \big(\lambda_\ell^\beta - \lambda_m^\beta\big) = -F_{n,x}\big(\lambda_\ell^\beta\big) + 2\beta F_n\big(\lambda_\ell^\beta\big) \\ &= -2iy\big(\hat{\lambda}_\ell^\beta\big) \end{aligned} \tag{1.204}$$

by (1.190). As in Lemma 1.15 this implies the following result for the β-variation of the eigenvalues $\lambda_\ell^\beta(x,t_r)$.

Lemma 1.39 *Let $(x,t_r,\beta) \in \Omega \times \mathcal{U}$, where $\Omega \subseteq \mathbb{R}^2$ is open and connected and $\mathcal{U} \subseteq \mathbb{R}$ is an open interval. Assume Hypothesis* 1.33 *and* (1.159), (1.160) *hold on Ω. Moreover, suppose that the zeros $\lambda_\ell^\beta(x,t_r)$, $\ell = 0, \dots, n$ of $K_{n+1}^\beta(\cdot\,, x, t_r)$ remain distinct for $(x,t_r,\beta) \in \Omega \times \mathcal{U}$. Then $\{\hat{\lambda}_\ell^\beta\}_{\ell=0,\dots,n}$, defined by* (1.190)*, satisfies the* *following first-order system of differential equations on* Ω

$$\partial_\beta \lambda_\ell^\beta = 2iy\big(\hat{\lambda}_\ell^\beta\big) \prod_{\substack{m=0 \\ m\neq\ell}}^{n} \big(\lambda_\ell^\beta - \lambda_m^\beta\big)^{-1}, \quad \ell = 0, \dots, n.$$

Proof Combine (1.203) and (1.204). □

Since the stationary trace formulas for KdV invariants in terms of symmetric functions of μ_j and λ_ℓ^β in Lemmas 1.16 and 1.17 extend line by line to the corresponding time-dependent setting, we next record their t_r-dependent analogs without proof.

Lemma 1.40 *Assume Hypothesis* 1.33 *and suppose that* (1.159)*,* (1.160) *hold. Then,*

$$\begin{aligned} u &= \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j, \\ u^2 - (1/2)u_{xx} &= \sum_{m=0}^{2n} E_m^2 - 2\sum_{j=1}^{n} \mu_j^2, \text{ etc.} \end{aligned}$$

Lemma 1.41 *Let* $\beta \in \mathbb{R}$*. Assume Hypothesis* 1.33 *and suppose that* (1.159)*,* (1.160) *hold. Then,*

$$\begin{aligned} &2\beta^2 - u = \sum_{m=0}^{2n} E_m - 2\sum_{\ell=0}^{n} \lambda_\ell^\beta, \\ &(1/2)u_{xx} - u(x,t_r)^2 + 2\beta u_x + 4\beta^2 u - 2\beta^4 = \sum_{m=0}^{2n} E_m^2 - 2\sum_{\ell=0}^{n} \big(\lambda_\ell^\beta\big)^2, \text{ etc.} \end{aligned}$$

Remark 1.42 We emphasize that instead of taking (1.159) and (1.160) as our starting point for solving (1.153), and subsequently deriving the first-order differential system (1.195), (1.196), one could have started directly with the system (1.195), (1.196) and derived (1.159) and (1.160) as well as the remaining facts of this section using the time-dependent trace formula for u in Lemma 1.40. This algebro-geometric initial value problem approach will be explicitly carried out in Theorem 1.48.

Clearly, Lemma 1.18 extends to the present time-dependent setting. We omit the corresponding details.

We also record the asymptotic properties of ϕ (whose proof is identical to that in Lemma 1.19).

Lemma 1.43 *Assume Hypothesis* 1.33 *and suppose that* (1.159), (1.160) *hold. Moreover, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}$. *Then, as* $P \to P_\infty$,

$$\phi(P) \underset{\zeta \to 0}{=} i\zeta^{-1} - (i/2)u\zeta + O(\zeta^2), \quad \zeta = \sigma/z^{1/2},\ \sigma = \pm 1.$$

As in Section 1.3 we continue with the theta function representation of ϕ, ψ, and u, assuming the affine part of $\mathcal{K}_n$ to be nonsingular. We start by introducing some of the necessary quantities.

Let $\omega^{(2)}_{P_\infty,2q}$ be a normalized differential of the second kind with a unique pole at P_∞ with principal part $\zeta^{-2q-2}d\zeta$ near P_∞ (cf. (A.20), (A.21), and (A.22)), and define

$$\widetilde{\Omega}^{(2)}_{P_\infty,2r} = \sum_{q=0}^{r}(2q+1)\tilde{c}_{r-q}\omega^{(2)}_{P_\infty,2q}, \quad \tilde{c}_0 = 1, \tag{1.205}$$

where $\tilde{c}_q$ are the constants introduced in the definition of $\widetilde{F}_r$. Thus, one infers as in (1.98)–(1.100),

$$\int_{a_j} \widetilde{\Omega}^{(2)}_{P_\infty,2r} = 0, \quad j = 1, \dots, n, \tag{1.206}$$

$$\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,2r} \underset{\zeta \to 0}{=} -\sum_{q=0}^{r} \tilde{c}_{r-q}\zeta^{-2q-1} + O(\zeta) \text{ as } P \to P_\infty, \tag{1.207}$$

choosing Q_0 to be one of the branch points. Moreover, define the vector of b-periods of $\widetilde{\Omega}^{(2)}_{P_\infty,2r}/(2\pi i)$ by

$$\underline{\widetilde{U}}^{(2)}_{2r} = \big(\widetilde{U}^{(2)}_{2r,1}, \dots, \widetilde{U}^{(2)}_{2r,n}\big), \quad \widetilde{U}^{(2)}_{2r,j} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}^{(2)}_{P_\infty,2r}, \quad j = 1, \dots, n. \tag{1.208}$$

By (B.33) one obtains

$$\widetilde{U}^{(2)}_{2r,j} = -2\sum_{q=0}^{r} \tilde{c}_{r-q} \sum_{k=1}^{n} c_j(k)\hat{c}_{k-n+q}(\underline{E}), \quad j = 1, \dots, n \tag{1.209}$$

with $\hat{c}_k(\underline{E})$ defined in (B.32). We also recall the definition of $\omega^{(3)}_{P_\infty,\hat{\lambda}_\beta(x,t_r)}$ and its properties in (1.93)–(1.97) with $\hat{\lambda}_\beta(x)$ replaced by $\hat{\lambda}_\beta(x, t_r)$, etc.

Recalling the abbreviation (1.103) and our choice of base point $Q_0 = (E_{m_0}, 0)$, we can now state one of the principal results of this section.

Theorem 1.44 *Suppose Hypothesis* 1.33 *and* (1.159), (1.160) *hold on* Ω, *and assume the affine part of* $\mathcal{K}_n$ *to be nonsingular. In addition, let* $P \in \mathcal{K}_n \setminus \{P_\infty\}$ *and*

(x,t_r), $(x_0,t_{0,r})\in\Omega$, where $\Omega\subseteq\mathbb{R}^2$ is open and connected. Moreover, suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\lambda}}^{\beta}(x,t)}$ is nonspecial for $(x,t_r)\in\Omega$. Then,[1]

$$\begin{aligned}\phi(P,x,t_r) = {}& -\beta + i\frac{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P,\underline{\hat{\lambda}}^{\beta}(x,t_r)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\lambda}}^{\beta}(x,t_r)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))} \\ &\times\exp\bigg(-\int_{Q_0}^{P}\omega^{(3)}_{P_\infty,\hat{\lambda}_0^{\beta}(x,t_r)} + (1/2)\ln\big(E_{m_0}-\lambda_0^{\beta}(x,t_r)\big)\bigg)\end{aligned} \tag{1.210}$$

and[2]

$$\begin{aligned}\psi(P,x,x_0,t_r,t_{0,r}) = {}& \frac{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x_0,t_{0,r})))\theta(\underline{z}(P,\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_r)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0,t_{0,r})))} \\ &\times\exp\bigg(-i(x-x_0)\int_{Q_0}^{P}\omega^{(2)}_{P_\infty,0} - i(t_r-t_{0,r})\int_{Q_0}^{P}\widetilde{\Omega}^{(2)}_{P_\infty,2r}\bigg)\end{aligned} \tag{1.211}$$

with the linearizing property of the Abel map

$$\begin{aligned}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) + i\underline{U}_0^{(2)}(x-x_0) \\ &\quad + i\underline{\widetilde{U}}_{2r}^{(2)}(t_r-t_{0,r}),\end{aligned} \tag{1.212}$$

$$\begin{aligned}\underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\lambda}_0^{\beta}(x,t_r)\underline{\hat{\lambda}}^{\beta}(x,t_r)}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\hat{\lambda}_0^{\beta}(x_0,t_{0,r})\underline{\hat{\lambda}}^{\beta}(x_0,t_{0,r})}) + i\underline{U}_0^{(2)}(x-x_0) \\ &\quad + i\underline{\widetilde{U}}_{2r}^{(2)}(t_r-t_{0,r}).\end{aligned} \tag{1.213}$$

The Its–Matveev formula for u finally reads

$$\begin{aligned}u(x,t_r) &= E_0 + \sum_{j=1}^{n}(E_{2j-1}+E_{2j}-2\lambda_j) \\ &\quad - 2\partial_x^2\ln\big(\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}))\big) \qquad (1.214)\\ &= E_0 + \sum_{j=1}^{n}(E_{2j-1}+E_{2j}-2\lambda_j) \\ &\quad - 2\partial_x^2\ln\big(\theta(\underline{\Xi}_{Q_0} + \underline{A}_{Q_0}(\hat{\lambda}_0^{\beta}(x,t_r)) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}^{\beta}(x,t_r)}))\big). \qquad (1.215)\end{aligned}$$

Proof The discussion with respect to the spatial variation of ϕ and ψ in Theorem 1.44 is identical to that of Theorem 1.20. Moreover, the proof of (1.210) carries over without any changes. The behavior of ψ, however, requires a more refined

[1] According to Remark 1.9, the right-hand side of (1.210) is symmetric with respect to $\hat{\lambda}_\ell^{\beta}$, $\ell=0,\dots,n$; hence, the pair $(\hat{\lambda}_0^{\beta},\underline{\hat{\lambda}}^{\beta})$ can be replaced by any of the pairs $(\hat{\lambda}_\ell^{\beta},\underline{\hat{\lambda}}^{\beta,\ell})$, $\ell=1,\dots,n$.

[2] To avoid multi-valued expressions in formulas such as (1.210), (1.211), etc., we agree always to choose the same path of integration connecting Q_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

treatment since we have an extra time-dependent term in the exponential. Let ψ be defined as in (1.168) and denote the right-hand side of (1.211) by $\tilde{\psi}$. We temporarily assume

$$\mu_j(x,t_r) \neq \mu_{j'}(x,t_r) \text{ for } j \neq j' \text{ and } (x,t_r) \in \widetilde{\Omega} \tag{1.216}$$

for appropriate $\widetilde{\Omega} \subseteq \Omega$. In order to prove that $\psi = \tilde{\psi}$, one first observes that (1.163), (1.165), (1.195), and (1.196) imply (cf. (1.111))

$$\begin{aligned}
\phi(P,x',t_r) &\underset{P\to\hat{\mu}_j(x',t_r)}{=} \partial_{x'} \ln(z-\mu_j(x',t_r)) + O(1),\\
\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s) &\underset{P\to\hat{\mu}_j(x_0,s)}{=} \partial_s \ln(z-\mu_j(x_0,s)) + O(1).
\end{aligned}$$

Together with (1.168) this yields

$$\begin{aligned}
&\psi(P,x,x_0,t_r,t_{0,r})\\
&\quad = \begin{cases} (z-\mu_j(x,t_r))O(1) & \text{as } P \to \hat{\mu}_j(x,t_r) \neq \hat{\mu}_j(x_0,t_{0,r}),\\ O(1) & \text{as } P \to \hat{\mu}_j(x,t_r) = \hat{\mu}_j(x_0,t_{0,r}),\\ (z-\mu_j(x_0,t_{0,r}))^{-1}O(1) & \text{as } P \to \hat{\mu}_j(x_0,t_{0,r}) \neq \hat{\mu}_j(x,t_r), \end{cases}\\
&\qquad\qquad\qquad\qquad P=(z,y)\in\mathcal{K}_n,\ (x,t_r),(x_0,t_{0,r})\in\widetilde{\Omega},
\end{aligned} \tag{1.217}$$

where $O(1) \neq 0$ in (1.217). Consequently, all zeros and poles of ψ and $\tilde{\psi}$ on $\mathcal{K}_n \setminus \{P_\infty\}$ are simple and coincide. To apply the Riemann–Roch theorem (Theorem A.13; cf. Lemma B.2), as in the stationary context in Theorem 1.20, it remains to identify the essential singularity of ψ and $\tilde{\psi}$ at P_∞. For this purpose we first observe that

$$\begin{aligned}
&\int_{t_{0,r}}^{t_r} ds\, \big(\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s) - (1/2)\widetilde{F}_{r,x}(z,x_0,s)\big)\\
&\quad = \sum_{q=0}^{r} \tilde{c}_{r-q} \int_{t_{0,r}}^{t_r} ds\, \big(\widehat{F}_q(z,x_0,s)\phi(P,x_0,s) - (1/2)\widehat{F}_{q,x}(z,x_0,s)\big),
\end{aligned}$$

and hence it suffices to treat the homogeneous case in which $\tilde{c}_0 = 1$, $\tilde{c}_q = 0$ for $q = 1,\dots,r$. Invoking (1.181) then yields from (1.167) and (1.92)

$$\begin{aligned}
&\int_{t_{0,r}}^{t_r} ds\, \big(\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s) - (1/2)\widetilde{F}_{r,x}(z,x_0,s)\big)\\
&\quad = \int_{t_{0,r}}^{t_r} ds\, \big(\widetilde{F}_r(z,x_0,s) iy F_n(z,x_0,s)^{-1} + (1/2)\partial_s \ln(F_n(z,x_0,s))\big)\\
&\quad \underset{\zeta\to 0}{=} iy \int_{t_{0,r}}^{t_r} ds\, \widetilde{F}_r(z,x_0,s) F_n(z,x_0,s)^{-1} + O(\zeta^2), \quad \zeta = \sigma/\sqrt{z}.
\end{aligned}$$

Comparing (1.11) (in the homogeneous case) and (1.92), one obtains

$$iy\widetilde{F}_r(z,x_0,s)F_n(z,x_0,s)^{-1} \underset{\zeta\to 0}{=} i\zeta^{-2r-1} + O(\zeta),$$

and hence (cf. (1.113))

$$\begin{aligned}
&\int_{x_0}^{x} dx'\phi(P,x',t_r) + \int_{t_{0,r}}^{t_r} ds\,\big(\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s) - (1/2)\widetilde{F}_{r,x}(z,x_0,s)\big) \\
&\underset{\zeta\to 0}{=} i\zeta^{-1}(x-x_0) + i\zeta^{-2r-1}(t_r - t_{0,r}) + O(\zeta).
\end{aligned} \tag{1.218}$$

A comparison of (1.168), (1.207) (recalling $\tilde{c}_0 = 1$, $\tilde{c}_q = 0$ for $q = 1, \dots, r$), the expression (1.211) for $\tilde{\psi}$, and (1.218) then identifies the t_r-dependent behavior of the exponentials of ψ and $\tilde{\psi}$ up to order $O(\zeta)$ near P_∞. This completes the proof of (1.211) subject to (1.216) and possibly the normalization of ψ. The latter is determined by (1.185) as in the stationary context (1.114).

Equations (1.196), the second part of (1.11) (with c_{n-j} replaced by $\tilde{c}_{r-q}$), and (F.5) as well as Langrange's interpolation theorem, Theorem E.1, yield

$$\begin{aligned}
\partial_{t_r}\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) &= \sum_{j=1}^{n}\mu_{j,t_r}\sum_{k=1}^{n}\underline{c}(k)\frac{\mu_j^{k-1}}{y(\hat{\mu}_j)} \\
&= -2i\sum_{j,k=1}^{n}\underline{c}(k)\frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j-\mu_\ell)}\,\widetilde{F}_r(\mu_j) \\
&= -2i\sum_{j,k=1}^{n}\underline{c}(k)\frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j-\mu_\ell)}\sum_{q=0}^{r}\tilde{c}_{r-q}\sum_{p=(q-n)\vee n}^{q}\hat{c}_p(\underline{E})\Phi_{q-p}^{(j)}(\underline{\mu}) \\
&= -2i\sum_{k=1}^{n}\underline{c}(k)\sum_{q=0}^{r}\sum_{p=(q-n)\vee 0}^{q}\tilde{c}_{r-q}\,\hat{c}_p(\underline{E})\sum_{j=1}^{n}\frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j-\mu_\ell)}\Phi_{q-p}^{(j)}(\underline{\mu}) \\
&= -2i\sum_{k=1}^{n}\sum_{q=0}^{r}\underline{c}(k)\,\tilde{c}_{r-q}\,\hat{c}_{k-n+q}(\underline{E}) = -\underline{\widetilde{U}}_{2r}^{(2)}, \quad r\in\mathbb{N},
\end{aligned} \tag{1.219}$$

using (B.33) and hence (1.212) subject to (1.216). (This computation is equivalent to that in (F.87).) The extension of these results from $\widetilde{\Omega}$ to Ω then follows by continuity of $\underline{\alpha}_{P_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}}$ being nonspecial on Ω. Finally, (1.213) immediately follows from (1.212) and the linear equivalence $\mathcal{D}_{\hat{\lambda}_0^\beta\underline{\hat{\lambda}}^\beta} \sim \mathcal{D}_{P_\infty\underline{\hat{\mu}}}$. $\square$

Combining (1.212) and (1.214) shows the remarkable linearity of the theta function with respect to x and t_r in the Its–Matveev formula for u. In fact, one can rewrite (1.214) as

$$u(x,t_r) = \Lambda_0 - 2\partial_x^2\ln(\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r)),$$

where

$$\underline{A} = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) - i\underline{U}_0^{(2)} x_0 - i\underline{\widetilde{U}}_{2r}^{(2)} t_{0,r} + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}),$$
$$\underline{B} = i\underline{U}_0^{(2)}, \quad \underline{C}_r = i\underline{\widetilde{U}}_{2r}^{(2)},$$
$$\Lambda_0 = E_0 + \sum_{j=1}^n (E_{2j-1} + E_{2j} - 2\lambda_j);$$

hence, the constants $\Lambda_0 \in \mathbb{C}$ and $\underline{B}, \underline{C}_r \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ and r, and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0, t_{0,r}) = (\hat{\mu}_1(x_0, t_{0,r}), \dots, \hat{\mu}_n(x_0, t_{0,r})) \in \mathrm{Sym}^n(\mathcal{K}_n)$ at the point $(x_0, t_{0,r})$ as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}$ is assumed to be nonspecial.

Remark 1.45 Remark 1.21 applies to the current time-dependent setting, that is, if $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial and $P_\infty \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$, then $\mathcal{D}_{\underline{\hat{\lambda}}^\beta}$ is nonspecial by Theorem A.31.

Remark 1.46 The explicit representation (1.211) for ψ again complements Lemma 1.34 and shows that ψ stays meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial (assuming the affine part of $\mathcal{K}_n$ to be nonsingular).

Remark 1.47 The linearization property (1.212) (and (1.106)) can also be obtained via an alternative procedure that we briefly sketch. One introduces the meromorphic differential

$$\Omega(x, x_0, t_r, t_{0,r}) = \partial_z \ln(\psi(\,\cdot\,, x, x_0, t_r, t_{0,r}))dz$$

and hence infers from the representation (1.211)

$$\Omega(x, x_0, t_r, t_{0,r}) = -i(x - x_0)\omega_{P_\infty,0}^{(2)} - (t_r - t_{0,r})\widetilde{\Omega}_{P_\infty,2r}^{(2)}$$
$$- \sum_{j=1}^n \omega^{(3)}_{\hat{\mu}_j(x_0,t_{0,r}),\hat{\mu}_j(x,t_r)} + \omega.$$

Here, ω denotes a holomorphic differential on $\mathcal{K}_n$, that is,

$$\omega = \sum_{j=1}^n c_j \omega_j$$

for some $c_j \in \mathbb{C}$, $j = 1, \dots, n$. Since $\psi(\,\cdot\,, x, x_0, t_r, t_{0,r})$ is single-valued on $\mathcal{K}_n$, all a- and b-periods of Ω are integer multiples of $2\pi i$; hence,

$$2\pi i m_k = \int_{a_k} \Omega(x, x_0, t_r, t_{0,r}) = \int_{a_k} \omega = c_k, \quad j = 1, \dots, n$$

for some $m_k \in \mathbb{Z}$ identifies c_k as integer multiples of $2\pi i$. Similarly, for some

$n_k \in \mathbb{Z}$,

$$\begin{aligned}
2\pi i n_k &= \int_{b_k} \Omega(x, x_0, t_r, t_{0,r}) \\
&= -i(x - x_0) \int_{b_k} \omega^{(2)}_{P_\infty,0} - i(t_r - t_{0,r}) \int_{b_k} \widetilde{\Omega}^{(2)}_{P_\infty,2r} \\
&\quad - \sum_{j=1}^{n} \int_{b_k} \omega^{(3)}_{\hat{\mu}_j(x_0,t_{0,r}),\hat{\mu}_j(x,t_r)} + 2\pi i \sum_{j=1}^{n} m_j \int_{b_k} \omega_j \\
&= 2\pi U^{(2)}_{0,k}(x - x_0) + 2\pi \widetilde{U}^{(2)}_{2r,k}(t_r - t_{0,r}) \qquad (1.220)\\
&\quad - 2\pi i \sum_{j=1}^{n} \underline{A}_{\hat{\mu}_j(x,t_r),k}(\hat{\mu}_j(x_0, t_{0,r})) + 2\pi i \sum_{j=1}^{n} m_j \tau_{j,k}, \quad k = 1, \dots, n,
\end{aligned}$$

using (1.101), (1.208), (A.14), and (A.26). By symmetry of τ (cf. (A.15)), (1.220) is equivalent to

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) + i\underline{U}^{(2)}_0(x - x_0) + i\underline{\widetilde{U}}^{(2)}_{2r}(t_r - t_{0,r}). \qquad (1.221)$$

For a systematic approach to the linearization property (1.221) along the lines used in (1.219), we refer to Appendix F.

Similarly, studying the meromorphic differential

$$\Omega^\beta(x, x_0, t_r, t_{0,r}) = \partial_z \ln\big(\psi^\beta(\,\cdot\,, x, x_0, t_r, t_{0,r})\big)dz,$$

where

$$\begin{aligned}
&\psi^\beta(P, x, x_0, t_r, t_{0,r}) \\
&\quad = \frac{\psi_x(P, x, x_0, t_r, t_{0,r}) + \beta\psi(P, x, x_0, t_r, t_{0,r})}{(\psi_x(P, x, x_0, t_r, t_{0,r}) + \beta\psi(P, x, x_0, t_r, t_{0,r}))|_{x=x_0,t_r=t_{0,r}}} \\
&\quad = \frac{\theta(\underline{z}(P_\infty, \underline{\hat{\lambda}}^\beta(x_0, t_{0,r})))\theta(\underline{z}(P, \underline{\hat{\lambda}}^\beta(x, t_r)))}{\theta(\underline{z}(P_\infty, \underline{\hat{\lambda}}^\beta(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\lambda}}^\beta(x_0, t_{0,r})))} \exp\bigg(-i(x - x_0) \int_{Q_0}^{P} \omega^{(2)}_{P_\infty,0} \\
&\qquad - i(t_r - t_{0,r}) \int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,2r} - \int_{Q_0}^{P} \omega^{(3)}_{\hat{\lambda}^\beta_0(x_0,t_{0,r}),\hat{\lambda}^\beta_0(x,t_r)}\bigg),
\end{aligned}$$

using (1.210) and (1.211), then proves (1.213).

The solution u in the Its–Matveev formula (1.214) is complex-valued in general. To obtain real-valued solutions, one argues as in Remark 1.24. In particular, the b-periods $\widetilde{U}^{(2)}_{2r,k}$, $k = 1, \dots, n$ of the second-order differential $\widetilde{\Omega}^{(2)}_{P_\infty,2r}$ also become purely imaginary, choosing $\tilde{c}_s \in \mathbb{R}$, $s = 1, \dots, r$. Moreover, the initial position of $\hat{\mu}_j(x_0, t_{0,r}) \in \mathcal{K}_n$ must be chosen in real position with its projections lying in the spectral gaps of H, that is,

$$\mu_j(x_0, t_{0,r}) \in [E_{2j-1}, E_{2j}], \quad j = 1, \dots, n$$

in order to render $\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})})$ purely imaginary $\pmod{\mathbb{Z}^n}$ as well and $u \in L^\infty(\mathbb{R}^2)$. The rest is completely analogous to the stationary discussion in Lemma 1.23 and Remark 1.24; hence, real-valued and smooth solutions u in (1.214), in general, will be quasi-periodic with respect to $x \in \mathbb{R}$ and $t_r \in \mathbb{R}$.

Up to this point we assumed Hypothesis 1.33 together with the basic equations (1.159) and (1.160). Next we will show that solvability of the Dubrovin equations (1.195) and (1.196) on $\Omega_\mu \subseteq \mathbb{R}^2$ in fact implies equations (1.159) and (1.160) on Ω_μ. In complete analogy to our discussion in Section 3.3 (cf. Remark 1.29), this amounts to solving the time-dependent algebro-geometric initial value problem (1.153), (1.154) on Ω_μ. In this context we recall the definition of $\widetilde{F}_r(\mu_j)$ in terms of $\mu_1, \dots, \mu_n$, introduced[1] in (F.16), (F.19),

$$\widetilde{F}_r(\mu_j) = \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(\underline{E})\Phi_k^{(j)}(\underline{\mu}), \quad r \in \mathbb{N}_0, \ \tilde{c}_0 = 1, \tag{1.222}$$

$$\tilde{d}_{r,k}(\underline{E}) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s}\hat{c}_s(\underline{E}), \quad k = 0, \dots, r \wedge n, \tag{1.223}$$

in terms of a given set of integration constants $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$.

Theorem 1.48 *Fix $n \in \mathbb{N}$ and assume the affine part of $\mathcal{K}_n$ to be nonsingular. Suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the Dubrovin equations* (1.195), (1.196) *on an open and connected set $\Omega_\mu \subseteq \mathbb{R}^2$ with $\widetilde{F}_r(\mu_j)$ in* (1.196) *expressed in terms of μ_k, $k = 1, \dots, n$, by* (1.222) *and* (1.223). *Moreover, assume that μ_j, $= 1, \dots, n$ remain distinct on Ω_μ. Then $u \in C^\infty(\Omega_\mu)$, defined by*

$$u = E_0 + \sum_{j=1}^{n} \big(E_{2j-1} + E_{2j} - 2\mu_j\big), \tag{1.224}$$

satisfies the rth KdV equation (1.159), *that is,*

$$\widetilde{\mathrm{KdV}}_r(u) = 0 \text{ on } \Omega_\mu \tag{1.225}$$

with initial values satisfying the nth stationary KdV equation (1.160).

Proof Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (1.195), (1.196), we define the polynomials F_n and H_{n+1} as in the stationary case (cf. (1.131), (1.133)) by

$$F_n(z) = \prod_{j=1}^{n} (z - \mu_j) \text{ on } \mathbb{C} \times \Omega_\mu \tag{1.226}$$

and

$$R_{2n+1}(z) + (1/4)F_{n,x}(z)^2 = F_n(z)H_{n+1}(z) \text{ on } \mathbb{C} \times \Omega_\mu.$$

[1] $m \wedge n = \min(m, n)$.

The proof that H_{n+1} satisfies (1.137),

$$H_{n+1}(z) = (1/2)F_{n,xx}(z) - (u-z)F_n(z) \text{ on } \mathbb{C} \times \Omega_\mu,$$

and that u satisfies (1.160) is identical to the stationary case treating t_r as a parameter. Hence, we exclusively focus on the proof of (1.159).

Next we prove (1.181), that is,

$$F_{n,t_r}(z) = F_{n,x}(z)\widetilde{F}_r(z) - F_n(z)\widetilde{F}_{r,x}(z) \text{ on } \mathbb{C} \times \Omega_\mu, \tag{1.227}$$

keeping in mind that $\widetilde{F}_r$ on $\mathbb{C} \times \Omega_\mu$ is defined in terms of μ_j, $j = 1, \dots, n$ by (F.10) or (F.12) in the special homogeneous case and by

$$\widetilde{F}_r = \sum_{s=0}^{r} \tilde{c}_{r-s} \widehat{F}_s, \quad \tilde{c}_0 = 1,$$

in general, with $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$ a given set of integration constants. To this end we compute from (1.195), (1.196),

$$F_{n,t_r}(z) = -F_n(z) \sum_{j=1}^{n} \widetilde{F}_r(\mu_j)\mu_{j,x}(z-\mu_j)^{-1},$$

$$\widetilde{F}_r(z)F_{n,x}(z) = -F_n(z)\widetilde{F}_r(z) \sum_{j=1}^{n} \mu_{j,x}(z-\mu_j)^{-1}$$

and hence infer (1.227) immediately from (F.74). Next we introduce

$$\widetilde{H}_{r+1}(z) = (1/2)\widetilde{F}_{r,xx}(z) - (u-z)\widetilde{F}_r(z) \text{ on } \mathbb{C} \times \Omega_\mu.$$

Then one computes

$$\begin{aligned} F_{n,xt_r} &= F_{n,t_r x} = \widetilde{F}_{r,x} F_{n,x} + \widetilde{F}_r F_{n,xx} - F_{n,x}\widetilde{F}_{r,x} - F_n \widetilde{F}_{r,xx} \\ &= \widetilde{F}_r(2H_{n+1} + 2(u-z)F_n) - F_n(2H_{r+1} + 2(u-z)\widetilde{F}_r), \end{aligned}$$

and consequently

$$F_{n,xt_r} = 2(H_{n+1}\widetilde{F}_r - F_n\widetilde{H}_{r+1}) \text{ on } \mathbb{C} \times \Omega_\mu, \tag{1.228}$$

that is, we obtained (1.182). Given F_n in (1.226) we define (as in (1.165)) on Ω_μ

$$\phi(P) = \frac{iy + (1/2)F_{n,x}(z)}{F_n(z)}, \quad P = (z,y) \in \mathcal{K}_n \tag{1.229}$$

and then observe

$$\begin{aligned} \big(\phi(P) - \phi(P^*)\big)_{t_r} &= -2iyF_{n,t_r}(z)F_n(z)^{-2} \\ &= 2iy(F_n(z)\widetilde{F}_{r,x}(z) - \widetilde{F}_r(z)F_{n,x}(z))F_n(z)^{-2} \\ &= \big(2iy\widetilde{F}_r(z)F_n(z)^{-1}\big)_x = \big(\widetilde{F}_r(z)(\phi(P) - \phi(P^*))\big)_x, \end{aligned}$$

using (1.227), and

$$
\begin{aligned}
\big(\phi(P)+\phi(P^*)\big)_{t_r} &= \big(F_{n,x}(z)F_n(z)^{-1}\big)_{t_r} \\
&= \big(F_n(z)F_{n,xt_r}(z) - F_{n,x}(z)F_{n,t_r}(z)\big)F_n(z)^{-2} \\
&= F_n(z)^{-2}\big(F_n(z)2(\widetilde F_r(z)H_{n+1}(z) - F_n(z)\widetilde H_{r+1}(z)) \\
&\quad - F_{n,x}(z)(\widetilde F_r(z)F_{n,x}(z) - \widetilde F_{r,x}(z)F_n(z))\big) \\
&= F_n(z)^{-2}\big(F_n(z)\widetilde F_r(z)F_{n,xx}(z) - F_n(z)^2\widetilde F_{r,xx}(z) \\
&\quad - F_{n,x}(z)^2\widetilde F_r(z) + F_{n,x}(z)F_n(z)\widetilde F_{r,x}(z)\big) \\
&= \big(\widetilde F_r(z)(\phi(P)+\phi(P^*))\big)_x - \widetilde F_{r,xx}(z),
\end{aligned}
$$

using (1.227) and (1.228). Thus, we proved (1.172), that is,

$$
\phi_{t_r}(P) = \big(\widetilde F_r(z)\phi(P) - (1/2)\widetilde F_{r,x}(z)\big)_x \text{ on } \mathcal{K}_n \times \Omega_\mu. \tag{1.230}
$$

Equations (1.160) and (1.229) then yield (1.171), that is,

$$
\phi_x(P) + \phi(P)^2 = u - z \text{ on } \mathcal{K}_n \times \Omega_\mu, \tag{1.231}
$$

and repeatedly combining (1.230) and (1.231) then implies

$$
\begin{aligned}
u_{t_r} &= \phi_{xt_r} + 2\phi\phi_{t_r} = (\widetilde F_r\phi - (1/2)\widetilde F_{r,x})_{xx} + 2\phi(\widetilde F_r\phi - (1/2)\widetilde F_{r,x})_x \\
&= -(1/2)\widetilde F_{r,xxx} + 2(u-z)\widetilde F_{r,x} + \widetilde F_r\phi_{xx} + 2\phi\phi_x\widetilde F_r \\
&= -(1/2)\widetilde F_{r,xxx} + 2(u-z)\widetilde F_{r,x} + u_x\widetilde F_r \text{ on } \mathbb{C}\times\Omega_\mu,
\end{aligned}
$$

and hence (1.159) on $\mathbb{C}\times\Omega_\mu$. □

Remark 1.49 The explicit theta function representation (1.214) of u on Ω_μ in (1.224) then permits one to extend u beyond Ω_μ as long as $\mathcal{D}_{\underline{\hat\mu}}$ remains nonspecial (cf. Remark 1.46).

Remark 1.50 Again we formulated Theorem 1.48 in terms of Dirichlet eigenvalues μ_j, $j = 1, \dots, n$ only. Clearly the analogous result can be proved in connection with all β-boundary conditions (1.54) in terms of λ_ℓ^β, $\ell = 0, \dots, n$, for each $\beta \in \mathbb{R}$.

The analog of Remark 1.29 directly extends to the current time-dependent setting.

As in our previous Section 1.3 we will end this section with a few examples illustrating the general results. Again we also consider some examples involving singular curves and/or singular (i.e., meromorphic) algebro-geometric KdV solutions u.

We start with *rational* KdV solutions.

Example 1.51 The case of rational KdV solutions. Let $(x, t_r) \in \Omega$ for some open connected subset $\Omega \subset \mathbb{R}^2$, $r \in \mathbb{N}$.
(i) The simplest nontrivial rational KdV_1 solution vanishing at infinity is the following,

$$\begin{aligned} &u_2(x, t_1) = 6\frac{x(x^3 - 6t_1)}{(x^3 + 3t_1)^2} = 2\sum_{\ell=1}^{3} \frac{1}{(x - x_\ell(t_1))^2}, \quad (x, t_1) \in \Omega, \\ &x \neq x_\ell(t_1), \quad x_\ell(t_1) = -(3t_1)^{1/3}\omega_\ell, \quad \omega_\ell = \exp(2\pi i\ell/3), \; \ell = 1, 2, 3, \\ &\text{s-}\widehat{\mathrm{KdV}}_m(u_2) = 0, \; m \geq 2, \quad \widehat{\mathrm{KdV}}_1(u_2) = 0, \end{aligned}$$

with associated curve given by (1.142).
(ii) More generally, generic rational KdV_1 solutions vanishing at infinity are of the type

$$u_n(x, t_1) = 2\sum_{k=1}^{N} \frac{1}{(x - x_k(t_1))^2}, \quad (x, t_1) \in \Omega, \; x \neq x_k(t_1), \; k = 1, \dots, N, \tag{1.232}$$

where $N \in \mathbb{N}$ must be of the form

$$N = n(n + 1)/2 \text{ for some } n \in \mathbb{N},$$

and the points $x_k(t_1)$ are pairwise distinct and satisfy the constraints

$$\begin{aligned} &\sum_{\substack{k'=1 \\ k' \neq k}}^{N} (x_k(t_1) - x_{k'}(t_1))^{-3} = 0, \quad k = 1, \dots, N, \\ &\frac{d}{dt_1} x_k(t_1) = -3\sum_{\substack{k'=1 \\ k' \neq k}}^{N} (x_k(t_1) - x_{k'}(t_1))^{-2}, \quad k = 1, \dots, N. \end{aligned}$$

The KdV solutions (1.232) satisfy

$$\text{s-}\widehat{\mathrm{KdV}}_m(u_n) = 0, \; m \geq n, \quad \widehat{\mathrm{KdV}}_1(u_n) = 0,$$

with associated curve given by (1.143).
(iii) Finally, generic rational KdV_r solutions are of the type

$$u_n(x, t_r) = u_0 + 2\sum_{k=1}^{N} \frac{1}{(x - x_k(t_r))^2}, \quad (x, t_r) \in \Omega, \; x \neq x_k(t_r), \; k = 1, \dots, N, \tag{1.233}$$

where $N \in \mathbb{N}$ must be of the form

$$N = n(n + 1)/2 \text{ for some } n \in \mathbb{N},$$

$u_0 \in \mathbb{C}$, and the points $x_k(t_r)$ are pairwise distinct and satisfy the constraints

$$\sum_{\substack{k'=1 \\ k'\neq k}}^{N} (x_k(t_r) - x_{k'}(t_r))^{-3} = 0, \quad k = 1, \dots, N,$$

$$\frac{d}{dt_r} x_k(t_r) = a_{r+1,k}(t_r), \quad k = 1, \dots, N.$$

Here $a_{s,j}$ are recursively defined by

$$\begin{aligned} a_{0,j}(t_r) &= 0, \ j = 1, \dots, N, \quad \tilde{c}_0 = 1, \\ a_{s+1,j}(t_r) &= a_{s,j}(t_r)u_0 - \tilde{c}_s - \sum_{p=1}^{s} \tilde{c}_{s-p}\alpha_p u_0^p \\ &\quad - \sum_{\substack{k=1 \\ k\neq j}}^{N} \big(a_{s,k}(t_r) + 2a_{s,j}(t_r)\big)(z_j(t_r) - z_k(t_r))^{-2}, \\ &\qquad\qquad s = 0, \dots, r, \ j = 1, \dots, N, \end{aligned}$$

with $\alpha_p = 2^{-2p}(p!)^{-2}(2p)!$, $p \in \mathbb{N}$.

The KdV solutions (1.233) satisfy

$$\text{s-KdV}_m(u_n) = 0, \ m \geq n, \quad \widetilde{\text{KdV}}_r(u_n) = 0,$$

with associated curve given by $y^2 + (z - u_0)^{2n+1} = 0$ and for a particular set of integration constants $\{c_\ell\}_{\ell=1,\dots,m}$ in $\text{s-KdV}_m(\,\cdot\,)$ and $\{\tilde{c}_\ell\}_{\ell=1,\dots,r}$ in $\widetilde{\text{KdV}}_r(\,\cdot\,)$.

Here (and in Example 1.53) the notion "generic" refers to the collisionless case in which all x_k remain pairwise distinct.

Our second example describes the *n-soliton* solutions of the KdV hierarchy.

Example 1.52 The case of n-soliton KdV solutions.
Let $n, r \in \mathbb{N}$. Then

$$\begin{aligned} &u_n(x, t_r) = -2\frac{d^2}{dx^2} \ln(\tau_n(x, t_r)), \quad (x, t_r) \in \mathbb{R}^2 \setminus \{(y, s) \in \mathbb{R}^2 \mid \tau_n(y, s) = 0\}, \\ &\tau_n(x, t_r) = \det(I_n + C_n(x, t_r)), \\ &C_n(x, t_r) = \left(\frac{c_j c_k}{\kappa_j + \kappa_k} \exp(-(\kappa_j + \kappa_k)x + (-1)^{r+1}(\kappa_j^{2r+1} + \kappa_k^{2r+1})t_r) \right)_{j,k=1,\dots,n}, \\ &\qquad\qquad c_j, \kappa_j \in \mathbb{C}, \ j = 1, \dots, n, \\ &\quad \text{s-}\widehat{\text{KdV}}_n(u) = 0, \quad \widehat{\text{KdV}}_r(u_n) = 0. \end{aligned}$$

Nonsingular soliton solutions are obtained upon imposing the restrictions $c_j > 0$, $\kappa_j > 0$, $j = 1, \dots, n$.

Finally, we consider *elliptic* KdV solutions. We recall our notation $\wp(\,\cdot\,) = \wp(\,\cdot\,|\,\omega_1, \omega_3) = \wp(\,\cdot\,; g_2, g_3)$ of the Weierstrass $\wp$-function with periods $2\omega_j$, $j = 1, 3$, invariants g_2 and g_3, and associated fundamental period parallelogram Δ (cf. Appendix H).

Example 1.53 The case of elliptic KdV solutions. Let $(x, t_r) \in \Omega$ for some open connected subset $\Omega \subset \mathbb{R}^2$, $r \in \mathbb{N}$.
(i) Generically, elliptic KdV_1 solutions are of the type

$$u_n(x, t_1) = u_0 + 2 \sum_{k=1}^{N} \wp(x - x_k(t_1)), \qquad (1.234)$$
$$(x, t_1) \in \Omega, \ x \neq x_k(t_1) \pmod{\Delta},\ k = 1, \dots, N,$$

where $N \in \mathbb{N}$ must be of the form

$$N = n(n+1)/2 \text{ for some } n \in \mathbb{N},$$

$u_0 \in \mathbb{C}$, and the points $x_k(t_1)$ are pairwise distinct $\pmod{\Delta}$ and satisfy the constraints

$$\sum_{\substack{k'=1 \\ k' \neq k}}^{N} \wp'(x_k(t_1) - x_{k'}(t_1)) = 0, \quad k = 1, \dots, N,$$
$$\frac{d}{dt_1} x_k(t_1) = -3 \sum_{\substack{k'=1 \\ k' \neq k}}^{N} \wp(x_k(t_1) - x_{k'}(t_1)), \quad k = 1, \dots, N.$$

The KdV solutions (1.234) with $u_0 = 0$ satisfy

$$\text{s-KdV}_n(u_n) = 0, \quad \widehat{\mathrm{KdV}}_1(u_n) = 0$$

for a particular set of integration constants $\{c_\ell\}_{\ell=1,\dots,n}$ in $\text{s-KdV}_n(\,\cdot\,)$.
(ii) More generally, generic elliptic KdV_r solutions are of the type

$$u_n(x, t_r) = u_0 + 2 \sum_{k=1}^{N} \wp(x - x_k(t_r)), \qquad (1.235)$$
$$(x, t_r) \in \Omega, \ x \neq x_k(t_r) \pmod{\Delta},\ k = 1, \dots, N,$$

where $N \in \mathbb{N}$ must be of the form

$$N = n(n+1)/2 \text{ for some } n \in \mathbb{N},$$

$u_0 \in \mathbb{C}$, and the points $x_k(t_r)$ are pairwise distinct (mod Δ) and satisfy the constraints

$$\sum_{\substack{k'=1\\k'\neq k}}^{N} \wp'(x_k(t_r) - x_{k'}(t_r)) = 0, \quad k = 1, \dots, N,$$
$$\frac{d}{dt_r} x_k(t_r) = a_{r+1,k}(t_r), \quad k = 1, \dots, N.$$

Here $a_{s,j}$ are recursively defined by

$$\begin{aligned} a_{0,j}(t_r) &= 0, \;\; j = 1, \dots, N, \quad \tilde c_0 = 1, \\ a_{s+1,j}(t_r) &= a_{s,j}(t_r) u_0 - \tilde c_s - \sum_{p=1}^{s} \tilde c_{s-p} \alpha_p u_0^p \\ &\quad - \sum_{\substack{k=1\\k\neq j}}^{N} \big(a_{s,k}(t_r) + 2a_{s,j}(t_r)\big) \wp(z_j(t_r) - z_k(t_r)), \\ &\qquad s = 0, \dots, r, \;\; j = 1, \dots, N, \end{aligned}$$

with $\alpha_p = 2^{-2p}(p!)^{-2}(2p)!$, $p \in \mathbb{N}$.
The KdV solutions (1.235) satisfy

$$\text{s-KdV}_n(u_n) = 0, \quad \widetilde{\text{KdV}}_r(u_n) = 0$$

for a particular set of integration constants $\{c_\ell\}_{\ell=1,\dots,n}$ in $\text{s-KdV}_n(\,\cdot\,)$ and $\{\tilde c_\ell\}_{\ell=1,\dots,r}$ in $\widetilde{\text{KdV}}_r(\,\cdot\,)$.

As in the stationary context described at the end of Section 1.3, the rational and soliton cases described in Example 1.51 and Example 1.52 are appropriate limiting cases of elliptic KdV solutions in (1.234).

1.5 General Trace Formulas

In this section we will extend the classical trace formula in the algebro-geometric case presented in Lemmas 1.16 and 1.17,

$$u = \sum_{m=0}^{2n} E_m - 2\sum_{j=1}^{n} \mu_j, \tag{1.236}$$
$$2\beta^2 - u = \sum_{m=0}^{2n} E_m - 2\sum_{\ell=0}^{n} \lambda_\ell^\beta, \tag{1.237}$$

as well as their higher-order analogs, to general $C^\infty(\mathbb{R})$-potentials u bounded from below. The key to our approach is to consider pairs of self-adjoint operators, for example (H_x^D, H), that are closely related in the sense that their resolvents differ only by a rank-one operator. For such pairs the associated spectral shift function takes a particularly simple form in terms of the diagonal of the Green's function of H, and this circle of ideas will be our main topic in this section. In addition to the pair of operators just mentioned, we will also do a complete analysis of the case with the more general boundary condition (1.54), that is, for the pair (H_x^β, H).

As an aside, we develop a recursive method for computing small-time heat kernel and asymptotic spectral parameter resolvent expansion coefficients associated with the general β-boundary conditions (1.54). (Additional expansions of Weyl–Titchmarsh m-functions as the spectral parameter tends to infinity are presented in Appendix J.)

Unlike Sections 1.3 and 1.4, in which we focused on the special case of algebro-geometric solutions of the KdV hierarchy, we now turn to the general situation and throughout this section consider smooth and real-valued potentials that are bounded from below.

Hypothesis 1.54 *Let* $u\colon \mathbb{R} \to \mathbb{R}$ *satisfy*

$$u \in C^\infty(\mathbb{R}), \quad u \geq c \tag{1.238}$$

for some $c \in \mathbb{R}$.

As in Section 1.3, we study the differential expression $L = -\frac{d^2}{dx^2} + u$ on $\mathbb{R}$ and associate with it operators H and H_x^D, H_x^β in analogy to Sections 1.3 and 1.4. The self-adjoint operator H_x^D is associated with the Dirichlet boundary condition at the point x, that is, with $g(x) = 0$, whereas the self-adjoint operator H_x^β corresponds to the boundary condition $g'(x) + \beta g(x) = 0$ with $\beta \in \mathbb{R}$, as in (1.54). See Appendix J for precise definitions.

Let $G(z, x, x')$ and $g(z, x) = G(z, x, x)$ denote the Green's function and diagonal Green's function of H, respectively, and recall formulas (J.20)–(J.22) for the resolvent of $H_{x_0}^\beta$. Defining

$$\Gamma^\beta(z, x) = \begin{cases} (\beta + \partial_{x_1})(\beta + \partial_{x_2})G(z, x_1, x_2)\big|_{x_1 = x, x_2 = x} & \text{for } \beta \in \mathbb{R}, \\ g(z, x) & \text{for } \beta = \infty, \end{cases} \tag{1.239}$$

(cf. the notation introduced in (J.23)–(J.24)) one computes for $\beta \in \mathbb{R} \cup \{\infty\}$

$$\operatorname{tr}\big((H_x^\beta - z)^{-1} - (H - z)^{-1}\big) = -\frac{d}{dz} \ln\big(\Gamma^\beta(z, x)\big), \tag{1.240}$$
$$z \in \mathbb{C} \setminus (\operatorname{spec}(H_x^\beta) \cup \operatorname{spec}(H)).$$

Combining (1.239), (1.240), (J.16), (J.23), (J.24), (J.27), and (J.29) yields the existence of asymptotic expansions of the type

$$\operatorname{tr}\big((H_x^\beta - z)^{-1} - (H - z)^{-1}\big) \underset{z \to i\infty}{=} \sum_{\ell=0}^{\infty} r_\ell^\beta(x) z^{-\ell-1}, \quad \beta \in \mathbb{R} \cup \{\infty\} \tag{1.241}$$

uniformly with respect to x varying in compact intervals. Moreover, one can derive the heat kernel expansion

$$\operatorname{tr}\left(e^{-\tau H_x^\infty} - e^{-\tau H}\right) \underset{\tau\downarrow 0}{\sim} \sum_{\ell=0}^{\infty} s_\ell^\infty(x)\tau^\ell, \quad x \in \mathbb{R}, \tag{1.242}$$

where

$$s_\ell^\infty(x) = \frac{(-1)^{\ell+1}}{\ell!} r_\ell^\infty(x), \quad \ell \in \mathbb{N}_0 \tag{1.243}$$

and s_ℓ^∞ and r_ℓ^∞ are the well-known invariants of the KdV hierarchy. More precisely, suppose u satisfies the rth KdV equation (for some choice of integration constants c_ℓ) and generates $r_\ell^\infty(x, t_r)$, replacing $u(x)$ by $u(x, t_r)$ (cf. (1.152)–(1.162)). Then

$$\frac{d}{dt_r} \int dx\, r_\ell^\infty(x, t_r) = 0, \quad \ell \in \mathbb{N},\ r \in \mathbb{N}_0, \tag{1.244}$$

where $\int dx$ denotes $\int_{\mathbb{R}} dx$ in the case of sufficient decrease of $u(x, t_r)$ as $|x| \to \infty$, $\int_0^\Omega dx$ in the case where $u(x + \Omega, t_r) = u(x, t_r)$ is Ω-periodic in x, and the ergodic mean $\lim_{\Omega\uparrow\infty} \frac{1}{\Omega} \int_0^\Omega dx$ for classes of almost periodic solutions $u(x, t_r)$ with respect to x. We will return to this circle of ideas at the end of this section where we briefly discuss the Hamiltonian approach to the KdV hierarchy in the case of spatially rapidly decaying solutions.

In the special case of algebro-geometric potentials considered in Section 1.3, the connection of Γ^β in (1.239) with our polynomial approach in Section 1.3 is clearly demonstrated by (J.45)–(J.48).

Before describing a recursive approach to the coefficients r_j^β, $\beta \in \mathbb{R}$, we recall the definition of the spectral shift function associated with the pair (H_x^β, H). The rank-one resolvent difference of H_x^β and H (cf. (J.21), (J.22)) is intimately connected with the fact that $\Gamma^\beta(\,\cdot\,, x)$ is a Herglotz function[1] with respect to z for each $x \in \mathbb{R}$, $\beta \in \mathbb{R} \cup \{\infty\}$. The exponential Herglotz representation for Γ^β (cf. (I.1)) then reads for each $x \in \mathbb{R}$,

$$\begin{aligned}
&\Gamma^\beta(z, x) = \exp\left(c^\beta(x) + \int_{\mathbb{R}} d\lambda \left(\frac{1}{(\lambda - z)} - \frac{\lambda}{(1 + \lambda^2)}\right)\left(\xi^\beta(\lambda, x) + \delta^\beta\right)\right), \\
&c^\beta(x) = \operatorname{Re}(\Gamma^\beta(i, x)),\ \beta \in \mathbb{R} \cup \{\infty\},\ \delta^\beta = \begin{cases} 1 & \text{for } \beta \in \mathbb{R}, \\ 0 & \text{for } \beta = \infty, \end{cases}
\end{aligned} \tag{1.245}$$

where, by Fatou's lemma,

$$\xi^\beta(\lambda, x) = \pi^{-1} \lim_{\varepsilon\downarrow 0} \operatorname{Im}\big(\ln(\Gamma^\beta(\lambda + i\varepsilon, x))\big) - \delta^\beta, \quad \beta \in \mathbb{R} \cup \{\infty\} \tag{1.246}$$

for a.e. $\lambda \in \mathbb{R}$. Moreover,

$$\begin{aligned}
&-1 \le \xi^\beta(\lambda, x) \le 0, \quad \xi^\beta(\lambda, x) = 0, \quad \lambda < \inf\operatorname{spec}(H_x^\beta), \quad \beta \in \mathbb{R}, \\
&0 \le \xi^\infty(\lambda, x) \le 1, \quad \xi^\infty(\lambda, x) = 0, \quad \lambda < \inf\operatorname{spec}(H)
\end{aligned}$$

[1] Herglotz functions are holomorphic maps $\mathbb{C}_+ \to \mathbb{C}_+$; see Appendix I.

for a.e. $\lambda \in \mathbb{R}$. As a consequence, one can prove (cf. (I.2))

$$\operatorname{tr}\left(f(H_x^\beta) - f(H)\right) = \int_{\mathbb{R}} d\lambda\, f'(\lambda)\xi^\beta(\lambda, x), \quad \beta \in \mathbb{R} \cup \{\infty\},\ x \in \mathbb{R} \quad (1.247)$$

for any $f \in C^2(\mathbb{R})$ with $(1+\lambda^2)f^{(j)} \in L^2((0,\infty))$ for $j = 1, 2$ and for $f(\lambda) = 1/(\lambda - z)$, $z \in \mathbb{C} \setminus [\inf \operatorname{spec}(H_x^\beta), \infty)$. In particular, (1.247) holds for traces of heat kernel and resolvent differences, that is, for any $\beta \in \mathbb{R} \cup \{\infty\}$, $x \in \mathbb{R}$,

$$\operatorname{tr}\left(e^{-\tau H_x^\beta} - e^{-\tau H}\right) = -\tau \int_{e_{x,0}^\beta}^\infty d\lambda\, e^{-\tau\lambda}\xi^\beta(\lambda, x), \quad \tau > 0, \quad (1.248)$$

$$\operatorname{tr}\left((H_x^\beta - z)^{-1} - (H - z)^{-1}\right) = -\int_{e_{x,0}^\beta}^\infty d\lambda\, (\lambda - z)^{-2}\xi^\beta(\lambda, x), \quad (1.249)$$
$$z \in \mathbb{C} \setminus (\operatorname{spec}\left(H_x^\beta\right) \cup \operatorname{spec}(H)),$$

where

$$e_{x,0}^\beta = \begin{cases} \inf \operatorname{spec}\left(H_x^\beta\right) & \text{for } \beta \in \mathbb{R}, \\ \inf \operatorname{spec}(H) & \text{for } \beta = \infty. \end{cases}$$

In the particularly simple case $u = 0$, one derives the following explicit formulas.

Example 1.55 Consider the case $u = 0$. Then

$$G(z, x, x') = (i/2)z^{-1/2} \exp(iz^{1/2}|x - x'|), \quad \operatorname{Im}(z^{1/2}) \geq 0$$

yields

$$\begin{aligned} \Gamma^\beta(z, x) &= (\beta + \partial_{x_1})(\beta + \partial_{x_2})G(z, x_1, x_2)\big|_{x_1 = x, x_2 = x} \\ &= (i/2)(\beta^2 z^{-1/2} + z^{1/2}), \quad \beta \in \mathbb{R}, \\ \Gamma^\infty(z, x) &= g(z, x) = (i/2)z^{-1/2}, \end{aligned}$$

and

$$\xi^\beta(\lambda, x) = \begin{cases} 0 & \text{for } \lambda < -\beta^2, \\ -1 & \text{for } -\beta^2 < \lambda < 0, \\ -\frac{1}{2} & \text{for } \lambda > 0, \end{cases} \quad \beta \in \mathbb{R} \setminus \{0\},$$

$$\xi^0(\lambda, x) = \begin{cases} 0 & \text{for } \lambda < 0, \\ -\frac{1}{2} & \text{for } \lambda > 0, \end{cases}$$

$$\xi^\infty(\lambda, x) = \begin{cases} 0 & \text{for } \lambda < 0, \\ \frac{1}{2} & \text{for } \lambda > 0. \end{cases}$$

Thus,

$$\begin{aligned}
&\operatorname{tr}\left((H_x^{\beta}-z)^{-1}-(H-z)^{-1}\right)=\frac{\beta^2-z}{2z(z+\beta^2)},\\
&\qquad\qquad \beta\in\mathbb{R},\ z\in\mathbb{C}\setminus(\{-\beta^2\}\cup[0,\infty)),\\
&\operatorname{tr}\left((H_x^{\infty}-z)^{-1}-(H-z)^{-1}\right)=\frac{1}{2z},\quad z\in\mathbb{C}\setminus[0,\infty),\\
&\operatorname{tr}\left(e^{-tH_x^{\beta}}-e^{-tH}\right)=-(1/2)+\exp(t\beta^2),\quad \beta\in\mathbb{R},\ t>0,\\
&\operatorname{tr}\left(e^{-tH_x^{\infty}}-e^{-tH}\right)=-1/2,\quad t>0,
\end{aligned}$$

where $H=-\frac{d^2}{dx^2}$ in $L^2(\mathbb{R})$ with domain $\operatorname{dom}(H)=H^{2,2}(\mathbb{R})$. One has

$$\begin{aligned}
\operatorname{spec}(H^{\beta})&=\{-\beta^2\}\cup[0,\infty),\quad \beta\in\mathbb{R},\\
\operatorname{spec}(H^{\infty})&=[0,\infty).
\end{aligned}$$

Returning to a recursive approach for the expansion coefficients r_j^{β} in (1.241), we first consider the expansion

$$\Gamma^{\beta}(z)\underset{z\to i\infty}{=}\frac{i}{2}\sum_{\ell=-\delta^{\beta}}^{\infty}\gamma_\ell^{\beta}z^{-\ell-1/2},\quad \beta\in\mathbb{R}\cup\{\infty\}. \tag{1.250}$$

(A comparison of (1.250) and (1.92) reveals that $\gamma_\ell^{\infty}=\hat{f}_\ell$, $\ell\in\mathbb{N}_0$ in the case $\beta=\infty$.) To obtain a recursion relation for γ_ℓ^{β}, one can use the following result.

Lemma 1.56 *Assume Hypothesis* 1.54 *and let* $z\in\mathbb{C}\setminus\operatorname{spec}(H)$, $x\in\mathbb{R}$.
(i) *Suppose* $\beta\in\mathbb{R}$. *Then* $\Gamma^{\beta}(z)$ *satisfies*

$$\begin{aligned}
&2(u-\beta^2-z)\Gamma_{xx}^{\beta}(z)\Gamma^{\beta}(z)-(u-\beta^2-z)\Gamma_x^{\beta}(z)^2-2u_x\Gamma_x^{\beta}(z)\Gamma^{\beta}(z)\\
&\quad-4\big((u-z)(u-\beta^2-z)-\beta u_x\big)\Gamma^{\beta}(z)^2\\
&=-(u-z-\beta^2)^3.
\end{aligned}\tag{1.251}$$

(ii) *Suppose* $\beta=\infty$. *Then* $\Gamma^{\infty}(z)$ *satisfies*

$$\Gamma_{xxx}^{\infty}(z)-4(u-z)\Gamma_x^{\infty}(z)-2u_x\Gamma^{\infty}(z)=0 \tag{1.252}$$

and

$$-2\Gamma_{xx}^{\infty}(z)\Gamma^{\infty}(z)+\Gamma_x^{\infty}(z)^2+4(u-z)\Gamma^{\infty}(z)^2=1. \tag{1.253}$$

Proof The derivations of (1.252) and (1.253) follow from straightforward calculations using (J.14), (J.15), whereas the proof of (1.251), although straightforward in principle, is very tedious. □

Insertion of expansion (1.250) into (1.251) and (1.253) in Lemma 1.56 yields the following result.

Lemma 1.57 *Assume Hypothesis* 1.54. *Then the coefficients* $\gamma_\ell^\beta(x)$ *in* (1.250) *satisfy the following recursion relation.*
(i) *Suppose* $\beta \in \mathbb{R}$. *Then,*

$$\begin{aligned}
\gamma_{-1}^\beta &= 1,\ \gamma_0^\beta = \beta^2 - \tfrac{1}{2}u,\ \gamma_1^\beta = \tfrac{1}{2}\beta^2 u + \tfrac{1}{2}\beta u_x - \tfrac{1}{8}u^2 + \tfrac{1}{8}u_{xx},\\
\gamma_2^\beta &= -\tfrac{1}{16}u^3 + \tfrac{3}{8}\beta^2 u^2 + \tfrac{3}{16}u_x(4\beta u + u_x) + \tfrac{1}{8}u_{xx}(u-\beta^2)\\
&\quad - \tfrac{1}{8}\beta u_{xxx} - \tfrac{1}{64}u_{xxxx},\\
\gamma_{\ell+1}^\beta &= \tfrac{1}{8}\sum_{k=1}^{\ell}\big(2(u-\beta^2)\gamma_{k-1}^\beta\gamma_{\ell-k,xx}^\beta - (u-\beta^2)\gamma_{k-1,x}^\beta\gamma_{\ell-k,x}^\beta\\
&\quad - 4\gamma_k^\beta\gamma_{\ell-k+1}^\beta - 4u(u-\beta^2)\gamma_{k-1}^\beta\gamma_{\ell-k}^\beta - 2u_x\gamma_{k-1}^\beta\gamma_{\ell-k,x}^\beta + \gamma_{k-1}^\beta\gamma_{\ell-k}^\beta\big)\\
&\quad + \tfrac{1}{8}\sum_{k=0}^{\ell}\big(\gamma_{k,x}^\beta\gamma_{\ell-k,x}^\beta - 2\gamma_k^\beta\gamma_{\ell-k,xx}^\beta - 4(\beta^2-2u)\gamma_k^\beta\gamma_{\ell-k}^\beta\big),\\
&\qquad \ell = 2,3,\dots.
\end{aligned} \tag{1.254}$$

(ii) *Suppose* $\beta = \infty$. *Then,*

$$\begin{aligned}
\gamma_0^\infty &= 1,\ \gamma_1^\infty = \tfrac{1}{2}u,\\
\gamma_{\ell+1}^\infty &= -\tfrac{1}{2}\sum_{k=1}^{\ell}\gamma_k^\infty\gamma_{\ell+1-k}^\infty + \tfrac{1}{2}\sum_{k=0}^{\ell}\big(u\gamma_k^\infty\gamma_{\ell-k}^\infty + \tfrac{1}{4}\gamma_{k,x}^\infty\gamma_{\ell-k,x}^\infty - \tfrac{1}{2}\gamma_{k,xx}^\infty\gamma_{\ell-k}^\infty\big),\\
&\qquad \ell \in \mathbb{N}.
\end{aligned} \tag{1.255}$$

By comparison with the recursion (D.8) for $\hat{f}_\ell$, one infers

$$\gamma_\ell^\infty = \hat{f}_\ell, \quad \ell \in \mathbb{N}_0.$$

The final result for r_ℓ^β then reads as follows.

Theorem 1.58 *Assume Hypothesis* 1.54. *Then the coefficients* r_ℓ^β *in* (1.241) *satisfy the following recursion relations.*
(i) *Suppose* $\beta \in \mathbb{R}$. *Then,*

$$\begin{aligned}
r_0^\beta &= -\tfrac{1}{2}, \quad r_1^\beta = \beta^2 - \tfrac{1}{2}u,\\
r_\ell^\beta &= \ell\gamma_{\ell-1}^\beta - \sum_{k=1}^{\ell-1}\gamma_{\ell-k-1}^\beta r_k^\beta, \quad \ell = 2,3,\dots.
\end{aligned}$$

(ii) Suppose $\beta = \infty$. *Then,*

$$r_0^\infty = \tfrac{1}{2}, \quad r_1^\infty = \tfrac{1}{2}u,$$
$$r_\ell^\infty = \ell\gamma_\ell^\infty - \sum_{k=1}^{\ell-1} \gamma_{\ell-k}^\infty r_k^\infty, \quad \ell = 2, 3, \dots.$$

Proof It suffices to combine (1.240), (1.241), (1.250), and the following well-known fact on asymptotic expansions:

$$F(z) \underset{|z|\to\infty}{=} \sum_{\ell=1}^{\infty} c_\ell z^{-\ell}$$

implies

$$\ln(1 + F(z)) \underset{|z|\to\infty}{=} \sum_{\ell=1}^{\infty} d_\ell z^{-\ell},$$

where

$$d_1 = c_1, \quad d_\ell = c_\ell - \sum_{k=1}^{\ell-1} \frac{k}{\ell} c_{\ell-k} d_k, \quad \ell = 2, 3, \dots. \qquad \square$$

Combined with (1.254), Theorem 1.58 (i) yields an efficient algorithm for computing r_ℓ^β, $\beta \in \mathbb{R}$.

The connection between r_ℓ^β and ξ^β is illustrated in the following result.

Theorem 1.59 *Assume Hypothesis* 1.54 *and let* $e_{x,0}^\beta = \inf \operatorname{spec}(H_x^\beta)$, $\beta \in \mathbb{R}$, *and* $e_0^\infty = \inf \operatorname{spec}(H)$.
(i) Suppose $\beta \in \mathbb{R}$. *Then,*

$$r_\ell^\beta(x) = -\frac{1}{2}\big(e_{x,0}^\beta\big)^\ell - \lim_{z\to i\infty} \int_{e_{x,0}^\beta}^{\infty} d\lambda\, z^{\ell+1}(\lambda - z)^{-\ell-1}\ell(-\lambda)^{\ell-1}\bigg(\frac{1}{2} + \xi^\beta(\lambda, x)\bigg),$$
$$\ell \in \mathbb{N}. \quad (1.256)$$

(ii) Suppose $\beta = \infty$. *Then,*

$$r_\ell^\infty(x) = \frac{1}{2}\big(e_0^\infty\big)^\ell + \lim_{z\to i\infty} \int_{e_0^\infty}^{\infty} d\lambda\, z^{\ell+1}(\lambda - z)^{-\ell-1}\ell(-\lambda)^{\ell-1}\bigg(\frac{1}{2} - \xi^\infty(\lambda, x)\bigg),$$
$$\ell \in \mathbb{N}. \quad (1.257)$$

Proof Since (1.256) and (1.257) are proved exactly along the same lines, we focus on a proof of (1.257). Combining (1.241) and (1.248) and introducing $w = 1/z$

yield

$$\int_{e_0^\infty}^{\infty} d\lambda\,(1-\lambda w)^{-2}\bigg(\frac{1}{2}-\xi^\infty(\lambda,x)\bigg) \underset{-iw\downarrow 0}{=} \sum_{\ell=0}^{\infty}\bigg(r_{\ell+1}^\infty(x)-\frac{1}{2}(e_0^\infty)^{\ell+1}\bigg)w^\ell. \tag{1.258}$$

As pointed out in the paragraph following (J.31), the asymptotic expansions for $m_{\pm,0}(z,x)$ in (J.29), and hence that for the diagonal Green's function $g(z,x)$, are valid uniformly with respect to $\arg(z)$ in cones in $\mathbb{C}_+$ with apex e_0^∞, symmetry axis parallel to the imaginary axis, and arbitrarily small angle $\varepsilon > 0$ with the real axis. Consequently, this property extends to (1.241) and hence to (1.258). Moreover, since $g(\,\cdot\,,x)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$, the asymptotic expansion (1.258) can be differentiated term by term and infinitely often. Hence, differentiating (1.258) $k-1$ times with respect to w finally yields

$$r_k^\infty(x)-\frac{1}{2}\big(e_0^\infty\big)^k = \lim_{-iw\downarrow 0}\int_{e_0^\infty}^{\infty} d\lambda\,(1-\lambda w)^{-k-2}(k+1)\lambda^k\bigg(\frac{1}{2}-\xi(\lambda,x)\bigg),\qquad k\in\mathbb{N},$$

which is equivalent to (1.257). $\square$

We conclude with an example that yields the higher-order trace formulas for (real-valued) periodic potentials and simultaneously applies to the (quasi-periodic) algebro-geometric potentials of Section 1.3.

Example 1.60 In addition to Hypothesis 1.54, assume that u is periodic with period $\Omega > 0$, that is, $u(x+\Omega) = u(x)$ for all $x\in\mathbb{R}$. Then Floquet theory implies

$$\operatorname{spec}(H) = \bigcup_{j=1}^{\infty}[E_{2j-2},E_{2j-1}],\qquad E_0 < E_1 \le E_2 < E_3 \le \cdots$$

(i) Suppose $\beta\in\mathbb{R}$. Then,

$$\begin{aligned}&\operatorname{spec}\big(H_x^\beta\big) = \big\{\lambda_\ell^\beta(x)\big\}_{\ell\in\mathbb{N}_0}\cup\operatorname{spec}(H),\\ &\lambda_0^\beta(x)\le E_0,\ \lambda_j^\beta(x)\in[E_{2j-1},E_{2j}],\qquad j\in\mathbb{N},\end{aligned}$$

$$\xi^\beta(\lambda,x) = \begin{cases} 0 & \text{for } \lambda<\lambda_0^\beta(x),\ E_{2j-1}<\lambda<\lambda_j^\beta(x),\ j\in\mathbb{N},\\ -1 & \text{for } \lambda_0^\beta(x)<\lambda<E_0,\ \lambda_j^\beta(x)<\lambda<E_{2j},\ j\in\mathbb{N},\\ -\frac{1}{2} & \text{for } E_{2j-2}<\lambda<E_{2j-1},\ j\in\mathbb{N}.\end{cases} \tag{1.259}$$

Inserting (1.259) into (1.256) then yields the higher-order periodic trace formulas

$$r_\ell^\beta = \frac{1}{2}E_0^\ell - \big(\lambda_0^\beta\big)^\ell + \frac{1}{2}\sum_{j=1}^{\infty}\big(E_{2j-1}^\ell + E_{2j}^\ell - 2\big(\lambda_j^\beta\big)^\ell\big),\qquad \ell\in\mathbb{N}. \tag{1.260}$$

(ii) Suppose $\beta = \infty$. Then,

$$\operatorname{spec}(H_x^\infty) = \{\mu_j(x)\}_{j\in\mathbb{N}} \cup \operatorname{spec}(H), \quad \mu_j(x) \in [E_{2j-1}, E_{2j}], \quad j \in \mathbb{N},$$

$$\xi^\infty(\lambda, x) = \begin{cases} 0 & \text{for } \lambda < E_0,\, \mu_j(x) < \lambda < E_{2j},\, j \in \mathbb{N}, \\ 1 & \text{for } E_{2j-1} < \lambda < \mu_j(x),\, j \in \mathbb{N}, \\ \frac{1}{2} & \text{for } E_{2j-2} < \lambda < E_{2j-1},\, j \in \mathbb{N}. \end{cases} \tag{1.261}$$

Insertion of (1.261) into (1.257) then yields

$$r_\ell^\infty = \frac{1}{2}E_0^\ell + \frac{1}{2}\sum_{j=1}^\infty \big(E_{2j-1}^\ell + E_{2j}^\ell - 2\mu_j^\ell\big), \quad \ell \in \mathbb{N}. \tag{1.262}$$

The results (1.259) and (1.261) remain valid in the algebro-geometric situation discussed in Section 1.3, where

$$E_{2j+1} = \lambda_j^\beta(x) = E_{2j+2}, \quad j \geq n+1,\ \beta \in \mathbb{R} \cup \{\infty\}.$$

Hence, (1.260) and (1.262) apply to the real-valued stationary KdV solutions of Section 1.3 (e.g., (1.262) and (1.260) for $\ell = 1$ coincide with (1.83) and (1.85), which are reproduced in (1.236) and (1.237)). In general, these stationary KdV solutions are quasi-periodic with respect to x.

We conclude this section with a brief description of the Hamiltonian approach to the KdV hierarchy in connection with spatially rapidly decaying solutions. Given the phase space $\mathcal{P} = \mathcal{S}_\mathbb{R}(\mathbb{R})$, the Schwartz space of rapidly decreasing real-valued functions $u\colon \mathbb{R} \to \mathbb{R}$, we will exhibit a symplectic structure Ω on $\mathcal{P} \times \mathcal{P}$ and Hamiltonian functions $\mathcal{H}_n\colon \mathcal{P} \to \mathbb{R}$ such that the nth KdV equation takes on the form $u_t = \partial_x(\nabla\mathcal{H}_n)_u$.

To set up the formalism, we define

$$\partial\colon \mathcal{P} \to \mathcal{P}, \quad (\partial u)(x) = u_x(x),$$
$$\partial^{-1}\colon \mathcal{P} \to C^\infty(\mathbb{R}), \quad (\partial^{-1}u)(x) = \int_{-\infty}^x dx'\, u(x')$$

and

$$\Omega\colon \mathcal{P} \times \mathcal{P} \to \mathbb{R}, \quad \Omega(u, v) = \frac{1}{2}\int_\mathbb{R} dx\, \big((\partial^{-1}u)(x)v(x) - u(x)(\partial^{-1}v)(x)\big).$$

One verifies

$$\Omega(\partial u, v) = \langle u, v\rangle, \quad u, v \in \mathcal{P}, \tag{1.263}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product in the real Hilbert space $L^2_{\mathbb{R}}(\mathbb{R})$

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon L^2_{\mathbb{R}}(\mathbb{R})\times L^2_{\mathbb{R}}(\mathbb{R})\to\mathbb{R},\quad \langle u,v\rangle=\int_{\mathbb{R}}dx\,u(x)v(x),\quad u,v\in L^2_{\mathbb{R}}(\mathbb{R}).$$

Thus, Ω is a weakly non-degenerate 2-form on $\mathcal{P}\times\mathcal{P}$ (i.e., $\Omega(u,v)=0$ for all $u\in\mathcal{P}$ implies $v=0$) since we may choose $u=\partial_x v$ and use (1.263). Moreover, Ω is exact (i.e., $\Omega=d\omega$ for some 1-form ω on $\mathcal{P}\times\mathcal{P}$) and closed, that is, $d\Omega=0$; hence, Ω is a symplectic form for $\mathcal{P}$ and one can view $\mathcal{P}$ as a symplectic manifold. If $\mathcal{F}\colon\mathcal{P}\to\mathbb{R}$ is a smooth functional, the differential $d\mathcal{F}$ of $\mathcal{F}$ is the 1-form on $\mathcal{P}$ defined by

$$(d\mathcal{F})_u(v)=\frac{d}{d\varepsilon}\mathcal{F}(u+\varepsilon v)\big|_{\varepsilon=0},\quad u,v\in\mathcal{P};$$

hence,

$$(d\mathcal{F})_u(v)=\langle(\nabla\mathcal{F})_u,v\rangle=\Omega(\partial_x(\nabla\mathcal{F})_u,v)=\Omega((\nabla_s\mathcal{F})_u,v).$$

Here $\nabla\mathcal{F}$ denotes the gradient with respect to the flat Riemannian structure on $\mathcal{P}$ defined by the $L^2_{\mathbb{R}}$ inner product, and the symplectic gradient $\nabla_s\mathcal{F}$ is given by

$$(\nabla_s\mathcal{F})_u=\partial_x(\nabla\mathcal{F})_u.$$

With attention focused on functionals $\mathcal{F}\colon\mathcal{P}\to\mathbb{R}$ of the type

$$\mathcal{F}(u)=\int_{\mathbb{R}}dx\,F\big(u,u_x,u_{xx},\dots,\partial_x^m u\big)$$

in the following, where $F\colon\mathbb{R}^{m+1}\to\mathbb{R}$ is a polynomial function (the density of $\mathcal{F}$) with $F(0)=0$, a standard integration by parts argument in the calculus of variations shows that

$$(\nabla\mathcal{F})_u=\frac{\delta F}{\delta u}=\sum_{k=0}^{m}(-\partial_x)^k\partial_{u^{(k)}}F,$$

where $u^{(0)}=u$, $u^{(k)}=\partial_x^k u$, $k\in\mathbb{N}$, and $\delta F/\delta u$ abbreviates the variational derivative of F. In particular,

$$(\nabla_s\mathcal{F})_u=\partial_x\frac{\delta F}{\delta u}=\sum_{k=0}^{m}(-1)^k\partial_x^{k+1}\partial_{u^{(k)}}F,\quad u\in\mathcal{P}$$

and

$$(d\mathcal{F})_u(v)=\int_{\mathbb{R}}dx\,\frac{\delta F}{\delta u}(x)\,v(x),\quad u,v\in\mathcal{P}.$$

Hence, any such functional $\mathcal{F}$ is a Hamiltonian function on $\mathcal{P}$ defining the Hamiltonian flow

$$u_t = (\nabla_s \mathcal{F})_u = \partial_x \frac{\delta F}{\delta u}, \tag{1.264}$$

where $u(t)$ denotes a smooth curve in $\mathcal{P}$. Writing $u(t)(x) = u(x,t)$, the ordinary differential equation (1.264) on $\mathcal{P}$ becomes the partial differential equation

$$u_t = \partial_x \partial_u F - \partial_x^2 \partial_{u_x} F + \partial_x^3 \partial_{u_{xx}} F + \cdots + (-1)^m \partial_x^{m+1} \partial_{u^{(m)}} F.$$

Concerning Poisson brackets in terms of the Riemann structure on $\mathcal{P}$, one obtains

$$\begin{aligned}\{\mathcal{F}_1, \mathcal{F}_2\} &= d\mathcal{F}_1(\nabla_s \mathcal{F}_2) = \Omega(\nabla_s \mathcal{F}_1, \nabla_s \mathcal{F}_2) = \Omega(\partial_x(\nabla \mathcal{F}_1), \partial_x(\nabla \mathcal{F}_2)) \\ &= \langle \nabla \mathcal{F}_1, \partial_x(\nabla \mathcal{F}_2)\rangle = \int_{\mathbb{R}} dx \, \frac{\delta F_1}{\delta u}(x) \left(\partial_x \frac{\delta F_2}{\delta u} \right)(x).\end{aligned}$$

This implies the Jacobi identity

$$\{\{\mathcal{F}_1, \mathcal{F}_2\}, \mathcal{F}_3\} + \{\{\mathcal{F}_2, \mathcal{F}_3\}, \mathcal{F}_1\} + \{\{\mathcal{F}_3, \mathcal{F}_1\}, \mathcal{F}_2\} = 0$$

and the Leibniz rule

$$\{\mathcal{F}_1, \mathcal{F}_2\mathcal{F}_3\} = \{\mathcal{F}_1, \mathcal{F}_2\}\mathcal{F}_3 + \mathcal{F}_2\{\mathcal{F}_1, \mathcal{F}_3\}.$$

If $\mathcal{F}$ is a smooth functional and u evolves according to a Hamiltonian flow with Hamiltonian $\mathcal{H}$ and density H, that is,

$$u_t = (\nabla_s \mathcal{H})_u = \partial_x(\nabla \mathcal{H})_u = \partial_x \frac{\delta H}{\delta u},$$

then

$$\begin{aligned}\frac{d\mathcal{F}}{dt} &= \frac{d}{dt} \int_{\mathbb{R}} dx \, F(u, u_x, u_{xx}, \dots, \partial_x^m u) \\ &= \int_{\mathbb{R}} dx \, \frac{\delta F}{\delta u}(x,t) u_t(x,t) = \int_{\mathbb{R}} dx \, \frac{\delta F}{\delta u}(x,t) \left(\partial_x \frac{\delta H}{\delta u} \right)(x,t) \\ &= \{\mathcal{F}, \mathcal{H}\}.\end{aligned} \tag{1.265}$$

In particular, any functional $\mathcal{G}$ in involution with the Hamiltonian $\mathcal{H}$,

$$\{\mathcal{G}, \mathcal{H}\} = 0,$$

will be conserved by the flow

$$\frac{d\mathcal{G}}{dt} = 0.$$

To apply this to the KdV hierarchy we first derive a few auxiliary results.

Lemma 1.61 *Suppose $u \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ and $x \in \mathbb{R}$.*
(i) Let $g(z,x) = G(z,x,x)$, $z \in \mathbb{C}$ denote the diagonal Green's function of H (if $z \in \operatorname{spec}(H)$, then we agree to take a nontangential limit toward z).

Then

$$-\partial_z\big(g(z,x)^{-1}\big) = 2g(z,x) \\ - \partial_x\left(\frac{g(z,x)^{-1}\big(g(z,x)^{-1}\big)_{zx} - \big(g(z,x)^{-1}\big)_x\big(g(z,x)^{-1}\big)_z}{g(z,x)^{-3}}\right). \tag{1.266}$$

(ii) The following asymptotic expansion holds as $|z| \to \infty$ *in a cone* C_ε *with apex at* $\inf(\operatorname{spec}(H))$*, symmetry axis the imaginary axis, and opening angle* $\pi - \varepsilon$ *for some* $0 < \varepsilon < \pi$*,*

$$g(z,x) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} \frac{i}{2z^{1/2}} \sum_{\ell=0}^{\infty} \hat{f}_\ell(x) z^{-\ell}, \quad \hat{f}_0(x) = 1. \tag{1.267}$$

The expansion (1.267) *is uniform with respect to* $\arg(z)$ *within the cone* C_ε *and uniform in x as long as x varies in compact intervals.*
(iii) Let $\hat{m}(z,x) = m_{+,0}(z,x)$ *be the Weyl–Titchmarsh m-function of H associated with H on the interval* $[x,\infty)$*, as discussed in* (J.29), (J.30)*. Then*

$$\hat{f}_\ell(x) = i(2\ell-1)\hat{m}_{2\ell-1}(x) + \hat{h}_{\ell,x}(x), \quad \ell \in \mathbb{N}, \tag{1.268}$$

where $\hat{m}_\ell(x) = m_{+,0,\ell}(x)$ *are the coefficients in the asymptotic expansion* (J.29) *of* $m_{+,0}(z,x)$ *as* $|z| \to \infty$ *in* C_ε*, and* $\hat{h}_\ell$ *is a differential polynomial in u without constant term.*
(iv) The quantities $\hat{f}_j \hat{f}_{k,x}$*,* $j,k \in \mathbb{N}_0$ *are x-derivatives of a differential polynomial. More precisely, there exist differential polynomials* $\hat{p}_{j,k}$ *in u without constant term such that*

$$\hat{f}_j \hat{f}_{k,x} = \hat{p}_{j,k,x}, \quad j,k \in \mathbb{N}_0. \tag{1.269}$$

(v) Regarding $\hat{f}_\ell(x) = \hat{f}_\ell(u, u_x, u_{xx}, \dots)$*,* $\ell \in \mathbb{N}$*, as a differential polynomial in u (by a slight abuse of notation), one obtains*

$$\frac{\delta \hat{f}_\ell}{\delta u} = \frac{2\ell-1}{2}\hat{f}_{\ell-1} = \frac{\partial \hat{f}_\ell}{\partial u}, \quad \ell \in \mathbb{N}. \tag{1.270}$$

Proof In order to prove (1.266), one first rewrites the nonlinear differential equation for $g = \Gamma^\infty$ in (1.253) in the form

$$2g(g^{-1})_{xx} - (3g_x^2 + 1)g^{-2} + 4(u-z) = 0. \tag{1.271}$$

Differentiating (1.271) with respect to z then yields (1.266) after a series of elementary (yet tedious) manipulations.
The existence of the asymptotic expansion (1.267) and its uniformity properties as stated in (ii) follows from

$$g(z,x) = \Gamma^\infty(z,x) = (m_{-,0}(z,x) - m_{+,0}(z,x))^{-1}$$

(cf. (J.29)) and the corresponding expansions (J.29), (J.30) for $m_{\pm,0}(z,x)$. In the present short-range case in which $u \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$, the existence of the latter can be obtained straightforwardly by iterating the Volterra integral equations for the Jost (i.e., Weyl–Titchmarsh) solutions of H. The actual expansion coefficients in (1.267) are identified by noticing that the recursion relation (1.255) for γ_ℓ^∞ coincides with that of $\hat{f}_\ell$ in (D.8).

Rewriting (1.266) in the form

$$g(z,x) = \frac{1}{2}\big(m_{+,0}(z,x) - m_{-,0}(z,x)\big)_z + (\dots)_x,$$

inserting the asymptotic expansions (1.267) for g and (J.29) for $m_{\pm,0}$, and taking into account (J.31) yield (1.268).

To prove (1.269), one first notes that the linear recursion (1.4) for $\hat{f}_\ell$ implies

$$\begin{aligned}\hat{f}_j\hat{f}_{k,x} = \hat{f}_{j+1}\hat{f}_{k-1,x} - \partial_x\big(\hat{f}_{j+1}\hat{f}_{k-1} + (1/4)\hat{f}_{j,xx}\hat{f}_{k-1} - (1/4)\hat{f}_{j,x}\hat{f}_{k-1,x}\\ + (1/4)\hat{f}_j\hat{f}_{k-1,xx} - u\hat{f}_j\hat{f}_{k-1}\big), \quad j,k\in\mathbb{N}.\end{aligned} \tag{1.272}$$

Iterating (1.272) while noticing $\hat{f}_{0,x} = 0$ then proves (1.269).

To prove

$$\frac{\partial \hat{f}_\ell}{\partial u} = \frac{2\ell-1}{2}\hat{f}_{\ell-1}, \quad \ell\in\mathbb{N} \tag{1.273}$$

one can proceed inductively upon ℓ as follows. First, (1.273) is easily verified for $\ell = 0, 1$. Next, assuming (1.273) for $\ell = 0,\dots,k$ for some $k\in\mathbb{N}$ and noticing

$$\partial_u(\partial_x P(u)) = \partial_x(\partial_u P(u)) \tag{1.274}$$

for any differential polynomial P with respect to u, the nonlinear recursion relation (D.8) yields after a straightforward calculation that

$$\frac{\partial \hat{f}_{k+1}}{\partial u} = \frac{2k+1}{2}\hat{f}_k,$$

proving (1.273).

Finally, we turn to a proof of the remarkable fact

$$\frac{\delta \hat{f}_\ell}{\delta u} = \frac{2\ell-1}{2}\hat{f}_{\ell-1}, \quad \ell\in\mathbb{N}. \tag{1.275}$$

To avoid technicalities with (formal) asymptotic power series, etc., and especially to be able to compute $\delta g/\delta u$, we assume for the remainder of this proof that g is of the special algebro-geometric form

$$g(z,x) = \frac{iF_n(z,x)}{2R_{2n+1}(z)^{1/2}} \tag{1.276}$$

associated with a compact hyperelliptic Riemann surface $\mathcal{K}_n\colon\ y^2 = R_{2n+1}(z)$ of genus g. The universal nonlinear second-order differential equation (1.253) for diagonal Green's functions $g = \Gamma^\infty$ of Schrödinger-type operators then reads

$$2gg_{xx} - g_x^2 + 4(z-u)g^2 = -1,$$

and its z-derivative becomes

$$g_z g_{xx} + gg_{xxz} - g_x g_{xz} + 2g^2 + 4(z-u)gg_z = 0. \tag{1.277}$$

Since by (1.276), g is a differential polynomial with respect to u, one thus computes

$$\begin{aligned}(dg)_u(v)g_{xx} &+ g(dg_{xx})_u(v) - g_x(dg_x)_u(v) - 2g^2 v \\ &+ 4(z-u)g(dg)_u(v) = 0, \quad u, v \in \mathcal{P}.\end{aligned} \tag{1.278}$$

Multiplying (1.278) by g_z/g^2 then results in

$$\begin{aligned}g^{-2}&\big(g_z g_{xx} + 4(z-u)gg_z\big)(dg)_u(v) + g^{-1}g_z(dg_{xx})_u(v) \\ &- g^{-2}g_x g_z(dg_x)_u(v) - 2g_z v = 0.\end{aligned} \tag{1.279}$$

Since

$$(d(\partial_x P(u)))_u(v) = \partial_x((dP)_u(v))$$

for any differential polynomial P with respect to u, one can rewrite (1.279) in the form

$$\begin{aligned}&\big(g^{-2}(g_z g_{xx} + 4(z-u)gg_z) + (g_z/g)_{xx} + (g_x g_z/g^2)_x\big)(dg)_u(v) \\ &\quad = g^{-2}\big(g_z g_{xx} + 4(z-u)gg_z + gg_{zxx} - g_x g_{zx}\big)(dg)_u(v) \\ &\quad = 2g_z v + \partial_x\{\cdots\},\end{aligned} \tag{1.280}$$

where $\{\cdots\}$ vanishes as $|x| \to \infty$ since $u, v \in \mathcal{P}$. By (1.277), the middle expression in (1.280) simplifies to $-2(dg)_u(v)$, and hence one obtains

$$(dg)_u(v) = -g_z v + \partial_x\{\cdots\};$$

thus,

$$\int_{\mathbb{R}} dx\, (dg)_u(v) = \int_{\mathbb{R}} dx\, \frac{\delta g}{\delta u} v = -\int_{\mathbb{R}} dx\, g_z v, \quad v \in \mathcal{P}.$$

Since $v \in \mathcal{P}$ is arbitrary, one finally concludes

$$\frac{\delta g}{\delta u} = -g_z \tag{1.281}$$

for any g of the type (1.276). To conclude the proof of (1.275), we invoke the following (convergent) asymptotic expansion of $g(z)$ as $|z| \to \infty$. By (D.1) and

(D.10), one computes

$$
\begin{aligned}
g(z) &= \frac{iF_n(z)}{2R_{2n+1}(z)^{1/2}} = \frac{iz^n}{2R_{2n+1}(z)^{1/2}} \sum_{\ell=0}^{n} \left(\sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{f}_k \right) z^{-\ell} \\
&= \frac{i}{2z^{1/2}} \left(\prod_{m=0}^{2n+1} (1-(E_m/z)) \right)^{-1/2} \sum_{\ell=0}^{n} \left(\sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{f}_k \right) z^{-\ell} \\
&= \frac{i}{2z^{1/2}} \left(\sum_{p=0}^{\infty} \hat{c}_p(\underline{E}) z^{-p} \right) \sum_{\ell=0}^{n} \left(\sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{f}_k \right) z^{-\ell} \\
&= \frac{i}{2z^{1/2}} \sum_{r=0}^{\infty} \left(\sum_{s=0}^{r\wedge n} \hat{c}_{r-s}(\underline{E}) \sum_{k=0}^{s} c_{s-k}(\underline{E}) \hat{f}_k \right) z^{-r} \\
&= \frac{i}{2z^{1/2}} \sum_{r=0}^{n} \hat{f}_r z^{-r} + \frac{i}{2z^{1/2}} \sum_{r=n+1}^{\infty} \left(\sum_{k=0}^{n} \hat{f}_k \sum_{s=0}^{n-k} \hat{c}_{r-s-k}(\underline{E}) c_s(\underline{E}) \right) z^{-r} \\
&= \frac{i}{2z^{1/2}} \sum_{r=0}^{n} \hat{f}_r z^{-r} + O(|z|^{-n-1}) \text{ as } |z| \to \infty, \qquad (1.282)
\end{aligned}
$$

where we used (D.16) to isolate the first term $\sum_{r=0}^{n} \dots$ in (1.282). As discussed in the proof of Theorem D.1, the homogeneous coefficients $\hat{f}_r$ in (1.282) are universal differential polynomials of u uniquely defined by the nonlinear recursion relation (D.8). The same computation then yields

$$
\frac{\delta g}{\delta u} = \frac{i}{2z^{1/2}} \sum_{r=1}^{n} \frac{\delta \hat{f}_r}{\delta u} z^{-r} + O(|z|^{-n-1}) \text{ as } |z| \to \infty \qquad (1.283)
$$

and

$$
g_z = -\frac{i}{2z^{1/2}} \sum_{r=1}^{n+1} (r-(1/2)) \hat{f}_{r-1} z^{-r} + O(|z|^{-n-2}) \text{ as } |z| \to \infty. \qquad (1.284)
$$

By inserting (1.283) and (1.284) into (1.281) and comparing powers of z^{-r}, one then proves

$$
\frac{\partial \hat{f}_\ell}{\partial u} = \frac{2\ell-1}{2} \hat{f}_{\ell-1}, \quad \ell = 1, \dots, n.
$$

Since n can be chosen arbitrarily large, this proves (1.275). $\square$

As a consequence of identity (1.268), one obtains

$$
\int_{\mathbb{R}} dx\, \hat{f}_{\ell+1}(x) = i(2\ell+1) \int_{\mathbb{R}} dx\, \hat{m}_{2\ell+1}(x), \quad \ell \in \mathbb{N}_0
$$

and hence introduces the functionals[1] $\widehat{\mathcal{I}}_\ell = \widehat{\mathcal{I}}_\ell(u, u_x, u_{xx}, \dots, \partial_x^m u)$,

$$\widehat{\mathcal{I}}_\ell = i \int_{\mathbb{R}} dx\, \hat{m}_{2\ell+1}(x) = \frac{1}{2\ell+1} \int_{\mathbb{R}} dx\, \hat{f}_{\ell+1}(x), \ell \in \mathbb{N}_0. \tag{1.285}$$

Thus, (1.275) implies

$$\begin{aligned}(\nabla \widehat{\mathcal{I}}_\ell)_u &= \frac{1}{2\ell+1}\left(\nabla \int_{\mathbb{R}} dx\, \hat{f}_{\ell+1}(x)\right)_u = \frac{1}{2\ell+1}\frac{\delta \hat{f}_{\ell+1}}{\delta u} \\ &= \frac{1}{2}\hat{f}_\ell(u), \quad \ell \in \mathbb{N}_0.\end{aligned} \tag{1.286}$$

Hence, we may rewrite the nth KdV equation (for some $n \in \mathbb{N}_0$) in a variety of ways,

$$\begin{aligned}0 = \mathrm{KdV}_n(u) &= u_{t_n} - 2f_{n+1,x}(u) \\ &= u_{t_n} - 2\sum_{\ell=1}^{n+1} c_{n+1-\ell}\hat{f}_{\ell,x}(u) \\ &= u_{t_n} - \partial_x \sum_{\ell=1}^{n+1} \frac{4c_{n+1-\ell}}{2\ell+1}\frac{\delta \hat{f}_{\ell+1}}{\delta u} \\ &= u_{t_n} - \partial_x \sum_{\ell=0}^{n} 4c_{n-\ell}(\nabla \widehat{\mathcal{I}}_{\ell+1})_u \\ &= u_{t_n} - (\nabla_s \mathcal{H}_n)_u \\ &= u_{t_n} - \partial_x(\nabla_s \mathcal{H}_n)_u,\end{aligned} \tag{1.287}$$

where $\{\mathcal{H}_n\}_{n\in\mathbb{N}_0}$ represents the sequence of KdV Hamiltonians,

$$\mathcal{H}_n = \sum_{\ell=0}^{n} c_{n-\ell}\widehat{\mathcal{H}}_\ell, \quad \widehat{\mathcal{H}}_\ell = 4\widehat{\mathcal{I}}_{\ell+1},\ \ell \in \mathbb{N}_0, \tag{1.288}$$

and we used again Hirota's notation of a separate time variable t_n for the nth KdV flow.

The following result sums up the principal aspects of the KdV equations as completely integrable Hamiltonian systems.

Theorem 1.62 *Suppose $u \in \mathcal{S}_{\mathbb{R}}(\mathbb{R})$ satisfies the nth KdV equation* (1.287) *(for some set of integration constants c_ℓ, $\ell = 1, \dots, n$ if $n \in \mathbb{N}$). Then $\widehat{\mathcal{I}}_\ell$, $\ell \in \mathbb{N}_0$ are conserved by this flow and hence represent the infinitely many KdV conservation laws*

$$\frac{d\widehat{\mathcal{I}}_\ell}{dt_n} = 0, \quad \ell \in \mathbb{N}_0.$$

[1] We note that by (J.30), $\hat{m}_{2\ell+1}$, $\ell \in \mathbb{N}_0$ are purely imaginary if u is real-valued.

In particular, since $n \in \mathbb{N}_0$ is arbitrary, $\{\widehat{\mathcal{I}}_\ell\}_{\ell\in\mathbb{N}_0}$ are conserved by any (higher-order) KdV flow, and all $\widehat{\mathcal{I}}_\ell$, $\ell \in \mathbb{N}_0$ are in involution,

$$\{\widehat{\mathcal{I}}_\ell, \widehat{\mathcal{I}}_m\} = 0, \quad \ell, m \in \mathbb{N}_0.$$

Proof Combining (1.265), (1.275), and (1.287), one computes

$$\begin{aligned}
\frac{d\widehat{\mathcal{I}}_\ell}{dt_n} &= \frac{1}{2\ell+1}\int_{\mathbb{R}} dx\, \frac{\delta \hat f_{\ell+1}}{\delta u}(x,t_n)\, u_{t_n}(x,t_n) \\
&= \frac{1}{2\ell+1}\sum_{m=1}^{n+1}\frac{4c_{n+1-m}}{2m+1}\int_{\mathbb{R}} dx\, \frac{\delta \hat f_{\ell+1}}{\delta u}(x,t_n)\bigg(\partial_x \frac{\delta \hat f_{m+1}}{\delta u}\bigg)(x,t_n) \\
&= \sum_{m=1}^{n+1} c_{n+1-m}\int_{\mathbb{R}} dx\, \hat f_\ell(x,t_n)\hat f_{m,x}(x,t_n) = 0, \quad \ell \in \mathbb{N}_0,
\end{aligned}$$

applying (1.269). The same argument also yields

$$\begin{aligned}
\{\widehat{\mathcal{I}}_\ell, \widehat{\mathcal{I}}_m\} &= \frac{1}{(2\ell+1)(2m+1)}\int_{\mathbb{R}} dx\, \frac{\delta \hat f_{\ell+1}}{\delta u}(x,t_n)\bigg(\partial_x \frac{\delta \hat f_{m+1}}{\delta u}\bigg)(x,t_n) \\
&= \frac{1}{4}\int_{\mathbb{R}} dx\, \hat f_\ell(x,t_n)\hat f_{m,x}(x,t_n) = 0, \quad \ell, m \in \mathbb{N}_0.
\end{aligned}$$ □

Rewriting the linear recursion (1.4) in the form

$$\partial_x \hat f_{\ell+1} = \mathcal{D}_x^{(3)} \hat f_\ell, \quad \ell \in \mathbb{N}_0, \tag{1.289}$$

where we abbreviated

$$\mathcal{D}_x^{(3)} = -\frac{1}{4}\partial_x^3 + u\partial_x + \frac{1}{2}u_x, \tag{1.290}$$

yields a second Hamiltonian structure for the KdV hierarchy. In fact, the second Poisson bracket $\{\{\cdot\,,\,\cdot\}\}$, defined by

$$\{\{\mathcal{F}_1, \mathcal{F}_2\}\} = \int_{\mathbb{R}} dx\, \frac{\delta F_1}{\delta u}(x)\bigg(\mathcal{D}_x^{(3)}\frac{\delta F_2}{\delta u}\bigg)(x), \tag{1.291}$$

is also skew-symmetric and satisfies the Jacobi identity. As in Theorem 1.62, one verifies that all $\widehat{\mathcal{I}}_\ell$, $\ell \in \mathbb{N}_0$ are in involution also with respect to the second Poisson bracket (1.291), that is,

$$\{\{\widehat{\mathcal{I}}_\ell, \widehat{\mathcal{I}}_m\}\} = 0, \quad \ell, m \in \mathbb{N}_0.$$

Finally, combining (1.285)–(1.287) and (1.289), (1.290) permits one to write the nth KdV equation in the two Hamiltonian forms

$$\begin{aligned}
0 = \mathrm{KdV}_n(u) &= u_{t_n} - \partial_x(\nabla\mathcal{H}_n)_u \\
&= u_{t_n} - \mathcal{D}_x^{(3)}(\nabla\mathcal{H}_{n-1})_u, \quad n \in \mathbb{N}_0
\end{aligned}$$

with the KdV Hamiltonians $\mathcal{H}_n$, $n \in \mathbb{N}_0$, defined in (1.288) and $\mathcal{H}_{-1} = 4\widehat{\mathcal{I}}_0$.

1.6 Notes

Section 1.1. There are many excellent sources for the highly interesting background and historical development of the Korteweg–de Vries (KdV) equation, starting with the observation and subsequent experiments by Russell in 1834 (Russell (1837; 1840; 1845)), the controversy with Airy and Stokes on the origin of the "great wave of translation," and the derivation by Korteweg and his student de Vries of the fundamental KdV equation (Korteweg and de Vries (1895)). Nice surveys of the early days of KdV and all that can be found in Bullough (1988) and Bullough and Caudrey (1995). This includes the fact that the KdV equation was already known to Boussinesq (1871a,b; 1872; 1877) as discussed in Pego (1998). We also refer to van der Blij (1978), Heyerhoff (1997), and Miles (1981) for a glimpse at the early history of the KdV equation.

For modern physical applications of the KdV equation in connection with shallow water waves, etc., we refer, for instance, to Ablowitz and Segur (1981, Sec. 4.1), Dodd et al. (1982, Ch. 5), Infeld and Rowlands (1990), Johnson (1997), and the literature cited therein.

In 1965, Zabusky and Kruskal (Zabusky and Kruskal (1965)) coined the term "soliton" while analyzing the important numerical results of Fermi et al. (1955) on the a priori unrelated problem of describing phonons in an anharmonic lattice. Their analysis prepared the ground for the breakthrough by Gardner et al. (1967). It was their fundamental insight in 1967 that brought the KdV equation to the forefront of modern mathematical physics. They showed that one could solve the KdV equation by relating it to the well-studied linear, one-dimensional Schrödinger operator. In particular, the Cauchy problem for the KdV equation, with sufficiently rapidly decaying initial data, was closely related to the inverse scattering problem of the Schrödinger equation with the KdV solution $u(\,\cdot\,,t)$ serving as the potential of the Schrödinger equation, which now depends on the parameter[1] t. Their key insight was to realize that the t-dependence of the scattering data of the Schrödinger equation with a potential that is a solution of the KdV equation was extremely simple and could be characterized explicitly. Moreover, they showed the isospectral property of one-parameter families of Schrödinger operators with potentials being KdV solutions depending on t. Soon thereafter Lax (1968) explained this magical isospectral property of the t-dependent family of Schrödinger operators by what is now called the Lax pair[2] and introduced the whole hierarchy of nonlinear evolution equations of KdV-type. In the same year an infinite sequence of polynomial conservation laws was established with the help of Miura's transformation (Miura (1968)) in Miura et al. (1968) (see also Kruskal et al. (1970)), and the tools to view the Korteweg–de Vries equation as a completely integrable system were

[1] We note that the time parameter t is *not* the (quantum mechanical) time associated with the time-dependent Schrödinger equation; rather, one considers the stationary Schrödinger equation with a potential $u(\,\cdot\,,t)$ depending on an additional (deformation) parameter $t \in \mathbb{R}$.

[2] He actually did view t as a time parameter in connection with an explicitly t-dependent Hamiltonian given by a third-order differential operator.

provided by Gardner (1971) and especially Zakharov and Faddeev (1971). The early stormy period up to about the mid-1970s dealing with solitons, conservation laws, Bäcklund transformations, Poisson brackets, canonical transformations, and the inverse scattering method is summarized, for instance, in Flaschka (1975a), Flaschka and McLaughlin (1976a,b), Gardner et al. (1974), Gel'fand and Dikii (1975), Kruskal (1975), Miura (1976), and Scott et al. (1973).

The Zakharov–Shabat (ZS) (Zakharov and Shabat (1972; 1973; 1974)) and Ablowitz, Kaup, Newell, and Segur (AKNS) (Ablowitz et al. (1973a,b; 1974)) approaches then extended the inverse scattering method to several other nonlinear partial differential equations of mathematical physics, as will be discussed in the notes to Chapters 2 and 3, but here we will focus on the development relevant to the KdV hierarchy.

The analogous application of the inverse scattering method to the case of periodic initial data was not immediately possible. It was well-known that the corresponding Schrödinger equation with periodic potential generically has a spectrum consisting of infinitely many bands separated by spectral gaps, the lengths of which decrease as the spectral parameter increases. The exceptional case, in which the actual number of gaps in the spectrum is finite, the so-called finite-gap case, is what we study in detail in this chapter. The extension of the inverse scattering method to periodic initial data, partly based on inverse spectral theory and partly relying on algebro-geometric methods, was developed by pioneers such as Dubrovin (1975b), Flaschka (1975b), Its and Matveev (1975b), Lax (1974; 1975), Marčenko (1974a,b), McKean and van Moerbeke (1975), and Novikov (1974), to name just a few. We will return to this in some detail in connection with the notes to Section 1.3.

For more recent reviews on the KdV equation we refer, for instance, to Bullough (1988), Bullough and Caudrey (1995), Lax (1996), Palais (1997), and Segal (1999). For textbook literature on the KdV equation, we refer to Ablowitz and Clarkson (1991, Ch. 2), Ablowitz and Segur (1981, Ch. 1), Asano and Kato (1990, Chs. 6, 7), Belokolos et al. (1994, Ch. 3), Calogero and Degasperis (1982), Cherednik (1996), Das (1989, Chs. 1–8), Dickey (1991, Chs. 3, 4, 12), Dodd et al. (1982, Ch. 8), Drazin and Johnson (1989, Chs. 1–5), Eckhaus and van Harten (1983, Chs. 1–4), Eilenberger (1983, Chs. 2, 3), Faddeev and Takhtajan (1987, Part 1), Miwa et al. (2000), Newell (1985, Ch. 3), Novikov et al. (1984, Sec. I.10), and Toda (1989a, Chs. 4–11).

Section 1.2. The approach presented in this section closely follows the one in Gesztesy et al. (1996a).

The construction of the KdV hierarchy using a recursive approach is due to Lax (1968). It has also been studied by Gel'fand and Dikii (1975), Lenard (unpublished, see Gardner et al. (1974, p. 130)), and McKean and van Moerbeke (1975) and later on especially by Al'ber (1979; 1981) (see also Dickey (1991, Ch. 12), Gel'fand and Dikii (1979), Gesztesy and Weikard (1993), Levitan (1987, Ch. 12), and Marchenko (1986, Ch. 4)). However, the recursion (1.4) was well-known to Burchnall and Chaundy (1923) and used by them to construct differential expressions commuting with L.

For explicit expressions of the KdV functions f_ℓ in terms of u and its x-derivatives, we refer to Avramidi and Schimming (2000), Rosenhouse and Katriel (1987), and Schimming (1988; 1995). That the functions f_ℓ given by (1.5) are polynomials with respect to u and its x-derivatives, is proved, for instance, in Eilenberger (1983, p. 19f) (see also Ohmiya (1988b), who attributes the proof presented to Tanaka). To describe the argument in Eilenberger (1983, pp. 19–20) briefly, one rewrites the recursion (1.5) in the form $f_{j+1,x} = R(f_j)$ and defines $\Phi(f,g) = -\frac{1}{4}(fg)_{xxx} + \frac{3}{4}f_x g_x + ufg$. An explicit calculation then shows that $R(f)g + fR(g) = \Phi(f,g)_x$. In addition, one verifies that

$$f_{j+1,x} = \sum_{\ell=0}^{m} \big(\Phi(f_{j-\ell}, f_\ell) - f_{j-\ell} f_{\ell+1}\big)_x + f_{j-m,x} f_{m+1},$$

where $m = 0, \dots, j,\ j \in \mathbb{N}_0$, and

$$f_{j+1} = c_{j+1} + \sum_{\ell=0}^{k} \big(\Phi(f_{j-\ell}, f_\ell) - f_{j-\ell} f_{\ell+1}\big) + \begin{cases} \frac{1}{2}\Phi(f_{k+1}, f_{k+1}), & j = 2k, \\ \frac{1}{2} f_{k+1}^2, & j = 2k+1 \end{cases} \tag{1.292}$$

with $c_{j+1} \in \mathbb{C},\ k \in \mathbb{N}_0$. By induction, relation (1.292) then shows that all f_ℓ are indeed differential polynomials with respect to u. An alternative argument of this fact is mentioned in Remark 1.2; Gel'fand and Dikii (1975, Ch. 2) contains additional results in this direction.

From a historical perspective it is interesting to remark that Appell was quite familiar with our fundamental equation (1.12) for $F_n(z,\,\cdot\,)$ in 1880. In fact, let y_1 and y_2 be linearly independent solutions of $-y'' + Uy = 0$. Then he showed that $y_1^2,\ y_1 y_2$, and y_2^2 are linearly independent solutions of

$$w''' - 4Uw' - 2U'w = 0 \tag{1.293}$$

(cf. Appell (1880)). This equation is easily integrated and yields

$$2gg'' - (g')^2 - 4Ug^2 = -W(y_1, y_2)^2, \tag{1.294}$$

where $g = y_1 y_2$ denotes the product of any two solutions y_1 and y_2 of $-y''(x) + U(x)y(x) = 0$, and $W(y_1, y_2) = y_1 y_2' - y_1' y_2$ denotes their (x-independent) Wronskian. Noticing that the formal Green's function $G(z, x, x')$ of $-d^2/dx^2 + u$ on the diagonal $x = x'$ is in fact of the type $g(z,x) = y_1(z,x)y_2(z,x)/W(y_1(z), y_2(z))$, (1.294) with $U = u - z$ is equivalent to the well-known universal nonlinear second-order differential equation satisfied by g (see also Gel'fand and Dikii (1975; 1979)). A comparison of Lemma 1.8, (1.12), (1.13) and (1.293), (1.294) shows that (1.13) and (1.294) are equivalent in the special algebro-geometric context of Section 1.3. Moreover, it should also be pointed out that Drach used (1.293) to derive a class of completely integrable systems now known as the stationary KdV hierarchy as early as 1918–19 (cf. Drach (1918; 1919a,b)). In fact, it appears he was the first to make the explicit connection between completely integrable systems and spectral theory. More than 55 years later, Gelfand and Dikii also based some of their celebrated work on the KdV hierarchy on (1.293) (Gel'fand and Dikii (1975; 1979)). Finally, it must be mentioned that the linear recursion relation for the coefficients f_ℓ in (1.4) was used by Burchnall and Chaundy in 1923 in their construction of odd-order differential expressions P_{2n+1} (cf. (1.7)) commuting with the second-order differential expression L in (1.3) (Burchnall and Chaundy (1923)).

Burchnall–Chaundy theory, the basic formalism underlying commuting differential expressions, has been pioneered by Burchnall and Chaundy in their seminal papers Burchnall and Chaundy (1923; 1928; 1932) (see also Baker (1928)). More recent treatments of this circle of ideas can be found, for instance, in Amitsur (1954; 1958), Bogoyavlenskii (1976), Carlson and Goodearl (1980), Chalykh (1993), Chalykh and Veselov (1990), Dehornoy (1981), Dubrovin (1975b), Dubrovin et al. (1976), Frentzen et al. (1993), Giertz et al. (1981), Krichever (1976a; 1977a,b; 1978), Latham and Previato (1994), Mulase (1984; 1990a,b), Mumford (1977), Nakayashiki (1994), Previato (1996; 1998), Previato and Wilson (1989; 1992), Race and Zettl (1990), Segal and Wilson (1985), Veselov (1979), Weikard (1998b; 1999; 2000; 2002), and Wilson (1985). Some of the extensions of the classical Burchnall–Chaundy theory in connection with formal pseudo-differential expressions were anticipated in Wallenberg (1903) and Schur (1905). The special case involving a second-order differential expression L in Theorem 1.3 then leads to hyperelliptic curves branched at infinity. Certain classes of commuting (non-self-adjoint) operators (as opposed to merely commuting differential expressions) and their connections with algebraic curves are also discussed in various works by Livšic, Kravitsky, Vinnikov, and others. An extensive account of these activities up to 1994 is presented in the monograph Livšic et al. (1995); more recent work in this direction can be found in Vinnikov (1998) and the references therein.

Section 1.3. Again the presentation of most of the material in this section follows the one in Gesztesy et al. (1996a).

The fundamental meromorphic function $\phi(\,\cdot\,, x_0)$ on $\mathcal{K}_n$ defined in (1.38) is in many respects the key object of our formalism. For instance, in the special self-adjoint case, where $u \in C^\infty(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and u and E_m, $m = 0, \dots, 2n$ are real-valued, its two branches represent the two Dirichlet half-line Weyl–Titchmarsh m-functions $m_{\pm,0}(z, x)$ associated with proper closed realizations of the differential expression $L = -d^2/dx^2 + u$ in $L^2((x_0, \pm\infty))$. In particular, the spectral properties of the self-adjoint realization of L in $L^2(\mathbb{R})$ (as well as those of the self-adjoint Dirichlet-type operators in $L^2((x_0, \pm\infty))$ can be inferred directly from $\phi(P, x_0)$. Completely analogous remarks apply to all other (non-Dirichlet) β-boundary conditions defined in (1.54). This clearly illustrates the distinguished role played by ϕ in the special self-adjoint case (for details we refer to Appendix J). However, as amply demonstrated in Sections 1.3 and 1.4, ϕ turns out to be the principal object of the algebro-geometric formalism independently of any self-adjointness considerations in connection with L.

A look at (1.38)–(1.40) shows that ϕ links the Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ of degree n and the Neumann divisor $\mathcal{D}_{\underline{\hat{\nu}}}$ of degree $n+1$. This is of course a direct consequence of the identity (1.32) together with the factorizations of F_n and H_{n+1} in (1.31) and (1.34). This construction of positive divisors of degree n and $n+1$ on hyperelliptic curves $\mathcal{K}_n$ of genus n apparently goes back at least to Jacobi (1846). It has been applied to the KdV case in Mumford (1984, Sec. III a).1), and subsequently in McKean (1985), and our presentation of algebro-geometric solutions of the stationary KdV hierarchy relies on these constructions.

The squared eigenfunction approach alluded to in connection with (1.53) is discussed, for instance, in Ablowitz and Segur (1981, pp. 42–52) and Eilenberger (1983, Sec. 3.5).

The Dubrovin equations (1.66) for Dirichlet-type eigenvalues in Lemma 1.10 appeared in papers by Dubrovin (1975a,b); (see also Dubrovin and Novikov (1974; 1975b)) around 1974–75. Additional discussions of these equations can be found, for instance, in Dubrovin et al. (1976), Levitan (1987, Chs. 8–12), Marchenko (1986, Ch. 4), McKean (1979a), and Trubowitz (1977).

The results in Lemma 1.10 can be extended to the case of colliding Dirichlet eigenvalues as long as the corresponding Dirichlet divisor remains nonspecial. The details are somewhat involved and have been worked out in Birnir (1986a,b). As pointed out in Remark 1.27, symmetric functions of the Dirichlet eigenvalues (such as u) are somewhat simpler to handle; in particular, they can be directly expressed in terms of the underlying Riemann theta function associated with $\mathcal{K}_n$.

Lemma 1.11 is well-known in the case of Neumann boundary conditions (cf., e.g., McKean and Trubowitz (1976) where the result is stated in the infinite genus context). The general case $\beta \in \mathbb{R} \setminus \{0\}$ can be found in Levitan and Savin (1988) (and in Gesztesy et al. (1996a)).

For more results concerning Remark 1.13, we refer to Dauge and Helffer (1993a,b), Kong and Zettl (1996a,b), Kong et al. (1999).

Lemma 1.14 is quite familiar in the context of periodic Schrödinger operators with even potentials, that is, $u(-x) = u(x)$, $x \in \mathbb{R}$; see, for instance, McKean and Trubowitz (1976).

Lemma 1.15 is due to Gesztesy et al. (1996a).

The trace formulas in Lemmas 1.16 and 1.17 have a rich history going back to Gelfand and Levitan. We defer a detailed discussion to the notes of Section 1.5.

The linearization property (1.106) of the Abel map, and hence the straightening out of the (higher-order) KdV flows on the Jacobi variety, is due to Dubrovin

(1975a) (see also Dubrovin (1975b), Dubrovin and Novikov (1974)). For connections with the recursion formalism of Section 1.2, we refer to Gesztesy and Holden (2002) and to Appendix F.

The celebrated expression (1.108) for u in terms of the Riemann theta function associated with $\mathcal{K}_n$ was found by Its and Matveev (1975a,b). Its and Matveev utilized previous work by Akhiezer (1961) (see also (Akhiezer (1960), Akhiezer and Tomčuk (1961)). By a remarkable coincidence around 1975, Dubrovin (1975a,b) (see also Dubrovin and Novikov (1974; 1975b)) had also noticed essentially simultaneously that Akhiezer's methods applied to the inverse spectral theory for finite-band potentials. In particular, he isolated the solution of Jacobi's inversion problem as a key tool in this context.

It should be remarked at this point that Novikov had noted earlier in 1974 that finite-band potentials are the natural analogs of multi-soliton solutions in the sense that they are solving stationary higher-order KdV equations (Novikov (1974)). He also observed that, generically, such solutions would be quasi-periodic rather than periodic with respect to x, and hence the solution of the inverse spectral problem associated with finitely many spectral bands should be posed within the class of quasi-periodic functions u. Moreover, assuming u to be real-valued, nonsingular on $\mathbb{R}$, and periodic, he showed that if u satisfies an nth-order stationary KdV equation, then the $L^2(\mathbb{R})$-spectrum associated with u consists of at most n compact intervals and an additional half-line (cf. (1.295)). Within a year, Dubrovin (1975b), Flaschka (1975b), and McKean and van Moerbeke (1975) also proved the converse of this assertion (a different proof of this result was also published in Goldberg (1976)) and hence it became possible to identify the set of real-valued periodic finite-band potentials with the set of real-valued periodic solutions of stationary higher-order KdV equations. It also became clear from Novikov's 1974 paper that solutions of the time-dependent higher-order KdV equations periodic with respect to x would generically be quasi-periodic in t. Independently, Lax (1975) (see also Lax (1976)) showed in 1975 that real-valued periodic solutions in x of the higher-order time-dependent KdV equations lie on finite-dimensional tori and that they depend on t in a quasi-periodic manner. In this paper Lax also gave a very simple proof of the fact that if a real-valued periodic potential u satisfies an nth stationary KdV equation, then its corresponding $L^2(\mathbb{R})$-spectrum consists of at most n compact intervals (see also Lax (1974)).

Finally, another milestone must be mentioned. As early as 1974, Marčenko (1974a,b) solved the periodic Cauchy problem for the KdV equation assuming real-valued and sufficiently smooth initial data. In particular, he also characterized the monodromy matrices of all real-valued periodic finite-band potentials.

Looking back at this period (and now with some distance from it), one cannot fail to be in awe of the amazing theory developed between 1974 and 1976 in Moscow, St. Petersburg, Kharkov, and New York by pioneers such as Dubrovin, Flaschka, Its, Lax, Marchenko, Matveev, McKean, Novikov, van Moerbeke, and others.

Since this initial period, many authors presented reviews and slightly varying approaches to algebro-geometric (respectively periodic) solutions of (stationary and time-dependent) equations of the KdV hierarchy. We mention, for instance, the very influential reviews by Dubrovin et al. (1976) and Matveev (1976), Date and Tanaka (1976), all in 1976, and especially Dubrovin's 1981 review (Dubrovin (1981); see also Dubrovin (1982b)), which is still regarded as a masterpiece of exposition and paved the way for a subsequent generation of scientists to enter this exciting field. A wealth of additional material can be found in Bobenko (1984), Bobenko and Kubenskii (1988), Cherednik (1978), Dubrovin (1977; 1983), Dubrovin et al. (1990), Gel'fand and Dikii (1979), Gesztesy (1992), Gesztesy et al. (1996a), Grinevich and Krichever (1990), Krichever (1976a,b; 1977a,b; 1978; 1995), Krichever and Novikov (1980b,a; 1981; 2000), Levitan (1977; 1984), McKean (1979a,b; 1985), Moser (1983), Novikov (1978a,b; 1980), and Taimanov (1997). Moreover, the subject is briefly treated in the monographs of Ablowitz and Segur (1981, Sec. 2.3), Asano and Kato (1990, Sec. 7.3), Dickey (1991, Ch. 12), Marchenko (1988, Ch. 4), Newell (1985, Sec. 3h), and Rodin (1988, Ch. 6), and dealt with at length in the monographs of Belokolos et al. (1994, Ch. 3), Levitan (1987, Ch. 8), Marchenko (1986, Ch. 4), and Novikov et al. (1984, Ch. II). A succinct overview of the algebro-geometric method can also be found in the introduction to Novikov et al. (1981) by Wilson.

It is perhaps worth mentioning that $\phi(P_0)$, $P_0 = (E_0, 0)$ in (1.104) is a solution of the corresponding stationary equation of the modified Korteweg–de Vries (mKdV) hierarchy related to u in (1.108) by the Miura transformation $u = \phi(P_0)^2 + \phi_x(P_0) + E_0$. We refer to Gesztesy (1989; 1991a,b; 1992), Gesztesy et al. (1991), Gesztesy and Simon (1990), Gesztesy and Svirsky (1995), Previato (1993) for details.

We have repeatedly emphasized that u was usually assumed to be real-valued (and nonsingular) in these early investigations. In this case the spectrum of the self-adjoint realization H in $L^2(\mathbb{R})$ of the differential expression $L = -d^2/dx^2 + u$ is given by a collection of bands of the type

$$\operatorname{spec}(H) = \bigcup_{j=1}^{n} [E_{2j-2}, E_{2j-1}] \cup [E_{2n}, \infty), \qquad E_0 < E_1 < \cdots < E_{2n} \tag{1.295}$$

for some $n \in \mathbb{N}$. Incidentally, this explains why potentials u with associated $L^2(\mathbb{R})$-spectrum of the type (1.295) are traditionally called finite-band (or finite-gap) potentials. For the conclusion that the isospectral torus of a KdV potential u associated with an $L^2(\mathbb{R})$-spectrum of the type (1.295) is an n-dimensional real torus $\mathbb{T}^n = \prod_{j=1}^n S^1$, real-valuedness of u is crucial. The isospectral torus $\mathbb{T}^n$ then comes about as follows. For a fixed $x_0 \in \mathbb{R}$, the Dirichlet divisors (data) $\hat{\mu}_j(x_0) = (\mu_j(x_0), (-i/2)F_{n,x_0}(\mu_j(x_0), x_0))$, $j = 1, \dots, n$ can be prescribed so that the projection $\mu_j(x_0)$ lies anywhere in the jth spectral gap $\mu_j(x_0) \in [E_{2j-1}, E_{2j}]$ and

$\hat{\mu}_j(x_0)$ lies on the upper or lower sheet of $\mathcal{K}_n$. Effectively, this yields $[E_{2j-1}, E_{2j}] \times \{\pm\} \simeq S^1$ for each spectral gap (*cum grano salis*) and thus leads to $\mathbb{T}^n$ (or equivalently, to the real part of the Jacobi variety of the underlying hyperelliptic curve) since we have n such spectral gaps available for Dirichlet eigenvalues.

Apart from the Its–Matveev realization of the n-dimensional isospectral torus $I_n(u_0)$ of a given real-valued nonsingular KdV potential u_0, with spectrum as in (1.295) in terms of the Riemann theta functions associated with $\mathcal{K}_n$ and positive Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ of degree n (Its and Matveev (1975a,b)), there also exist explicit realizations of $I_n(u_0)$ in terms of $2n$ Darboux transformations representable as a $2n \times 2n$ Wronski determinant of certain Baker–Akhiezer functions. The relevant papers (in case u_0 is also periodic) are Buys and Finkel (1984) and Finkel et al. (1987) with pertinent results also in McKean (1985), McKean and van Moerbeke (1975), McKean and Trubowitz (1976), Ralston and Trubowitz (1988), and Trubowitz (1977). A complete spectral theoretic characterization of this method (including the effects of isospectral deformations on Weyl m-functions and spectral functions for the associated Schrödinger operators on a half-line and on $\mathbb{R}$) was presented in Gesztesy et al. (1996b) (see also Gesztesy et al. (1996c)). In particular, their methods were applied to general one-dimensional Schrödinger operators with gaps in their essential spectrum without assuming periodicity of the underlying base potential u_0. Hence, these results (like the Its–Matveev theta-function representation) describe an explicit realization of the isospectral torus $\mathbb{T}^n$ for general quasi-periodic, finite-band potentials with associated $L^2(\mathbb{R})$-spectrum of the type (1.295). For yet another characterization of $I_n(u_0)$ for periodic potentials u_0, in terms of Fredholm determinants, we refer to Iwasaki (1987).

Equations (1.128) single out those spectral band edges $E_0, \dots, E_{2n}$ in (1.125) that correspond to periodic potentials, that is, they separate periodic from quasi-periodic (real-valued) algebro-geometric potentials u. This criterion, reformulated in terms of conformal mapping techniques involving Schwarz–Christoffel integrals, appeared already in Marčenko and Ostrovskiĭ (1975a,b). The case of real-valued, periodic algebro-geometric potentials and KdV solutions is exhaustively discussed in the literature. We refer, for instance, to Ablowitz and Segur (1981, Sec. 2.3), Belokolos et al. (1994, Sec. 3.6), Buys and Finkel (1984), Date and Tanaka (1976), Dubrovin (1975a,b), Dubrovin et al. (1976), Dubrovin and Novikov (1974; 1975b), Ercolani et al. (1986b), Finkel et al. (1987), Flaschka (1975a,b), Goldberg (1976), Hochstadt (1965), Its and Matveev (1975a,b), Johnson (1982), Krichever and Novikov (2000), Lax (1974; 1975; 1976), Marčenko (1974a,b), Marchenko (1986, Secs. 3.4, 4.3, 4.4), Marčenko and Ostrovskiĭ (1975a,b), Marchenko and Ostrovsky (1987), McKean (1979a; 1985), McKean and van Moerbeke (1975), Moser (1983), Meiman (1977), Newell (1985, Sec. 3h), Novikov (1974; 1978a,b; 1980), Novikov et al. (1984, Ch. II), and Trubowitz (1977).

In the complex-valued algebro-geometric periodic case much of the theory goes through, but there are some characteristic changes apart from the corresponding isospectral manifold of u no longer being described by a real torus $\mathbb{T}^n$. In this complex-valued situation, the spectrum of the associated closed realization H of $L = -d^2/dx^2 + u$ in $L^2(\mathbb{R})$ is given by n regular analytic arcs in the complex plane and a semi-infinite arc tending to ∞. (This is why it seems much more appropriate to use the notion of finite-band as opposed to finite-gap KdV potentials.) Locally, these arcs may exhibit multiple crossings and hence exhibit a much more intricate picture than in the real-valued case, although for sufficiently large energies the picture resembles again that of the self-adjoint situation (cf. Gesztesy and Weikard (1995b), Pastur and Tkachenko (1991a), Rofe-Beketov (1963), Sansuc and Tkachenko (1996a,b; 1997), Serov (1960), Tkachenko (1992; 1994; 1996), and Weikard (1998a,c)). The corresponding Dirichlet eigenvalues $\mu_j(x)$ now are no longer confined to certain regions of $\mathbb{C}$ (as opposed to their trapping in the spectral gaps $[E_{2j-1}, E_{2j}]$ in the self-adjoint case), and the simple torus $\mathbb{T}^n$ now turns into a complex torus, the Jacobian of the underlying curve $\mathcal{K}_n$. This is discussed in Dubrovin and Novikov (1974) and in great detail in Birnir (1986a,b; 1987). From a spectral theoretic point of view, however, the distinction between real-valued and complex-valued potentials is crucial in this finite-band context, as will be pointed out next. In fact, Gasymov analyzed examples of the type $u_0(x) = \exp(ix)$ (actually, he analyzed a whole class of generalizations of this example) and proved that the spectrum of the corresponding closed $L^2(\mathbb{R})$-realization H_0 associated with the differential expression $-d^2/dx^2 + u_0$ is of the finite-band-type (cf. Gasymov (1980)) $\operatorname{spec}(H_0) = [0, \infty)$. This spectrum is not only of the finite-band type but is even a subset of the real line although H_0 is clearly not self-adjoint. Further extensions of this type of result where obtained in Guillemin and Uribe (1983) and Pastur and Tkachenko (1988; 1991b). It is not difficult to see that this example, despite its being of the finite-band spectral type, is not algebro-geometric, and the underlying Riemann surface encountered from Floquet theoretic considerations is, in fact, of infinite genus. The reason for this surprise is actually quite simple. Although all spectral bands touch their neighboring spectral bands and so all spectral gaps close (with the exception of only the spectral gap $(-\infty, 0)$), the corresponding Dirichlet eigenvalues are by no means trapped between these bands but can roam freely in the complex plane once one varies the reference point $x \in \mathbb{R}$ at which the Dirichlet boundary condition is imposed. It is the number of movable Dirichlet eigenvalues that decides the genus of the underlying curve. In the algebro-geometric case, only finitely many movable Dirichlet eigenvalues are available, whereas the remaining infinite Dirichlet eigenvalues are pinned down (i.e., they are immovable with respect to variations of x). These facts (and others on complex-valued singular periodic potentials) are discussed in detail in Gesztesy (2001), Gesztesy and Weikard (1996), and Weikard (1998a,b). The upshot of this analysis is that the notion of "finite-band" (and much less that of "finite-gap") is not a sufficient characterization of complex-valued, algebro-geometric (stationary) KdV solutions. The safe alternative adopted in this monograph is to talk about algebro-geometric KdV potentials (or simply KdV potentials), whenever an underlying finite genus case is implied.

Finally, for those not yet sufficiently convinced by these arguments, we recommend the paper Chulaevsky and Sinai (1989), in which the spectrum of a self-adjoint discrete Schrödinger (Jacobi) operator with two basic (rationally independent) frequencies is shown to be a compact interval (i.e., without bounded spectral gaps) consisting of a dense pure point spectrum. The corresponding quasi-periodic potential term is real-valued, and yet it is clearly not an algebro-geometric situation. In particular, the corresponding Dirichlet eigenvalues are certainly real-valued and interlace with the eigenvalues of the discrete Schrödinger operator on $\mathbb{Z}$. What went "wrong" with this example is a bit different. The essential spectrum of this operator is not absolutely continuous. Moreover, it is associated with a positive Lyapunov exponent. Put differently, this potential is not reflectionless – a terminology briefly explained in Appendix J. The property of being reflectionless turns out to be a necessary (though not sufficient) condition for a real-valued potential to be of the algebro-geometric type. The interested reader can find more about this, for instance, in Belokolos (1990), Carmona and Lacroix (1990), Craig (1986), Johnson (1982; 1983), Johnson and Moser (1982), Kotani (1984; 1988), Kotani and Krishna (1988), and Sodin and Yuditskiĭ (1995a,b; 1996).

Theorem 1.26 is certainly known, although, a detailed proof in the generality we formulated it seems difficult to locate in the literature. That u extends meromorphically to $\mathbb{C}$ as stated in Remark 1.29 is a special case of Theorem 6.10 in Segal and Wilson (1985).

Examples 1.30–1.32 can be found, for instance, in Dickson et al. (1999), Duistermaat and Grünbaum (1986), Gardner et al. (1974), Gesztesy et al. (1992; 2000; 2003), Kay and Moses (1956), and Ohmiya (1988b). In particular, the rational KdV potentials summarized in Example 1.30 (v) have been analyzed in detail by Duistermaat and Grünbaum (1986) in their study of bispectral pairs of differential operators. A new approach to this circle of ideas that exploits results due to Halphen (1885) and permits an extension to elliptic KdV potentials, was developed in Gesztesy et al. (2000; 2003). The n-soliton solutions in Example 1.31 employ the determinant approach used in Kay and Moses (1956) and later in Gardner et al. (1974). Alternative representations of n-soliton solutions were found in Hirota (1971; 1980); (see also Miwa et al. (2000, Ch. 3), Newell (1985, Ch. 4), and the references therein). Our notation in connection with elliptic functions in Example 1.32 follows Abramowitz and Stegun (1972, Ch. 18); their basic properties are summarized in Appendix H. A large body of literature on Lamé potentials can be found, for instance, in the classical monographs Burkhardt (1906), Halphen (1888), Krause (1897), Picard (1928), and Whittaker and Watson (1986). A key result in their analysis is a theorem due to Picard (1879; 1880; 1881), with important contributions to this circle of ideas by Hermite (1877; 1912), Floquet (1884a,b,c), Mittag-Leffler (1880), and Halphen (1884; 1885). For applications of these ideas to completely integrable systems, we refer to Gesztesy and Sticka (1998), Gesztesy and Weikard (1995a,c,d,e; 1996; 1998a; 1999), Weikard (1998b,c; 1999; 2000), and especially to Gesztesy and Weikard (1998b), and the extensive literature cited therein.

The classical literature on Lamé potentials and alike amassed by the French school in the latter part of the 19th century with very few exceptions – such as Baker (1928), Burchnall and Chaundy (1923; 1928; 1932), and Drach (1918; 1919a,b), Guerritore (1909), and Strutt (1967)–went out of fashion for about the first 75 years of the 20th century. A notable exception was a paper by Ince in 1940 (cf. Ince (1940)). He investigated the special case in which u is real-valued and nonsingular on $\mathbb{R}$ and is given by $u_n(x) = n(n+1)\wp(x+\omega_3)$, with $-i\omega_3 > 0$ from a Floquet-theoretic point of view and established that such a potential exhibits a finite-band (gap) structure in the associated self-adjoint Schrödinger operator H_n on $L^2(\mathbb{R})$ if and only if n is an integer. In particular, if $n \in \mathbb{N}$, the spectrum $\operatorname{spec}(H_n)$ of H_n is of the type (1.295); that is, it consists of n compact intervals and one half-line. It took 35 more years until an explicit expression of another finite-band potential (i.e., not of Lamé-type) was found in Dubrovin and Novikov (1975b). The actual history of the elliptic finite-band potentials (as solutions of some of the stationary KdV equations) is not without interest, and hence we record it here to a certain extent. Dubrovin and Novikov (1975b) explicitly integrated the KdV flow $u_t = -\frac{1}{4}u_{xxx} + \frac{3}{2}uu_x$, with initial condition $u(x,0) = 6\wp(x+\omega_3)$ (see also Enol'skii (1983; 1984a,b), Its and Enol'skii (1986)) and found it to be of the type

$$u(x,t) = 2\sum_{j=1}^{3} \wp(x - x_j(t)) \tag{1.296}$$

for appropriate $\{x_j(t)\}_{j=1,2,3}$. Due to the time evolution operator U_n constructed with the help of P_{2n+1} via $U_{n,t} = P_{2n+1}U_n$, all potentials $u(\cdot\,,t)$ in (1.296) are isospectral to $u(\cdot\,,0) = 6\wp(\cdot+\omega_3)$. In 1977, Airault, McKean, and Moser, in their seminal paper Airault et al. (1977) presented the first systematic study of the isospectral torus $I_{\mathbb{R}}(u_0)$ of real-valued smooth potentials u_0 of the type

$$u_0(x) = 2\sum_{j=1}^{M} \wp(x - x_j) \tag{1.297}$$

with a finite-gap spectrum. Among a variety of results, they proved that any element u of $I_{\mathbb{R}}(u_0)$ is an elliptic function of the type (1.297) (with different x_j) with M constant throughout $I_{\mathbb{R}}(u_0)$ and $\dim I_{\mathbb{R}}(u_0) \leq M$. In particular, if u_0 evolves according to any equation of the KdV hierarchy it remains an elliptic finite-gap potential. However, explicit new examples of elliptic KdV potentials remained elusive even though it was clear from the Its–Matveev formula that there existed a whole torus of elliptic potentials isospectral to a given elliptic base potential (e.g., the base potential can be taken as the Lamé potential $n(n+1)\wp(\cdot)$, $n \in \mathbb{N}$).

The potential (1.297) is intimately connected with completely integrable many-body systems of the Calogero–Moser-type (Calogero (1975), Moser (1975) (see also Calogero (1978), Choodnovsky and Choodnovsky (1977), Chudnovsky (1979), Olshanetsky and Perelomov (1981), and Ruijsenaars (1987)). This connection with integrable particle systems was subsequently exploited in Krichever (1980) (see also Krichever (1983; 1990)) in his construction of elliptic algebro-geometric solutions of the Kadomtsev–Petviashvili equation. In the KdV context of (1.297), Krichever's approach relies on the ansatz

$$\psi(z, x) = e^{\kappa(z)x} \sum_{j=1}^{M} A_j(z)\Phi(x - x_j, \rho(z)), \tag{1.298}$$

for the Floquet solutions of $L = -d^2/dx^2 + u_0(x)$, where

$$\Phi(x, \rho) = \frac{\sigma(x-\rho)}{\sigma(x)\sigma(-\rho)} e^{\zeta(\rho)x}$$

(assuming for simplicity the generic case $x_j \neq x_k \pmod{\Delta}$ for $j \neq k$, where Δ denotes the fundamental period parallelogram associated with $\wp(\cdot)$). Applying L_0 to (1.298) then yields an M-sheeted covering of the torus associated with the fundamental periods $2\omega_1, 2\omega_3$ and hence a description of the underlying algebraic curve. The next breakthrough occurred when Verdier (1988) published new explicit examples of elliptic finite-gap potentials. Verdier's examples spurred a flurry of activities and inspired Belokolos and Enol'skii (1989a,b), Smirnov (1989), and subsequently Taimanov (1990a) and Kostov and Enol'skii (1993) to find further such examples by combining the reduction process of Abelian integrals to elliptic integrals (see Babich et al. (1983; 1986), Belokolos et al. (1994, Ch. 7; 1986)) with the aforementioned techniques in Krichever (1980; 1983). This development finally culminated in a series of papers by Treibich and Verdier (1990a,b; 1992), in which it was shown that a general complex-valued potential of the form

$$u(x) = \sum_{j=1}^{4} d_j \, \wp(x - \omega_j)$$

($\omega_2 = \omega_1 + \omega_3, \ \omega_4 = 0$) is a finite-gap potential if and only if $d_j/2$ are triangular numbers, that is, if and only if

$$d_j = s_j(s_j + 1) \text{ for some } s_j \in \mathbb{Z}, \ 1 \leq j \leq 4.$$

The methods of Treibich and Verdier are based on hyperelliptic tangent covers of the torus $\mathbb{C}/\Lambda$, Λ being the period lattice generated by $2\omega_1$ and $2\omega_3$ (cf. also Colombo et al. (1994) and Treibich (1989; 1994)).

The state of the art of elliptic finite-gap solutions, until around 1993, was reviewed in Issues 1 and 2 of Volume 36 of *Acta Applicandae Mathematicae*, which appeared in 1994, and we refer, for instance, to Belokolos and Enol'skii (1994), Enol'skii and Kostov (1994), Krichever (1994), Smirnov (1994a), Taimanov (1994), and Treibich (1994) therein. For more recent results see Belokolos et al. (2001a), Buchstaber et al. (to appear; 1997a,b,c; 1999), Eilbeck and Enol'skii (1994a,b; 2000), Eilbeck et al. (2000; 2001), Enol'skii and Eilbeck (1995), and Smirnov (1994b). Moreover, Chapter 7 of the monograph Belokolos et al. (1994), and the detailed review Belokolos and Enol'skii (2002a,b) present the connection between reduction theory of Abelian functions and completely integrable systems. Here we just add the comment that if one is interested in spatially elliptic (not necessarily real-valued) solutions u in the Its–Matveev formula (1.108) with periods $2\omega_1$, $2\omega_3$, with $\mathrm{Im}(\omega_3/\omega_1) > 0$ (cf. Appendix H), one necessarily needs to impose the constraints

$$2i\omega_p \underline{U}_0^{(2)} \in \mathbb{Z}^n + \tau\mathbb{Z}^n, \quad p = 1, 3.$$

More about these constraints and a discussion of sufficiency is provided in Belokolos et al. (1994, Sec. 7.7).

The whole development up to this point, however, missed the fundamental connection between elliptic algebro-geometric KdV solutions and Picard's theorem mentioned a bit earlier. This connection was the starting point of a complete characterization of all elliptic finite-band solutions of the KdV hierarchy by Gesztesy and Weikard alluded to above. A detailed treatment of this approach will be the subject of a forthcoming monograph, and so we refer the interested reader to Gesztesy and Weikard (1996; 1998b) for now.

Darboux-type transformations and a complete account of their effect on the hyperelliptic curve $\mathcal{K}_n$ (possibly with a singular affine part) associated with algebro-geometric KdV potentials are discussed in Gesztesy and Holden (2000c) (see Appendix G for details).

Section 1.4. As in Sections 1.2 and 1.3, the approach presented closely follows the one in Gesztesy et al. (1996a).

Since most of the references provided in connection with Section 1.3 treat the time-dependent KdV equation and not just stationary KdV solutions, we will now mainly focus on issues different from stationary ones and topics not yet covered.

In analogy to its stationary analog in Section 1.3, the role of ϕ defined in (1.165) is still central to Section 1.4, and the corresponding facts recorded in the notes to Section 1.3 apply accordingly in the present time-dependent setting.

The Dubrovin equations (1.195), (1.196) in Lemma 1.37 and their β-dependent analogs in Lemma 1.38 were found simultaneously with their stationary counterparts, as discussed in the notes to Section 1.3. However, they are often discussed in connection with the simplest cases $r = 0, 1$ only.

Since the proofs of Lemmas 1.39, 1.40, and 1.41 are identical to those in the corresponding stationary cases, what was said in connection with their stationary counterparts in the notes to Section 1.3 again applies line by line.

The linearization properties (1.212), (1.213) of the Abel map and the Its–Matveev formula (1.214) for u in terms of the Riemann theta function associated with $\mathcal{K}_n$ were again found simultaneously with their stationary counterparts, and thus the historical development sketched in this connection in the notes to Section 1.3 remains valid in the context of Theorem 1.44.

The argument presented in Remark 1.47 can be found in the monograph Novikov et al. (1984, pp. 139–144).

As in Section 1.3, we remark that $\phi(P_0)$, $P_0 = (E_0, 0)$ in (1.210) is a solution of the corresponding equation of the modified Korteweg–de Vries (mKdV) hierarchy and is related to u in (1.214) by Miura's transformation $u = \phi(P_0)^2 + \phi_x(P_0) + E_0$.

The solution of the algebro-geometric initial value problem in Theorem 1.48 is rarely presented in this detail. Again the result is well-known to experts, but discussions in the literature (such as a related one in Dubrovin et al. (1976, p. 139)) are usually restricted to the first KdV equation (i.e., to $r = 1$). An attempt to solve the periodic KdV initial value problem numerically (for $r = 1$) based on the coupled system of Dubrovin equations (1.195), (1.196) is undertaken in Osborne and Segré (1990a,b,c). For connections between the Dubrovin equations and the Hamiltonian formalism, including action-angle variables for algebro-geometric solutions, we refer the reader, for instance, to Al'ber (1979), Al'ber and Al'ber (1985; 1987a), Alber and Marsden (1992; 1994a), Bättig et al. (1993a; 1995; 1997), Kappeler (1991), Kappeler and Makarov (2000), Kappeler and Mityagin (1999), McKean (1997), and Vanhaecke (1992).

The differences between real-valued and complex-valued KdV solutions pointed out in Section 1.3 remain of course valid in the present time-dependent context. However, there are additional difficulties due to the presence of time variables t_r. A special divisor $\mathcal{D}_{\hat{\mu}(x,t_r)}$ may become trapped in the theta divisor, rendering the Its–Matveev formula meaningless. This cannot happen for translations (i.e., $r = 0$) but can occur for higher-order KdV flows in the presence of certain symmetries in the distribution of the branch points of $\mathcal{K}_n$, as discussed in Birnir (1986b). A careful discussion of these issues is somewhat involved, and we refer the interested reader to Birnir (1986a,b).

Examples 1.51–1.53 are taken from Gesztesy et al. (1992; 2000; 2003), and the references therein. Example 1.52 describing n-soliton solutions employs the determinant approach, as used in Kay and Moses (1956) and later in Gardner et al. (1974). For Hirota's alternative representation of n-soliton solutions Hirota (1971; 1980), we also refer the reader to the monographs Miwa et al. (2000, Ch. 3), Newell (1985, Ch. 4), and the literature cited therein.

The connection between the Neumann system of constrained harmonic oscillators to a sphere and algebro-geometric solution of the KdV hierarchy is treated in Ercolani and Flaschka (1985).

Although we mentioned rational and n-soliton KdV solutions at the end of Sections 1.3 and 1.4, we did not explicitly study degenerations of quasi-periodic algebro-geometric solutions as certain periods tend to infinity, or alternatively, systematically apply Darboux-type transformations (i.e., various commutation methods) that can lead to Bäcklund transformations between the KdV and mKdV hierarchies as well as auto-Bäcklund transformations for the KdV hierarchy. This singularization procedure creates solitons and certain meromorphic solutions relative to a remaining algebro-geometric background solution (whose associated Riemann surface has appropriately diminished genus) and leads to a singular curve for the combined soliton and background KdV solution, etc. The possibility of obtaining soliton solutions from degenerating hyperelliptic curves (i.e., pinching handles in the language of Fay (1973, Ch. III)) was recognized already in Krichever (1975) (without, however, providing any details) and in the special case of an elliptic genus one background solution by Kuznetsov and Mikhailov (1975). Matveev (1976), and subsequently McKean (1979b), degenerated the quasi-periodic algebro-geometric solutions into the class of soliton solutions by explicitly exhibiting Hirota's KdV soliton representation Hirota (1971) as the result of singularizing the Its–Matveev formula (1.214). Although this procedure amounts to a complete degeneration of $\mathcal{K}_n$ into the Riemann sphere with additional double points, partial degenerations were also mentioned by Matveev, and more details appear in Appendix 1 of Dubrovin et al. (1976) and in Levitan (1989); the corresponding scattering matrix is described in Firsova (1989). Additional degenerations into the Riemann sphere and one multiple point on it then led to the class of rational KdV solutions (derived in Ablowitz and Airault (1981), Ablowitz and Satsuma (1978), and Adler and Moser (1978)) as shown in Ehlers and Knörrer (1982) using Darboux transformations. These singularization procedures (especially, their practical implementation in terms of Darboux-type transformations) have been intensively studied in the literature, and the interested reader can find plenty of additional results in Bikbaev (1989; 1994), Bikbaev and Sharipov (1989), Gesztesy (1991a; 1993; 2001), Gesztesy and Holden (2000c), Gesztesy et al. (1991; 1996b), Gesztesy and Svirsky (1995), Gesztesy and Teschl (1996), Grinevich (1989; 1994), Jaulent et al. (1989), Matveev and Salle (1991, Ch. 3), McRae and Weikard (1997), Rybin and Sall (1985), Sharipov (1986; 1987), Trlifaj (1989), Veselov and Shabat (1993), and Zagrodziński (1984; 1991).

Section 1.5. Again our presentation follows Gesztesy et al. (1996a) to some extent; the material on general trace formulas is taken from Gesztesy et al. (1995b), Gesztesy and Simon (1996a,b).

For $\beta \in \mathbb{R}$, the fundamental Herglotz function $\Gamma^{\beta}(\,\cdot\,, x)$ in (1.239) and its associated spectral shift function $\xi^{\beta}(\,\cdot\,, x)$, $x \in \mathbb{R}$, in (1.246) were introduced in Gesztesy et al. (1995b). The corresponding quantities in the Dirichlet context (where $\beta = \infty$) were originally introduced in Gesztesy and Simon (1996b). Equation (1.245) follows from the exponential representation theorem for Herglotz functions as proven in Aronszajn and Donoghue (1957). For more details, we refer to Appendix I.

In addition to (1.241) for $\beta = \infty$ and (1.255), one can derive the following asymptotic expansion for $\frac{d}{dx}\operatorname{tr}\big((H_x^\infty - z)^{-1} - (H-z)^{-1}\big)$. From

$$\frac{d}{dx}\ln(g(z,x)) = m_{-,0}(z,x) + m_{+,0}(z,x)$$

(cf. Johnson and Moser (1982)) and (J.28)–(J.30) one observes (see Gesztesy et al. (1995b))

$$\begin{aligned}\frac{d}{dx}\operatorname{tr}\big((H_x^\infty - z)^{-1} - (H-z)^{-1}\big) &= -\frac{d}{dz}\big(m_{-,0}(z,x) + m_{+,0}(z,x)\big)\\ &\underset{z\to i\infty}{=} 2\sum_{\ell=1}^{\infty} \ell\, m_{\pm,0,2\ell}(x) z^{-\ell-1}.\end{aligned}$$

Equations (1.242) and (1.243) are proved in Gesztesy et al. (1995b).

The trace formula (1.247) for f in various function classes has been discussed a great deal in the literature. It originates in works of Lifshits (1952; 1956), who had in mind applications to solid state physics. The first rigorous approach to the spectral shift function and a proof of (1.247) is due to Kreĭn (1953), Krein (1962; 1983). Since then this circle of ideas has been repeatedly revisited by many authors. We refer, for instance, to Baumgärtel and Wollenberg (1983, Ch. 19), Peller (1985; 1990), Simon (1995; 1998), Sinha and Mohapatra (1994), Yafaev (1992, Ch. 8).

Lemma 1.56 is well-known in the Dirichlet case see, for instance, Gel'fand and Dikii (1975), and the case of general β boundary condition can be found in Gesztesy et al. (1996a). The special Neumann case $\beta = 0$ was derived in Gesztesy and Weikard (1996).

Lemma 1.57 is an elementary consequence of Lemma 1.56 (cf. Gesztesy et al. (1996a)).

Theorem 1.58 (i) was derived using a different strategy of proof in Gesztesy et al. (1995b). The current derivation, based on the universal differential equation (1.251) is due to Gesztesy et al. (1996a).

Theorem 1.59 is due to Gesztesy et al. (1995b). The case $\beta = \infty$, $\ell = 1$ was first proved in Gesztesy and Simon (1996b). For simplicity we confined ourselves to the resolvent regularization trace formulas in Theorem 1.59. The analogous approach using a heat kernel regularization for the coefficients s_ℓ^∞ in (1.243) reads (cf. Gesztesy et al. (1995b))

$$s_\ell^\infty(x) = \frac{(-1)^{\ell+1}}{\ell!}\left(\frac{E_0^\ell}{2} + \ell \lim_{t\downarrow 0}\int_{e_0^\infty}^{\infty} d\lambda\, e^{-t\lambda}\lambda^{\ell-1}\left(\frac{1}{2} - \xi(\lambda,x)\right)\right), \quad \ell \in \mathbb{N}.$$

The original motivation that led to (1.257) (actually, its analog applying a heat kernel regularization method) in Gesztesy and Simon (1996b), was to extend the well-known trace formula for certain classes of reflectionless potentials, considered, for instance, in Craig (1989a), Kotani and Krishna (1988), and Levitan (1982; 1985; 1987, Chs. 9, 11), to general non-reflectionless potentials. In the special scattering theoretic context, a trace formula equivalent to (1.257) stated in terms of a reflection coefficient rather than a spectral shift function had been found earlier by Deift and Trubowitz (1979). This formula was rediscovered in slightly different terms in Venakides (1988) as a result of studying the periodic trace formula (1.262) for $\ell = 1$ in the limit where the period tends to infinity. For various connections between trace formulas in terms of Krein's spectral shift function and trace formulas in terms of scattering theoretic objects (such as reflection coefficients, etc.; cf. Gesztesy and Holden (1994) and the literature therein) we refer to Gesztesy (1995). Additional results on trace formulas in terms of spectral shift functions can be found in Gesztesy (1995), Gesztesy and Holden (1995; 1997), Gesztesy et al. (1995a; 1993), Gesztesy and Makarov (2000), and Rybkin (2001a,b; 2002).

Historically, trace formulas for Schrödinger operators on compact intervals with self-adjoint boundary conditions at the end points go back at least to the paper by Gel'fand and Levitan (1953) with subsequent work by Gel'fand (1956), Dikiĭ (1961), Halberg and Kramer (1960), and Gilbert and Kramer (1964) (see also Magnus and Winkler (1979, Ch. VI)).

In the real-valued algebro-geometric context (i.e., in the case of only finitely many gaps in the spectrum), the periodic Dirichlet-type trace formula (1.262) for $j = 1$ (or rather, (1.236)), had been noticed in Hochstadt (1965) and later in Dubrovin (1975b). The general case $j \in \mathbb{N}$ appeared in Flaschka (1975b), McKean and van Moerbeke (1975). The general real-valued periodic case with infinitely many gaps in the spectrum can be found in Trubowitz (1977). The Neumann case $\beta = 0$ in (1.260) is due to McKean and Trubowitz (1976). The general case $\beta \in \mathbb{R}$ is taken from Gesztesy et al. (1995b). In particular, (1.260) and (1.262) also extend to certain classes of real-valued almost periodic potentials; see, for instance, Craig (1989a), Kotani and Krishna (1988), and Levitan (1982; 1985; 1987, Chs. 9, 11). For a (perturbative) trace formula approach to Schrödinger operators on a finite interval with periodic boundary conditions in terms of the trace of the corresponding semigroup (i.e., the trace of the heat kernel) see Kac and van Moerbeke (1974), Novitskii (1995), and Sunada (1980).

It should also be noted that the distinction between real-valued and complex-valued potentials is of some importance in connection with trace formulas. The trace formula (1.247) heavily relies on a pair of self-adjoint operators and hence so do the results stated in Theorem 1.59. On the other hand, in the algebro-geometric context, the trace formulas (1.236) and (1.237) (as well as their higher-order analogs) remain valid for complex-valued potentials, as proven in Lemmas 1.16 and 1.17. Yet the periodic and almost periodic trace formulas associated with infinitely many gaps in the spectrum mentioned in the preceding paragraph all assume real-valuedness of the potential u. In the special complex-valued periodic case, however, it is not too difficult to prove that the trace formulas (1.262) remain valid whether or not the associated $L^2(\mathbb{R})$-spectrum consists of finitely many spectral bands (cf. Gesztesy (2001)). We hope to return to this topic in a later book project.

For the origins of the Hamiltonian KdV formalism presented at the end of Section 1.5, the interested reader might consult Gardner (1971) and Zakharov and Faddeev (1971). Our presentation follows Dickey (1991, Ch. 12), Gel'fand and Dikii (1975), Lax (1975; 1976; 1978; 1996), Palais (1997), and Sattinger and Weaver (1986). Related material can be found, for instance, in Dubrovin (1975b), Dubrovin et al. (1976), Flaschka (1975b), Gel'fand and Dikii (1979), McKean and van Moerbeke (1975), Novikov (1974; 1980), Novikov et al. (1984, Chs. I, II), and Serre (1981). The remarkable identity (1.275) is derived in Dickey (1991, Sec. 12.1), Gel'fand and Dikii (1975), and Yusin (1978), using a somewhat different approach based on formal power series. In addition to identity (1.266), we also mention the closely related identity (cf. Carmona and Lacroix (1990, p. 369)),

$$\begin{aligned}-\partial_z\big(g(z,x)^{-1}\big) = 2g(z,x) - \partial_x\bigg(g(z,x)\bigg(&f_+(z,x)^{-2}\int_x^\infty dx'\, f_+(z,x')^2 \\ &- f_-(z,x)^{-2}\int_{-\infty}^x dx'\, f_-(z,x')^2\bigg)\bigg), \quad z\in\mathbb{C}\setminus\mathbb{R}, \quad (1.299)\end{aligned}$$

where $f_\pm(z,x)$ denote the Jost (Weyl–Titchmarsh) solutions associated with H, that is,

$$f_\pm(z,\,\cdot\,)\in L^2((R,\pm\infty)), \quad z\in\mathbb{C}\setminus\mathbb{R},\ R\in\mathbb{R}.$$

2

The Combined sine-Gordon and Modified KdV Hierarchy

Someone told me that each equation I included in the book would halve the sales. I therefore resolved not to have any equations at all.

Stephen W. Hawking[1]

2.1 Contents

The sine-Gordon (sG) equation,

$$u_{xt} - \sin(u) = 0,$$

for a function $u = u(x, t)$, which has its origins in the 19th-century analysis of surfaces of constant negative curvature, has a long and interesting history.[2] However, it got its name as a pun on the well-known Klein–Gordon equation when it became a prominent model in elementary particle and condensed matter physics. This chapter focuses on a relatively recent development since the mid-1970s – the construction of algebro-geometric solutions of a combined sine-Gordon and modified Korteweg–de Vries (sGmKdV) hierarchy whose first nontrivial element reads

$$u_{xt} + (i/8)\big(u_x^3 + 2u_{xxx}\big)_x - \sin(u) = 0.$$

Below we briefly summarize the principal content of each section. A more detailed discussion, using the KdV hierarchy as a model, has been provided in the introduction to this volume.

Section 2.2.

- polynomial recursion formalism, zero-curvature pairs (U, V_n)
- stationary and time-dependent sGmKdV hierarchy
- hyperelliptic curve $\mathcal{K}_n$

[1] *A Brief History of Time*, Bantam Books, Toronto, 1988, p. vi.
[2] A guide to the literature can be found in the detailed notes at the end of this chapter.

Section 2.3. (stationary)

- properties of ϕ and the Baker–Akhiezer vector Ψ
- Dubrovin equations for auxiliary divisors
- trace formulas for u
- theta function representations for ϕ, ψ, and u
- the algebro-geometric initial value problem

Section 2.4. (time-dependent)

- properties of ϕ and the Baker–Akhiezer vector Ψ
- Dubrovin equations for auxiliary divisors
- trace formulas for u
- theta function representations for ϕ, Ψ, and u
- the algebro-geometric initial value problem

This chapter relies on terminology and notions developed in connection with compact Riemann surfaces. A brief summary of key results as well as definitions of some of the main quantities can be found in Appendices A, B, and F.

2.2 The sGmKdV Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section we provide the construction of a hierarchy of integrable equations, the sGmKdV hierarchy, which combines the sine-Gordon (sG) equation and the modified Korteweg–de Vries (mKdV) hierarchy. Using a polynomial recursion formalism, we derive the corresponding sequence of zero-curvature pairs and introduce the underlying hyperelliptic curve in connection with the stationary sGmKdV hierarchy.

Throughout this section we suppose the following hypothesis.

Hypothesis 2.1 *In the stationary case we assume*[1] *that*

$$u \in C^\infty(\mathbb{R}). \tag{2.1}$$

In the time-dependent case we suppose[2]

$$u(\,\cdot\,,t) \in C^\infty(\mathbb{R}),\ t \in \mathbb{R}, \quad u_x(x,\,\cdot\,) \in C^1(\mathbb{R}),\ x \in \mathbb{R}. \tag{2.2}$$

To set up a zero-curvature formalism for the combined sine-Gordon and mKdV equations, one can proceed as follows. One defines recursion relations for $\{f_\ell\}_{\ell\in\mathbb{N}_0}$,

[1] Alternatively, we could suppose that $u\colon \mathbb{C} \to \mathbb{C}_\infty$ is meromorphic.

[2] Again one could assume that for fixed $t \in \mathbb{R}$, $u(\,\cdot\,,t)$ is meromorphic, etc.

$\{g_\ell\}_{\ell\in\mathbb{N}_0}$, and $\{h_\ell\}_{\ell\in\mathbb{N}_0}$ by

$$f_0 = h_0 = 1. \tag{2.3}$$

$$g_\ell = (i/2)(f_{\ell,x} + iu_x f_\ell), \quad \ell \in \mathbb{N}_0, \tag{2.4}$$

$$g_\ell = (i/2)(-h_{\ell,x} + iu_x h_\ell), \quad \ell \in \mathbb{N}_0, \tag{2.5}$$

$$g_{\ell,x} = i(h_{\ell+1} - f_{\ell+1}), \quad \ell \in \mathbb{N}_0. \tag{2.6}$$

From (2.4) (or (2.5)) and (2.3), one immediately infers that $g_0 = -u_x/2$. To show that one can indeed solve this recursion for all $\ell \in \mathbb{N}_0$, one first subtracts equations (2.4) and (2.5) and then uses (2.6) to obtain

$$f_\ell + h_\ell = \int^x dx\, u_x g_{\ell-1,x}, \quad \ell \in \mathbb{N}. \tag{2.7}$$

By adding and subtracting equations (2.6) and (2.7), one finds

$$f_\ell = \frac{1}{2}\int^x dx\, u_x g_{\ell-1,x} + \frac{i}{2} g_{\ell-1,x}, \tag{2.8}$$

$$h_\ell = \frac{1}{2}\int^x dx\, u_x g_{\ell-1,x} + \frac{i}{2} g_{\ell-1,x}, \quad \ell \in \mathbb{N}. \tag{2.9}$$

Inserting any of these expressions into (2.4), one obtains

$$g_\ell = -\frac{1}{4}\bigg(g_{\ell-1,xx} + u_x \int^x dx\, u_x g_{\ell-1,x}\bigg), \quad \ell \in \mathbb{N}. \tag{2.10}$$

This proves that the set of recursion relations (2.3)–(2.6) can be solved.

Remark 2.2 One can show that the quantities f_ℓ, g_ℓ, and h_ℓ, $\ell \in \mathbb{N}_0$, are all differential polynomials in u (i.e., polynomials in u and (some of) its x-derivatives). Starting from (2.8), one infers that

$$\begin{aligned} f_{\ell,x} &= (1/2)u_x g_{\ell-1,x} + (i/2)g_{\ell-1,xx} = (i/2)(\partial_x - iu_x)g_{\ell-1,x} \\ &= (i/2)(\partial_x - iu_x)\partial_x\big((i/2)(\partial + iu_x)\big)f_{\ell-1} \\ &= -(1/4)f_{\ell-1,xxx} + w_+ f_{\ell-1,x} + (1/2)w_{+,x} f_{\ell-1}, \end{aligned} \tag{2.11}$$

where we used (2.4) and introduced

$$w_+ = -((u_x)^2 + 2iu_{xx})/4. \tag{2.12}$$

However, the recursion (2.11) is nothing but the KdV recursion (1.4) with the KdV potential u replaced by w_+. Thus, f_ℓ, $\ell \in \mathbb{N}_0$ are differential polynomials in u (see Remark 1.2). Similarly, one finds that h_ℓ in (2.9) satisfies

$$h_{\ell,x} = -(1/4)h_{\ell-1,xxx} + w_- h_{\ell-1,x} + (1/2)w_{-,x} h_{\ell-1}, \tag{2.13}$$

with

$$w_- = -((u_x)^2 - 2iu_{xx})/4, \tag{2.14}$$

which is again the KdV recursion but this time with potential w_-. Finally, by (2.4), g_ℓ are obviously differential polynomials in u since f_ℓ are.

Explicitly, one computes

$$\begin{aligned}
f_0 &= 1,\\
f_1 &= \tfrac{1}{2}w_+ + c_1 = -\tfrac{1}{8}\big(u_x^2 + 2iu_{xx}\big) + c_1,\\
f_2 &= -\tfrac{1}{8}w_{+,xx} + \tfrac{3}{8}w_+^2 + c_1\tfrac{1}{2}w_+ + c_2\\
&= -\tfrac{1}{32}u_{xx}^2 + \tfrac{1}{16}u_xu_{xxx} + \tfrac{i}{16}u_{xxxx} + \tfrac{3}{128}u_x^4 + \tfrac{3i}{32}u_x^2u_{xx}\\
&\quad - \tfrac{c_1}{8}\big(u_x^2 + 2iu_{xx}\big) + c_2,\\
f_3 &= \tfrac{1}{32}w_{+,xxxx} - \tfrac{5}{16}w_+w_{+,xx} - \tfrac{5}{32}w_{+,x}^2 + \tfrac{5}{16}w_+^3 + c_1\big(-\tfrac{1}{8}w_{+,xx} + \tfrac{3}{8}w_+^2\big)\\
&\quad + c_2\tfrac{1}{2}w_+ + c_3, \text{ etc.},\\
g_0 &= -\tfrac{1}{2}u_x,\\
g_1 &= \tfrac{1}{16}u_x^3 + \tfrac{1}{8}u_{xxx} - \tfrac{c_1}{2}u_x,\\
g_2 &= -\tfrac{3}{256}u_x^5 - \tfrac{1}{32}u_{xxxxx} - \tfrac{i}{32}u_{xx}u_{xxx} - \tfrac{5}{64}u_x^2u_{xxx} - \tfrac{5}{64}u_xu_{xx}^2\\
&\quad + c_1\big(\tfrac{1}{16}u_x^3 + \tfrac{1}{8}u_{xxx}\big) - \tfrac{c_2}{2}u_x, \text{ etc.},\\
h_0 &= 1,\\
h_1 &= \tfrac{1}{2}w_- + c_1 = -\tfrac{1}{8}\big(u_x^2 - 2iu_{xx}\big) + c_1,\\
h_2 &= -\tfrac{1}{8}w_{-,xx} + \tfrac{3}{8}w_-^2 + c_1\tfrac{1}{2}w_- + c_2\\
&= -\tfrac{1}{32}u_{xx}^2 + \tfrac{1}{16}u_xu_{xxx} - \tfrac{i}{16}u_{xxxx} + \tfrac{3}{128}u_x^4 - \tfrac{3i}{32}u_x^2u_{xx}\\
&\quad - \tfrac{c_1}{8}\big(u_x^2 - 2iu_{xx}\big) + c_2,\\
h_3 &= \tfrac{1}{32}w_{-,xxxx} - \tfrac{5}{16}w_-w_{-,xx} - \tfrac{5}{32}w_{-,x}^2 + \tfrac{5}{16}w_-^3 + c_1\big(-\tfrac{1}{8}w_{-,xx} + \tfrac{3}{8}w_-^2\big)\\
&\quad + c_2\tfrac{1}{2}w_- + c_3, \text{ etc.}
\end{aligned}$$

Here $\{c_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}$ denote integration constants. For subsequent purposes, it is convenient also to introduce the corresponding homogeneous coefficients $\hat{f}_\ell$, $\hat{g}_\ell$, and $\hat{h}_\ell$ defined by the vanishing of the integration constants c_k, for $k = 1, \dots, \ell$,

$$\hat{f}_0 = f_0 = 1, \quad \hat{f}_\ell = f_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{2.15}$$

$$\hat{g}_0 = g_0 = -\tfrac{1}{2}u_x, \quad \hat{g}_\ell = g_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{2.16}$$

$$\hat{h}_0 = h_0 = 1, \quad \hat{h}_\ell = h_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{2.17}$$

Hence,

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{f}_k, \quad g_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{g}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{h}_k, \quad \ell \in \mathbb{N}_0,$$

introducing

$$c_0 = 1.$$

Next, we establish the zero-curvature formalism for the sGmKdV hierarchy. Given Hypothesis 2.1, one introduces the 2×2 matrix U by

$$U(z) = -i \begin{pmatrix} \frac{1}{2}u_x & 1 \\ z & -\frac{1}{2}u_x \end{pmatrix}, \tag{2.18}$$

and for each $n \in \mathbb{N}_0$ the following 2×2 matrix V_n by

$$V_n(z) = \begin{pmatrix} -G_{n-1}(z) & (1/z)F_n(z) \\ H_n(z) & G_{n-1}(z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{0\}, n \in \mathbb{N}_0, \tag{2.19}$$

assuming F_n, H_n, and G_{n-1} to be polynomials of degree n and $n-1$ with C^∞ (or meromorphic) coefficients with respect to x. Postulating the stationary zero-curvature condition

$$-V_{n,x} + [U, V_n] = 0, \tag{2.20}$$

equation (2.20) yields the following fundamental relationships between the polynomials F_n, H_n, and G_{n-1},

$$F_{n,x} = -iu_x F_n - 2izG_{n-1}, \tag{2.21}$$

$$H_{n,x} = iu_x(x)H_n + 2izG_{n-1}, \tag{2.22}$$

$$G_{n-1,x} = i(H_n - F_n). \tag{2.23}$$

From (2.21)–(2.23) one infers that

$$\partial_x \det(V_n(z,x)) = -(1/z)\partial_x\big(zG_{n-1}(z,x)^2 + F_n(z,x)H_n(z,x)\big) = 0,$$

and hence

$$zG_{n-1}^2 + F_nH_n = Q_{2n}, \tag{2.24}$$

where the monic polynomial Q_{2n} of degree $2n$ is x-independent. It turns out that it is more convenient to define

$$R_{2n+1}(z) = zQ_{2n}(z) = \prod_{m=0}^{2n}(z - E_m), \quad E_0 = 0,\ E_1, \dots, E_{2n} \in \mathbb{C} \tag{2.25}$$

so that (2.24) becomes

$$z^2G_{n-1}^2 + zF_nH_n = R_{2n+1}. \tag{2.26}$$

Moreover, computing the characteristic equation[1] of $i V_n$

$$\det(wI_2 - iV_n(z)) = w^2 - \det(V_n(z)) = w^2 + G_{n-1}(z)^2 + z^{-1}F_n(z)H_n(z)$$
$$= w^2 + z^{-2}R_{2n+1}(z) = 0, \tag{2.27}$$

one is naturally led to introduce the hyperelliptic curve $\mathcal{K}_n$ of arithmetic genus n (possibly with a singular affine part) defined by

$$\mathcal{K}_n\colon \mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0. \tag{2.28}$$

To establish the connection between the zero-curvature formalism and the recursion relations (2.3)–(2.6), we now make the following polynomial ansatz with respect to the spectral parameter z,

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell, \quad H_n(z) = \sum_{\ell=0}^{n} h_{n-\ell} z^\ell, \tag{2.29}$$

$$G_{n-1}(z) = \sum_{\ell=0}^{n-1} g_{n-1-\ell} z^\ell, \quad G_{-1}(z) = 0. \tag{2.30}$$

Insertion of (2.29) and (2.30) into (2.21)–(2.23) then yields the recursion relation (2.3)–(2.6) for $\ell = 0, \dots, n-1$ (assuming $n \in \mathbb{N}$ to avoid cumbersome case distinctions in connection with the trivial case $n = 0$, which can easily be handled directly, cf. (2.37)) as well as

$$f_{n,x} = -iu_x f_n, \tag{2.31}$$
$$h_{n,x} = iu_x h_n \tag{2.32}$$

and the constraint

$$f_n h_n = Q_{2n}(0) = \prod_{m=1}^{2n} E_m, \tag{2.33}$$

taking $z = 0$ in (2.24). Thus, one obtains

$$f_n = \alpha e^{-iu}, \quad \alpha \in \mathbb{C}, \tag{2.34}$$
$$h_n = \beta e^{iu}, \quad \beta \in \mathbb{C}. \tag{2.35}$$

The corresponding homogeneous coefficients $\hat{f}_n$ and $\hat{h}_n$ are then also defined by

$$\hat{f}_n = f_n = \alpha e^{-iu}, \quad \hat{h}_n = h_n = \beta e^{iu}. \tag{2.36}$$

The trivial case $n = 0$ leads to

$$\hat{f}_0 = f_0 = \alpha e^{-iu}, \quad g_{-1} = 0, \quad \hat{h}_0 = h_0 = \beta e^{iu}. \tag{2.37}$$

[1] I_2 denotes the identity matrix in $\mathbb{C}^2$.

The last equation in (2.6) (i.e., the case $\ell = n-1$) then defines the nth stationary sGmKdV equation[1]

$$\text{s-sGmKdV}_n(u) = 2ig_{n-1,x}(u) + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \quad n \in \mathbb{N}_0 \tag{2.38}$$

subject to the constraint (cf. (2.33))

$$\alpha\beta = Q_{2n}(0) = \prod_{m=1}^{2n} E_m. \tag{2.39}$$

Varying $n \in \mathbb{N}_0$ in (2.38) then defines the stationary sGmKdV hierarchy. We record the first few equations explicitly,

$$\begin{aligned}
&\text{s-sGmKdV}_0(u) = 2(\beta e^{iu} - \alpha e^{-iu}) = 0,\\
&\text{s-sGmKdV}_1(u) = -iu_{xx} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0,\\
&\text{s-sGmKdV}_2(u) = (i/8)\big(u_x^3 + 2u_{xxx}\big)_x - c_1 iu_{xx} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \text{ etc.}
\end{aligned} \tag{2.40}$$

In particular, for $\alpha = \beta \neq 0$, the first equation in (2.40) yields the stationary sine-Gordon equation (in light-cone coordinates), that is,

$$\sin(u) = 0.$$

In the special case $\alpha = \beta = 0$, one obtains $f_n = h_n = 0$, and hence the $(n-1)$th stationary KdV equation is satisfied for the potential $w_\pm$. Introducing

$$v = -(i/2)u_x,$$

we see that $w_\pm$ and v are related by the Miura transformation

$$w_\pm = v^2 \pm v_x,$$

and we may conclude that $g_{n-1,x} = 0, n \in \mathbb{N}$, equals the $(n-1)$th stationary mKdV equation with solution v.

By definition, the set of solutions of (2.38), with n ranging in $\mathbb{N}_0$ and c_ℓ in $\mathbb{C}$, $\ell \in \mathbb{N}$, represents the class of algebro-geometric sGmKdV solutions. If u satisfies one of the stationary sGmKdV equations in (2.38) for a particular value of n, then it satisfies infinitely many such equations of order higher than n for certain choices of integration constants c_ℓ (one can follow the argument in Remark 1.5). At times it will be convenient to abbreviate algebro-geometric stationary sGmKdV solutions u simply as sGmKdV *potentials*.

In the following we will frequently assume that u satisfies the nth stationary sGmKdV equation. By this we mean it satisfies one of the nth stationary sGmKdV equations after a particular choice of integration constants $c_\ell \in \mathbb{C}$, $\ell = 1, \dots, n-1$, $n \geq 2$, has been made.

[1] In a slight abuse of notation we will occasionally stress the functional dependence of f_ℓ, g_ℓ, h_ℓ on u, writing $f_\ell(u), g_\ell(u), h_\ell(u)$, etc.

For subsequent purposes we also introduce the corresponding homogeneous polynomials $\widehat{F}_\ell$, $\widehat{G}_{\ell-1}$, and $\widehat{H}_\ell$ defined by

$$\begin{aligned}
&\widehat{F}_\ell(z) = F_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{f}_{\ell-k} z^k, \quad \ell = 0,\dots,n-1,\\
&\widehat{F}_n(z) = \alpha e^{-iu} + \sum_{k=1}^{n} \hat{f}_{n-k} z^k = \alpha e^{-iu} + z\widehat{F}_{n-1}(z),
\end{aligned} \tag{2.41}$$

$$\begin{aligned}
&\widehat{G}_{-1}(z) = G_{-1}(z) = 0,\\
&\widehat{G}_{\ell-1}(z) = G_{\ell-1}(z)\big|_{c_k=0,\, k=1,\dots,\ell-1} = \sum_{k=0}^{\ell-1} \hat{g}_{\ell-1-k} z^k, \quad \ell = 0,\dots,n,
\end{aligned} \tag{2.42}$$

$$\begin{aligned}
&\widehat{H}_\ell(z) = H_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{h}_{\ell-k} z^k, \quad \ell = 0,\dots,n-1,\\
&\widehat{H}_n(z) = \beta e^{iu} + \sum_{k=1}^{n} \hat{h}_{n-k} z^k = \beta e^{iu} + z\widehat{H}_{n-1}(z).
\end{aligned} \tag{2.43}$$

In accordance with our notation introduced in (2.15)–(2.17) and (2.41)–(2.43), the corresponding homogeneous stationary sGmKdV equations are then defined by

$$\begin{aligned}
&\text{s-}\widehat{\text{sGmKdV}}_0(u) = 2(\beta e^{iu} - \alpha e^{-iu}) = 0,\\
&\text{s-}\widehat{\text{sGmKdV}}_n(u) = \text{s-sGmKdV}_n(u)\big|_{c_\ell=0,\, \ell=1,\dots,n-1} = 0, \quad n \in \mathbb{N}.
\end{aligned}$$

Using equations (2.21)–(2.23) one can derive individual differential equations for F_n and H_n as follows. From (2.21) and (2.23), one infers

$$F_{n,xx} = -iu_{xx}F_n + 2z(H_n - F_n) - u_x^2 F_n - 2zu_x G_{n-1}. \tag{2.44}$$

Multiplying (2.44) by F_n, one can eliminate G_{n-1} to find

$$F_n F_{n,xx} - (1/2)F_{n,x}^2 + \big(2z + (1/2)u_x^2 + iu_{xx}\big)F_n^2 = 2R_{2n+1}, \tag{2.45}$$

and differentiating with respect to x finally yields

$$F_{n,xxx} + \big(4z + u_x^2 + 2iu_{xx}\big)F_{n,x} + (u_x u_{xx} + iu_{xxx})F_n = 0. \tag{2.46}$$

A similar analysis for H_n results in

$$H_n H_{n,xx} - (1/2)H_{n,x}^2 + \big(2z + (1/2)u_x^2 - iu_{xx}\big)H_n^2 = 2R_{2n+1} \tag{2.47}$$

and

$$H_{n,xxx} + \big(4z + u_x^2 - 2iu_{xx}\big)H_{n,x} + (u_x u_{xx} - iu_{xxx})H_n = 0. \tag{2.48}$$

Recalling (2.12), (2.14), one observes that equations (2.46) and (2.48) take on the form

$$-(1/4)F_{n,xxx} + (w_+ - z)F_{n,x} + (1/2)w_{+,x}F_n = 0, \tag{2.49}$$

$$-(1/4H_{n,xxx} + (w_- - z)H_{n,x} + (1/2)w_{-,x}H_n = 0, \tag{2.50}$$

which are identical to the corresponding equations for the KdV hierarchy (see (1.4) and footnote on this page) with KdV potential $u = w_\pm$. Analogous assertions apply to (2.45) and (2.47).

Equations (2.45) and (2.47) can be used to derive nonlinear recursion relations for the homogeneous coefficients $\hat{f}_\ell$ and $\hat{h}_\ell$ (i.e., the ones satisfying (2.15), (2.17) in the case of vanishing integration constants) as proved in Theorem D.1 inAppendix D. In addition, as proven in Theorem D.1, (2.45) leads to an explicit determination of the integration constants $c_1, \dots, c_n$ in

$$\text{s-sGmKdV}_n(u) = 2ig_{n-1,x}(u) + 2(\beta e^{iu} - \alpha e^{-iu}) = 0$$

in terms of the zeros $E_0 = 0, E_1, \dots, E_{2n}$ of the associated polynomial R_{2n+1} in (2.25). In fact, one can prove (cf. (D.9))

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n-1, \tag{2.51}$$

where

$$c_0(\underline{E}) = 1,$$

$$c_k(\underline{E}) = \sum_{\substack{j_1,\dots,j_{2n}=0 \\ j_1+\dots+j_{2n}=k}}^{k} \frac{(2j_1)!\cdots(2j_{2n})!}{2^{2k}(j_1!)^2\cdots(j_{2n}!)^2(2j_1-1)\cdots(2j_{2n}-1)} E_1^{j_1}\cdots E_{2n}^{j_{2n}},$$

$$k = 1, \dots, n-1. \tag{2.52}$$

Finally, we turn to the time-dependent sGmKdV hierarchy. Introducing a deformation parameter $t_n \in \mathbb{R}$ into u (i.e., replacing $u(x)$ by $u(x, t_n)$), the definitions (2.18), (2.19), and (2.29), (2.30) of U, V_n, and F_n and H_n, respectively, still apply. The corresponding zero-curvature relation reads

$$U_{t_n} - V_{n,x} + [U, V_n] = 0, \quad n \in \mathbb{N}_0,$$

which results in the following set of equations

$$u_{xt_n} = -2iG_{n-1,x} - 2(H_n - F_n), \tag{2.53}$$

$$F_{n,x} = -iu_xF_n - 2izG_{n-1}, \tag{2.54}$$

$$H_{n,x} = iu_xH_n + 2izG_{n-1}. \tag{2.55}$$

Inserting the polynomial expressions for F_n, H_n, and G_{n-1} into (2.54) and (2.55), respectively, first yields

$$f_{n,x} = -iu_xf_n, \quad h_{n,x} = iu_xh_n, \quad n \in \mathbb{N}_0.$$

In the general case we find

$$\begin{aligned}
&f_0 = 1, \quad n \in \mathbb{N}, \\
&f_{\ell,x} = -(1/4) f_{\ell-1,xxx} + w_+ f_{\ell-1,x} + (1/2) w_{+,x} f_{\ell-1}, \quad \ell = 1, \dots, n-1, \ n \geq 2,
\end{aligned}$$

and

$$\begin{aligned}
&h_0 = 1, \quad n \in \mathbb{N}, \\
&h_{\ell,x} = -(1/4) h_{\ell-1,xxx} + w_- h_{\ell-1,x} + (1/2) w_{-,x} h_{\ell-1}, \quad \ell = 1, \dots, n-1, \ n \geq 2,
\end{aligned}$$

and (recalling our convention $g_{-1} = 0$, cf. (2.42))

$$u_{xt_n} = -2i g_{n-1,x} - 2(h_n - f_n), \quad n \in \mathbb{N}_0,$$

in addition to equations (2.4), (2.5), (2.34), (2.35), and (2.6) (the latter equation for $\ell = 0, \dots, n-2$ and only for $n \geq 2$). Varying $n \in \mathbb{N}_0$ then defines the time-dependent sGmKdV hierarchy by

$$\begin{aligned}
\mathrm{sGmKdV}_n(u) = u_{xt_n} + 2i g_{n-1,x}(u) + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \qquad (2.56) \\
(x, t_n) \in \mathbb{R}^2, \ n \in \mathbb{N}_0.
\end{aligned}$$

Explicitly, the first few equations read

$$\begin{aligned}
&\mathrm{sGmKdV}_0(u) = u_{xt_0} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \\
&\mathrm{sGmKdV}_1(u) = u_{xt_1} - i u_{xx} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \qquad (2.57) \\
&\mathrm{sGmKdV}_2(u) = u_{xt_2} + \tfrac{i}{8}\big(u_x^3 + 2u_{xxx}\big)_x - c_1 i u_{xx} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \text{ etc.}
\end{aligned}$$

Similarly, one introduces the corresponding homogeneous sGmKdV hierarchy by

$$\begin{aligned}
&\mathrm{s}\widehat{\mathrm{GmKd}}\mathrm{V}_0(u) = u_{xt_0} + 2(\beta e^{iu} - \alpha e^{-iu}) = 0, \\
&\mathrm{s}\widehat{\mathrm{GmKd}}\mathrm{V}_n(u) = \mathrm{sGmKdV}_n(u)\big|_{c_\ell = 0,\, \ell = 0, \dots, n-1} = 0, \quad n \in \mathbb{N}.
\end{aligned}$$

In contrast to the stationary case, the constraint (2.39) does not apply in the t_n-dependent context (2.56) (since the left-hand side of (2.53) is nonvanishing).

Remark 2.3 Choosing $\alpha = \beta = i/4$, $\mathrm{sGmKdV}_0(u) = \mathrm{sG}(u) = 0$ is the sine-Gordon equation in light-cone coordinates,

$$u_{xt_0} = \sin(u). \qquad (2.58)$$

In general, by introducing the scaled function

$$v(x, t_0) = u((i/2)(\alpha\beta)^{-1/2} x, t_0) - (i/2) \ln(\beta/\alpha), \quad \alpha\beta \in \mathbb{C} \setminus \{0\}$$

the equation $\mathrm{sGmKdV}_0(u) = 0$ is equivalent to

$$v_{xt_0} = \sin(v).$$

Introducing $v = iu$ and rewriting $\mathrm{sGmKdV}_0(u) = 0$ in (2.57) in terms of v, setting $\alpha = \beta = i/4$, one finds

$$v_{xt_0} = \sinh(v),$$

that is, the sinh-Gordon equation. Similarly, introducing $\xi = x + t_0$, $\eta = x - t_0$ produces the (hyperbolic) sine-Gordon and (hyperbolic) sinh-Gordon equations in laboratory coordinates

$$u_{\xi\xi} - u_{\eta\eta} = \sin(u), \quad v_{\xi\xi} - v_{\eta\eta} = \sinh(v),$$

and $\xi = x + t_0$, $\tau = i(x - t_0)$ produces the elliptic sine-Gordon and elliptic sinh-Gordon equations

$$u_{\xi\xi} + u_{\tau\tau} = \sin(u), \quad v_{\xi\xi} + v_{\tau\tau} = \sinh(v).$$

Moreover, the case $\alpha = 0$ yields

$$u_{xt_0} = -2\beta e^{iu}$$

and a similar equation in the case $\beta = 0$. Hence, writing $v = iu$ and changing coordinates $(x, t_0) \to (\xi, \tau)$ yield the Liouville hierarchy for v starting with

$$v_{\xi\xi} + v_{\tau\tau} = 2i\beta e^{-v}. \tag{2.59}$$

In particular, the results in Sections 2.3 and 2.4 extend to these hierarchies, and hence we omit further distinctions and focus on the generalized sGmKdV hierarchy in light-cone coordinates for the rest of this chapter. Finally, in the case $\alpha = \beta = 0$, we define

$$v(x, t_n) = -(i/2)u_x(x, it_n),$$

and the sG hierarchy reduces to

$$v_{t_n} - ig_{n-1,x} = 0, \quad n \in \mathbb{N},$$

which equals the $(n - 1)$th modified KdV equation in terms of v.

Remark 2.4 The relation between the KdV and the modified KdV hierarchy on the one side and the local sGmKdV hierarchy on the other side alluded to in equations (2.12), (2.14), (2.49), and (2.50) can be made more precise as follows. The equation

$$\Psi_x = U\Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

is equivalent to

$$\psi_{1,xx} = (w_+ - z)\psi_1, \quad \psi_{2,xx} = (w_- - z)\psi_2$$

with

$$w_{\pm} = v^2 \pm v_x, \quad v = -(i/2)u_x. \tag{2.60}$$

Equations (2.60) represent the familiar Miura transformation between solutions $w_{\pm}$ of the KdV hierarchy and v of the modified KdV hierarchy. Since $\mathrm{sGmKdV}_n(u) = 0$ reduces to $\mathrm{mKdV}_n(v) = 0$, $n \in \mathbb{N}$ for $\alpha = \beta = 0$, as discussed in Remark 2.3, $\mathrm{sGmKdV}_n(u) = 0$ for general $\alpha, \beta \in \mathbb{C}$ represents a combination of the $\mathrm{sGmKdV}_0(u) = 0$ equation and the $\mathrm{mKdV}_{n-1}(u) = 0$ equation. This corresponds to our choice (2.19) of $zV_n(z)$ being a polynomial in z. The usually considered nonlocal sG hierarchy then corresponds to a choice of $zV_n(z)$ rational in z with a nontrivial principal part.

2.3 The Stationary sGmKdV Formalism

This section is devoted to a detailed study of the stationary sGmKdV hierarchy and its algebro-geometric solutions. Our principal tools are derived from combining the polynomial recursion formalism introduced in Section 2.2 and a fundamental meromorphic function ϕ on a hyperelliptic curve $\mathcal{K}_n$. With the help of ϕ we study the Baker–Akhiezer vector Ψ, Dubrovin-type equations governing the motion of auxiliary divisors on $\mathcal{K}_n$, trace formulas, and theta function representations of ϕ, Ψ, and u. We also discuss the algebro-geometric initial value problem of constructing u from the Dubrovin equations and auxiliary divisors as initial data.

For major parts of this section we suppose

$$u \in C^{\infty}(\mathbb{R}) \tag{2.61}$$

(which could be replaced by $u\colon \mathbb{C} \to \mathbb{C}_{\infty}$ meromorphic) and assume (2.3)–(2.6), (2.29), (2.30), (2.38), (2.39) (respectively, (2.21)–(2.23)), and freely employ the formalism developed in (2.18)–(2.50), keeping $n \in \mathbb{N}_0$ fixed.

We recall the hyperelliptic curve

$$\begin{aligned} &\mathcal{K}_n\colon \mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0, \\ &R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad E_0 = 0,\ E_1, \dots, E_{2n} \in \mathbb{C}, \end{aligned} \tag{2.62}$$

as introduced in (2.28). The curve $\mathcal{K}_n$ is compactified by joining the point P_{∞}, but for notational simplicity the compactification is also denoted by $\mathcal{K}_n$. Points P on $\mathcal{K}_n \setminus \{P_{\infty}\}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = 0$. The complex structure on $\mathcal{K}_n$ is then defined in the usual way (see Appendix B). Hence, $\mathcal{K}_n$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_n$ (i.e., by fixing $E_0 = 0, E_1, \dots, E_{2n}$), the integration constants $c_1, \dots, c_{n-1}$ in $g_{n-1,x}$ (and hence in the corresponding stationary sGmKdV$_n$ equation) are uniquely determined, as is clear from (2.51), (2.52), which establish the integration constants c_ℓ as symmetric functions of $E_1, \dots, E_{2n}$.

To avoid numerous case distinctions in connection with the case $n = 0$, we will assume $n \in \mathbb{N}$ for the remainder of this section. (The trivial case $n = 0$ is explicitly treated in Example 2.13.) Moreover, to simplify our presentation in the following, we will subsequently focus on sG-type equations and hence assume

$$\alpha,\ \beta \in \mathbb{C} \setminus \{0\}.$$

By (2.39) this is equivalent to

$$Q_{2n}(0) = \prod_{m=1}^{2n} E_m = \alpha\beta \neq 0.$$

Hence, from this point on, we suppose

$$E_0 = 0,\ \ E_m \in \mathbb{C} \setminus \{0\},\ \ m = 1, \dots, 2n. \tag{2.63}$$

In the following, the roots of the polynomials F_n and H_n will play a special role, and hence we introduce

$$F_n(z) = \prod_{j=1}^{n} (z - \mu_j), \quad H_n(z) = \prod_{j=1}^{n} (z - \nu_j). \tag{2.64}$$

Moreover, we introduce (lifting μ_j and ν_j to $\mathcal{K}_n$)

$$\hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x) G_{n-1}(\mu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n,\ x \in \mathbb{R}, \tag{2.65}$$

$$\hat{\nu}_j(x) = (\nu_j(x), \nu_j(x) G_{n-1}(\nu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n,\ x \in \mathbb{R} \tag{2.66}$$

and

$$P_0 = (0, 0).$$

Due to the $C^\infty(\mathbb{R})$ assumption (2.61) on u, $F_n(z, \cdot\,), H_n(z, \cdot\,) \in C^\infty(\mathbb{R})$ by (2.11), (2.13) and (2.29). Thus, one concludes

$$\mu_j, \nu_k \in C(\mathbb{R}),\ \ j, k = 1, \dots, n, \tag{2.67}$$

taking multiplicities (and appropriate renumbering) of the zeros of F_n and H_n into account. (Away from collisions of zeros, μ_j and ν_k are of course C^∞.) In addition, (2.34) and (2.35) imply

$$\mu_j(x), \nu_k(x) \neq 0,\ \ j, k = 1, \dots, n,\ \ x \in \mathbb{R},$$

(cf. also (2.91) and (2.92)).

Next, define the fundamental meromorphic function $\phi(\,\cdot\,, x)$ on $\mathcal{K}_n$ by

$$\phi(P, x) = \frac{y - zG_{n-1}(z, x)}{F_n(z, x)} \tag{2.68}$$

$$= \frac{zH_n(z, x)}{y + zG_{n-1}(z, x)}, \tag{2.69}$$

$$P = (z, y) \in \mathcal{K}_n,\; x \in \mathbb{R}$$

with divisor $(\phi(\,\cdot\,, x))$ of $\phi(\,\cdot\,, x)$ given by

$$(\phi(\,\cdot\,, x)) = \mathcal{D}_{P_0\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_\infty\underline{\hat{\mu}}(x)}, \tag{2.70}$$

using (2.64) and (2.67). Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n).$$

The stationary Baker–Akhiezer vector

$$\Psi(P, x, x_0) = \begin{pmatrix} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{pmatrix}, \quad P \in \mathcal{K}_n \setminus \{P_\infty\},\; (x, x_0) \in \mathbb{R}^2, \tag{2.71}$$

is defined by

$$\psi_1(P, x, x_0) = \exp\left(-(i/2)(u(x) - u(x_0)) + i\int_{x_0}^{x} dx'\, \phi(P, x')\right), \tag{2.72}$$

$$\psi_2(P, x, x_0) = -\psi_1(P, x, x_0)\phi(P, x). \tag{2.73}$$

We summarize the fundamental properties of ϕ and Ψ in the following lemma.

Lemma 2.5 *Suppose that $u \in C^\infty(\mathbb{R})$ satisfies the nth stationary sGmKdV equation* (2.38) *subject to the constraints* (2.39) *and* (2.63). *Moreover, let $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}$, $(x, x_0) \in \mathbb{R}^2$. Then ϕ satisfies the Riccati-type equation*

$$-i\phi_x(P) + \phi(P)^2 - u_x\phi(P) = z, \tag{2.74}$$

as well as

$$\phi(P)\phi(P^*) = -z\frac{H_n(z)}{F_n(z)}, \tag{2.75}$$

$$\phi(P) + \phi(P^*) = -2z\frac{G_{n-1}(z)}{F_n(z)}, \tag{2.76}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}, \tag{2.77}$$

whereas Ψ satisfies

$$\Psi_x(P) = U(z)\Psi(P), \tag{2.78}$$

$$-\,y\Psi(P) = zV_n(z)\Psi(P), \tag{2.79}$$

$$\psi_1(P,x,x_0) = \left(\frac{F_n(z,x)}{F_n(z,x_0)}\right)^{1/2} \exp\left(iy\int_{x_0}^{x} dx' F_n(z,x')^{-1}\right), \tag{2.80}$$

$$\psi_1(P,x,x_0)\psi_1(P^*,x,x_0) = \frac{F_n(z,x)}{F_n(z,x_0)}, \tag{2.81}$$

$$\psi_2(P,x,x_0)\psi_2(P^*,x,x_0) = -z\frac{H_n(z,x)}{F_n(z,x_0)}, \tag{2.82}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0) + \psi_1(P^*,x,x_0)\psi_2(P,x,x_0) = 2z\frac{G_{n-1}(z,x)}{F_n(z,x_0)}, \tag{2.83}$$

$$\psi_1(P,x,x_0)\psi_2(P^*,x,x_0) - \psi_1(P^*,x,x_0)\psi_2(P,x,x_0) = \frac{2y}{F_n(z,x_0)}. \tag{2.84}$$

In addition, as long as the zeros of $F_n(\,\cdot\,,x)$ are all simple for $x\in\Omega$, $\Omega\subseteq\mathbb{R}$ an open interval, $\Psi(\,\cdot\,,x,x_0)$ is meromorphic on $\mathcal{K}_n\setminus\{P_\infty\}$ for $x,x_0\in\Omega$.

Proof Equation (2.74) follows using the definition (2.68) of ϕ as well as relations (2.21)–(2.23). The other relations, (2.75)–(2.77), are easy consequences of $y(P^*)=-y(P)$, (2.68), and (2.69). By (2.72) and (2.73), Ψ is meromorphic on $\mathcal{K}_n\setminus\{P_\infty\}$ away from the poles $\hat\mu_j(x')$ of $\phi(\,\cdot\,,x')$. By (2.21), (2.65), and (2.68),

$$i\phi(P,x') \underset{P\to\hat\mu_j(x')}{=} \partial_{x'}\ln(F_n(z,x')) + O(1) \text{ as } z\to\mu_j(x'), \tag{2.85}$$

and hence ψ_1 is meromorphic on $\mathcal{K}_n\setminus\{P_\infty\}$ by (2.72) as long as the zeros of $F_n(\,\cdot\,,x)$ are all simple. This follows from (2.72) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j$ of $\{\hat\mu_j(x')\in\mathcal{K}_n \mid x'\in\Omega,\ x'\in[x_0,x]\}$ such that $\hat\mu_k(x')\notin\mathcal{U}_j$ for all $x'\in[x_0,x]$ and all $k\in\{1,\dots,n\}\setminus\{j\}$. Since ϕ is meromorphic on $\mathcal{K}_n$ by (2.68), ψ_2 is meromorphic on $\mathcal{K}_n\setminus\{P_\infty\}$ by (2.73). The remaining properties of Ψ can be verified by using the definition (2.71)–(2.73) as well as the relations (2.74)–(2.77). In particular, (2.80) follows by inserting the definition of ϕ, (2.68), into (2.72), using (2.21). $\square$

Equations (2.81)–(2.84) show that the basic identity (2.26), $z^2G_{n-1}^2+zF_nH_n=R_{2n+1}$, is equivalent to the elementary fact

$$(\psi_{1,+}\psi_{2,-}+\psi_{1,-}\psi_{2,+})^2 - 4\psi_{1,+}\psi_{1,-}\psi_{2,+}\psi_{2,-} = (\psi_{1,+}\psi_{2,-}-\psi_{1,-}\psi_{2,+})^2,$$

identifying $\psi_1(P)=\psi_{1,+}$, $\psi_1(P^*)=\psi_{1,-}$, $\psi_2(P)=\psi_{2,+}$, $\psi_2(P^*)=\psi_{2,-}$. This provides the intimate link between our approach and the squared function systems also employed in the literature in connection with algebro-geometric solutions of the sine-Gordon hierarchy.

Next, we derive Dubrovin-type equations, that is, first-order systems of nonlinear differential equations that govern the dynamics of μ_j and ν_j with respect to variations of x. Since in the remainder of this section we will frequently assume the affine part of $\mathcal{K}_n$ to be nonsingular, we list all restrictions on $\mathcal{K}_n$ in this case,

$$E_0 = 0, \; E_m \in \mathbb{C} \setminus \{0\}, \; E_m \neq E_{m'}, \; m, m' = 1, \dots, 2n. \tag{2.86}$$

Lemma 2.6 *Suppose that $u \in C^\infty(\widetilde{\Omega}_\mu)$ satisfies the nth stationary sGmKdV equation* (2.38) *on an open interval $\widetilde{\Omega}_\mu \subseteq \mathbb{R}$ subject to the constraint* (2.39). *Moreover, assume that the zeros μ_j, $j = 1, \dots, n$, of $F_n(\,\cdot\,)$ remain distinct on $\widetilde{\Omega}_\mu$. Then $\{\hat{\mu}_j\}_{j=1,\dots,n}$, defined by* (2.65)*, satisfies the following first-order system of differential equations on $\widetilde{\Omega}_\mu$*

$$\mu_{j,x} = -2iy(\hat{\mu}_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, n. \tag{2.87}$$

Next, assume the affine part of $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{2.88}$$

for some $x_0 \in \mathbb{R}$, where $\mu_j(x_0)$, $j = 1, \dots, n$, are assumed to be distinct. Then there exists an open interval $\Omega_\mu \subseteq \mathbb{R}$, with $x_0 \in \Omega_\mu$, such that the initial value problem (2.87), (2.88) *has a unique solution $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{2.89}$$

and μ_j, $j = 1, \dots, n$, remain distinct on Ω_μ.
For the zeros ν_j, $j = 1, \dots, n$, of $H_n(\,\cdot\,)$ identical statements hold with μ_j and Ω_μ replaced by ν_j and Ω_ν, etc. In particular, $\{\hat{\nu}_j\}_{j=1,\dots,n}$, defined by (2.66)*, satisfies the system*

$$\nu_{j,x} = -2iy(\hat{\nu}_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} (\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, n. \tag{2.90}$$

Proof It suffices to prove (2.87) and (2.89) since the proof of (2.90) is analogous to that of (2.87). Inserting $z = \mu_j$ into equation (2.21), one concludes from (2.65) that

$$F_{n,x}(\mu_j) = -\mu_{j,x} \prod_{\substack{k=1 \\ k \neq j}}^{n} (\mu_j - \mu_k) = -2i\mu_j G_{n-1}(\mu_j) = 2iy(\hat{\mu}_j),$$

proving (2.87). The smoothness assertion (2.89) is clear as long as $\hat{\mu}_j$ stays away from the branch points $(E_m, 0)$. Hence, we suppose

$$\mu_{j_0}(x) \to E_{m_0} \text{ as } x \to x_0 \in \Omega_\mu,$$

for some $j_0 \in \{1, \dots, n\}$, $m_0 \in \{0, \dots, 2n\}$. Introducing

$$\zeta_{j_0}(x) = \sigma(\mu_{j_0}(x) - E_{m_0})^{1/2}, \ \sigma = \pm 1, \quad \mu_{j_0}(x) = E_{m_0} + \zeta_{j_0}(x)^2$$

for x in an open interval centered around x_0, the Dubrovin equation (2.87) for μ_{j_0} becomes

$$\begin{aligned}\zeta_{j_0,x}(x) \underset{x\to x_0}{=} c(\sigma) &\left(\prod_{\substack{m=0\\m\neq m_0}}^{2n} (E_{m_0} - E_m)\right)^{1/2} \\ &\times \left(\prod_{\substack{k=1\\k\neq j_0}}^{n} \big(E_{m_0} - \mu_k(x)\big)^{-1}\right) \big(1 + O(\zeta_{j_0}(x)^2)\big)\end{aligned}$$

for some $|c(\sigma)| = 1$, and one concludes (2.89). $\square$

Combining the polynomial approach in Section 2.2 with (2.64) readily yields trace formulas for the sGmKdV invariants, that is, expressions of f_ℓ and h_ℓ in terms of symmetric functions of the zeros μ_j and ν_j of F_n and H_n, respectively. For simplicity, we just record the simplest case.

Lemma 2.7 *Suppose that $u \in C^\infty(\mathbb{R})$ satisfies the nth stationary sGmKdV equation* (2.38) *subject to the constraints* (2.39) *and* (2.63). *Then,*

$$u = i \ln\left((-1)^n \alpha^{-1} \prod_{j=1}^{n} \mu_j\right) \tag{2.91}$$

$$= -i \ln\left((-1)^n \beta^{-1} \prod_{j=1}^{n} \nu_j\right), \tag{2.92}$$

where $\alpha\beta = \prod_{m=1}^{2n} E_m \neq 0$. In particular, one infers the constraint

$$\prod_{j=1}^{n} \mu_j \nu_j = \prod_{m=1}^{2n} E_m. \tag{2.93}$$

Proof Equation (2.91) follows by considering the constant term in F_n in (2.29) combined with (2.34) and (2.64). Equation (2.92) can be deduced in a similar way by studying the polynomial H_n. Equation (2.93) follows upon exponentiating (2.91) and (2.92). $\square$

Remark 2.8 If $\alpha = 0$ (as in Liouville-type equations (2.59)), the fact that $f_n(x) = (-1)^n \prod_{j=1}^{n} \mu_j(x) = 0$ forces (at least) one $\hat{\mu}_{j_0}(x)$ to coincide with $P_0 = (0, 0)$ and hence to be x-independent. Similarly, if $\alpha = \beta = 0$, then $\hat{\mu}_{j_1}(x) = \hat{\nu}_{j_2}(x) = P_0$ for some $j_1, j_2 \in \{1, \dots, n\}$.

Now we turn to asymptotic properties of ϕ and ψ_j, $j = 1, 2$.

Lemma 2.9 *Suppose that $u \in C^\infty(\mathbb{R})$ satisfies the nth stationary sGmKdV equation* (2.38) *subject to the constraints* (2.39) *and* (2.63). *Moreover, let $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}$, $(x, x_0) \in \mathbb{R}^2$. Then,*

$$\phi(P) \underset{\zeta \to 0}{=} \zeta^{-1} + (1/2)u_x + \big((1/8)u_x^2 + (i/4)u_{xx}\big)\zeta + O(\zeta^2) \text{ as } P \to P_\infty, \tag{2.94}$$

$$\psi_1(P, x, x_0) \underset{\zeta \to 0}{=} \exp(i\zeta^{-1}(x - x_0))(1 + O(\zeta)) \text{ as } P \to P_\infty, \tag{2.95}$$

$$\psi_2(P, x, x_0) \underset{\zeta \to 0}{=} -\exp(i\zeta^{-1}(x - x_0))(\zeta^{-1} + O(1)) \text{ as } P \to P_\infty, \tag{2.96}$$

$$\zeta = \sigma/z^{1/2}, \ \sigma = \pm 1.$$

Proof The existence of the asymptotic expansion of ϕ in terms of the local coordinate $\zeta = \sigma/z^{1/2}$, $\sigma = \pm 1$ near P_∞ (cf. (B.7)–(B.11)) is clear from the explicit form of ϕ in (2.68). Insertion of the polynomial F_n into (2.68) then yields the explicit expansion coefficients in (2.94). Alternatively, one can insert the ansatz

$$\phi \underset{z \to \infty}{=} \phi_{-1} z^{1/2} + \phi_0 + \phi_1 z^{-1/2} + O(z^{-1}) \tag{2.97}$$

into the Riccati-type equation (2.74). A comparison of powers of $z^{-1/2}$ then proves (2.94). Equation (2.95) then follows from inserting (2.94) into (2.72) and (2.96) is clear from (2.73), (2.94), and (2.95). □

Next, we derive representations of ϕ, ψ, as well as u in terms of the Riemann theta function of $\mathcal{K}_n$, assuming the affine part of $\mathcal{K}_n$ to be nonsingular. We will freely use the notation established in Appendices A and C. To avoid the trivial case $n = 0$ (considered in Example 2.13), we assume $n \in \mathbb{N}$ for the remainder of this argument.

We choose a fixed base point Q_0 on $\mathcal{K}_n \setminus \{P_0, P_\infty\}$ to be one of the remaining branch points. Let $\omega^{(3)}_{P_\infty, P_0}$ be a normal differential of the third kind holomorphic on $\mathcal{K}_n \setminus \{P_\infty, P_0\}$ with simple poles at P_∞ and P_0 and residues 1 and -1, respectively (cf. (A.23)–(A.26)),

$$\omega^{(3)}_{P_\infty, P_0} \underset{\zeta \to 0}{=} (\zeta^{-1} + O(1))d\zeta \text{ as } P \to P_\infty, \tag{2.98}$$

$$\omega^{(3)}_{P_\infty, P_0} \underset{\zeta \to 0}{=} (-\zeta^{-1} + O(1))d\zeta \text{ as } P \to P_0, \tag{2.99}$$

where the local coordinates are given by

$$\zeta = \sigma/z^{1/2} \text{ for } P \text{ near } P_\infty, \quad \zeta = \sigma' z^{1/2} \text{ for } P \text{ near } P_0, \quad \sigma, \sigma' \in \{1, -1\}. \tag{2.100}$$

In particular,

$$\int_{a_j} \omega^{(3)}_{P_\infty, P_0} = 0, \quad j = 1, \dots, n, \tag{2.101}$$

and with $Q_0 = (E_{m_0}, 0)$,

$$\int_{Q_0}^{P} \omega^{(3)}_{P_\infty, P_0} \underset{\zeta \to 0}{=} \ln(\zeta) + (1/2)\ln(E_{m_0}) + O(\zeta) \text{ as } P \to P_\infty, \tag{2.102}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_\infty, P_0} \underset{\zeta \to 0}{=} -\ln(\zeta) + (1/2)\ln(E_{m_0}) + O(\zeta) \text{ as } P \to P_0 \tag{2.103}$$

in analogy to (1.96), (1.97).

Next, let $\omega^{(2)}_{P_\infty,0}$ be the normalized differential of the second kind holomorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ such that (cf. (A.20), (A.21), and (A.22))

$$\int_{a_j} \omega^{(2)}_{P_\infty,0} = 0, \quad j = 1, \dots, n \tag{2.104}$$

$$\omega^{(2)}_{P_\infty,0} \underset{\zeta \to 0}{=} (\zeta^{-2} + O(1))d\zeta \text{ as } P \to P_\infty, \tag{2.105}$$

$$\int_{Q_0}^{P} \omega^{(2)}_{P_\infty,0} \underset{\zeta \to 0}{=} -\zeta^{-1} + O(\zeta) \text{ as } P \to P_\infty \tag{2.106}$$

in analogy to (1.98)–(1.100). Denoting the vector of b-periods of $\omega^{(2)}_{P_\infty,0}/(2\pi i)$ by $\underline{U}^{(2)}_0$, one obtains

$$\underline{U}^{(2)}_0 = \big(U^{(2)}_{0,1}, \dots, U^{(2)}_{0,n}\big), \quad U^{(2)}_{0,j} = \frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_\infty,0} = -2c_j(n), \quad j = 1, \dots, n, \tag{2.107}$$

applying (B.33). In the following, it will be convenient to introduce the abbreviations

$$\begin{aligned} \underline{z}(P, \underline{Q}) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ \underline{z}(P, \underline{Q}, \underline{\Delta}) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}) + \underline{\Delta}, \\ P \in \mathcal{K}_n, \ \underline{Q} &= \{Q_1, \dots, Q_n\} \in \operatorname{Sym}^n(\mathcal{K}_n). \end{aligned} \tag{2.108}$$

We note that by (A.52) and (A.53), $\underline{z}(\,\cdot\,, \underline{Q})$ and $\underline{z}(\,\cdot\,, \underline{Q}, \underline{\Delta})$ are independent of the choice of base point Q_0.

Theorem 2.10 *Suppose that $u \in C^\infty(\Omega)$ satisfies the nth stationary sGmKdV equation* (2.38) *on Ω subject to the constraints* (2.39) *and* (2.86). *In addition,* *let* $P \in \mathcal{K}_n \setminus \{P_\infty\}$ *and* $x, x_0 \in \Omega$, *where* $\Omega \subseteq \mathbb{R}$ *is an open interval. Moreover, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, *or equivalently,* $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ *is nonspecial for* $x \in \Omega$. *Then* ϕ *admits the representation*

$$\phi(P,x) = \frac{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x), \underline{\Delta}))}{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x), \underline{\Delta}))\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))} \exp\bigg(- \int_{Q_0}^{P} \omega^{(3)}_{P_\infty, P_0} + \tfrac{1}{2}\ln(E_{m_0})\bigg), \tag{2.109}$$

where $\underline{\Delta}$ *is a half-period defined as*

$$\underline{\Delta} = \underline{A}_{P_0}(P_\infty), \quad 2\underline{\Delta} = 0 \pmod{L_n}. \tag{2.110}$$

The components ψ_j, $j = 1, 2$ *of the Baker–Akhiezer function* Ψ *are given by*[1]

$$\begin{aligned}\psi_1(P, x, x_0) = {} & \frac{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0)))} \\ & \times \exp\bigg(-i(x - x_0)\int_{Q_0}^{P} \omega^{(2)}_{P_\infty, 0}\bigg)\end{aligned} \tag{2.111}$$

and

$$\begin{aligned}\psi_2(P, x, x_0) = {} & -\frac{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x), \underline{\Delta}))}{\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}(x), \underline{\Delta}))\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0)))} \\ & \times \exp\bigg(- \int_{Q_0}^{P} \omega^{(3)}_{P_\infty, P_0} + \tfrac{1}{2}\ln(E_{m_0}) - i(x - x_0)\int_{Q_0}^{P} \omega^{(2)}_{P_\infty, 0}\bigg).\end{aligned} \tag{2.112}$$

The Abel map linearizes the auxiliary divisors in the sense that (cf. (2.107)*)*

$$\underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat{\mu}}(x)}\big) = \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}\big) + i\underline{U}_0^{(2)}(x - x_0), \tag{2.113}$$

$$\underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat{\nu}}(x)}\big) = \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat{\nu}}(x_0)}\big) + i\underline{U}_0^{(2)}(x - x_0). \tag{2.114}$$

Finally, u *is of the form*

$$u(x) = -(i/2)\ln(\alpha/\beta) \pmod{2\pi\mathbb{Z}} + 2i\ln\left(\frac{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}(x), \underline{\Delta})\big)}{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}(x))\big)}\right). \tag{2.115}$$

Proof First, we assume temporarily that

$$\mu_j(x) \neq \mu_{j'}(x), \ \nu_k(x) \neq \nu_{k'}(x) \text{ for } j \neq j', k \neq k' \text{ and } x \in \widetilde{\Omega} \tag{2.116}$$

for appropriate $\widetilde{\Omega} \subseteq \Omega$. Since by (2.70), $\mathcal{D}_{P_0\underline{\hat{\nu}}} \sim \mathcal{D}_{P_\infty\underline{\hat{\mu}}}$, and $P_\infty = (P_\infty)^* \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ by hypothesis, one can apply Theorem A.31 to conclude that $\mathcal{D}_{\underline{\hat{\nu}}} \in$

[1] To avoid multi-valued expressions in formulas such as (2.109), (2.111), (2.112), etc., we agree always to choose the same path of integration connecting Q_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

$\mathrm{Sym}^n(\mathcal{K}_n)$ is nonspecial. This argument is of course symmetric with respect to $\underline{\hat{\mu}}$ and $\underline{\hat{\nu}}$. Thus, $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial if and only if $\mathcal{D}_{\underline{\hat{\nu}}}$ is. Next, we denote the right-hand side of (2.109) by $\tilde{\phi}$. To prove that $\phi = \tilde{\phi}$ by applying the Riemann–Roch theorem (Theorem A.13), it suffices to show that ϕ and $\tilde{\phi}$ have the same poles and zeros as well as the same value at one point on $\mathcal{K}_n$. From the definition (2.68) of $\phi(\,\cdot\,, x)$ one concludes that it has simple zeros at $\underline{\hat{\nu}}(x)$ and P_0 and simple poles at $\underline{\hat{\mu}}(x)$ and P_∞. By (2.102), (2.103), the expression (2.109) for $\tilde{\phi}$, and a special case of Riemann's vanishing theorem (cf. Theorem A.26), $\tilde{\phi}$ also shares these properties. As a consequence of the linear equivalence $\mathcal{D}_{P_\infty\underline{\hat{\mu}}} \sim \mathcal{D}_{P_0\underline{\hat{\nu}}}$, that is,

$$\underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = \underline{A}_{Q_0}(P_0) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}), \tag{2.117}$$

one obtains

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{\Delta}, \tag{2.118}$$

with

$$\underline{\Delta} = \underline{A}_{P_0}(P_\infty). \tag{2.119}$$

Since P_0 and P_∞ are branch points of $\mathcal{K}_n$, the right-hand side of (2.119) is a half-period, proving (2.110). By the Riemann–Roch theorem (Theorem A.13) and since ϕ and $\tilde{\phi}$ share common zeros, one concludes that $\tilde{\phi}/\phi = c$ for some constant $c \in \mathbb{C}$. Using (2.94), one infers together with (2.102) that

$$\frac{\tilde{\phi}}{\phi} \underset{\zeta\to 0}{=} \frac{(1+O(\zeta))(\zeta^{-1}+O(1))}{\zeta^{-1}+O(1)} \underset{\zeta\to 0}{=} 1 + O(\zeta),$$

and hence $c = 1$. This proves $\phi = \tilde{\phi}$ subject to (2.116).

Next we turn to the proof of (2.111). Denote by $\tilde{\psi}$ the right-hand side of (2.111). To prove $\psi_1 = \tilde{\psi}_1$, with ψ_1 given by (2.72), one first observes, using (2.65), the definition (2.68) of ϕ and the Dubrovin equations (2.87) that

$$i\phi(P, x') \underset{P\to\hat{\mu}_j(x')}{=} \partial_{x'} \ln(z - \mu_j(x')) + O(1). \tag{2.120}$$

Together with (2.72), this yields

$$\psi_1(P, x, x_0) = \begin{cases} (z-\mu_j(x))O(1) & \text{as } P \to \hat{\mu}_j(x) \neq \hat{\mu}_j(x_0), \\ O(1) & \text{as } P \to \hat{\mu}_j(x) = \hat{\mu}_j(x_0), \\ (z-\mu_j(x_0))^{-1}O(1) & \text{as } P \to \hat{\mu}_j(x_0) \neq \hat{\mu}_j(x), \end{cases} \tag{2.121}$$
$$P = (z, y) \in \mathcal{K}_n, \;\; x, x_0 \in \widetilde{\Omega},$$

where $O(1) \neq 0$ in (2.121). Consequently, all zeros and poles of ψ_1 and $\tilde{\psi}_1$ on $\mathcal{K}_n \setminus \{P_\infty\}$ are simple and coincide. Hence one concludes by Theorem A.23 that ψ_1 contains a factor $\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))/\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0)))$. It remains to identify the essential singularity of ψ_1 and $\tilde{\psi}_1$ at P_∞. The asymptotic spectral parameter expansion (2.94) of ϕ yields

$$-\frac{i}{2}(u(x)-u(x_0))+i\int_{x_0}^{x} dx'\,\phi(P,x') \underset{\zeta\to 0}{=} i(x-x_0)(\zeta^{-1}+O(\zeta)) \text{ as } P\to P_\infty, \tag{2.122}$$

and thus comparing (2.72), (2.106), the expression (2.111) for $\tilde{\psi}_1$, and (2.122) then shows that ψ_1 and $\tilde{\psi}_1$ have identical exponential behavior up to order $O(\zeta)$ near P_∞. Consequently, ψ_1 and $\tilde{\psi}_1$ share the same singularities and zeros, and the Riemann–Roch-type uniqueness result in Lemma B.2 (taking $t_r = t_{0,r}$) then proves that ψ_1 and $\tilde{\psi}_1$ coincide up to normalization. The latter is determined by (2.81), implying

$$\psi_1(P,x,x_0)\psi_1(P^*,x,x_0) \underset{P\to P_\infty}{=} 1. \tag{2.123}$$

Hence (2.111) holds subject to (2.116). The corresponding expression (2.112) for ψ_2 is then obvious from (2.73), (2.109), and (2.111).

Next we prove the linearization property (2.113) using Lemma 2.6 (and still assuming (2.116)). Equation (2.114) then follows from (2.113) and (2.117). From

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = \left(\sum_{j=1}^{n}\int_{Q_0}^{\hat{\mu}_j}\underline{\omega}\right) \pmod{L_n}$$

and the Dubrovin equations (2.87), one infers on Ω

$$\partial_x\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = \sum_{j=1}^{n}\mu_{j,x}\sum_{k=1}^{n}\underline{c}(k)\frac{\mu_j^{k-1}}{y(\hat{\mu}_j)} = -2i\sum_{j,k=1}^{n}\underline{c}(k)\frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j-\mu_\ell)}.$$

Lagrange's interpolation formula (cf. Theorem E.1)

$$\sum_{j=1}^{n}\mu_j^{k-1}\prod_{\substack{\ell=1\\ \ell\neq j}}^{n}(\mu_j-\mu_\ell)^{-1} = \delta_{k,n}, \quad k=1,\dots,n$$

then yields

$$\partial_x\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = -2i\underline{c}(n) = i\underline{U}_0^{(2)}, \quad x\in\widetilde{\Omega} \tag{2.124}$$

and hence (2.113), subject to (2.116). In particular, applying (B.33) establishes the relation between the left-hand side of (2.124) and the vector $\underline{U}_0$ of b-periods of $\omega_{P_\infty,0}^{(2)}/(2\pi i)$ introduced in (2.107).

To prove formula (2.115) for u we employ an asymptotic spectral parameter expansion for ϕ and the Riccati equation (2.74). First we conclude from (B.31)

$$\underline{\omega} \underset{\zeta\to 0}{=} (-2\underline{c}(n)+O(\zeta^2))d\zeta \text{ as } P\to P_\infty,$$

and hence

$$\underline{A}_{Q_0}(P) = \int_{Q_0}^{P} \underline{\omega} \quad (\mathrm{mod}\ L_n) \underset{\zeta\to 0}{=} \underline{A}_{Q_0}(P_\infty) - 2\underline{c}(n)\zeta + O(\zeta^3)$$
$$= \underline{A}_{Q_0}(P_\infty) + \underline{U}_0^{(2)}\zeta + O(\zeta^2),$$

using (B.33) and (2.107). Expanding the subsequent ratios of Riemann theta functions in (2.109) one finds

$$\frac{\theta(P, \underline{\hat{\mu}})}{\theta(P_\infty, \underline{\hat{\mu}})}$$
$$\underset{\zeta\to 0}{=} 1 - \frac{\sum_{j=1}^{n} U_{0,j}^{(2)} \partial_{w_j} \theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{w} + \underline{\alpha}_{Q_0}(\underline{\hat{\mu}})\big)|_{\underline{w}=0}}{\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\underline{\hat{\mu}})\big)} \zeta + O(\zeta^3) \tag{2.125}$$

and the same formula for the theta function ratio involving the additional half-period $\underline{\Delta}$. Here $\sum_{j=1}^{n} U_{0,j}^{(2)} \partial_{w_j}$ denotes the directional derivative in $\underline{U}_0^{(2)}$-direction. Given (2.107), we may write

$$\sum_{j=1}^{n} i U_{0,j}^{(2)} \partial_{w_j} \theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{w} + \underline{\alpha}_{Q_0}(\underline{\hat{\mu}}(x_0)) + i\underline{U}_0^{(2)}(x - x_0)\big)\big|_{\underline{w}=0}$$
$$= \partial_x \theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\underline{\hat{\mu}}(x_0)) + i\underline{U}_0^{(2)}(x - x_0)\big)$$

and hence obtain from (2.125)

$$\frac{\theta\big(\underline{z}(P, \underline{\hat{\mu}})\big)}{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}})\big)} \underset{\zeta\to 0}{=} 1 + i\partial_x \ln\big(\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}})\big)\big)\zeta + O(\zeta^3) \text{ as } P \to P_\infty,$$

and the identical formula for the theta function ratio involving $\underline{\Delta}$. Together with (2.102) this shows that

$$\phi(P) \underset{\zeta\to 0}{=} \zeta^{-1} + i\partial_x \ln\left(\frac{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}, \underline{\Delta})\big)}{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}})\big)}\right) + O(\zeta) \text{ as } P \to P_\infty.$$

Expanding the Riccati equation (2.74) for P near P_∞ then yields

$$u_x = 2i\partial_x \ln\left(\frac{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}, \underline{\Delta})\big)}{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}})\big)}\right), \quad x \in \widetilde{\Omega}$$

and hence

$$u = 2i \ln\left(\frac{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}, \underline{\Delta})\big)}{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}})\big)}\right) + C \text{ on } \widetilde{\Omega}$$

for some integration constant $C \in \mathbb{C}$. The equalities

$$f_n = \alpha e^{-iu} = (-1)^n \prod_{j=1}^n \mu_j = \alpha e^{-iC} \left(\frac{\theta\big(\underline{z}(P_\infty, \hat{\underline{\nu}})\big)}{\theta\big(\underline{z}(P_\infty, \hat{\underline{\mu}})\big)} \right)^2,$$

$$h_n = \beta e^{iu} = (-1)^n \prod_{j=1}^n \nu_j = \beta e^{iC} \left(\frac{\theta\big(\underline{z}(P_\infty, \hat{\underline{\mu}})\big)}{\theta\big(\underline{z}(P_\infty, \hat{\underline{\nu}})\big)} \right)^2,$$

and the facts that $\hat{\underline{\mu}}$ and $\hat{\underline{\nu}}$ satisfy identical Dubrovin-type equations and the constraint $\prod_{j=1}^n \mu_j \nu_j = \prod_{m=1}^{2n} E_m$ is symmetric with respect to $\hat{\underline{\mu}}$ and $\hat{\underline{\nu}}$ then yield

$$\alpha e^{-iC} = \beta e^{iC}$$

and hence (2.115), assuming (2.116). The extension of all these results from $\widetilde{\Omega}$ to Ω then simply follows from the continuity of $\underline{\alpha}_{Q_0}$ and the hypothesis of $\mathcal{D}_{\hat{\underline{\mu}}}$ being nonspecial on Ω. $\square$

Combining (2.113), (2.114), (2.118), and (2.115) shows the remarkable linearity of the theta function arguments with respect to x in the formula for u. In fact, one can rewrite (2.115) as

$$u(x) = c_0 + 2i \ln \left(\frac{\theta(\underline{A} + \underline{B}x + \underline{\Delta})}{\theta(\underline{A} + \underline{B}x)} \right),$$

where

$$\begin{aligned}
&\underline{A} = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) - i\underline{U}_0^{(2)} x_0 + \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\hat{\underline{\mu}}(x_0)}\big),\\
&\underline{B} = i\underline{U}_0^{(2)},\\
&\underline{\Delta} = \underline{A}_{P_0}(P_\infty), \quad c_0 = -(i/2)\ln(\alpha/\beta) \pmod{2\pi\mathbb{Z}},
\end{aligned}$$

and hence the constants $\underline{\Delta}, \underline{B} \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ (and its homology basis), and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\hat{\underline{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \mathrm{Sym}^n(\mathcal{K}_n)$ at the point x_0 as long as the divisor $\mathcal{D}_{\hat{\underline{\mu}}(x_0)}$ is assumed to be nonspecial.

Remark 2.11 Although this approach to the algebro-geometric solutions of the sGmKdV hierarchy resembles that of the AKNS hierarchy (which includes the mKdV hierarchy) in many ways, there are, however, some characteristic differences. In particular, the branch point $P_0 = (0, 0)$ is an unusual necessity in the sG context, and P_∞ (as in the KdV context) is a branch point as opposed to the AKNS case. This shows that sGmKdV curves are actually special KdV curves (with $(0, 0)$ a branch point).

Remark 2.12 The explicit expressions (2.111), (2.112) for ψ_j, $j = 1, 2$ complement Lemma 2.5 and show that Ψ stays meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ as long as $\mathcal{D}_{\hat{\underline{\mu}}}$ is nonspecial (assuming the affine part of $\mathcal{K}_n$ to be nonsingular).

The sGmKdV potential u in (2.115) is complex-valued in general. A discussion of the isospectral set of real-valued (quasi-periodic and periodic) algebro-geometric sGmKdV solutions will be deferred to the next section (cf. Lemmas 2.30 and 2.31 and Remark 2.32).

Next we briefly consider the trivial case $n = 0$ excluded in Theorem 2.10.

Example 2.13 Assume $n = 0$, $P = (z, y) \in \mathcal{K}_0 \setminus \{P_\infty\}$ and let $(x, x_0) \in \mathbb{R}^2$. Then

$$\begin{aligned}
&\mathcal{K}_0\colon \mathcal{F}_0(z, y) = y^2 - R_1(z) = y^2 - z = 0, \quad E_0 = 0,\\
&u(x) = -i \ln(\alpha), \quad \alpha \in \mathbb{C} \setminus \{0\},\\
&F_0(z, x) = 1, \quad G_{-1}(z, x) = 0, \quad H_1(z, x) = 1,\\
&\phi(P, x) = y,\\
&\psi_1(P, x, x_0) = \exp(iy(x - x_0)),\\
&\psi_2(P, x, x_0) = -y \exp(iy(x - x_0)).
\end{aligned}$$

Up to this point we assumed $u \in C^\infty(\mathbb{R})$ satisfies the stationary sGmKdV equation (2.38) for some fixed $n \in \mathbb{N}_0$. Next we will show that solvability of the Dubrovin equations (2.87) on $\widehat{\Omega}_\mu \subseteq \Omega_\mu$, such that $\mu_j \neq 0$ on $\widehat{\Omega}_\mu$, $j = 1, \dots, n$, in fact implies equation (2.38) on $\widehat{\Omega}_\mu$. As pointed out in Remark 2.17, this amounts to solving the algebro-geometric initial value problem in the stationary case.

Theorem 2.14 *Fix $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$, and assume* (2.86)*. Suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the stationary Dubrovin equations* (2.87) *on an open interval $\widehat{\Omega}_\mu \subseteq \mathbb{R}$ such that μ_j, $j = 1, \dots, n$, remain distinct and nonzero on $\widehat{\Omega}_\mu$. Then $u \in C^\infty(\widehat{\Omega}_\mu)$, defined by*

$$u = i \ln\biggl((-1)^n \alpha^{-1} \prod_{j=1}^n \mu_j\biggr), \tag{2.126}$$

satisfies the nth stationary sGmKdV equation (2.38)*, that is,*

$$\text{s-sGmKdV}_n(u) = 0 \text{ on } \widehat{\Omega}_\mu \tag{2.127}$$

subject to the constraint $\alpha\beta = \prod_{m=1}^{2n} E_m \neq 0$.

Proof Given the solutions $\hat{\mu}_j = (\mu_j, y(\mu_j)) \in C^\infty(\widehat{\Omega}_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (2.87), we introduce

$$F_n(z) = \prod_{j=1}^n (z - \mu_j) \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu \tag{2.128}$$

and define $u \in C^\infty(\widehat{\Omega}_\mu)$ by (2.126). In addition, we introduce

$$G_{n-1}(z) = \frac{i}{2z}\bigl(F_{n,x}(z) + iu_x F_n(z)\bigr) \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu, \tag{2.129}$$

(cf. (2.21)), where $G_{n-1}(0)$ is well-defined because of (2.128) and (2.129). We also note that on $\widehat{\Omega}_\mu$,

$$\hat{\mu}_j = (\mu_j, -(i/2)F_{n,x}(\mu_j)) = (\mu_j, -\mu_j G_{n-1}(\mu_j)), \quad j = 1, \dots, n \tag{2.130}$$

by (2.87) and $F_{n,x}(\mu_j) = -\mu_{j,x} \prod_{\substack{k=1 \\ k \neq j}}^n (\mu_j - \mu_k)$. Next, we define a monic polynomial H_n of degree n on $\mathbb{C} \times \widehat{\Omega}_\mu$ such that (2.26) holds, that is,

$$R_{2n+1}(z) - z^2 G_{n-1}(z)^2 = z F_n(z) H_n(z) \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu. \tag{2.131}$$

The polynomial H_n exists since the left-hand side of (2.131) vanishes at $z = 0$, and $z = \mu_j(x)$, $j = 1, \dots, n$ by (2.130). Introducing the polynomial P_{n-1} of degree $n - 1$ on $\mathbb{C} \times \widehat{\Omega}_\mu$ by

$$P_{n-1}(z) = H_n(z) - F_n(z) + i G_{n-1,x}(z) \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu$$

and differentiating (2.131) with respect to x yields

$$2z G_{n-1}(z) G_{n-1,x}(z) + F_{n,x}(z) H_n(z) + F_n(z) H_{n,x}(z) = 0, \tag{2.132}$$

and hence

$$\mu_j G_{n-1}(\mu_j) P_{n-1}(\mu_j) = 0, \quad j = 1, \dots, n$$

on $\widehat{\Omega}_\mu$. From this point on one can follow the proof of Theorem 1.26 step by step; hence, one concludes that

$$P_{n-1} = 0 \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu,$$

and therefore (2.23), that is,

$$H_n(z) = F_n(z) - i G_{n-1,x}(z) \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu. \tag{2.133}$$

Combining (2.132) and (2.133) then yields (2.22),

$$H_{n,x}(z) = i u_x H_n(z) + 2iz G_{n-1}(z) \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu;$$

thus, we have derived the fundamental equations (2.21)–(2.23) and (2.26) on $\mathbb{C} \times \widehat{\Omega}_\mu$. One can now mimic our analysis in (2.26)–(2.35) to arrive at the nth stationary sGmKdV equation (2.38) satisfied by u, subject to the constraint (2.39), $\alpha\beta = \prod_{m=1}^{2n} E_m \neq 0$. $\square$

Remark 2.15 The explicit theta function representation (2.115) of u on Ω_μ in (2.126) then permits one to extend u beyond Ω_μ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ remains nonspecial (cf. Theorem A.31).

Remark 2.16 Although we formulated Theorem 2.14 in terms of Dirichlet eigenvalues μ_j, $j = 1, \dots, n$ only, the analogous result (and strategy of proof) obviously works in terms of ν_j, $j = 1, \dots, n$.

Remark 2.17 A closer look at Theorem 2.14 reveals that u is uniquely determined in an open neighborhood Ω of x_0 by $\mathcal{K}_n$ and the initial condition $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \operatorname{Sym}^n(\mathcal{K}_n)$, or equivalently, by the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)} \in \operatorname{Sym}^n(\mathcal{K}_n)$ at $x = x_0$. Conversely, given $\mathcal{K}_n$ and u in an open neighborhood Ω of x_0, one can construct the corresponding polynomials $F_n(\,\cdot\,, x)$, $G_{n-1}(\,\cdot\,, x)$, and $H_n(\,\cdot\,, x)$ for $x \in \Omega$ (using the recursion relation (2.3)–(2.6) to determine the homogeneous elements $\hat{f}_\ell, \hat{g}_\ell, \hat{h}_\ell$, and (D.9) to determine $c_\ell = c_\ell(\underline{E})$, $\ell = 0, \dots, n$) and then recover the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ for $x \in \Omega$ from the zeros of $F_n(\,\cdot\,, x)$ and from (2.65). This remark is of relevance in connection with determining the isospectral set of sGmKdV potentials u in the sense that once the curve $\mathcal{K}_n$ is fixed, elements of the isospectral class of potentials are parametrized by (nonspecial) auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ (cf. Remark 2.32).

2.4 The Time-Dependent sGmKdV Formalism

In this section we extend the algebro-geometric analysis of Section 2.3 to the time-dependent sGmKdV hierarchy.

For most of this section, we assume the following hypothesis.

Hypothesis 2.18 *Suppose that* $u\colon \mathbb{R}^2 \to \mathbb{C}$ *satisfies*

$$u(\,\cdot\,, t) \in C^\infty(\mathbb{R}),\ t \in \mathbb{R}, \quad u_x(x,\,\cdot\,) \in C^1(\mathbb{R}),\ x \in \mathbb{R}. \tag{2.134}$$

The basic problem in the analysis of algebro-geometric solutions of the sGmKdV hierarchy consists in solving the time-dependent rth sGmKdV flow with initial data a stationary solution of the nth equation in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, consider a solution $u^{(0)}$ of the nth stationary sGmKdV equation $\text{s-sGmKdV}_n(u^{(0)}) = 0$ subject to the constraint $\alpha\beta \neq 0$, associated with $\mathcal{K}_n$ and a given set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to construct a solution u of the rth sGmKdV flow $\text{sGmKdV}_r(u) = 0$ with $u(t_{0,r}) = u^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the integration constants in the definitions of the stationary and the time-dependent sGmKdV equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence, we employ the notation $\widetilde{V}_r, \widetilde{F}_r, \widetilde{G}_{r-1}, \widetilde{H}_r, \tilde{f}_s, \tilde{g}_s, \tilde{h}_s, \tilde{c}_s, \tilde{\alpha}, \tilde{\beta}$ to distinguish them from $V_n, F_n, G_{n-1}, H_n, f_\ell, g_\ell, h_\ell, c_\ell, \alpha, \beta$ in the following. In addition, we follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth sGmKdV flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution u of the time-dependent algebro-geometric initial value problem

$$\begin{aligned} &\widetilde{\text{sGmKdV}}_r(u) = u_{xt_r} + 2i\tilde{g}_{r-1,x}(u) + 2(\tilde{\beta}e^{iu} - \tilde{\alpha}e^{-iu}) = 0, \\ &u\big|_{t_r = t_{0,r}} = u^{(0)}, \end{aligned} \tag{2.135}$$

$$\text{s-sGmKdV}_n\big(u^{(0)}\big) = 2ig_{n-1,x}\big(u^{(0)}\big) + 2\big(\beta e^{iu^{(0)}} - \alpha e^{-iu^{(0)}}\big) = 0 \tag{2.136}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$, where $u = u(x, t_r)$ satisfies (2.134) and a fixed curve $\mathcal{K}_n$ is associated with the stationary solution $u^{(0)}$ in (2.136). Actually, relying on the isospectral property of the sGmKdV flows, we will go a step further and assume (2.136) not only at $t_r = t_{0,r}$ but for all $t_r \in \mathbb{R}$. Hence, we start with

$$U_{t_r} - \widetilde{V}_{r,x} + [U, \widetilde{V}_r] = 0, \tag{2.137}$$

$$-V_{n,x} + [U, V_n] = 0, \tag{2.138}$$

where (cf. (2.18), (2.19), (2.29), (2.30))

$$\begin{aligned} U(z) &= -i \begin{pmatrix} \frac{1}{2}u_x & 1 \\ z & -\frac{1}{2}u_x \end{pmatrix}, \\ V_n(z) &= i \begin{pmatrix} -G_{n-1}(z) & (1/z)F_n(z) \\ H_n(z) & G_{n-1}(z) \end{pmatrix}, \\ \widetilde{V}_r(z) &= i \begin{pmatrix} -\widetilde{G}_{r-1}(z) & (1/z)\widetilde{F}_r(z) \\ \widetilde{H}_r(z) & \widetilde{G}_{r-1}(z) \end{pmatrix}, \end{aligned} \tag{2.139}$$

and

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell = \prod_{j=1}^{n} (z - \mu_j), \tag{2.140}$$

$$G_{n-1}(z) = \sum_{\ell=0}^{n-1} g_{n-1-\ell} z^\ell, \quad G_{-1}(z) = 0, \tag{2.141}$$

$$H_n(z) = \sum_{\ell=0}^{n} h_{n-\ell} z^\ell = \prod_{j=1}^{n} (z - \nu_j), \tag{2.142}$$

$$\widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s} z^s, \tag{2.143}$$

$$\widetilde{G}_{r-1}(z) = \sum_{s=0}^{r-1} \tilde{g}_{r-1-s} z^s, \quad \widetilde{G}_{-1}(z) = 0, \tag{2.144}$$

$$\widetilde{H}_r(z) = \sum_{s=0}^{r} \tilde{h}_{r-s} z^s \tag{2.145}$$

for fixed $n, r \in \mathbb{N}_0$. Here f_ℓ, $\tilde{f}_s$, g_ℓ, $\tilde{g}_s$, h_ℓ, and $\tilde{h}_s$, $\ell = 0, \dots, n-1$, $s = 0, \dots, r-1$, are defined as in (2.3)–(2.6) with appropriate sets of integration constants. Explicitly, (2.137) and (2.138) are equivalent to (cf. (2.53)–(2.55))

$$u_{xt_r} = -2i\widetilde{G}_{r-1,x} - 2(\widetilde{H}_r - \widetilde{F}_r), \tag{2.146}$$

$$\widetilde{F}_{r,x} = -iu_x\widetilde{F}_r - 2iz\widetilde{G}_{r-1}, \tag{2.147}$$

$$\widetilde{H}_{r,x} = iu_x\widetilde{H}_r + 2iz\widetilde{G}_{r-1} \tag{2.148}$$

and (cf. (2.21)–(2.23)),

$$F_{n,x} = -iu_x F_n - 2izG_{n-1}, \tag{2.149}$$

$$H_{n,x} = iu_x H_n + 2izG_{n-1}, \tag{2.150}$$

$$G_{n-1,x} = i(H_n - F_n), \tag{2.151}$$

respectively. Taking $z = 0$ in (2.147)–(2.150), yields

$$\tilde{f}_r = \tilde{\alpha} e^{-iu}, \quad \tilde{h}_r = \tilde{\beta} e^{iu}, \tag{2.152}$$

$$f_n = \alpha e^{-iu}, \quad h_n = \beta e^{iu}, \tag{2.153}$$

as discussed in Section 2.2. It will turn out later (cf. Remark 2.21) that $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ in (2.135) and (2.136) are not independent of each other but constrained by

$$f_n \tilde{h}_r = \tilde{f}_r h_n \text{ or } \alpha\tilde{\beta} = \tilde{\alpha}\beta. \tag{2.154}$$

We also recall our conventions $g_{-1} = G_{-1} = 0$ in (2.42) and set

$$\widetilde{G}_{-1}(z, x, t_0) = 0, \quad \tilde{g}_{-1}(x, t_0) = 0 \text{ for } r = 0. \tag{2.155}$$

Hence, (2.3)–(2.50) apply to F_n, G_{n-1}, H_n, f_j, g_j, and h_j and (2.3)–(2.6), (2.29), (2.30) with $n \to r$, $c_\ell \to \tilde{c}_\ell$, $\alpha \to \tilde{\alpha}$, $\beta \to \tilde{\beta}$ apply to $\widetilde{F}_r$, $\widetilde{G}_{r-1}$, $\widetilde{H}_r$, $\tilde{f}_j$, $\tilde{g}_j$, and $\tilde{h}_j$. In particular, the fundamental identity (2.26) holds,

$$z^2 G_{n-1}^2 + zF_n H_n = R_{2n+1}, \tag{2.156}$$

and the hyperelliptic curve $\mathcal{K}_n$ is still given by (2.62) assuming (2.63) for the remainder of this section, that is,

$$E_0 = 0,\ E_m \in \mathbb{C} \setminus \{0\},\ m = 1, \dots, 2n. \tag{2.157}$$

First we will assume the existence of a solution u of equations (2.146)–(2.151), (2.154) and derive an explicit formula for u in terms of Riemann theta functions. In addition, we will show in Theorem 2.34 that (2.146)–(2.151), (2.154) and hence the algebro-geometric initial value problem (2.135), (2.136) has a solution at least locally, that is for $(x, t_r) \in \Omega$ for some open and connected set $\Omega \subseteq \mathbb{R}^2$.

In analogy to equations (2.65) and (2.66) we define

$$\begin{aligned} \hat{\mu}_j(x, t_r) = (\mu_j(x, t_r), -\mu_j(x, t_r) G_{n-1}(\mu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \\ j = 1, \dots, n,\ (x, t_r) \in \mathbb{R}^2, \end{aligned} \tag{2.158}$$

$$\begin{aligned} \hat{\nu}_j(x, t_r) = (\nu_j(x, t_r), \nu_j(x, t_r) G_{n-1}(\nu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \\ j = 1, \dots, n,\ (x, t_r) \in \mathbb{R}^2. \end{aligned} \tag{2.159}$$

As in Section 2.3, the regularity assumptions (2.134) on u imply analogous regularity properties of F_n, H_n, μ_j, and ν_k. Moreover, (2.153) implies

$$\mu_j(x, t_r), \nu_k(x, t_r) \neq 0,\ j, k = 1, \dots, n,\ (x, t_r) \in \mathbb{R}^2.$$

Next, in accordance with (2.68), one defines the meromorphic function $\phi(\,\cdot\,, x, t_r)$ on $\mathcal{K}_n$ by

$$\phi(P, x, t_r) = \frac{y - zG_{n-1}(z, x, t_r)}{F_n(z, x, t_r)} \tag{2.160}$$

$$= \frac{zH_n(z, x, t_r)}{y + zG_{n-1}(z, x, t_r)}, \tag{2.161}$$

$$P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}, \ (x, t_r) \in \mathbb{R}^2,$$

and hence the divisor $(\phi(\,\cdot\,, x, t_r))$ of $\phi(\,\cdot\,, x, t_r)$ reads

$$(\phi(\,\cdot\,, x, t_r)) = \mathcal{D}_{P_0 \underline{\hat{\nu}}(x,t_r)} - \mathcal{D}_{P_\infty \underline{\hat{\mu}}(x,t_r)},$$

with

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n).$$

The time-dependent Baker–Akhiezer vector

$$\Psi(P, x, x_0, t_r, t_{0,r}) = \begin{pmatrix} \psi_1(P, x, x_0, t_r, t_{0,r}) \\ \psi_2(P, x, x_0, t_r, t_{0,r}) \end{pmatrix}, \tag{2.162}$$

$$P \in \mathcal{K}_n \setminus \{P_\infty, P_0\}, \ (x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$$

is defined by

$$\begin{aligned} \psi_1(P, x, x_0, t_r, t_{0,r}) = \exp\bigg(&- \int_{t_{0,r}}^{t_r} ds\, \big(z^{-1}\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) \\ &+ \widetilde{G}_{r-1}(z, x_0, s)\big) - (i/2)(u(x, t_r) - u(x_0, t_r)) + i\int_{x_0}^{x} dx'\, \phi(P, x', t_r)\bigg), \end{aligned} \tag{2.163}$$

$$\psi_2(P, x, x_0, t_r, t_{0,r}) = -\psi_1(P, x, x_0, t_r, t_{0,r})\phi(P, x, t_r). \tag{2.164}$$

Basic properties of ϕ are summarized next.

Lemma 2.19 *Assume Hypothesis* 2.18 *and suppose that* (2.137)*,* (2.138) *hold. In addition, let $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}$ and $(x, t_r) \in \mathbb{R}^2$. Then ϕ satisfies*

$$-i\phi_x(P) + \phi(P)^2 - u_x\phi(P) = z, \tag{2.165}$$

$$\phi_{t_r}(P) = \widetilde{F}_r(z) - \widetilde{H}_r(z) + (i/z)(\phi(P)\widetilde{F}_r(z))_x \tag{2.166}$$

$$= (1/z)\widetilde{F}_r(z)\phi(P)^2 + 2\widetilde{G}_{r-1}(z)\phi(P) - \widetilde{H}_r(z), \tag{2.167}$$

$$\phi(P)\phi(P^*) = -z\frac{H_n(z)}{F_n(z)}, \tag{2.168}$$

$$\phi(P) + \phi(P^*) = -2z\frac{G_{n-1}(z)}{F_n(z)}, \tag{2.169}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}. \tag{2.170}$$

Proof Equations (2.165) and (2.168)–(2.170) are proved as in Lemma 2.5. To prove (2.166), we first observe that

$$\big(\partial_x + i(2\phi - u_x)\big)\big(\phi_{t_r} + (\widetilde{H}_r - \widetilde{F}_r) - (i/z)(\phi\widetilde{F}_r)_x\big) = 0,$$

using (2.165) and relations (2.146)–(2.148) repeatedly. Thus,

$$\phi_{t_r} + (\widetilde{H}_r - \widetilde{F}_r) - (i/z)(\phi\widetilde{F}_r)_x = C\exp\left(-i\int^x dx'\,(2\phi - u_{x'})\right),$$

where the left-hand side is meromorphic in a neighborhood of P_∞, whereas the right-hand side is meromorphic near P_∞ only if $C = 0$. This proves (2.166). Equation (2.167) is then clear from (2.147), (2.165), and (2.166). □

Next we prove that relations (2.146)–(2.148) and (2.149)–(2.151) determine the time development of F_n, G_n, and H_n.

Lemma 2.20 *Assume Hypothesis* 2.18 *and suppose that* (2.137), (2.138) *hold. Then,*

$$F_{n,t_r} = 2(G_{n-1}\widetilde{F}_r - F_n\widetilde{G}_{r-1}), \tag{2.171}$$

$$zG_{n-1,t_r} = F_n\widetilde{H}_r - H_n\widetilde{F}_r, \tag{2.172}$$

$$H_{n,t_r} = 2(H_n\widetilde{G}_{r-1} - G_{n-1}\widetilde{H}_r). \tag{2.173}$$

Equations (2.171)–(2.173) *are equivalent to*

$$-V_{n,t_r} + [\widetilde{V}_r, V_n] = 0.$$

Proof We prove (2.171) by using (2.170), which shows that

$$(\phi(P) - \phi(P^*))_{t_r} = -2y\frac{F_{n,t_r}}{F_n^2}. \tag{2.174}$$

However, the left-hand side of (2.174) also equals

$$\phi(P)_{t_r} - \phi(P^*)_{t_r} = \frac{4y}{F_n^2}(\widetilde{G}_{r-1}F_n - \widetilde{F}_rG_{n-1}) \tag{2.175}$$

by means of (2.166), (2.170), (2.149), and (2.147). Combining (2.174) and (2.175) proves (2.171). Similarly, to prove (2.172), we use (2.169) to write

$$(\phi(P) + \phi(P^*))_{t_r} = -\frac{2z}{F_n^2}\big(G_{n-1,t_r}F_n - G_{n-1}F_{n,t_r}\big). \tag{2.176}$$

Here we can express the left-hand side as

$$\phi(P)_{t_r} + \phi(P^*)_{t_r} = -2\widetilde{H}_r + 2\frac{H_n}{F_n}\widetilde{F}_r + 4z\Big(\frac{G_{n-1}}{F_n}\Big)^2\widetilde{F}_r - 4z\frac{G_{n-1}}{F_n}\widetilde{G}_{r-1}, \tag{2.177}$$

using (2.147) and (2.149). Combining (2.176) and (2.177) by means of (2.171) proves (2.172). Finally, (2.173) follows by differentiating (2.156), that is, $(zG_{n-1})^2 + zF_n H_n = R_{2n+1}$, with respect to t_r, and using (2.171) and (2.172). $\square$

Remark 2.21 Taking $z = 0$ in (2.172), one infers the compatibility relation

$$f_n \tilde{h}_r = \tilde{f}_r h_n,$$

or equivalently, using $f_n = \alpha e^{-iu}$, $\tilde{f}_r = \tilde{\alpha} e^{-iu}$, $h_n = \beta e^{iu}$, and $\tilde{h}_r = \tilde{\beta} e^{iu}$ (cf. (2.152), (2.153)), one obtains the constraint

$$\alpha\tilde{\beta} = \tilde{\alpha}\beta.$$

Lemmas 2.19 and 2.20 permit one to characterize Ψ.

Lemma 2.22 *Assume Hypothesis* 2.18 *and suppose that* (2.137), (2.138) *hold. In addition, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty, P_0\}$ *and* $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$. *Then the Baker–Akhiezer vector* Ψ *satisfies*

$$\Psi_x(P) = U(z)\Psi(P), \tag{2.178}$$

$$-y\Psi(P) = zV_n(z)\Psi(P), \tag{2.179}$$

$$\Psi_{t_r}(P) = \widetilde{V}_r(z)\Psi(P), \tag{2.180}$$

$$\begin{aligned} \psi_1(P, x, x_0, t_r, t_{0,r}) = {} & \left(\frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})}\right)^{1/2} \\ & \times \exp\biggl(-(y/z)\int_{t_{0,r}}^{t_r} ds\, \widetilde{F}_r(z, x_0, s) F_n(z, x_0, s)^{-1} \\ & \qquad + iy\int_{x_0}^{x} dx' F_n(z, x', t_r)^{-1}\biggr), \end{aligned} \tag{2.181}$$

$$\psi_1(P, x, x_0, t_r, t_{0,r})\psi_1(P^*, x, x_0, t_r, t_{0,r}) = \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})}, \tag{2.182}$$

$$\psi_2(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) = -z\frac{H_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})}, \tag{2.183}$$

$$\begin{aligned} & \psi_1(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) \\ & \quad + \psi_1(P^*, x, x_0, t_r, t_{0,r})\psi_2(P, x, x_0, t_r, t_{0,r}) = 2z\frac{G_{n-1}(z, x, t_r)}{F_n(z, x_0, t_{0,r})}, \end{aligned} \tag{2.184}$$

$$\begin{aligned} & \psi_1(P, x, x_0, t_r, t_{0,r})\psi_2(P^*, x, x_0, t_r, t_{0,r}) \\ & \quad - \psi_1(P^*, x, x_0, t_r, t_{0,r})\psi_2(P, x, x_0, t_r, t_{0,r}) = \frac{2y}{F_n(z, x_0, t_{0,r})}. \end{aligned} \tag{2.185}$$

In addition, as long as the zeros of $F_n(\,\cdot\,, x, t_r)$ *are all simple for* $(x, t_r), (x_0, t_{0,r}) \in \Omega$, $\Omega \subseteq \mathbb{R}^2$ *open and connected,* $\Psi(\,\cdot\,, x, x_0, t_r, t_{0,r})$ *is meromorphic on* $\mathcal{K}_n \setminus \{P_\infty\}$ *for* $(x, t_r), (x_0, t_{0,r}) \in \Omega$.

Proof By (2.163), $\psi_1(\,\cdot\,, x, x_0, t_r, t_{0,r})$ is meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ away from the poles $\hat{\mu}_j(x_0, s)$ of $\phi(\,\cdot\,, x_0, s)$ and $\hat{\mu}_k(x', t_r)$ of $\phi(\,\cdot\,, x', t_r)$. That $\psi_1(\,\cdot\,, x, x_0, t_r, t_{0,r})$ is meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ if $F_n(\,\cdot\,, x, t_r)$ has only simple zeros is a consequence of (cf. (2.85))

$$i\phi(P, x', t_r) \underset{P \to \hat{\mu}_j(x', t_r)}{=} \partial_{x'} \ln(F_n(z, x', t_r)) + O(1) \text{ as } z \to \mu_j(x', t_r),$$

and

$$-z^{-1}\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) \underset{P \to \hat{\mu}_j(x_0, s)}{=} \partial_s \ln\big(F_n(z, x_0, s)\big) + O(1) \quad \text{as } z \to \mu_j(x_0, s)$$

by means of (2.158), (2.160), and (2.171). This follows from (2.163) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j(x_0)$ of $\{\hat{\mu}_j(x_0, s) \in \mathcal{K}_n \,|\, (x_0, s) \in \Omega,\ s \in [t_{0,r}, t_r]\}$ such that $\hat{\mu}_k(x_0, s) \notin \mathcal{U}_j(x_0)$ for all $s \in [t_{0,r}, t_r]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$ and by simultaneously restricting P to a sufficiently small neighborhood $\mathcal{U}_j(t_r)$ of $\{\hat{\mu}_j(x', t_r) \in \mathcal{K}_n \,|\, (x', t_r) \in \Omega,\ x' \in [x_0, x]\}$ such that $\hat{\mu}_k(x', t_r) \notin \mathcal{U}_j(t_r)$ for all $x' \in [x_0, x]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$. By (2.164) and since ϕ is meromorphic on $\mathcal{K}_n$, one concludes that ψ_2 is meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ as well. Relations (2.178) and (2.179) follow as in Lemma 2.5, and the time evolution (2.180) is a consequence of the definition of Ψ in (2.163), (2.164) as well as (2.166) and (2.167). To prove (2.181), we recall (2.163), that is,

$$\begin{aligned}
\psi_1(P, x, x_0, t_r, t_{0,r}) &= \exp\bigg(-(i/2)(u(x, t_r) - u(x_0, t_r)) + i\int_{x_0}^{x} dx'\, \phi(P, x', t_r) \\
&\quad - \int_{t_{0,r}}^{t_r} ds\, (z^{-1}\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \widetilde{G}_{r-1}(z, x_0, s))\bigg), \\
&= \exp\bigg(iy\int_{x_0}^{x} dx'\, F_n(z, x', t_r)^{-1} - (y/z)\int_{t_{0,r}}^{t_r} ds\, \widetilde{F}_r(z, x_0, s)F_n(z, x_0, s)^{-1} \\
&\quad - (i/2)(u(x, t_r) - u(x_0, t_r)) - iz\int_{x_0}^{x} dx'\, G_{n-1}(z, x', t_r)F_n(z, x', t_r)^{-1} \\
&\quad + \int_{t_{0,r}}^{t_r} ds\, \big(\widetilde{F}_r(z, x_0, s)F_n(z, x_0, s)^{-1}G_{n-1}(z, x_0, s) - \widetilde{G}_{r-1}(z, x_0, s)\big)\bigg)
\end{aligned}$$

by means of (2.160). By first invoking (2.149) and subsequently (2.171), we obtain relation (2.181). Evaluating (2.181) at the points P and P^* and multiplying the resulting expressions yield (2.182). The remaining statements (2.183)–(2.185) are direct consequences of (2.169), (2.170), and (2.181). $\square$

The stationary Dubrovin-type equations in Lemma 2.6 have analogs for each sGmKdV$_r$ flow (indexed by the parameter t_r) that govern the dynamics of μ_j and ν_j with respect to variations of x and t_r. In this context the stationary case simply corresponds to the special case $r = 0$, as described in the following result.

Lemma 2.23 *Assume Hypothesis* 2.18 *and* (2.137), (2.138) *hold on an open and connected set* $\widetilde{\Omega}_\mu \subseteq \mathbb{R}^2$, *and suppose that the zeros* μ_j, $j = 1, \dots, n$, *of* $F_n(\,\cdot\,)$ *remain distinct and nonzero on* $\widetilde{\Omega}_\mu$. *Then* $\{\hat{\mu}_j\}_{j=1,\dots,n}$, *defined by* (2.158), *satisfies the following first-order system of differential equations on* $\widetilde{\Omega}_\mu$

$$\mu_{j,x} = -2iy(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k)^{-1}, \tag{2.186}$$

$$\mu_{j,t_r} = 2\widetilde{F}_r(\mu_j)\mu_j^{-1}y(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, n. \tag{2.187}$$

Next, assume the affine part of $\mathcal{K}_n$ *to be nonsingular and introduce the initial condition*

$$\{\hat{\mu}_j(x_0, t_{0,r})\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{2.188}$$

for some $(x_0, t_{0,r}) \in \mathbb{R}^2$, *where* $\mu_j(x_0, t_{0,r}) \neq 0$, $j = 1, \dots, n$, *are assumed to be distinct. Then there exists an open and connected set* $\Omega_\mu \subseteq \mathbb{R}^2$, *with* $(x_0, t_{0,r}) \in \Omega_\mu$, *such that the initial value problem* (2.186)–(2.188) *has a unique solution* $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ *satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{2.189}$$

and μ_j, $j = 1, \dots, n$, *remain distinct and nonzero on* Ω_μ.
For the zeros ν_j, $j = 1, \dots, n$, *of* $H_n(\,\cdot\,)$ *identical statements hold with* μ_j *and* Ω_μ *replaced by* ν_j *and* Ω_ν, *etc. In particular,* $\{\hat{\nu}_j\}_{j=1,\dots,n}$, *defined by* (2.159), *satisfies the system*

$$\nu_{j,x} = -2iy(\hat{\nu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\nu_j - \nu_k)^{-1}, \tag{2.190}$$

$$\nu_{j,t_r} = 2\widetilde{H}_r(\nu_j)\nu_j^{-1}y(\hat{\nu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, n. \tag{2.191}$$

Proof It suffices to prove (2.187) and (2.189) since the argument for (2.191) is analogous to that of (2.187) and that for (2.186) and (2.190) has been given in the proof of Lemma 2.6. Inserting $z = \mu_j$ into (2.171) with (2.158) observed yields

$$F_{n,t_r}(\mu_j) = -\mu_{j,t_r}\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k) = 2\frac{\widetilde{F}_r(\mu_j)}{\mu_j}\mu_j G_{n-1}(\mu_j) = -2\frac{\widetilde{F}_r(\mu_j)}{\mu_j}y(\hat{\mu}_j)$$

and hence (2.187). For the proof of (2.189) one invokes again the charts (B.3)–(B.6) and (B.12)–(B.15). As in the stationary case, the only nontrivial issue to check is the case in which $\hat{\mu}_j$ hits one of the branch points $(E_m, 0) \in \mathcal{B}(\mathcal{K}_n)$, $m \neq 0$, and hence the right-hand sides of (2.186) and (2.187) vanish. Thus, we suppose

$$\mu_{j_0}(x, t_r) \to E_{m_0} \text{ as } (x, t_r) \to (x_0, t_{0,r}) \in \Omega_\mu$$

for some $j_0 \in \{1, \dots, n\}$, $m_0 \in \{1, \dots, 2n\}$. Upon introduction of

$$\begin{aligned} &\zeta_{j_0}(x, t_r) = \sigma(\mu_{j_0}(x, t_r) - E_{m_0})^{1/2}, \ \sigma = \pm 1, \\ &\mu_{j_0}(x, t_r) = E_{m_0} + \zeta_{j_0}(x, t_r)^2 \end{aligned}$$

for (x, t_r) in a sufficiently small neighborhood of $(x_0, t_{0,r})$, the Dubrovin equations (2.186), (2.187) for μ_{j_0} become

$$\begin{aligned} \zeta_{j_0,x}(x, t_r) \underset{(x,t_r)\to(x_0,t_{0,r})}{=} & c(\sigma) \left(\prod_{\substack{m=0 \\ m\neq m_0}}^{2n} (E_{m_0} - E_m) \right)^{1/2} \\ & \times \left(\prod_{\substack{k=1 \\ k\neq j_0}}^{n} \big(E_{m_0} - \mu_k(x, t_r)\big)^{-1} \right) \big(1 + O(\zeta_{j_0}(x, t_r)^2)\big), \\ \zeta_{j_0,t_r}(x, t_r) \underset{(x,t_r)\to(x_0,t_{0,r})}{=} & -ic(\sigma)\widetilde{F}_r(E_{m_0}, x_0, t_{0,r})E_{m_0}^{-1} \times \left(\prod_{\substack{m=0 \\ m\neq m_0}}^{2n} (E_{m_0} - E_m) \right)^{1/2} \\ & \times \left(\prod_{\substack{k=1 \\ k\neq j_0}}^{n} \big((E_{m_0} - \mu_k(x, t_r)\big)^{-1} \right) \big(1 + O(\zeta_{j_0}(x, t_r)^2)\big) \end{aligned}$$

for some $|c(\sigma)| = 1$, and one concludes (2.189). $\square$

Since the stationary trace formulas for sGmKdV invariants in terms of symmetric functions of μ_j and ν_j in Lemma 2.7 extend line by line to the corresponding time-dependent setting, we next record their t_r-dependent analogs without proof.

Lemma 2.24 *Assume Hypothesis* 2.18 *and* (2.157) *and suppose that* (2.137), (2.138) *hold. Then,*

$$u = i \ln\left((-1)^n \alpha^{-1} \prod_{j=1}^{n} \mu_j \right) \tag{2.192}$$

$$= -i \ln\left((-1)^n \beta^{-1} \prod_{j=1}^{n} \nu_j \right), \tag{2.193}$$

where $\alpha\beta = \prod_{m=1}^{2n} E_m \neq 0$. *In particular, one infers the constraint*

$$\prod_{j=1}^{n} \mu_j \nu_j = \prod_{m=1}^{2n} E_m. \tag{2.194}$$

Proof The proof is identical to that of relations (2.91)–(2.93). □

Remark 2.25 Consider the case $n \in \mathbb{N}$, and $r = 0$, and $\alpha = \beta = i/4$. Then

$$u(x,t_0) = i \ln\Bigg((-1)^n \alpha^{-1} \prod_{j=1}^{n} \mu_j(x,t_0)\Bigg)$$

satisfies

$$u_{xt_0} = \sin(u)$$

whenever $\{\hat{\mu}_j(x,t_0)\}_{j=1,\dots,n}$ satisfies (2.186) and (2.187) for $r = 0$. Conversely, when solving the Dubrovin-type equations (2.186), (2.187), the trace relation (2.192) for $u(x,t_r)$ yields algebro-geometric solutions of (higher-order) sGmKdV equations as will be shown in Theorem 2.34.

Now we turn to asymptotic properties of ϕ (which are proven as in Lemma 2.9).

Lemma 2.26 *Assume Hypothesis* 2.18 *and* (2.157) *and suppose that* (2.137), (2.138) *hold. Moreover, let* $P = (z,y) \in \mathcal{K}_n \setminus \{P_\infty\}$. *Then, as* $P \to P_\infty$,

$$\phi(P) \underset{\zeta\to 0}{=} \zeta^{-1} + (1/2)u_x + \big((1/8)u_x^2 + (i/4)u_{xx}\big)\zeta + O(\zeta^2),$$
$$\zeta = \sigma/z^{1/2},\ \sigma = \pm 1.$$

Next we turn to one of the principal results of this section, the representation of ϕ, Ψ, and u in terms of the Riemann theta function associated with $\mathcal{K}_n$ assuming (2.86). We start by introducing some notation.

Let $\omega^{(2)}_{P_\infty,2q}$ be a normalized differential of the second kind with unique pole at P_∞ and principal part $\zeta^{-2q-2}d\zeta$ near P_∞ (cf. (A.20), (A.21), and (A.22)) and define

$$\widetilde{\Omega}^{(2)}_{P_\infty,r} = \begin{cases} \sum_{q=0}^{r-1}(2q+1)\tilde{c}_{r-1-q}\,\omega^{(2)}_{P_\infty,2q} & \text{for } r \in \mathbb{N}, \\ 0 & \text{for } r = 0, \end{cases} \quad \tilde{c}_0 = 1, \tag{2.195}$$

where $\tilde{c}_q$ are the constants introduced in the definition of $\widetilde{F}_r$. Thus, one infers

$$\int_{a_j} \widetilde{\Omega}^{(2)}_{P_\infty,r} = 0, \quad j = 1, \dots, n, \tag{2.196}$$

$$\int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,r} \underset{\zeta\to 0}{=} \begin{cases} -\sum_{q=0}^{r-1} \tilde{c}_{r-1-q}\zeta^{-2q-1} + O(\zeta), & r \in \mathbb{N} \\ 0, & r = 0, \end{cases} \text{ as } P \to P_\infty, \tag{2.197}$$

choosing Q_0 to be a branch point different from P_0 and P_∞. The corresponding vector of b-periods of $\widetilde{\Omega}^{(2)}_{P_\infty,2r-2}/(2\pi i)$ is then denoted by $\underline{\widetilde{U}}^{(2)}_{2r-2}$,

$$\underline{\widetilde{U}}^{(2)}_{2r-2} = \big(\widetilde{U}^{(2)}_{2r-2,1}, \dots, \widetilde{U}^{(2)}_{2r-2,n}\big), \quad \widetilde{U}^{(2)}_{2r-2,j} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}^{(2)}_{P_\infty,2r-2}, \quad j = 1, \dots, n. \tag{2.198}$$

One computes from (B.33),

$$\widetilde{U}^{(2)}_{2r-2,j} = -2\sum_{q=0}^{r-1} \tilde{c}_{r-1-q} \sum_{k=1}^{n} c_j(k)\hat{c}_{k-n+q}(\underline{E}), \quad j = 1, \dots, n, \tag{2.199}$$

with $\hat{c}_k(\underline{E})$ defined in (B.32), using the convention that $\hat{c}_{-k}(\underline{E}) = 0$ for $k \in \mathbb{N}$. Moreover, let $\omega^{(2)}_{P_0,0}$ be the normalized differential of the second kind, holomorphic on $\mathcal{K}_n \setminus \{P_0\}$. Then (cf. (B.34)–(B.36)),

$$\omega^{(2)}_{P_0,0} = \frac{Q^{1/2}}{2}\frac{dz}{zy} \underset{\zeta\to 0}{=} (\zeta^{-2} + O(1))d\zeta \text{ as } P \to P_0 \tag{2.200}$$

and

$$\frac{\tilde{\alpha}}{\alpha} Q^{1/2} \int_{Q_0}^{P} \omega^{(2)}_{P_0,0} \underset{\zeta\to 0}{=} -\frac{\tilde{\alpha}}{\alpha} Q^{1/2}\zeta^{-1} + O(\zeta) \text{ as } P \to P_0, \tag{2.201}$$

with local coordinate $\zeta = \sigma' z^{1/2}$, $\sigma' \in \{1, -1\}$ near P_0. Since by (2.200), $\int_{Q_0}^{P} \omega^{(2)}_{P_0,0} + \int_{Q_0}^{P^*} \omega^{(2)}_{P_0,0} = 0$, choosing the same path of integration on both sheets $\Pi_\pm$, the right-hand side of (2.201) is odd with respect to ζ and hence contains no constant term. The vector of b-periods of $(\tilde{\alpha}/\alpha)Q^{1/2}\omega^{(2)}_{P_0,0}/(2\pi i)$ will be denoted by $\underline{W}^{(2)}_0$ in the following, and one computes (cf. (B.36)),

$$\underline{W}^{(2)}_0 = \big(W^{(2)}_{0,1}, \dots, W^{(2)}_{0,n}\big), \quad W^{(2)}_{0,j} = \frac{\tilde{\alpha}}{\alpha}\frac{Q^{1/2}}{2\pi i}\int_{b_j} \omega^{(2)}_{P_0,0} = 2\frac{\tilde{\alpha}}{\alpha}c_j(1), \quad j = 1, \dots, n. \tag{2.202}$$

Recalling the abbreviations in (2.108) and our choice of base point $Q_0 = (E_{m_0}, 0)$, we can now state one of the principal results of this section.

Theorem 2.27 *Assume Hypothesis* 2.18 *and* (2.137), (2.138) *hold on* Ω *subject to the constraint* (2.86). *In addition, let* $P \in \mathcal{K}_n \setminus \{P_\infty, P_0\}$ *and* (x, t_r), $(x_0, t_{0,r}) \in \Omega$, *where $\Omega \subseteq \mathbb{R}^2$ is open and connected. Moreover, suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ is nonspecial for $(x,t_r) \in \Omega$. Then ϕ admits the representation*

$$\phi(P,x,t_r) = \frac{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}(x,t_r))\big)}{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}(x,t_r), \underline{\Delta})\big)} \frac{\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x,t_r), \underline{\Delta})\big)}{\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x,t_r))\big)} \times \exp\bigg(-\int_{Q_0}^{P} \omega^{(3)}_{P_\infty,P_0} + (1/2)\ln(E_{m_0})\bigg) \tag{2.203}$$

with the half-period $\underline{\Delta}$ defined in (2.110). *The components ψ_j, $j=1,2$ of the Baker–Akhiezer vector Ψ are given by*[1]

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r}) = {} & \frac{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}(x_0,t_{0,r}))\big)\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x,t_r))\big)}{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}(x,t_r))\big)\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x_0,t_{0,r}))\big)} \\ & \times \exp\bigg(-i(x-x_0)\int_{Q_0}^{P} \omega^{(2)}_{P_\infty,0} \\ & \quad + (t_r - t_{0,r})\bigg(\frac{\tilde{\alpha}}{\alpha} Q^{1/2} \int_{Q_0}^{P} \omega^{(2)}_{P_0,0} + \int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,r}\bigg)\bigg)\end{aligned} \tag{2.204}$$

and

$$\begin{aligned}\psi_2(P,x,x_0,t_r,t_{0,r}) = {} & -\frac{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}(x_0,t_{0,r}))\big)\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x,t_r), \underline{\Delta})\big)}{\theta\big(\underline{z}(P_\infty, \underline{\hat{\mu}}(x,t_r), \underline{\Delta})\big)\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x_0,t_{0,r}))\big)} \\ & \times \exp\bigg(-\int_{Q_0}^{P} \omega^{(3)}_{P_\infty,P_0} + (1/2)\ln(E_{m_0}) - i(x-x_0)\int_{Q_0}^{P} \omega^{(2)}_{P_\infty,0} \\ & + (t_r - t_{0,r})\bigg(\frac{\tilde{\alpha}}{\alpha} Q^{1/2} \int_{Q_0}^{P} \omega^{(2)}_{P_0,0} + \int_{Q_0}^{P} \widetilde{\Omega}^{(2)}_{P_\infty,r}\bigg)\bigg).\end{aligned} \tag{2.205}$$

The Abel map linearizes the auxiliary divisors in the sense that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) + i\underline{U}_0^{(2)}(x-x_0) + \underline{d}_r(t_r - t_{0,r}), \tag{2.206}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})}) + i\underline{U}_0^{(2)}(x-x_0) + \underline{d}_r(t_r - t_{0,r}), \tag{2.207}$$

where $\underline{U}_0^{(2)}$ is defined in (2.107) *and $\underline{d}_r$ is given by (cf.* (2.199)*)*

$$\underline{d}_r = \begin{cases} -\underline{\widetilde{U}}^{(2)}_{2r-2} - 2(\tilde{\alpha}/\alpha)\underline{c}(1) & \text{for } r \in \mathbb{N}, \\ -2(\tilde{\alpha}/\alpha)\underline{c}(1) & \text{for } r = 0. \end{cases} \tag{2.208}$$

[1] To avoid multi-valued expressions in formulas such as (2.203)–(2.205), etc., we agree to always choose the same path of integration connecting Q_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

Finally, u is of the form

$$u(x,t_r) = -(i/2)\ln(\alpha/\beta) \pmod{2\pi\mathbb{Z}} + 2i\ln\left(\frac{\theta\big(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_r),\underline{\Delta})\big)}{\theta\big(\underline{z}(P_\infty,\underline{\hat{\mu}}(x,t_r))\big)}\right). \tag{2.209}$$

Proof First one observes that the proofs of (2.203) and (2.209) carry over without changes from the stationary situation described in Theorem 2.10 since they are based on the Riccati-type equation (2.165) in both cases with essentially the same expression for ϕ. Hence, we turn to the Baker–Akhiezer vector Ψ whose first component ψ_1 is given by (2.181). We temporarily assume

$$\mu_j(x,t_r) \neq \mu_{j'}(x,t_r) \text{ for } j \neq j' \text{ and } (x,t_r) \in \widetilde{\Omega} \tag{2.210}$$

for appropriate $\widetilde{\Omega} \subseteq \Omega$. The time-dependent term in the exponential of (2.181) has two potential singularities, one at P_∞, and the other at P_0. We first study this term as $P \to P_\infty$. First assume that $\widetilde{F}_r$ is homogeneous, that is, $\widetilde{F}_r = \widehat{F}_r$. Then, using the asymptotic spectral parameter expansion (1.92) at $x = x_0$ (identifying the KdV potential u with the expression $-(1/4)(u_x^2 + 2iu_{xx})$ in the sGmKdV context, cf. (2.12)), equation (2.41) at $\ell = r$, as well as (2.36) (with n, α, β replaced by $r, \tilde{\alpha}, \tilde{\beta}$), we obtain

$$\begin{aligned}\frac{y\widehat{F}_r(z,x_0,t_r)}{zF_n(z,x_0,t_r)} &= \zeta^2\left(\tilde{f}_r + \sum_{q=1}^{r}\hat{f}_{r-q}(x_0,t_r)z^q\right)\frac{y}{F_n(z,x_0,t_r)}\\ &= \zeta^{-2r+1} + (\tilde{f}_r(x_0,t_r) - \hat{f}_r(x_0,t_r))\zeta + O(\zeta^3) \text{ as } P \to P_\infty.\end{aligned}$$

Here $\hat{f}_r$ is computed from the KdV recursion (2.11), whereas $\tilde{f}_r = \tilde{\alpha}e^{-iu}$. Similarly,

$$\frac{y\widehat{F}_q(z,x_0,t_r)}{zF_n(z,x_0,t_r)} \underset{\zeta\to 0}{=} \zeta^{-2q+1} + O(\zeta^3) \text{ as } P \to P_\infty, \quad q = 0,\dots,r-1.$$

Hence, one concludes

$$\begin{aligned}\frac{y\widetilde{F}_r(z,x_0,t_r)}{zF_n(z,x_0,t_r)} &= \sum_{q=0}^{r}\tilde{c}_{r-q}\frac{y\widehat{F}_q(z,x_0,t_r)}{zF_n(z,x_0,t_r)}\\ &\underset{\zeta\to 0}{=} \begin{cases}\sum_{q=1}^{r}\tilde{c}_{r-q}\zeta^{-2q+1} + O(\zeta) & \text{for } r \in \mathbb{N},\\ \zeta + O(\zeta^3) & \text{for } r = 0,\end{cases}\end{aligned}$$

and thus

$$\begin{aligned}&\frac{y}{z}\int_{t_{0,r}}^{t_r} ds\,\frac{\widetilde{F}_r(z,x_0,s)}{F_n(z,x_0,s)}\\ &\underset{\zeta\to 0}{=} \begin{cases}\big(\sum_{q=0}^{r-1}\tilde{c}_{r-1-q}\zeta^{-2q-1}\big)(t_r - t_{0,r}) + O(\zeta) & \text{for } r \in \mathbb{N},\\ \zeta(t_r - t_{0,r}) + O(\zeta^3) & \text{for } r = 0.\end{cases}\end{aligned} \tag{2.211}$$

Secondly, we study the behavior near P_0 (using the local coordinate $\zeta = \sigma z^{1/2} \to 0$, $\sigma = \pm 1$, cf. (2.100)). One finds

$$\frac{y\widetilde{F}_r(z, x_0, t_r)}{zF_n(z, x_0, t_r)} \underset{\zeta\to 0}{=} \frac{\tilde{\alpha}}{\alpha} Q^{1/2}\zeta^{-1} + O(\zeta) \text{ as } P \to P_0,$$

and hence

$$\frac{y}{z}\int_{t_{0,r}}^{t_r} ds\, \frac{\widetilde{F}_r(z, x_0, s)}{F_n(z, x_0, s)} \underset{\zeta\to 0}{=} \frac{\tilde{\alpha}}{\alpha} Q^{1/2}\zeta^{-1}(t_r - t_{0,r}) + O(\zeta) \text{ as } P \to P_0, \tag{2.212}$$

where $Q^{1/2} = \big(\prod_{m=1}^{2n} E_m\big)^{1/2}$ and the sign of $Q^{1/2}$ is determined by the compatibility of the charts. A comparison of (2.197), the expression (2.204) for $\tilde{\psi}$, (2.181), (2.211), and (2.212) then identifies the t_r-dependent behavior of the exponentials of ψ and $\tilde{\psi}$ up to order $O(\zeta)$ near P_∞ and P_0. The x-dependent exponential behavior of ψ and $\tilde{\psi}$ can be discussed as in the stationary context of Theorem 2.10 (cf. (2.122)). Next we turn to the local behavior of ψ and $\tilde{\psi}$ and compare their zeros and poles. One first observes that (2.158), (2.160), and (2.187) imply (cf. (2.120))

$$\begin{aligned} i\phi(P, x', t_r) &\underset{P\to\hat{\mu}_j(x',t_r)}{=} \partial_{x'} \ln(z - \mu_j(x', t_r)) + O(1), \\ -(1/z)\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) &\underset{P\to\hat{\mu}_j(x_0,s)}{=} \partial_s \ln(z - \mu_j(x_0, s)) + O(1). \end{aligned}$$

Together with (2.163), this yields

$$\begin{aligned} &\psi(P, x, x_0, t_r, t_{0,r}) \\ &= \begin{cases} (z - \mu_j(x, t_r))O(1) & \text{as } P \to \hat{\mu}_j(x, t_r) \neq \hat{\mu}_j(x_0, t_{0,r}), \\ O(1) & \text{as } P \to \hat{\mu}_j(x, t_r) = \hat{\mu}_j(x_0, t_{0,r}), \\ (z - \mu_j(x_0, t_{0,r}))^{-1}O(1) & \text{as } P \to \hat{\mu}_j(x_0, t_{0,r}) \neq \hat{\mu}_j(x, t_r), \end{cases} \\ &\qquad\qquad P = (z, y) \in \mathcal{K}_n,\ (x, t_r), (x_0, t_{0,r}) \in \widetilde{\Omega}, \end{aligned} \tag{2.213}$$

where $O(1) \neq 0$ in (2.213). Consequently, all zeros and poles of ψ and $\tilde{\psi}$ on $\mathcal{K}_n \setminus \{P_\infty\}$ are simple and coincide. Thus, ψ and $\tilde{\psi}$ share all singularities and zeros, and an application of the Riemann–Roch-type uniqueness result in Lemma B.2 then proves that ψ and $\tilde{\psi}$ coincide up to normalization. The latter is determined by (2.182), as in the stationary context (2.123). This proves (2.204) subject to (2.210).

Next we indicate the proof of (2.206) using Lemma 2.23 and following the corresponding argument in the proof of Theorem 2.10. Equations (2.187), (F.8), Lagrange's interpolation theorem, Theorem E.1, and (B.33) then yield the following for $r \in \mathbb{N}$,

$$
\begin{aligned}
\partial_{t_r} \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) &= \sum_{j=1}^{n} \mu_{j,t_r} \sum_{k=1}^{n} \underline{c}(k) \frac{\mu_j^{k-1}}{y(\hat{\mu}_j)} \\
&= 2 \sum_{j,k=1}^{n} \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} (\mu_j - \mu_\ell)} \frac{\widetilde{F}_r(\mu_j)}{\mu_j} \\
&= 2 \sum_{j,k=1}^{n} \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} (\mu_j - \mu_\ell)} \Bigg(\sum_{q=0}^{r-1} \sum_{p=(q-n)\vee 0}^{q} \tilde{c}_{r-1-q} \, \hat{c}_p(\underline{E}) \\
&\quad \times \Phi_{q-p}^{(j)}(\underline{\mu}) - \frac{\tilde{\alpha}}{\alpha} \Phi_{n-1}^{(j)}(\underline{\mu}) \Bigg) \\
&= 2 \sum_{k=1}^{n} \underline{c}(k) \Bigg(\sum_{q=0}^{r-1} \sum_{p=(q-n)\vee 0}^{q} \tilde{c}_{r-1-q} \, \hat{c}_p(\underline{E}) \sum_{j=1}^{n} \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} (\mu_j - \mu_\ell)} \\
&\quad \times \Phi_{q-p}^{(j)}(\underline{\mu}) - \frac{\tilde{\alpha}}{\alpha} \sum_{j=1}^{n} \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} (\mu_j - \mu_\ell)} \Phi_{n-1}^{(j)}(\underline{\mu}) \Bigg) \\
&= 2 \sum_{k=1}^{n} \underline{c}(k) \Bigg(\sum_{q=0}^{r-1} \sum_{p=(q-n)\vee 0}^{q} \tilde{c}_{r-1-q} \, \hat{c}_p(\underline{E}) \delta_{k,n-(q-p)} - \frac{\tilde{\alpha}}{\alpha} \delta_{k,1} \Bigg) \\
&= \underline{d}_r
\end{aligned}
$$

and hence (2.206). (This computation is equivalent to that in (F.87); see also Corollary F.11.) The extension of all these results from $\widetilde{\Omega}$ to Ω then follows by continuity of $\underline{\alpha}_{Q_0}$ and nonspecialty of $\mathcal{D}_{\underline{\hat{\mu}}}$ on Ω. Equation (2.207) then follows from (2.206) and the linear equivalence of $\mathcal{D}_{P_\infty \underline{\hat{\mu}}}$ and $\mathcal{D}_{P_0 \underline{\hat{\nu}}}$, that is,

$$
\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{\Delta}. \tag{2.214}
$$

□

Combining (2.206), (2.207), (2.214), and (2.209) shows the remarkable linearity of the theta function arguments with respect to x and t_r in the formula for u. In fact, one can rewrite (2.209) as

$$
u(x, t_r) = c_0 + 2i \ln \left(\frac{\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r + \underline{\Delta})}{\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r)} \right), \tag{2.215}
$$

where

$$
\underline{A} = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) - i \underline{U}_0^{(2)} x_0 - \underline{d}_r t_{0,r} + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0, t_{0,r})}), \tag{2.216}
$$

$$
\underline{B} = i \underline{U}_0^{(2)}, \quad \underline{C}_r = \underline{d}_r, \tag{2.217}
$$

$$
\underline{\Delta} = \underline{A}_{P_0}(P_\infty), \quad c_0 = -(i/2) \ln(\alpha/\beta) \pmod{2\pi\mathbb{Z}}, \tag{2.218}
$$

and hence the constants $\underline{\Delta}, \underline{B}, \underline{C}_r \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ and r, and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0, t_{0,r}) = (\hat{\mu}_1(x_0, t_{0,r}), \dots, \hat{\mu}_n(x_0, t_{0,r})) \in \operatorname{Sym}^n(\mathcal{K}_n)$ at the point $(x_0, t_{0,r})$ as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}$ is assumed to be nonspecial.

Remark 2.28 The explicit expressions (2.204), (2.205) for ψ_j, $j = 1, 2$ again complement Lemma 2.19 and show that Ψ stays meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial (assuming the affine part of $\mathcal{K}_n$ to be nonsingular).

Remark 2.29 The linearization property (2.206) (and (2.113)) can also be obtained as follows. One introduces the meromorphic differential

$$\Omega_1(x, x_0, t_r, t_{0,r}) = \partial_z \ln(\psi_1(\,\cdot\,, x, x_0, t_r, t_{0,r}))dz$$

and hence infers from the representation (2.204) that

$$\begin{aligned}\Omega_1(x, x_0, t_r, t_{0,r}) = &-i(x - x_0)\omega^{(2)}_{P_\infty,0} + (t_r - t_{0,r})\widetilde{\Omega}^{(2)}_{P_\infty,2r-2} \\ &+ (t_r - t_{0,r})\frac{\tilde{\alpha}}{\alpha} Q^{1/2}\omega^{(2)}_{P_0,0} - \sum_{j=1}^n \omega^{(3)}_{\hat{\mu}_j(x_0,t_{0,r}),\hat{\mu}_j(x,t_r)} + \omega.\end{aligned}$$

Here ω denotes a holomorphic differential on $\mathcal{K}_n$, that is,

$$\omega = \sum_{j=1}^n c_j \omega_j$$

for some $c_j \in \mathbb{C}$, $j = 1, \dots, n$. Since $\psi_1(\,\cdot\,, x, x_0, t_r, t_{0,r})$ is single-valued on $\mathcal{K}_n$, all a- and b-periods of Ω_1 are integer multiples of $2\pi i$, and hence

$$2\pi i m_k = \int_{a_k} \Omega_1(x, x_0, t_r, t_{0,r}) = \int_{a_k} \omega = c_k, \quad j = 1, \dots, n$$

for some $m_k \in \mathbb{Z}$ identifies c_k as integer multiples of $2\pi i$. Similarly, for some $n_k \in \mathbb{Z}$,

$$\begin{aligned}2\pi i n_k &= \int_{b_k} \Omega_1(x, x_0, t_r, t_{0,r}) \\ &= -i(x - x_0)\int_{b_k} \omega^{(2)}_{P_\infty,0} + (t_r - t_{0,r})\frac{\tilde{\alpha}}{\alpha} Q^{1/2}\int_{b_k} \omega^{(2)}_{P_0,0} \\ &\quad + (t_r - t_{0,r})\int_{b_k} \widetilde{\Omega}^{(2)}_{P_\infty,2r-2} - \sum_{j=1}^n \int_{b_k} \omega^{(3)}_{\hat{\mu}_j(x_0,t_{0,r}),\hat{\mu}_j(x,t_r)} + 2\pi i \sum_{j=1}^n m_j \int_{b_k} \omega_j \\ &= 2\pi U^{(2)}_{0,k}(x - x_0) + 2\pi i\big(\widetilde{U}^{(2)}_{2r-2,k} + W^{(2)}_{0,k}\big)(t_r - t_{0,r}) \\ &\quad - 2\pi i \sum_{j=1}^n \underline{A}_{\hat{\mu}_j(x,t_r),k}(\hat{\mu}_j(x_0, t_{0,r})) + 2\pi i \sum_{j=1}^n m_j \tau_{j,k}, \quad k = 1, \dots, n,\end{aligned} \tag{2.219}$$

by use of (2.107), (2.198), (2.202), and (A.26). By symmetry of τ (cf. (A.15)), (2.219) is equivalent to

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) + i\underline{U}_0^{(2)}(x-x_0) \\ - \big(\underline{\widetilde{U}}_{2r-2}^{(2)} + \underline{W}_0^{(2)}\big)(t_r - t_{0,r}).$$

The algebro-geometric solution u in (2.209) is complex-valued in general. To obtain real-valued solutions, one needs to impose a certain symmetry on $\mathcal{K}_n$ and additional constraints on $\underline{A}$ in (2.215), (2.216). Since the cases $r=0$ and $r \in \mathbb{N}$ in (2.209) are quite different (cf. (2.208)), we only focus on the important case of the sine-Gordon equation, where

$$\begin{gathered} \tilde{\alpha} = \tilde{\beta} = i/4, \quad \alpha = \beta = \bigg(\prod_{m=1}^{2n} E_m\bigg)^{1/2} \in \mathbb{R}, \\ r = 0 \text{ in (2.135)}, \quad \text{and} \quad \underline{d}_0 = (2i\alpha)^{-1}\underline{c}(1) \end{gathered} \tag{2.220}$$

(recalling $\tilde{g}_{-1} = 0$, cf. (2.155)). Then the corresponding symmetry of $\mathcal{K}_n$ results in the constraint that the projections of the branch points of $\mathcal{K}_n$, different from P_0 and P_∞, occur either in pairs on $(-\infty, 0)$ or else in complex conjugate pairs. Hence, we list the corresponding symmetry constraints on the zeros of R_{2n+1} as[1]

$$\begin{aligned} &\{\widehat{E}_p\}_{p=1,\dots,2k}, \quad \widehat{E}_1 < \widehat{E}_2 < \cdots < \widehat{E}_{2k-1} < \widehat{E}_{2k} < 0, \\ &E_0 = 0, \quad \{\widetilde{E}_q, \overline{\widetilde{E}_q}\}_{q=1,\dots,\ell}, \quad k + \ell = n \end{aligned} \tag{2.221}$$

with the convention that $k=0$ corresponds to the absence of pairs on $(-\infty, 0)$ and $\ell = 0$ corresponds to the absence of complex conjugate pairs in (2.221). The occurrence of possibly complex conjugate pairs indicates the inherent non-self-adjoint character of the underlying linear (pseudo)differential expression (the analog of the Schrödinger differential expression L in the KdV case) and points to marked differences with the far simpler reality discussion in the KdV context.

We start by recalling that real-valued algebro-geometric sine-Gordon solutions are smooth (cf. the notes of this section for a pertinent reference). For simplicity we abbreviate t_0 with t since we only discuss the case $r=0$ in the following.

Lemma 2.30 *Assume* (2.220) *and* (2.221) *and suppose u is a real-valued algebro-geometric sine-Gordon solution*

$$\mathrm{sG}(u) = u_{xt} - \sin(u) = 0$$

of the type (2.209). *Then,*

$$u \in C^\infty(\mathbb{R}^2).$$

[1] Of course we still assume the affine part of $\mathcal{K}_n$ to be nonsingular; cf. (2.86).

Lemma 2.31 *Assume* (2.220) *and* (2.221), *suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}$ *is nonspecial for some* $(x_0, t_0) \in \mathbb{R}^2$, *and choose the homology basis* $\{a_j, b_j\}_{j=1}^n$ *according to Theorem* A.36 *(i). Then the algebro-geometric solution* u *in* (2.209) *is real-valued if and only if* $\underline{A}$ *in* (2.216) *(with* $\underline{d}_0$ *given by* (2.220)*) satisfies the constraint*

$$\operatorname{Re}(\underline{A}) = (1/2)\big((1/2)\underline{\operatorname{diag}}(R) - \underline{\Delta} + (0, \dots, 0, \chi_1, \dots, \chi_k)\big) \pmod{\mathbb{Z}^n},$$
$$\chi_j \in \{0, 1\}, \ j = 1, \dots, k. \tag{2.222}$$

In particular, under the present hypotheses, the set of real-valued algebro-geometric sG solutions u *in* (2.215) *consists of* 2^k *connected components indexed by* $(\chi_1, \dots, \chi_k)$, $\chi_j \in \{0, 1\}$, $j = 1, \dots, k$, *and all such solutions* u *are smooth,* $u \in C^\infty(\mathbb{R}^2)$.

Proof Define the antiholomorphic involution $\rho_+ \colon (z, y) \mapsto (\overline{z}, \overline{y})$ as in in Example A.35 (iii). For brevity we only treat the case $1 \le k \le n-1$, where $(\mathcal{K}_n, \rho_+)$ is of nondividing type, in some detail. The case $k = n$ is slightly simpler and commented on below. By Example A.35 (iii), Theorem A.36 (cf. (A.66), (A.69)–(A.71)), one infers

$$r = k + 1, \quad \overline{\tau} = R - \tau, \quad \underline{\operatorname{diag}}(R) = (\underbrace{1 \dots, 1}_{\ell}, \underbrace{0, \dots, 0}_{k}),$$
$$\overline{\theta(\underline{z})} = \theta(\underline{\overline{z}} + (1/2)\underline{\operatorname{diag}}(R)), \ \underline{z} \in \mathbb{C}^n,$$
$$\rho_+(a_j) = a_j, \quad \rho_+(b_j) = (\underline{a}R)_j - b_j, \ j = 1, \dots, n,$$
$$\underline{U}_0^{(2)} \in \mathbb{R}^n.$$

Thus,

$$\overline{\underline{B}} = -\underline{B},$$

by (2.217) and hence real-valuedness of u in (2.215) is equivalent to the condition

$$1 = \frac{\theta(\underline{A} + \underline{B}x + \underline{\Delta})\overline{\theta(\underline{A} + \underline{B}x + \underline{\Delta})}}{\theta(\underline{A} + \underline{B}x)\overline{\theta(\underline{A} + \underline{B}x)}}$$
$$= \frac{\theta(\underline{A} + \underline{B}x + \underline{\Delta})\theta(-\overline{\underline{A}} + \underline{B}x - \overline{\underline{\Delta}} + (1/2)\underline{\operatorname{diag}}(R))}{\theta(\underline{A} + \underline{B}x)\theta(-\overline{\underline{A}} + \underline{B}x + (1/2)\underline{\operatorname{diag}}(R))}.$$

This in turn is equivalent to

$$\begin{aligned} \underline{A} &= -\overline{\underline{A}} + (1/2)\underline{\operatorname{diag}}(R) - \underline{\Delta} + \underline{m}_1 + \underline{n}_1\tau, \\ \underline{A} &= -\overline{\underline{A}} + (1/2)\underline{\operatorname{diag}}(R) - \overline{\underline{\Delta}} + \underline{m}_2 + \underline{n}_2\tau \end{aligned} \tag{2.223}$$

for some $\underline{n}_1, \underline{n}_2 \in \mathbb{Z}^n$ and arbitrary $\underline{m}_1, \underline{m}_2 \in \mathbb{Z}^n$. Taking real and imaginary parts in (2.223) then yields $\underline{m}_1 = \underline{m}_2$, $\underline{n}_1 = \underline{n}_2$ and

$$\operatorname{Re}(\underline{A}) = (1/2)\big((1/2)\underline{\operatorname{diag}}(R) - \underline{\Delta} + \underline{m}_1 + \underline{n}_1 R\big), \quad \underline{m} \in \mathbb{Z}^n,$$
$$0 = \pm\operatorname{Im}(\underline{\Delta}) + \underline{n}_1\operatorname{Im}(\tau). \tag{2.224}$$

Noticing that

$$\underline{\Delta} \in \mathbb{R}^n,$$

since $\lim_{\varepsilon\downarrow 0} y(\lambda \pm i\varepsilon)$ is real-valued for $\lambda \in [0,\infty)$, one finds that (2.224) implies $\underline{n}_1 = 0$. Replacing $\underline{A}$ by $\underline{A} + \underline{m} + \underline{n}\tau$ with $\underline{m}, \underline{n} \in \mathbb{Z}^n$, then yields

$$\operatorname{Re}(\underline{A}) = (1/2)\big((1/2)\underline{\operatorname{diag}}(R) - \underline{\Delta} + \underline{m}_1 + (n_1, \dots, n_\ell, 0, \dots, 0)\big) - \underline{m},$$
$$\underline{m}_1, \underline{m} \in \mathbb{Z}^n, \;\; n_j \in \mathbb{Z}, \;\; j = 1, \dots, \ell,$$

and hence (2.222). In the case $k = n$, $\ell = 0$, where $(\mathcal{K}_n, \rho_+)$ is of dividing type, one infers $r = n+1$ and $R = 0$ according to (A.65), simplifying the formulas just presented. Finally, $u \in C^\infty(\mathbb{R}^2)$ by Lemma 2.30. □

The assumption $\widehat{E}_p < 0$ in (2.221) is crucial for the solvability of (2.224), for otherwise $\underline{\Delta}$ acquires a nontrivial imaginary part.

Remark 2.32 A careful analysis of the sine-Gordon case (pertinent references are provided in the notes to this section) reveals the following additional facts concerning the 2^k connected components of real-valued algebro-geometric sG solutions described in Lemma 2.31: In contrast to the self-adjoint KdV case, the motion of the auxiliary divisors[1] $\hat{\mu}_j(x,t)$ is not constrained to certain fixed curves on $\mathcal{K}_n$ (in addition, collisions between them may occur). Moreover, and again in sharp contrast to the KdV case, the initial projections $\mu_j(x_0,t_0)$ cannot be chosen independently from each other due to the following additional constraint they need to satisfy:

$$\prod_{j=1}^{n} |\mu_j(x_0,t_0)|^2 = \prod_{m=1}^{2n} E_m. \tag{2.225}$$

This follows from (2.194), $\alpha\beta = \alpha^2 = \prod_{m=1}^{2n} E_m > 0$, and because $g_\ell(x)$, $\ell = 0, \dots, n-1$, are real-valued, and $f_\ell(x) = \overline{h_\ell(x)}$, $\ell = 0, \dots, n$. Still, the motion of the $\hat{\mu}_j(x,t)$ can be shown to remain homologous to some linear combinations of appropriate a_j and b_j cycles. In particular, the constraint (2.225) holds for all $(x,t) \in \mathbb{R}^2$,

$$\prod_{j=1}^{n} |\mu_j(x,t)|^2 = \prod_{m=1}^{2n} E_m.$$

By Lemma 2.31, the isospectral class of real algebro-geometric sG solutions (i.e., all real algebro-geometric sG solutions corresponding to a fixed curve $\mathcal{K}_n$ constrained by (2.221)) consists of 2^k connected components. It can be shown that each such connected component is given by an n-dimensional real torus $\mathbb{T}^n$. In

[1] We recall our convention to abbreviate t_0 with t for simplicity, since we only discuss the case $r = 0$ in this remark.

other words, each pair $(E_p, \widehat{E}_p)$ on $(-\infty, 0)$ carries two degrees of freedom (resembling a kink and antikink) as opposed to a complex conjugate pair $(\widetilde{E}_q, \overline{\widetilde{E}_q})$, which carries one degree of freedom (resembling a breather). In general, the elements in each isospectral torus represent smooth quasi-periodic functions of x and t_r.

If, in addition, one is interested in spatially periodic solutions with a real period $\Omega > 0$, the additional periodicity constraints

$$i\Omega \underline{U}_0^{(2)} \in \mathbb{Z}^n \setminus \{0\}$$

must be imposed. (By (B.45) this is equivalent to $2i\Omega\underline{c}(n) \in \mathbb{Z}^n \setminus \{0\}$.)

In sharp contrast to this discussion, the corresponding sinh-Gordon reality problem turns out to be a self-adjoint one with the absence of complex conjugate pairs in (2.221). The resulting constraint on $\mathcal{K}_n$ then results in all projections of the branch points (being different from P_∞) to be in nonnegative position, and hence one can list them as

$$0 = E_0 < E_1 < \cdots < E_{2n}. \tag{2.226}$$

Moreover, as in the KdV context, the μ_j are real-valued and confined to the spectral gaps $[E_{2j-1}, E_{2j}]$. The whole discussion then parallels the one in the KdV chapter, and the corresponding isospectral set of smooth, real-valued, algebro-geometric sinh-Gordon solutions $u \in C^\infty(\mathbb{R}^2)$ associated with a fixed curve $\mathcal{K}_n$ (constrained by (2.226)) turns out to be a real n-dimensional torus $\mathbb{T}^n$ (cf. Remark 1.24).

Finally, we describe an interesting property of the time-dependent sGmKdV hierarchy in connection with its algebro-geometric solutions. In fact, equations (2.135) and (2.136) can be rewritten so that u satisfies a differential equation with a pure first-order time derivative.

Remark 2.33 The solution u of equations (2.135) and (2.136) satisfies

$$u_{t_r} = 2i\alpha^{-1}(\tilde{\alpha} g_{n-1}(u) - \alpha \tilde{g}_{r-1}(u)), \quad u\big|_{t_r = t_{0,r}} = u^{(0)}. \tag{2.227}$$

Indeed, (2.34) yields

$$f_n = \alpha e^{-iu}, \quad \tilde{f}_r = \tilde{\alpha} e^{-iu},$$

which, inserted into the constant term (i.e., the coefficient of z^0) in (2.171) results in

$$-i\alpha u_{t_r} e^{-iu} = 2(g_{n-1}\tilde{f}_r - \tilde{g}_{r-1} f_n).$$

Remark 2.33 can be illustrated as follows:

(i) Consider the case $n = 1$.

(ia) $r = 0$. Then (2.227) becomes

$$u_{t_0} + i(\tilde{\alpha}/\alpha)u_x = 0$$

with solution

$$u(x, t_0) = u^{(0)}(x - i(\tilde{\alpha}/\alpha)t_0).$$

(ib) $r = 1$. Then (2.227) reads

$$u_{t_1} + i(\tilde{\alpha}/\alpha) - 1)u_x = 0,$$

with solution

$$u(x, t_1) = u^{(0)}(x - i((\tilde{\alpha}/\alpha) - 1)t_1).$$

(ii) Consider the case $n = 2$.
(iia) $r = 0$. Then (2.227) becomes

$$u_{t_0} = (i/8)(\tilde{\alpha}/\alpha)\big(u_x^3 + 2u_{xxx}\big) - i(\tilde{\alpha}/\alpha)c_1u_x, \quad u(x, t_{0,0}) = u^{(0)}(x).$$

(iib) $r = 1$. Then (2.227) reads

$$\begin{aligned}&u_{t_1} = (i/8)(\tilde{\alpha}/\alpha)\big(u_x^3 + 2u_{xxx}\big) + i(1 - (c_1/8)(\tilde{\alpha}/\alpha))u_x,\\ &u(x, t_{0,1}) = u^{(0)}(x).\end{aligned}$$

(iic) $r = 2$. Then (2.227) becomes

$$\begin{aligned}&u_{t_2} = -(i/8)((\tilde{\alpha}/\alpha) - 1)\big(u_x^3 + 2u_{xxx}\big) + i(\tilde{c}_1 - (\tilde{\alpha}/\alpha)c_1)u_x,\\ &u(x, t_{0,2}) = u^{(0)}(x).\end{aligned}$$

Up to this point we assumed Hypothesis 2.18 together with the basic equations (2.137) and (2.138). Next we will show that solvability of the Dubrovin equations (2.186), (2.187) on $\Omega_\mu \subseteq \mathbb{R}^2$ in fact implies equations (2.137) and (2.138) on Ω_μ. In complete analogy to our discussion in Section 2.3 (cf. Remark 2.17), this amounts to solving the time-dependent algebro-geometric initial value problem (2.135), (2.136) on Ω_μ. In this context we recall the definition of $\widehat{F}_r(\mu_j)/\mu_j$ introduced in (F.8) in the homogeneous case[1]

$$\frac{\widehat{F}_r(\mu_j)}{\mu_j} = \sum_{s=(r-1-n)\vee 0}^{r-1} \hat{c}_s(\underline{E})\Phi_{r-1-s}^{(j)}(\underline{\mu}) - \frac{\tilde{\alpha}}{\alpha}\Phi_{n-1}^{(j)}(\underline{\mu}), \tag{2.228}$$

with $\Phi_k^{(j)}(\underline{\mu})$ given by (E.2). The expression $\widetilde{F}_r(\mu_j)/\mu_j$ is then defined by[2]

$$\frac{\widetilde{F}_r(\mu_j)}{\mu_j} = \sum_{s=0}^{r-1} \tilde{c}_{r-s}\frac{\widehat{F}_s(\mu_j)}{\mu_j} + \frac{\widetilde{\alpha}}{\alpha}(-1)^n \prod_{\substack{k=1\\k\neq j}}^{n} \mu_k \tag{2.229}$$

[1] $m \vee n = \max\{m, n\}$.
[2] Since r is independent of n, one obtains $\hat{f}_r = \tilde{f}_r = \tilde{\alpha}e^{-iu}$, $\hat{h}_r = \tilde{h}_r = \tilde{\beta}e^{iu}$ with $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}$ independent of α, β, and $\hat{f}_q, \hat{h}_q$, $q = 1, \dots, r-1$ constructed as in (2.31) and (2.32).

in terms of a given set of integration constants $\{\tilde{c}_1, \dots, \widetilde{c}_r\} \subset \mathbb{C}$, $\tilde{\alpha} \in \mathbb{C} \setminus \{0\}$, subject to the constraint $\alpha\tilde{\beta} = \tilde{\alpha}\beta$.

Theorem 2.34 *Fix $n \in \mathbb{N}$, $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in \mathbb{C} \setminus \{0\}$ with $\alpha\tilde{\beta} = \tilde{\alpha}\beta$ and assume* (2.86)*. Suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the Dubrovin equations* (2.186), (2.187) *on an open and connected set $\Omega_\mu \subseteq \mathbb{R}^2$ with $\widetilde{F}_r(\mu_j)/\mu_j$ in* (2.187) *expressed in terms of μ_k, $k = 1, \dots, n$, by* (2.228) *and* (2.229)*. Moreover, assume that μ_j, $j = 1, \dots, n$, remain distinct and nonzero on Ω_μ. Then $u \in C^\infty(\Omega_\mu)$, defined by*

$$u = i \ln\left((-1)^n \alpha^{-1} \prod_{j=1}^{n} \mu_j\right), \tag{2.230}$$

satisfies the rth sGmKdV equation (2.135)*, that is,*

$$\widetilde{\mathrm{sGmKdV}}_r(u) = 0 \tag{2.231}$$

with initial values satisfying the nth stationary sGmKdV equation (2.136) *subject to the constraint $\alpha\beta \neq 0$.*

Proof Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (2.186), (2.187), we define u by (2.230) and F_n on $\mathbb{C} \times \Omega_\mu$ by

$$F_n(z) = \prod_{j=1}^{n} (z - \mu_j).$$

Following the proof of Theorem 2.14 in the stationary case, we introduce polynomials G_{n-1} and H_n satisfying (2.149)–(2.151) (subject to $\alpha\beta = \prod_{m=1}^{2n} E_m$), and (2.156), that is,

$$F_{n,x}(z) = -iu_x F_n(z) - 2izG_{n-1}(z), \tag{2.232}$$

$$H_{n,x}(z) = iu_x H_n(z) + 2izG_{n-1}(z), \tag{2.233}$$

$$G_{n-1,x}(z) = i(H_n(z) - F_n(z)), \tag{2.234}$$

$$R_{2n+1}(z) = zF_n(z)H_n(z) + z^2 G_{n-1}(z)^2 \tag{2.235}$$

on $\mathbb{C} \times \Omega_\mu$, treating t_r as a parameter. In particular, u satisfies (2.138) on Ω_μ. Hence, it suffices to focus on the proof of (2.146)–(2.148), and (2.154).
Next, we define $\widetilde{F}_r$ on $\mathbb{C} \times \Omega_\mu$ in terms of the homogeneous polynomial $\widehat{F}_r$ and integration constants $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$, $\tilde{\alpha} \in \mathbb{C} \setminus \{0\}$, by

$$\widetilde{F}_r = \sum_{s=0}^{r} \tilde{c}_{r-s} \widehat{F}_s \text{ on } \mathbb{C} \times \Omega_\mu, \quad \tilde{c}_0 = 1,$$

where

$$\widehat{F}_r(z) = z\widehat{F}_{r-1}(z) + \hat{f}_r, \quad \hat{f}_r = \tilde{\alpha} e^{-iu} \text{ on } \mathbb{C} \times \Omega_\mu, \tag{2.236}$$

and $\widehat{F}_{r-1}$ is defined by (F.10) or (F.12). The function $\widetilde{G}_{r-1}$ is then introduced by

$$\widetilde{G}_{r-1}(z) = (i/2)z^{-1}\big(\widetilde{F}_{r,x}(z) + iu_x\widetilde{F}_r(z)\big) \text{ on } \mathbb{C}\times\Omega_\mu,$$

and remains well-defined at $z=0$ by (2.236) (which implies (2.154)). In particular, this yields (2.147). Next we claim that (cf. (2.171))

$$F_{n,t_r}(z) = 2\big(G_{n-1}(z)\widetilde{F}_r(z) - F_n(z)\widetilde{G}_{r-1}(z)\big) \tag{2.237}$$

$$= iz^{-1}\big(F_{n,x}(z)\widetilde{F}_r(z) - F_n(z)\widetilde{F}_{r,x}(z)\big) \text{ on } \mathbb{C}\times\Omega_\mu. \tag{2.238}$$

To prove (2.237), we compute from (2.186), (2.187),

$$F_{n,t_r}(z) = -iF_n(z)\sum_{j=1}^{n}\widetilde{F}_r(\mu_j)\mu_j^{-1}\mu_{j,x}(z-\mu_j)^{-1},$$

$$iz^{-1}F_{n,x}(z)\widetilde{F}_r(z) = -iF_n(z)\sum_{j=1}^{n}\widetilde{F}_r(z)z^{-1}\mu_{j,x}(z-\mu_j)^{-1}.$$

Thus, (2.237) is equivalent to

$$\sum_{j=1}^{n}\widetilde{F}_r(\mu_j)\mu_j^{-1}\mu_{j,x}(z-\mu_j)^{-1} = \sum_{j=1}^{n}\widetilde{F}_r(z)z^{-1}\mu_{j,x}(z-\mu_j)^{-1} + \widetilde{F}_{r,x}(z)z^{-1}. \tag{2.239}$$

It suffices to prove (2.239) in the homogeneous case, that is, with $\widetilde{F}_r$ replaced by $\widehat{F}_r$. Inserting (2.236) into (2.239) and applying (F.74) then reduces (2.239) to

$$\sum_{j=1}^{n}\hat{f}_r\mu_j^{-1}\mu_{j,x}(z-\mu_j)^{-1} = \sum_{j=1}^{n}\hat{f}_r z^{-1}\mu_{j,x}(z-\mu_j)^{-1} + \hat{f}_{r,x}z^{-1}. \tag{2.240}$$

By (F.14), (2.240) in turn is equivalent to

$$-z\sum_{j=1}^{n}\Phi_{n-1}^{(j)}(\underline{\mu})\mu_{j,x}(z-\mu_j)^{-1} = \sum_{j=1}^{n}\Psi_n(\underline{\mu})\mu_{j,x}(z-\mu_j)^{-1} - \sum_{j=1}^{n}\Phi_{n-1}^{(j)}(\underline{\mu})\mu_{j,x}.$$

Since

$$-z\Phi_{n-1}^{(j)}(\underline{\mu})(z-\mu_j)^{-1} = \Psi_n(\underline{\mu})(z-\mu_j)^{-1} - \Phi_{n-1}^{(j)}(\underline{\mu})$$

is equivalent to

$$\Psi_n(\underline{\mu}) = -\mu_j\Phi_{n-1}^{(j)}(\underline{\mu}),$$

and the latter is clearly true by the definitions (E.1), (E.2) of $\Psi_k(\underline{\mu})$ and $\Phi_k^{(j)}(\underline{\mu})$ in Appendix E, we proved (2.237) and (2.238). Next, we define the monic polynomial $\widetilde{H}_r$ of degree r by

$$u_{xt_r} = -2i\widetilde{G}_{r-1,x}(z) - 2\big(\widetilde{H}_r(z) - \widetilde{F}_r(z)\big) \text{ on } \mathbb{C}\times\Omega_\mu, \tag{2.241}$$

that is, by postulating (2.146). Differentiating (2.149) with respect to t_r, inserting (2.241), the x-derivative of (2.237), (2.232), and (2.234) into the resulting expression then yields (2.172), that is,

$$zG_{n-1,t_r}(z) = F_n(z)\widetilde{H}_r(z) - H_n(z)\widetilde{F}_r(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{2.242}$$

Differentiating (2.235) with respect to t_r, inserting (2.237) and (2.242), then yields (2.173),

$$H_{n,t_r}(z) = 2\big(H_n(z)\widetilde{G}_{r-1}(z) - G_{n-1}(z)\widetilde{H}_r(z)\big) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{2.243}$$

Finally, differentiating (2.242) with respect to x, observing (2.234), (2.237), and (2.243), yields

$$\begin{aligned} zG_{n-1,t_r x} &= iz(H_{n,t_r} - F_{n,t_r}) \\ &= 2iz(H_n\widetilde{G}_{r-1} - G_{n-1}\widetilde{H}_r + F_n\widetilde{G}_{r-1} - G_{n-1}\widetilde{F}_r). \end{aligned} \tag{2.244}$$

On the other hand, differentiating (2.242) with respect to x, using (2.232) and (2.233), we find

$$\begin{aligned} zG_{n-1,t_r x} &= F_n\widetilde{H}_{r,x} + F_{n,x}\widetilde{H}_r - H_n\widetilde{F}_{r,x} - H_{n,x}\widetilde{F}_r \\ &= F_n\widetilde{H}_{r,x} + (-iu_x F_n - 2izG_{n-1})\widetilde{H}_r - H_n\widetilde{F}_{r,x} \\ &\quad - (iu_x H_n + 2izG_{n-1})\widetilde{F}_r. \end{aligned} \tag{2.245}$$

Combining (2.244) and (2.245), using (2.238), we infer (2.148), that is,

$$\widetilde{H}_{r,x}(z) = iu_x\widetilde{H}_r(z) + 2iz\widetilde{G}_{r-1}(z) \text{ on } \mathbb{C} \times \Omega_\mu.$$

Thus, we derived (2.154), (2.146)–(2.148) and incidentally also (2.171)–(2.173) on $\mathbb{C} \times \Omega_\mu$. Hence, we proved (2.231). $\square$

Remark 2.35 The explicit theta function representation (2.209) of u on Ω_μ in (2.230) then permits one to extend u beyond Ω_μ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ remains nonspecial (cf. Theorem A.31).

Remark 2.36 Again we formulated Theorem 2.34 in terms of Dirichlet eigenvalues μ_j, $j = 1, \dots, n$ only. Obviously, the analogous result (and strategy of proof) works in terms of ν_j, $j = 1, \dots, n$.

The analog of Remark 2.17 directly extends to the current time-dependent setting.

2.5 Notes

Most of the material presented in this chapter closely follows Gesztesy and Holden (2000b).

Section 2.1. Solvability of the initial value problem for the sine-Gordon equation by the inverse scattering transform method was first shown in Ablowitz et al. (1973a) (see also Ablowitz et al. (1973b; 1974)) and soon after in Takhtadzhyan (1974) and Zakharov et al. (1975).

For a discussion of the early studies of the sine-Gordon equation in the context of surfaces of constant negative curvature, we refer to Eisenhart (1909).

In physics one finds the sine-Gordon model isolated, for instance, in elementary particle physics as a relativistically invariant integrable model (Ablowitz and Segur (1981, Sec. 4.5), Cherednik (1996), Dodd et al. (1982, Secs. 7.1–7.5), and Zakharov and Mikhailov (1978)), quantum optics (Ablowitz and Segur (1981, Sec. 4.4) and Dodd et al. (1982, Sec. 7.8)), Josephson junctions (Dodd et al. (1982, Sec. 7.8.1)), nonlinear excitations in condensed matter physics (Borisov and Kiseliev (1988; 1989)), and vortex structures in fluids and plasmas (Ting et al. (1987)).

In connection with the reduction process of Abelian integrals on hyperelliptic curves to elliptic functions, we refer to Babich (1985), Babich et al. (1986), Belokolos et al. (1994, Sec. 7.9; 1986), Belokolos and Enol'skii (1982), Bobenko (1984), Smirnov (1991), and Taimanov (1990b).

In spite of its popularity, relatively little effort has been spent on deriving solutions that simultaneously satisfy a hierarchy of sine-Gordon equations. The generally accepted hierarchy in the sine-Gordon case, as originally derived in Sasaki and Bullough (1980) (see also Sasaki and Bullough (1981)), is nonlocal in u for all but the first element (2.58) in the hierarchy. There were other attempts to introduce a (nonlocal) sG hierarchy. For instance, Newell (1985, Sec. 5k) introduced a nonlocal sG hierarchy using an extension of the AKNS hierarchy, Tracy and Widom (1996) discussed the sG hierarchy in close connection with the mKdV hierarchy, Gu (1986) derived a generalized mKdV-sG hierarchy, and Al'ber and Al'ber (1987b) considered a hierarchy starting from the so called $\underline{\mu}$-representation of the algebro-geometric sGmKdV solutions (cf. Remark 2.25). The hierarchy described in this chapter, as originally derived in Gesztesy and Holden (2000b), in a certain sense embeds the sine-Gordon equation into the mKdV hierarchy.

The sG equation as a completely integrable Hamiltonian system and an infinite sequence of conservation laws for (2.58) polynomial in u and its x-derivatives (see, e.g., Sanuki and Konno (1974)) was established around 1974. Its integrability in light-cone coordinates was discussed early on in Ablowitz et al. (1973a,b; 1974), Lamb (1974); its integrability in laboratory coordinates is treated in Takhtadzhyan and Faddeev (1974), Kaup (1975), Kaup and Newell (1978), Tahtadžjan and Faddeev (1979), Zakharov et al. (1975). This is further discussed in Faddeev and Takhtajan (1987, Part II, Secs. II.6, II.7) and Novikov et al. (1984, Sec. I.11).

It is an interesting fact that the usual zero-curvature representation for (2.58) is gauge equivalent to that of the nonlinear Schrödinger (nS) equation, as discussed in Faddeev and Takhtajan (1987, Part II, Sec. II.7) (see also Bobenko (1991b, Sec. 5) and Pinkall and Sterling (1989)). Moreover, dimensional reductions of the self-dual Yang–Mills equations to the (elliptic) sG equation (see, e.g., Uhlenbeck (1992) and Ward (1986)) should also be mentioned in this connection.

For early reviews on the sG equation covering the period up to 1978, we refer, for instance, to Flaschka and Newell (1975), Newell (1978).

Classes of relativistically invariant integrable systems containing the sine-Gordon and Thirring models as special cases are discussed in Cherednik (1996) and Zakharov and Mikhailov (1978).

For textbook literature on the KdV equation we refer to Ablowitz and Segur (1981, Ch. 1), Cherednik (1996), Dodd et al. (1982, Ch. 7), Drazin and Johnson (1989, Ch. 6), Eilenberger (1983, Ch. 5), Faddeev and Takhtajan (1987, Part 2, Ch. II), Newell (1985, Ch. 5), and Novikov et al. (1984, Sec. I.11).

Section 2.2. To the best of our knowledge, the zero-curvature condition utilized in Section 2.2 (cf. (2.18)–(2.20)) was first introduced in Gesztesy and Holden (2000b).

The construction of the sGmKdV hierarchy using a recursive approach is patterned after work by Al'ber and Al'ber (1985; 1987b).

The intimate connections between the KdV, mKdV, and sine-Gordon (respectively sinh-Gordon) equations involving the Miura-type transformations displayed in equations (2.49) and (2.50) have been known for a long time, see, for instance, Cherednik (1979), Case and Roos (1982), Chodos (1980), Drin'feld and Sokolov (1985), Gu (1986), and more recently, Tracy and Widom (1996).

Section 2.3. As indicated in Remark 2.3, our formalism not only combines the sine-Gordon (sG) equation and the modified Korteweg–de Vries (mKdV) hierarchies, but it easily can be adapted to the sinh-Gordon, Liouville, elliptic sine-Gordon, and elliptic sinh-Gordon equations. To simplify matters a bit, the notes below, for the most part, refer to treatments of the sG equation only.

As in all other chapters, the fundamental meromorphic function ϕ on $\mathcal{K}_n$ defined in (2.68) is still the key object of our algebro-geometric formalism. By (2.68)–(2.70), ϕ again links the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and its counterpart, $\mathcal{D}_{\underline{\hat{\nu}}}$. This is of course a direct consequence of the identity (2.27) together with the factorizations of F_n and H_n in (2.64). Thus, our construction of positive divisors of degree n (respectively, $n+1$, since the points P_0 and P_∞ are also involved) on the hyperelliptic curve $\mathcal{K}_n$ of genus n again follows the recipe of Jacobi (1846), Mumford (1984, Sec. III a).1), and McKean (1985).

The Dubrovin equations (2.87) and (2.90) in Lemma 2.6, in connection with the auxiliary divisors, and the corresponding trace formulas in Lemma 2.7 (perhaps, determinant formulas might be more appropriate in this particular instance) are well-known in the sine-Gordon context. We refer, for instance, to Al'ber and Al'ber (1987b), Forest and McLaughlin (1982), and McKean (1981).

Algebro-geometric solutions for the sine-Gordon equation are usually discussed directly in the time-dependent context. Hence, we defer their discussion to the notes in the following section.

Section 2.4. As in the notes of Section 2.3, the notes below, for the most part, refer to treatments of the sG equation only.

In analogy to its stationary analog in in Section 2.3, the role of ϕ defined in (2.160) is again central to Section 2.4, and the corresponding facts recorded in the notes to Section 2.3 still apply.

The Dubrovin equations (2.187) in Lemma 2.23 were found simultaneously with their stationary counterparts, as discussed in the notes to Section 2.3. As in the corresponding KdV context, they are usually discussed in connection with the simplest cases $r = 0, 1$ only.

Since the proof of Lemma 2.24 is identical to that in the corresponding stationary case, the remarks pertaining to the trace formulas in Lemma 2.7 in the notes to Section 2.3 apply again.

The linearization property (2.206), (2.207) of the Abel map and formula (2.209) for u in terms of the Riemann theta function associated with $\mathcal{K}_n$ was first published by Kozel and Kotlyarov (1976). A simplified derivation of this result, due to Its, is presented in the review Matveev (1976, Sec. 11).

Since then, many authors have presented reviews and slightly varying approaches to algebro-geometric (respectively periodic) solutions of the sG equation (and some of its close relatives such as the sinh-Gordon equation, etc.). We mention, for instance, Babich (1991; 1992), Cherednik (1978; 1980; 1981; 1983; 1996, Ch. 4), Date (1980; 1982), Dubrovin (1982a; 1983), Dubrovin et al. (1990), Dubrovin and Natanzon (1982), Dubrovin and Novikov (1975a), Ercolani (1989), Ercolani and Forest (1985), Ercolani et al. (1984; 1986a,b; 1987), Forest and McLaughlin (1982; 1983), Harnad (1993), Harnad and Wisse (1993), Krichever (1983), Larson and Tracy (1988), McKean (1981), Novikov (1985), Taimanov (1990c), Ting et al. (1984a,b; 1987), and the monographs Belokolos et al. (1994, Chs. 4, 5), Cherednik (1996, Sec. I.4). Algebro-geometric solutions and their theta function representations for higher-order sGmKdV equations were derived in Gesztesy and Holden (2000b).

The smoothness of real-valued sine-Gordon solutions stated in Lemma 2.30 was discussed in detail in Taimanov (1990c).

As indicated in Remark 2.32, the characterization of real-valued algebro-geometric sG solutions proved to be more difficult than in the earlier settled KdV case. Although most of the references cited in the previous paragraph address the reality problem of sG solutions in one form or another, the problem was finally settled by Dubrovin, Natanzon, and Novikov, and in great detail by Ercolani and Forest in the early- to mid-1980s. We refer to Dubrovin (1982a; 1983), Dubrovin and Natanzon (1982), Dubrovin and Novikov (1975a), Novikov (1985), Ercolani and Forest (1985), and Taimanov (1990c). (The genus $n = 2$ case had also been discussed early on by Belokolos and Enol'skii (1982) and Dubrovin and Natanzon (1982).) In particular, Ercolani and Forest (1985) provide a careful discussion of isospectral sG manifolds in the real and complex-valued cases. For more recent textbook discussions of these results, we refer to Belokolos et al. (1994, Sec. 4.3) and Cherednik (1996, Ch. 4). Pertinent remarks on the reality problem for the sinh-Gordon equation can be found in Forest and McLaughlin (1982; 1983) and McKean (1981). A computation of topological charges (and a detailed review of the reality problem) for the sG equation can be found in Grinevich and Novikov (2001).

The symplectic structure and action-angle variables for the periodic sG equation are discussed, for instance, in Al'ber and Al'ber (1985; 1987b), Ercolani et al. (1986b), and Novikov (1985).

Theorem 2.34 (in the sG context) is well-known in the case $r = 1$. It has been extensively used in the sG literature. Readers are referred to Al'ber and Al'ber (1985; 1987b), Ercolani and Forest (1985), Ercolani et al. (1986b), Forest and McLaughlin (1982; 1983), Ting et al. (1984a,b; 1987), and Tracy et al. (1986).

A multiscale spectral averaging method applied to the sG equation yields the nonlinear Schrödinger equation, its spectral data, conservation laws, and solutions in terms of theta functions, as shown in Larson and Tracy (1988).

Special cases of elliptic and genus two (respectively three) sG solutions are discussed, for instance, in Belokolos et al. (1994, Ch. 7), Dubrovin and Natanzon (1982), Ercolani and Forest (1985), Forest and McLaughlin (1982), Smirnov (1991; 1997a,c), and Taimanov (1990b,c).

A completely integrable system related to the algebro-geometric solutions of the sG hierarchy (similar to the connection between the Neumann system of constrained harmonic oscillators to a sphere and the KdV equation) was treated in Previato (1986).

Degenerations of the underlying hyperelliptic curve and solitons relative to algebro-geometric background sG solutions are discussed in Borisov and Kiseliev (1989), Kotlyarov (1989), and Zagrodziński and Jaworski (1982).

The 1990s experienced renewed interest in algebro-geometric solutions of the sG equation owing to its relevance in connection with integrable surfaces, Willmore tori, etc. The interested reader can find much pertinent information in Babich and Bobenko (1993), Bobenko (1990a,b; 1991a,b; to appear), Dorfmeister and Haak (1998a,b), Ercolani et al. (1993), Korotkin (1999), Melko and Sterling (1993), Pinkall and Sterling (1989), and Taimanov (1998), and the literature therein.

3
The AKNS Hierarchy

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research.
>
> *Eugene P. Wigner*[1]

3.1 Contents

In 1974 Ablowitz, Kaup, Newell, and Segur introduced a new system of integrable nonlinear evolution equations, later called the AKNS system,

$$p_t + \frac{i}{2} p_{xx} - i p^2 q = 0,$$
$$q_t - \frac{i}{2} q_{xx} + i p q^2 = 0,$$

for functions $p = p(x, t)$, $q = q(x, t)$, which can be viewed as a complexified, nonlinear Schrödinger (nS) equation

$$q_t - \tfrac{i}{2} q_{xx} \pm i|q|^2 q = 0$$

under the assumption $p = \pm\overline{q}$. A Lax pair for the nS equation had previously been found by Zakharov and Shabat (ZS) in 1972, and its integrability as a Hamiltonian system[2] had been established by Zakharov and Manakov in 1974. This chapter focuses on the construction of algebro-geometric solutions of the AKNS hierarchy. Below we briefly summarize the principal content of each section. A more detailed discussion, using the KdV hierarchy as a model, has been provided in the introduction to this volume.

[1] The unreasonable effectiveness of mathematics in the natural sciences, *Comm. Pure Appl. Math.* **13** (1960), 1–14.
[2] A guide to the literature can be found in the detailed notes at the end of this chapter.

Section 3.2.

- polynomial recursion formalism, zero-curvature pairs (U, V_{n+1})
- stationary and time-dependent AKNS hierarchy
- Burchnall–Chaundy polynomial, hyperelliptic curve $\mathcal{K}_n$

Section 3.3. (stationary)

- properties of ϕ and the Baker–Akhiezer vector Ψ
- Dubrovin equations for auxiliary divisors
- trace formulas for p, q, and higher-order AKNS invariants
- theta function representations for ϕ, Ψ, and p, q
- the algebro-geometric initial value problem

Section 3.4. (time-dependent)

- properties of ϕ and the Baker–Akhiezer vector Ψ
- Dubrovin equations for auxiliary divisors
- trace formulas for p, q and higher-order AKNS invariants
- theta function representations for ϕ, Ψ, and p, q
- the algebro-geometric initial value problem

Section 3.5.

- Gauge equivalence of AKNS and classical Boussinesq (cBsq) hierarchies
- polynomial recursion formalism, cBsq zero-curvature pairs $(\overline{U}, \overline{V}_{n+1})$
- theta function representations for cBsq solutions u, v

This chapter relies on terminology and notions developed in connection with compact Riemann surfaces. A brief summary of key results as well as definitions of some of the main quantities can be found in Appendices A, C, and F.

3.2 The AKNS Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section we provide the construction of the AKNS hierarchy using a polynomial recursion formalism and derive the associated sequence of AKNS Lax pairs. Moreover, we discuss the Burchnall–Chaundy polynomial in connection with the stationary AKNS hierarchy and the underlying hyperelliptic curve.

Throughout this section we suppose the following hypothesis.

Hypothesis 3.1 *In the stationary case we assume that* $p\colon \mathbb{R} \to \mathbb{C}$ *and* $q\colon \mathbb{R} \to \mathbb{C}$ *are smooth nonvanishing functions,*[1]

$$p, q \in C^\infty(\mathbb{R}), \quad p(x) \neq 0, q(x) \neq 0, x \in \mathbb{R}. \tag{3.1}$$

[1] Alternatively, one could suppose $p, q\colon \mathbb{C} \to \mathbb{C}_\infty$ to be meromorphic.

In the time-dependent case we suppose that $p\colon \mathbb{R}^2 \to \mathbb{C}$ *and* $q : \mathbb{R}^2 \to \mathbb{C}$ *satisfy*[1]

$$\begin{aligned} &p(\,\cdot\,,t), q(\,\cdot\,,t) \in C^\infty(\mathbb{R}), t \in \mathbb{R}, \quad p(x,\,\cdot\,), q(x,\,\cdot\,) \in C^1(\mathbb{R}), x \in \mathbb{R}, \\ &p(x,t) \neq 0, q(x,t) \neq 0, (x,t) \in \mathbb{R}^2. \end{aligned} \tag{3.2}$$

Actually, up to (3.34) our analysis will be time-independent, and hence only the space variation of u will matter. Consider the one-dimensional 2×2 matrix-valued differential expression

$$M = i \begin{pmatrix} \frac{d}{dx} & -q \\ p & -\frac{d}{dx} \end{pmatrix} \tag{3.3}$$

of Dirac-type. To construct the AKNS hierarchy we will need another 2×2 matrix-valued differential expression of order $n+1$ denoted by Q_{n+1}, $n \in \mathbb{N}_0$, which is defined recursively as follows. We take the quickest route to the construction of Q_{n+1} and hence to that of the AKNS hierarchy by starting from the recursion relation (3.4)–(3.7) below. Subsequently, we will offer the motivation behind this approach (cf. Remark 3.4).

Define $\{f_\ell\}_{\ell\in\mathbb{N}_0}$, $\{g_\ell\}_{\ell\in\mathbb{N}_0}$, and $\{h_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$f_0 = -iq, \quad g_0 = 1, \quad h_0 = ip, \tag{3.4}$$

$$f_{\ell+1} = (i/2) f_{\ell,x} - iqg_{\ell+1}, \quad \ell \in \mathbb{N}_0, \tag{3.5}$$

$$g_{\ell+1,x} = pf_\ell + qh_\ell, \quad \ell \in \mathbb{N}_0, \tag{3.6}$$

$$h_{\ell+1} = -(i/2)h_{\ell,x} + ipg_{\ell+1}, \quad \ell \in \mathbb{N}_0. \tag{3.7}$$

Explicitly, one computes

$$\begin{aligned} f_0 &= -iq, \\ f_1 &= \tfrac{1}{2}q_x + c_1(-iq), \\ f_2 &= \tfrac{i}{4}q_{xx} - \tfrac{i}{2}pq^2 + c_1\big(\tfrac{1}{2}q_x\big) + c_2(-iq), \quad \text{etc.}, \\ g_0 &= 1, \\ g_1 &= c_1, \\ g_2 &= \tfrac{1}{2}pq + c_2, \\ g_3 &= -\tfrac{i}{4}(p_x q - pq_x) + c_1\left(\tfrac{1}{2}pq\right) + c_3, \quad \text{etc.}, \\ h_0 &= ip, \\ h_1 &= \tfrac{1}{2}p_x + c_1(ip), \\ h_2 &= -\tfrac{i}{4}p_{xx} + \tfrac{i}{2}p^2 q + c_1\big(\tfrac{1}{2}p_x\big) + c_2(ip), \quad \text{etc.}, \end{aligned} \tag{3.8}$$

where $\{c_\ell\}_{\ell\in\mathbb{N}_0} \subset \mathbb{C}$ are integration constants. Subsequently, it will be convenient to introduce also the corresponding homogeneous coefficients $\hat{f}_\ell$, $\hat{g}_\ell$, and $\hat{h}_\ell$, defined

[1] Again one could assume that for fixed $t \in \mathbb{R}$, $p(\,\cdot\,,t), q(\,\cdot\,,t)$ are meromorphic, etc.

by the vanishing of the integration constants c_k for $k = 1, \dots, \ell$,

$$\hat{f}_0 = -iq, \quad \hat{f}_\ell = f_\ell\big|_{c_k=0,k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{3.9}$$

$$\hat{g}_0 = 1, \quad \hat{g}_\ell = g_\ell\big|_{c_k=0,k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{3.10}$$

$$\hat{h}_0 = ip, \quad \hat{h}_\ell = h_\ell\big|_{c_k=0,k=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{3.11}$$

Hence,

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{f}_k, \quad g_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{g}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{h}_k,$$

introducing

$$c_0 = 1.$$

Remark 3.2 Using the nonlinear recursions (D.24) and (D.25) in Theorem D.3, one infers inductively that all homogeneous elements $\hat{f}_\ell$, $\hat{h}_\ell$ (and hence all f_ℓ and h_ℓ), $\ell \in \mathbb{N}_0$, are differential polynomials in p and q, that is, polynomials with respect to p and q and (some of) their x-derivatives. By (3.6), $g_{\ell,x}$ are also differential polynomials in p and q, and by (3.5) (respectively (3.7)) the same applies to qg_ℓ (respectively pg_ℓ). Combining these facts readily proves that g_ℓ, $\ell \in \mathbb{N}_0$, are differential polynomials in p and q.

Next, we define the 2×2 matrix-valued differential expression Q_{n+1} by

$$Q_{n+1} = i\sum_{\ell=0}^{n+1} \begin{pmatrix} -g_{n+1-\ell} & f_{n-\ell} \\ -h_{n-\ell} & g_{n+1-\ell} \end{pmatrix} M^\ell, \quad n \in \mathbb{N}_0, \quad f_{-1} = h_{-1} = 0. \tag{3.12}$$

We record the first few Q_{n+1},

$$Q_1 = \begin{pmatrix} \frac{d}{dx} - ic_1 & 0 \\ 0 & \frac{d}{dx} + ic_1 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} i\frac{d^2}{dx^2} - \frac{ipq}{2} + c_1\frac{d}{dx} - c_2 i & -iq\frac{d}{dx} - \frac{i}{2}q_x \\ ip\frac{d}{dx} - \frac{i}{2}p_x & -i\frac{d^2}{dx^2} + \frac{ipq}{2} + c_1\frac{d}{dx} + c_2 i \end{pmatrix}, \text{ etc.}$$

Introducing the corresponding homogeneous differential expressions $\widehat{Q}_{n+1}$, defined by

$$\widehat{Q}_{\ell+1} = Q_{\ell+1}\big|_{c_k=0,k=1,\dots,\ell+1}, \quad \ell \in \mathbb{N}_0, \tag{3.13}$$

one finds

$$Q_{n+1} = \sum_{\ell=0}^{n} c_{n-\ell}\widehat{Q}_{\ell+1}.$$

From the recursion relation (3.4)–(3.7), the commutator of Q_{n+1} and M can be explicitly computed and yields[1]

$$[Q_{n+1}, M] = \begin{pmatrix} 0 & -2if_{n+1} \\ 2ih_{n+1} & 0 \end{pmatrix}, \quad n \in \mathbb{N}_0. \tag{3.14}$$

In particular, (M, Q_{n+1}) represents the Lax pair of the AKNS hierarchy. Varying $n \in \mathbb{N}_0$, the stationary AKNS hierarchy is then defined in terms of the vanishing of the commutator of Q_{n+1} and M in (3.14) by

$$[Q_{n+1}, M] = 0, \quad n \in \mathbb{N}_0, \tag{3.15}$$

or equivalently, by[2]

$$\text{s-AKNS}_n(p, q) = -2 \begin{pmatrix} h_{n+1}(p, q) \\ f_{n+1}(p, q) \end{pmatrix} = 0, \quad n \in \mathbb{N}_0. \tag{3.16}$$

Explicitly,

$$\text{s-AKNS}_0(p, q) = \begin{pmatrix} -p_x + c_1(-2ip) \\ -q_x + c_1(2iq) \end{pmatrix} = 0,$$

$$\text{s-AKNS}_1(p, q) = \begin{pmatrix} \frac{i}{2}p_{xx} - ip^2q + c_1(-p_x) + c_2(-2ip) \\ -\frac{i}{2}q_{xx} + ipq^2 + c_1(-q_x) + c_2(2iq) \end{pmatrix} = 0,$$

$$\text{s-AKNS}_2(p, q)$$
$$= \begin{pmatrix} \frac{1}{4}p_{xxx} - \frac{3}{2}pp_xq + c_1(\frac{i}{2}p_{xx} - ip^2q) + c_2(-p_x) + c_3(-2ip) \\ \frac{1}{4}q_{xxx} - \frac{3}{2}pqq_x + c_1(-\frac{i}{2}q_{xx} + ipq^2) + c_2(-q_x) + c_3(2iq) \end{pmatrix} = 0, \text{ etc.,}$$

represent the first few equations of the stationary KdV hierarchy. By definition, the set of solutions of (3.16), with n ranging in $\mathbb{N}_0$ and c_ℓ in $\mathbb{C}$, $\ell \in \mathbb{N}$, represents the class of algebro-geometric AKNS solutions. If p, q satisfy one of the stationary AKNS equations in (3.16) for a particular value of n, then they satisfy infinitely many such equations of order higher than n for certain choices of integration constants c_ℓ (one can follow the argument in Remark 1.5). At times it will be convenient to abbreviate algebro-geometric stationary AKNS solutions p, q simply as AKNS *potentials*.

In the following we will frequently assume that p, q satisfy the nth stationary AKNS equations. By this we mean they satisfy one of the nth stationary AKNS equations after a particular choice of integration constants $c_\ell \in \mathbb{C}$, $\ell = 1, \dots, n+1$, $n \in \mathbb{N}_0$, has been made.

[1] The recursion (3.4)–(3.7) is constructed so that the commutator of Q_{n+1} and M ceases to be a higher-order 2×2 matrix-valued differential expression but results in multiplication by $\left(\begin{smallmatrix} 0 & -2if_{n+1} \\ 2ih_{n+1} & 0 \end{smallmatrix}\right)$ only.

[2] In a slight abuse of notation we will occasionally stress the functional dependence of f_ℓ and h_ℓ on p, q, writing $f_\ell(p, q)$ and $h_\ell(p, q)$, etc.

In accordance with our notation introduced in (3.9)–(3.11) and (3.13), the corresponding homogeneous stationary AKNS equations are defined by

$$\text{s-}\widehat{\text{AKNS}}_n(p,q) = \text{s-AKNS}_n(p,q)\big|_{c_\ell=0,\ell=1,\dots,n+1} = 0, \quad n \in \mathbb{N}_0.$$

Next, we introduce polynomials F_n, G_{n+1}, and H_n with respect to the spectral parameter $z \in \mathbb{C}$ by

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell = \sum_{\ell=0}^{n} c_{n-\ell} \widehat{F}_\ell(z), \tag{3.17}$$

$$G_{n+1}(z) = \sum_{\ell=0}^{n+1} g_{n+1-\ell} z^\ell = \sum_{\ell=-1}^{n} c_{n-\ell} \widehat{G}_{\ell+1}(z), \tag{3.18}$$

$$H_n(z) = \sum_{\ell=0}^{n} h_{n-\ell} z^\ell = \sum_{\ell=0}^{n} c_{n-\ell} \widehat{H}_\ell(z), \tag{3.19}$$

with $\widehat{F}_\ell$, $\widehat{G}_{\ell+1}$, and $\widehat{H}_\ell$ denoting the corresponding homogeneous polynomials defined by

$$\begin{aligned}
\widehat{F}_0(z) &= F_0(z) = -iq,\\
\widehat{F}_\ell(z) &= F_\ell(z)\big|_{c_k=0,k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{f}_{\ell-k} z^k, \quad \ell \in \mathbb{N},\\
\widehat{G}_0(z) &= 1,\\
\widehat{G}_{\ell+1}(z) &= G_{\ell+1}(z)\big|_{c_k=0,k=1,\dots,\ell+1} = \sum_{k=0}^{\ell+1} \hat{g}_{\ell+1-k} z^k, \quad \ell \in \mathbb{N},\\
\widehat{H}_0(z) &= H_0(z) = ip,\\
\widehat{H}_\ell(z) &= H_\ell(z)\big|_{c_k=0,k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{h}_{\ell-k} z^k, \quad \ell \in \mathbb{N}.
\end{aligned}$$

Explicitly, one obtains

$$\begin{aligned}
F_0 &= -iq,\\
F_1 &= -iqz + \tfrac{1}{2}q_x + c_1(-iq),\\
F_2 &= -iqz^2 + \tfrac{1}{2}q_x z + \tfrac{i}{4}q_{xx} - \tfrac{i}{2}pq^2 + c_1\big(-iqz + \tfrac{1}{2}q_x\big) + c_2(-iq), \quad \text{etc.},\\
G_1 &= z + c_1,\\
G_2 &= z^2 + \tfrac{1}{2}pq + c_1 z + c_2,\\
G_3 &= z^3 + \tfrac{1}{2}pqz - \tfrac{i}{4}(p_x q - pq_x) + c_1\big(z^2 + \tfrac{1}{2}pq\big) + c_2 z + c_3, \quad \text{etc.},\\
H_0 &= ip,\\
H_1 &= ipz + \tfrac{1}{2}p_x + c_1(ip),\\
H_2 &= ipz^2 + \tfrac{1}{2}p_x z - \tfrac{i}{4}p_{xx} + \tfrac{i}{2}p^2 q + c_1\big(ipz + \tfrac{1}{2}p_x\big) + c_2(ip), \quad \text{etc.}
\end{aligned}$$

We note that (3.15), or equivalently (3.16), becomes

$$F_{n,x} = -2izF_n + 2qG_{n+1}, \tag{3.20}$$

$$G_{n+1,x} = pF_n + qH_n, \tag{3.21}$$

$$H_{n,x} = 2izH_n + 2pG_{n+1}. \tag{3.22}$$

Moreover, (3.20)–(3.22) yield

$$\left(G_{n+1}^2 - F_n H_n\right)_x = 0,$$

and hence $G_{n+1}^2 - F_n H_n$ is x-independent, implying

$$G_{n+1}^2 - F_n H_n = R_{2n+2}, \tag{3.23}$$

where the integration constant R_{2n+2} is a monic polynomial of degree $2n+2$. If $\{E_m\}_{m=0,\dots,2n+1}$ denote its zeros, then

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \ \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}. \tag{3.24}$$

One can use (3.20)–(3.22) and (3.23) to derive differential equations for F_n and H_n separately by eliminating G_{n+1}. We obtain for F_n

$$F_n F_{n,xx} - \frac{q_x}{q} F_n F_{n,x} - \frac{1}{2} F_{n,x}^2 + \left(2z^2 - 2iz\frac{q_x}{q} - 2pq\right) F_n^2 = -2q^2 R_{2n+2}, \tag{3.25}$$

and upon dividing (3.25) by q^2 and differentiating the result with respect to x,

$$\begin{aligned} &F_{n,xxx} - 3\frac{q_x}{q} F_{n,xx} + \left(4z^2 - 4iz\frac{q_x}{q} - 4pq - \frac{q_{xx}}{q} + 3\frac{q_x^2}{q^2}\right) F_{n,x} \\ &\quad + \left(-4z^2\frac{q_x}{q} + 6iz\frac{q_x^2}{q^2} - 2iz\frac{q_{xx}}{q} + 2pq_x - 2p_x q\right) F_n = 0. \end{aligned} \tag{3.26}$$

Similarly, one obtains for H_n,

$$H_n H_{n,xx} - \frac{p_x}{p} H_n H_{n,x} - \frac{1}{2} H_{n,x}^2 + \left(2z^2 + 2iz\frac{p_x}{p} - 2pq\right) H_n^2 = -2p^2 R_{2n+2}, \tag{3.27}$$

and

$$\begin{aligned} &H_{n,xxx} - 3\frac{p_x}{p} H_{n,xx} + \left(4z^2 + 4iz\frac{p_x}{p} - 4pq - \frac{p_{xx}}{p} + 3\frac{p_x^2}{p^2}\right) H_{n,x} \\ &\quad + \left(-4z^2\frac{p_x}{p} - 6iz\frac{p_x^2}{p^2} + 2iz\frac{p_{xx}}{p} + 2p_x q - 2pq_x\right) H_n = 0. \end{aligned} \tag{3.28}$$

Equations (3.25) and (3.27) can be used to derive recursion relations for the homogeneous coefficients $\hat{f}_\ell$, $\hat{g}_\ell$, and $\hat{h}_\ell$ (i.e., the ones in (3.9)–(3.11) in the case of vanishing integration constants), as proved in Theorem D.3 in Appendix D. This has interesting applications to the asymptotic expansion of the Green's matrix of M as the spectral parameter tends to infinity, as briefly discussed in Remark D.4, and also yields a proof that f_ℓ, g_ℓ, and h_ℓ are differential polynomials in p, q (cf. Remark 3.2). In addition, as proven in Theorem D.3, (3.25) leads to an explicit determination of the integration constants $c_1, \dots, c_{n+1}$ in

$$\text{s-AKNS}_n(p,q) = -2\begin{pmatrix} h_{n+1}(p,q) \\ f_{n+1}(p,q) \end{pmatrix} = 0,$$

in terms of the zeros $E_0, \dots, E_{2n+1}$ of the associated polynomial R_{2n+2} in (3.24). In fact, one can prove (cf. (D.26))

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n+1, \tag{3.29}$$

where

$$\begin{aligned} &c_0(\underline{E}) = 1, \\ &c_k(\underline{E}) \\ &= \sum_{\substack{j_0,\dots,j_{2n+1}=0 \\ j_0+\cdots+j_{2n+1}=k}}^{k} \frac{(2j_0)!\cdots(2j_{2n+1})!}{2^{2k}(j_0!)^2\cdots(j_{2n+1}!)^2(2j_0-1)\cdots(2j_{2n+1}-1)} E_0^{j_0}\cdots E_{2n+1}^{j_{2n+1}}, \\ &\hspace{20em} k = 1,\dots,n+1. \end{aligned} \tag{3.30}$$

Next, we study the restriction of the differential expression Q_{n+1} to the two-dimensional kernel (i.e., the null space in an algebraic sense as opposed to the functional analytic one) of $(M - z)$. More precisely, let[1]

$$\ker(M-z) = \left\{ \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} : \mathbb{R} \to \mathbb{C}^2_\infty \,\middle|\, (M-z)\Psi = 0 \right\}, \quad z \in \mathbb{C}, \tag{3.31}$$

then (3.12) implies

$$Q_{n+1}\big|_{\ker(M-z)} = i\begin{pmatrix} -G_{n+1}(z) & F_n(z) \\ -H_n(z) & G_{n+1}(z) \end{pmatrix}\Bigg|_{\ker(M-z)}. \tag{3.32}$$

We emphasize that the result (3.32) is valid independently of whether Q_{n+1} and M commute. However, if one makes the additional assumption that Q_{n+1} and M commute, we will now prove that this implies an algebraic relationship between Q_{n+1} and M. This is the matrix-valued analog of the celebrated result of Burchnall and Chaundy discussed in Theorem 1.3. The following theorem details this relationship.

[1] If p, q are considered on $\mathbb{C}$, then Ψ in (3.31) should be considered on $\mathbb{C}$ too.

Theorem 3.3 *Assume that* Q_{n+1} *and* M *commute,* $[Q_{n+1}, M] = 0$*, or equivalently, suppose* s-AKNS$(p, q) = 0$ *(i.e.,* $f_{n+1}(p, q) = h_{n+1}(p, q) = 0$*) for some* $n \in \mathbb{N}_0$*. Then* M *and* Q_{n+1} *satisfy an algebraic relationship of the type (cf.* (3.24)*)*

$$\begin{aligned} &\mathcal{F}_n(M, Q_{n+1}) = -Q_{n+1}^2 - R_{2n+2}(M) = 0, \\ &R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad z \in \mathbb{C}. \end{aligned} \tag{3.33}$$

Proof The commutativity of Q_{n+1} and M, the definition of R_{2n+2}, (3.23), as well as the expression for Q_{n+1} on the kernel of $M - z$, (3.32), imply

$$\begin{aligned} Q_{n+1}^2\big|_{\ker(M-z)} &= -\begin{pmatrix} G_{n+1}^2 - F_n H_n & 0 \\ 0 & G_{n+1}^2 - F_n H_n \end{pmatrix}\bigg|_{\ker(M-z)} \\ &= -R_{2n+2}(z)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\bigg|_{\ker(M-z)} = -R_{2n+2}(M)\big|_{\ker(M-z)}. \end{aligned}$$

Hence, Q_{n+1}^2 and $-R_{2n+2}(M)$ coincide on $\ker(M - z)$, and since $z \in \mathbb{C}$ is arbitrary, one infers (3.33). $\square$

One calls $\mathcal{F}_n(M, Q_{n+1})$ the Burchnall–Chaundy polynomial of the pair (M, Q_{n+1}). Equation (3.33) naturally leads to the hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part), where

$$\begin{aligned} &\mathcal{K}_n \colon \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0, \\ &R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}. \end{aligned} \tag{3.34}$$

Remark 3.4 At this point it is easy to motivate the recursion relation (3.4)–(3.7) used as our starting point for constructing the AKNS hierarchy. If one is interested in determining 2×2 matrix-valued differential expressions Q commuting with M (other than simply polynomials of M or the case in which Q and M are polynomials of a third matrix-valued differential expression), one can proceed as follows. Restricting Q to the two-dimensional null space, $\ker(M - z)$, of $(M - z)$, one can systematically replace $\Psi_x = (\psi_{1,x}, \psi_{2,x})^\top$ by $(q\psi_2 - iz\psi_1, p\psi_1 + iz\psi_2)^\top$ and hence effectively reduce Q on $\ker(M - z)$ to multiplication by $Q\big|_{\ker(M-z)} = i\left(\begin{smallmatrix} -G & F \\ -H & G \end{smallmatrix}\right)\big|_{\ker(M-z)}$, where F, G, and H are polynomials. Imposing commutativity of Q and M on $\ker(M - z)$ then yields relations (3.20)–(3.22). Moreover, we reproduced identity (3.32). Making the polynomial ansatz (3.17)–(3.19) for F, G, and H and inserting it into (3.20)–(3.22) then readily yields the recursion relation (3.4)–(3.7) for $f_0, \dots, f_n, g_0, \dots, g_{n+1}, h_0, \dots, h_n$, together with $(i/2)f_{n,x} - iqg_{n+1} = 0$, $-(i/2)h_{n,x} + ipg_{n+1} = 0$. In other words, one obtains the beginning of the recursion relation (3.4)–(3.7) as well as relation (3.16) defining the nth stationary AKNS equations.

We end this section by introducing the time-dependent AKNS hierarchy. This means that p, q are now considered as functions of both space and time. For each equation in the hierarchy – that is, for each n – we introduce a deformation parameter $t_n \in \mathbb{R}$ in p and q, replacing $p(x), q(x)$ by $p(x, t_n), q(x, t_n)$. The matrix differential expression M now reads (cf. (3.3)),

$$M(t_n) = i \begin{pmatrix} \frac{d}{dx} & -q(\,\cdot\,, t_n) \\ p(\,\cdot\,, t_n) & -\frac{d}{dx} \end{pmatrix}.$$

The quantities $\{f_\ell\}_{\ell\in\mathbb{N}_0}$, $\{g_\ell\}_{\ell\in\mathbb{N}_0}$, $\{h_\ell\}_{\ell\in\mathbb{N}_0}$, and Q_{n+1}, $n \in \mathbb{N}$, are still defined by (3.4)–(3.7) and (3.12), respectively. The time-dependent AKNS hierarchy is obtained by imposing the Lax commutator equations

$$\frac{d}{dt_n} M - [Q_{n+1}, M] = 0, \quad t_n \in \mathbb{R}, \tag{3.35}$$

varying $n \in \mathbb{N}_0$, or equivalently, by

$$\begin{aligned} &\begin{pmatrix} 0 & -iq_{t_n} - F_{n,x} - 2izF_n + 2qG_{n+1} \\ ip_{t_n} - H_{n,x} + 2izH_n + 2pG_{n+1} & 0 \end{pmatrix} \\ &\quad = \begin{pmatrix} 0 & -iq_{t_n} + 2if_{n+1} \\ ip_{t_n} - 2ih_{n+1} & 0 \end{pmatrix} = 0, \quad (x, t_n) \in \mathbb{R}^2, \quad n \in \mathbb{N}_0. \end{aligned} \tag{3.36}$$

The latter are equivalent to the collection of evolution equations

$$\mathrm{AKNS}_n(p, q) = \begin{pmatrix} p_{t_n} - 2h_{n+1}(p, q) \\ q_{t_n} - 2f_{n+1}(p, q) \end{pmatrix} = 0, \quad (x, t_n) \in \mathbb{R}^2, \quad n \in \mathbb{N}_0. \tag{3.37}$$

Explicitly,

$$\mathrm{AKNS}_0(p, q) = \begin{pmatrix} p_{t_0} - p_x + c_1(-2ip) \\ q_{t_0} - q_x + c_1(2iq) \end{pmatrix} = 0,$$

$$\mathrm{AKNS}_1(p, q) = \begin{pmatrix} p_{t_1} + \frac{i}{2} p_{xx} - ip^2 q + c_1\,(-p_x) + c_2(-2ip) \\ q_{t_1} - \frac{i}{2} q_{xx} + ipq^2 + c_1\,(-q_x) + c_2(2iq) \end{pmatrix} = 0,$$

$$\begin{aligned} &\mathrm{AKNS}_2(p, q) \\ &\quad = \begin{pmatrix} p_{t_2} + \frac{1}{4} p_{xxx} - \frac{3}{2} pp_x q + c_1(\frac{i}{2} p_{xx} - ip^2 q) + c_2(-p_x) + c_3(-2ip) \\ q_{t_2} + \frac{1}{4} q_{xxx} - \frac{3}{2} pqq_x + c_1(-\frac{i}{2} q_{xx} + ipq^2) + c_2(-q_x) + c_3(2iq) \end{pmatrix} = 0, \end{aligned}$$

etc.,

represent the first few equations of the time-dependent AKNS hierarchy. The system of equations $\mathrm{AKNS}_1(p, q) = 0$ (with $c_1 = c_2 = 0$) represents *the* AKNS system. Similarly, one introduces the corresponding homogeneous AKNS hierarchy by

$$\widehat{\mathrm{AKNS}}_n(p, q) = \mathrm{AKNS}_n(p, q)\big|_{c_\ell = 0, \ell = 1, \dots, n+1} = 0, \quad n \in \mathbb{N}_0.$$

We conclude this section by pointing out an alternative construction of the AKNS hierarchy using a zero-curvature approach instead of the Lax pairs (M, Q_{n+1}).

Remark 3.5 Frequently, the AKNS hierarchy is introduced by developing its zero-curvature formalism. To this end one defines

$$U(z) = \begin{pmatrix} -iz & q \\ p & iz \end{pmatrix}, \tag{3.38}$$

$$V_{n+1}(z) = i \begin{pmatrix} -G_{n+1}(z) & F_n(z) \\ -H_n(z) & G_{n+1}(z) \end{pmatrix}, \quad n \in \mathbb{N}_0. \tag{3.39}$$

Then (3.32) implies

$$-i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} [Q_{n+1}, M]\bigg|_{\ker(M-z)} = \big(- V_{n+1,x}(z) + [U(z), V_{n+1}(z)]\big)\big|_{\ker(M-z)},$$

and the stationary part of this section, being a consequence of $[Q_{n+1}, M] = 0$, can equivalently be based on the stationary zero-curvature equation

$$\begin{aligned} 0 &= -V_{n+1,x} + [U, V_{n+1}] \\ &= \begin{pmatrix} iG_{n+1,x} - ipF_n - iqH_n & -iF_{n,x} + 2zF_n + 2iqG_{n+1} \\ iH_{n,x} + 2zH_n - 2ipG_{n+1} & -iG_{n+1,x} + ipF_n + iqH_n \end{pmatrix} \\ &= \begin{pmatrix} 0 & -2f_{n+1} \\ -2h_{n+1} & 0 \end{pmatrix}. \end{aligned}$$

In particular, the hyperelliptic curve $\mathcal{K}_n$ in (3.34) is then obtained from the characteristic equation of iV_{n+1} by[1]

$$\begin{aligned} &\det(yI_2 - iV_{n+1}(z)) = y^2 - \det(V_{n+1}(z)) \\ &\quad = y^2 - G_{n+1}(z)^2 + F_n(z)H_n(z) = y^2 - R_{2n+2}(z) = 0. \end{aligned}$$

Similarly, the time-dependent part (3.35)–(3.37), being based on the Lax equation

[1] I_2 denotes the identity matrix in $\mathbb{C}^2$.

(3.35), can equivalently be developed from the zero-curvature equation

$$
\begin{aligned}
0 &= U_{t_n} - V_{n+1,x} + [U, V_{n+1}] \\
&= \begin{pmatrix} 0 & q_{t_n} - iF_{n,x} + 2zF_n + 2iqG_{n+1} \\ p_{t_n} + iH_{n,x} + 2zH_n - 2ipG_{n+1} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & q_{t_n} - 2f_{n+1} \\ p_{t_n} - 2h_{n+1} & 0 \end{pmatrix}.
\end{aligned} \tag{3.40}
$$

In fact, since the latter approach (3.40) is almost universally adopted in the contemporary literature on the AKNS hierarchy, we thought it might be worthwhile to recall the alternative approach using the Lax pair (M, Q_{n+1}) instead.

Finally we show that the AKNS equations (3.37) are invariant with respect to certain scale transformations. More precisely, one has the following result.

Lemma 3.6 *Suppose p, q satisfy one of the AKNS equations* (3.37) *for some $n \in \mathbb{N}_0$,*

$$\mathrm{AKNS}_n(p, q) = 0.$$

Consider the scale transformation

$$(p(x, t_n), q(x, t_n)) \to (\breve{p}(x, t_n), \breve{q}(x, t_n)) = (Ap(x, t_n), A^{-1}q(x, t_n)), \quad A \in \mathbb{C} \setminus \{0\}. \tag{3.41}$$

Then,

$$\mathrm{AKNS}_n(\breve{p}, \breve{q}) = 0. \tag{3.42}$$

Proof Let (M, Q_{n+1}) and $(\breve{M}, \breve{Q}_{n+1})$ be associated with (p, q) and $(\breve{p}, \breve{q})$, respectively, and defined according to (3.3) and (3.12). Defining the matrix T in $\mathbb{C}^2$ by

$$T = \begin{pmatrix} (A^{1/2})^{-1} & 0 \\ 0 & A^{1/2} \end{pmatrix}$$

(fixing a particular square root branch $A^{1/2}$), one computes

$$
\begin{aligned}
TMT^{-1} &= \breve{M}, \\
TQ_{n+1}T^{-1} &= i\sum_{\ell=0}^{n+1} \begin{pmatrix} -g_{n+1-\ell} & A^{-1}f_{n-\ell} \\ -Ah_{n-\ell} & g_{n+1-\ell} \end{pmatrix} \breve{M}^{\ell} = \breve{Q}_{n+1}.
\end{aligned} \tag{3.43}
$$

A comparison of (3.43) with

$$\breve{Q}_{n+1} = i \sum_{\ell=0}^{n+1} \begin{pmatrix} -\breve{g}_{n+1-\ell} & \breve{f}_{n-\ell} \\ -\breve{h}_{n-\ell} & \breve{g}_{n+1-\ell} \end{pmatrix} \breve{M}^{\ell}$$

yields

$$\breve{f}_{n-\ell} = A^{-1} f_{n-\ell}, \quad \breve{g}_{n+1-\ell} = g_{n+1-\ell}, \quad \breve{h}_{n-\ell} = A^{-1} h_{n-\ell}, \quad \ell = 0, \dots, n+1$$

and hence (3.42) if (3.37) and (3.41) are taken into account. □

In the particular case of the nonlinear Schrödinger ($\mathrm{nS}_{\pm}$) hierarchy, where

$$p(x, t_n) = \pm \overline{q(x, t_n)}, \quad n \in \mathbb{N}_0, \quad c_\ell \in \mathbb{R}, \ell \in \mathbb{N}, \tag{3.44}$$

(3.41) further restricts A to be unimodular, that is,

$$|A| = 1. \tag{3.45}$$

We remark that the plus sign in (3.44), denoted by nS_+, corresponds to the *defocusing* case in which M is formally self-adjoint,

$$M = i \begin{pmatrix} \frac{d}{dx} & -q \\ \overline{q} & -\frac{d}{dx} \end{pmatrix}, \quad M^* = M. \tag{3.46}$$

On the other hand, the minus sign in (3.44), denoted by nS_-, corresponds to the *focusing* case, in which

$$M = i \begin{pmatrix} \frac{d}{dx} & -q \\ -\overline{q} & -\frac{d}{dx} \end{pmatrix}, \quad M = -\mathcal{C}_2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{C}_2, \tag{3.47}$$

with $\mathcal{C}_2$ the antilinear conjugation map

$$\mathcal{C}_2 \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix}, \quad a, b \in \mathbb{C},$$

and

$$M^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq M. \tag{3.48}$$

Hence, M is formally non-self-adjoint (but shares certain symmetries with M^*). In analogy to our notation in (3.37), the corresponding $\mathrm{nS}_{\pm}$ hierarchies are denoted by

$$\mathrm{nS}_{\pm,n}(q) = 0, \quad (x, t_n) \in \mathbb{R}^2, \quad n \in \mathbb{N}_0.$$

The first few equations explicitly read

$$
\begin{aligned}
\mathrm{nS}_{\pm,0}(q) &= q_{t_0} - q_x + c_1(2iq) = 0,\\
\mathrm{nS}_{\pm,1}(q) &= q_{t_1} - \tfrac{i}{2}q_{xx} \pm i|q|^2 q + c_1(-q_x) + c_2(2iq) = 0,\\
\mathrm{nS}_{\pm,2}(q) &= q_{t_2} + \tfrac{1}{4}q_{xxx} \mp \tfrac{3}{2}|q|^2 q_x + c_1\big(-\tfrac{i}{2}q_{xx} \pm i|q|^2 q\big)\\
&\quad + c_2(-q_x) + c_3(2iq) = 0, \quad \text{etc.}
\end{aligned}
$$

In the special case of the modified Korteweg–de Vries ($\mathrm{mKdV}_\pm$) hierarchies, where

$$
p(x,t_n) = \pm q(x,t_n), \quad n \in 2\mathbb{N}_0, \quad c_{2\ell+1} = 0, \quad \ell \in \mathbb{N}_0,
$$

(3.41) implies the additional restriction

$$
A \in \{1, -1\}.
$$

3.3 The Stationary AKNS Formalism

As shown in Section 3.2, the stationary AKNS hierarchy is intimately connected with pairs of commuting 2×2 matrix-valued differential expressions Q_{n+1} and M of orders $n+1$ and 1, respectively, and a hyperelliptic curve $\mathcal{K}_n$. In this section we study this relationship more closely and present a detailed study of the stationary AKNS hierarchy and its algebro-geometric solutions p, q. Our principal tools are derived from combining the polynomial recursion formalism introduced in Section 1.2 and a fundamental meromorphic function ϕ on $\mathcal{K}_n$, the analog of the Weyl–Titchmarsh function of M. With the help of ϕ we study the Baker–Akhiezer vector Ψ, the common eigenfunction of Q_{n+1} and M, Dubrovin-type equations governing the motion of auxiliary divisors on $\mathcal{K}_n$, trace formulas, and theta function representations of ϕ, Ψ, and p, q. We also discuss the algebro-geometric initial value problem of constructing p, q from the Dubrovin equations and auxiliary divisors as initial data.

For major parts of this section we suppose that

$$
p, q \in C^\infty(\mathbb{R}), \quad p(x) \neq 0, q(x) \neq 0, x \in \mathbb{R} \tag{3.49}
$$

(which could be replaced by $p, q\colon \mathbb{C} \to \mathbb{C}_\infty$ meromorphic) and assume (3.4)–(3.7), (3.16), (3.17)–(3.22), and (3.34) and freely employ the formalism in (3.1)–(3.34), keeping $n \in \mathbb{N}_0$ fixed.

We recall the hyperelliptic curve

$$
\begin{aligned}
&\mathcal{K}_n\colon \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0,\\
&R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C},
\end{aligned} \tag{3.50}
$$

as introduced in (3.34). The curve $\mathcal{K}_n$ is compactified by joining two points at infinity, $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_n$. Points P on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = 0$. The complex structure on $\mathcal{K}_n$ is then defined in the usual way (see Appendix C). Hence, $\mathcal{K}_n$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) in the standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_n$ (i.e., by fixing $E_0, \ldots, E_{2n+1}$), the integration constants $c_1, \ldots, c_{n+1}$ in f_{n+1} and h_{n+1} (and hence in the corresponding stationary AKNS$_n$ equations) are uniquely determined, as is clear from (3.29), (3.30), which establish the integration constants c_ℓ as symmetric functions of $E_0, \ldots, E_{2n+1}$.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$. (The trivial case $n = 0$ is explicitly treated in Example 3.20.)

The two most frequently discussed cases in applications are the case of real roots, where $E_m \in \mathbb{R}$, $m = 0, \ldots, 2n+1$, and the case of complex conjugate roots, where $\{E_m\}_{m=0,\ldots,2n+1} = \{E_{2m'}, \overline{E}_{2m'}\}_{m'=0,\ldots,n}$. These cases are treated in detail in Appendix C.

Let $\{\mu_j(x)\}_{j=1,\ldots,n}$ and $\{\nu_j(x)\}_{j=1,\ldots,n}$ denote the zeros of $F_n(\,\cdot\,, x)$ and $H_n(\,\cdot\,, x)$, respectively (cf. (3.17), (3.19)). We may then write

$$F_n(z) = -iq \prod_{j=1}^{n} (z - \mu_j) \tag{3.51}$$

and

$$H_n(z) = ip \prod_{j=1}^{n} (z - \nu_j), \tag{3.52}$$

and define $\{\hat{\mu}_j(x)\}_{j=1,\ldots,n} \subset \mathcal{K}_n$ and $\{\hat{\nu}_j(x)\}_{j=1,\ldots,n} \subset \mathcal{K}_n$ by

$$\hat{\mu}_j(x) = (\mu_j(x), G_{n+1}(\mu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \ldots, n, \quad x \in \mathbb{R}, \tag{3.53}$$

$$\hat{\nu}_j(x) = (\nu_j(x), -G_{n+1}(\nu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \ldots, n, \quad x \in \mathbb{R}, \tag{3.54}$$

lifting μ_j and ν_j to $\mathcal{K}_n$. Due to the $C^\infty(\mathbb{R})$ assumption (3.49) on p, q, $F_n(z, \,\cdot\,)$, $H_n(z, \,\cdot\,) \in C^\infty(\mathbb{R})$ by (3.5) and (3.7). Thus, one concludes

$$\mu_j, \nu_k \in C(\mathbb{R}),\ j, k = 1, \ldots, n, \tag{3.55}$$

taking multiplicities (and appropriate renumbering) of the zeros of F_n and H_n into account. (Away from collisions of zeros, μ_j and ν_k are of course C^∞.)

Next, recalling identity (3.23), we introduce the fundamental meromorphic function $\phi(\,\cdot\,, x)$ on $\mathcal{K}_n$,

$$\phi(P, x) = \frac{y + G_{n+1}(z, x)}{F_n(z, x)} \tag{3.56}$$

$$= \frac{-H_n(z, x)}{y - G_{n+1}(z, x)}, \tag{3.57}$$

$$P = (z, y) \in \mathcal{K}_n, x \in \mathbb{R}$$

with divisor $(\phi(\,\cdot\,, x))$ of $\phi(\,\cdot\,, x)$ given by

$$(\phi(\,\cdot\,, x)) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\nu}}(x)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(x)}, \tag{3.58}$$

using (3.51), (3.52), and (3.55). Here we used the abbreviations

$$\underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\}, \underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n)$$

and our convention (A.47) as well as additive notation for divisors. Equivalently, $P_{\infty_+}, \hat{\nu}_1(x), \dots, \hat{\nu}_n(x)$, are the $n+1$ zeros and $P_{\infty_-}, \hat{\mu}_1(x), \dots, \hat{\mu}_n(x)$, the $n+1$ poles of $\phi(P, x)$. Clearly $\mu_j(x)$ and $\nu_j(x)$ play the analogous role of Dirichlet and Neumann eigenvalues in comparison with the KdV case. In particular, $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ represent the corresponding analogs of Dirichlet and Neumann divisors.

Given $\phi(\,\cdot\,, x)$, one defines the stationary Baker–Akhiezer vector $\Psi(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ by

$$\Psi(P, x, x_0) = \begin{pmatrix} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{pmatrix}, \tag{3.59}$$

$$P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}, (x, x_0) \in \mathbb{R}^2,$$

where

$$\psi_1(P, x, x_0) = \exp\left(\int_{x_0}^{x} dx'(-iz + q(x')\phi(P, x'))\right), \tag{3.60}$$

$$\psi_2(P, x, x_0) = \phi(P, x)\psi_1(P, x, x_0). \tag{3.61}$$

Note that

$$\psi_1(P, x_0, x_0) = 1, \quad P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}.$$

Next we summarize a variety of properties of ϕ and Ψ.

Lemma 3.7 *Suppose $p, q \in C^\infty(\mathbb{R})$ satisfy the nth stationary AKNS system* (3.16). *Moreover, let $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$, $(x, x_0) \in \mathbb{R}^2$. Then ϕ satisfies the Riccati-type equation*

$$\phi_x(P) + q\phi(P)^2 - 2iz\phi(P) = p, \tag{3.62}$$

as well as

$$\phi(P)\phi(P^*) = \frac{H_n(z)}{F_n(z)}, \tag{3.63}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_{n+1}(z)}{F_n(z)}, \tag{3.64}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}. \tag{3.65}$$

Moreover, Ψ *satisfies the first-order system* (*cf.* (3.38), (3.39))

$$\Psi_x(P) = U(z)\Psi(P), \tag{3.66}$$

$$iy\Psi(P) = V_{n+1}(z)\Psi(P), \tag{3.67}$$

or equivalently,

$$(M - z(P))\Psi(P) = 0, \quad (Q_{n+1} - iy(P))\Psi(P) = 0. \tag{3.68}$$

In addition,

$$\psi_1(P, x, x_0) = \left(\frac{F_n(z, x)}{F_n(z, x_0)}\right)^{1/2} \exp\left(y \int_{x_0}^x dx' q(x') F_n(z, x')^{-1}\right), \tag{3.69}$$

$$\psi_1(P, x, x_0)\psi_1(P^*, x, x_0) = \frac{F_n(z, x)}{F_n(z, x_0)}, \tag{3.70}$$

$$\psi_2(P, x, x_0)\psi_2(P^*, x, x_0) = \frac{H_n(z, x)}{F_n(z, x_0)}, \tag{3.71}$$

$$\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) + \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = 2\frac{G_{n+1}(z, x)}{F_n(z, x_0)}, \tag{3.72}$$

$$\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) - \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = -\frac{2y}{F_n(z, x_0)}. \tag{3.73}$$

Moreover, as long as the zeros of $F_n(\,\cdot\,, x)$ *are all simple for* $x \in \Omega$, $\Omega \subseteq \mathbb{R}$ *an open interval,* $\Psi(\,\cdot\,, x, x_0)$ *is meromorphic on* $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *for* $x, x_0 \in \Omega$.

Proof Equation (3.62) follows from (3.20), (3.21), (3.56), and (3.57). Relations (3.63)–(3.65) are clear from (3.56), (3.57). By (3.60) and (3.61), $\Psi(\,\cdot\,, x, x_0)$ is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ away from the poles $\hat{\mu}_j(x')$ of $\phi(\,\cdot\,, x')$. By (3.20), (3.53), and (3.56),

$$q(x')\phi(P, x') \underset{P \to \hat{\mu}_j(x')}{=} \partial_{x'} \ln(F_n(z, x')) + O(1) \text{ as } z \to \mu_j(x'), \tag{3.74}$$

and hence ψ_1 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ by (3.60) as long as the zeros of $F_n(\,\cdot\,, x)$ are all simple. This follows from (3.60) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j$ of $\{\hat{\mu}_j(x') \in \mathcal{K}_n \,|\, x' \in \Omega, x' \in [x_0, x]\}$ such that $\hat{\mu}_k(x') \notin \mathcal{U}_j$ for all $x' \in [x_0, x]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$. Since ϕ is meromorphic on

$\mathcal{K}_n$ by (3.56), ψ_2 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ by (3.61). Equation (3.68) is an immediate consequence of (3.57), (3.60), and (3.61). Equation (3.69) is a consequence of (3.20), (3.60), (3.64), (3.65), and

$$\begin{aligned}\phi(P) &= \frac{1}{2}(\phi(P)+\phi(P^*)) + \frac{1}{2}(\phi(P)-\phi(P^*))\\ &= \frac{G_{n+1}}{F_n} + \frac{y}{F_n} = \frac{1}{q}\left(\frac{F_{n,x}}{F_n} + iz\right) + \frac{y}{F_n}.\end{aligned}$$

Equation (3.70) is clear from (3.69), and (3.71) is a consequence of (3.61), (3.63), and (3.70). Equation (3.72) is a consequence of (3.61), (3.64), and (3.70). Finally, (3.73) follows from (3.61), (3.65), and (3.70). □

Equations (3.70)–(3.73) show that the basic identity (3.23), $G_{n+1}^2 - F_n H_n = R_{2n+2}$, is equivalent to the elementary fact

$$(\psi_{1,+}\psi_{2,-} + \psi_{1,-}\psi_{2,+})^2 - 4\psi_{1,+}\psi_{1,-}\psi_{2,+}\psi_{2,-} = (\psi_{1,+}\psi_{2,-} - \psi_{1,-}\psi_{2,+})^2, \tag{3.75}$$

identifying $\psi_1(P) = \psi_{1,+}$, $\psi_1(P^*) = \psi_{1,-}$, $\psi_2(P) = \psi_{2,+}$, $\psi_2(P^*) = \psi_{2,-}$. This provides the intimate link between our approach and the squared function systems also employed in the literature in connection with algebro-geometric solutions of the AKNS hierarchy.

Next, we derive Dubrovin-type equations, that is, first-order coupled systems of differential equations that govern the dynamics of μ_j and ν_j with respect to variations of x. We recall that the affine part of $\mathcal{K}_n$ is nonsingular if

$$\{E_m\}_{m=0,,\dots,2n+1} \subset \mathbb{C},\ E_m \neq E_{m'} \text{ for } m \neq m',\, m, m' = 0, \dots, 2n+1. \tag{3.76}$$

Lemma 3.8 *Suppose that $p, q \in C^\infty(\widetilde{\Omega}_\mu)$ are nonzero and satisfy the nth stationary AKNS system* (3.16) *on an open interval $\widetilde{\Omega}_\mu \subseteq \mathbb{R}$. Moreover, assume that the zeros μ_j, $j = 1, \dots, n$, of $F_n(\,\cdot\,)$ remain distinct on $\widetilde{\Omega}_\mu$. Then $\{\hat{\mu}_j\}_{j=1,\dots,n}$, defined by* (3.53)*, satisfies the following first-order system of differential equations on $\widetilde{\Omega}_\mu$*

$$\mu_{j,x} = -2iy(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, n. \tag{3.77}$$

Next, assume the affine part of $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{3.78}$$

for some $x_0 \in \mathbb{R}$, where $\mu_j(x_0)$, $j = 1, \dots, n$, are distinct. Then there exists an open interval $\Omega_\mu \subseteq \mathbb{R}$, with $x_0 \in \Omega_\mu$, such that the initial value problem (3.77), (3.78) *has a unique solution $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{3.79}$$

and μ_j, $j = 1, \dots, n$, remain distinct on Ω_μ.

For the zeros ν_j, $j = 1, \dots, n$, of $H_n(\,\cdot\,)$ identical statements hold with μ_j and Ω_μ replaced by ν_j and Ω_ν, etc. In particular, $\{\hat{\nu}_j\}_{j=1,\dots,n}$, defined by (3.54)*, satisfies the system*

$$\nu_{j,x} = -2iy(\hat{\nu}_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} (\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, n. \tag{3.80}$$

Proof It suffices to prove (3.77) and (3.79) since the proof of (3.80) is analogous to that of (3.77). Equations (3.20), (3.51), and (3.53) readily yield

$$F_{n,x}(\mu_j) = iq\mu_{j,x} \prod_{\substack{k=1 \\ k \neq j}}^{n} (\mu_j - \mu_k) = 2qG_{n+1}(\mu_j) = 2qy(\hat{\mu}_j)$$

and hence (3.77). The smoothness assertion (3.79) is clear as long as $\hat{\mu}_j$ stays away from the branch points $(E_m, 0)$. In case $\hat{\mu}_j$ hits such a branch point, one can use the local chart around $(E_m, 0)$ (with local coordinate $\zeta = \sigma(z - E_m)^{1/2}$, $\sigma = \pm 1$) to verify (3.79), as in the proof of Lemma 1.10. $\square$

Combining the polynomial approach of Section 3.2 with (3.51) and (3.52) readily yields trace formulas for the AKNS invariants, that is, expressions of f_ℓ and h_ℓ in terms of symmetric functions of the zeros μ_j and ν_j of F_n and H_n, respectively. For simplicity we just record the simplest case. We explicitly indicate the first few of these below.

Lemma 3.9 *Suppose that $p, q \in C^\infty(\mathbb{R})$ are nonzero and satisfy the nth stationary AKNS system* (3.16)*. Then,*

$$i\frac{p_x}{p} - 2c_1 = 2\sum_{j_1=1}^{n} \nu_{j_1}, \tag{3.81}$$

$$\frac{1}{4}\frac{p_{xx}}{p} - \frac{1}{2}pq + c_1\left(\frac{i}{2}\frac{p_x}{p}\right) - c_2 = -\sum_{\substack{j_1, j_2 = 1 \\ j_1 < j_2}}^{n} \nu_{j_1}\nu_{j_2}, \ \textit{etc.}, \tag{3.82}$$

$$i\frac{q_x}{q} + 2c_1 = -2\sum_{j_1=1}^{n} \mu_{j_1}, \tag{3.83}$$

$$\frac{1}{4}\frac{q_{xx}}{q} - \frac{1}{2}pq + c_1\left(-\frac{i}{2}\frac{q_x}{q}\right) - c_2 = -\sum_{\substack{j_1, j_2 = 1 \\ j_1 < j_2}}^{n} \mu_{j_1}\mu_{j_2}, \ \textit{etc.}, \tag{3.84}$$

$$\frac{(pq)_x}{pq} = 2i\sum_{j_1=1}^{n} \left(\mu_{j_1} - \nu_{j_1}\right). \tag{3.85}$$

Here

$$c_1 = -\frac{1}{2}\sum_{m_1=0}^{2n+1} E_{m_1}, \quad c_2 = \frac{1}{2}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{2n+1} E_{m_1}E_{m_2} - \frac{1}{8}\bigg(\sum_{m_1=0}^{2n+1} E_{m_1}\bigg)^2, \text{ etc.} \tag{3.86}$$

Proof Relations (3.81)–(3.84) follow by comparison of powers of z equating the corresponding expressions (3.51) and (3.52) for F_n and H_n with those in (3.17), (3.19) and with (3.8) taken into account. Equation (3.86) follows in exactly the same way from (3.8), (3.17), (3.19), (3.23), and (3.24). Adding (3.81) and (3.83) yields (3.85). □

Now we turn to asymptotic properties of ϕ and ψ_j, $j = 1, 2$.

Lemma 3.10 *Suppose that $p, q \in C^\infty(\mathbb{R})$ are nonzero and satisfy the nth stationary AKNS system* (3.16). *Moreover let $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$, $(x, x_0) \in \mathbb{R}^2$. Then,*

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} (i/2)p\zeta + (p_x/4)\zeta^2 + O(\zeta^3) & \text{as } P \to P_{\infty_+}, \\ (2i/q)\zeta^{-1} + (q_x/q^2) + O(\zeta) & \text{as } P \to P_{\infty_-}, \end{cases} \tag{3.87}$$

$$\psi_1(P, x, x_0) \underset{\zeta\to 0}{=} \begin{cases} \exp(-i\zeta^{-1}(x - x_0) + O(\zeta)) & \text{as } P \to P_{\infty_+}, \\ \big(\frac{q(x)}{q(x_0)} + O(\zeta)\big)\exp(i\zeta^{-1}(x - x_0) + O(\zeta)) & \text{as } P \to P_{\infty_-}, \end{cases} \tag{3.88}$$

$$\psi_2(P, x, x_0) \tag{3.89}$$
$$\underset{\zeta\to 0}{=} \begin{cases} \big((i/2)p(x)\zeta + O(\zeta^2)\big)\exp(-i\zeta^{-1}(x - x_0) + O(\zeta)) & \text{as } P \to P_{\infty_+}, \\ \big((2i/q(x_0))\zeta^{-1} + O(1)\big)\exp(i\zeta^{-1}(x - x_0) + O(\zeta)) & \text{as } P \to P_{\infty_-}. \end{cases}$$

Proof The existence of the asymptotic expansion of ϕ in terms of the local coordinate $\zeta = 1/z$, near $P_{\infty_\pm}$ (cf. (C.7)–(C.11)) is clear from the explicit form of ϕ in (3.56). Insertion of the polynomial F_n into (3.56) then yields the explicit expansion coefficients in (3.87). Alternatively, and more efficiently, one can insert the ansatz

$$\phi \underset{z\to\infty}{=} \phi_1 z^{-1} + \phi_2 z^{-2} + O(z^{-3})$$

into the Riccati-type equation (3.62). A comparison of powers of z^{-1} then proves the first line in (3.87). Similarly, after the ansatz is inserted

$$\phi \underset{z\to\infty}{=} \phi_{-1} z + \phi_0 + \phi_1 z^{-1} + O(z^{-2})$$

into the Riccati-type equation (3.62), comparing powers of z^{-1} proves the second line in (3.87). Equation (3.88) then follows from inserting (3.87) into (3.60), and (3.89) is clear from (3.61), (3.87), and (3.88). □

For subsequent purpose we note the following asymptotic spectral parameter expansion of F_n/y as $P \to P_{\infty_\pm}$:

$$\frac{F_n(z)}{y} \underset{\zeta\to 0}{=} \mp\zeta \sum_{\ell=0}^{\infty} \hat{f}_\ell \zeta^\ell \text{ as } P \to P_{\infty_\pm},\ \zeta = 1/z. \tag{3.90}$$

Here $\hat{f}_\ell$ denote the homogeneous coefficients in (3.9) (i.e., the ones satisfying (3.4)–(3.7) with vanishing integration constants). In particular, $\hat{f}_\ell$ can be computed from a nonlinear recursion relation, as proven in Theorem D.3 in Appendix D. Analogous expansions exist for G_{n+1}/y and H_n/y.

Next, we provide an explicit representation of ϕ, Ψ, p, and q in terms of the Riemann theta function associated with $\mathcal{K}_n$, assuming the affine part of $\mathcal{K}_n$ to be nonsingular. We freely employ the notation established in Appendices A and C. To avoid the trivial case $n = 0$ (considered in Example 3.20), we assume $n \in \mathbb{N}$ for the remainder of this argument.

Without loss of generality, we choose the branch point $P_0 = (E_0, 0)$ as a convenient base point in the following. Let $\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}$ be the normal differential of the third kind holomorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ with simple poles at P_{∞_+} and P_{∞_-} and residues $+1$ and -1, respectively (cf. (A.23)–(A.26), (C.45)),

$$\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = \frac{1}{y}\prod_{j=1}^{n}(z-\lambda_j)dz \underset{\zeta\to 0}{=} (\pm\zeta^{-1} + O(\zeta))d\zeta \text{ as } P \to P_{\infty_\pm}. \tag{3.91}$$

Here the constants, $\lambda_j \in \mathbb{C}$, $j = 1, \dots, n$, are determined by employing the normalization

$$\int_{a_j} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = 0, \quad j = 1, \dots, n, \tag{3.92}$$

and ζ in (3.91) denotes the local coordinate

$$\zeta = 1/z \text{ for } P \text{ near } P_{\infty_\pm}.$$

Moreover,

$$\int_{P_0}^{P} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} \underset{\zeta\to 0}{=} \pm(\ln(\zeta) - \ln(\omega_0) + O(\zeta)) \text{ as } P \to P_{\infty_\pm} \tag{3.93}$$

for some constant $\omega_0 \in \mathbb{C}$.

Next, let $\omega^{(2)}_{P_{\infty_\pm},0}$ be normalized differentials of the second kind (cf. (A.20), (A.21), and (A.22)) satisfying

$$\int_{a_j} \omega^{(2)}_{P_{\infty_\pm},0} = 0, \quad j = 1, \dots, n, \tag{3.94}$$

$$\omega^{(2)}_{P_{\infty_\pm},0} \underset{\zeta\to 0}{=} (\zeta^{-2} + O(1))d\zeta \text{ as } P \to P_{\infty_\pm} \tag{3.95}$$

and introduce

$$\Omega_0^{(2)} = \omega^{(2)}_{P_{\infty_+},0} - \omega^{(2)}_{P_{\infty_-},0}. \tag{3.96}$$

Then

$$\int_{P_0}^{P} \Omega_0^{(2)} \underset{\zeta\to 0}{=} \mp(\zeta^{-1} + e_{0,0} + e_{0,1}\zeta + O(\zeta^2)) \text{ as } P \to P_{\infty_\pm}. \tag{3.97}$$

In addition, the vector of b-periods of $\Omega_0^{(2)}/(2\pi i)$ is denoted by

$$\underline{U}_0^{(2)} = (U_{0,1}^{(2)}, \dots, U_{0,n}^{(2)}), \quad U_{0,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \Omega_0^{(2)}, \quad j = 1, \dots, n \tag{3.98}$$

in the following. If $\mathcal{D}_{\underline{Q}}$ is assumed to be nonspecial, that is, $i(\mathcal{D}_{\underline{Q}}) = 0$ with $\underline{Q} = (Q_1, \dots, Q_n)$, a special case of Riemann's vanishing theorem (cf. Theorem A.26) yields

$$\theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}})) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_n\}. \tag{3.99}$$

Hence the divisor (3.58) of $\phi(\,\cdot\,, x)$ suggests considering expressions of the type

$$C(x)\frac{\theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}))}{\theta(\underline{\Xi}_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}))} \exp\left(\int_{P_0}^{P} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}}\right), \tag{3.100}$$

where $C(x)$ is independent of $P \in \mathcal{K}_n$. In the following it is convenient to use the abbreviation

$$\begin{aligned} &\underline{z}(P, \underline{Q}) = \underline{\Xi}_{P_0} - \underline{A}_{P_0}(P) + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{Q}}), \\ &P \in \mathcal{K}_n, \underline{Q} = \{Q_1, \dots, Q_n\} \in \mathrm{Sym}^n(\mathcal{K}_n). \end{aligned} \tag{3.101}$$

We note that by (A.52) and (A.53), $\underline{z}(\,\cdot\,, \underline{Q})$ is independent of the choice of base point P_0.

Given these preparations, one obtains the following theta function representation for ϕ, Ψ, p, and q.

Theorem 3.11 *Suppose that $p, q \in C^\infty(\Omega)$ are nonzero and satisfy the nth stationary AKNS system* (3.16) *on Ω. In addition, assume the affine part of $\mathcal{K}_n$ to be nonsingular and let $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ and $x, x_0 \in \Omega$, where $\Omega \subseteq \mathbb{R}$ is an open interval. Moreover, suppose $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ is nonspecial for $x \in \Omega$. Then,*[1]

$$\begin{aligned} \phi(P, x) = C_0 &\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x)))\theta(\underline{z}(P, \underline{\hat{\nu}}(x)))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}(x)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x)))} \\ &\times \exp\left(\int_{P_0}^{P} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} - 2ie_{0,0}x\right), \end{aligned} \tag{3.102}$$

[1] To avoid multi-valued expressions in formulas such as (3.102)–(3.104) etc., we agree always to choose the same path of integration connecting P_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

$$\psi_1(P,x,x_0) = \frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0)))} \times \exp\left(i\left(\int_{P_0}^{P} \Omega_0^{(2)} + e_{0,0}\right)(x-x_0)\right), \tag{3.103}$$

$$\psi_2(P,x,x_0) = C_0 \exp(-2ie_{0,0}x_0)\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P,\underline{\hat{\nu}}(x)))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x_0)))} \tag{3.104}$$
$$\times \exp\left(\int_{P_0}^{P} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} + i\left(\int_{P_0}^{P} \Omega_0^{(2)} - e_{0,0}\right)(x-x_0)\right),$$

where

$$C_0 = \frac{2i}{q(x_0)\omega_0}\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0)))}\exp(2ie_{0,0}x_0).$$

The Abel map linearizes the auxiliary divisors in the sense that

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) - i\underline{U}_0^{(2)}(x-x_0), \tag{3.105}$$
$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0)}) - i\underline{U}_0^{(2)}(x-x_0). \tag{3.106}$$

Finally, p, q are of the form

$$p(x) = p(x_0)\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x_0)))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x)))}\exp(-2ie_{0,0}(x-x_0)), \tag{3.107}$$
$$q(x) = q(x_0)\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x)))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x)))}\exp(2ie_{0,0}(x-x_0)), r \tag{3.108}$$
$$p(x_0)q(x_0) = \frac{4}{\omega_0^2}\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x_0)))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0)))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0)))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0)))}. \tag{3.109}$$

Proof First, we assume temporarily that

$$\mu_j(x) \neq \mu_{j'}(x), \nu_k(x) \neq \nu_{k'}(x) \text{ for } j \neq j', k \neq k' \text{ and } x \in \widetilde{\Omega} \tag{3.110}$$

for appropriate $\widetilde{\Omega} \subseteq \Omega$. Since by (3.58), $\mathcal{D}_{P_{\infty_+}\underline{\hat{\nu}}} \sim \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}}$ and $P_{\infty_+} = (P_{\infty_-})^* \notin \{\hat{\mu}_1,\dots,\hat{\mu}_n\}$ by hypothesis, one can apply Theorem A.31 to conclude that $\mathcal{D}_{\underline{\hat{\nu}}} \in \operatorname{Sym}^n(\mathcal{K}_n)$ is nonspecial. This argument is of course symmetric with respect to $\underline{\hat{\mu}}$ and $\underline{\hat{\nu}}$. Thus, $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial if and only if $\mathcal{D}_{\underline{\hat{\nu}}}$ is. Next, we define the right-hand side of (3.103) to be $\tilde{\psi}_1$. We intend to prove $\psi_1 = \tilde{\psi}_1$ with ψ_1 given by (3.60). For that purpose we first investigate the local zeros and poles of ψ_1. Since they can only come from zeros of $F_n(z,x')$ in (3.60), one computes using (3.53), the definition (3.56) of ϕ and the Dubrovin equations (3.77),

$$q(x')\phi(P,x') \underset{P\to\hat{\mu}_j(x')}{=} q(x')\frac{2y(\hat{\mu}_j(x'))}{-iq(x')\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j(x')-\mu_k(x'))}\frac{1}{z-\mu_j(x')} + O(1)$$
$$\underset{P\to\hat{\mu}_j(x')}{=} \partial_{x'}\ln(z-\mu_j(x')) + O(1). \tag{3.111}$$

Together with (3.60) this yields

$$\psi_1(P,x,x_0) = \begin{cases} (z-\mu_j(x))O(1) & \text{as } P \to \hat{\mu}_j(x) \neq \hat{\mu}_j(x_0), \\ O(1) & \text{as } P \to \hat{\mu}_j(x) = \hat{\mu}_j(x_0), \\ (z-\mu_j(x_0))^{-1}O(1) & \text{as } P \to \hat{\mu}_j(x_0) \neq \hat{\mu}_j(x), \end{cases}$$
$$P=(z,y)\in\mathcal{K}_n,\ x,x_0\in\widetilde{\Omega}$$

with $O(1)\neq 0$. Consequently, ψ_1 and $\tilde{\psi}_1$ have identical zeros and poles on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$, which are all simple by hypothesis (3.110). Next, comparing the behavior of ψ_1 and $\tilde{\psi}_1$ near $P_{\infty_\pm}$, taking into account (3.60), (3.97), the expression (3.103) for $\tilde{\psi}_1$, and (3.88), we observe that ψ_1 and $\tilde{\psi}_1$ have identical exponential behavior up to order $O(\zeta)$ near P_{∞_+} and identical exponential behavior up to order $O(1)$ near P_{∞_-}. Thus, ψ_1 and $\tilde{\psi}_1$ share the same singularities and zeros, and the Riemann–Roch-type uniqueness result in Lemma C.2 (taking $t_r = t_{0,r}$) then proves that ψ_1 and $\tilde{\psi}_1$ coincide up to normalization. By (3.97) one infers from the right-hand side of (3.103) that

$$\tilde{\psi}_1(P,x,x_0) \underset{\zeta\to 0}{=} e^{-i\zeta^{-1}(x-x_0)}(1+O(\zeta)) \text{ as } P\to P_{\infty_+}. \tag{3.112}$$

A comparison of (3.112) and (3.88) then yields (3.103) subject to (3.110). Equation (3.99) also immediately yields that ϕ equals (3.100) which, together with (3.87), implies

$$p = \frac{2C}{i\omega_0}\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}))}, \quad q = \frac{2i}{C\omega_0}\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}))}. \tag{3.113}$$

On the other hand (3.88) (near P_{∞_-}) and (3.103) yield (3.108). A comparison of (3.108) and (3.113) determines C and p, as in (3.107), (3.109). Given C, one determines ϕ in (3.102) using (3.100) and hence ψ_2, as in (3.104), using $\psi_2=\phi\psi_1$ (all subject to (3.110)). By (3.77) and a special case of Lagrange's interpolation formula (cf. Theorem E.1),

$$\sum_{j=1}^n \mu_j^{k-1}\prod_{\substack{\ell=1\\ \ell\neq j}}^n(\mu_j-\mu_\ell)^{-1} = \delta_{k,n},\ \mu_j\in\mathbb{C},\ j,k=1,\dots,n,$$

one infers by (3.96) and (C.37)

$$\partial_x \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}}) = -2i\underline{c}(n) = -i\underline{U}_0^{(2)}$$

and hence (3.105). Linear equivalence of $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ then yields (3.106) (subject to (3.110)). The extension of all these results from $\widetilde{\Omega}$ to Ω then simply follows from the continuity of $\underline{\alpha}_{P_0}$ and the hypothesis of $\mathcal{D}_{\underline{\hat{\mu}}}$ being nonspecial on Ω. $\square$

An alternative derivation of (3.107), (3.108) (more precisely, a theta function representation of p_x/p, q_x/q) is sketched in (C.49)–(C.52).

Remark 3.12 Since by (3.58) $\mathcal{D}_{P_{\infty_+}\underline{\hat{\nu}}}$ and $\mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}}$ are linearly equivalent, one infers

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{\Delta}, \quad \underline{\Delta} = \underline{A}_{P_{\infty_+}}(P_{\infty_-}).$$

Hence, one can eliminate $\mathcal{D}_{\underline{\hat{\nu}}}$ in (3.102), (3.104), (3.107), etc., in terms of $\mathcal{D}_{\underline{\hat{\mu}}}$ using

$$\begin{aligned} \underline{z}(P, \underline{\hat{\nu}}) &= \underline{z}(P, \underline{\hat{\mu}}) + \underline{\Delta}, \quad P \in \mathcal{K}_n, \\ \underline{z}(P_{\infty_\pm}, \underline{\hat{\nu}}) &= \underline{z}(P_{\infty_\pm}, \underline{\hat{\mu}}) + \underline{\Delta}. \end{aligned} \tag{3.114}$$

Combining (3.105), (3.106), (3.114), and (3.107), (3.108) shows the remarkable linearity of the theta function arguments with respect to x in the formulas for p, q. In fact, one can rewrite (3.107), (3.108) as

$$p(x) = C_p \frac{\theta(\underline{A} + \underline{B}x + \underline{\Delta})}{\theta(\underline{A} + \underline{B}x)} \exp(-ie_0 x), \tag{3.115}$$

$$q(x) = C_q \frac{\theta(\underline{A} + \underline{B}x - \underline{\Delta})}{\theta(\underline{A} + \underline{B}x)} \exp(ie_0 x), \tag{3.116}$$

where

$$\underline{A} = \underline{\Xi}_{P_0} - \underline{A}_{P_0}(P_{\infty_+}) + i\underline{U}_0^{(2)} x_0 + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}), \tag{3.117}$$

$$\underline{B} = -i\underline{U}_0^{(2)}, \tag{3.118}$$

$$\underline{\Delta} = \underline{A}_{P_{\infty_+}}(P_{\infty_-}), \tag{3.119}$$

and hence the constants $e_0 \in \mathbb{C}$ and $\underline{\Delta}, \underline{B} \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ (and its homology basis), the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \operatorname{Sym}^n(\mathcal{K}_n)$ at the point x_0 as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ is assumed to be nonspecial. The constants $C_p, C_q \in \mathbb{C}$ satisfy constraints analogous to (3.109).

Remark 3.13 The explicit expressions (3.103), (3.104) for ψ_j, $j = 1, 2$ complement Lemma 3.7 and show that Ψ stays meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial (assuming the affine part of $\mathcal{K}_n$ to be nonsingular).

For completeness we also mention another theta function representation for the product pq that will be useful in connection with the classical Boussinesq equation in Section 3.5.

Corollary 3.14 *Assume the hypotheses of Theorem* 3.11. *Then,*

$$p(x)q(x) = -e_{0,1} - \partial_x^2 \ln(\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x)))). \tag{3.120}$$

Proof The proof is exactly along the lines of the derivation of the Its–Matveev formula for the KdV hierarchy (see Theorem 1.20). Eliminating ψ_2 in (3.68) results in

$$\psi_{1,xx}(P) = (q_x/q)\psi_{1,x}(P) + \big(pq + iz(q_x/q) - z^2\big)\psi_1(P).$$

Next, using

$$\psi_1(P, x, x_0) \underset{\zeta\to 0}{=} \exp(-i(\zeta^{-1} + e_{0,1}\zeta + O(\zeta^2))(x - x_0)) \\ \times (1 + c_1(x)\zeta + c_2(x)\zeta^2 + O(\zeta^3)) \text{ as } P \to P_{\infty_+},$$

one infers

$$\begin{aligned} 0 &= -\psi_{1,xx}(P, x, x_0) + (q_x(x)/q(x))\psi_{1,x}(P, x, x_0) \\ &\quad + \big(p(x)q(x) + i(q_x(x)/q(x))\zeta^{-1} - \zeta^{-2}\big)\psi_1(P, x, x_0) \\ &\underset{\zeta\to 0}{=} \exp(-i(\zeta^{-1} + e_{0,1}\zeta + O(\zeta^2))(x - x_0)) \\ &\quad \times (e_{0,1} + 2ic_{1,x}(x) + p(x)q(x) + O(\zeta)) \text{ as } P \to P_{\infty_+}. \end{aligned} \tag{3.121}$$

By the uniqueness of ψ_1, as discussed in Lemma C.2, one concludes that

$$pq = -e_{0,1} - 2ic_{1,x}. \tag{3.122}$$

It remains to determine $c_{1,x}$. First, we recall from (C.35) that

$$\underline{\omega} \underset{\zeta\to 0}{=} (\underline{c}(n) + O(\zeta))d\zeta \text{ near } P_{\infty_+}$$

and hence

$$\underline{A}_{P_0}(P) \underset{\zeta\to 0}{=} \underline{A}_{P_0}(P_{\infty_+}) + \underline{c}(n)\zeta + O(\zeta^2) \underset{\zeta\to 0}{=} \underline{A}_{P_0}(P_{\infty_+}) + \frac{1}{2}\underline{U}_0^{(2)}\zeta + O(\zeta^2) \\ \text{as } P \to P_{\infty_+},$$

where we combined (3.96) and $\underline{U}_0^{(2)} = 2\underline{c}(n)$ (cf. (3.98), (C.37)) in the second equality. Since pq only depends on $c_{1,x}$ as opposed to c_1 itself, it suffices to consider the following expansion near P_{∞_+},

$$\begin{aligned} \frac{\theta(\underline{z}(P, \underline{\hat{\mu}}))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}))} &\underset{\zeta\to 0}{=} 1 - \frac{1}{2}\frac{\sum_{j=1}^n U_{0,j}^{(2)}\partial_{w_j}\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}) + \underline{w}\big)\big|_{\underline{w}=0}\zeta}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}))} + O(\zeta^2) \\ &= 1 - (i/2)\partial_x \ln(\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}})))\zeta + O(\zeta^2). \end{aligned} \tag{3.123}$$

Here we used (3.105) to arrive at the last equality in (3.123). A comparison of (3.121) and (3.123) then yields

$$c_{1,x} = -(i/2)\partial_x^2 \ln(\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}))),$$

which finally yields (3.120) via (3.122). $\square$

We note that the free constant $q(x_0)$ in (3.107) (and in (3.108) using (3.109)) cannot be determined since the AKNS equations (3.37) are invariant with respect to scale transformations, as discussed in Lemma 3.6.

The algebro-geometric potentials p, q in (3.107), (3.108) represent stationary solutions of the AKNS hierarchy, a complexified hierarchy of the hierarchy of nonlinear Schrödinger equations, denoted by $\mathrm{nS}_\pm$. Next, we comment on the isospectral sets for the focusing, nS_-, and defocusing, nS_+, cases, as introduced at the end of Section 3.2. In both cases one needs to impose certain symmetry constraints on $\mathcal{K}_n$ and additional constraints on $\underline{A}$ in (3.115)–(3.117), which we will discuss in the following.

We start with the defocusing nS_+ case and hence suppose $p = \overline{q}$: As observed in (3.46), M is formally self-adjoint in this case, and hence the branch points of $\mathcal{K}_n$ must all be in real position. This leads to the reality constraints

$$E_0 < E_1 < \cdots < E_{2n+1} \tag{3.124}$$

for the zeros of R_{2n+2}, that is, all branch points of $\mathcal{K}_n$ are assumed to be in real position.

Lemma 3.15 *Assume* (3.124)*, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ *is nonspecial for some* $x_0 \in \mathbb{R}$*, and choose the homology basis* $\{a_j, b_j\}_{j=1}^n$ *according to Theorem* A.36 *(i) (compare with Figure* C.2*, implementing the constraint* (3.124)*). Then the meromorphic solution* q *in* (3.108) *equals* $\overline{p}$ *in* (3.107) *and hence represents a stationary* nS_+ *potential if and only if* $\underline{A}$ *in* (3.117) *satisfies the constraint*

$$\mathrm{Re}(\underline{A}) = (1/2)\underline{\chi} \pmod{\mathbb{Z}^n}, \quad \underline{\chi} = (\chi_1, \dots, \chi_n), \chi_j \in \{0,1\}, j = 1, \dots, n. \tag{3.125}$$

In particular, under the present hypotheses, the set of stationary algebro-geometric nS_+ *potentials* q *in* (3.116) *consists of* 2^n *connected components indexed by* $\underline{\chi} = (\chi_1, \dots, \chi_n)$*,* $\chi_j \in \{0,1\}$*,* $j = 1, \dots, n$*, and the component associated with* $\underline{\chi} = 0$ *comprises all real-valued smooth potentials* $q \in C^\infty(\mathbb{R})$*.*

Proof Define the antiholomorphic involution $\rho_+ : (z, y) \mapsto (\overline{z}, \overline{y})$ as in Example A.35 (i). By Example A.35 (i), Theorem A.36 (cf. (A.65), (A.69)–(A.71)), (C.31), (C.32), (C.33), (C.37), and (C.38)–(C.40) one infers that $(\mathcal{K}_n, \rho_+)$ is of dividing type, and hence

$$\begin{aligned}
&r = n+1, \quad \overline{\tau} = -\tau, \quad R = 0, \quad \overline{\theta(\underline{z})} = \theta(\overline{\underline{z}}), \underline{z} \in \mathbb{C}^n, \\
&\rho_+(a_j) = a_j, \quad \rho_+(b_j) = -b_j, j = 1, \dots, n, \\
&\underline{c}(k) \in \mathbb{R}^n, k = 1, \dots, n, \quad \underline{U}_0^{(2)} \in \mathbb{R}^n, \\
&\rho_+^* \Omega_0^{(2)} = \Omega_0^{(2)}, \quad e_0 \in \mathbb{R}.
\end{aligned}$$

Thus,

$$\overline{\underline{B}} = -\underline{B}$$

by (3.118), and hence $p = \overline{q}$ in (3.115), (3.116) is equivalent to

$$\frac{C_p}{\overline{C_q}} = \frac{\theta(\underline{A} + \underline{B}x)\overline{\theta(\underline{A} + \underline{B}x - \underline{\Delta})}}{\theta(\underline{A} + \underline{B}x + \underline{\Delta})\overline{\theta(\underline{A} + \underline{B}x)}} = \frac{\theta(\underline{A} + \underline{B}x)\theta(-\overline{\underline{A}} + \underline{B}x + \overline{\underline{\Delta}})}{\theta(\underline{A} + \underline{B}x + \underline{\Delta})\theta(-\overline{\underline{A}} + \underline{B}x)}. \tag{3.126}$$

Given that

$$\underline{\Delta} = \underline{A}_{P_{\infty_+}}(P_{\infty_-}) \in \mathbb{R}^n,$$

equation (3.126) is equivalent to

$$\underline{A} = -\overline{\underline{A}} + \underline{m}_1 + \underline{n}_1 \tau$$

for some $\underline{n}_1 \in \mathbb{Z}^n$ and arbitrary $\underline{m}_1 \in \mathbb{Z}^n$ and hence to

$$\mathrm{Re}(\underline{A}) = (1/2)\underline{m}_1, \quad \underline{m}_1 \in \mathbb{Z}^n$$

and $\underline{n}_1 = 0$. Replacing $\underline{A}$ by $\underline{A} + m + \underline{n}\tau$ with $\underline{m}, \underline{n} \in \mathbb{Z}^n$ then yields

$$\mathrm{Re}(\underline{A}) = (1/2)\underline{m}_1 - \underline{m}, \quad \underline{m}_1, \underline{m} \in \mathbb{Z}^n$$

and hence (3.125). Finally, since q is of the type

$$q(x) = C_q \frac{\theta(\underline{A} + \underline{B}x - \underline{\Delta})}{\theta(\underline{A} + \underline{B}x)} \exp(ie_0 x), \tag{3.127}$$

$q \in C^\infty(\mathbb{R})$ if and only if $\underline{\chi} = 0$. This follows from (A.73) (with $\ell = 0$) since the denominator in (3.127) is of the type

$$\theta(i\mathrm{Im}(\underline{A}) + i\mathrm{Im}(\underline{B})x + (1/2)(\chi_1, \dots, \chi_n)), \quad \chi_j \in \{0, 1\}, j = 1, \dots, n,$$

which has zeros if and only if $\underline{\chi} \neq 0$. $\square$

Remark 3.16 We briefly take a closer look at the connected component of nonsingular nS_+ potentials $q \in C^\infty(\mathbb{R})$ in (3.116) associated with $\underline{\chi} = 0$ in Lemma 3.15. Even though the Lax differential expression $M = i\begin{pmatrix} \frac{d}{dx} & -q \\ \overline{q} & -\frac{d}{dx} \end{pmatrix}$ is formally self-adjoint in the nS_+ context, the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ are not constrained to be in real position since the corresponding eigenvalue problem is easily seen to be non-self-adjoint. Indeed, assuming $q \in L^\infty(\mathbb{R})$, the auxiliary eigenvalues $\mu_j(\xi)$ and $\nu_j(\xi)$ (cf. (3.51)–(3.54), (3.59)–(3.61), (3.70), (3.71)) are associated with self-adjoint Dirac-type operators in $L^2(\mathbb{R})^2$ of the form[1]

$$\begin{aligned} &D_{1,\xi} f = Mf, \\ &f \in \mathrm{dom}(D_{1,\xi}) = \big\{g = (g_1, g_2)^\top \in L^2(\mathbb{R})^2 \big| g_1 \in AC_{\mathrm{loc}}(\mathbb{R}), \\ &\qquad\qquad g_2 \in AC_{\mathrm{loc}}(\mathbb{R} \setminus \{\xi\}), g_1(\xi) = 0, Mg \in L^2(\mathbb{R})^2\big\} \end{aligned}$$

[1] Here $AC_{\mathrm{loc}}(\mathcal{I})$, $\mathcal{I} \subseteq \mathbb{R}$ an open interval, denotes the set of locally absolutely continuous functions on $\mathcal{I}$.

and

$$\begin{aligned} &D_{2,\xi} f = Mf, \\ &f \in \mathrm{dom}(D_{2,\xi}) = \big\{g = (g_1, g_2)^\top \in L^2(\mathbb{R})^2 \big| g_1 \in AC_{\mathrm{loc}}(\mathbb{R} \setminus \{0\}), \\ &\qquad\qquad g_2 \in AC_{\mathrm{loc}}(\mathbb{R}),\, g_2(\xi) = 0,\, Mg \in L^2(\mathbb{R})^2\big\}, \end{aligned}$$

respectively. One readily verifies

$$D_{1,\xi}^* = D_{2,\xi}$$

and hence the non-self-adjoint character of $D_{j,\xi}$, $j = 1, 2$. That the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ are not constrained to be in real position also follows from the factor i on the right-hand side of (3.77) and the fact that for $\mu_j \in [E_{2j-1}, E_{2j}]$, $y(\hat{\mu}_j) \in \mathbb{R}$ for all $j = 1, \dots, n$ since by (C.19) $R_{2n+2}(\lambda)^{1/2}$ is real-valued for $\lambda \in (E_{2j-1}, E_{2j})$, $j = 1, \dots, n$.

It can be shown (pertinent references are provided in the notes to this section) that the corresponding isospectral set of all smooth algebro-geometric nS$_+$ potentials $q \in C^\infty(\mathbb{R})$ corresponding to a fixed curve $\mathcal{K}_n$ constrained by (3.124), that is, the connected component associated with $\underline{\chi} = 0$ in Lemma 3.15, can be identified with an $(n+1)$-dimensional real torus $\mathbb{T}^{n+1}$. This isospectral torus is of dimension $n+1$ (rather than n) due to the additional scaling invariance discussed in (3.45) after Lemma 3.6 involving an arbitrary constant multiple of absolute value equal to one.

Next, we turn to the focusing nS$_-$ case and hence suppose $p = -\overline{q}$: As observed in (3.48), M is not formally self-adjoint but formally unitarily equivalent to its formal adjoint M^*. Further investigations (especially in the spatially periodic case using Floquet theoretic methods) then lead to the assumption of complex conjugate pairs of nS$_-$ branch points. Thus, the constraints on the set of zeros of R_{2n+2} can be written as[1]

$$\{E_m, \overline{E_m}\}_{m=0,\dots,n}. \tag{3.128}$$

We start by recalling that algebro-geometric nS$_-$ potentials are smooth.

Lemma 3.17 *Assume that q is an algebro-geometric stationary* nS$_-$ *potential of the type* (3.108). *Then,*[2]

$$q \in C^\infty(\mathbb{R}).$$

Proof Nonsingularity of the algebro-geometric nS$_-$ solutions q follows from (3.116), the theta functions in (3.116) being entire with respect to x, and the equations of the stationary nS$_-$ hierarchy being incompatible with any pole behavior of q of the type $(x - x_1)^{-n}$, $n \in \mathbb{N}$, $x_1 \in \mathbb{R}$. $\square$

[1] Of course we still assume the affine part of $\mathcal{K}_n$ to be nonsingular (cf. (3.76)).
[2] We are not explicitly assuming (3.128).

Lemma 3.18 *Assume* (3.128), *suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ *is nonspecial for some* $x_0 \in \mathbb{R}$, *and choose the homology basis* $\{a_j, b_j\}_{j=1}^n$ *according to Theorem* A.36 *(i). Then the solution* q *in* (3.108) *equals* $-\overline{p}$ *in* (3.107) *and hence represents a stationary* nS_- *potential if and only if* $\underline{A}$ *in* (3.117) *satisfies the constraint*

$$\operatorname{Re}(\underline{A}) = 0 \pmod{\mathbb{Z}^n}. \tag{3.129}$$

In particular, under the present hypotheses, the set of stationary algebro-geometric nS_- *potentials* q *in* (3.116) *consists of one connected component, all of whose elements are smooth,* $q \in C^\infty(\mathbb{R})$.

Proof The proof closely parallels that of Lemma 3.15, and hence we mainly focus on those points that differ in the current case. By Example A.35 (i), Theorem A.36 (cf. (A.65), (A.69)–(A.71)), one infers that $(\mathcal{K}_n, \rho_+)$ is of dividing type; hence,

$$r = 1 \text{ if } n \text{ is even}, \; r = 2 \text{ if } n \text{ is odd}, \quad \overline{\tau} = R - \tau,$$

$$R = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_1 \end{pmatrix} \text{ if } r = 1 \text{ (and hence } n \text{ is even)},$$

$$R = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_1 & \\ & & & 0 \end{pmatrix} \text{ if } r = 2 \text{ (and hence } n \text{ is odd)},$$

$$\overline{\theta(\underline{z})} = \theta(\overline{\underline{z}}), \; \underline{z} \in \mathbb{C}^n, \quad \underline{\operatorname{diag}}(R) = 0,$$

$$\rho_+(a_j) = a_j, \quad \rho_+(b_j) = (\underline{a}R)_j - b_j, \; j = 1, \dots, n,$$

$$\rho_+^* \Omega_0^{(2)} = \Omega_0^{(2)}, \quad e_0 \in \mathbb{R}, \quad \underline{U}_0^{(2)} \in \mathbb{R}^n.$$

Thus,

$$\overline{\underline{B}} = -\underline{B}$$

by (3.118), and hence $p = -\overline{q}$ in (3.115), (3.116) is equivalent to

$$-\frac{C_p}{\overline{C_q}} = \frac{\theta(\underline{A} + \underline{B}x)\overline{\theta(\underline{A} + \underline{B}x - \underline{\Delta})}}{\theta(\underline{A} + \underline{B}x + \underline{\Delta})\overline{\theta(\underline{A} + \underline{B}x)}} = \frac{\theta(\underline{A} + \underline{B}x)\theta(-\overline{\underline{A}} + \underline{B}x + \overline{\underline{\Delta}})}{\theta(\underline{A} + \underline{B}x + \underline{\Delta})\theta(-\overline{\underline{A}} + \underline{B}x)}.$$

Thus, one infers

$$\underline{A} = -\overline{\underline{A}} + \underline{m}_1 + \underline{n}_1 \tau$$

for some $\underline{n}_1 \in \mathbb{Z}^n$ and arbitrary $\underline{m}_1 \in \mathbb{Z}^n$, and hence

$$\operatorname{Re}(\underline{A}) = (1/2)\underline{m}_1, \quad \underline{m}_1 \in \mathbb{Z}^n$$

and $\underline{n}_1 = 0$. Similarly, one obtains

$$\underline{A} + \underline{\Delta} = -\overline{\underline{A}} + \overline{\underline{\Delta}} + \underline{m}_2 + \underline{n}_2 \tau$$

for some $\underline{n}_2 \in \mathbb{Z}^n$ and arbitrary $\underline{m}_2 \in \mathbb{Z}^n$, and hence

$$2\mathrm{Im}(\underline{\Delta}) = \underline{n}_2 \mathrm{Im}(\tau),$$
$$\underline{m}_2 - \underline{m}_1 + (1/2)\underline{n}_2 R = 0.$$

Replacing $\underline{A}$ by $\underline{A} + \underline{m} + \underline{n}\tau$ with $\underline{m}, \underline{n} \in \mathbb{Z}^n$ then yields

$$\begin{aligned} \mathrm{Re}(\underline{A}) &= (1/2)\underline{m}_1 - \underline{m} - (1/2)\underline{n}R, \quad \underline{m}_1, \underline{m}, \underline{n} \in \mathbb{Z}^n \\ &= \begin{cases} (\underbrace{0, \dots, 0}_{n}), & r = 1, \\ (1/2)(\underbrace{0, \dots, 0}_{n-1}, \chi_n),\ \chi_n \in \{0, 1\}, & r = 2. \end{cases} \end{aligned} \tag{3.130}$$

For $r = 1$, this yields (3.129). For $r = 2$, (3.130) appears to yield two connected components corresponding to $\chi_n = 0$ and $\chi_n = 1$. However, all algebro-geometric nS_- potentials are smooth $q \in C^\infty(\mathbb{R})$ by Lemma 3.17. On the other hand, since q is of the type

$$q(x) = C_q \frac{\theta(\underline{A} + \underline{B}x - \underline{\Delta})}{\theta(\underline{A} + \underline{B}x)} \exp(ie_0 x), \tag{3.131}$$

the component corresponding to $\chi_n = 1$ would consist of singular solutions since the denominator in (3.131) is of the type

$$\theta(i\mathrm{Im}(\underline{A}) + i\mathrm{Im}(\underline{B})x + (1/2)(0, \dots, 0, 1)),$$

which has zeros by (A.73) (with $\ell = n - 1$). Hence, the component corresponding to $\chi_n = 1$ is empty, and we again arrive at (3.129). $\square$

Remark 3.19 A careful analysis of the nS_- case (pertinent references are provided in the notes to this section) reveals the following facts: The motion of the auxiliary divisors $\hat{\mu}_j$ is not constrained to certain fixed curves on $\mathcal{K}_n$; moreover, collisions between them may occur. Still, the motion of the $\hat{\mu}_j$ can be shown to remain homologous to some linear combinations of appropriate a_j and b_j cycles. An analysis of (3.107)–(3.109) yields that the connected component of algebro-geometrical nS_- potentials identified in Lemma 3.18, or equivalently, the isospectral set of algebro-geometric nS_- potentials q associated with the curve $\mathcal{K}_n$ constrained by (3.128), is given by a real torus $\mathbb{T}^{n+1}$. (Again the dimension of the torus is $n + 1$ (rather than n) due to the additional scaling invariance in connection with a unimodular constant, as in the nS_+ case.) Moreover, all elements in the isospectral torus $\mathbb{T}^{n+1}$ consist of smooth nS_- potentials $q \in C^\infty(\mathbb{R})$.

Next, we briefly consider the trivial case $n = 0$ excluded in Theorem 3.11.

Example 3.20 Assume $n = 0$, $P = (z, y) \in \mathcal{K}_0 \setminus \{P_{\infty_+}, P_{\infty_-}\}$, and let $(x, x_0) \in \mathbb{R}^2$. Then

$$\begin{aligned}
&\mathcal{K}_0\colon \mathcal{F}_0(z, y) = y^2 - R_2(z) = y^2 - (z - E_0)(z - E_1) = 0,\\
&c_1 = -(E_0 + E_1)/2, \quad E_0, E_1 \in \mathbb{C},\\
&p(x) = p(x_0)\exp(-2ic_1(x - x_0)),\\
&q(x) = q(x_0)\exp(2ic_1(x - x_0)),\\
&p(x)q(x) = (E_0 - E_1)^2/4,\\
&\text{s-AKNS}_0(p, q) = 0,\\
&F_0(z, x) = -iq(x), \quad G_1(z, x) = z + c_1, \quad H_0(z, x) = ip(x),\\
&\phi(P, x) = \frac{y + z + c_1}{-iq(x)} = \frac{ip(x)}{y - z - c_1},\\
&\psi_1(P, x, x_0) = \exp(i(y + c_1)(x - x_0)),\\
&\psi_2(P, x, x_0) = \frac{y + z + c_1}{-iq(x_0)}\exp(i(y - c_1)(x - x_0)).
\end{aligned}$$

Up to this point we assumed $p, q \in C^\infty(\mathbb{R})$ and the stationary AKNS equation (3.16) for some fixed $n \in \mathbb{N}_0$. Next we will show that solvability of the Dubrovin equations (3.77) on $\Omega_\mu \subseteq \mathbb{R}$ in fact implies equation (3.16) on Ω_μ. As pointed out in Remark 3.24, this amounts to solving the algebro-geometric initial value problem in the stationary case.

Theorem 3.21 *Fix $n \in \mathbb{N}$, assume the affine part of $\mathcal{K}_n$ to be nonsingular, and suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the stationary Dubrovin equations* (3.77) *on an open interval $\Omega_\mu \subseteq \mathbb{R}$ such that μ_j, $j = 1, \dots, n$, remain distinct on Ω_μ. Define $p, q \in C^\infty(\Omega_\mu)$ by*

$$q(x) = q(x_0)\exp\bigg(-i(x - x_0)\sum_{m=0}^{2n+1} E_m + i\sum_{j=1}^{n}\int_{x_0}^{x} dx'\mu_j(x')\bigg), \tag{3.132}$$

$$q(x_0) \neq 0$$

and[1]

$$\begin{aligned}
p(x) = {}& \frac{1}{2q(x)\prod_{j=1}^n \mu_j(x)}\bigg(\frac{1}{q(x)}\bigg(q(x)\prod_{j=1}^n \mu_j(x)\bigg)_x\bigg)_x\\
&- \frac{1}{4q(x)^3\prod_{j=1}^n \mu_j(x)^2}\bigg(\bigg(q(x)\prod_{j=1}^n \mu_j(x)\bigg)_x\bigg)^2 - \frac{\prod_{m=0}^{2n+1} E_m}{q(x)\prod_{j=1}^n \mu_j(x)^2},\\
&\text{if } \mu_j \neq 0 \text{ on } \Omega_\mu,\ j = 1, \dots, n,
\end{aligned} \tag{3.133}$$

[1] If $\mu_{j_0}(x_0) = 0$ for some $j_0 \in \{1, \dots, n\}$ and some $x_0 \in \Omega_\mu$, one can use (3.145) to define p at x_0. Since the explicit formula for p in terms of $\{\mu_j\}_{j=1,\dots,n}$ is straightforward but rather cumbersome in this case, we omit further details.

with q defined by (3.132). *Then p is nonzero on Ω_μ and p, q satisfy the nth stationary AKNS equation* (3.16)*, that is,*

$$\text{s-AKNS}_n(p, q) = 0 \text{ on } \Omega_\mu. \tag{3.134}$$

Proof Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (3.77), we introduce

$$F_n(z) = -iq \prod_{j=1}^{n} (z - \mu_j) \text{ on } \mathbb{C} \times \Omega_\mu \tag{3.135}$$

with q defined by (3.132) up to a multiplicative constant (due to the scale invariance of the AKNS hierarchy; cf. Lemma 3.6). Given F_n and q, one defines the monic polynomial G_{n+1} of degree $n + 1$ by

$$G_{n+1}(z) = (2q)^{-1} F_{n,x}(z) + izq^{-1} F_n(z) \text{ on } \mathbb{C} \times \Omega_\mu \tag{3.136}$$

(cf. (3.20)) and then infers on Ω_μ

$$\hat{\mu} = (\mu_j, (2q)^{-1} F_{n,x}(\mu_j)) = (\mu_j, G_{n+1}(\mu_j)), \quad j = 1, \dots, n \tag{3.137}$$

as a consequence of (3.77). We note that G_{n+1} is uniquely defined independently of the scaling freedom in q since F_n contains a factor of q. Next, we define a polynomial H_n on $\mathbb{C} \times \Omega_\mu$ such that (3.23) holds, that is, by

$$R_{2n+2}(z) - G_{n+1}(z)^2 = -F_n(z) H_n(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{3.138}$$

Such a polynomial H_n exists since the left-hand side of (3.138) vanishes at $z = \mu_j$, $j = 1, \dots, n$ by (3.137). To determine the degree of H_n, one computes

$$G_{n+1}(z)^2 \underset{|z|\to\infty}{=} z^{2n+2} - \left(\sum_{m=0}^{2n+1} E_m \right) z^{2n+1} + O(z^{2n}) \tag{3.139}$$

using (3.132) with $O(z^{2n})$ depending on x by inspection. Thus, combining (3.138) and (3.139), H_n has degree n with respect to z. Hence, we may write

$$H_n(z) = ip \prod_{j=1}^{n} (z - \nu_j) \text{ on } \mathbb{C} \times \Omega_\mu \tag{3.140}$$

for some $p \in C^\infty(\Omega_\mu)$. Equation (3.140) defines $p \in C^\infty(\Omega_\mu)$ up to a multiplicative constant and in accordance with the AKNS scale invariance discussed in Lemma 3.6. With the polynomial P_{n-1} introduced by

$$P_{n-1}(z) = pF_n(z) + qH_n(z) - G_{n+1,x}(z) \text{ on } \mathbb{C} \times \Omega_\mu,$$

equations (3.135), (3.139), and (3.140) show that P_{n-1} is a polynomial of degree

$n-1$. Differentiating (3.138) with respect to x yields

$$2G_{n+1}(z)G_{n+1,x}(z) - F_{n,x}(z)H_n(z) - F_n(z)H_{n,x}(z) = 0 \text{ on } \mathbb{C}\times\Omega_\mu, \quad (3.141)$$

and hence

$$G_{n+1}(\mu_j)P_{n-1}(\mu_j) = 0, \quad j = 1,\dots,n$$

on Ω_μ. From this point on one can follow the proof of Theorem 1.26 line by line and thus conclude that

$$P_{n-1} = 0 \text{ on } \mathbb{C}\times\Omega_\mu.$$

Hence, (3.21) holds, that is,

$$G_{n+1,x}(z) = pF_n(z) + qH_n(z) \text{ on } \mathbb{C}\times\Omega_\mu. \quad (3.142)$$

Combining (3.136), (3.141), and (3.142) then yields (3.22),

$$H_{n,x}(z) = 2izH_n(z) + 2pG_{n+1}(z) \text{ on } \mathbb{C}\times\Omega_\mu, \quad (3.143)$$

and thus we have derived the fundamental equations (3.20)–(3.22) and (3.23) on $\mathbb{C}\times\Omega_\mu$. One can now mimic the analysis in (3.4)–(3.15) and thus conclude that p, q satisfy the stationary AKNS equations (3.134). Equation $iq_x/q = \sum_{m=0}^{2n+1} E_m - 2\sum_{j=1}^n \mu_j$ (cf. (3.132)) and a comparison of powers of z in (3.135), (3.138), (3.140), and (3.143) then yield $ip_x/p = -\sum_{m=0}^{2n+1} E_m + 2\sum_{j=1}^n \nu_j$ (cf. (3.81)) and hence show that $p \neq 0$ on Ω_μ. Finally, using (3.142), (3.138), and (3.136), one obtains

$$\begin{aligned} p &= \frac{G_{n+1,x}(z) - qH_n(z)}{F_n(z)} = \frac{F_n(z)G_{n+1,x}(z) + q(R_{2n+2}(z) - G_{n+1}(z)^2)}{F_n(z)^2} \\ &= \frac{1}{F_n(z)}\bigg(\frac{1}{2q}F_{n,x}(z) + \frac{iz}{q}F_n(z)\bigg)_x \\ &\quad + \frac{q}{F_n(z)^2}\Bigg(R_{2n+2}(z) - \bigg(\frac{1}{2q}F_{n,x}(z) + \frac{iz}{q}F_n(z)\bigg)^2\Bigg). \end{aligned} \quad (3.144)$$

Insertion of (3.135) into (3.144), taking $z = 0$, then yields expression (3.133) for p as long as $\prod_{j=1}^n \mu_j \neq 0$. If $\mu_{j_0}(x_0) = 0$ for some $j_0 \in \{1,\dots,n\}$ and some $x_0 \in \Omega_\mu$ (we note that at most one μ_j can vanish at x_0 by our hypothesis that the μ_j are distinct on Ω_μ), one uses

$$p(x_0) = \frac{(d/dz)(G_{n+1,x}(z) - qH_n(z))}{(d/dz)F_n(z)}\bigg|_{z=\mu_{j_0}(x_0)} \quad (3.145)$$

to define $p(x_0)$ and then eliminates H_n in terms of F_n and G_{n+1} using (3.138) and finally eliminates G_{n+1} and $G_{n+1,x}$ in terms of F_n and its x-derivatives using (3.136). $\square$

Remark 3.22 The explicit theta function representations (3.107), (3.108) of p, q on Ω_μ in (3.132), (3.133) then permit one to extend p and q beyond Ω_μ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ remains nonspecial (cf. Theorem A.31).

Remark 3.23 We singled out q and $\{\mu_j\}_{j=1,\dots,n}$ in Theorem 3.21, but of course one can prove the analogous result in terms of p and $\{\nu_j\}_{j=1,\dots,n}$.

Remark 3.24 A closer look at Theorem 3.21 reveals that p, q are uniquely determined in an open neighborhood Ω of x_0 by $\mathcal{K}_n$ and the initial condition $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \operatorname{Sym}^n(\mathcal{K}_n)$ or, equivalently, by the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)} \in \operatorname{Sym}^n(\mathcal{K}_n)$ at $x = x_0$. Conversely, given $\mathcal{K}_n$ and p, q in an open neighborhood Ω of x_0, one can construct the corresponding polynomials $F_n(\,\cdot\,, x)$, $G_{n+1}(\,\cdot\,, x)$, $H_n(\,\cdot\,, x)$ for $x \in \Omega$ (using the recursion relation (3.4)–(3.7) to determine the homogeneous elements $\hat{f}_\ell$, $\hat{g}_\ell$, $\hat{h}_\ell$ and (D.26) to determine $c_\ell = c_\ell(\underline{E})$, $\ell = 0, \dots, n$) and then recover the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ for $x \in \Omega$ from the zeros of $F_n(\,\cdot\,, x)$ and from (3.53). This remark is of relevance in connection with determining the isospectral set of AKNS potentials p, q in the sense that once the curve $\mathcal{K}_n$ is fixed, elements of the isospectral class of potentials are parametrized by (nonspecial) auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ (cf. Remark 3.16).

We will end this section by providing a few examples that are analyzed in detail in the references given in the notes to this section. We recall our convention abbreviating algebro-geometric stationary solutions p, q of some (and hence infinitely many such) stationary AKNS equations as AKNS potentials. By $\wp(\,\cdot\,) = \wp(\,\cdot\,|\omega_1, \omega_3) = \wp(\,\cdot\,; g_2, g_3)$ we denote the Weierstrass $\wp$-function with periods $2\omega_j$, $j = 1, 3$, $\operatorname{Im}(\omega_3/\omega_1) \neq 0$, $\omega_2 = \omega_1 + \omega_3$, invariants g_2 and g_3, and associated fundamental period parallelogram Δ (cf. Appendix H). Similarly, $\zeta(\,\cdot\,) = \zeta(\,\cdot\,|\omega_1, \omega_3) = \zeta(\,\cdot\,; g_2, g_3)$ denotes the corresponding Weierstrass ζ-function.

Example 3.25 A few elliptic algebro-geometric AKNS potentials follow.

$$p(x) = q(x) = n(\zeta(x) - \zeta(x - \omega_2) - \zeta(\omega_2)), \quad n \in \mathbb{N},$$
$$p(x) = 1, \quad q(x) = n(n+1)\wp(x), \quad n \in \mathbb{N},$$
$$p(x) = 3 - \wp'(x)/(2e_1), \quad q(x) = -\wp'(x - \omega_2)/(2e_1) \text{ assuming } e_2 = 0,$$
$$p(x) = \frac{2}{3}(\wp''(x) - e_1^2), \quad q(x) = -\wp(x - \omega_2)/e_1^2 \text{ assuming } e_2 = 0.$$

Incidentally, if q is an elliptic algebro-geometric potential of the KdV hierarchy, then $p = 1, q$ are algebro-geometric AKNS potentials. Conversely, if p, q are algebro-geometric AKNS potentials with $p = 1$, then q is an algebro-geometric potential of the KdV hierarchy. In particular, $q(x) = n(n+1)\wp(x)$ is the well-known class of Lamé potentials associated with the KdV hierarchy (cf. Example 1.32).

3.4 The Time-Dependent AKNS Formalism

In this section we extend the algebro-geometric analysis of Section 3.3 to the time-dependent AKNS hierarchy.

For most of this section we assume the following hypothesis.

Hypothesis 3.26 *Suppose that* $p\colon \mathbb{R}^2 \to \mathbb{C}$ *and* $q\colon \mathbb{R}^2 \to \mathbb{C}$ *satisfy*

$$\begin{aligned} &p(\,\cdot\,,t), q(\,\cdot\,,t) \in C^\infty(\mathbb{R}), t \in \mathbb{R}, \quad p(x,\,\cdot\,), q(x,\,\cdot\,) \in C^\infty(\mathbb{R}), x \in \mathbb{R}, \\ &p(x,t) \neq 0, q(x,t) \neq 0, (x,t) \in \mathbb{R}^2. \end{aligned} \tag{3.146}$$

The basic problem in the analysis of algebro-geometric solutions of the AKNS hierarchy consists in solving the time-dependent rth AKNS flow with initial data a stationary solution of the nth equation in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, consider a solution $p^{(0)}, q^{(0)}$ of the nth stationary AKNS equation s-AKNS$_n(p^{(0)}, q^{(0)}) = 0$ associated with $\mathcal{K}_n$ and a given set of integration constants $\{c_\ell\}_{\ell=1,\dots,n+1} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to construct a solution p, q of the rth AKNS flow $\mathrm{AKNS}_r(p,q) = 0$ with $p(t_{0,r}) = p^{(0)}, q(t_{0,r}) = q^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the integration constants in the definitions of the stationary and the time-dependent AKNS equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence, we employ the notation $\widetilde{Q}_{r+1}$, $\widetilde{V}_{r+1}$, $\widetilde{F}_r$, $\widetilde{G}_{r+1}$, $\widetilde{H}_r$, $\tilde{f}_s$, $\tilde{g}_s$, $\tilde{h}_s$, $\tilde{c}_s$ to distinguish them from Q_{n+1}, V_{n+1}, F_n, G_{n+1}, H_n, f_ℓ, g_ℓ, h_ℓ, c_ℓ in the following. In addition, we follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth AKNS flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution p, q of the time-dependent algebro-geometric initial value problem

$$\begin{aligned} &\widetilde{\mathrm{AKNS}}_r(p,q) = \begin{pmatrix} p_{t_r} - 2\tilde{h}_{r+1}(p,q) \\ q_{t_r} - 2\tilde{f}_{r+1}(p,q) \end{pmatrix} = 0, \\ &(p,q)\big|_{t_r = t_{0,r}} = (p^{(0)}, q^{(0)}), \end{aligned} \tag{3.147}$$

$$\text{s-AKNS}_n(p^{(0)}, q^{(0)}) = -2 \begin{pmatrix} h_{n+1}(p^{(0)}, q^{(0)}) \\ f_{n+1}(p^{(0)}, q^{(0)}) \end{pmatrix} = 0 \tag{3.148}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$, where $p = p(x,t_r)$, $q = q(x,t_r)$ satisfy (3.146), and a fixed curve $\mathcal{K}_n$ is associated with the stationary solutions $p^{(0)}, q^{(0)}$ in (3.148). In terms of Lax pairs, this amounts to solving

$$\frac{d}{dt_r} M(t_r) - [\widetilde{Q}_{r+1}(t_r), M(t_r)] = 0, \quad t_r \in \mathbb{R}, \tag{3.149}$$

$$[Q_{n+1}(t_{0,r}), M(t_{0,r})] = 0. \tag{3.150}$$

Anticipating that the AKNS flows are isospectral deformations of $M(t_{0,r})$, we are going a step further, replacing (3.150) by

$$[Q_{n+1}(t_r), M(t_r)] = 0, \quad t_r \in \mathbb{R}. \tag{3.151}$$

This then implies

$$Q_{n+1}(t_r)^2 = -R_{2n+2}(M(t_r)) = -\prod_{m=0}^{2n+1}(M(t_r) - E_m), \quad t_r \in \mathbb{R}.$$

Actually, instead of working directly with (3.149), (3.150), and (3.151), it is more convenient to take the zero-curvature equations (3.40) as our point of departure, that is, we start from

$$U_{t_r} - \widetilde{V}_{r+1,x} + [U, \widetilde{V}_{r+1}] = 0, \tag{3.152}$$

$$-V_{n+1,x} + [U, V_{n+1}] = 0, \tag{3.153}$$

where (cf. (3.17)–(3.19), (3.38), (3.39))

$$\begin{aligned} U(z) &= \begin{pmatrix} -iz & q \\ p & iz \end{pmatrix}, \\ V_{n+1}(z) &= i\begin{pmatrix} -G_{n+1}(z) & F_n(z) \\ -H_n(z) & G_{n+1}(z) \end{pmatrix}, \\ \widetilde{V}_{r+1}(z) &= i\begin{pmatrix} -\widetilde{G}_{r+1}(z) & \widetilde{F}_r(z) \\ -\widetilde{H}_r(z) & \widetilde{G}_{r+1}(z) \end{pmatrix}, \end{aligned} \tag{3.154}$$

and

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell = -iq\prod_{j=1}^{n}(z - \mu_j), \quad f_0 = -iq, \tag{3.155}$$

$$G_{n+1}(z) = \sum_{\ell=0}^{n+1} g_{n+1-\ell} z^\ell, \quad g_0 = 1, \tag{3.156}$$

$$H_n(z) = \sum_{\ell=0}^{n} h_{n-\ell} z^\ell = ip\prod_{j=1}^{n}(z - \nu_j), \quad h_0 = ip, \tag{3.157}$$

$$\widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s} z^s, \quad \tilde{f}_0 = -iq, \tag{3.158}$$

$$\widetilde{G}_{r+1}(z) = \sum_{s=0}^{r+1} \tilde{g}_{r+1-s} z^s, \quad \tilde{g}_0 = 1, \tag{3.159}$$

$$\widetilde{H}_r(z) = \sum_{s=0}^{r} \tilde{h}_{r-s} z^s, \quad \tilde{h}_0 = ip, \tag{3.160}$$

for fixed $n, r \in \mathbb{N}_0$. Here, $f_\ell, \tilde{f}_s, g_\ell, \tilde{g}_s, h_\ell$, and $\tilde{h}_s$, $\ell = 0, \dots, n$, $s = 0, \dots, r$ are defined as in (3.4)–(3.7) with appropriate sets of integration constants. Explicitly, (3.152) and (3.153) are equivalent to (cf. (3.40))

$$p_{t_r} = -i\widetilde{H}_{r,x} - 2z\widetilde{H}_r + 2ip\widetilde{G}_{r+1}, \tag{3.161}$$
$$q_{t_r} = i\widetilde{F}_{r,x} - 2z\widetilde{F}_r - 2iq\widetilde{G}_{r+1}, \tag{3.162}$$
$$\widetilde{G}_{r+1,x} = p\widetilde{F}_r + q\widetilde{H}_r \tag{3.163}$$

and (cf. (3.20)–(3.22))

$$F_{n,x} = -2izF_n + 2qG_{n+1}, \tag{3.164}$$
$$G_{n+1,x} = pF_n + qH_n, \tag{3.165}$$
$$H_{n,x} = 2izH_n + 2pG_{n+1}, \tag{3.166}$$

respectively. In particular, (3.23) holds in the present t_r-dependent setting, that is,

$$G_{n+1}^2 - F_n H_n = R_{2n+2}. \tag{3.167}$$

First we will assume the existence of a solution p, q of (3.161)–(3.166) and derive an explicit formula for p, q in terms of Riemann theta functions. In addition, we will show in Theorem 3.37 that (3.161)–(3.166), and hence the algebro-geometric initial value problem (3.147), (3.148) has a solution at least locally, that is, for $(x, t_r) \in \Omega$ for some open and connected set $\Omega \subseteq \mathbb{R}^2$.

As in (3.53) and (3.54), one introduces

$$\hat{\mu}_j(x, t_r) = (\mu_j(x, t_r), G_{n+1}(\mu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \quad j = 1, \dots, n, (x, t_r) \in \mathbb{R}^2, \tag{3.168}$$

$$\hat{\nu}_j(x, t_r) = (\nu_j(x, t_r), -G_{n+1}(\nu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \quad j = 1, \dots, n, (x, t_r) \in \mathbb{R}^2 \tag{3.169}$$

and notes in accordance with Section 3.3 that the regularity assumptions (3.146) on p, q imply analogous regularity properties of F_n, H_n, μ_j, and ν_k.

In analogy to (3.56), (3.57) one defines the following meromorphic function $\phi(\,\cdot\,, x, t_r)$ on $\mathcal{K}_n$,

$$\phi(P, x, t_r) = \frac{y + G_{n+1}(z, x, t_r)}{F_n(z, x, t_r)} \tag{3.170}$$
$$= \frac{-H_n(z, x, t_r)}{y - G_{n+1}(z, x, t_r)}, \tag{3.171}$$
$$P = (z, y) \in \mathcal{K}_n, \ (x, t_r) \in \mathbb{R}^2$$

and infers that the divisor $(\phi(\,\cdot\,, x, t_r))$ of $\phi(\,\cdot\,, x, t_r)$ is then given by

$$(\phi(\,\cdot\,, x, t_r)) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\nu}}(x,t_r)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(x,t_r)} \tag{3.172}$$

with

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \sigma^n \mathcal{K}_n.$$

Introduce the time-dependent Baker–Akhiezer function $\Psi(\,\cdot\,, x, x_0, t_r, t_{0,r})$ by

$$\Psi(P, x, x_0, t_r, t_{0,r}) = \begin{pmatrix} \psi_1(P, x, x_0, t_r, t_{0,r}) \\ \psi_2(P, x, x_0, t_r, t_{0,r}) \end{pmatrix}, \tag{3.173}$$

$$\begin{aligned} \psi_1(P, x, x_0, t_r, t_{0,r}) = \exp\bigg(i \int_{t_{0,r}}^{t_r} & ds (\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) \\ & - \widetilde{G}_{r+1}(z, x_0, s)) + \int_{x_0}^{x} dx' (-iz + q(x', t_r)\phi(P, x', t_r)) \bigg), \end{aligned} \tag{3.174}$$

$$\begin{aligned} \psi_2(P, x, x_0, t_r, t_{0,r}) &= \phi(P, x, t_r)\psi_1(P, x, x_0, t, t_{0,r}), \\ & P \in \mathcal{K}_n \setminus \{P_{\infty_\pm}\}, (x, t_r) \in \mathbb{R}^2, \end{aligned} \tag{3.175}$$

with fixed $(x_0, t_{0,r}) \in \mathbb{R}^2$. The following lemma records properties of ϕ and Ψ in analogy to the stationary case discussed in Lemma 3.7.

Lemma 3.27 *Assume Hypothesis* 3.26 *and suppose that* (3.152), (3.153) *hold. In addition, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *and* $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$. *Then* ϕ *satisfies*

$$\phi_x(P) + q\phi(P)^2 - 2iz\phi(P) = p, \tag{3.176}$$

$$(q\phi(P))_{t_r} = i\partial_x\big(\widetilde{F}_r(z)\phi(P) - \widetilde{G}_{r+1}(z)\big), \tag{3.177}$$

$$\begin{aligned} \phi_{t_r}(P) &= 2i\widetilde{G}_{r+1}(z)\phi(P) \\ &\quad + q^{-1}\big(-i\widetilde{G}_{r+1,x}(z) + i\widetilde{F}_r(z)\phi_x(P) + 2z\widetilde{F}_r(z)\phi(P)\big) \end{aligned} \tag{3.178}$$

$$= -i\widetilde{F}_r(z)\phi(P)^2 + 2i\widetilde{G}_{r+1}(z)\phi(P) - i\widetilde{H}_r(z), \tag{3.179}$$

$$\phi(P)\phi(P^*) = \frac{H_n(z)}{F_n(z)}, \tag{3.180}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_{n+1}(z)}{F_n(z)}, \tag{3.181}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}. \tag{3.182}$$

Moreover, Ψ *satisfies*

$$\psi_{1,x}(P) = (q\phi(P) - iz)\psi_1(P), \tag{3.183}$$

$$\psi_{1,t_r}(P) = i(\widetilde{F}_r(z)\phi(P) - \widetilde{G}_{r+1}(z))\psi_1(P), \tag{3.184}$$

$$\psi_{2,x}(P) = (p\phi(P)^{-1} + iz)\psi_1(P), \tag{3.185}$$

$$\psi_{2,t_r}(P) = -i(\widetilde{H}_r(z)\phi(P)^{-1} - \widetilde{G}_{r+1}(z))\psi_2(P), \tag{3.186}$$

or, equivalently,

$$\Psi_x(P) = U(z)\Psi(P), \tag{3.187}$$
$$iy\Psi(P) = V_{n+1}(z)\Psi(P), \tag{3.188}$$
$$\Psi_{t_r}(P) = \widetilde{V}_{r+1}(z)\Psi(P) \tag{3.189}$$

and hence

$$(M - z(P))\Psi(P) = 0, \quad (Q_{n+1} - iy(P))\Psi(P) = 0. \tag{3.190}$$

In addition, as long as the zeros of $F_n(\,\cdot\,, x, t_r)$ *are all simple for* $(x, t_r) \in \Omega$, $\Omega \subseteq \mathbb{R}^2$ *open and connected,* $\Psi(\,\cdot\,, x, x_0, t_0, t_{0,r})$ *is meromorphic on* $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *for* $(x, t_r), (x_0, t_{0,r}) \in \Omega$.

Proof Equation (3.176) follows from (3.153) and (3.170). Relation (3.177) can be proven as follows. Using (3.147) and (3.176), one infers by a straightforward (but rather lengthy) calculation that

$$\big(\partial_x + 2q\phi - 2iz - (q_x/q)\big)\big((q\phi)_{t_r} - i(\widetilde{F}_r\phi - \widetilde{G}_{r+1})_x\big) = 0.$$

Thus,

$$(q\phi)_{t_r} - i(\widetilde{F}_r\phi - \widetilde{G}_{r+1})_x = C\exp\bigg(\int^x dx'\big(2iz + (q_x/q) - 2q\phi\big)\bigg), \tag{3.191}$$

where C is independent of x (but may depend on P and t_r). By inspection of (3.170), the left-hand side of (3.191) is meromorphic on $\mathcal{K}_n$, whereas the right-hand side of (3.191) is not meromorphic near P_{∞_+} and P_{∞_-} unless $C = 0$. Hence, one infers $C = 0$ and thus (3.177). Equation (3.178) is then an immediate consequence of (3.147) (more precisely, the AKNS equation for q_{t_r}) and (3.177). Similarly, (3.179) is an immediate consequence of (3.163), (3.176), and (3.178). Relation (3.183) is clear from (3.174), and (3.185) is obvious from (3.175), (3.176), and (3.183). Equation (3.184) follows from (3.174), and (3.186) is a straightforward consequence of (3.175), and (3.184). Moreover, (3.180)–(3.182) are proved as in Lemma 3.7. By (3.174), $\psi_1(\,\cdot\,, x, x_0, t_r, t_{0,r})$ is meromorphic away from the poles $\hat{\mu}_j(x_0, s)$ of $\phi(\,\cdot\,, x_0, s)$ and $\hat{\mu}_k(x', t_r)$ of $\phi(\,\cdot\,, x', t_r)$. That ψ_1 is actually meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ if $F_n(\,\cdot\,, x, t_r)$ has only simple zeros is a consequence of

$$q(x')\phi(P, x', t_r) \underset{P\to\hat{\mu}_j(x',t_r)}{=} \partial_{x'} \ln\big(F_n(z, x', t_r)\big) + O(1) \text{ as } z \to \mu_j(x', t_r)$$

(cf. (3.74)) and

$$i\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) \underset{P\to\hat{\mu}_j(x_0,s)}{=} \partial_s \ln\big(F_n(z, x_0, s)\big) + O(1) \text{ as } z \to \mu_j(x_0, s)$$

via (3.170), (3.168), and (3.192). (Equation (3.192) in Lemma 3.28 only requires (3.178), (3.181), and (3.182), which have already been proven.) This follows from (3.174) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j(x_0)$ of $\{\hat{\mu}_j(x_0, s) \in \mathcal{K}_n \mid (x_0, s) \in \Omega, s \in [t_{0,r}, t_r]\}$ such that $\hat{\mu}_k(x_0, s) \notin \mathcal{U}_j(x_0)$ for all $s \in [t_{0,r}, t_r]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$ and by simultaneously restricting P to a sufficiently small neighborhood $\mathcal{U}_j(t_r)$ of $\{\hat{\mu}_j(x', t_r) \in \mathcal{K}_n \mid (x', t_r) \in \Omega, x' \in [x_0, x]\}$ such that $\hat{\mu}_k(x', t_r) \notin \mathcal{U}_j(t_r)$ for all $x' \in [x_0, x]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$. The function ψ_2 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ by (3.175) and since ϕ is meromorphic on $\mathcal{K}_n$ by (3.170). $\square$

Next we consider the t_r-dependence of F_n, G_{n+1}, and H_n.

Lemma 3.28 *Assume Hypothesis* 3.26 *and suppose that* (3.152), (3.153) *hold. Then,*

$$F_{n,t_r} = 2i(G_{n+1}\widetilde{F}_r - F_n\widetilde{G}_{r+1}), \tag{3.192}$$

$$G_{n+1,t_r} = i(H_n\widetilde{F}_r - F_n\widetilde{H}_r), \tag{3.193}$$

$$H_{n,t_r} = 2i(H_n\widetilde{G}_{r+1} - G_{n+1}\widetilde{H}_r). \tag{3.194}$$

In particular, (3.192)–(3.194) *are equivalent to*

$$-V_{n+1,t_r} + [\widetilde{V}_{r+1}, V_{n+1}] = 0.$$

Proof By (3.170), (3.178), (3.181), and (3.182) one infers

$$\phi_{t_r}(P) - \phi_{t_r}(P^*) = -\frac{2yF_{n,t_r}}{F_n^2} = \frac{4iy}{F_n^2}(\widetilde{G}_{r+1}F_n - \widetilde{F}_r G_{n+1}),$$

which proves (3.192). Similarly, differentiating (3.181) with respect to t_r, using (3.176), (3.178), (3.180)–(3.182), and (3.163), proves (3.193). Relation (3.194) finally follows from $(G_{n+1}^2 - F_nH_n)_{t_r} = 0$ (cf. (3.167)), (3.192), and (3.193). $\square$

The remaining items (3.69)–(3.73) of Lemma 3.7 in the present time-dependent setting then read as follows.

Lemma 3.29 *Assume Hypothesis* 3.26 *and suppose that* (3.152), (3.153) *hold. In addition, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *and* $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$. *Then*

$$\begin{aligned}\psi_1(P, x, x_0, t_r, t_{0,r}) = &\left(\frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})}\right)^{1/2} \\ &\times \exp\left(iy\int_{t_{0,r}}^{t_r} ds\,\widetilde{F}_r(z, x_0, s)F_n(z, x_0, s)^{-1}\right. \\ &\left.+\, y\int_{x_0}^{x} dx'q(x', t_r)F_n(z, x', t_r)^{-1}\right),\end{aligned} \tag{3.195}$$

$$\psi_1(P, x, x_0, t_r, t_{0,r})\psi_1(P^*, x, x_0, t_r, t_{0,r}) = \frac{F_n(z, x, t_r)}{F_n(z, x_0, t_{0,r})}, \tag{3.196}$$

$$\psi_2(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) = \frac{H_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}, \tag{3.197}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r})\\ &\quad + \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = 2\frac{G_{n+1}(z,x,t_r)}{F_n(z,x_0,t_{0,r})},\end{aligned} \tag{3.198}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r})\\ &\quad - \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = -\frac{2y}{F_n(z,x_0,t_{0,r})}.\end{aligned} \tag{3.199}$$

Proof Relation (3.195) follows from (3.174), (3.170), (3.164), and (3.192). Equation (3.196) follows from (3.164), (3.174), (3.181), and (3.192). Equation (3.197) is a consequence of (3.175), (3.180), and (3.196). Finally, (3.198) and (3.199) are clear from (3.175), (3.181), (3.182), and (3.196). □

The stationary Dubrovin-type equations in Lemma 3.8 have analogs for each AKNS$_r$ flow (indexed by the parameter t_r) that govern the dynamics of μ_j and ν_j with respect to variations of x and t_r. In this context the stationary case simply corresponds to the special case $r = 0$, as described in the following result.

Lemma 3.30 *Assume Hypothesis* 3.26 *and* (3.152), (3.153) *hold on an open and connected set* $\widetilde{\Omega}_\mu \subseteq \mathbb{R}^2$. *Moreover, suppose that the zeros* μ_j, $j = 1,\dots,n$, *of* $F_n(\,\cdot\,)$ *remain distinct on* $\widetilde{\Omega}_\mu$. *Then* $\{\hat{\mu}_j\}_{j=1,\dots,n}$, *defined by* (3.168), *satisfies the following first-order system of differential equations on* $\widetilde{\Omega}_\mu$

$$\mu_{j,x} = -2iy(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j-\mu_k)^{-1}, \tag{3.200}$$

$$\mu_{j,t_r} = 2\widetilde{F}_r(\mu_j)q^{-1}y(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j-\mu_k)^{-1}, \quad j=1,\dots,n. \tag{3.201}$$

Next, assume the affine part of $\mathcal{K}_n$ *to be nonsingular and introduce the initial condition*

$$\{\hat{\mu}_j(x_0,t_{0,r})\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{3.202}$$

for some $(x_0,t_r) \in \mathbb{R}^2$, *where* $\mu_j(x_0,t_{0,r})$, $j=1,\dots,n$, *are assumed to be distinct. Then there exists an open and connected set* $\Omega_\mu \subseteq \mathbb{R}^2$, *with* $(x_0,t_{0,r}) \in \Omega_\mu$, *such that the initial value problem* (3.200)–(3.202) *has a unique solution* $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset$

$\mathcal{K}_n$ satisfying

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{3.203}$$

and μ_j, $j = 1, \dots, n$, remain distinct on Ω_μ.
For the zeros ν_j, $j = 1, \dots, n$, of $H_n(\,\cdot\,)$, identical statements hold with μ_j and Ω_μ replaced by ν_j and Ω_ν, etc. In particular, $\{\hat{\nu}_j\}_{j=1,\dots,n}$, defined by (3.169)*, satisfies the system*

$$\nu_{j,x} = -2iy(\hat{\nu}_j)\prod_{\substack{k=1\\k\neq j}}^n(\nu_j - \nu_k)^{-1}, \tag{3.204}$$

$$\nu_{j,t_r} = -2\widetilde{H}_r(\nu_j)p^{-1}y(\hat{\nu}_j)\prod_{\substack{k=1\\k\neq j}}^n(\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, n. \tag{3.205}$$

Proof For obvious reasons it suffices to focus on (3.200), (3.201), and (3.203). But the proof of (3.200) is identical to that in Lemma 3.8, and equation (3.201), follows from (3.155), (3.168), and (3.192) since

$$F_{n,t_r}(\mu_j) = iq\mu_{j,t_r}\prod_{\substack{k=1\\k\neq j}}^n(\mu_j - \mu_k) = 2i\widetilde{F}_r(\mu_j)G_{n+1}(\mu_j) = 2i\widetilde{F}_r(\mu_j)y(\hat{\mu}_j).$$

The smoothness assertion (3.203) is clear as long as $\hat{\mu}_j$ stays away from the branch points $(E_m, 0)$. In case $\hat{\mu}_j$ hits such a branch point, one can use the local chart around $(E_m, 0)$ (with local coordinate $\zeta = \sigma(z - E_m)^{1/2}, \sigma \pm 1$) to verify (3.203), as in the proof of Lemma 1.37. □

Since the stationary trace formulas for AKNS invariants in terms of symmetric functions of μ_j and ν_j in Lemma 3.9 extend line by line to the corresponding time-dependent setting, we next record their t_r-dependent analogs without proof. For simplicity we confine ourselves to the simplest ones only.

Lemma 3.31 *Assume that $p, q \in C^\infty(\mathbb{R}^2)$ are nonzero and suppose that* (3.152)*,* (3.153) *hold. Then,*

$$i\frac{p_x}{p} = 2c_1 + 2\sum_{j=1}^n \nu_j,$$

$$i\frac{q_x}{q} = -2c_1 - 2\sum_{j=1}^n \mu_j,$$

where

$$c_1 = -\frac{1}{2}\sum_{m=0}^{2n+1} E_m.$$

Now we turn to asymptotic properties of ϕ (which are proven as in Lemma 3.10).

Lemma 3.32 *Suppose that $p, q \in C^\infty(\mathbb{R}^2)$ are nonzero and assume that* (3.152), (3.153) *hold. Moreover, let $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$. Then,*

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} (i/2)p\zeta + (p_x/4)\zeta^2 + O(\zeta^3) & \text{as } P \to P_{\infty_+}, \\ (2i/q)\zeta^{-1} + (q_x/q^2) + O(\zeta) & \text{as } P \to P_{\infty_-}. \end{cases} \tag{3.206}$$

Next, we provide the explicit theta function representations of Ψ, ϕ, p, and q. We rely on the notation established in Section 3.3 and Appendix C in the following, assuming the affine part of $\mathcal{K}_n$ to be nonsingular as in (3.76).

In addition to (3.91)–(3.97), let $\omega^{(2)}_{P_{\infty_\pm},r}$ be normalized differentials of the second kind (cf. (A.20), (A.21), and (A.22)) with a unique pole at $P_{\infty_\pm}$ and principal part $\zeta^{-2-r}d\zeta$ near $P_{\infty_\pm}$ and define

$$\widetilde{\Omega}^{(2)}_r = \sum_{q=0}^{r}(q+1)\tilde{c}_{r-q}\big(\omega^{(2)}_{P_{\infty_+},q} - \omega^{(2)}_{P_{\infty_-},q}\big), \quad \tilde{c}_0 = 1, \tag{3.207}$$

where $\tilde{c}_q$ are the constants introduced in the definition of $\widetilde{F}_r$. Thus, one infers

$$\int_{a_j} \widetilde{\Omega}^{(2)}_r = 0, \quad j = 1, \dots, n, \tag{3.208}$$

$$\int_{P_0}^{P} \widetilde{\Omega}^{(2)}_r \underset{\zeta\to 0}{=} \mp\left(\sum_{q=0}^{r}\tilde{c}_{r-q}\zeta^{-1-q} + \tilde{e}_{r,0} + O(\zeta)\right) \text{ as } P \to P_{\infty_\pm} \tag{3.209}$$

for some constants $\tilde{e}_{r,0} \in \mathbb{C}$. The corresponding vector of b-periods of $\widetilde{\Omega}^{(2)}_r/(2\pi i)$ is then denoted by

$$\underline{\widetilde{U}}^{(2)}_r = \big(\widetilde{U}^{(2)}_{r,1}, \dots, \widetilde{U}^{(2)}_{r,n}\big), \quad \widetilde{U}^{(2)}_{r,j} = \frac{1}{2\pi i}\int_{b_j} \widetilde{\Omega}^{(2)}_r, \quad j = 1, \dots, n. \tag{3.210}$$

Moreover, if one writes

$$\omega_j = \left(\sum_{m=0}^{\infty} d_{j,m}(P_{\infty_\pm})\zeta^m\right)d\zeta = \pm\left(\sum_{m=0}^{\infty} d_{j,m}(P_{\infty_+})\zeta^m\right)d\zeta \text{ near } P_{\infty_\pm},$$

relation (A.22) yields

$$\widetilde{U}_{r,j}^{(2)} = 2\sum_{q=0}^{r} \tilde{c}_{r-q} d_{j,q}(P_{\infty_+}), \quad j = 1, \dots, n. \tag{3.211}$$

Recalling the abbreviation (3.101), one of the principal results of this section, the theta function representations for ϕ, Ψ, p, and q, then reads as follows.

Theorem 3.33 *Assume Hypothesis* 3.26 *and* (3.152), (3.153) *hold on* Ω, *and suppose the affine part of* $\mathcal{K}_n$ *to be nonsingular. In addition, let* $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *and* $(x, t_r), (x_0, t_{0,r}) \in \Omega$, *where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$, *or equivalently,* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *is nonspecial for* $(x, t_r) \in \Omega$. *Then,*[1]

$$\begin{aligned}\phi(P, x, t_r) &= C_0 \frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\nu}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x, t_r)))} \\ &\quad \times \exp\bigg(\int_{P_0}^{P} \omega_{P_{\infty_+},P_{\infty_-}}^{(3)} - 2ie_{0,0}x - 2i\tilde{e}_{r,0}t_r\bigg), \end{aligned} \tag{3.212}$$

$$\begin{aligned}\psi_1(P, x, x_0, t_r, t_{0,r}) &= \frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x_0, t_{0,r})))\theta(\underline{z}(P, \underline{\hat{\mu}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0, t_{0,r})))} \\ \times \exp\bigg(i\bigg(\int_{P_0}^{P} \Omega_0^{(2)} &+ e_{0,0}\bigg)(x - x_0) + i\bigg(\int_{P_0}^{P} \widetilde{\Omega}_r^{(2)} + \tilde{e}_{r,0}\bigg)(t_r - t_{0,r})\bigg),\end{aligned} \tag{3.213}$$

$$\begin{aligned}\psi_2(P, x, x_0, t_r, t_{0,r}) &= C_0 \exp(-2ie_{0,0}x_0 - 2i\tilde{e}_{r,0}t_{0,r}) \\ &\quad \times \frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x_0, t_{0,r})))\theta(\underline{z}(P, \underline{\hat{\nu}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x_0, t_{0,r})))} \\ &\quad \times \exp\bigg(\int_{P_0}^{P} \omega_{P_{\infty_+},P_{\infty_-}}^{(3)} + i\bigg(\int_{P_0}^{P} \Omega_0^{(2)} - e_{0,0}\bigg)(x - x_0) \\ &\qquad + i\bigg(\int_{P_0}^{P} \widetilde{\Omega}_r^{(2)} - \tilde{e}_{r,0}\bigg)(t_r - t_{0,r})\bigg),\end{aligned} \tag{3.214}$$

where

$$C_0 = \frac{2i}{q(x_0, t_{0,r})\omega_0} \frac{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x_0, t_{0,r})))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x_0, t_{0,r})))} \exp(2ie_{0,0}x_0 + 2i\tilde{e}_{r,0}t_{0,r}).$$

The Abel map linearizes the auxiliary divisors in the sense that

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}) - i\underline{U}_0^{(2)}(x - x_0) - i\underline{\widetilde{U}}_r^{(2)}(t_r - t_{0,r}), \tag{3.215}$$

$$\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) = \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})}) - i\underline{U}_0^{(2)}(x - x_0) - i\underline{\widetilde{U}}_r^{(2)}(t_r - t_{0,r}). \tag{3.216}$$

[1] To avoid multi-valued expressions in formulas such as (3.212)–(3.214), etc., we agree always to choose the same path of integration connecting P_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

Finally, p, q are of the form

$$p(x,t_r) = p(x_0,t_{0,r})\frac{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0,t_{0,r})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x_0,t_{0,r})))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t_r)))}$$
$$\times \exp(-2ie_{0,0}(x-x_0) - 2i\tilde{e}_{r,0}(t_r - t_{0,r})), \tag{3.217}$$
$$q(x,t_r) = q(x_0,t_{0,r})\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_{0,r})))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t_r)))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_{0,r})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t_r)))}$$
$$\times \exp(2ie_{0,0}(x-x_0) + 2i\tilde{e}_{r,0}(t_r - t_{0,r})), \tag{3.218}$$
$$p(x_0,t_{0,r})q(x_0,t_{0,r}) = \frac{4}{\omega_0^2}\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x_0,t_{0,r})))\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_{0,r})))}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0,t_{0,r})))\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_{0,r})))}. \tag{3.219}$$

Proof We start with the proof of the theta function representation (3.213) for ψ_1. Without loss of generality it suffices to treat the homogeneous case $\hat{c}_0 = 1, \hat{c}_q = 0$, $q = 1, \dots, r$. As in the corresponding stationary case we temporarily assume

$$\mu_j(x,t_r) \neq \mu_{j'}(x,t_r) \text{ for } j \neq j' \text{ and } (x,t_r) \in \widetilde{\Omega} \tag{3.220}$$

for appropriate $\widetilde{\Omega} \subseteq \Omega$, and define the right-hand side of (3.213) to be $\tilde{\psi}_1$. We intend to prove $\psi_1 = \tilde{\psi}_1$, with ψ_1 given by (3.174). For that purpose we first investigate the local zeros and poles of ψ_1 and note (cf. (3.111))

$$\begin{aligned}
q(x',t_r)\phi(P,x',t_r) &\underset{P\to\hat{\mu}_j(x',t_r)}{=} \partial_{x'}\ln(z-\mu_j(x',t_r)) + O(1),\\
i\widehat{F}_r(z,x_0,s)\phi(P,x_0,s) &\underset{P\to\hat{\mu}_j(x_0,s)}{=} \frac{2i\widehat{F}_r(z,x_0,s)y(\hat{\mu}_j(x_0,s))}{-iq(x_0,s)\prod_{\substack{k=1\\k\neq j}}^n(\mu_j(x_0,s)-\mu_k(x_0,s))}\\
&\qquad\qquad \times \frac{1}{z-\mu_j(x_0,s)} + O(1)\\
&\underset{P\to\hat{\mu}_j(x_0,s)}{=} \partial_s\ln(z-\mu_j(x_0,s)) + O(1),
\end{aligned}$$

using (3.168), (3.170), (3.200), and (3.201). Together with (3.174), this implies

$$\psi_1(P,x,x_0,t_r,t_{0,r}) = \begin{cases} (z-\mu_j(x,t_r))O(1), & P\to\hat{\mu}_j(x,t_r)\neq\hat{\mu}_j(x_0,t_{0,r}),\\ O(1), & P\to\hat{\mu}_j(x,t_r)=\hat{\mu}_j(x_0,t_{0,r}),\\ (z-\mu_j(x_0,t_r))^{-1}O(1), & P\to\hat{\mu}_j(x_0,t_{0,r})\neq\hat{\mu}_j(x,t_r),\end{cases}$$
$$P=(z,y)\in\mathcal{K}_n,\ (x,t_r),(x_0,t_{0,r})\in\widetilde{\Omega},$$

with $O(1)\neq 0$, and hence ψ_1 and $\tilde{\psi}_1$ have identical zeros and poles on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$, which are all simple by hypothesis (3.220). It remains to study the behavior of ψ_1 near $P_{\infty_\pm}$. By (3.170), (3.192), (3.206), and (3.90) one computes

$$
\begin{aligned}
&\int_{x_0}^{x} dx'(-i\zeta^{-1} + q(x',t_r)\phi(P,x',t_r)) \\
&\qquad + i\int_{t_{0,r}}^{t_r} ds(\widetilde{F}_r(\zeta^{-1},x_0,s)\phi(P,x_0,s) - \widetilde{G}_{r+1}(\zeta^{-1},x_0,s)) \\
&\underset{\zeta\to 0}{=} \mp i\zeta^{-1}(x-x_0) + \begin{cases} O(\zeta) & \text{for } P \to P_{\infty_+} \\ O(1) & \text{for } P \to P_{\infty_-} \end{cases} \\
&\qquad + \int_{t_{0,r}}^{t_r} ds \left(\frac{iy\widetilde{F}_r(\zeta^{-1},x_0,s)}{F_n(\zeta^{-1},x_0,s)} + \frac{1}{2}\frac{F_{n,t_r}(\zeta^{-1},x_0,s)}{F_n(\zeta^{-1},x_0,s)} \right) \\
&\underset{\zeta\to 0}{=} \mp i\zeta^{-1}(x-x_0) + \begin{cases} O(\zeta) & \text{for } P \to P_{\infty_+} \\ O(1) & \text{for } P \to P_{\infty_-} \end{cases} \\
&\qquad + \int_{t_{0,r}}^{t_r} ds \left(\mp i\zeta^{-r-1}\frac{\sum_{m=0}^{r}\tilde{f}_m(x_0,s)\zeta^m}{\sum_{\ell=0}^{\infty}\tilde{f}_\ell(x_0,s)\zeta^\ell} + \frac{1}{2}\frac{q_{t_r}(x_0,s)}{q(x_0,s)} + O(\zeta) \right) \\
&\underset{\zeta\to 0}{=} \mp i\zeta^{-1}(x-x_0) + \begin{cases} O(\zeta) & \text{for } P \to P_{\infty_+} \\ O(1) & \text{for } P \to P_{\infty_-} \end{cases} \\
&\qquad + \int_{t_{0,r}}^{t_r} ds \left(\mp i\zeta^{-r-1} \pm \frac{i\tilde{f}_{r+1}(x_0,s)}{\tilde{f}_0(x_0,s)} + \frac{1}{2}\frac{q_{t_r}(x_0,s)}{q(x_0,s)} + O(\zeta) \right) \\
&\underset{\zeta\to 0}{=} \mp i\zeta^{-1}(x-x_0) \mp i\zeta^{-r-1}(t_r - t_{0,r}) + \begin{cases} O(\zeta) & \text{for } P \to P_{\infty_+}, \\ O(1) & \text{for } P \to P_{\infty_-}, \end{cases}
\end{aligned}
\tag{3.221}
$$

where we used $\tilde{f}_0 = -iq$ and

$$q_{t_r} = 2\tilde{f}_{r+1}$$

(cf. (3.147)) in the homogeneous case $\tilde{c}_0 = 1, \tilde{c}_q = 0, q = 1, \dots, r$. A comparison of ψ_1 and $\tilde{\psi}_1$ near $P_{\infty_\pm}$, taking into account (3.174), (3.209) (recalling $\tilde{c}_0 = 1$, $\tilde{c}_q = 0$ for $q = 1, \dots, r$), the expression (3.213) for $\tilde{\psi}$, and (3.221) then show that ψ_1 and $\tilde{\psi}_1$ have identical exponential behavior up to order $O(\zeta)$ near P_{∞_+} and identical exponential behavior up to order $O(1)$ near P_{∞_-}. Thus, ψ_1 and $\tilde{\psi}_1$ share the same singularities and zeros, and the Riemann–Roch-type uniqueness result in Lemma C.2 then proves that ψ_1 and $\tilde{\psi}_1$ coincide up to normalization. The latter is determined as in the stationary context (3.112) by inserting (3.209) into (3.213). This results in

$$\tilde{\psi}_1(P,x,x_0,t_r,t_{0,r}) \underset{\zeta\to 0}{=} e^{-i\zeta^{-1}(x-x_0)-i\zeta^{-r-1}(t_r-t_{0,r})}(1+O(\zeta)) \text{ as } P \to P_{\infty_+}. \tag{3.222}$$

Similarly, inserting (3.221) into (3.174) yields the identical asymptotic behavior

(3.222) of ψ_1 near P_{∞_+}. Hence (3.213) holds subject to (3.220). The expression (3.172) for the divisor of ϕ then yields

$$\phi(P) = C\frac{\theta(\underline{z}(P, \underline{\hat{\nu}}))}{\theta(\underline{z}(P, \underline{\hat{\mu}}))} \exp\left(\int_{P_0}^{P} \omega^{(3)}_{P_{\infty_+}, P_{\infty_-}}\right), \tag{3.223}$$

where $C = C(x, t_r)$ is independent of $P \in \mathcal{K}_n$. Thus, (3.206) implies

$$p = \frac{2C}{i\omega_0}\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}))}, \quad q = \frac{2i}{C\omega_0}\frac{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}))}. \tag{3.224}$$

Re-examining the asymptotic behavior (3.221) of ψ_1 near P_{∞_-}, taking into account (3.88), yields

$$\begin{aligned}\psi_1(P, x, x_0, t_r, t_{0,r}) &\underset{\zeta\to 0}{=} \frac{q(x, t_r)}{q(x_0, t_r)} \exp(i\zeta^{-1}(x - x_0) + O(\zeta)) \\ &\quad \times \frac{q(x_0, t_r)}{q(x_0, t_{0,r})} \exp(i\zeta^{-1-r}(t_r - t_{0,r}) + O(\zeta)) \\ &\underset{\zeta\to 0}{=} \frac{q(x, t_r)}{q(x_0, t_{0,r})} \exp(i\zeta^{-1}(x - x_0) \\ &\quad + i\zeta^{-1-r}(t_r - t_{0,r}) + O(\zeta)) \text{ as } P \to P_{\infty_-}.\end{aligned} \tag{3.225}$$

A comparison of (3.213), (3.224), and (3.225) then proves (3.218). A further comparison of (3.218) and (3.224) then determines $C(x, t_r)$ and hence yields (3.217) and (3.219). Given $C(x, t_r)$, one determines ϕ in (3.212) from (3.223) and hence ψ_2 in (3.214) from $\psi_2 = \phi\psi_1$ (all subject to (3.220)). Finally, the linearization property of the Abel map in (3.215) and (3.216) follows from Corollary F.11 and Remark F.12. The extension of all these results from $\widetilde{\Omega}$ to Ω then follows by continuity of $\underline{\alpha}_{Q_0}$ and nonspecialty of $\mathcal{D}_{\underline{\hat{\mu}}}$ on Ω. $\square$

Of course, Remark 3.12 applies in the present time-dependent context as well.

Combining (3.215), (3.216), (3.114), and (3.217), (3.218) shows the remarkable linearity of the theta function arguments with respect to x and t_r in the formulas for p, q. In fact, one can rewrite (3.217), (3.218) as

$$p(x, t_r) = C_p\frac{\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r + \underline{\Delta})}{\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r)} \exp(-ie_0 x - ie_1 t_r), \tag{3.226}$$

$$q(x, t_r) = C_q\frac{\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r - \underline{\Delta})}{\theta(\underline{A} + \underline{B}x + \underline{C}_r t_r)} \exp(ie_0 x + ie_1 t_r), \tag{3.227}$$

where

$$\underline{A} = \underline{\Xi}_{P_0} - \underline{A}_{P_0}(P_{\infty_+}) + i\underline{U}_0^{(2)} x_0 + i\underline{\widetilde{U}}_r^{(2)} t_{0,r} + \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0, t_{0,r})}), \tag{3.228}$$

$$\underline{B} = -i\underline{U}_0^{(2)}, \quad \underline{C}_r = -i\underline{\widetilde{U}}_r^{(2)}, \tag{3.229}$$

$$\underline{\Delta} = \underline{A}_{P_{\infty_+}}(P_{\infty_-}), \tag{3.230}$$

and hence the constants $e_0, e_1 \in \mathbb{C}$ and $\underline{\Delta}, \underline{B}, \underline{C}_r \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$ and r, and the constant $\underline{A} \in \mathbb{C}^n$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0, t_{0,r}) = (\hat{\mu}_1(x_0, t_{0,r}), \dots, \hat{\mu}_n(x_0, t_{0,r})) \in \mathrm{Sym}^n(\mathcal{K}_n)$ at the point $(x_0, t_{0,r})$, as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}$ is assumed to be nonspecial. The constants $C_p, C_q \in \mathbb{C}$ satisfy constraints analogous to (3.219).

Remark 3.34 The explicit expressions (3.213), (3.214) for ψ_j, $j = 1, 2$ again complement Lemma 3.7 and show that Ψ stays meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial (assuming the affine part of $\mathcal{K}_n$ to be nonsingular).

Since Corollary 3.14 extends to the present time-dependent setting in a straightforward manner, we record the corresponding result without proof.

Corollary 3.35 *Assume the hypotheses of Theorem* 3.33. *Then*

$$p(x, t_r)q(x, t_r) = -e_{0,1} - \partial_x^2 \ln(\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x, t_r)))).$$

The constant $q(x_0, t_{0,r})$ in (3.217)–(3.219) is inherent to the AKNS formalism due to its scale invariance, as discussed in Lemma 3.6.

The functions p, q in (3.217), (3.218) represent solutions, of the AKNS system, a complexified nonlinear Schrödinger equation (nS). To obtain algebro-geometric nS$_\pm$ solutions, one can proceed as discussed in Lemmas 3.15, 3.17, and 3.18 and Remarks 3.16 and 3.19, which all extend to the present time-dependent situation.

In analogy to Example 3.20, the special case $n = 0$ (excluded in Theorem 3.33) yields solutions (p, q) as in (3.217), (3.218), replacing the theta quotients by 1. For simplicity we just consider the elementary cases $n = 0$, $r = 0, 1$.

Example 3.36 Assume $n = 0$, $P = (z, y) \in \mathcal{K}_0 \setminus \{P_{\infty_+}, P_{\infty_-}\}$, and let (x, t_r), $(x_0, t_{0,r}) \in \mathbb{R}^2$, $r = 0, 1$. Then,

$$\begin{aligned}
&\mathcal{K}_0\colon \mathcal{F}_0(z, y) = y^2 - R_2(z) = y^2 - (z - E_0)(z - E_1) = 0,\\
&c_1 = -(E_0 + E_1)/2, \quad E_0, E_1 \in \mathbb{C},\\
&p(x, t_0) = p(x_0, t_{0,0}) \exp(-2ic_1(x - x_0) + 2i(\tilde{c}_1 - c_1)(t_0 - t_{0,0})), r = 0,\\
&q(x, t_0) = q(x_0, t_{0,0}) \exp(2ic_1(x - x_0) - 2i(\tilde{c}_1 - c_1)(t_0 - t_{0,0})), r = 0,\\
&p(x, t_1) = p(x_0, t_{0,1}) \exp(-2ic_1(x - x_0) + i(2c_1^2 + ((E_0 - E_1)^2/4)\\
&\qquad\qquad - 2c_1\tilde{c}_1 + 2\tilde{c}_2)(t_1 - t_{0,1})), \quad r = 1,\\
&q(x, t_1) = q(x_0, t_{0,1}) \exp(2ic_1(x - x_0) - i(2c_1^2 + ((E_0 - E_1)^2/4)\\
&\qquad\qquad - 2c_1\tilde{c}_1 + 2\tilde{c}_2)(t_1 - t_{0,1})), \quad r = 1,\\
&p(x, t_r)q(x, t_r) = (E_0 - E_1)^2/4, \quad r = 0, 1.
\end{aligned}$$

Up to this point we assumed Hypothesis 3.26 together with the basic equations

(3.152) and (3.153). Next, we will show that solvability of the Dubrovin equations (3.200), (3.201) on $\Omega_\mu \subseteq \mathbb{R}^2$ in fact implies (3.152) and (3.153) on Ω_μ. In complete analogy to our discussion in Section 3.3 (cf. Remark 3.24), this amounts to solving the time-dependent algebro-geometric initial value problem (3.147), (3.148) on Ω_μ. In this context we recall the definition of $\widetilde{F}_r(\mu_j)$ in terms of $\mu_1, \dots, \mu_n$, introduced[1] in (F.16), (F.19),

$$\widetilde{F}_r(\mu_j) = \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(\underline{E})\Phi_k^{(j)}(\underline{\mu}), \quad r \in \mathbb{N}_0, \tilde{c}_0 = 1, \tag{3.231}$$

$$\tilde{d}_{r,k}(\underline{E}) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s}\hat{c}_s(\underline{E}), \quad k = 0, \dots, r \wedge n, \tag{3.232}$$

in terms of a given set of integration constants $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$.

Theorem 3.37 *Fix $n \in \mathbb{N}$ and assume the affine part of $\mathcal{K}_n$ to be nonsingular. Suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the Dubrovin equations* (3.200), (3.201) *on an open and connected set $\Omega_\mu \subseteq \mathbb{R}^2$, with $\widetilde{F}_r(\mu_j)$ in* (3.201) *expressed in terms of μ_k, $k = 1, \dots, n$, by* (3.231) *and* (3.232) *and $q \in C^\infty(\Omega_\mu)$ defined by*

$$\begin{aligned} &q(x, t_r) = q(x_0, t_r) \exp\bigg(-i(x - x_0) \sum_{m=0}^{2n+1} E_m + i \sum_{j=1}^{n} \int_{x_0}^{x} dx' \mu_j(x', t_r)\bigg), \\ &q(x_0, t_{0,r}) \neq 0. \end{aligned} \tag{3.233}$$

Moreover, assume that μ_j, $j = 1, \dots, n$, remain distinct on Ω_μ. Next, define $p \in C^\infty(\Omega_\mu)$ by[2]

$$\begin{aligned} p(x, t_r) = &\frac{1}{2q(x, t_r)\prod_{j=1}^n \mu_j(x, t_r)}\bigg(\frac{1}{q(x, t_r)}\bigg(q(x, t_r)\prod_{j=1}^n \mu_j(x, t_r)\bigg)_x\bigg)_x \\ &- \frac{1}{4q(x, t_r)^3\prod_{j=1}^n \mu_j(x, t_r)^2}\bigg(\bigg(q(x, t_r)\prod_{j=1}^n \mu_j(x, t_r)\bigg)_x\bigg)^2 \\ &- \frac{\prod_{m=0}^{2n+1} E_m}{q(x, t_r)\prod_{j=1}^n \mu_j(x, t_r)^2}, \quad \text{if } \mu_j \neq 0 \text{ on } \Omega_\mu, \quad j = 1, \dots, n \end{aligned} \tag{3.234}$$

with q defined by (3.233)*. Then $p \neq 0$ on Ω_μ and p, q satisfy the the rth AKNS equation* (3.147)*, that is,*

$$\widetilde{\mathrm{AKNS}}_r(p, q) = 0 \text{ on } \Omega_\mu \tag{3.235}$$

with initial values satisfying the nth stationary AKNS equation (3.148).

[1] $m \wedge n = \min(m, n)$.

[2] If $\mu_{j_0}(x_0, t_{0,r}) = 0$ for some $j_0 \in \{1, \dots, n\}$ and some $(x_0, t_{0,r}) \in \Omega_\mu$, one can use (3.145) to define p at $(x_0, t_{0,r})$. Since the explicit formula for p in terms of $\{\mu_j\}_{j=1,\dots,n}$ is straightforward but rather cumbersome in this case, we omit further details.

Proof Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (3.200), (3.201), we introduce polynomials F_n, G_{n+1}, and H_n exactly as in the proof of Theorem 3.21 in the stationary case, treating t_r as a parameter. In particular, we have the following on Ω_μ,

$$\begin{aligned} F_n(z) &= -iq \prod_{j=1}^n (z - \mu_j), \\ 2qG_{n+1}(z) &= F_{n,x}(z) + 2izF_n(z), \\ H_n(z) &= ip \prod_{j=1}^n (z - \nu_j), \\ G_{n+1,x}(z) &= pF_n(z) + qH_n(z), \\ R_{2n+1}(z) &= G_{n+1}(z)^2 - F_n(z)H_n(z), \\ H_{n,x}(z) &= 2izH_n(z) + 2pG_{n+1}(z). \end{aligned}$$

Hence, it suffices to focus on the proof of (3.147).
We define the monic polynomial $\widetilde{G}_{r+1}$ of degree $r + 1$ by

$$q_{t_r} = i\widetilde{F}_{r,x}(z) - 2z\widetilde{F}_r(z) - 2iq\widetilde{G}_{r+1}(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{3.236}$$

Next we want to establish

$$F_{n,t_r}(z) = 2i(G_{n+1}(z)\widetilde{F}_r(z) - F_n(z)\widetilde{G}_{r+1}(z)) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{3.237}$$

Here $\widetilde{F}_r$ is defined on $\mathbb{C} \times \Omega_\mu$ in terms of the homogeneous quantities $\widehat{F}_r$ and integration constants $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$ by

$$\widetilde{F}_r = \sum_{s=0}^r \tilde{c}_{r-s}\widehat{F}_s, \quad \tilde{c}_0 = 1,$$

and $\widehat{F}_s$ is given by (F.10) or (F.12) times $-iq$. Temporarily introducing

$$\begin{aligned} \check{F}_n(z) &= \prod_{j=1}^n (z - \mu_j) = iq^{-1}F_n(z), \\ \check{\widetilde{F}}_r(z) &= iq^{-1}\widetilde{F}_r(z) \end{aligned}$$

on $\mathbb{C} \times \Omega_\mu$, (3.237) is equivalent to

$$\check{F}_{n,t_r}(z) = \check{F}_{n,x}(z)\check{\widetilde{F}}_r(z) - \check{F}_n(z)\check{\widetilde{F}}_{r,x}(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{3.238}$$

Identifying $\check{F}_n$, $\check{\widetilde{F}}_r$ with F_n, $\widetilde{F}_r$ in the KdV context, (3.238) is identical to (1.227) and hence is an immediate consequence of (F.74). This proves (3.238) and thus (3.237). Next, we define the polynomial $\widetilde{H}_r$ of degree r by

$$\widetilde{G}_{r+1,x}(z) = p\widetilde{F}_r(z) + q\widetilde{H}_r(z) \text{ on } \mathbb{C} \times \Omega_\mu \tag{3.239}$$

in accordance with (3.163). Differentiating (3.164) with respect to t_r, inserting (3.236), the x-derivative of (3.237), (3.238), and (3.239) into the resulting expression then yields (3.193), that is,

$$G_{n+1,t_r}(z) = i(H_n(z)\widetilde{F}_r(z) - F_n(z)\widetilde{H}_r(z)) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{3.240}$$

Differentiating (3.167) with respect to t_r, inserting (3.237) and (3.240), then yields (3.194),

$$H_{n,t_r}(z) = 2i(H_n(z)\widetilde{G}_{r+1}(z) - G_{n+1}(z)\widetilde{H}_r(z)) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{3.241}$$

Finally, differentiating (3.239) with respect to t_r, inserting (3.236), (3.237), the x-derivative of (3.240), (3.241), (3.164), and (3.166) into the resulting expression then proves (3.161), that is,

$$p_{t_r} = -i\widetilde{H}_{r,x}(z) - 2z\widetilde{H}_r(z) + 2ip\widetilde{G}_{r+1}(z) \text{ on } \mathbb{C} \times \Omega_\mu.$$

Thus, we have proved (3.161)–(3.166) and (3.192)–(3.194) on $\mathbb{C} \times \Omega_\mu$ and hence (taking $z = 0$) conclude that p, q satisfy the rth AKNS equation (3.147) on $\mathbb{C} \times \Omega_\mu$. That $p \neq 0$ on Ω_μ and that p is given by (3.234) if $\prod_{j=1}^n \mu_j \neq 0$ then follows precisely, as in the proof of Theorem 3.21. The case in which one of the μ_j vanishes at some $x_0 \in \Omega_\mu$ is also treated, as in the proof of Theorem 3.21, using (3.145). We omit the details. $\square$

Remark 3.38 The explicit theta function representations (3.217), (3.218) of p, q on Ω_μ in (3.233), (3.234) then permit one to extend p and q beyond Ω_μ as long as $\mathcal{D}_{\underline{\hat{\mu}}}$ remains nonspecial (cf. Theorem A.31).

Remark 3.39 Again we singled out q and $\{\mu_j\}_{j=1,\dots,n}$ in Theorem 3.37. The analogous results can of course be proven in terms of p and $\{\nu_j\}_{j=1,\dots,n}$.

The analog of Remark 3.24 directly extends to the current time-dependent setting.

3.5 The Classical Boussinesq Hierarchy

In this section we show that the classical Boussinesq hierarchy is gauge equivalent to the AKNS hierarchy and apply the techniques used for the latter to derive algebro-geometric solutions for the classical Boussinesq hierarchy.

Fix[1] $\alpha \in \mathbb{C} \setminus \{0\}$, $\beta \in \mathbb{C}$, and define the 2×2 matrix

$$\breve{U}(z) = \begin{pmatrix} -iz - \alpha v & u + \beta v_x \\ -1 & iz + \alpha v \end{pmatrix}, \quad z \in \mathbb{C}. \tag{3.242}$$

[1] The constants α and β remain fixed in the following and will not be emphasized in the notation.

Define recursively $\{\breve{f}_\ell\}_{\ell\in\mathbb{N}_0}$, $\{\breve{g}_\ell\}_{\ell\in\mathbb{N}_0}$, and $\{\breve{h}_\ell\}_{\ell\in\mathbb{N}_0}$ by

$$\breve{f}_0 = -i(u+\beta v_x), \quad \breve{g}_0 = 1, \quad \breve{h}_0 = -i, \tag{3.243}$$

$$\breve{f}_{\ell+1} = (i/2)\breve{f}_{\ell,x} + i\alpha v \breve{f}_\ell - i(u+\beta v_x)\breve{g}_{\ell+1}, \tag{3.244}$$

$$\breve{g}_{\ell+1,x} = (u+\beta v_x)\breve{h}_\ell - \breve{f}_\ell, \tag{3.245}$$

$$\breve{h}_{\ell+1} = -(i/2)\breve{h}_{\ell,x} + i\alpha v \breve{h}_\ell - i\breve{g}_{\ell+1}, \quad \ell\in\mathbb{N}_0. \tag{3.246}$$

Explicitly, the first few elements read

$$\begin{aligned}
\breve{f}_0 &= -i(u+\beta v_x),\\
\breve{f}_1 &= \tfrac{1}{2}(u+\beta v_x)_x + \alpha v(u+\beta v_x) + c_1(-i)(u+\beta v_x), \quad \text{etc.},\\
\breve{g}_0 &= 1,\\
\breve{g}_1 &= c_1,\\
\breve{g}_2 &= -\tfrac{1}{2}(u+\beta v_x) + c_2, \text{ etc.},\\
\breve{h}_0 &= -i,\\
\breve{h}_1 &= \alpha v - ic_1, \text{ etc.},
\end{aligned}$$

where $\{c_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$ are integration constants.

Next, define the 2×2 matrix

$$\breve{V}_{n+1}(z) = i\begin{pmatrix} -\breve{G}_{n+1}(z) & \breve{F}_n(z) \\ -\breve{H}_n(z) & \breve{G}_{n+1}(z)\end{pmatrix}, \quad z\in\mathbb{C}, \tag{3.247}$$

where $\breve{F}_n$, $\breve{G}_{n+1}$, and $\breve{H}_n$ are polynomials of the type

$$\begin{aligned}
\breve{F}_n(z) &= \sum_{\ell=0}^{n} \breve{f}_{n-\ell} z^\ell = -i(u+\beta v_x)\prod_{j=1}^{n}(z-\breve{\mu}_j),\\
\breve{G}_{n+1}(z) &= \sum_{\ell=0}^{n+1} \breve{g}_{n+1-\ell} z^\ell,\\
\breve{H}_n(z) &= \sum_{\ell=0}^{n} \breve{h}_{n-\ell} z^\ell = -i\prod_{j=1}^{n}(z-\breve{\nu}_j).
\end{aligned} \tag{3.248}$$

Using the recursion (3.243)–(3.246), one verifies

$$\breve{F}_{n,x} = -2(iz+\alpha v)\breve{F}_n + 2(u+\beta v_x)\breve{G}_{n+1}, \tag{3.249}$$

$$\breve{G}_{n+1,x} = (u+\beta v_x)\breve{H}_n - \breve{F}_n, \tag{3.250}$$

$$\breve{H}_{n,x} = 2(iz+\alpha v)\breve{H}_n - 2\breve{G}_{n+1}, \tag{3.251}$$

implying

$$\left(\breve{G}_{n+1}^2 - \breve{F}_n\breve{H}_n\right)_x = 0$$

and hence

$$\breve{G}_{n+1}^2 - \breve{F}_n \breve{H}_n = R_{2n+2},$$

where R_{2n+2} is a monic polynomial of degree $2n + 2$ with zeros $\{E_0, \dots, E_{2n+1}\} \subset \mathbb{C}$. Thus,

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1} (z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}.$$

Again the corresponding hyperelliptic curve $\mathcal{K}_n$ of genus n is naturally obtained from the characteristic equation of $i\breve{V}_{n+1}$,

$$\begin{aligned}\det\big(yI_2 - i\breve{V}_{n+1}(z,x)\big) &= y^2 - \breve{G}_{n+1}(z,x)^2 + \breve{F}_n(z,x)\breve{H}_n(z,x) \\ &= y^2 - R_{2n+2}(z) = 0.\end{aligned}$$

The corresponding stationary zero-curvature relation then reads

$$-\breve{V}_{n+1,x} + [\breve{U}, \breve{V}_{n+1}] = 0,$$

implying

$$\begin{aligned}0 &= -\breve{V}_{n+1,x} + [\breve{U}, \breve{V}_{n+1}] \\ &= i \begin{pmatrix} \breve{G}_{n+1,x} + \breve{F}_n & -\breve{F}_{n,x} - 2(iz+\alpha v)\breve{F}_n \\ -(u+\beta v_x)\breve{H}_n & +2(u+\beta v_x)\breve{G}_{n+1} \\ \breve{H}_{n,x} + 2\breve{G}_{n+1} & -\breve{G}_{n+1,x} - \breve{F}_n \\ -2(iz+\alpha v)\breve{H}_n & +(u+\beta v_x)\breve{H}_n \end{pmatrix}. \end{aligned} \tag{3.252}$$

Using the recursion (3.243)–(3.246) to compute $\breve{f}_j$, $\breve{g}_j$, and $\breve{h}_j$ for $j = 0, \dots, n$, (3.252) equals

$$\begin{aligned}&-\breve{V}_{n+1,x} + [\breve{U}, \breve{V}_{n+1}] \\ &= i \begin{pmatrix} \breve{g}_{n+1,x} + \breve{f}_n - (u+\beta v_x)\breve{h}_n & -\breve{f}_{n,x} - 2\alpha v\breve{f}_n + 2(u+\beta v_x)\breve{g}_{n+1} \\ \breve{h}_{n,x} - 2\alpha v\breve{h}_n + 2\breve{g}_{n+1} & -\breve{g}_{n+1,x} - \breve{f}_n + (u+\beta v_x)\breve{h}_n \end{pmatrix} = 0.\end{aligned} \tag{3.253}$$

Next, let

$$\breve{g}_{n+1} = -(1/2)\breve{h}_{n,x} + \alpha v\breve{h}_n \tag{3.254}$$

(consistent with $\breve{h}_{n+1} = 0$ in (3.246)). Inserting (3.254) into (3.253), the stationary cBsq hierarchy is given by

$$\begin{pmatrix} \breve{f}_{n,x} + 2\alpha v\breve{f}_n + (u+\beta v_x)(\breve{h}_{n,x} - 2\alpha v\breve{h}_n) \\ -(1/2)\breve{h}_{n,xx} + \alpha(v\breve{h}_n)_x + \breve{f}_n - (u+\beta v_x)\breve{h}_n \end{pmatrix} = 0, \quad n \in \mathbb{N}_0. \tag{3.255}$$

Remark 3.40 As a consequence of (3.254), the nth stationary cBsq system will contain only integration constants $c_1, \dots, c_n$, $n \in \mathbb{N}$, coming from integrating (3.245). Since $\breve{g}_{n+1,x} + \breve{f}_n - (u + \beta v_x)\breve{h}_n = 0$ by (3.253), our definition (3.254) is consistent with the definition of $\breve{g}_{n+1}$ given by the recursion (3.245). However, no new integration constant is introduced in this context.

The first few equations read (after some additional simplifications)

$$
\begin{aligned}
n = 0: &\quad \begin{pmatrix} u_x \\ v_x \end{pmatrix} = 0, \\
n = 1: &\quad \begin{pmatrix} (u + \beta v_x)_{xx} + 4\alpha(v(u + \beta v_x))_x - c_1 i(u + \beta v_x)_x \\ u_x + (\beta - \alpha)v_{xx} + 2\alpha^2(v^2)_x - 2ic_1\alpha v_x \end{pmatrix} = 0, \text{ etc.}
\end{aligned}
$$

In the special homogeneous case, the latter set of equations, the stationary classical Boussinesq system, can be rewritten in the more familiar form

$$u + (\beta - \alpha)v_x + 2\alpha^2 v^2 = 0, \tag{3.256}$$

$$(2\alpha\beta - \beta^2)v_{xxx} + (\alpha - \beta)u_{xx} + 4\alpha^2(uv)_x = 0. \tag{3.257}$$

By inserting (3.256) into (3.257), the latter can also be rewritten as

$$v_{xxx} - 12\alpha^2(v^2)_x = 0.$$

To discuss the time-dependent hierarchy of classical Boussinesq systems, we follow the AKNS case and introduce a deformation parameter $t_n \in \mathbb{C}$ in the functions u and v, that is, $u = u(x, t_n)$, $v = v(x, t_n)$. The time-dependent zero-curvature relation then reads

$$\breve{U}_{t_n} - \breve{V}_{n+1,x} + [\breve{U}, \breve{V}_{n+1}] = 0,$$

implying

$$
\begin{aligned}
0 &= \breve{U}_{t_n} - \breve{V}_{n+1,x} + [\breve{U}, \breve{V}_{n+1}] \\
&= \begin{pmatrix} -\alpha v_{t_n} + i\breve{G}_{n+1,x} & (u + \beta v_x)_{t_n} - 2i(iz + \alpha v)\breve{F}_n \\ + i\breve{F}_n - i(u + \beta v_x)\breve{H}_n & - i\breve{F}_{n,x} + 2i(u + \beta v_x)\breve{G}_{n+1} \\ i\breve{H}_{n,x} + 2i\breve{G}_{n+1} & \alpha v_{t_n} - i\breve{G}_{n+1,x} \\ - 2i(iz + \alpha v)\breve{H}_n & - i\breve{F}_n + i(u + \beta v_x)\breve{H}_n \end{pmatrix}.
\end{aligned} \tag{3.258}
$$

Using the recursion (3.243)–(3.246) to compute $\breve{f}_j$, $\breve{g}_j$, and $\breve{h}_j$ for $j = 0, \dots, n$, we find that (3.258) reduces to

$$\breve U_{t_n} - \breve V_{n+1,x} + [\breve U, \breve V_{n+1}]$$
$$= \begin{pmatrix} -\alpha v_{t_n} + i\breve g_{n+1,x} & (u+\beta v_x)_{t_n} - i\breve f_{n,x} \\ +i\breve f_n - i(u+\beta v_x)\breve h_n & -2i\alpha v\breve f_n + 2i(u+\beta v_x)\breve g_{n+1} \\ i\breve h_{n,x} - 2i\alpha v\breve h_n & \alpha v_{t_n} - i\breve g_{n+1,x} \\ +2i\breve g_{n+1} & -i\breve f_n + i(u+\beta v_x)\breve h_n \end{pmatrix} = 0,$$

or equivalently, to

$$\alpha v_{t_n} - i\breve g_{n+1,x} - i\breve f_n + i(u+\beta v_x)\breve h_n = 0, \tag{3.259}$$
$$(u+\beta v_x)_{t_n} - i\breve f_{n,x} - 2i\alpha v\breve f_n + 2i(u+\beta v_x)\breve g_{n+1} = 0, \tag{3.260}$$
$$\breve h_{n,x} - 2\alpha v\breve h_n + 2\breve g_{n+1} = 0. \tag{3.261}$$

Using $\breve h_{n,x} - 2\alpha v\breve h_n + 2\breve g_{n+1} = 0$ to eliminate $\breve g_{n+1}$ in (3.259) and (3.260) then yields the following expressions for the time-dependent classical Boussinesq hierarchy,

$$\begin{aligned} &\alpha u_{t_n} - (i/2)\beta\breve h_{n,xxx} + i\alpha\beta v\breve h_{n,xx} - i\big((\beta+\alpha)u + \beta(\beta-\alpha)v_x\big)\breve h_{n,x} \\ &\quad - i\big(\beta(u+(\beta-\alpha)v_x)_x - 2\alpha^2(u+\beta v_x)v\big)\breve h_n + i(\beta-\alpha)\breve f_{n,x} - 2i\alpha^2 v\breve f_n = 0, \\ &\quad \alpha v_{t_n} + (i/2)\breve h_{n,xx} - i\alpha v\breve h_{n,x} + i(u+(\beta-\alpha)v_x)\breve h_n - i\breve f_n = 0, \quad n\in\mathbb{N}_0. \end{aligned} \tag{3.262}$$

For brevity, equations (3.262) will be denoted by

$$\mathrm{cBsq}_n(u,v) = 0, \quad n\in\mathbb{N}_0.$$

Remark 3.41 One observes that $\breve g_{n+1}$ defined by (3.261) does not satisfy (3.245) but rather (3.259). This is in contrast to the stationary case as well as the corresponding definitions for the AKNS hierarchy. As in the stationary case, the nth cBsq system contains n integration constants $c_1,\dots,c_n$, $n\in\mathbb{N}$.

Explicitly, the first few equations read

$$\mathrm{cBsq}_0(u,v) = \begin{pmatrix} \alpha u_{t_0} - \alpha u_x \\ \alpha v_{t_0} - \alpha v_x \end{pmatrix} = 0,$$
$$\mathrm{cBsq}_1(u,v) = \begin{pmatrix} \alpha u_{t_1} - \frac{i}{2}\alpha\beta v_{xxx} + \frac{i}{2}(\beta-\alpha)(u+\beta v_x)_{xx} \\ -2i\alpha^2(uv)_x + c_1(-\alpha u_x) \\ \alpha v_{t_1} - \frac{i}{2}(u+\beta v_x)_x \\ -2i\alpha^2 vv_x + c_1(-\alpha v_x) \end{pmatrix} = 0, \text{ etc.}$$

In the homogeneous case $\mathrm{cBsq}_1(u, v) = 0$ can be rewritten as

$$\begin{aligned}\alpha u_{t_1} &= (i/2)\beta(2\alpha - \beta)v_{xxx} + (i/2)(\alpha - \beta)u_{xx} + 2i\alpha^2(uv)_x,\\ \alpha v_{t_1} &= (1/2)(\beta - \alpha)v_{xx} + 2i\alpha^2 v v_x + (i/2)u_x.\end{aligned}$$

Finally, specializing to $\alpha = \beta$, one obtains the classical Boussinesq system

$$u_{t_1} = (i/2)\alpha v_{xxx} + 2i\alpha(uv)_x, \quad \alpha v_{t_1} = 2i\alpha^2 v v_x + (i/2)u_x.$$

We now prove that the AKNS and the cBsq hierarchies are gauge equivalent by exhibiting an explicit gauge transformation between them. We first recall the effect of gauge transformations on zero-curvature equations. Starting with the time-dependent equations

$$\Psi_x = U\Psi, \quad \Psi_t = V\Psi, \quad \Psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \tag{3.263}$$

whose compatibility relation $\Psi_{xt} = \Psi_{tx}$ yields the zero-curvature equation

$$U_t - V_x + [U, V] = 0,$$

we introduce the gauge transformation

$$\breve{\Psi} = S\Psi, \quad S \text{ invertible.} \tag{3.264}$$

Then one derives,

$$\breve{\Psi}_x = \breve{U}\breve{\Psi}, \quad \breve{\Psi}_t = \breve{V}\breve{\Psi},$$

with

$$\breve{U} = S_x S^{-1} + SUS^{-1}, \quad \breve{V} = S_t S^{-1} + SVS^{-1}$$

and hence

$$\breve{U}_t - \breve{V}_x + [\breve{U}, \breve{V}] = 0. \tag{3.265}$$

The corresponding stationary formalism starts from

$$\Psi_x = U\Psi, \quad iy\Psi = V\Psi, \quad y \in \mathbb{C} \tag{3.266}$$

and

$$-V_x + [U, V] = 0. \tag{3.267}$$

The gauge transformation (3.264) then effects

$$\breve{\Psi}_x = \breve{U}\breve{\Psi}, \quad iy\breve{\Psi} = \breve{V}\breve{\Psi}, \tag{3.268}$$

with

$$\breve{U} = S_x S^{-1} + SUS^{-1}, \quad \breve{V} = SVS^{-1} \tag{3.269}$$

and hence

$$-\check{V}_x + [\check{U}, \check{V}] = 0. \tag{3.270}$$

Introducing the particular choice

$$S = \begin{pmatrix} (-p)^{1/2} & 0 \\ 0 & 1/(-p)^{1/2} \end{pmatrix}, \quad p \in C^\infty(\mathbb{R}),\, p \neq 0 \tag{3.271}$$

and applying it to (3.266)–(3.270) in the case of the stationary AKNS hierarchy, thus identifying (U, V) and (U, V_{n+1}), then yield the following result.

Theorem 3.42 *The stationary AKNS and cBsq hierarchies are gauge equivalent in the sense that*

$$\begin{aligned} \check{U} &= S_x S^{-1} + S U S^{-1}, \\ \check{V}_{n+1} &= S V_{n+1} S^{-1}, \end{aligned}$$

where (U, V_{n+1}) *and* $(\check{U}, \check{V}_{n+1})$ *are given by* (3.38)*,* (3.39) *and* (3.242)*,* (3.247)*, respectively, and S is defined by* (3.271)*. In particular,* u, v *given by*

$$u = -pq + \frac{\beta}{2\alpha}\left(\frac{p_x}{p}\right)_x, \quad v = -\frac{1}{2\alpha}\frac{p_x}{p}, \tag{3.272}$$

satisfy the nth stationary cBsq system if and only if p, q*, given by*

$$\begin{aligned} p(x) &= \exp\left(-2\alpha \int^x dx'\, v(x')\right), \\ q(x) &= -(u(x) + \beta v_x(x)) \exp\left(2\alpha \int^x dx'\, v(x')\right), \end{aligned} \tag{3.273}$$

satisfy the nth stationary AKNS system with identical sets of integration constants $c_j \in \mathbb{C}$*,* $j = 1, \dots, n$ *for* $n \in \mathbb{N}$*.*

Proof $\check{U} = S_x S^{-1} + S U S^{-1}$ is easily seen to be equivalent to (3.272). Similarly, $\check{V}_{n+1} = S V_{n+1} S^{-1}$ is equivalent to

$$\check{F}_n = -pF_n, \quad \check{G}_{n+1} = G_{n+1}, \quad \check{H}_n = -\frac{1}{p}H_n. \tag{3.274}$$

Next suppose that p, q solve the nth stationary AKNS system, that is, equations (3.20)–(3.22) hold. Define $\check{F}_n$, $\check{G}_{n+1}$, and $\check{H}_n$ by (3.274) and u, v by (3.272). Then clearly, (3.252) is satisfied, proving that u, v satisfy the nth stationary cBsq system. Conversely, starting with u, v solving the nth stationary cBsq system (3.252), we can define p, q and F_n, G_{n+1}, and H_n using (3.273) and (3.274), respectively. One then easily verifies that (3.20)–(3.22) hold, and thus p, q solve the nth stationary AKNS system. □

We note that the ambiguity inherent to (3.273), resulting from an arbitrary integration constant, corresponds to the scale invariance of the AKNS hierarchy, as discussed in Lemma 3.6.

The time-dependent analog of Theorem 3.42 reads as follows.

Theorem 3.43 *The time-dependent AKNS and cBsq hierarchies are gauge equivalent in the sense that*

$$\breve{U} = S_x S^{-1} + SUS^{-1}, \quad \breve{V}_{n+1} = S_{t_n} S^{-1} + SV_{n+1}S^{-1}$$

with (U, V_{n+1}) *and* $(\breve{U}, \breve{V}_{n+1})$ *given by* (3.38)*,* (3.39)*, and* (3.242)*,* (3.247)*, respectively, and* S *defined by* (3.271)*. In particular,* u, v *given by*

$$u = -pq + \frac{\beta}{2\alpha}\left(\frac{p_x}{p}\right)_x, \quad v = -\frac{1}{2\alpha}\frac{p_x}{p}, \tag{3.275}$$

satisfy the nth cBsq system $\mathrm{cBsq}_n(u, v) = 0$ *if and only if* p, q *given by*

$$\begin{aligned} p(x, t_n) &= \exp\left(-2\alpha \int^x dx' v(x', t_n)\right), \\ q(x, t_n) &= -(u(x, t_n) + \beta v_x(x, t_n)) \exp\left(2\alpha \int^x dx' v(x', t_n)\right) \end{aligned} \tag{3.276}$$

satisfy the nth AKNS system $\mathrm{AKNS}_n(p, q) = 0$ *with identical sets of integration constants* $c_j \in \mathbb{C}$, $j = 1, \dots, n$ *for* $n \in \mathbb{N}$.

Proof $\breve{U} = S_x S^{-1} + SUS^{-1}$ is equivalent to (3.275), as noted in the proof of Theorem 3.42. By a direct calculation, $\breve{V}_{n+1} = S_{t_n} S^{-1} + SV_{n+1}S^{-1}$ is equivalent to

$$\breve{F}_n = -pF_n, \quad \breve{G}_{n+1} = G_{n+1} + \frac{i}{2}\frac{p_{t_n}}{p}, \quad \breve{H}_n = -p^{-1}H_n. \tag{3.277}$$

Next, assume that p, q solve the nth AKNS system. Define $\breve{F}_n$, $\breve{G}_{n+1}$ and $\breve{H}_n$ by (3.277) and u, v by (3.275). Then clearly, (3.258) is satisfied, proving that u, v satisfy the nth cBsq system. Conversely, starting with u, v solving the nth cBsq system (3.258), we can define p, q and F_n, G_{n+1} and H_n using (3.276) and (3.277), respectively. Again one verifies that p, q solve the nth AKNS system. □

Finally we derive the theta function representation of algebro-geometric cBsq solutions utilizing the gauge equivalence of the cBsq and AKNS hierarchies.

Let $u^{(0)}$, $v^{(0)}$ be stationary solutions of the nth classical Boussinesq system, that is, we assume they satisfy

$$\begin{aligned}
&\breve{f}_{n,x}(u^{(0)}, v^{(0)}) + 2\alpha v \breve{f}_n(u^{(0)}, v^{(0)}) + (u + \beta v_x)(\breve{h}_{n,x}(u^{(0)}, v^{(0)}) \\
&\quad - 2\alpha v \breve{h}_n(u^{(0)}, v^{(0)})) = 0, \\
&\breve{f}_n(u^{(0)}, v^{(0)}) - (u + \beta v_x)\breve{h}_n(u^{(0)}, v^{(0)}) - (1/2)\breve{h}_{n,xx}(u^{(0)}, v^{(0)}) \\
&\quad + \alpha(v\breve{h}_n(u^{(0)}, v^{(0)}))_x = 0,
\end{aligned}$$

for a given set of integration constants $\{c_j\}_{j=1,\dots,n} \subset \mathbb{C}$. Fix $r \in \mathbb{N}_0$ and corresponding integration constants $\{\tilde{c}_j\}_{j=1,\dots,r} \subset \mathbb{C}$. The aim in this section is to construct a solution (u, v) of

$$\mathrm{cBsq}_r(u, v) = 0, \quad (u, v)|_{t_r = t_{0,r}} = (u^{(0)}, v^{(0)}).$$

The function $\breve{\phi}$ and the Baker–Akhiezer function $\breve{\Psi}$ associated with the classical Boussinesq hierarchy can be obtained as follows.

Lemma 3.44 *Consider* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *and* $(z, x, x_0, t_r, t_{0,r}) \in \mathbb{C} \times \mathbb{R}^4$. *Let* ϕ, Ψ, *and* S *be given by* (3.170), (3.173), *and* (3.271), *respectively. Define*

$$\breve{\Psi} = \begin{pmatrix} \breve{\psi}_1 \\ \breve{\psi}_2 \end{pmatrix} = S\Psi, \quad \breve{\phi} = -\frac{\phi}{p}. \tag{3.278}$$

Then $\breve{\phi}$ *satisfies* $\breve{\phi} = \breve{\psi}_2/\breve{\psi}_1$ *and*

$$v\breve{\phi}(P) - \frac{1}{2\alpha}\breve{\phi}_x(P) + \frac{1}{2\alpha}(u + \beta v_x)\breve{\phi}(P)^2 - \frac{iz}{\alpha}\breve{\phi}(P) - \frac{1}{2\alpha} = 0. \tag{3.279}$$

In addition, $\breve{\Psi}$ *satisfies*

$$\breve{\Psi}_x(P) = \breve{U}(z)\breve{\Psi}(P), \tag{3.280}$$

$$iy\breve{\Psi}(P) = \breve{V}_{n+1}(z)\breve{\Psi}(P), \tag{3.281}$$

$$\breve{\Psi}_{t_r}(P) = \widetilde{\breve{V}}_{r+1}(z)\breve{\Psi}(P). \tag{3.282}$$

Moreover, as long as the zeros of $\breve{F}_n(\,\cdot\,, x, t_r)$ *are all simple for* $(x, t_r) \in \Omega$, $\Omega \subseteq \mathbb{R}^2$ *open and connected,* $\breve{\Psi}(\,\cdot\,, x, x_0, t_0, t_{0,r})$ *is meromorphic on* $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *for* $(x, t_r), (x_0, t_{0,r}) \in \Omega$.

Proof Immediate from Lemma 3.27 and (3.263)–(3.265). □

The explicit representation of algebro-geometric solutions of the classical Boussinesq hierarchy in terms of the Riemann theta function associated with $\mathcal{K}_n$ then reads as follows.

Theorem 3.45 *Assume the hypotheses of Theorem* 3.33 *and suppose that* u, v *satisfy*

$$\widetilde{\mathrm{cBsq}}_r(u, v) = 0 \text{ on } \Omega, \quad (u, v)\big|_{t_r = t_{0,r}} = (u^{(0)}, v^{(0)})$$

with $u^{(0)}, v^{(0)}$ *satisfying the nth stationary cBsq system* (3.255). *Then the theta function representation of* u, v *is given by*

$$\begin{aligned} u(x, t_r) = {} & e_{0,1} + \partial_x^2 \ln(\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x, t_r)))) \\ & + \frac{\beta}{2\alpha}\partial_x^2 \ln\left(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}(x, t_r)))}\right), \end{aligned} \tag{3.283}$$

$$v(x, t_r) = \frac{i}{\alpha} e_{0,0} - \frac{1}{2\alpha}\partial_x \ln\left(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}(x, t_r)))}\right), \quad (x, t_r) \in \Omega. \tag{3.284}$$

Proof Combine Theorem 3.33, Corollary 3.35, and (3.275). □

Obviously one can derive formulas similar to (3.212)–(3.214) for the functions $\breve{\phi}$ and $\breve{\Psi}$ using the explicit relation (3.278). Moreover, one can replace $\mathcal{D}_{\underline{\hat{\nu}}}$ in (3.283), (3.284) by $\mathcal{D}_{\underline{\hat{\mu}}}$ according to Remark 3.12.

3.6 Notes

Most of the material presented in Sections 3.1–3.4 closely follows Gesztesy and Ratnaseelan (1998), whereas Section 3.5 is taken from Gesztesy and Holden (2000a).

Section 3.1. Zakharov and Shabat established a Lax pair for the nonlinear Schrödinger (nS) equation in 1972 and reduced the construction of spatially decaying solutions to the inverse scattering problem of a Dirac-type operator (Zakharov and Shabat (1972; 1973)). Complete integrability of the nS equation as a Hamiltonian system and action and angle variables were subsequently established in Zakharov and Manakov (1974). In the same year the AKNS system was introduced by Ablowitz et al. (1974). In particular, Ablowitz et al. (1974) proved that the inverse scattering method applied to the AKNS system and studied the case of spatially decaying solutions in some detail. A general scheme of integrating nonlinear soliton-type evolution equations was presented by Zakharov and Shabat (1974) and then continued in Zakharov and Shabat (1979).

For reviews of the early period up to 1978, we refer to Ablowitz et al. (1974), Flaschka and Newell (1975), and Newell (1978).

Originally, the equations were motivated by applications to nonlinear optics. Since even an attempt of a bibliography on this topic is beyond the scope of this monograph, we refer the interested reader to Abdullaev et al. (1993), Ablowitz and Segur (1981, Sec. 4.3), Dodd et al. (1982, Ch. 8), Hasegawa (1990), Hasegawa and

Kodama (1995), Kodama (1999), Newell and Moloney (1992), and the references therein.

For textbook literature on the nS equation and the AKNS system, we refer, for instance, to Ablowitz and Segur (1981, Ch. 1), Asano and Kato (1990, Ch. 5), Cherednik (1996), Dodd et al. (1982, Ch. 6), Drazin and Johnson (1989, Ch. 6), Eckhaus and van Harten (1983, Chs. 5, 6), Faddeev and Takhtajan (1987, Part 1), Newell (1985, Chs. 3, 5), and Novikov et al. (1984, Sec. I.10); for a recent review we refer to Palais (1997).

Section 3.2. Our recursive approach to the Lax and zero-curvature pairs of the AKNS hierarchy follows Alber's treatment of the KdV and nonlinear Schrödinger hierarchies in Al'ber (1979; 1981), S. J. Al'ber and Al'ber (1987b) (as well as Dickey (1991, Ch. 12), Gel'fand and Dikii (1979), Gesztesy and Weikard (1993), and Gesztesy et al. (1996a)).

The original Burchnall–Chaundy theory has been developed in Burchnall and Chaundy (1923; 1928; 1932) (and Baker (1928)). More recent presentations can be found, for instance, in Carlson and Goodearl (1980), Previato (1996; 1998), and Wilson (1985).

Global existence and longtime behavior of nS solutions recently attracted much activity. Since a detailed list of these activities is far beyond the scope of this monograph, we just refer to Bourgain (1999) and the references therein.

Connections between the motion of closed curves in $\mathbb{R}^3$ guided by the filament equation (describing the motion of thin isolated filament vortices in a fluid) and the nS hierarchy are discussed in Grinevich and Schmidt (1999) and Grinevich (2001). Finally, we mention an interesting gauge equivalence between the nS equation and the continuous isotropic Heisenberg ferromagnet model observed in Zakharov and Takhtadzhyan (1979) and further explored, for instance, in Faddeev and Takhtajan (1987, Part 2, Ch. I).

Section 3.3. The theory of commuting matrix-valued differential expressions and, more generally, the algebro-geometric approach to matrix hierarchies of soliton equations has been developed in great generality by Dubrovin and Krichever. Corresponding authoritative accounts can be found, for instance, in Belokolos et al. (1994, Chs. 3–4), Dubrovin (1977; 1983), Dubrovin et al. (1990), Its (1981; 1986), Krichever (1977a; 1983), and Previato (1985). In contrast to these references, our approach relies on two basic ingredients, an elementary polynomial recursive approach to Lax pairs (or zero-curvature pairs) of the AKNS hierarchy and its explicit connection with the fundamental meromorphic function ϕ (cf. (3.56), (3.57)) that allows for a unified algebro-geometric treatment of the entire AKNS hierarchy.

As in all other chapters, the meromorphic function ϕ on $\mathcal{K}_n$ defined in (3.56), is the key object of our algebro-geometric formalism. By (3.56)–(3.58), ϕ again links the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and its counterpart, $\mathcal{D}_{\underline{\hat{\nu}}}$. This is of course a direct consequence of the identity (3.23) together with the factorizations of F_n and H_n in (3.51) and (3.52). Thus, our construction of positive divisors of degree n (respectively $n+1$ since the points $P_{\infty_\pm}$ are also involved) on the hyperelliptic curve $\mathcal{K}_n$ of genus n again follows the recipe of Jacobi (1846), Mumford (1984, Sec. III a).1), and McKean (1985).

The Dubrovin equations (3.77) and (3.80) in Lemma 3.8, in connection with the auxiliary divisors, and the corresponding trace formulas in Lemma 3.9 are well-known in the AKNS context. We refer, for instance, to Al'ber and Al'ber (1987b), Alber and Marsden (1994b), De Concini and Johnson (1987), Elgin (1990), Forest and Lee (1986), Lee (1990; 1991), Ma and Ablowitz (1981), Mertsching (1987), and Tracy et al. (1984; 1988).

The analog of the theta function representations (3.107), (3.108) in connection with the nS equation was first published by Its and Kotlyarov (1976).

Since then, many authors presented reviews and slightly varying approaches to algebro-geometric (respectively periodic) solutions of the nS and AKNS equations. The linearization property (3.105), (3.106) of the Abel map and formulas (3.107)–(3.109) for p, q in terms of the Riemann theta function associated with $\mathcal{K}_n$ in Theorem 3.11 can be found, for instance, in Adams et al. (1990; 1993), Alber (1993), Dubrovin (1977; 1983), Fedorov and Ma (2002), Harnad (1993), Its (1981; 1986), Lee (1991), Ma and Ablowitz (1981), Matveev (1976, Sec. 9), Mertsching (1987), Previato (1985), Wisse (1992), and the monographs Belokolos et al. (1994, Chs. 4, 5), Cherednik (1996, Sec. I.4). Corollary 3.14 can be found in Its (1981).

The characterization of the isospectral sets of algebro-geometric $\mathrm{nS}_\pm$ solutions, as discussed in Remark 3.16, proved to be more difficult than the earlier settled KdV case and turned out to be somewhat similar to that of the characterization of the isospectral set of all real-valued sG solutions. In fact, the corresponding $\mathrm{nS}_\pm$ and sG cases were settled more or less simultaneously in the first half of the 1980s. After being first discussed in Cherednik (1980) (see also Cherednik (1983)), the problem was later settled by Dubrovin and Novikov and in great detail by Previato. We refer to Dubrovin (1982a; 1983), Dubrovin and Novikov (1975a), Lee (1986), Novikov (1985), Previato (1983; 1985) and the textbook accounts in Belokolos et al. (1994, Sec. 4.3) and Cherednik (1996, Ch. 4) for further details.

We also mention an interesting characterization of all algebro-geometric AKNS potentials by De Concini and Johnson (1987) in the special case in which the 2×2 matrix differential expression M generates a self-adjoint Dirac-type operator D in $L^2(\mathbb{R}) \otimes \mathbb{C}^2$. In this case the algebro-geometric potentials are characterized by the corresponding spectrum consisting of finitely many intervals and the Lyapunov exponent vanishing a.e. on the spectrum. The corresponding 2×2 matrix-valued spectral function of such self-adjoint, algebro-geometric, Dirac-type operators is discussed in Levitan and Mamatov (1993). In this context we also mention a detailed study of Floquet theory for periodic and self-adjoint AKNS operators (not necessarily of algebro-geometric type) generated by D in Grébert and Guillot (1993) and a study of almost periodic, self-adjoint, Dirac-type operators in Giachetti and Johnson (1984). Both references also treat the analog of Borg-type theorems for D when the spectrum of D is either $\mathbb{R}$ or contains one gap (see also Clark and Gesztesy (2002) and Gesztesy et al. (1991)).

The symplectic structure and action-angle variables for the periodic nS and AKNS equations are discussed in Bättig et al. (1993b; 1995), Ercolani and McLaughlin (1991), Grébert and Kappeler (1999), McKean (1997), and McKean and Vaninsky (1997a,b).

For variants of Theorem 3.21, refer, for instance, to Elgin (1990), Ma and Ablowitz (1981), and Tracy et al. (1984; 1988).

The elliptic AKNS potentials in Example 3.25 were analyzed in Gesztesy and Weikard (1998a) (see also Gesztesy and Weikard (1998b)). These references provide a complete characterization of all elliptic algebro-geometric AKNS potentials on the basis of Picard's theorem in analogy to the KdV case, which was mentioned in some detail in the notes to Chapter 1 on the KdV hierarchy. Additional special cases of elliptic AKNS solutions are discussed, for instance, in A'lfimov et al. (1990), Babich et al. (1986), Christiansen et al. (1995; 2000), Its (1981), Lee and Tsui (1990), Ma and Ablowitz (1981), Matveev and Smirnov (1993), Mertsching (1987), Osborne and Boffetta (1990), Pavlov (1987), and Smirnov (1995a,b; 1996; 1997b). For a discussion of rational AKNS potentials bounded at infinity and simply periodic AKNS potentials bounded at the end of the period strip, see Weikard (2002).

Darboux-type transformations and a complete account of their effect on the hyperelliptic curve $\mathcal{K}_n$ (possibly with a singular affine part) associated with algebro-geometric AKNS potentials are discussed in Gesztesy and Holden (2000c); see Appendix G for details.

Section 3.4. Since almost all of the references provided in connection with Section 3.4 treat the time-dependent AKNS system and not just stationary AKNS equations, we will now mainly focus on issues markedly different from stationary ones and topics not yet covered.

In analogy to its stationary analog in Section 3.3, the role of ϕ defined in (3.170) is again central to Section 3.4, and the corresponding facts recorded in the notes to Section 3.3 still apply.

The Dubrovin equations (3.201) in Lemma 3.30 were found simultaneously with their stationary counterparts, as discussed in the notes to Section 3.3. As in the corresponding KdV and sG contexts, equations (3.201) are typically discussed in connection with the simplest cases $r = 0, 1$.

Since the proof of Lemma 3.31 is identical to that in the corresponding stationary case, the remarks in connection with the trace formulas in Lemma 3.9 in the notes to Section 3.3 apply again.

The linearization property (3.215), (3.216) of the Abel map and formulas (3.217)–(3.219) for p, q in terms of the Riemann theta function associated with $\mathcal{K}_n$ in Theorem 3.33 were again found simultaneously with their stationary counterparts, and thus the historical development sketched in this connection in the notes to Section 3.3 remains valid in the context of Theorem 3.33.

The solution of the algebro-geometric initial value problem in Theorem 3.37 is rarely presented in this generality. The result is well-known in the special case $r = 1$, refer to Elgin (1990), Ma and Ablowitz (1981), Tracy et al. (1984), and Tracy et al. (1988).

The Cauchy problem and leading longtime asymptotics of solutions of the nS equation with algebro-geometric behavior as $x \to \pm\infty$ (belonging to different hyperelliptic curves at $-\infty$ and $+\infty$) are studied in Bikbaev (1991).

Completely integrable systems related to the algebro-geometric solutions of the nS and AKNS hierarchies (similar to the connection between the Neumann system of constrained harmonic oscillators to a sphere and the KdV equation) are treated in Previato (1985) and Schilling (1992).

Flows on the moduli space of hyperelliptic curves preserving the periods of nS solutions are studied in Grinevich and Schmidt (1995). These flows are used to provide a complete description of the moduli space of algebraic curves corresponding to spatially periodic nS solutions.

Degenerations of the underlying hyperelliptic curve (resulting in soliton solutions) and solitons relative to algebro-geometric background AKNS solutions are discussed in Belokolos et al. (1994, Secs. 4.4–5), Its (1986), Its et al. (1988), and Previato (1985).

Algebro-geometric solutions of the modified nonlinear Schrödinger equation (including the singularizations of $\mathcal{K}_n$ yielding soliton solutions) are described in Its and Matveev (1983).

Various aspects of the coupled nonlinear Schrödinger equations such as quasi-periodic solutions, Dubrovin-type equations, and elliptic solutions are studied in Alber et al. (1997), Christiansen et al. (1995; 2000), and Eilbeck et al. (2000).

Section 3.5. The equivalence of the cBsq and AKNS hierarchies, on the basis of the transformation (3.275), has been noted in Jaulent and Miodek (1977) and later in Matveev and Yavor (1979). It has been further discussed and linked to Hirota's bilinear formalism in Sachs (1988; 1989).

Algebro-geometric solutions of the time-dependent classical Boussinesq system $\mathrm{cBsq}_1(u, v) = 0$ and their theta function representations were originally derived in Matveev and Yavor (1979). The case of real-valued smooth solutions and additional reductions to elliptic solutions (in the case of genus $n \leq 3$) was subsequently studied by Smirnov (1986). Theta function representations of algebro-geometric solutions of $\mathrm{cBsq}_r(u, v) = 0$ in the special case $r \leq 3$ recently appeared in Geng and Wu (1999).

4

The Classical Massive Thirring System

> And let it be noted that there is no more delicate matter to take in hand, nor more dangerous to conduct, nor more doubtful in its success, than to set up as a leader in the introduction of changes. For he who innovates will have for his enemies all those who are well off under the existing order of things, and only lukewarm supporters in those who might be better off under the new.
>
> *Niccolò Machiavelli (1469–1527)*[1]

4.1 Contents

Integrability of the classical massive Thirring model,[2]

$$\begin{aligned} -iu_x + 2v + 2|v|^2 u &= 0, \\ -iv_t + 2u + 2|u|^2 v &= 0 \end{aligned} \tag{4.1}$$

for functions $u = u(x,t)$, $v = v(x,t)$ was originally established by Mikhailov in 1976. This chapter focuses on the construction of algebro-geometric solutions of the classical massive Thirring system,

$$\begin{aligned} -iu_x + 2v + 2vv^*u &= 0, \\ iu_x^* + 2v^* + 2vv^*u^* &= 0, \\ -iv_t + 2u + 2uu^*v &= 0, \\ iv_t^* + 2u^* + 2uu^*v^* &= 0, \end{aligned}$$

a complexified version of the classical massive Thirring model (4.1). Below we briefly summarize the principal content of each section.

Section 4.2.

- polynomial recursion formalism, zero-curvature triples $(U, V_{n+1}, \widetilde{V})$
- hyperelliptic curve $\mathcal{K}_n$

[1] *The Prince*, Dover, New York, 1992, p. 13.
[2] A guide to the literature can be found in the detailed notes at the end of this chapter.

Section 4.3.

- properties of ϕ and the Baker–Akhiezer vector Ψ
- Dubrovin equations for auxiliary divisors
- trace formulas for u, v, u^*, v^*
- the algebro-geometric initial value problem

Section 4.4.

- theta function representations for ϕ, ψ_1, and u, v, u^*, v^*

This chapter relies on terminology and notions developed in connection with compact Riemann surfaces. A brief summary of key results as well as definitions of some of the main quantities can be found in Appendices A, C, and F.

4.2 The Classical Massive Thirring System, Recursion Relations, and Hyperelliptic Curves

In this section we provide the zero-curvature setup for the classical massive Thirring system by developing a polynomial recursion relation formalism. Moreover, we introduce the underlying hyperelliptic curve $\mathcal{K}_n$ needed subsequently in the construction of algebro-geometric solutions of the Thirring system.

Throughout this section we suppose the following hypothesis.

Hypothesis 4.1 *Suppose* $u, v, u^*, v^* \colon \mathbb{R}^2 \to \mathbb{C}$ *satisfy*[1]

$$\begin{aligned}
&u(\,\cdot\,,t), u^*(\,\cdot\,,t) \in C^1(\mathbb{R}), \quad v(\,\cdot\,,t), v^*(\,\cdot\,,t) \in C^\infty(\mathbb{R}), \quad t \in \mathbb{R},\\
&u(x,\,\cdot\,), u^*(x,\,\cdot\,) \in C(\mathbb{R}), \quad v(x,\,\cdot\,), v^*(x,\,\cdot\,) \in C^1(\mathbb{R}), \quad x \in \mathbb{R},\\
&u(x,t) \neq 0, \quad u^*(x,t) \neq 0, \quad v(x,t) \neq 0, \quad v^*(x,t) \neq 0, \ (x,t) \in \mathbb{R}^2.
\end{aligned} \tag{4.2}$$

To set up a zero-curvature formalism for the classical massive Thirring system, one can proceed as follows. One defines recursion relations for $\{f_\ell\}_{\ell\in\mathbb{N}_0}$, $\{g_\ell\}_{\ell\in\mathbb{N}_0}$, and $\{h_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$f_{-1} = 0, \quad g_0 = 1, \quad h_{-1} = 0, \tag{4.3}$$

$$f_\ell = (2i)^{-1} f_{\ell-1,x} + vv^* f_{\ell-1} - 2v g_\ell, \quad \ell \in \mathbb{N}_0, \tag{4.4}$$

$$g_{\ell,x} = 2iv^* f_\ell + 2iv h_\ell, \quad \ell \in \mathbb{N}_0, \tag{4.5}$$

$$h_\ell = -(2i)^{-1} h_{\ell-1,x} + vv^* h_{\ell-1} + 2v^* g_\ell, \quad \ell \in \mathbb{N}_0, \tag{4.6}$$

Manipulating (4.3)–(4.6), one can replace (4.5) by

$$g_{\ell,x} = v^* f_{\ell-1,x} - v h_{\ell-1,x} + 2iv(v^*)^2 f_{\ell-1} + 2iv^2 v^* h_{\ell-1}, \quad \ell \in \mathbb{N}_0. \tag{4.7}$$

[1] Again one could assume that for fixed $t \in \mathbb{R}$, $u(\,\cdot\,,t)$, $v(\,\cdot\,,t)$, $u^*(\,\cdot\,,t)$, $v^*(\,\cdot\,,t)$ are meromorphic, etc.

Explicitly, the first few coefficients read

$$
\begin{aligned}
f_0 &= -2v, \\
f_1 &= iv_x + 2v^2v^* + c_1(-2v), \quad \text{etc.}, \\
g_0 &= 1, \\
g_1 &= -2vv^* + c_1, \quad \text{etc.}, \\
h_0 &= 2v^*, \\
h_1 &= iv_x^* - 2v(v^*)^2 + c_1 2v^*, \quad \text{etc.},
\end{aligned} \tag{4.8}
$$

where $\{c_\ell\}_{\ell\in\mathbb{N}} \subset \mathbb{C}$ denote integration constants. For subsequent use we also introduce the corresponding homogeneous coefficients $\hat{f}_\ell$, $\hat{g}_\ell$, and $\hat{h}_\ell$ defined by the vanishing of the integration constants c_k for $k = 1, \dots, \ell$

$$\hat{f}_0 = f_0 = -2v, \quad \hat{f}_\ell = f_\ell\big|_{c_k=0,\ k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{4.9}$$

$$\hat{g}_0 = g_0 = 1, \quad \hat{g}_\ell = g_\ell\big|_{c_k=0,\ k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{4.10}$$

$$\hat{h}_0 = h_0 = 2v^*, \quad \hat{h}_\ell = h_\ell\big|_{c_k=0,\ k=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{4.11}$$

One then obtains

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{f}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{h}_k, \quad g_\ell = \sum_{k=0}^{\ell} c_{\ell-k}\hat{g}_k, \quad \ell \in \mathbb{N}_0,$$

defining

$$c_0 = 1.$$

Remark 4.2 Using the nonlinear recursions (D.43) and (D.44) in Theorem D.5, one infers inductively that all homogeneous elements $\hat{f}_\ell$, $\hat{h}_\ell$ (and hence all f_ℓ and h_ℓ), $\ell \in \mathbb{N}_0$, are differential polynomials in v and v^*, that is, polynomials with respect to v and v^* and (some of) their x-derivatives. By (4.5), $g_{\ell,x}$ are also differential polynomials in v and v^*, and by (4.4) (respectively (4.6)) the same applies to vg_ℓ (respectively v^*g_ℓ). Combining these facts readily proves that g_ℓ, $\ell \in \mathbb{N}_0$, are differential polynomials in v and v^*.

Next, one introduces the 2×2 matrices

$$U(\xi) = i\begin{pmatrix} z - vv^* & 2\xi v \\ 2\xi v^* & -z + vv^* \end{pmatrix}, \tag{4.12}$$

$$V_{n+1}(\xi) = i\begin{pmatrix} -G_{n+1}(z) & \xi F_n(z) \\ -\xi H_n(z) & G_{n+1}(z) \end{pmatrix}, \quad n \in \mathbb{N}_0, \tag{4.13}$$

$$\xi \in \mathbb{C} \setminus \{0\},\ z = \xi^2$$

assuming F_n, H_n, and G_{n+1} to be polynomials of degree n and $n+1$ with respect to z. Postulating the zero-curvature representation

$$-V_{n+1,x} + [U, V_{n+1}] = 0, \tag{4.14}$$

one finds

$$F_{n,x} = -2i(vv^* - z)F_n + 4ivG_{n+1}, \tag{4.15}$$

$$G_{n+1,x} = 2iz(v^*F_n + vH_n), \tag{4.16}$$

$$H_{n,x} = 2i(vv^* - z)H_n + 4iv^*G_{n+1}. \tag{4.17}$$

By (4.15)–(4.17), one infers that

$$\big(G_{n+1}^2 - zF_nH_n\big)_x = 0$$

and hence

$$G_{n+1}^2 - zF_nH_n = R_{2n+2}, \tag{4.18}$$

where the integration constant R_{2n+2} is a monic polynomial of degree $2n+2$, that is,

$$R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}, \tag{4.19}$$

since we chose $g_0 = 1$. Moreover, (4.18) implies

$$G_{n+1}(0)^2 = \prod_{m=0}^{2n+1} E_m,$$

and we subsequently choose

$$G_{n+1}(0) \neq 0, \text{ that is, } E_m \neq 0, m = 0, \dots, 2n+1. \tag{4.20}$$

The identity (4.18) then yields for the characteristic equation[1] of iV_{n+1}

$$\begin{aligned}\det(yI_2 - iV_{n+1}(z)) &= y^2 - \det(V_{n+1}(z)) \\ &= y^2 - G_{n+1}(z)^2 + zF_n(z)H_n(z) = y^2 - R_{2n+2}(z) = 0.\end{aligned}$$

This naturally leads to a hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part), where

$$\mathcal{K}_n \colon \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0. \tag{4.21}$$

To establish the connection between the zero-curvature formalism and the recursion relations (4.3)–(4.6) we now make the following polynomial ansatz with

[1] I_2 denotes the identity matrix in $\mathbb{C}^2$.

respect to the spectral parameter z,

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell = f_0 \prod_{j=1}^{n} (z - \mu_j), \tag{4.22}$$

$$G_{n+1}(z) = \sum_{\ell=0}^{n+1} g_{n+1-\ell} z^\ell, \tag{4.23}$$

$$H_n(z) = \sum_{\ell=0}^{n} h_{n-\ell} z^\ell = h_0 \prod_{j=1}^{n} (z - \nu_j). \tag{4.24}$$

Imposing equations (4.15)–(4.17) then amounts to the additional constraints

$$f_{n+1} = 0, \quad g_{n+1,x} = 0, \quad h_{n+1} = 0$$

together with the recursion (4.3)–(4.6), respectively (4.7), for $\ell = 0, \dots, n$. Equations (4.15)–(4.17) permit one to derive differential equations for F_n and H_n separately. One obtains

$$\begin{aligned} &F_{n,xx} F_n - 2^{-1} F_{n,x}^2 - (v_x/v) F_{n,x} F_n \\ &\quad + 2\big(z^2 + 2zvv^* + iz(v_x/v) + v^2(v^*)^2 + ivv_x^*\big) F_n^2 = 8v^2 R_{2n+2}, \end{aligned} \tag{4.25}$$

$$\begin{aligned} &H_{n,xx} H_n - 2^{-1} H_{n,x}^2 - (v_x^*/v^*) H_{n,x} H_n \\ &\quad + 2\big(z^2 + 2zvv^* - iz(v_x^*/v^*) + v^2(v^*)^2 - iv_x v^*\big) H_n^2 = 8(v^*)^2 R_{2n+2}. \end{aligned} \tag{4.26}$$

Equations (4.25) and (4.26) can be used to derive nonlinear recursion relations for the homogeneous coefficients $\hat{f}_\ell$, g_ℓ, and $\hat{h}_\ell$ (i.e., the ones satisfying (4.9)–(4.11) in the case of vanishing integration constants), as proved in Theorem D.5 in Appendix D. Moreover, equations (4.25) and (4.26) also yield a proof that f_ℓ, g_ℓ, and h_ℓ are differential polynomials in v, v^* (cf. Remark 4.2). In addition, as proven in Theorem D.5, (4.25) leads to an explicit determination of the integration constants $c_1, \dots, c_n$ in F_n in terms of the zeros $E_0, \dots, E_{2n+1}$ of the associated polynomial R_{2n+2} in (4.19). In fact, one can prove (cf. (D.45))

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n, \tag{4.27}$$

where

$$\begin{aligned} &c_0(\underline{E}) = 1, \\ &c_k(\underline{E}) \\ &\quad = \sum_{\substack{j_0, \dots, j_{2n+1} = 0 \\ j_0 + \dots + j_{2n+1} = k}}^{k} \frac{(2j_0)! \cdots (2j_{2n+1})!}{2^{2k} (j_0!)^2 \cdots (j_{2n+1}!)^2 (2j_0 - 1) \cdots (2j_{2n+1} - 1)} E_0^{j_0} \cdots E_{2n+1}^{j_{2n+1}}, \\ &\qquad\qquad k = 1, \dots, n+1. \end{aligned} \tag{4.28}$$

With the introduction of the 2×2 matrix

$$\widetilde{V}(\xi) = i \begin{pmatrix} z^{-1} - uu^* & 2\xi^{-1}u \\ 2\xi^{-1}u^* & -z^{-1} + uu^* \end{pmatrix}, \tag{4.29}$$

the time-dependent zero-curvature condition equals

$$U_t - \widetilde{V}_x + [U, \widetilde{V}] = 0, \tag{4.30}$$

which yields the first-order system

$$\begin{aligned} -iu_x + 2v + 2vv^*u &= 0, \\ iu_x^* + 2v^* + 2vv^*u^* &= 0, \\ -iv_t + 2u + 2uu^*v &= 0, \\ iv_t^* + 2u^* + 2uu^*v^* &= 0. \end{aligned} \tag{4.31}$$

One observes that (4.31) implies the relation

$$\big(uu^*\big)_x + \big(vv^*\big)_t = 0.$$

Equations (4.31) represent the classical massive Thirring system in light cone coordinates. It should be emphasized that the original Thirring model equations are given by

$$\begin{aligned} -iu_x + 2v + 2|v|^2u &= 0, \\ -iv_t + 2u + 2|u|^2v &= 0. \end{aligned} \tag{4.32}$$

In fact, equations (4.32) result from the system (4.31) by imposing the constraints

$$u^* = \overline{u}, \quad v^* = \overline{v}, \tag{4.33}$$

where the bar denotes the operation of complex conjugation. Hence the system (4.31) can be viewed as a complexified classical massive Thirring model. For most of this chapter, however, we will not impose the constraints (4.33) but rather study the system (4.31).

Remark 4.3 The zero-curvature formalism for the classical massive Thirring system markedly differs from that of the KdV, AKNS, CH, and sGmKdV equations in the following sense: In the KdV and AKNS cases, U coincides with V_1; in the CH case, U coincides with V_0. Similarly, $\widetilde{V}_{r+1}$ (respectively $\widetilde{V}_r$) are constructed as V_{r+1} (respectively V_r), the only difference being a priori different sets of integration constants $\tilde{c}_\ell$ and c_ℓ. The sGmKdV case already deviates from this scheme since U differs from all V_n, but $\tilde{V}_r$ is still constructed like V_r (apart from independent sets of integration constants $\tilde{c}_\ell$ and c_ℓ). The Thirring system finally shows one additional difference in the sense that U differs from all V_{n+1} but in addition $\widetilde{V}$ also differs from U and is constructed differently from all the V_{n+1}. The latter property complicates the construction of a classical massive Thirring hierarchy, and hence we omit its discussion.

The stationary Thirring system, characterized by $v_t = v_t^* = 0$, reduces to

$$\begin{aligned}-iu_x + 2v + 2vv^*u &= 0,\\ iu_x^* + 2v^* + 2vv^*u^* &= 0,\\ 2u + 2uu^*v &= 0,\\ 2u^* + 2uu^*v^* &= 0.\end{aligned}$$

Hence, one either obtains the trivial solution

$$u = v = u^* = v^* = 0$$

or else constant solutions of the type

$$u = c, \quad v = -1/c^*, \quad u^* = c^*, \quad v^* = -1/c^*, \quad c, c^* \in \mathbb{C} \setminus \{0\}.$$

Thus, we will ignore this special case in the following.

Finally, we note the elementary fact that the Thirring system (4.31) is invariant under the scaling transformation

$$(u, v, u^*, v^*) \to (Au, Av, A^{-1}u^*, A^{-1}v^*), \quad A \in \mathbb{C} \setminus \{0\}. \tag{4.34}$$

In the special case of the classical massive Thirring model (4.32), where $u^* = \overline{u}$, $v^* = \overline{v}$, A in (4.34) is further constrained by

$$|A| = 1. \tag{4.35}$$

Remark 4.4 If $v^* \equiv 0$, then $H_n \equiv 0$. The recursion (4.5) yields that G_{n+1} is constant in x in this case. The Thirring equations (4.31) reduce to

$$-iu_x + 2v = 0, \quad -iv_t + 2u = 0. \tag{4.36}$$

In particular, $u^* \equiv 0$. These equations can be obtained directly from the zero-curvature relation

$$U_t - \widetilde{V}_x + [U, \widetilde{V}] = 0$$

for matrices[1]

$$U(z) = i\begin{pmatrix} z & 2v \\ 0 & -z\end{pmatrix}, \quad \widetilde{V}(z) = i\begin{pmatrix} z^{-1} & 2u \\ 0 & -z^{-1}\end{pmatrix}.$$

The hyperelliptic curve is determined by

$$R_{2n+2}(z) = G_{n+1}(z)^2 = \prod_{\ell=0}^{n}(z - \widetilde{E}_\ell)^2, \quad \{\widetilde{E}_\ell\}_{\ell=0,\dots,n} \subseteq \mathbb{C},$$

making the curve singular. Similarly, one could start with $u^* \equiv 0$, then $F_n \equiv 0$, etc., and one again obtains (4.36).

[1] One can put $\xi = 1$ in (4.12), (4.29).

4.3 The Basic Algebro-Geometric Formalism

This section is devoted to a detailed study of the algebro-geometric setup for the classical massive Thirring system. Our principal tools are derived from combining the polynomial recursion formalism introduced in Section 4.2 and a fundamental meromorphic function ϕ on a hyperelliptic curve $\mathcal{K}_n$. With the help of ϕ we study the Baker–Akhiezer vector Ψ, Dubrovin-type equations governing the motion of auxiliary divisors on $\mathcal{K}_n$ and trace formulas. We also discuss the algebro-geometric initial value problem of constructing u, v, u^*, v^* from the Dubrovin equations and auxiliary divisors as initial data.

We recall the hyperelliptic curve

$$\mathcal{K}_n \colon \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0,$$
$$R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C} \setminus \{0\}, \tag{4.37}$$

as introduced in (4.21). The curve $\mathcal{K}_n$ is compactified by joining two points at infinity, $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_n$. Points P on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = 0$. The complex structure on $\mathcal{K}_n$ is then defined in the usual way (see Appendix C). Hence, $\mathcal{K}_n$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) in a standard manner.

We also emphasize that by fixing the curve $\mathcal{K}_n$ (i.e., by fixing $E_0, \dots, E_{2n+1}$), the integration constants $c_1, \dots, c_n$ in f_n are uniquely determined, as is clear from (4.27), (4.28), which establish the integration constants c_ℓ as symmetric functions of $E_0, \dots, E_{2n+1}$.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$. (The trivial case $n = 0$ is explicitly discussed in Example 4.31.

Next, we define the fundamental meromorphic function $\phi(\,\cdot\,, x, t)$ on $\mathcal{K}_n$ by

$$\phi(P, x, t) = \frac{y + G_{n+1}(z, x, t)}{F_n(z, x, t)} \tag{4.38}$$
$$= \frac{-zH_n(z, x, t)}{y - G_{n+1}(z, x, t)}, \tag{4.39}$$
$$P = (z, y) \in \mathcal{K}_n,\ (x, t) \in \mathbb{R}^2,$$

where we used (4.18) to obtain (4.39). In addition, we introduce

$$\hat{\mu}_j(x, t) = (\mu_j(x, t), G_{n+1}(\mu_j(x, t), x, t)) \in \mathcal{K}_n, \quad j = 1, \dots, n,\ (x, t) \in \mathbb{R}^2, \tag{4.40}$$
$$\hat{\nu}_j(x, t) = (\nu_j(x, t), -G_{n+1}(\nu_j(x, t), x, t)) \in \mathcal{K}_n, \quad j = 1, \dots, n,\ (x, t) \in \mathbb{R}^2, \tag{4.41}$$

lifting μ_j and ν_j to $\mathcal{K}_n$, and

$$P_{0,\pm} = (0, \pm G_{n+1}(0)) = (0, \pm g_{n+1}) \in \mathcal{K}_n, \tag{4.42}$$

where

$$y(P_{0,\pm}) = \pm g_{n+1}, \quad g_{n+1}^2 = \prod_{m=0}^{2n+1} E_m. \tag{4.43}$$

We emphasize that $P_{0,\pm}$ and $P_{\infty_\pm}$ are not necessarily on the same sheet of $\mathcal{K}_n$. The actual sheet on which $P_{0,\pm}$ lie depends on the sign of g_{n+1}. The branch of $y(\,\cdot\,)$ near $P_{\infty_\pm}$ is fixed according to

$$\lim_{\substack{|z(P)|\to\infty \\ P\to P_{\infty_\pm}}} \frac{y(P)}{G_{n+1}(z(P),x,t)} = \lim_{\substack{|z(P)|\to\infty \\ P\to P_{\infty_\pm}}} \frac{y(P)}{z(P)^{n+1}} = \mp 1.$$

Next we collect a few characteristic properties of ϕ.

Lemma 4.5 *Assume* (4.2)*,* (4.14)*,* (4.30)*, and* (4.37) *hold. In addition, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ *and* $(x, t) \in \mathbb{R}^2$*. Then* ϕ *satisfies the Riccati-type equations*

$$\phi_x(P) + 2iv\phi(P)^2 + 2i(z - vv^*)\phi(P) = 2izv^*, \tag{4.44}$$

$$\phi_t(P) + 2iz^{-1}u\phi(P)^2 + 2i(z^{-1} - uu^*)\phi(P) = 2iu^*. \tag{4.45}$$

Moreover,

$$\phi(P)\phi(P^*) = z\frac{H_n(z)}{F_n(z)}, \tag{4.46}$$

$$\phi(P) + \phi(P^*) = 2\frac{G_{n+1}(z)}{F_n(z)}, \tag{4.47}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}. \tag{4.48}$$

Proof Equation (4.44) follows from (4.15)–(4.17), (4.18), and (4.38). A direct calculation, using (4.31) and (4.44), shows that

$$\big(\partial_x + 2i(2v\phi + z - vv^*)\big)\big(\phi_t + 2iz^{-1}u\phi^2 + 2i(z^{-1} - uu^*)\phi - 2iu^*\big) = 0,$$

which implies

$$\begin{aligned} &\phi_t + 2iz^{-1}u\phi^2 + 2i(z^{-1} - uu^*)\phi - 2iu^* \\ &\quad = C\exp\bigg(-2i\int^x dx\,(2v\phi + z - vv^*)\bigg). \end{aligned} \tag{4.49}$$

Since by (4.38) the left-hand side of (4.49) is meromorphic near $P_{\infty_\pm}$, whereas the right-hand side has an essential singularity at $P_{\infty_\pm}$ unless $C = 0$, one infers that (4.45) holds. Relations (4.46)–(4.48) are obvious from (4.18), (4.38), and (4.39). $\square$

Next we determine the time evolution of F_n, G_{n+1}, and H_n.

Lemma 4.6 *Assume* (4.2), (4.14), (4.30), *and* (4.37) *hold. Then,*

$$F_{n,t} = -2i(uu^* - z^{-1})F_n + 4iz^{-1}uG_{n+1}, \tag{4.50}$$

$$G_{n+1,t} = 2i(u^*F_n + uH_n), \tag{4.51}$$

$$H_{n,t} = 2i(uu^* - z^{-1})H_n + 4iz^{-1}u^*G_{n+1}. \tag{4.52}$$

Equations (4.50)–(4.52) *are equivalent to*

$$-V_{n+1,t} + [\widetilde{V}, V_{n+1}] = 0. \tag{4.53}$$

Proof To prove (4.50), we note that (4.48) implies

$$\big(\phi(P) - \phi(P^*)\big)_t = -2yF_n^{-2}F_{n,t}. \tag{4.54}$$

However, the left-hand side of (4.54) also equals

$$\phi(P)_t - \phi(P^*)_t = 2yF_n^{-2}\big(-4iuz^{-1}G_{n+1} - 2i(z^{-1} - uu^*)F_n\big) \tag{4.55}$$

by means of (4.45), (4.47), and (4.48). Combining (4.54) and (4.55) proves (4.50). Similarly, to prove (4.51), we use (4.47) to write

$$\big(\phi(P) + \phi(P^*)\big)_t = 2F_n^{-2}\big(G_{n+1,t}F_n - G_{n+1}F_{n,t}\big). \tag{4.56}$$

Now the left-hand side equals

$$\phi(P)_t + \phi(P^*)_t = -2G_{n+1}F_n^{-2}F_{n,t} + 4iF_n^{-1}\big(uH_n + u^*F_n\big) \tag{4.57}$$

by means of (4.45), (4.46), (4.47), and (4.50). Equations (4.56) and (4.57) yield (4.51). Finally, (4.52) follows by differentiating (4.18) with respect to t, using (4.50) and (4.51). □

By (4.50) and (4.52), a comparison of coefficients of z^{-1} then yields

$$f_n = -2g_{n+1}u, \tag{4.58}$$

$$h_n = 2g_{n+1}u^*. \tag{4.59}$$

At this point we briefly return to properties of ϕ. Due to the regularity assumptions (4.2) on u, v, u^*, and v^*, one infers analogous regularity properties of F_n, H_n, μ_j, and ν_k. Moreover, since $u \neq 0$, $u^* \neq 0$ by (4.2), equations (4.20), (4.22), (4.24), (4.58), and (4.59) imply

$$\mu_j(x,t), \nu_k(x,t) \neq 0, \;\; j,k = 1,\dots,n, \;\; (x,t) \in \mathbb{R}^2 \tag{4.60}$$

and

$$\mu_j, \nu_k \in C(\mathbb{R}^2) \tag{4.61}$$

with multiplicities (and appropriate renumbering) of the zeros of F_n and H_n taken into account. (Away from collisions of zeros, μ_j and ν_k are of course C^∞.) Combining (4.38), (4.39), (4.58), (4.59), (4.60), and (4.61), the divisor $(\phi(\,\cdot\,,x,t))$ of $\phi(\,\cdot\,,x,t)$ thus reads

$$(\phi(\,\cdot\,,x,t)) = \mathcal{D}_{P_{0,-}\underline{\hat{\nu}}(x,t)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(x,t)}, \quad (x,t)\in\mathbb{R}^2 \tag{4.62}$$

with

$$\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_n\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1,\dots,\hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n).$$

Given $\phi(\,\cdot\,,x,t)$, we can define the Baker–Akhiezer vector $\Psi(\,\cdot\,,\xi,x,x_0,t,t_0)$ by

$$\Psi(P,\xi,x,x_0,t,t_0) = \begin{pmatrix} \psi_1(P,x,x_0,t,t_0) \\ \psi_2(P,\xi,x,x_0,t,t_0)\end{pmatrix}, \tag{4.63}$$

$$\begin{aligned}
&P=(z,y)\in\mathcal{K}_n\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ z=\xi^2,\ (x,t),(x_0,t_0)\in\mathbb{R}^2,\\
&\psi_1(P,x,x_0,t,t_0)\\
&\quad=\exp\bigg(i\int_{t_0}^t ds\,\big(z^{-1}-u(x_0,s)u^*(x_0,s)+2z^{-1}u(x_0,s)\phi(P,x_0,s)\big)\\
&\qquad\qquad + i\int_{x_0}^x dx'\,\big(z-v(x',t)v^*(x',t)+2v(x',t)\phi(P,x',t)\big)\bigg), \qquad (4.64)\\
&\psi_2(P,\xi,x,x_0,t,t_0) = \xi^{-1}\psi_1(P,x,x_0,t,t_0)\phi(P,x,t). \qquad (4.65)
\end{aligned}$$

Properties of Ψ are summarized in the following result.

Lemma 4.7 *Assume* (4.2), (4.14), (4.30), *and* (4.37) *hold. In addition, let* $P=(z,y)\in\mathcal{K}_n\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\}$ *and* $(x,x_0,t,t_0)\in\mathbb{R}^4$. *Then* Ψ *satisfies*

$$\Psi_x(P,\xi) = U(\xi)\Psi(P,\xi), \tag{4.66}$$

$$\Psi_t(P,\xi) = \widetilde{V}(\xi)\Psi(P,\xi), \tag{4.67}$$

$$iy\Psi(P,\xi) = V_{n+1}(\xi)\Psi(P,\xi). \tag{4.68}$$

Moreover, if the zeros of $F_n(\,\cdot\,,x,t)$ *are all simple for* $(x,t)\in\Omega$, $\Omega\subseteq\mathbb{R}^2$ *open and connected, then* $\psi_1(\,\cdot\,,x,x_0,t,t_0)$ *is meromorphic on* $\mathcal{K}_n\setminus\{P_{\infty_+},P_{\infty_-}\}$ *for* $(x,t),(x_0,t_0)\in\Omega$. *In addition,*

$$\begin{aligned}
\psi_1(P,x,x_0,t,t_0) &= \bigg(\frac{F_n(z,x,t)}{F_n(z,x_0,t_0)}\bigg)^{1/2}\\
&\quad\times\exp\bigg(2i(y/z)\int_{t_0}^t ds\,u(x_0,s)F_n(z,x_0,s)^{-1} \qquad (4.69)\\
&\qquad + 2iy\int_{x_0}^x dx'\,v(x',t)F_n(z,x',t)^{-1}\bigg),
\end{aligned}$$

$$\psi_1(P,x,x_0,t,t_0)\psi_1(P^*,x,x_0,t,t_0) = \frac{F_n(z,x,t)}{F_n(z,x_0,t_0)}, \tag{4.70}$$

$$\psi_2(P,\xi,x,x_0,t,t_0)\psi_2(P^*,\xi,x,x_0,t,t_0) = \frac{H_n(z,x,t)}{F_n(z,x_0,t_0)}, \tag{4.71}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t,t_0)\psi_2(P^*,\xi,x,x_0,t,t_0) + \psi_1(P^*,x,x_0,t,t_0)\psi_2(P,\xi,x,x_0,t,t_0)\\ &\quad = 2\xi^{-1}\frac{G_{n+1}(z,x,t)}{F_n(z,x_0,t_0)},\end{aligned} \tag{4.72}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t,t_0)\psi_2(P^*,\xi,x,x_0,t,t_0) - \psi_1(P^*,x,x_0,t,t_0)\psi_2(P,\xi,x,x_0,t,t_0)\\ &\quad = -\frac{2\xi^{-1}y}{F_n(z,x_0,t_0)}.\end{aligned} \tag{4.73}$$

Proof Equations (4.66), (4.67) are verified using (4.15)–(4.17), (4.50)–(4.52), (4.44), (4.45), (4.64), and (4.65). Equation (4.68) follows by combining (4.13), (4.38), (4.39), (4.64), and (4.65). By (4.64), ψ_1 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, \hat{\mu}_1(x,t), \dots, \hat{\mu}_n(x,t)\}$. Since

$$\begin{aligned}2iv(x',t)\phi(P,x',t) &\underset{P\to\hat{\mu}_j(x',t)}{=} \partial_{x'}\ln\big(F_n(z,x',t)\big) + O(1)\\ &\qquad\qquad \text{as } z \to \mu_j(x',t),\\ 2iz^{-1}u(x_0,s)\phi(P,x_0,s) &\underset{P\to\hat{\mu}_j(x_0,s)}{=} \partial_s\ln\big(F_n(z,x_0,s)\big) + O(1)\\ &\qquad\qquad \text{as } z \to \mu_j(x_0,s),\end{aligned} \tag{4.74}$$

one infers that ψ_1 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ if the zeros of $F_n(\,\cdot\,,x,t)$ are all simple. This follows from (4.64) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j(x_0)$ of $\{\hat{\mu}_j(x_0,s) \in \mathcal{K}_n \mid (x_0,s) \in \Omega,\ s \in [t_0,t]\}$ such that $\hat{\mu}_k(x_0,s) \notin \mathcal{U}_j(x_0)$ for all $s \in [t_0,t]$ and all $k \in \{1,\dots,n\}\setminus\{j\}$ and simultaneously restricting P to a sufficiently small neighborhood $\mathcal{U}_j(t)$ of $\{\hat{\mu}_j(x',t) \in \mathcal{K}_n \mid (x',t) \in \Omega,\ x' \in [x_0,x]\}$ such that $\hat{\mu}_k(x',t) \notin \mathcal{U}_j(t)$ for all $x' \in [x_0,x]$ and all $k \in \{1,\dots,n\}\setminus\{j\}$. Equation (4.69) follows from (4.64) after replacing ϕ by the right-hand side of (4.38) and utilizing (4.15) in the x'-integral and (4.50) in the s-integral. Equations (4.70)–(4.73) immediately follow from (4.46)–(4.48) and (4.65). $\square$

Equations (4.70)–(4.73) show that the basic identity (4.18), $G_{n+1}^2 - zF_nH_n = R_{2n+2}$, is equivalent to the elementary fact

$$(\psi_{1,+}\psi_{2,-} + \psi_{1,-}\psi_{2,+})^2 - 4\psi_{1,+}\psi_{1,-}\psi_{2,+}\psi_{2,-} = (\psi_{1,+}\psi_{2,-} - \psi_{1,-}\psi_{2,+})^2 \tag{4.75}$$

identifying $\psi_1(P) = \psi_{1,+}$, $\psi_1(P^*) = \psi_{1,-}$, $\psi_2(P) = \psi_{2,+}$, $\psi_2(P^*) = \psi_{2,-}$. This provides the intimate link between our approach and the squared function systems also employed in the literature in connection with algebro-geometric solutions of the classical massive Thirring system.

Next we discuss the asymptotic behavior of $\phi(P,x,t)$ as $P \to P_{0,\pm}, P_{\infty_\pm}$ in some detail since this will turn out to be a crucial ingredient for the theta function representation to be derived in Section 4.4.

Lemma 4.8 *Assume* (4.2), (4.14), (4.30), *and* (4.37) *hold. In addition, let* $P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$. *Then,*

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} -v^{-1}\zeta^{-1} + (i/2)\big(v^{-1}\big)_x + O(\zeta) & \text{as } P \to P_{\infty_-}, \\ v^* + (i/2)v^*_x\zeta + O\big(\zeta^2\big) & \text{as } P \to P_{\infty_+}, \end{cases} \qquad \zeta = 1/z, \tag{4.76}$$

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} u^*\zeta + (i/2)u^*_t\zeta^2 + O(\zeta^3) & \text{as } P \to P_{0,-}, \\ -u^{-1} + (i/2)\big(u^{-1}\big)_t\zeta + O\big(\zeta^2\big) & \text{as } P \to P_{0,+}, \end{cases} \qquad \zeta = z. \tag{4.77}$$

Proof The existence of these asymptotic expansions in terms of local coordinates $\zeta = 1/z$ near $P_{\infty_\pm}$ and $\zeta = z$ near $P_{0,\pm}$ is clear from the explicit form of ϕ in (4.38). Insertion of the polynomials F_n, H_n, and G_{n+1} then, in principle, yields the explicit expansion coefficients in (4.76) and (4.77). However, this is a cumbersome procedure, especially with regard to the next-to-leading coefficients in (4.76) and (4.77). Much more efficient is the actual computation of these coefficients utilizing the Riccati-type equations (4.44) and (4.45). Indeed, inserting the ansatz

$$\phi \underset{z\to\infty}{=} z\phi_{-1} + \phi_0 + O(z^{-1})$$

into (4.44) and comparing the first two leading powers of z immediately yield the first line of (4.76). Similarly, the ansatz

$$\phi \underset{z\to\infty}{=} \phi_0 + \phi_1 z^{-1} + O(z^{-2})$$

inserted into (4.44) immediately produces the second line of (4.76). In exactly the same manner, inserting the ansatz

$$\phi \underset{z\to 0}{=} \phi_1 z + \phi_2 z^2 + O(z^3)$$

and the ansatz

$$\phi \underset{z\to 0}{=} \phi_0 + \phi_1 z + O(z^2)$$

into (4.45) immediately yields (4.77). $\square$

We follow up with a similar asymptotic analysis of $\Psi(P, \xi, x, x_0, t, t_0)$.

Lemma 4.9 *Assume* (4.2), (4.14), (4.53), *and* (4.37) *hold. In addition, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ *and* $(x, x_0, t, t_0) \in \mathbb{R}^4$. *Then,*

$$\begin{aligned}
\psi_1(P, x, x_0, t, t_0) &\underset{\zeta\to 0}{=} \exp\big(\mp i\zeta^{-1}(x - x_0)\big) \\
&\quad\times \big(\psi_{1,\infty_\mp,0}(x, x_0, t, t_0) + \zeta\,\psi_{1,\infty_\mp,1}(x, x_0, t, t_0) + O(\zeta^2)\big), \qquad (4.78)\\
&\qquad\qquad \text{as } P \to P_{\infty_\mp},\ \zeta = 1/z,\\
\psi_2(P, \xi, x, x_0, t, t_0) &\underset{\zeta\to 0}{=} \xi^{-1}\exp\big(\mp i\zeta^{-1}(x - x_0)\big) \\
&\quad\times \begin{cases} \zeta^{-1}\psi_{2,\infty_-,-1}(x, x_0, t, t_0) + \zeta\,\psi_{2,\infty_-,0}(x, x_0, t, t_0) + O(\zeta) \\ \psi_{2,\infty_+,0}(x, x_0, t, t_0) + \zeta\,\psi_{2,\infty_+,1}(x, x_0, t, t_0) + O(\zeta^2) \end{cases} \qquad (4.79)\\
&\qquad\qquad \text{as } P \to P_{\infty_\mp},\ \zeta = 1/z,\\
\psi_1(P, x, x_0, t, t_0) &\underset{\zeta\to 0}{=} \exp\big(\pm i\zeta^{-1}(t - t_0)\big) \\
&\quad\times \big(\psi_{1,0,\mp,0}(x, x_0, t, t_0) + \zeta\,\psi_{1,0,\mp,1}(x, x_0, t, t_0) + O(\zeta^2)\big), \qquad (4.80)\\
&\qquad\qquad \text{as } P \to P_{0,\mp},\ \zeta = z,\\
\psi_2(P, \xi, x, x_0, t, t_0) &\underset{\zeta\to 0}{=} \xi^{-1}\exp\big(\pm i\zeta^{-1}(x - x_0)\big) \\
&\quad\times \begin{cases} \zeta\,\psi_{2,0,-,1}(x, x_0, t, t_0) + \zeta^2\psi_{2,0,-,2}(x, x_0, t, t_0) + O(\zeta^3) \\ \psi_{2,0,+,0}(x, x_0, t, t_0) + \zeta\,\psi_{2,0,+,1}(x, x_0, t, t_0) + O(\zeta^2) \end{cases} \qquad (4.81)\\
&\qquad\qquad \text{as } P \to P_{0,\mp},\ \zeta = z
\end{aligned}$$

and

$$u = -\psi_{1,0,+,0}/\psi_{2,0,+,0}, \tag{4.82}$$

$$v = -\psi_{1,\infty_-,0}/\psi_{2,\infty_-,-1}, \tag{4.83}$$

$$u^* = \psi_{2,0,-,1}/\psi_{1,0,-,0}, \tag{4.84}$$

$$v^* = \psi_{2,\infty_+,0}/\psi_{1,\infty_+,0}, \tag{4.85}$$

as well as

$$\frac{\psi_{1,\infty_+,0,x}}{\psi_{1,\infty_+,0}} = -\frac{\psi_{2,\infty_-,-1,x}}{\psi_{2,\infty_-,-1}} = \frac{\psi_{2,0,+,0,x}}{\psi_{2,0,+,0}} = -\frac{\psi_{1,0,-,0,x}}{\psi_{1,0,-,0}} = ivv^*, \tag{4.86}$$

$$\frac{\psi_{2,\infty_-,-1,t}}{\psi_{2,\infty_-,-1}} = -\frac{\psi_{1,\infty_+,0,t}}{\psi_{1,\infty_+,0}} = \frac{\psi_{1,0,-,0,t}}{\psi_{1,0,-,0}} = -\frac{\psi_{2,0,+,0,t}}{\psi_{2,0,+,0}} = iuu^*. \tag{4.87}$$

Proof Equations (4.78) and (4.80) follow from (4.64) by noting that

$$i(z^{-1} - u(x_0, s)u^*(x_0, s)) + 2iz^{-1}u(x_0, s)\phi(P, x_0, s) \underset{z\to\infty}{=} O(1) \quad \text{as } P \to P_{\infty_\mp},$$

$$i(z - v(x, t)v^*(x, t)) + 2iv(x, t)\phi(P, x, t) \underset{z\to\infty}{=} \mp iz + O(1) \text{ as } P \to P_{\infty_\mp},$$

$$i(z^{-1} - u(x_0, s)u^*(x_0, s)) + 2iz^{-1}u(x_0, s)\phi(P, x_0, s) \underset{z\to 0}{=} \pm iz^{-1} + O(1) \quad \text{as } P \to P_{0,\mp},$$

$$i(z - v(x, t)v^*(x, t)) + 2iv(x, t)\phi(P, x, t) \underset{z\to 0}{=} O(1) \text{ as } P \to P_{0,\mp}.$$

Similarly, (4.79) and (4.81) follow from (4.65), (4.76), (4.77), (4.78), and (4.80). Equations (4.65), (4.76), (4.77), (4.78), and (4.80) imply (4.82)–(4.85). Insertion of (4.78)–(4.81) into $\Psi_x = U\Psi$ and $\Psi_t = \widetilde{V}\Psi$, collecting leading and next-to-leading order terms (utilizing (4.82)–(4.85)) then yields (4.86) and (4.87). □

Although (4.79), (4.81), and (4.82)–(4.87) are not needed in our derivation of the theta function representations of u, v, u^*, v^* in Section 4.4, they play a crucial role in the context of isospectral set considerations (cf. Lemma 4.29).

In some of the following considerations it is appropriate to assume that the affine part of $\mathcal{K}_n$ is nonsingular, and hence we then assume

$$E_m \neq E_{m'} \text{ for } m \neq m', \ m, m' = 0, \dots, 2n+1 \tag{4.88}$$

in addition to (4.37).

Next, we turn to Dubrovin-type equations for $\mu_j, \nu_j, j = 1, \dots, n$, that is, we derive the nonlinear first-order system of partial differential equations governing their (x, t)-dynamics.

Lemma 4.10 *Assume that* (4.2) *and* (4.14), (4.30) *hold on an open connected set* $\widetilde{\Omega}_\mu \subseteq \mathbb{R}^2$, *suppose* (4.37), *and assume that the zeros* μ_j, $j = 1, \dots, n$, *of* $F_n(\,\cdot\,)$ *remain distinct on* $\widetilde{\Omega}_\mu$. *Then* $\{\hat{\mu}_j\}_{j=1,\dots,n}$, *defined by* (4.40), *satisfies the following first-order system of differential equations on* $\widetilde{\Omega}_\mu$

$$\mu_{j,x} = 2iy(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k)^{-1}, \tag{4.89}$$

$$\mu_{j,t} = (-1)^n g_{n+1}^{-1}\left(\prod_{\substack{\ell=1\\\ell\neq j}}^{n}\mu_\ell\right)2iy(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, n. \tag{4.90}$$

Next, assume the affine part of $\mathcal{K}_n$ *to be nonsingular and introduce the initial* *condition*

$$\{\hat{\mu}_j(x_0, t_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{4.91}$$

for some $(x_0, t_0) \in \mathbb{R}^2$*, where* $\mu_j(x_0, t_0) \neq 0,\ j = 1, \dots, n$*, are assumed to be distinct. Then there exists an open and connected set* $\Omega_\mu \subseteq \mathbb{R}^2$ *with* $(x_0, t_0) \in \Omega_\mu$*, such that the initial value problem* (4.89)–(4.91) *has a unique solution* $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ *satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{4.92}$$

and $\mu_j,\ j = 1, \dots, n$*, remain distinct and nonzero on* Ω_μ*.*
For the zeros $\{\nu_j\}_{j=1,\dots,n}$ *of* $H_n(\,\cdot\,)$ *identical statements hold with* μ *and* $\widetilde{\Omega}_\mu$ *replaced by* ν *and* $\widetilde{\Omega}_\nu$*, etc. In particular,* $\{\hat{\nu}_j\}_{j=1,\dots,n}$*, defined by* (4.41)*, satisfies the system*

$$\nu_{j,x} = 2iy(\hat{\nu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\nu_j - \nu_k)^{-1}, \tag{4.93}$$

$$\nu_{j,t} = (-1)^n g_{n+1}^{-1}\left(\prod_{\substack{\ell=1\\\ell\neq j}}^{n}\nu_\ell\right)2iy(\hat{\nu}_j)\prod_{\substack{k=1\\k\neq j}}^{n}(\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, n. \tag{4.94}$$

Proof Equations (4.15), (4.22), and (4.40) imply

$$F_{n,x}(\mu_j) = -\mu_{j,x} f_0 \prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k) = 4ivG_{n+1}(\mu_j) = 4ivy(\hat{\mu}_j).$$

Using $f_0 = -2v$ by (4.8), one concludes (4.89). Similarly, one derives from (4.22), (4.50), and (4.40),

$$F_{n,t}(\mu_j) = -\mu_{j,t} f_0 \prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k) = (4iu/\mu_j)G_{n+1}(\mu_j) = (4iu/\mu_j)y(\hat{\mu}_j).$$

Since

$$-4iu/f_0 = 2if_n/(f_0 g_{n+1}) = 2i(-1)^n\left(\prod_{k=1}^{n}\mu_k\right)/g_{n+1}$$

by (4.8), (4.58), and (4.22), one arrives at (4.90). Equations (4.93) and (4.94) are derived analogously. To conclude (4.92), one invokes the charts (B.3)–(B.6) and (B.12)–(B.15). In particular, the only nontrivial issue to investigate concerns the case in which $\hat{\mu}_j(x, t)$ hits one of the branch points $(E_m, 0) \in \mathcal{B}(\mathcal{K}_n)$, and hence the right-hand sides of (4.89) and (4.90) vanish. Thus, we suppose that

$$\mu_{j_0}(x, t) \to E_{m_0} \text{ as } (x, t) \to (\tilde{x}_0, \tilde{t}_0)$$

for some $j_0 \in \{1, \dots, n\}$, $m_0 \in \{0, \dots, 2n+1\}$, and some $(\tilde{x}_0, \tilde{t}_0) \in \Omega_\mu$. By introducing

$$\zeta_{j_0}(x,t) = \sigma(\mu_{j_0}(x,t) - E_{m_0})^{1/2}, \ \sigma = \pm 1, \quad \mu_{j_0}(x,t) = E_{m_0} + \zeta_{j_0}(x,t)^2$$

for (x,t) in an open neighborhood of $(\tilde{x}_0, \tilde{t}_0) \in \Omega_\mu$, equations (4.89) and (4.90) become

$$\begin{aligned}
\zeta_{j_0,x}(x,t) \underset{(x,t)\to(\tilde{x}_0,\tilde{t}_0)}{=} c(\sigma) &\left(\prod_{\substack{m=0\\ m\neq m_0}}^{2n+1} \big(E_{m_0} - E_m\big) \right)^{1/2} \\
&\times \left(\prod_{\substack{k=1\\ k\neq j_0}}^{n} \big(E_{m_0} - \mu_k(x,t)\big)^{-1} \right) \big(1 + O(\zeta_{j_0}(x,t)^2)\big),
\end{aligned}$$

$$\begin{aligned}
\zeta_{j_0,t}(x,t) &\underset{(x,t)\to(\tilde{x}_0,\tilde{t}_0)}{=} c(\sigma)(-1)^n g_{n+1}^{-1} \left(\prod_{\substack{m=0\\ m\neq m_0}}^{2n+1} \big(E_{m_0} - E_m\big) \right)^{1/2} \\
&\times \left(\prod_{\substack{k=1\\ k\neq j_0}}^{n} \big(E_{m_0} - \mu_k(x,t)\big)^{-1} \right) \left(\prod_{\substack{\ell=1\\ \ell\neq j_0}}^{n} \mu_\ell(x,t) \right) \big(1 + O(\zeta_{j_0}(x,t)^2)\big)
\end{aligned}$$

for some $|c(\sigma)| = 1$, and one arrives at (4.92). $\square$

Combining the polynomial approach in Section 4.2 with (4.22), (4.24), we next derive a few trace formulas involving u, v, u^*, v^* and some of their x-derivatives in terms of symmetric functions of the zeros μ_j and ν_j of F_n and H_n, respectively.

Lemma 4.11 *Assume* (4.2)*,* (4.14)*,* (4.30)*, and* (4.37) *hold. Then,*

$$i\frac{v_x}{v} + 2vv^* - 2c_1 = 2\sum_{j=1}^{n} \mu_j, \tag{4.95}$$

$$i\frac{v_x}{v} - 2vv^* = -i\sum_{j=1}^{n} \frac{\mu_{j,x}}{\mu_j} + \frac{2(-1)^n g_{n+1}}{\prod_{j=1}^n \mu_j}, \tag{4.96}$$

$$\frac{v}{u} = \frac{(-1)^n g_{n+1}}{\prod_{j=1}^n \mu_j}, \tag{4.97}$$

$$i\frac{v_x^*}{v^*} - 2vv^* + 2c_1 = -2\sum_{j=1}^{n} \nu_j, \tag{4.98}$$

$$i\frac{v_x^*}{v^*} + 2vv^* = -i\sum_{j=1}^{n} \frac{\nu_{j,x}}{\nu_j} - \frac{2(-1)^n g_{n+1}}{\prod_{j=1}^n \nu_j}, \tag{4.99}$$

$$\frac{v^*}{u^*} = \frac{(-1)^n g_{n+1}}{\prod_{j=1}^n \nu_j}, \tag{4.100}$$

$$\partial_x \ln\big(vv^*\big) = -2i \sum_{j=1}^n \big(\mu_j - \nu_j\big), \tag{4.101}$$

$$\partial_x \ln\big(uu^*\big) = -2i(-1)^n g_{n+1}\bigg(\prod_{j=1}^n \mu_j^{-1} - \prod_{j=1}^n \nu_j^{-1}\bigg), \tag{4.102}$$

$$i\frac{u_t}{u} + 2uu^* - \frac{p_1}{g_{n+1}^2} = 2\sum_{j=1}^n \mu_j^{-1}, \tag{4.103}$$

$$i\frac{u_t^*}{u^*} - 2uu^* + \frac{p_1}{g_{n+1}^2} = -2\sum_{j=1}^n \nu_j^{-1}, \tag{4.104}$$

$$\partial_t \ln\big(vv^*\big) = -2i\frac{(-1)^n}{g_{n+1}}\bigg(\prod_{j=1}^n \mu_j - \prod_{j=1}^n \nu_j\bigg), \tag{4.105}$$

$$\partial_t \ln\big(uu^*\big) = -2i \sum_{j=1}^n \big(\mu_j^{-1} - \nu_j^{-1}\big). \tag{4.106}$$

Here

$$c_1 = -\frac{1}{2}\sum_{m=0}^{2n+1} E_m, \quad p_1 = -\bigg(\prod_{m=0}^{2n+1} E_m\bigg)\sum_{m=0}^{2n+1} E_m^{-1}$$

(cf. (D.6) *and* (D.45)*), and g_{n+1} has been introduced in* (4.42).

Proof Equations (4.95) and (4.98) follow from (4.22), (4.24) by comparing powers of z^n and z^{n-1}, using (4.8). Equations (4.96) and (4.99) follow from taking $z = 0$ in (4.15) and (4.17), using (4.8), and (4.22), (4.24). Next, (4.97) and (4.100) follow from $f_n = f_0 \prod_{j=1}^n(-\mu_j)$, $h_n = h_0 \prod_{j=1}^n(-\nu_j)$ and from (4.58) and (4.59). Adding (4.95) and (4.96) yields (4.101). Divide the first two equations in (4.31) by u and u^*, respectively, and add the results. Applying (4.97) and (4.100) yields (4.102). A similar argument using the two last equations in (4.31) results in (4.105). Taking the $z \to 0$ limit in (4.50) and using (4.58), (4.18) (the first-order term in z) and (4.22), one arrives at

$$\begin{aligned}
i\frac{u_t}{u} &= i\frac{f_{n,t}}{f_n} = \frac{i}{f_n}(-2iuu^* f_n + 2if_{n-1} + 4iug_n) \\
&= 2uu^* - 2\frac{f_{n-1}}{f_n} - \frac{4u}{f_n}\bigg(-2g_{n+1}uu^* + \frac{p_1}{2g_{n+1}}\bigg) \\
&= -2uu^* + \frac{p_1}{g_{n+1}^2} - 2\frac{f_{n-1}}{f_n},
\end{aligned}$$

which is (4.103). An analogous argument using H_n rather than F_n yields (4.104). Adding equations (4.103) and (4.104) results in (4.106). $\square$

Up to this point we assumed that u, v, u^*, v^* satisfy the zero-curvature equations (4.14) and (4.30), or equivalently, (4.15)–(4.17), (4.50)–(4.52) and, as a consequence, derived the corresponding algebro-geometric formalism. In the remainder of this section we will study the algebro-geometric initial value problem, that is, starting from the Dubrovin equations (4.89)–(4.91) and the trace formulas (4.95)–(4.97), derive (4.15)–(4.17), (4.50)–(4.52), and hence the zero-curvature equations (4.14) and (4.30) at least locally, that is, for $(x,t) \in \Omega$ for some open and connected set $\Omega \subseteq \mathbb{R}^2$.

We start with an elementary result extending the scaling transformation mentioned in (4.34).

Lemma 4.12 *Assume* (4.2)*, suppose that* u, v, u^*, v^* *satisfy the Thirring system* (4.31)*, and let* $(x,t,t_0) \in \mathbb{R}^3$*. Assume* $B(t) = A \exp\big(\int_{t_0}^t ds\, b(s)\big)$*, with* $b \in C(\mathbb{R})$*,* $A \in \mathbb{C} \setminus \{0\}$ *and consider the time-dependent scaling transformation*

$$(u, v, u^*, v^*) \to (\breve{u}, \breve{v}, \breve{u}^*, \breve{v}^*) = (Bu, Bv, B^{-1}u^*, B^{-1}v^*). \tag{4.107}$$

Then $\breve{u}, \breve{v}, \breve{u}^*, \breve{v}^*$ *satisfy the corresponding extended massive Thirring system*

$$\begin{aligned}
-i\breve{u}_x + 2\breve{v} + 2\breve{v}\breve{v}^*\breve{u} &= 0,\\
i\breve{u}^*_x + 2\breve{v}^* + 2\breve{v}\breve{v}^*\breve{u}^* &= 0,\\
-i\breve{v}_t + 2\breve{u} + 2\breve{u}\breve{u}^*\breve{v} + ib\breve{v} &= 0,\\
i\breve{v}^*_t + 2\breve{u}^* + 2\breve{u}\breve{u}^*\breve{v}^* - ib\breve{v}^* &= 0.
\end{aligned} \tag{4.108}$$

Proof It suffices to insert (4.107) into the system (4.31). □

In the special case in which $u^* = \overline{u}$, $v^* = \overline{v}$, the function B in Lemma 4.12 is further constrained by

$$|B(t)| = 1, \quad t \in \mathbb{R}.$$

Next we provide the basic setup for the algebro-geometric initial value problem. We start from the following assumptions.

Hypothesis 4.13 *Given the hyperelliptic curve* $\mathcal{K}_n$ *in* (4.37) *and a constant* $g_{n+1} \in \mathbb{C} \setminus \{0\}$ *with* $g_{n+1}^2 = \prod_{m=0}^{2n+1} E_m$*, consider the Dubrovin-type system of differential equations* (4.89), (4.90) *on* Ω_μ*, for some initial conditions* (4.91)*. Here* $\Omega_\mu \subseteq \mathbb{R}^2$ *is assumed to be open and connected and such that the projections* μ_j *of* $\hat{\mu}_j$ *onto* $\mathbb{C}$ *remain distinct and nonzero on* Ω_μ*, that is,*

$$\begin{aligned}
&\mu_j(x,t) \neq \mu_{j'}(x,t), \quad j \neq j',\ j, j' = 1, \dots, n,\ (x,t) \in \Omega_\mu,\\
&\mu_j(x,t) \neq 0, \quad j = 1, \dots, n,\ (x,t) \in \Omega_\mu.
\end{aligned} \tag{4.109}$$

Assuming Hypothesis 4.13 in the following, we will next define u, v, u^*, v^* and the polynomials F_n, G_{n+1}, H_n in steps (S1)–(S4) below.

(S1). Use the trace formulas (4.95)–(4.97) on Ω_μ, that is,

$$i\frac{v_x}{v} + 2vv^* - 2c_1 = 2\sum_{j=1}^n \mu_j, \tag{4.110}$$

$$i\frac{v_x}{v} - 2vv^* = -i\sum_{j=1}^n \frac{\mu_{j,x}}{\mu_j} + \frac{2(-1)^n g_{n+1}}{\prod_{j=1}^n \mu_j}, \tag{4.111}$$

$$u = (-1)^n g_{n+1}^{-1} v \prod_{j=1}^n \mu_j, \tag{4.112}$$

to define $u, v, v^* \in C^\infty(\Omega_\mu)$ up to a possibly t-dependent multiplicative factor according to the scale transformation described in Lemma 4.12. (More precisely, adding (4.110) and (4.111) yields a first-order differential equation for v. Given v, we obtain v^* and u from (4.110) and (4.112), respectively.) Moreover, v is nonvanishing on Ω_μ by adding (4.110) and (4.111), and hence u is also nonvanishing on Ω_μ by (4.112).

(S2). Define the polynomial F_n of degree n by

$$F_n(z) = -2v\prod_{j=1}^n (z-\mu_j) \text{ on } \mathbb{C}\times\Omega_\mu \tag{4.113}$$

and define the polynomial G_{n+1} of degree $n+1$ by

$$F_{n,x}(z) = -2i(vv^* - z)F_n(z) + 4ivG_{n+1}(z) \text{ on } \mathbb{C}\times\Omega_\mu. \tag{4.114}$$

Both F_n and G_{n+1} have $C^\infty(\Omega_\mu)$ coefficients. One then verifies from

$$2iy(\hat{\mu}_j) = \mu_{j,x}\prod_{\substack{k=1\\k\neq j}}^n (\mu_j - \mu_k) = \frac{F_{n,x}(\mu_j)}{2v}, \quad j=1,\dots,n$$

and (4.114) that

$$y(\hat{\mu}_j) = \frac{F_{n,x}(\mu_j)}{4iv} = G_{n+1}(\mu_j), \quad j=1,\dots,n \tag{4.115}$$

on Ω_μ, and hence

$$\big(G_{n+1}(z)^2 - R_{2n+2}(z)\big)\big|_{z=\mu_j} = 0, \quad j=1,\dots,n \tag{4.116}$$

on Ω_μ.

(S3). Taking $z = 0$ in (4.114), using (4.113), results in

$$\frac{2(-1)^n G_{n+1}(0)}{\prod_{j=1}^n \mu_j} = i\frac{v_x}{v} - 2vv^* + i\sum_{j=1}^n \frac{\mu_{j,x}}{\mu_j}$$

and hence a comparison with (4.111) yields

$$G_{n+1}(0, x, t) = g_{n+1}, \quad (x, t) \in \Omega_\mu, \tag{4.117}$$

and thus,

$$\big(G_{n+1}(z)^2 - R_{2n+2}(z)\big)\big|_{z=0} = 0 \text{ on } \Omega_\mu. \tag{4.118}$$

Because of (4.116) and (4.118) we can define a polynomial H_n of degree n with $C^\infty(\Omega_\mu)$ coefficients by

$$G_{n+1}(z)^2 - R_{2n+2}(z) = zF_n(z)H_n(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{4.119}$$

(S4). Given H_n, we finally define $u^* \in C^\infty(\Omega_\mu)$ by

$$u^* = \frac{H_n(0)}{2g_{n+1}}. \tag{4.120}$$

Again u^* is unique up to a possibly t-dependent factor in accordance with Lemma 4.12.

Now we are ready to prove that, starting from the Dubrovin equations (4.89)–(4.91) and the trace formulas (4.95)–(4.97) for u, v, u^*, v^*, one can derive (4.15)–(4.17), (4.50)–(4.52), and hence the zero-curvature equations (4.14) and (4.53), at least locally, that is, for $(x, t) \in \Omega$ for some open and connected set $\Omega \subseteq \mathbb{R}^2$. In particular u, v, u^*, v^* so constructed satisfy the classical massive Thirring system (4.31) (apart from some additional scaling terms). As pointed out in Remark 4.19, this amounts to solving the algebro-geometric initial value problem.

Theorem 4.14 *Fix $n \in \mathbb{N}$, assume Hypothesis* 4.13*, define u, v, u^*, v^* and the polynomials F_n, G_{n+1}, H_n as in (S1)–(S4), and let $(z, x, t) \in \mathbb{C} \times \Omega_\mu$. Then u and v are nonvanishing on Ω_μ and there exists a function $b \in C^\infty(\Omega_\mu)$, independent of x ($b_x|_{\Omega_\mu} = 0$), such that*

$$F_{n,x} = -2i(vv^* - z)F_n + 4ivG_{n+1}, \tag{4.121}$$

$$G_{n+1,x} = 2iz(v^*F_n + vH_n), \tag{4.122}$$

$$H_{n,x} = 2i(vv^* - z)H_n + 4iv^*G_{n+1}, \tag{4.123}$$

$$F_{n,t} = -2i(uu^* - z^{-1})F_n + bF_n + 4iz^{-1}uG_{n+1}, \tag{4.124}$$

$$G_{n+1,t} = 2i(u^*F_n + uH_n), \tag{4.125}$$

$$H_{n,t} = 2i(uu^* - z^{-1})H_n - bH_n + 4iz^{-1}u^*G_{n+1}. \tag{4.126}$$

In particular, u, v, u^*, v^* satisfy the extended massive Thirring system (4.108) *on* Ω_μ,

$$\begin{aligned} -iu_x + 2v + 2vv^*u &= 0, \\ iu_x^* + 2v^* + 2vv^*u^* &= 0, \\ -iv_t + 2u + 2uu^*v + ibv &= 0, \\ iv_t^* + 2u^* + 2uu^*v^* - ibv^* &= 0. \end{aligned} \tag{4.127}$$

Proof Define the polynomial P_n by

$$P_n(z) = 2izv^*F_n(z) + 2izvH_n(z) - G_{n+1,x}(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{4.128}$$

Using (4.115) and $2G_{n+1}G_{n+1,x} = z(F_{n,x}H_n + F_nH_{n,x})$ (by differentiating (4.119) with respect to x), one then computes

$$\begin{aligned} G_{n+1}(\mu_j)P_n(\mu_j) &= 2i\mu_j vH_n(\mu_j)G_{n+1}(\mu_j) - G_{n+1}(\mu_j)G_{n+1,x}(\mu_j) \\ &= (1/2)\mu_j H_n(\mu_j)F_{n,x}(\mu_j) - (1/2)\mu_j F_{n,x}(\mu_j)H_n(\mu_j) = 0, \quad j = 1, \dots, n. \end{aligned} \tag{4.129}$$

To investigate the leading-order term with respect to z of P_n, we first study the leading-order z-behavior of F_n, G_{n+1}, and H_n. Writing (cf. (4.22)–(4.24))

$$\begin{aligned} &F_n(z) = \sum_{j=0}^{n} f_{n-j}z^j, \quad H_n(z) = \sum_{j=0}^{n} h_{n-j}z^j, \\ &G_{n+1}(z) = \sum_{j=0}^{n+1} g_{n+1-j}z^j, \;\; g_0 = 1, \end{aligned} \tag{4.130}$$

a comparison of leading powers with respect to z in (4.113), (4.114), and (4.119) yields

$$f_0 = -2v, \tag{4.131}$$

$$g_0 = 1, \tag{4.132}$$

$$v_x + 2iv^2v^* + if_1 + 2ivg_1 = 0, \tag{4.133}$$

$$2g_1 + 2vh_0 + \sum_{m=0}^{2n+1} E_m = 0. \tag{4.134}$$

Since (4.110) can be rewritten in the form

$$f_1 = iv_x + 2v^2v^* + v\sum_{m=0}^{2n+1} E_m, \tag{4.135}$$

a comparison of (4.133) and (4.135) implies

$$g_1 = -2vv^* - \frac{1}{2}\sum_{m=0}^{2n+1} E_m \tag{4.136}$$

and hence

$$h_0 = 2v^*. \tag{4.137}$$

Insertion of (4.131), (4.132), and (4.137) into (4.128) then yields

$$P_n(z) = O(z^n) \text{ as } |z| \to \infty. \tag{4.138}$$

Thus, (4.129) and (4.138) prove

$$P_n(z) = cF_n(z) \text{ on } \mathbb{C} \times \Omega_\mu$$

for some $c \in C^\infty(\Omega_\mu)$ (independent of z), implying

$$G_{n+1,x}(z) = 2izv^* F_n(z) + 2izvH_n(z) - cF_n(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{4.139}$$

Taking $z = 0$ in (4.139), observing that $G_{n+1}(0, x, t_r)$ is independent of $(x, t) \in \Omega_\mu$ by (4.117), then shows that

$$0 = -cF_n(0) \text{ on } \Omega_\mu,$$

and hence $c = 0$ on Ω_μ because of (4.109). Thus,

$$G_{n+1,x}(z) = 2izv^* F_n(z) + 2izvH_n(z) \text{ on } \mathbb{C} \times \Omega_\mu. \tag{4.140}$$

Differentiating (4.119) with respect to x, inserting (4.114) and (4.140), then yields

$$H_{n,x}(z) = 2i(vv^* - z)H_n(z) + 4iv^* G_{n+1}(z) \text{ on } \mathbb{C} \times \Omega_\mu, \tag{4.141}$$

and we have proved (4.121)–(4.123).

Next, combining (4.90), (4.112), and (4.115), one computes

$$\begin{aligned} F_{n,t}(\mu_j) &= 2v \frac{(-1)^n}{g_{n+1}} \left(\prod_{\substack{k=1\\k\neq j}}^n \mu_k \right) 2iy(\hat{\mu}_j) = \frac{(-1)^n}{g_{n+1}} \left(\prod_{k=1}^n \mu_k \right) \frac{4iv}{\mu_j} G_{n+1}(\mu_j) \\ &= \frac{4iu}{\mu_j} G_{n+1}(\mu_j), \quad j = 1, \dots, n. \end{aligned} \tag{4.142}$$

Since clearly

$$F_{n,t}(z) - \big(-2i(uu^* - z^{-1})F_n(z) + 4iz^{-1}uG_{n+1}(z) \big) = O(z^n) \text{ as } |z| \to \infty, \tag{4.143}$$

a comparison of (4.142) and (4.143) yields

$$F_{n,t}(z) - \big(-2i(uu^* - z^{-1})F_n(z) + 4iz^{-1}uG_{n+1}(z) \big) = bF_n(z) \text{ on } \mathbb{C} \times \Omega_\mu \tag{4.144}$$

for some $b \in C^\infty(\Omega_\mu)$ (independent of z), and hence (4.124), except for $b_x = 0$. A comparison of powers of z^n in (4.144) then yields the equation involving v_t in (4.127).

Next, we further restrict Ω_μ and introduce $\widehat{\Omega}_\mu \subseteq \Omega_\mu$ by the requirement that μ_j remain distinct and also distinct from $\{E_m\}_{m=0,\dots,2n+1} \cup \{0\}$ on $\widehat{\Omega}_\mu$, that is, we suppose

$$\begin{aligned} &\mu_j(x,t) \neq \mu_{j'}(x,t) \quad j \neq j', \ j, j' = 1, \dots, n, \ (x,t) \in \widehat{\Omega}_\mu, \\ &\mu_j(x,t) \notin \{E_0, \dots, E_{2n+1}, 0\}, \quad j = 1, \dots, n, \ (x,t) \in \widehat{\Omega}_\mu. \end{aligned} \tag{4.145}$$

Differentiating (4.119) with respect to t and inserting (4.144) then yields

$$\begin{aligned} 2G_{n+1}(z)G_{n+1,t}(z) = zF_n(z)\big(&-2i(uu^* - z^{-1})H_n(z) \\ &+ bH_n(z) + H_{n,t}(z)\big) + 4iuG_{n+1}H_n(z). \end{aligned} \tag{4.146}$$

Since the zeros of F_n and G_{n+1} are distinct by hypothesis (4.146) (cf. (4.18)), $zH_{n,t}(z)$ necessarily must be of the form

$$zH_{n,t}(z) = 2i(zuu^* - 1)H_n(z) - bzH_n(z) + 4idG_{n+1}(z) \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu \tag{4.147}$$

for some $d \in C^\infty(\widehat{\Omega}_\mu)$ (independent of z), and (4.147) inserted into (4.146) then yields

$$G_{n+1,t}(z) = 2iuH_n(z) + 2idF_n(z) \text{ on } \mathbb{C} \times \widehat{\Omega}_\mu. \tag{4.148}$$

Since

$$u = -\frac{F_n(0)}{2g_{n+1}} \text{ on } \widehat{\Omega}_\mu, \tag{4.149}$$

combining (4.112) and (4.113), taking $z = 0$ in (4.148) and observing (4.117) and (4.120), results in

$$0 = 2iu2g_{n+1}u^* + 2id(-2g_{n+1}u)$$

and hence in

$$d = u^* \text{ on } \widehat{\Omega}_\mu. \tag{4.150}$$

Using property (4.92), (4.147)–(4.150) then extend by continuity from $\widehat{\Omega}_\mu$ to Ω_μ. This proves (4.125) and (4.126), except for $b_x = 0$. A comparison of powers of z^n in (4.126) then yields the equation involving v_t^* in (4.127). Taking $z = 0$ in (4.121) and (4.123), observing (4.120) and (4.149), then proves the equations involving u_x and u_x^* in (4.127). Finally, computing the partial t-derivative of $F_{n,x}$ and separately the partial x-derivative of $F_{n,t}$, utilizing (4.121), (4.122), (4.124), (4.125), and (4.127) then shows

$$F_{n,xt} - F_{n,tx} = -b_x F_n(z) \text{ on } \mathbb{C} \times \Omega_\mu,$$

and hence

$$b_x = 0 \text{ on } \Omega_\mu.$$

Furthermore, u and v are nonvanishing on Ω_μ by the construction in step $(S1)$. $\square$

Remark 4.15 One can factorize the polynomial $H_n(z)$ as (cf. (4.130), (4.137))

$$H_n(z) = 2v^* \prod_{j=1}^{n} (z - \nu_j). \tag{4.151}$$

Considering the coefficient of z^n in relation (4.141), one finds, using (4.151) and (4.136), that

$$i v_x^* = \big(2vv^* - 2\sum_{j=1}^{n} \nu_j - 2c_1\big)v^*. \tag{4.152}$$

Since $\nu_j \in C(\Omega_\mu)$, $j = 1, \dots, n$, (4.152) yields that either v^* is nonzero on Ω_μ or it vanishes identically on Ω_μ. In the latter case, however, one also infers $u^* = 0$ on Ω_μ by the last equation in (4.127). By Remark 4.4, this in turn contradicts the fact that the affine part of $\mathcal{K}_n$ is nonsingular. Hence v^* is nonzero on Ω_μ. Whether or not u^* can have isolated zeros in Ω_μ appears to be unknown.

Remark 4.16 That the system of Dubrovin equations (4.89)–(4.91) cannot uniquely determine the solutions u, v, u^*, v^* of the massive Thirring system (4.31), as is evident from the occurrence of $b(t)$ in the equations involving v_t and v_t^* in (4.127), is of course due to the scale invariance displayed explicitly in Lemma 4.12. In particular, once a certain $b(t)$ has been identified, a scaling transformation of the type (4.107) (with $B(t)$ replaced by $1/B(t)$) will restore the extended massive Thirring system (4.127) to its original form in (4.31).

Remark 4.17 The explicit theta function representations (4.194)–(4.197) for u, v, u^*, v^* to be proven in Section 4.4 (this approach is independent of that used to prove Theorem 4.14) permit one to extend the principal assertions (4.121)–(4.127) of Theorem 4.14 by continuity beyond Ω_μ as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ remains nonspecial (cf. Theorem A.31).

Remark 4.18 Although we formulated Theorem 4.14 in terms of $\{\hat{\mu}_j\}_{j=1,\dots,n}$ and (4.89)–(4.91) only, there exists of course a completely analogous approach starting with $\{\hat{\nu}_j\}_{j=1,\dots,n}$ and the system (4.93), (4.94) instead.

Remark 4.19 A closer look at Theorem 4.14 reveals that, up to scaling (cf. Lemma 4.12), u, v, u^*, v^* are uniquely determined in an open neighborhood Ω of (x_0, t_0) by $\mathcal{K}_n$ and the initial data $\underline{\hat{\mu}}(x_0, t_0) = (\hat{\mu}_1(x_0, t_0), \dots, \hat{\mu}_n(x_0, t_0)) \in \operatorname{Sym}^n(\mathcal{K}_n)$ or, equivalently, by the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)} \in \operatorname{Sym}^n(\mathcal{K}_n)$ at $(x, t) = (x_0, t_0)$. Conversely, given $\mathcal{K}_n$ and u, v, u^*, v^* in an open neighborhood Ω of (x_0, t_0), one can construct the polynomials $F_n(\,\cdot\,, x, t)$, $G_{n+1}(\,\cdot\,, x, t)$, $H_n(\,\cdot\,, x, t)$ for $(x, t) \in \Omega$ (using the recursion relation (4.3)–(4.6) to determine the homogeneous elements $\hat{f}_\ell$, $\hat{g}_\ell$, $\hat{h}_\ell$ and (D.45) to determine $c_\ell = c_\ell(\underline{E})$, $\ell = 0, \dots, n$) and then recover the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ for $(x,t) \in \Omega$ from the zeros of $F_n(\,\cdot\,,x,t)$ and from (4.40). This remark is of relevance in connection with determining the isospectral set of Thirring potentials u, v, u^*, v^* in the sense that once the curve $\mathcal{K}_n$ is fixed, elements of the isospectral class of potentials are parametrized by (nonspecial) auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ (cf. Lemma 4.30).

4.4 Theta Function Representations of u, v, u^*, v^*

In this final section we complete the algebro-geometric approach initiated in Section 4.3 and now derive theta function representations of the principal objects such as ϕ, ψ_1, u, v, u^*, and v^*.

According to our shift in emphasis from the Baker–Akhiezer vector Ψ to our fundamental meromorphic function ϕ on $\mathcal{K}_n$, we first aim at the theta function representation of ϕ.

Assuming the affine part of $\mathcal{K}_n$ to be nonsingular for the remainder of this section (i.e., $E_m \neq E_{m'}$ for $m \neq m'$, $m, m' = 0, \dots, 2n+1$) and $n \in \mathbb{N}$ for simplicity (to avoid repeated case distinctions; see Example 4.31 for the case $n = 0$), we next recall the formula for a normal differential of the third kind, which has simple poles at $P_{0,-}$ and P_{∞_-}, corresponding residues $+1$ and -1, vanishing a-periods, and is holomorphic otherwise on $\mathcal{K}_n$. One computes

$$\omega^{(3)}_{P_{0,-},P_{\infty_-}} = \frac{y + y_{0,-}}{2z}\,\frac{dz}{y} + \frac{1}{2y}\prod_{j=1}^{n}(z-\lambda_j)dz, \quad P_{0,-} = (0, y_{0,-}) = (0, -g_{n+1}), \tag{4.153}$$

where $\{\lambda_j\}_{j=1,\dots,n}$ are uniquely determined by the normalization

$$\int_{a_j} \omega^{(3)}_{P_{0,-},P_{\infty_-}} = 0, \quad j = 1, \dots, n. \tag{4.154}$$

The explicit formula (4.153) then implies (using the local coordinate $\zeta = z$ near $P_{0,\mp}$)

$$\omega^{(3)}_{P_{0,-},P_{\infty_-}}(P) \underset{\zeta\to 0}{=} \begin{Bmatrix} \zeta^{-1} \\ 0 \end{Bmatrix} d\zeta \pm \left(\sum_{q=0}^{\infty}(q+1)\omega^0_{q+1}\zeta^q\right) d\zeta \text{ as } P \to P_{0,\mp}, \tag{4.155}$$

and similarly (using the local coordinate $\zeta = 1/z$ near $P_{\infty_\mp}$),

$$\omega^{(3)}_{P_{0,-},P_{\infty_-}}(P) \underset{\zeta\to 0}{=} \begin{Bmatrix} -\zeta^{-1} \\ 0 \end{Bmatrix} d\zeta \pm \left(\sum_{q=0}^{\infty}(q+1)\omega^\infty_{q+1}\zeta^q\right) d\zeta \text{ as } P \to P_{\infty_\mp}. \tag{4.156}$$

In particular,

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} \ln(\zeta) \\ 0 \end{Bmatrix} + \omega_0^{0,\mp} \pm \omega_1^0 \zeta \pm \omega_2^0 \zeta^2 + O(\zeta^3) \text{ as } P \to P_{0,\mp}, \tag{4.157}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} \underset{\zeta\to 0}{=} \begin{Bmatrix} -\ln(\zeta) \\ 0 \end{Bmatrix} + \omega_0^{\infty_\mp} \pm \omega_1^\infty \zeta \pm \omega_2^\infty \zeta^2 + O(\zeta^3) \text{ as } P \to P_{\infty_\mp}. \tag{4.158}$$

Here $Q_0 \in \mathcal{B}(\mathcal{K}_n)$ is an appropriate base point, and we agree to choose the same path of integration from Q_0 to P in all Abelian integrals in this section.

A comparison of (4.155), (4.156) with (4.153), (C.41), and (C.42) then yields

$$\omega_1^0 = \frac{1}{4}\sum_{m=0}^{2n+1} \frac{1}{E_m} - \frac{(-1)^n}{2g_{n+1}} \prod_{j=1}^{n} \lambda_j, \quad \omega_1^\infty = -\frac{1}{4}\sum_{m=0}^{2n+1} E_m + \frac{1}{2}\sum_{j=1}^{n} \lambda_j. \tag{4.159}$$

Next, we intend to go a step further and derive alternative expressions for the expansion coefficients $\omega_0^{0,\pm}$, ω_1^0, $\omega_0^{\infty_\pm}$, and ω_1^∞ in (4.157) and (4.158). To begin these calculations we first recall the notion of a nonsingular odd half-period Υ defined by

$$2\underline{\Upsilon} = 0 \pmod{L_n}, \quad \theta(\underline{\Upsilon}) = 0, \ \partial_{z_j}\theta(\underline{z})\big|_{\underline{z}=\underline{\Upsilon}} \neq 0 \text{ for some } j \in \{1, \dots, n\}.$$

In addition, we need to introduce the normalized differentials $\omega^{(2)}_{P_{0,+},0}$ and $\omega^{(2)}_{P_{\infty_+},0}$ with unique poles at $P_{0,+}$ and P_{∞_+} and principal parts

$$\omega^{(2)}_{P_{0,+},0} \underset{\zeta\to 0}{=} (\zeta^{-2} + O(1))d\zeta \quad \text{as } P \to P_{0,+}, \tag{4.160}$$

$$\omega^{(2)}_{P_{\infty_+},0} \underset{\zeta\to 0}{=} (\zeta^{-2} + O(1))d\zeta \quad \text{as } P \to P_{\infty_+}, \tag{4.161}$$

respectively. In particular,

$$\int_{a_j} \omega^{(2)}_{P_{0,+},0} = 0, \quad \int_{a_j} \omega^{(2)}_{P_{\infty_+},0} = 0, \quad j = 1, \dots, n. \tag{4.162}$$

In the following we find it convenient to use the notations

$$\underline{\Delta}_0 = \underline{A}_{Q_0}(P_{0,+}), \quad \underline{\Delta}_\infty = \underline{A}_{Q_0}(P_{\infty_+}), \tag{4.163}$$

$$\underline{W}_1^0 = \big(W_{1,1}^0, \dots, W_{1,n}^0\big), \ W_{1,j}^0 = \frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_{0,+},0} = \frac{c_j(1)}{g_{n+1}}, \ j = 1, \dots, n, \tag{4.164}$$

$$\underline{W}_2^0 = \big(W_{2,1}^0, \dots, W_{2,n}^0\big), \ \ W_{2,j}^0 = \frac{c_j(1)}{4g_{n+1}} \sum_{m=0}^{2n+1} E_m^{-1} + \frac{c_j(2)}{2g_{n+1}}, \ \ j = 1, \dots, n, \tag{4.165}$$

$$\underline{W}_1^\infty = \big(W_{1,1}^\infty, \dots, W_{1,n}^\infty\big), \ \ W_{1,j}^\infty = \frac{1}{2\pi i} \int_{b_j} \omega_{P_{\infty_+},0}^{(2)} = c_j(n), \ \ j = 1, \dots, n, \tag{4.166}$$

$$\underline{W}_2^\infty = \big(W_{2,1}^\infty, \dots, W_{2,n}^\infty\big), \ \ W_{2,j}^\infty = \frac{c_j(n)}{4} \sum_{m=0}^{2n+1} E_m + \frac{c_j(n-1)}{2}, \ \ j = 1, \dots, n. \tag{4.167}$$

$$(\partial_{\underline{W}} f)(\underline{z}) = \sum_{j=1}^n W_j (\partial_{z_j} f)(\underline{z}), \quad \big(\partial_{\underline{W}}^2 f\big)(\underline{z}) = \sum_{j,k=1}^n W_j W_k \big(\partial_{z_j z_k}^2 f\big)(\underline{z}),$$
$$\underline{z} = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Then one obtains the following result.

Lemma 4.20 *Given* (4.153)–(4.166) *one obtains*

$$\omega_0^{0,+} = \ln\left(\frac{\theta(\underline{\Upsilon} - 2\underline{\Delta}_0)\theta(\underline{\Upsilon} - \underline{\Delta}_\infty)}{\theta(\underline{\Upsilon} - \underline{\Delta}_0 - \underline{\Delta}_\infty)\theta(\underline{\Upsilon} - \underline{\Delta}_0)}\right), \tag{4.168}$$

$$\omega_0^{0,-} = \ln\left(\frac{\big(\partial_{\underline{W}_1^0}\theta\big)(\underline{\Upsilon})\theta(\underline{\Upsilon} - \underline{\Delta}_\infty)}{\theta(\underline{\Upsilon} + \underline{\Delta}_0 - \underline{\Delta}_\infty)\theta(\underline{\Upsilon} - \underline{\Delta}_0)}\right), \tag{4.169}$$

$$\omega_1^0 = -\partial_{\underline{W}_1^0} \ln(\theta(\underline{\Upsilon} + \underline{\Delta}_0 - \underline{\Delta}_\infty)) + \frac{\big(\partial_{\underline{W}_2^0}\theta\big)(\underline{\Upsilon}) + 2^{-1}\big(\partial_{\underline{W}_1^0}^2\theta\big)(\underline{\Upsilon})}{\big(\partial_{\underline{W}_1^0}\theta\big)(\underline{\Upsilon})} \tag{4.170}$$

$$= \partial_{\underline{W}_1^0} \ln\left(\frac{\theta(\underline{\Upsilon} - 2\underline{\Delta}_0)}{\theta(\underline{\Upsilon} - \underline{\Delta}_0 - \underline{\Delta}_\infty)}\right), \tag{4.171}$$

$$\omega_0^{\infty_+} = \ln\left(\frac{\theta(\underline{\Upsilon} - \underline{\Delta}_0 - \underline{\Delta}_\infty)\theta(\underline{\Upsilon} - \underline{\Delta}_\infty}{\theta(\underline{\Upsilon} - 2\underline{\Delta}_\infty)\theta(\underline{\Upsilon} - \underline{\Delta}_0)}\right), \tag{4.172}$$

$$\omega_0^{\infty_-} = -\ln\left(\frac{\big(\partial_{\underline{W}_1^\infty}\theta\big)(\underline{\Upsilon})\theta(\underline{\Upsilon} - \underline{\Delta}_0)}{\theta(\underline{\Upsilon} - \underline{\Delta}_0 + \underline{\Delta}_\infty)\theta(\underline{\Upsilon} - \underline{\Delta}_\infty)}\right), \tag{4.173}$$

$$\omega_1^\infty = \partial_{\underline{W}_1^\infty} \ln(\theta(\underline{\Upsilon} - \underline{\Delta}_0 + \underline{\Delta}_\infty)) - \frac{\big(\partial_{\underline{W}_2^\infty}\theta\big)(\underline{\Upsilon}) + 2^{-1}\big(\partial_{\underline{W}_1^\infty}^2\theta\big)(\underline{\Upsilon})}{\big(\partial_{\underline{W}_1^\infty}\theta\big)(\underline{\Upsilon})} \tag{4.174}$$

$$= \partial_{\underline{W}_1^\infty} \ln\left(\frac{\theta(\underline{\Upsilon} - \underline{\Delta}_0 - \underline{\Delta}_\infty)}{\theta(\underline{\Upsilon} - 2\underline{\Delta}_\infty)}\right). \tag{4.175}$$

Proof Abbreviating

$$\underline{w}(P, Q) = \underline{\Upsilon} - \underline{A}_{Q_0}(P) + \underline{A}_{Q_0}(Q) \pmod{L_n}, \quad P, Q \in \mathcal{K}_n,$$

one infers from

$$\underline{A}_{Q_0}(P) \underset{\zeta\to 0}{=} \underline{A}_{Q_0}(P_{0,\pm}) \pm \underline{W}_1^0\zeta \pm \underline{W}_2^0\zeta^2 + O(\zeta^3) \text{ as } P \to P_{0,\pm}, \tag{4.176}$$

$$\underline{A}_{Q_0}(P) \underset{\zeta\to 0}{=} \underline{A}_{Q_0}(P_{\infty_\pm}) \pm \underline{W}_1^\infty\zeta \pm \underline{W}_2^\infty\zeta^2 + O(\zeta^3) \text{ as } P \to P_{\infty_\pm} \tag{4.177}$$

(cf. (C.35) and (C.41)), and (4.164), (4.166), that

$$\begin{aligned}\theta(\underline{w}(P,Q)) &\underset{\zeta\to 0}{=} \theta(\underline{w}(P_{0,\pm},Q)) \mp \big(\partial_{\underline{W}_1^0}\theta\big)(\underline{w}(P_{0,\pm},Q))\zeta \\ &\mp \big(\partial_{\underline{W}_2^0}\theta\big)(\underline{w}(P_{0,\pm},Q))\zeta^2 + 2^{-1}\big(\partial^2_{\underline{W}_1^0}\theta\big)(\underline{w}(P_{0,\pm},Q))\zeta^2 + O(\zeta^3) \text{ as } P \to P_{0,\pm},\end{aligned} \tag{4.178}$$

$$\begin{aligned}\theta(\underline{w}(P,Q)) &\underset{\zeta\to 0}{=} \theta(\underline{w}(P_{\infty_\pm},Q)) \mp (\partial_{\underline{W}_1^\infty}\theta)(\underline{w}(P_{\infty_\pm},Q))\zeta \\ &\mp \big(\partial_{\underline{W}_2^\infty}\theta\big)(\underline{w}(P_{\infty_\pm},Q))\zeta^2 + 2^{-1}\big(\partial^2_{\underline{W}_1^\infty}\theta\big)(\underline{w}(P_{\infty_\pm},Q))\zeta^2 + O(\zeta^3) \text{ as } P \to P_{\infty_\pm}.\end{aligned} \tag{4.179}$$

Next, by observing that

$$\omega^{(3)}_{P_{0,-},P_{\infty_-}} = d\,\ln\left(\frac{\theta(\underline{w}(\,\cdot\,,P_{0,-}))}{\theta(\underline{w}(\,\cdot\,,P_{\infty_-}))}\right), \tag{4.180}$$

it becomes a straightforward matter to derive (4.168)–(4.175). For simplicity we just focus on the expansion of $\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}$ as $P \to P_{0,\pm}$; the rest is completely analogous. Using

$$\begin{aligned}&\underline{w}(Q_0,P_{0,\pm}) = \underline{\Upsilon} \pm \underline{\Delta}_0, \quad \underline{w}(Q_0,P_{\infty_\pm}) = \underline{\Upsilon} \pm \underline{\Delta}_\infty, \quad \underline{w}(Q,Q) = \underline{\Upsilon}, \\ &\underline{w}(P_{\infty_\sigma},P_{0,\sigma'}) = \underline{\Upsilon} + \sigma'\underline{\Delta}_0 - \sigma\underline{\Delta}_\infty, \quad \underline{w}(P_{0,\sigma'},P_{\infty_\sigma}) = \underline{\Upsilon} - \sigma'\underline{\Delta}_0 + \sigma\underline{\Delta}_\infty, \\ &\underline{w}(P_{0,\sigma},P_{0,\sigma'}) = \underline{\Upsilon} + (\sigma'-\sigma)\underline{\Delta}_0, \quad \underline{w}(P_{0,\sigma'},P_{0,\sigma}) = \underline{\Upsilon} + (\sigma-\sigma')\underline{\Delta}_0, \\ &\underline{w}(P_{\infty_\sigma},P_{\infty_{\sigma'}}) = \underline{\Upsilon} + (\sigma'-\sigma)\underline{\Delta}_\infty, \quad \underline{w}(P_{\infty_{\sigma'}},P_{\infty_\sigma}) = \underline{\Upsilon} + (\sigma-\sigma')\underline{\Delta}_\infty, \\ &\qquad\qquad\qquad Q \in \mathcal{K}_n, \quad \sigma,\sigma' \in \{1,-1\}\end{aligned} \tag{4.181}$$

and (4.178)–(4.180), one computes the following by comparison with (4.157),

$$\begin{aligned}\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}} &= \int_{Q_0}^P d\,\ln\left(\frac{\theta(\underline{w}(P',P_{0,-}))}{\theta(\underline{w}(P',P_{\infty_-}))}\right) \\ &= \ln\left(\frac{\theta(\underline{w}(P,P_{0,-}))}{\theta(\underline{w}(P,P_{\infty_-}))}\right) - \ln\left(\frac{\theta(\underline{w}(Q_0,P_{0,-}))}{\theta(\underline{w}(Q_0,P_{\infty_-}))}\right) \\ &= \ln\left(\frac{\theta(\underline{w}(P,P_{0,-}))}{\theta(\underline{w}(P,P_{\infty_-}))}\right) - \ln\left(\frac{\theta(\underline{\Upsilon}-\underline{\Delta}_0)}{\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}\right) \\ &= \ln\left(\frac{\theta(\underline{\Upsilon}-2\underline{\Delta}_0)\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}{\theta(\underline{\Upsilon}-\underline{\Delta}_0-\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_0)}\right) \\ &\quad - \partial_{\underline{W}_1^0}\ln\left(\frac{\theta(\underline{\Upsilon}-2\underline{\Delta}_0)}{\theta(\underline{\Upsilon}-\underline{\Delta}_0-\underline{\Delta}_\infty)}\right)\zeta + O(\zeta^2) \\ &= \omega_0^{0,+} - \omega_1^0\zeta + O(\zeta^2) \text{ as } P \to P_{0,+}.\end{aligned}$$

This proves (4.168) and (4.171). Similarly, one calculates,

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{0,-},P_{\infty_-}} = \ln\left(\frac{\theta(\underline{w}(P,P_{0,-}))}{\theta(\underline{w}(P,P_{\infty_-}))}\right) - \ln\left(\frac{\theta(\underline{\Upsilon}-\underline{\Delta}_0)}{\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}\right)$$
$$= \ln(\zeta) + \ln\left(\frac{(\partial_{\underline{W}^0_1}\theta)(\underline{\Upsilon})\theta(\underline{\Upsilon}-\underline{\Delta}_\infty)}{\theta(\underline{\Upsilon}+\underline{\Delta}_0-\underline{\Delta}_\infty)\theta(\underline{\Upsilon}-\underline{\Delta}_0)}\right) - \partial_{\underline{W}^0_1}\ln(\theta(\underline{\Upsilon}+\underline{\Delta}_0-\underline{\Delta}_\infty))\zeta$$
$$+ \frac{(\partial_{\underline{W}^0_2}\theta)(\underline{\Upsilon}) + 2^{-1}(\partial^2_{\underline{W}^0_1}\theta)(\underline{\Upsilon})}{(\partial_{\underline{W}^0_1}\theta)(\underline{\Upsilon})}\zeta + O(\zeta^2)$$
$$= \ln(\zeta) + \omega_0^{0,-} + \omega_1^0\zeta + O(\zeta^2) \text{ as } P \to P_{0,-},$$

proving (4.169) and (4.170). □

The results of Lemma 4.20 can conveniently be reformulated in terms of theta functions with characteristics associated with the vector $\underline{\Upsilon}$, but we omit further details at this point.

Next, we now turn to the theta function representation of ϕ and refer to Appendices A and C for our notational conventions concerning Abel maps $\underline{A}_{Q_0}$, $\underline{\alpha}_{Q_0}$, and θ-functions. Here $Q_0 \in \mathcal{K}_n \setminus \{P_{0,\pm}, P_{\infty_\pm}\}$ is a fixed base point that we will always choose among the branch points of $\mathcal{K}_n$ (e.g., $Q_0 = (E_0, 0)$).

Combining (4.62) and Theorem A.26, the theta function representation of ϕ must be of the following form

$$\phi(P) = C\frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{0,-},P_{\infty_-}}\right),$$
$$P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\},\ (x,t) \in \Omega, \tag{4.182}$$

assuming $\mathcal{D}_{\underline{\hat{\mu}}}$ (or equivalently $\mathcal{D}_{\underline{\hat{\nu}}}$) to be nonspecial on Ω, where $\Omega \subseteq \mathbb{R}^2$ is open and connected. Indeed, by (4.62), (4.157), (4.158), and Theorem A.26, ϕ and

$$\frac{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}))}{\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{P_{0,-},P_{\infty_-}}\right) \tag{4.183}$$

have the same singularity structure with respect to $P \in \mathcal{K}_n$. By (A.26), (A.38), and (A.39), the expression (4.183) is single-valued and hence meromorphic on $\mathcal{K}_n$. Since by (4.62), $\mathcal{D}_{P_{0,-}\underline{\hat{\nu}}} \sim \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}}$, and $P_{\infty_+} = (P_{\infty_-})^* \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ by hypothesis, one can apply Theorem A.31 to conclude that $\mathcal{D}_{\underline{\hat{\nu}}} \in \operatorname{Sym}^n(\mathcal{K}_n)$ is nonspecial. This argument is of course symmetric with respect to $\underline{\hat{\mu}}$ and $\underline{\hat{\nu}}$. Thus, $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial if and only if $\mathcal{D}_{\underline{\hat{\nu}}}$ is. Nonspecialty of $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ then yields (4.182).

It remains to analyze the function $C = C(x,t)$ in (4.182) (which is P-independent), and in the course of that we will also obtain the theta function representations of u, v, u^*, v^*.

In the following it will occasionally be convenient to use a short-hand notation for the arguments of the theta functions in (4.182), and hence we introduce the abbreviation

$$\begin{aligned} &\underline{z}(P, \underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ &P \in \mathcal{K}_n, \underline{Q} = \{Q_1, \ldots, Q_n\} \in \mathrm{Sym}^n(\mathcal{K}_n). \end{aligned} \tag{4.184}$$

We note that by (A.52) and (A.53), $\underline{z}(\cdot\,, \underline{Q})$ is independent of the choice of base point Q_0.

Next we show that the Abel map linearizes the auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ and $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$.

Lemma 4.21 *Assume* (4.2)*,* (4.14)*,* (4.53) *hold on* Ω*. In addition, suppose* (4.37) *holds and let* (x, t), $(x_0, t_0) \in \Omega$*, where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose that the affine part of* $\mathcal{K}_n$ *is nonsingular and that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$*, or equivalently,* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *is nonspecial for* $(x, t) \in \Omega$*. Then,*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}) + 2i(x - x_0)\underline{c}(n) - 2ig_{n+1}^{-1}(t - t_0)\underline{c}(1), \tag{4.185}$$

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_0)}) + 2i(x - x_0)\underline{c}(n) - 2ig_{n+1}^{-1}(t - t_0)\underline{c}(1). \tag{4.186}$$

Proof Given the expansions (C.35) and (C.41) of ω near $P_{\infty_\pm}$ and $P_{0,\pm}$, (4.185) and (4.186) are standard facts following from Lagrange interpolation results of the type

$$\sum_{j=1}^{n} \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{n}(\mu_j - \mu_\ell)} = \delta_{k,n},$$

$$\sum_{j=1}^{n} \frac{\mu_j^{k-1}\Big(\prod_{\substack{m=1 \\ m \neq j}}^{n} \mu_m\Big)}{\prod_{\substack{\ell=1 \\ \ell \neq j}}^{n}(\mu_j - \mu_\ell)} = (-1)^{n+1}\delta_{k,1}, \quad k = 1, \ldots, n. \qquad \square$$

In Lemma 4.8 we determined the asymptotic behavior of $\phi(P, x, t)$ as $P \to P_{\infty_\pm}, P_{0,\pm}$ comparing (4.38) with (4.44) and (4.45). Now we will recompute the asymptotics of ϕ starting from (4.182).

Lemma 4.22 *Assume* (4.2)*,* (4.14)*,* (4.53) *hold on* Ω*. In addition, suppose* (4.37) *holds, assume that the affine part of* $\mathcal{K}_n$ *is nonsingular, and let* $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *and* $(x, t) \in \Omega$*, where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose that*

$\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$, *or equivalently,* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *is nonspecial for* $(x,t) \in \Omega$. *Then,*[1]

$$\begin{aligned}\phi(P,x,t) \underset{\zeta\to 0}{=} & \, C(x,t)e^{\omega_0^{\infty_-}} \frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)}\zeta^{-1} \\ & + C(x,t)\omega_1^{\infty}e^{\omega_0^{\infty_-}} \frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)} \\ & - C(x,t)e^{\omega_0^{\infty_-}} \frac{i}{2}\,\partial_x\left(\frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)}\right) + O(\zeta) \text{ as } P \to P_{\infty_-},\end{aligned} \tag{4.187}$$

$$\begin{aligned}\phi(P,x,t) \underset{\zeta\to 0}{=} & \, C(x,t)e^{\omega_0^{\infty_+}} \frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)} \\ & - C(x,t)\omega_1^{\infty}e^{\omega_0^{\infty_+}} \frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)}\zeta \\ & + C(x,t)e^{\omega_0^{\infty_+}} \frac{i}{2}\,\partial_x\left(\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)}\right)\zeta + O(\zeta^2) \text{ as } P \to P_{\infty_+},\end{aligned} \tag{4.188}$$

$$\begin{aligned}\phi(P,x,t) \underset{\zeta\to 0}{=} & \, C(x,t)e^{\omega_0^{0,-}} \frac{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))\big)}\zeta \\ & + C(x,t)\omega_1^{0}e^{\omega_0^{0,-}} \frac{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))\big)}\zeta^2 \\ & + C(x,t)e^{\omega_0^{0,-}} \frac{i}{2}\,\partial_t\left(\frac{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))\big)}\right)\zeta^2 + O(\zeta^3) \text{ as } P \to P_{0,-},\end{aligned} \tag{4.189}$$

$$\begin{aligned}\phi(P,x,t) \underset{\zeta\to 0}{=} & \, C(x,t)e^{\omega_0^{0,+}} \frac{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t))\big)} \\ & - C(x,t)\omega_1^{0}e^{\omega_0^{0,+}} \frac{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t))\big)}\zeta \\ & - C(x,t)e^{\omega_0^{0,+}} \frac{i}{2}\,\partial_t\left(\frac{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t))\big)}\right)\zeta + O(\zeta^2) \text{ as } P \to P_{0,+}.\end{aligned} \tag{4.190}$$

Proof Using (4.176) and (4.177) (cf. (C.35) and (C.41)), one obtains

$$\underline{z}(P,\underline{\hat{\mu}}) \underset{\zeta\to 0}{=} \underline{z}(P_{\infty_\pm},\underline{\hat{\mu}}) \mp \underline{c}(n)\zeta + O(\zeta^2) \text{ as } P \to P_{\infty_\pm},$$
$$\underline{z}(P,\underline{\hat{\mu}}) \underset{\zeta\to 0}{=} \underline{z}(P_{0,\pm},\underline{\hat{\mu}}) \mp \underline{c}(1)g_{n+1}^{-1}\zeta + O(\zeta^2) \text{ as } P \to P_{0,\pm}$$

[1] To avoid multi-valued expressions in formulas such as (4.187), etc., we agree always to choose the same path of integration connecting Q_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

and hence

$$\theta\big(\underline{z}(P,\underline{\hat{\mu}})\big) \underset{\zeta\to 0}{=} \theta\big(\underline{z}(P_{\infty_\pm},\underline{\hat{\mu}})\big) \pm \frac{i}{2}\,\partial_x\theta\big(\underline{z}(P_{\infty_\pm},\underline{\hat{\mu}})\big)\zeta + O(\zeta^2) \text{ as } P \to P_{\infty_\pm}, \tag{4.191}$$

$$\theta\big(\underline{z}(P,\underline{\hat{\mu}})\big) \underset{\zeta\to 0}{=} \theta\big(\underline{z}(P_{0,\pm},\underline{\hat{\mu}})\big) \mp \frac{i}{2}\,\partial_t\theta\big(\underline{z}(P_{0,\pm},\underline{\hat{\mu}})\big)\zeta + O(\zeta^2) \text{ as } P \to P_{0,\pm}. \tag{4.192}$$

Here we used (4.185) to convert the directional derivatives $\sum_{j=1}^n c_j(n)\partial_{w_j}$ and $\sum_{j=1}^n c_j(1)\partial_{w_j}$, $\underline{w} = (w_1,\dots,w_n) \in \mathbb{C}^n$ into ∂_x and ∂_t derivatives. Since by (4.186) exactly the same formulas (4.191) and (4.192) apply to $\mathcal{D}_{\underline{\hat{\nu}}}$, insertion of (4.157), (4.158), (4.191), and (4.192) (and their $\mathcal{D}_{\underline{\hat{\nu}}}$ analogs) into (4.182) proves (4.187)–(4.190). $\square$

Lemma 4.22 may seem to be just another asymptotic result; however, a comparison with Lemma 4.8 reveals that in passing we have actually derived the theta function representations for u, v, u^*, and v^*.

Theorem 4.23 *Assume* (4.2), (4.14)*, and* (4.53) *hold on* Ω. *In addition, suppose* (4.37) *holds, assume that the affine part of* $\mathcal{K}_n$ *is nonsingular, and let* $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ *and* $(x,t) \in \Omega$*, where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ *or equivalently,* $\mathcal{D}_{\underline{\hat{\nu}}(x,t)}$ *is nonspecial for* $(x,t)\in\Omega$. *Then* $\phi(P,x,t)$ *admits the representation*[1]

$$\phi(P,x,t) = C_0 e^{2i(\omega_1^\infty x - \omega_1^0 t)} \frac{\theta\big(\underline{z}(P,\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x,t))\big)} \exp\left(\int_{Q_0}^P \omega^{(3)}_{P_{0,-},P_{\infty_-}}\right) \tag{4.193}$$

for some constant $C_0 \in \mathbb{C}\setminus\{0\}$ *and the theta function representations for the algebro-geometric solutions* u*,* u^**,* v*, and* v^* *of the classical massive Thirring system* (4.31) *read*

$$u(x,t) = -C_0^{-1} e^{-\omega_0^{0,+}} \frac{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t))\big)} e^{-2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.194}$$

$$v(x,t) = -C_0^{-1} e^{-\omega_0^{\infty_-}} \frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)} e^{-2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.195}$$

$$u^*(x,t) = C_0 e^{\omega_0^{0,-}} \frac{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))\big)} e^{2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.196}$$

$$v^*(x,t) = C_0 e^{\omega_0^{\infty_+}} \frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)} e^{2i(\omega_1^\infty x - \omega_1^0 t)}, \tag{4.197}$$

[1] To avoid multi-valued expressions in formulas such as (4.193), etc., we agree always to choose the same path of integration connecting P_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

with $\omega_0^{0,\pm}, \omega_1^0, \omega_0^{\infty_\pm}$, *and* ω_1^∞ *given by* (4.168)–(4.175) (*cf. also* (4.157)–(4.159)).

Proof A comparison of (4.76), (4.77), and (4.187)–(4.190) yields

$$C_x = 2i\omega_1^\infty C, \quad C_t = -2i\omega_1^0 C.$$

Hence,

$$C(x,t) = C_0 e^{2i(\omega_1^\infty x - \omega_1^0 t)} \tag{4.198}$$

proves (4.193). Insertion of (4.198) into the leading asymptotic term of (4.187)–(4.190) then yields (4.194)–(4.197). □

Remark 4.24 (i) The constant C_0 in (4.193)–(4.197) remains undetermined due to the scaling invariance (4.34) of the Thirring system. One can rewrite (4.194)–(4.197) in the form

$$\begin{aligned} u(x,t) &= u(x_0,t_0)\frac{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,+},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,+},\underline{\hat{\nu}}(x,t))\big)} \\ &\quad \times \exp\big(-2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\big), \end{aligned} \tag{4.199}$$

$$\begin{aligned} v(x,t) &= v(x_0,t_0)\frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)} \\ &\quad \times \exp\big(-2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\big), \end{aligned} \tag{4.200}$$

$$\begin{aligned} u^*(x,t) &= u^*(x_0,t_0)\frac{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,-},\underline{\hat{\nu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,-},\underline{\hat{\mu}}(x,t))\big)} \\ &\quad \times \exp\big(2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\big), \end{aligned} \tag{4.201}$$

$$\begin{aligned} v^*(x,t) &= v^*(x_0,t_0)\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)} \\ &\quad \times \exp\big(2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\big), \end{aligned} \tag{4.202}$$

where

$$\underline{z}(Q,\underline{\hat{\mu}}(x,t)) = \underline{z}(Q,\underline{\hat{\mu}}(x_0,t_0)) + 2i(x-x_0)\underline{c}(n) + 2ig_{n+1}^{-1}(t-t_0)\underline{c}(1), \tag{4.203}$$

$$\underline{z}(Q,\underline{\hat{\nu}}(x,t)) = \underline{z}(Q,\underline{\hat{\nu}}(x_0,t_0)) + 2i(x-x_0)\underline{c}(n) + 2ig_{n+1}^{-1}(t-t_0)\underline{c}(1), \tag{4.204}$$

by (4.184), (4.185), and (4.186).
(ii) Since the divisors $\mathcal{D}_{P_{0,-}\underline{\hat{\nu}}}$ and $\mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}}$ are linearly equivalent by (4.62), one infers

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{\Delta} \text{ on } \Omega, \quad \underline{\Delta} = \underline{A}_{P_{0,-}}(P_{\infty_-}).$$

Hence one can replace $\underline{z}(Q, \underline{\hat{\nu}})$ in (4.193)–(4.197), (4.199)–(4.202), (4.204) in terms of $\underline{z}(Q, \underline{\hat{\mu}})$ according to

$$\underline{z}(Q, \underline{\hat{\nu}}) = \underline{z}(Q, \underline{\hat{\mu}}) + \underline{\Delta}. \tag{4.205}$$

Combining (4.185),(4.186), (4.205), and (4.194)–(4.197) shows the remarkable linearity of the theta function arguments with respect to x and t in the formulas for u, v, u^*, v^*. In fact, one can rewrite (4.194)–(4.197) as

$$u(x,t) = C_u \frac{\theta(\underline{A} + \underline{B}x + \underline{C}t)}{\theta(\underline{A} + \underline{B}x + \underline{C}t + \underline{\Delta})} \exp(-i\omega_0 x - i\omega_1 t), \tag{4.206}$$

$$v(x,t) = C_v \frac{\theta(\underline{\widetilde{A}} + \underline{B}x + \underline{C}t)}{\theta(\underline{\widetilde{A}} + \underline{B}x + \underline{C}t + \underline{\Delta})} \exp(-i\omega_0 x - i\omega_1 t), \tag{4.207}$$

$$u^*(x,t) = C_{u^*} \frac{\theta(\underline{A} + \underline{B}x + \underline{C}t + \underline{\Delta} - \underline{A}_{P_{0,+}}(P_{0,-}))}{\theta(\underline{A} + \underline{B}x + \underline{C}t - \underline{A}_{P_{0,+}}(P_{0,-}))} \exp(i\omega_0 x + i\omega_1 t), \tag{4.208}$$

$$v^*(x,t) = C_{v^*} \frac{\theta(\underline{\widetilde{A}} + \underline{B}x + \underline{C}t + \underline{\Delta} - \underline{A}_{P_{\infty_-}}(P_{\infty_+}))}{\theta(\underline{\widetilde{A}} + \underline{B}x + \underline{C}t - \underline{A}_{P_{\infty_-}}(P_{\infty_+}))} \exp(i\omega_0 x + i\omega_1 t), \tag{4.209}$$

where

$$\underline{A} = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{0,+}) - 2i\underline{c}(n)x_0 - 2ig_{n+1}^{-1}\underline{c}(1)t_0 + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}), \tag{4.210}$$

$$\underline{\widetilde{A}} = \underline{A} + \underline{A}_{P_{\infty_-}}(P_{0,+}), \quad \underline{B} = 2i\underline{c}(n), \quad \underline{C} = 2ig_{n+1}^{-1}\underline{c}(1), \tag{4.211}$$

$$\underline{\Delta} = \underline{A}_{P_{\infty_+}}(P_{0,+}) = \underline{A}_{P_{0,-}}(P_{\infty_-}), \tag{4.212}$$

and hence the constants $\omega_0, \omega_1 \in \mathbb{C}$ and $\underline{\Delta}, \underline{B}, \underline{C} \in \mathbb{C}^n$ are uniquely determined by $\mathcal{K}_n$; the constant $\underline{A} \in \mathbb{C}^n$ (and hence $\underline{\widetilde{A}} \in \mathbb{C}^n$) is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(x_0, t_0) = (\hat{\mu}_1(x_0, t_0), \dots, \hat{\mu}_n(x_0, t_0)) \in \operatorname{Sym}^n(\mathcal{K}_n)$ at the point (x_0, t_0), as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}$ is assumed to be nonspecial. The constants $C_u, C_{u^*}, C_v, C_{v^*} \in \mathbb{C}$ satisfy certain constraints that we omit.

In principle, Theorem 4.23 completes the primary aim of this chapter, the derivation of the theta function representation of algebro-geometric solutions of the classical massive Thirring system (4.31). The reader will have noticed that our approach thus far is nontraditional in the sense that we did not use the Baker–Akhiezer vector Ψ at all but instead put all emphasis on the meromorphic ϕ on $\mathcal{K}_n$. Just for completeness we finally derive the theta function representation for ψ_1 in (4.64).

The singularity structure of $\psi_1(\,\cdot\,, x, x_0, t, t_0)$ near $P_{\infty_\pm}$ displayed in Lemma 4.9 suggests introducing Abelian differentials $\omega_{Q,0}^{(2)}$ of the second kind, normalized by the vanishing of their a-periods,

$$\int_{a_j} \omega_{Q,0}^{(2)} = 0, \quad j = 1, \dots, n, \tag{4.213}$$

with a second-order pole at Q of the type

$$\omega^{(2)}_{Q,0} \underset{\zeta\to 0}{=} (\zeta^{-2} + O(1))d\zeta \text{ as } P \to Q, \tag{4.214}$$

and holomorphic on $\mathcal{K}_n \setminus \{Q\}$. More precisely, we introduce

$$\Omega^{(2)}_{\infty,0} = \omega^{(2)}_{P_{\infty_+},0} - \omega^{(2)}_{P_{\infty_-},0}, \tag{4.215}$$

$$\Omega^{(2)}_{0,0} = \omega^{(2)}_{P_{0,+},0} - \omega^{(2)}_{P_{0,-},0} \tag{4.216}$$

and note that

$$\int_{Q_0}^{P} \Omega^{(2)}_{\infty,0} \underset{\zeta\to 0}{=} \pm(\zeta^{-1} + e_{\infty,0} + e_{\infty,1}\zeta + O(\zeta^2)) \text{ as } P \to P_{\infty_\mp}, \tag{4.217}$$

$$\int_{Q_0}^{P} \Omega^{(2)}_{0,0} \underset{\zeta\to 0}{=} \pm(\zeta^{-1} + e_{0,0} + e_{0,1}\zeta + O(\zeta^2)) \text{ as } P \to P_{0,\mp}. \tag{4.218}$$

Theorem 4.25 *Assume* (4.2)*,* (4.14)*,* (4.53) *hold on* Ω*. In addition, suppose that* (4.37) *holds, assume that the affine part of* $\mathcal{K}_n$ *is nonsingular, and let* $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ *and* $(x,t), (x_0,t_0) \in \Omega$*, where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}$ *is nonspecial for* $(x,t) \in \Omega$*. Then* ψ_1 *admits the representation*[1]

$$\begin{aligned}
\psi_1(P,x,x_0,t,t_0) &= \left(\frac{\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}(x,t))\big)}\right)^{1/2} \\
&\quad \times \frac{\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x_0,t_0))\big)} \\
&\quad \times \exp\left(-i\left(\omega_1^\infty + \int_{Q_0}^{P} \Omega^{(2)}_{\infty,0}\right)(x-x_0) + i\left(\omega_1^0 + \int_{Q_0}^{P} \Omega^{(2)}_{0,0}\right)(t-t_0)\right)
\end{aligned} \tag{4.219}$$

or equivalently,

$$\begin{aligned}
\psi_1(P,x,x_0,t,t_0) &= \left(\frac{\theta\big(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(x,t))\big)}\right)^{1/2} \\
&\quad \times \frac{\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P, \underline{\hat{\mu}}(x_0,t_0))\big)} \\
&\quad \times \exp\left(-i\left(\omega_1^\infty + \int_{Q_0}^{P} \Omega^{(2)}_{\infty,0}\right)(x-x_0) + i\left(\omega_1^0 + \int_{Q_0}^{P} \Omega^{(2)}_{0,0}\right)(t-t_0)\right).
\end{aligned} \tag{4.220}$$

[1] To avoid multi-valued expressions in formulas such as (4.219), (4.220), etc., we agree always to choose the same path of integration connecting Q_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

Proof Introducing

$$\begin{aligned}\hat{\psi}_1(P,x,x_0,t,t_0) &= \frac{C(x,t)}{C(x_0,t_0)}\,\frac{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P,\underline{\hat{\mu}}(x_0,t_0))\big)}\\ &\quad\times\exp\bigg(-i(x-x_0)\int_{Q_0}^{P}\Omega^{(2)}_{\infty,0}+i(t-t_0)\int_{Q_0}^{P}\Omega^{(2)}_{0,0}\bigg),\\ &\quad P\in\mathcal{K}_n\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (x,t),(x_0,t_0)\in\Omega\end{aligned} \tag{4.221}$$

with an appropriate normalization $C(x,t)$ (which is P-independent) to be determined later, we next intend to prove that

$$\begin{aligned}&\psi_1(P,x,x_0,t,t_0)=\hat{\psi}_1(P,x,x_0,t,t_0),\\ &P\in\mathcal{K}_n\setminus\{P_{\infty_+},P_{\infty_-},P_{0,+},P_{0,-}\},\ (x,t),(x_0,t_0)\in\Omega.\end{aligned} \tag{4.222}$$

A comparison of (4.8), (4.58), (4.69), (4.78), (4.80), (4.217), (4.218), and (4.221) shows that ψ_1 and $\hat{\psi}_1$ share the identical essential singularity near $P_{0,\pm}$ and $P_{\infty_\pm}$. Next we turn to the local behavior of ψ_1 with respect to its zeros and poles. We temporarily restrict Ω to $\widetilde{\Omega}\subseteq\Omega$ such that for all $(x',s)\in\widetilde{\Omega}$, $\mu_j(x',s)\neq\mu_k(x',s)$ for all $j\neq k$, $j,k=1,\dots,n$. Then, arguing as in the paragraph following (4.74), one infers from (4.64) that

$$\psi_1(P,x,x_0,t,t_0)=\begin{cases}(\mu_j(x,t)-z)O(1) & \text{as } P\to\hat{\mu}_j(x,t)\neq\hat{\mu}(x_0,t_0),\\ O(1) & \text{as } P\to\hat{\mu}_j(x,t)=\hat{\mu}(x_0,t_0),\\ (\mu_j(x_0,t_0)-z)^{-1}O(1) & \text{as } P\to\hat{\mu}_j(x_0,t_0)\neq\hat{\mu}(x,t),\end{cases}$$
$$P=(z,y)\in\mathcal{K}_n,\ \ (x,t),(x_0,t_0)\in\widetilde{\Omega},$$

where $O(1)\neq 0$. Applying Lemma C.3 then proves (4.222) for $(x,t),(x_0,t_0)\in\widetilde{\Omega}$. By continuity this extends to $(x,t),(x_0,t_0)\in\Omega$ as long as $\mathcal{D}_{\underline{\hat{\mu}}(x,t)}\in\mathrm{Sym}^n(\mathcal{K}_n)$ remains nonspecial. Finally, we determine $C(x,t)/C(x_0,t_0)$. A comparison of (4.8), (4.22), (4.58), (4.70), (4.199), (4.200), and (4.221) yield for $P=(z,y)$,

$$\begin{aligned}&\psi_1(P,x,x_0,t,t_0)\psi_1(P^*,x,x_0,t,t_0)\\ &\underset{z\to\infty}{=}\frac{C(x,t)^2}{C(x_0,t_0)^2}\,\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_0))\big)}\\ &\qquad\times\exp\big(-2i(e_{\infty,0}(x-x_0)-e_{0,0}(t-t_0))\big)\\ &=\frac{v(x,t)}{v(x_0,t_0)}\\ &=\frac{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\nu}}(x,t))\big)}\\ &\qquad\times\exp\big(-2i(\omega_1^{\infty}(x-x_0)-\omega_1^{0}(t-t_0))\big)\end{aligned} \tag{4.223}$$

and

$$
\begin{aligned}
&\psi_1(P, x, x_0, t, t_0)\psi_1(P^*, x, x_0, t, t_0) \\
&\underset{z\to 0}{=} \frac{C(x,t)^2}{C(x_0,t_0)^2} \frac{\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(x,t))\big)}{\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(x_0,t_0))\big)} \\
&\qquad \times \exp\big(-2i(e_{\infty,0}(x-x_0) - e_{0,0}(t-t_0))\big) \\
&= \frac{u(x,t)}{u(x_0,t_0)} \\
&= \frac{\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(x,t))\big)} \\
&\quad \times \exp\big(-2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\big).
\end{aligned} \tag{4.224}
$$

Thus, (4.223) implies

$$
\begin{aligned}
\frac{C(x,t)^2}{C(x_0,t_0)^2} &= \frac{\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\nu}}(x,t))\big)} \\
&\quad \times \exp\big(-2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\big),
\end{aligned} \tag{4.225}
$$

and (4.224) yields

$$
\begin{aligned}
\frac{C(x,t)^2}{C(x_0,t_0)^2} &= \frac{\theta\big(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(x_0,t_0))\big)\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(x_0,t_0))\big)}{\theta\big(\underline{z}(P_{0,-}, \underline{\hat{\mu}}(x,t))\big)\theta\big(\underline{z}(P_{0,+}, \underline{\hat{\nu}}(x,t))\big)} \\
&\quad \times \exp\big(-2i(\omega_1^\infty(x-x_0) - \omega_1^0(t-t_0))\big).
\end{aligned} \tag{4.226}
$$

To reconcile the two expressions (4.225) and (4.226) we obtained for $C(x,t)^2/C(x_0,t_0)^2$, it suffices to recall the linear dependence of the divisors $\mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}}$ and $\mathcal{D}_{P_{0,-}\underline{\hat{\nu}}}$, that is,

$$
\underline{A}_{Q_0}(P_{\infty_-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t)}) = \underline{A}_{Q_0}(P_{0,-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t)}),
$$

and

$$
\underline{A}_{Q_0}(P_{0,-}) = -\underline{A}_{Q_0}(P_{0,+}), \quad \underline{A}_{Q_0}(P_{\infty_-}) = -\underline{A}_{Q_0}(P_{\infty_+})
$$

to conclude that

$$
\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}) = \underline{z}(P_{0,+}, \underline{\hat{\nu}}), \quad \underline{z}(P_{0,-}, \underline{\hat{\mu}}) = \underline{z}(P_{\infty_-}, \underline{\hat{\nu}})
$$

and hence equality of the right-hand sides of (4.225) and (4.226). This proves (4.219) and (4.220). □

Remark 4.26 The explicit representation (4.219) for ψ_1 complements Lemma 4.7 and shows that ψ_1 stays meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ as long as

$\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial (assuming the affine part of $\mathcal{K}_n$ to be nonsingular). An analogous theta function derivation can be performed for $\xi\psi_2$; we omit further details.

Remark 4.27 We emphasize that ϕ and ψ_1 are naturally defined on the two-sheeted Riemann surface $\mathcal{K}_n$, whereas ψ_2 requires a four-sheeted Riemann surface because of the additional factor $1/z^{1/2}$ in (4.65). In particular, the Baker–Akhiezer vector Ψ in (4.63) requires a four-sheeted Riemann surface, which is a disadvantage when compared with our use of ϕ.

The algebro-geometric solutions u, v, u^*, v^* in (4.199)–(4.202) represent solutions of the classical massive Thirring system (4.31), a complexified version of the classical massive Thirring model (4.32) denoted by Th henceforth. To comment on the isospectral set for the Th model, one needs to impose certain symmetry constraints on $\mathcal{K}_n$ and additional constraints on $\underline{A}$ in (4.206)–(4.210), which we discuss in the following.

Multiplying (4.97) and (4.100) in the Th model case, where $u^* = \bar{u}$, $v^* = \bar{v}$, one infers

$$\frac{|v|^2}{|u|^2} = \frac{g_{n+1}^2}{\prod_{j=0}^n |\mu_j|^2} \tag{4.227}$$

and hence $g_{n+1}^2 = \prod_{m=0}^{2n+1} E_m > 0$. Thus, the set $\{E_m\}_{m=0,\dots,2n+1}$ consists of complex conjugate pairs, strictly negative pairs, and strictly positive pairs. However, the symmetry constraints we are imposing on $\mathcal{K}_n$ will be of the following more restrictive form.[1]

Hypothesis 4.28 *There exists at least one real pair of branch points* (E_0, E_1) *with either* $0 < E_0$ *and then no further branch points on* $(-\infty, E_0)$ *or with* $0 > E_1$ *and then no further branch points on* (E_1, ∞). *Moreover, we assume that no branch cut connecting complex conjugate pairs crosses* $(-\infty, E_0)$ *if* $0 < E_0$*, respectively,* (E_1, ∞) *if* $0 > E_1$. *The number of real pairs of branch points is denoted by k, and the number of complex conjugate pairs by ℓ (i.e., $k + \ell = n + 1$, $1 \leq k \leq n + 1$).*

We start by recalling that algebro-geometric Th solutions are smooth.

Lemma 4.29 *Assume u, v are algebro-geometric solutions of the classical massive Thirring model* (4.32) *of the type* (4.199), (4.200) *(satisfying $\overline{u} = u^*$ and $\overline{v} = v^*$ with u^*, v^* given by* (4.201), (4.202)*). Then,*[2]

$$u, v \in C^\infty(\mathbb{R}^2). \tag{4.228}$$

[1] Of course we still assume the affine part of $\mathcal{K}_n$ to be nonsingular (cf. (4.37) and (4.88)).
[2] We are not explicitly assuming Hypothesis 4.28.

Proof Identifying $\overline{u} = u^*$ and $\overline{v} = v^*$ in (4.82)–(4.87), one infers

$$\partial_x \ln(\psi_{2,0,+,0}(x,t)) = -\partial_x \ln(\psi_{2,\infty_-,-1}(x,t)) = i|v(x,t)|^2,$$
$$\partial_t \ln(\psi_{2,0,+,0}(x,t)) = -\partial_t \ln(\psi_{2,\infty_-,-1}(x,t)) = -i|u(x,t)|^2,$$

and hence

$$\partial_x \ln(|\psi_{2,0,+,0}(x,t)|) = \partial_t \ln(|\psi_{2,0,+,0}(x,t)|) = 0,$$
$$\partial_x \ln(|\psi_{2,\infty_-,-1}(x,t)|) = \partial_t \ln(|\psi_{2,\infty_-,-1}(x,t)|) = 0.$$

Since

$$u = -\psi_{1,0,+,0}/\psi_{2,0,+,0}, \quad v = -\psi_{1,\infty_-,0}/\psi_{2,\infty_-,-1},$$

with $\psi_{2,0_+,,0}$ and $\psi_{2,\infty_-,-1}$ entire with respect to x and t, u and v are free of local singularities and (4.228) holds. □

Lemma 4.30 *Assume Hypothesis* 4.28*, suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}$ is nonspecial for some $(x_0,t_0) \in \mathbb{R}^2$, and choose the homology basis $\{a_j, b_j\}_{j=1}^n$ according to Theorem* A.36 *(i). Then the solutions u, v in* (4.199)*,* (4.200) *equal $\overline{u^*}$, $\overline{v^*}$, in* (4.201)*,* (4.202)*, respectively, and hence represent algebro-geometric Th solutions if and only if $\underline{A}$ in* (4.210) *satisfies the constraint*

$$\begin{aligned}\operatorname{Re}(\underline{A}) = (1/2)\big(&\underline{A}_{P_{0,+}}(P_{0,-}) - \underline{\Delta} + (1/2)\underline{\operatorname{diag}}(R)\\ &+ (0,\dots,0,\chi_1,\dots,\chi_{k-1})\big) \pmod{\mathbb{Z}^n},\\ \underline{\chi} = (\chi_1,\dots,\chi_{k-1}),\ \ &\chi_j \in \{0,1\},\ j = 1,\dots,k-1.\end{aligned} \tag{4.229}$$

In particular, under the present hypotheses, the set of algebro-geometric Th solutions u, v in (4.206)*,* (4.207) *consist of 2^{k-1} connected components indexed by $(\chi_1,\dots,\chi_{k-1})$, $\chi_j \in \{0,1\}$, $j = 1,\dots,k-1$, and all such solutions u, v are smooth, $u, v \in C^\infty(\mathbb{R}^2)$.*

Proof Define the antiholomorphic involution $\rho_+ \colon (z,y) \mapsto (\overline{z},\overline{y})$ as in in Example A.35 (i). For brevity we only treat the case $1 \le k \le n$, where $(\mathcal{K}_n, \rho_+)$ is of nondividing type, in some detail. The case $k = n+1$ is slightly simpler and is commented on below. By Example A.35 (i), Theorem A.36 (cf. (A.66), (A.69)–(A.71)), one infers

$$r = k, \quad \overline{\tau} = R - \tau, \quad \underline{\operatorname{diag}}(R) = (\underbrace{1\dots,1}_{\ell}, \underbrace{0,\dots,0}_{k-1}),$$

$$\begin{aligned}&\overline{\theta(\underline{z})} = \theta(\overline{\underline{z}} + (1/2)\underline{\operatorname{diag}}(R)), \ \underline{z} \in \mathbb{C}^n,\\ &\rho_+(a_j) = a_j, \quad \rho_+(b_j) = (\underline{a}R)_j - b_j, \ j = 1,\dots,n,\\ &\rho^*\omega^{(2)}_{P_{0,+},0} = \omega^{(2)}_{P_{0,+},0}, \quad \underline{c}(1) \in \mathbb{R}^n,\\ &\rho^*\omega^{(2)}_{P_{\infty_+},0} = \omega^{(2)}_{P_{\infty_+},0}, \quad \underline{c}(n) \in \mathbb{R}^n,\\ &\rho_+^*\omega^{(3)}_{P_{0,-},P_{\infty_-}} = \omega^{(3)}_{P_{0,-},P_{\infty_-}}, \quad \omega_0,\ \omega_1 \in \mathbb{R}.\end{aligned}$$

Thus,

$$\overline{\underline{B}} = -\underline{B}, \quad \overline{\underline{C}} = -\underline{C},$$

by (4.211), and hence $u^* = \overline{u}$ in (4.206), (4.208) is equivalent to

$$\begin{aligned}\frac{C_{u^*}}{\overline{C}_u} &= \frac{\theta(\underline{A} + \underline{B}x + \underline{C}t - \underline{A}_{P_{0,+}}(P_{0,-}))\overline{\theta(\underline{A} + \underline{B}x + \underline{C}t)}}{\theta(\underline{A} + \underline{B}x + \underline{C}t + \underline{\Delta} - \underline{A}_{P_{0,+}}(P_{0,-}))\overline{\theta(\underline{A} + \underline{B}x + \underline{C}t + \underline{\Delta})}} \\ &= \frac{\theta(\underline{A} + \underline{B}x + \underline{C}t - \underline{A}_{P_{0,+}}(P_{0,-}))\theta(-\overline{\underline{A}} + \underline{B}x + \underline{C}t + (1/2)\underline{\mathrm{diag}}(R))}{\theta(\underline{A} + \underline{B}x + \underline{C}t + \underline{\Delta} - \underline{A}_{P_{0,+}}(P_{0,-}))\theta(-\overline{\underline{A}} + \underline{B}x + \underline{C}t - \overline{\underline{\Delta}} + (1/2)\underline{\mathrm{diag}}(R))}.\end{aligned} \tag{4.230}$$

Equation (4.230) is equivalent to

$$\underline{A} = -\overline{\underline{A}} + \underline{A}_{P_{0,+}}(P_{0,-}) - \overline{\underline{\Delta}} + (1/2)\underline{\mathrm{diag}}(R) + \underline{m}_1 + \underline{n}_1\tau, \tag{4.231}$$

$$\underline{A} = -\overline{\underline{A}} + \underline{A}_{P_{0,+}}(P_{0,-}) - \underline{\Delta} + (1/2)\underline{\mathrm{diag}}(R) + \underline{m}_2 + \underline{n}_2\tau \tag{4.232}$$

for some $\underline{n}_1, \underline{n}_2 \in \mathbb{Z}^n$ and arbitrary $\underline{m}_1, \underline{m}_2 \in \mathbb{Z}^n$. If one takes into account

$$\begin{gathered}\underline{\Delta} = \underline{A}_{P_{\infty_+}}(P_{0,+}) = \underline{A}_{P_{0,-}}(P_{\infty_-}) \in \mathbb{R}^n, \\ \underline{A}_{P_{0,+}}(P_{0,-}) \in \mathbb{R}^n, \quad \underline{A}_{P_{\infty_-}}(P_{\infty_+}) \in \mathbb{R}^n\end{gathered}$$

and uses $\overline{\tau} = R - \tau$, equations (4.231) and (4.232) are equivalent to

$$\mathrm{Re}(\underline{A}) = (1/2)\big(\underline{A}_{P_{0,+}}(P_{0,-}) - \underline{\Delta} + (1/2)\underline{\mathrm{diag}}(R) + \underline{m}_1\big), \quad \underline{m}_1 \in \mathbb{Z}^n$$

and $\underline{n}_1 = \underline{n}_2 = 0$, $\underline{m}_1 = \underline{m}_2$. Replacing $\underline{A}$ by $\underline{A} + m + \underline{n}\tau$ with $\underline{m}, \underline{n} \in \mathbb{Z}^n$ then yields

$$\mathrm{Re}(\underline{A}) = (1/2)\big(\underline{A}_{P_{0,+}}(P_{0,-}) - \underline{\Delta} + (1/2)\underline{\mathrm{diag}}(R) + \underline{m}_1\big) - \underline{m} - (1/2)\underline{n}R, \\ \underline{m}_1, \underline{m}, \underline{n} \in \mathbb{Z}^n \tag{4.233}$$

and hence (4.229). Exactly the same analysis applies to v and v^*. Due to the simple relationship (4.211) between $\underline{A}$ and $\underline{\widetilde{A}}$, one then verifies that the resulting reality constraint on $\underline{\widetilde{A}}$ is equivalent to that on $\underline{A}$ in (4.233). In the case $k = n + 1$, $\ell = 0$, where $(\mathcal{K}_n, \rho_+)$ is of dividing type, one infers $r = n + 1$ and $R = 0$ according to (A.65), simplifying the formulas just presented. Finally, $u, v \in C^\infty(\mathbb{R}^2)$ by Lemma 4.29. □

Even though (4.227) permits complex conjugate pairs of branch points and real pairs on either side of 0, it appears impossible to satisfy both equations (4.231) and (4.232) in such more general circumstances.

We conclude with the elementary genus zero example (i.e., $n = 0$), which is a case thus far excluded in this section.

Example 4.31 Assume $n = 0$, $P = (z, y) \in \mathcal{K}_0 \setminus \{P_{\infty_+}, P_{\infty_-}\}$, and let (x, t), $(x_0, t_0) \in \mathbb{R}^2$. Then

$$
\begin{aligned}
&\mathcal{K}_0\colon \mathcal{F}_0(z, y) = y^2 - R_2(z) = y^2 - (z - E_0)(z - E_1) = 0,\\
&c_1 = -(E_0 + E_1)/2, \quad g_1 = (E_0 E_1)^{1/2},\\
&\omega_1^\infty = (g_1 + c_1)/2, \quad \omega_1^0 = -\omega_1^\infty/(E_0 E_1),\\
&v(x, t) = v(x_0, t_0)\exp(-2i(\omega_1^\infty(x - x_0) - \omega_1^0(t - t_0))) = g_1 u(x, t),\\
&v^*(x, t) = v^*(x_0, t_0)\exp(2i(\omega_1^\infty(x - x_0) - \omega_1^0(t - t_0))) = g_1 u^*(x, t),\\
&v(x, t)v^*(x, t) = (c_1 - g_1)/2 = g_1^2 u(x, t)u^*(x, t),\\
&\phi(P, x, t) = \frac{y + z + g_1}{-2v(x, t)} = \frac{-2v^*(x, t)z}{y - z - g_1},\\
&\psi_1(P, x, x_0, t, t_0) = \exp(-i(y + \omega_1^\infty)(x - x_0) - i(g_1^{-1}z^{-1}y - \omega_1^0)(t - t_0)).
\end{aligned}
$$

4.5 Notes

This chapter follows Enolskii et al. (2000).

Section 4.1. Formal integrability of the classical massive Thirring model (4.32) was originally established by Mikhaĭlov (1976) utilizing a corresponding commutator representation (cf. (4.14) and (4.53)). Actually, one can replace (4.32) by the more general system (4.31) without losing formal integrability, and we decided to focus on the latter. Both (4.31) and (4.32) have been studied by numerous authors who derived the inverse scattering approach (Kaup and Lakoba (1996), Kawata et al. (1979), Kuznetsov and Mikhailov (1977), and Villarroel (1991)), considered soliton solutions (Barashenkov and Getmanov (1987), Barashenkov et al. (1993), Barashenkov and Getmanov (1993), Date (1979; 1982), David et al. (1984), and Vaklev (1996)), investigated Bäcklund transformations and close connections with other integrable equations (especially the sine-Gordon equation) (Alonso (1984), Kaup and Newell (1977), Lee (1993; 1994), Nijhoff et al. (1983), Prikarpatskii (1981), Prikarpatskii and Golod (1979), Tsuchida and Wadati (1996), and Wadati and Sogo (1983)), and considered monodromy deformations (Chowdhury and Naskar (1988)). Classes of relativistically invariant integrable systems containing the sine-Gordon and Thirring models as special cases are discussed, for instance, in Cherednik (1996) and Zakharov and Mikhailov (1978).

Originally, the Thirring model (Thirring (1958)) was constructed as a solvable model in quantum field theory. Although the quantum field theory aspects of this model are far beyond the scope of this monograph, it must be underscored that the model kept its fascination for two generations of mathematical physicists, as is witnessed by the incredible amount of attention paid to it since 1958 and by the interest it continues to generate (see, e.g., Ilieva and Thirring (1999) for a recent review).

Section 4.2. The polynomial recursion formalism presented follows Date (1978) and his explicit realization of the commutator representation of Mikhaĭlov (1976) of the classical massive Thirring system in terms of polynomials in the spectral parameter. Similar material can be found in papers by Holod and Prikarpatsky (1978) and Prikarpatskii and Golod (1979).

Section 4.3. As in all other chapters, the fundamental meromorphic function ϕ on $\mathcal{K}_n$ defined in (4.38) is still the key object of our algebro-geometric formalism. By (4.40)–(4.62), ϕ again links the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and its counterpart $\mathcal{D}_{\underline{\hat{\nu}}}$. This is of course a direct consequence of the identity (4.18) together with the factorizations of F_n and H_n in (4.22) and (4.24). Thus, our construction of positive divisors of degree n (respectively $n+1$ since the points $P_{0,-}$ and P_{∞_-} are also involved) on the hyperelliptic curve $\mathcal{K}_n$ of genus n again follows the recipe of Jacobi (1846), Mumford (1984, Sec. III a).1), and McKean (1985).

Dubrovin equations of the type (4.89) and (4.90) were discussed by Holod and Prikarpatsky (1978) and Prikarpatskii and Golod (1979) in the spatially periodic case, and by Date (1978) in the algebro-geometric context. They also derived the trace formulas collected in Lemma 4.11. The solution of the algebro-geometric initial value problem, as presented in Theorem 4.14, was first discussed in Enolskii et al. (2000).

Section 4.4. First attempts to derive algebro-geometric solutions of (4.32) were made by Date (1978) and almost simultaneously by Prikarpatskii and Golod (1979) (see also Holod and Prikarpatsky (1978)). Both papers are remarkably similar in strategy. In particular, both groups discuss theta function representations of appropriate symmetric functions associated with auxiliary divisors, but neither derives explicit theta function representations of u and v. An attempt at theta function representations of u, v, u^*, v^* for the general massive Thirring system (4.31) was made by Bikbaev (1985), following a different strategy than ours. More recently, algebro-geometric solutions of (4.32) were also briefly considered by Wisse (1993), but without explicitly deriving theta function representations for u and v.

In connection with Lemma 4.20, discussions of even and odd half-periods (singular and nonsingular ones) can be found, for instance, in Fay (1973, pp. 12–15) and Lewittes (1964).

The explicit theta function representations of u, v, u^*, and v^* in Theorem 4.23 complement the papers by Date (1978) and Prikarpatskii and Golod (1979) in which theta function representations were derived for appropriate symmetric functions associated with auxiliary divisors but not explicitly for u, v, u^*, and v^*. The treatment in Enolskii et al. (2000) appears to offer the first complete solution of the problem at hand.

The smoothness of solutions of the classical massive Thirring model (4.32) stated in Lemma 4.29 was first discussed in Bikbaev (1985).

Reality constraints of the type (4.33) have also been discussed by Bikbaev (1985); see also Date (1982). In analogy to the sGmKdV and $\mathrm{nS}_{\pm}$ cases, one expects the 2^{k-1} connected components alluded to in Lemma 4.30 to be real tori. Due to the scaling property discussed in (4.35), one then expects these tori to be of dimension $n+1$ (in analogy to the $\mathrm{nS}_{\pm}$ case). However, as far as we know, the precise nature of these connected components has not been investigated.

5

The Camassa–Holm Hierarchy

The motion proceeds in straight lines – but not at constant speed . . . Why, nobody knows.

Adrian Constantin and Henry P. McKean[1]

5.1 Contents

The Camassa–Holm equation, also known as the dispersive shallow water equation,

$$4u_t - u_{xxt} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x = 0$$

for a function $u = u(x, t)$, was established as an integrable evolution equation[2] in 1993. This chapter focuses on the construction of algebro-geometric solutions of the Camassa–Holm (CH) hierarchy. Below we briefly summarize the principal content of each section. A more detailed discussion, using the KdV hierarchy as a model, has been provided in the introduction to this volume.

Section 5.2.

- polynomial recursion formalism, zero-curvature pairs (U, V_n)
- stationary and time-dependent CH hierarchy
- hyperelliptic curve $\mathcal{K}_n$

Section 5.3. (stationary)

- properties of ϕ and the Baker–Akhiezer vector Ψ
- Dubrovin equations for auxiliary divisors
- trace formula for u
- theta function representations for ϕ and u
- the algebro-geometric initial value problem

[1] Constantin and McKean (1999, p. 954).
[2] A guide to the literature can be found in the detailed notes at the end of this chapter.

Section 5.4. (time-dependent)

- properties of ϕ and the Baker–Akhiezer vector Ψ
- Dubrovin equations for auxiliary divisors
- trace formula for u
- theta function representations for ϕ and u
- the algebro-geometric initial value problem

This chapter relies on terminology and notions developed in connection with compact Riemann surfaces. A brief summary of key results as well as definitions of some of the main quantities can be found in Appendices A, C, and F.

5.2 The CH Hierarchy, Recursion Relations, and Hyperelliptic Curves

In this section we provide the construction of the CH hierarchy, a completely integrable hierarchy of nonlinear evolution equations in which the Camassa–Holm (CH) equation, or dispersive shallow water equation, is the first element in the hierarchy (the higher-order CH equations will turn out to be nonlocal with respect to u). Using a polynomial recursion formalism we derive the corresponding sequence of zero-curvature pairs and introduce the underlying hyperelliptic curve in connection with the stationary CH hierarchy.

Throughout this section we suppose the following hypothesis.

Hypothesis 5.1 *In the stationary case we assume that*

$$u \in C^\infty(\mathbb{R}),\ \partial_x^m u \in L^\infty(\mathbb{R}),\ m \in \mathbb{N}_0. \tag{5.1}$$

In the time-dependent case we suppose

$$\begin{aligned} &u(\,\cdot\,,t) \in C^\infty(\mathbb{R}),\ \partial_x^m u(\,\cdot\,,t) \in L^\infty(\mathbb{R}),\ m \in \mathbb{N}_0,\ t \in \mathbb{R}, \\ &u(x,\,\cdot\,),\ u_{xx}(x,\,\cdot\,) \in C^1(\mathbb{R}),\ x \in \mathbb{R}. \end{aligned} \tag{5.2}$$

We start by formulating the basic polynomial set-up. Define $\{f_\ell\}_{\ell\in\mathbb{N}_0}$ recursively by

$$f_0 = 1, \quad f_{\ell,x} = -2\mathcal{G}\big(2(4u - u_{xx})f_{\ell-1,x} + (4u_x - u_{xxx})f_{\ell-1}\big), \quad \ell \in \mathbb{N}, \tag{5.3}$$

where $\mathcal{G}$ is given by

$$\mathcal{G}\colon L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R}), \quad (\mathcal{G}v)(x) = \frac{1}{4}\int_{\mathbb{R}} dy\, e^{-2|x-y|}v(y), \quad x \in \mathbb{R},\ v \in L^\infty(\mathbb{R}). \tag{5.4}$$

One observes that $\mathcal{G}$ is the resolvent of minus the one-dimensional Laplacian at spectral parameter equal to -4, that is,

$$\mathcal{G} = \Big(-\frac{d^2}{dx^2} + 4 \Big)^{-1}.$$

The first coefficient reads

$$f_1 = -2u + c_1,$$

where c_1 is an integration constant. Subsequent coefficients are nonlocal with respect to u. At each level a new integration constant, denoted by c_ℓ, is introduced. Moreover, we introduce coefficients $\{g_\ell\}_{\ell\in\mathbb{N}_0}$ and $\{h_\ell\}_{\ell\in\mathbb{N}_0}$ by

$$g_\ell = f_\ell + (1/2) f_{\ell,x}, \quad \ell \in \mathbb{N}_0, \tag{5.5}$$

$$h_\ell = (4u - u_{xx}) f_\ell - g_{\ell+1,x}, \quad \ell \in \mathbb{N}_0. \tag{5.6}$$

Explicitly, one computes

$$\begin{aligned}
f_0 &= 1,\\
f_1 &= -2u + c_1,\\
f_2 &= 2u^2 + 2\mathcal{G}\big(u_x^2 + 8u^2\big) + c_1(-2u) + c_2,\\
g_0 &= 1,\\
g_1 &= -2u - u_x + c_1,\\
g_2 &= 2u^2 + 2uu_x + 2\mathcal{G}\big(u_x^2 + u_x u_{xx} + 8uu_x + 8u^2\big) + c_1(-2u - u_x) + c_2,\\
h_0 &= 4u + 2u_x,\\
h_1 &= -2u_x^2 - 4uu_x - 8u^2\\
&\quad - 2\mathcal{G}\big(u_x u_{xxx} + u_{xx}^2 + 2u_x u_{xx} + 8uu_{xx} + 8u_x^2 + 16uu_x\big)\\
&\quad + c_1(4u + 2u_x), \quad \text{etc.}
\end{aligned}$$

For later use it is convenient also to introduce the corresponding homogeneous coefficients $\hat{f}_\ell$, $\hat{g}_\ell$, and $\hat{h}_\ell$ defined by the vanishing of the integration constants c_k, $k = 1, \dots, \ell$,

$$\hat{f}_0 = f_0 = 1, \quad \hat{f}_\ell = f_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{5.7}$$

$$\hat{g}_0 = g_0 = 1, \quad \hat{g}_\ell = g_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}, \tag{5.8}$$

$$\hat{h}_0 = h_0 = (4u + 2u_x), \quad \hat{h}_\ell = h_\ell\big|_{c_k=0,\, k=1,\dots,\ell}, \quad \ell \in \mathbb{N}. \tag{5.9}$$

Hence,

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{f}_k, \quad g_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{g}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k} \hat{h}_k, \quad \ell \in \mathbb{N}_0,$$

defining

$$c_0 = 1.$$

Next, given Hypothesis 5.1, one introduces the 2×2 matrix U by

$$U(z) = \begin{pmatrix} -1 & 1 \\ z^{-1}(4u - u_{xx}) & 1 \end{pmatrix}, \tag{5.10}$$

and for each $n \in \mathbb{N}_0$ the following 2×2 matrix V_n by

$$V_n(z) = \begin{pmatrix} -G_n(z) & F_n(z) \\ z^{-1}H_n(z) & G_n(z) \end{pmatrix}, \quad z \in \mathbb{C} \setminus \{0\}, \; n \in \mathbb{N}_0, \tag{5.11}$$

assuming F_n, G_n, and H_n to be polynomials of degree n with C^∞ coefficients with respect to x. Postulating the zero-curvature condition

$$-V_{n,x} + [U, V_n] = 0, \tag{5.12}$$

one finds

$$F_{n,x} = 2(G_n - F_n), \tag{5.13}$$

$$zG_{n,x} = (4u - u_{xx})F_n - H_n, \tag{5.14}$$

$$H_{n,x} = 2H_n - 2(4u - u_{xx})G_n. \tag{5.15}$$

From (5.13)–(5.15), one infers that

$$\frac{d}{dx}\det(V_n(z,x)) = -\frac{1}{z}\frac{d}{dx}\big(zG_n(z,x)^2 + F_n(z,x)H_n(z,x)\big) = 0,$$

and hence

$$zG_n^2 + F_nH_n = Q_{2n+1}, \tag{5.16}$$

where the polynomial Q_{2n+1} of degree $2n+1$ is x-independent. Actually it turns out that it is more convenient to define

$$R_{2n+2}(z) = zQ_{2n+1}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad E_0 = 0, \; E_1, \dots, E_{2n+1} \in \mathbb{C} \tag{5.17}$$

so that (5.16) becomes

$$z^2G_n^2 + zF_nH_n = R_{2n+2}. \tag{5.18}$$

Moreover, computing the characteristic equation[1] of iV_n

$$\begin{aligned} \det(wI_2 - iV_n(z)) &= w^2 - \det(V_n(z)) = w^2 + G_n(z)^2 + z^{-1}F_n(z)H_n(z) \\ &= w^2 + z^{-2}R_{2n+2}(z) = 0, \end{aligned}$$

[1] I_2 denotes the identity matrix in $\mathbb{C}^2$.

one is naturally led to introducing the hyperelliptic curve $\mathcal{K}_n$ of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) defined by

$$\mathcal{K}_n \colon \mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0. \tag{5.19}$$

To establish the connection between the zero-curvature formalism and the recursion relations (5.3), (5.5), (5.6), we now make the following polynomial ansatz with respect to the spectral parameter z,

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell, \tag{5.20}$$

$$G_n(z) = \sum_{\ell=0}^{n} g_{n-\ell} z^\ell, \tag{5.21}$$

$$H_n(z) = \sum_{\ell=0}^{n} h_{n-\ell} z^\ell. \tag{5.22}$$

Insertion of (5.20)–(5.22) into (5.13)–(5.15) then yields the recursion relations (5.3)–(5.4) and (5.5) for f_ℓ and g_ℓ for $\ell = 0, \dots, n$. For fixed $n \in \mathbb{N}$ we obtain the recursion (5.6) for h_ℓ for $\ell = 0, \dots, n-1$ and

$$h_n = (4u - u_{xx}) f_n. \tag{5.23}$$

(When $n = 0$ one directly gets $h_0 = (4u - u_{xx})$.) Moreover, taking $z = 0$ in (5.18) yields

$$f_n h_n = -\prod_{m=1}^{2n+1} E_m.$$

In addition, one finds

$$h_{n,x} - 2h_n + 2(4u - u_{xx}) g_n = 0, \quad n \in \mathbb{N}_0. \tag{5.24}$$

Using the relations (5.5) and (5.23) permits one to write (5.24) as[1]

$$\text{s-CH}_n(u) = (u_{xxx} - 4u_x) f_n(u) - 2(4u - u_{xx}) f_{n,x}(u) = 0, \quad n \in \mathbb{N}_0. \tag{5.25}$$

Varying $n \in \mathbb{N}_0$ in (5.25) then defines the stationary CH hierarchy. We record the first few equations explicitly,

$$\begin{aligned}
\text{s-CH}_0(u) &= u_{xxx} - 4u_x = 0,\\
\text{s-CH}_1(u) &= -2uu_{xxx} - 4u_x u_{xx} + 24uu_x + c_1(u_{xxx} - 4u_x) = 0,\\
\text{s-CH}_2(u) &= 2u^2 u_{xxx} - 8uu_x u_{xx} - 40u^2 u_x + 2(u_{xxx} - 4u_x)\mathcal{G}\big(u_x^2 + 8u^2\big)\\
&\quad - 8(4u - u_{xx})\mathcal{G}(u_x u_{xx} + 8uu_x)\\
&\quad + c_1(-2uu_{xxx} - 4u_x u_{xx} + 24uu_x) + c_2(u_{xxx} - 4u_x) = 0, \text{ etc.}
\end{aligned}$$

[1] In a slight abuse of notation we will occasionally stress the functional dependence of f_ℓ on u, writing $f_\ell(u)$.

By definition, the set of solutions of (5.25), with n ranging in $\mathbb{N}_0$ and c_ℓ in $\mathbb{C}$, $\ell \in \mathbb{N}$, represents the class of algebro-geometric CH solutions. If u satisfies one of the stationary CH equations in (5.25) for a particular value of n, then it satisfies infinitely many such equations of order higher than n for certain choices of integration constants c_ℓ (one can follow the argument in Remark 1.5). At times it will be convenient to abbreviate algebro-geometric stationary CH solutions u simply as CH *potentials*.

In the following we will frequently assume that u satisfies the nth stationary CH equation. By this we mean it satisfies one of the nth stationary CH equations after a particular choice of integration constants $c_\ell \in \mathbb{C}$, $\ell = 1, \dots, n$, $n \geq 1$, has been made.

For later use we also introduce the corresponding homogeneous polynomials $\widehat{F}_\ell$, $\widehat{G}_\ell$, and $\widehat{H}_\ell$ defined by

$$\widehat{F}_\ell(z) = F_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{f}_{\ell-k} z^k, \quad \ell = 0, \dots, n, \tag{5.26}$$

$$\widehat{G}_\ell(z) = G_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{g}_{\ell-k} z^k, \quad \ell = 0, \dots, n, \tag{5.27}$$

$$\widehat{H}_\ell(z) = H_\ell(z)\big|_{c_k=0,\, k=1,\dots,\ell} = \sum_{k=0}^{\ell} \hat{h}_{\ell-k} z^k, \quad \ell = 0, \dots, n-1, \tag{5.28}$$

$$\widehat{H}_n(z) = (4u - u_{xx})\hat{f}_n + \sum_{k=1}^{n} \hat{h}_{n-k} z^k. \tag{5.29}$$

In accordance with our notation introduced in (5.7)–(5.9) and (5.26)–(5.29), the corresponding homogeneous stationary CH equations are then defined by

$$\text{s-}\widehat{\text{CH}}_n(u) = \text{s-CH}_n(u)\big|_{c_\ell=0,\, \ell=1,\dots,n} = 0, \quad n \in \mathbb{N}_0.$$

Using equations (5.13)–(5.15) one can also derive individual differential equations for F_n and H_n. Focusing on F_n only, one obtains

$$F_{n,xxx} - 4(z^{-1}(4u - u_{xx}) + 1)F_{n,x} - 2z^{-1}(4u_x - u_{xxx})F_n = 0. \tag{5.30}$$

This is of course consistent with (5.20) and (5.3) (applying $\mathcal{G}^{-1}$ to (5.3)). Multiplying (5.30) with F_n and integrating the result yields

$$F_{n,xx}F_n - 2^{-1}F_{n,x}^2 - 2F_n^2 - 2z^{-1}(4u - u_{xx})F_n^2 = C$$

for some $C = C(z)$, constant with respect to x. Differentiating (5.13), inserting (5.14) into the resulting equation, and comparing with (5.13) and (5.18) then yields

$$C(z) = -2z^{-2}R_{2n+2}(z).$$

Thus,

$$-(z^2/2)F_{n,xx}F_n + (z^2/4)F_{n,x}^2 + z^2F_n^2 + z(4u-u_{xx})F_n^2 = R_{2n+2}. \quad (5.31)$$

Equation (5.31) can be used to derive a nonlinear recursion relation for the homogeneous coefficients $\hat{f}_\ell$ (i.e., the ones satisfying (5.3) in the case of vanishing integration constants) as proved in Theorem D.6 in Appendix D. In addition, as proven in Theorem D.6, (5.31) leads to an explicit determination of the integration constants $c_1, \dots, c_n$ in

$$\text{s-CH}_n(u) = (u_{xxx} - 4u_x)f_n(u) - 2(4u-u_{xx})f_{n,x}(u) = 0, \quad n \in \mathbb{N}_0.$$

in terms of the zeros $E_0 = 0, E_1, \dots, E_{2n+1}$ of the associated polynomial R_{2n+2} in (5.17). In fact, one can prove (cf. (D.59))

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n, \quad (5.32)$$

where

$$\begin{aligned} c_0(\underline{E}) =& 1, \\ c_k(\underline{E}) =& -\sum_{\substack{j_1,\dots,j_{2n+1}=0 \\ j_1+\dots+j_{2n+1}=k}}^{k} \frac{(2j_1)!\cdots(2j_{2n+1})!}{2^{2k}(j_1!)^2\cdots(j_{2n+1}!)^2(2j_1-1)\cdots(2j_{2n+1}-1)} \\ & \times E_1^{j_1}\cdots E_{2n+1}^{j_{2n+1}}, \quad k=1,\dots,n. \end{aligned} \quad (5.33)$$

Next, we turn to the time-dependent CH hierarchy. We introduce a deformation parameter $t_n \in \mathbb{R}$ into u (replacing $u(x)$ by $u(x,t_n)$), and note that the definitions (5.10), (5.11), and (5.20)–(5.22) of U, V_n, and F_n, G_n, and H_n still apply. The corresponding zero-curvature relation reads

$$U_{t_n} - V_{n,x} + [U, V_n] = 0, \quad n \in \mathbb{N}_0,$$

which results in the following set of equations

$$4u_{t_n} - u_{xxt_n} - H_{n,x} + 2H_n - 2(4u-u_{xx})G_n = 0, \quad (5.34)$$

$$F_{n,x} = 2(G_n - F_n), \quad (5.35)$$

$$zG_{n,x} = (4u-u_{xx})F_n - H_n. \quad (5.36)$$

Inserting the polynomial expressions for F_n, H_n, and G_n into (5.35) and (5.36), respectively, first yields recursion relations (5.3) and (5.5) for f_ℓ and g_ℓ for $\ell = 0, \dots, n$. For fixed $n \in \mathbb{N}$ we obtain from (5.34) the recursion (5.6) for h_ℓ for $\ell = 0, \dots, n-1$ and

$$h_n = (4u-u_{xx})f_n. \quad (5.37)$$

(When $n=0$ one directly gets $h_0 = (4u-u_{xx})$.) In addition, one finds

$$4u_{t_n} - u_{xxt_n} - h_{n,x} + 2h_n - 2(4u-u_{xx})g_n = 0, \quad n \in \mathbb{N}_0. \quad (5.38)$$

Using relations (5.5) and (5.37) permits one to write (5.38) as

$$\mathrm{CH}_n(u) = 4u_{t_n} - u_{xxt_n} + (u_{xxx} - 4u_x)f_n(u) - 2(4u - u_{xx})f_{n,x}(u) = 0, \quad (x,t_n) \in \mathbb{R}^2,\ n \in \mathbb{N}_0. \tag{5.39}$$

Varying $n \in \mathbb{N}_0$ in (5.39) then defines the time-dependent CH hierarchy. We record the first few equations explicitly,

$$\begin{aligned}
\mathrm{CH}_0(u) &= 4u_{t_0} - u_{xxt_0} + u_{xxx} - 4u_x = 0,\\
\mathrm{CH}_1(u) &= 4u_{t_1} - u_{xxt_1} - 2uu_{xxx} - 4u_xu_{xx} + 24uu_x + c_1(u_{xxx} - 4u_x) = 0,\\
\mathrm{CH}_2(u) &= 4u_{t_2} - u_{xxt_2} + 2u^2u_{xxx} - 8uu_xu_{xx} - 40u^2u_x\\
&\quad + 2(u_{xxx} - 4u_x)\mathcal{G}\big(u_x^2 + 8u^2\big) - 8(4u - u_{xx})\mathcal{G}(u_xu_{xx} + 8uu_x)\\
&\quad + c_1(-2uu_{xxx} - 4u_xu_{xx} + 24uu_x) + c_2(u_{xxx} - 4u_x) = 0, \ \text{etc.}
\end{aligned} \tag{5.40}$$

Similarly, one introduces the corresponding homogeneous CH hierarchy by

$$\widehat{\mathrm{CH}}_n(u) = \mathrm{CH}_n(u)\big|_{c_\ell=0,\,\ell=1,\dots,n} = 0, \quad n \in \mathbb{N}_0.$$

Up to inessential scaling of the (x, t_1) variables, $\widehat{\mathrm{CH}}_1(u) = 0$ represents *the* Camassa–Holm equation as discussed in the references cited in the notes to this section.

Our recursion formalism was introduced under the assumption of a sufficiently smooth function u in Hypothesis 5.1. The actual existence of smooth global solutions of the initial value problem associated with the CH hierarchy (5.40) is a nontrivial issue and is discussed in the references cited in the notes to Sections 5.1 and 5.2.

5.3 The Stationary CH Formalism

This section is devoted to a detailed study of the stationary CH hierarchy and its algebro-geometric solutions. Our principal tools are derived from combining the polynomial recursion formalism introduced in Section 5.2 and a fundamental meromorphic function ϕ on a hyperelliptic curve $\mathcal{K}_n$. With the help of ϕ we study the Baker–Akhiezer vector Ψ, Dubrovin-type equations governing the motion of auxiliary divisors on $\mathcal{K}_n$, trace formulas, and theta function representations of ϕ, Ψ, and u. We also discuss the algebro-geometric initial value problem of constructing u from the Dubrovin equations and auxiliary divisors as initial data.

For major parts of this section we suppose

$$u \in C^\infty(\mathbb{R}),\ \frac{d^m u}{dx^m} \in L^\infty(\mathbb{R}),\ m \in \mathbb{N}_0, \tag{5.41}$$

and assume (5.3), (5.4), (5.5), (5.6), (5.10)–(5.12), (5.17), (5.18), (5.20)–(5.22), (5.23)–(5.25), keeping $n \in \mathbb{N}_0$ fixed.

We recall the hyperelliptic curve

$$\begin{aligned} &\mathcal{K}_n\colon \mathcal{F}_n(z,y) = y^2 - R_{2n+2}(z) = 0, \\ &R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad E_0 = 0,\ E_1, \dots, E_{2n+1} \in \mathbb{C}, \end{aligned} \tag{5.42}$$

as introduced in (5.19). The curve $\mathcal{K}_n$ is compactified by joining two points at infinity, $P_{\infty_\pm}$, $P_{\infty_+} \neq P_{\infty_-}$, but for notational simplicity the compactification is also denoted by $\mathcal{K}_n$. Points P on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ are represented as pairs $P = (z, y)$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_n$ satisfying $\mathcal{F}_n(z, y) = 0$. The complex structure on $\mathcal{K}_n$ is then defined in the usual way (see Appendix C). Hence, $\mathcal{K}_n$ becomes a two-sheeted hyperelliptic Riemann surface of (arithmetic) genus $n \in \mathbb{N}_0$ (possibly with a singular affine part) in a standard manner. In the following we will occasionally impose further constraints on the zeros E_m of R_{2n+2} introduced in (5.42) and assume that

$$E_0 = 0,\ E_1, \dots, E_{2n+1} \in \mathbb{C} \setminus \{0\}. \tag{5.43}$$

We also emphasize that by fixing the curve $\mathcal{K}_n$ (i.e., by fixing $E_0 = 0, E_1, \dots, E_{2n+1}$), the integration constants $c_1, \dots, c_n$ in f_n (and hence in the corresponding stationary CH_n equation) are uniquely determined, as is clear from (5.32), (5.33), which establish the integration constants c_ℓ as symmetric functions of $E_1, \dots, E_{2n+1}$.

For notational simplicity we will usually tacitly assume that $n \in \mathbb{N}$. (The case $n = 0$ is explicitly treated in Example 5.13.)

In the following the roots of the polynomials F_n and H_n will play a special role, and hence we write

$$F_n(z) = \prod_{j=1}^{n}(z - \mu_j), \quad H_n(z) = h_0 \prod_{j=1}^{n}(z - \nu_j). \tag{5.44}$$

Moreover, we introduce

$$\hat{\mu}_j(x) = (\mu_j(x), -\mu_j(x)G_n(\mu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n,\ x \in \mathbb{R}, \tag{5.45}$$

$$\hat{\nu}_j(x) = (\nu_j(x), \nu_j(x)G_n(\nu_j(x), x)) \in \mathcal{K}_n, \quad j = 1, \dots, n,\ x \in \mathbb{R}, \tag{5.46}$$

lifting μ_j and ν_j to $\mathcal{K}_n$, and

$$P_0 = (0, 0).$$

The branch of $y(\,\cdot\,)$ near $P_{\infty_\pm}$ is fixed according to

$$\lim_{\substack{|z(P)|\to\infty \\ P\to P_{\infty_\pm}}} \frac{y(P)}{z(P)G_n(z(P), x)} = \mp 1.$$

Due to assumption (5.41), u is smooth and bounded, and hence $F_n(z, \,\cdot\,)$ and

$H_n(z, \cdot\,)$ share the same property. Thus, one concludes

$$\mu_j, \nu_k \in C(\mathbb{R}), \ \ j, k = 1, \dots, n,$$

taking multiplicities (and appropriate renumbering) of the zeros of F_n and H_n into account. (Away from collisions of zeros, μ_j and ν_k are of course C^∞.)

Next, define the fundamental meromorphic function $\phi(\,\cdot\,, x)$ on $\mathcal{K}_n$ by

$$\phi(P, x) = \frac{y - zG_n(z, x)}{F_n(z, x)} \tag{5.47}$$

$$= \frac{zH_n(z, x)}{y + zG_n(z, x)}, \tag{5.48}$$

$$P = (z, y) \in \mathcal{K}_n, \ x \in \mathbb{R}.$$

Assuming (5.43), the divisor $(\phi(\,\cdot\,, x))$ of $\phi(\,\cdot\,, x)$ is given by[1]

$$(\phi(\,\cdot\,, x)) = \mathcal{D}_{P_0 \underline{\hat{\nu}}(x)} - \mathcal{D}_{P_{\infty_+} \underline{\hat{\mu}}(x)}. \tag{5.49}$$

Here we abbreviated

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}, \ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n).$$

Given $\phi(\,\cdot\,, x)$, one then defines the associated vector $\Psi(\,\cdot\,, x, x_0)$ on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$ by

$$\Psi(P, x, x_0) = \begin{pmatrix} \psi_1(P, x, x_0) \\ \psi_2(P, x, x_0) \end{pmatrix}, \quad P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}, \ (x, x_0) \in \mathbb{R}^2, \tag{5.50}$$

where

$$\psi_1(P, x, x_0) = \exp\bigg(- z^{-1} \int_{x_0}^{x} dx' \, \phi(P, x') - (x - x_0) \bigg), \tag{5.51}$$

$$\psi_2(P, x, x_0) = -\psi_1(P, x, x_0)\phi(P, x)/z. \tag{5.52}$$

Although Ψ is formally the analog of the stationary Baker–Akhiezer vector of the stationary CH hierarchy when compared with analogous definitions in the context of the KdV or AKNS hierarchies, its actual properties in a neighborhood of its essential singularity will feature characteristic differences from standard Baker–Akhiezer vectors (cf. Remark 5.6). We summarize the fundamental properties of ϕ and Ψ in the following result.

[1] If h_0 is permitted to vanish at a point $x_1 \in \mathbb{R}$, then for $x = x_1$, the polynomial $H_n(\,\cdot\,, x_1)$ is at most of degree $n - 1$ (cf. (5.22)), and (5.49) is altered to

$$(\phi(\,\cdot\,, x_1)) = \mathcal{D}_{P_0 P_{\infty_-} \hat{\nu}_1(x_1), \dots, \hat{\nu}_{n-1}(x_1)} - \mathcal{D}_{P_{\infty_+} \underline{\hat{\mu}}(x)},$$

that is, one of the $\hat{\nu}_j(x)$ tends to P_{∞_-} as $x \to x_1$ (cf. also (5.71)). Analogously, one can discuss the case of several $\hat{\nu}_j$ approaching P_{∞_-}. Since this can be viewed as a limiting case of (5.49), we will henceforth not particularly distinguish the case $h_0 \neq 0$ from the more general situation in which h_0 is permitted to vanish.

Lemma 5.2 *Suppose* (5.41) *and assume that u satisfies the nth stationary CH equation* (5.25). *Moreover, let $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$, $(x, x_0) \in \mathbb{R}^2$. Then ϕ satisfies the Riccati-type equation*

$$\phi_x(P) - z^{-1}\phi(P)^2 - 2\phi(P) = u_{xx} - 4u, \tag{5.53}$$

as well as

$$\phi(P)\phi(P^*) = -\frac{zH_n(z)}{F_n(z)}, \tag{5.54}$$

$$\phi(P) + \phi(P^*) = -2\frac{zG_n(z)}{F_n(z)}, \tag{5.55}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}, \tag{5.56}$$

whereas Ψ fulfills

$$\Psi_x(P) = U(z)\Psi(P), \tag{5.57}$$

$$-y\Psi(P) = zV_n(z)\Psi(P), \tag{5.58}$$

$$\psi_1(P, x, x_0) = \left(\frac{F_n(z, x)}{F_n(z, x_0)}\right)^{1/2} \exp\left(-(y/z)\int_{x_0}^{x} dx' F_n(z, x')^{-1}\right), \tag{5.59}$$

$$\psi_1(P, x, x_0)\psi_1(P^*, x, x_0) = \frac{F_n(z, x)}{F_n(z, x_0)}, \tag{5.60}$$

$$\psi_2(P, x, x_0)\psi_2(P^*, x, x_0) = -\frac{H_n(z, x)}{zF_n(z, x_0)}, \tag{5.61}$$

$$\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) + \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = 2\frac{G_n(z, x)}{F_n(z, x_0)}, \tag{5.62}$$

$$\psi_1(P, x, x_0)\psi_2(P^*, x, x_0) - \psi_1(P^*, x, x_0)\psi_2(P, x, x_0) = \frac{2y}{zF_n(z, x_0)}. \tag{5.63}$$

In addition, as long as the zeros of $F_n(\,\cdot\,, x)$ are all simple for $x \in \Omega$, $\Omega \subseteq \mathbb{R}$ an open interval, $\Psi(\,\cdot\,, x, x_0)$ is meromorphic on $\mathcal{K}_n \setminus \{P_0\}$ for $x, x_0 \in \Omega$.

Proof Equation (5.53) follows using the definition (5.47) of ϕ as well as relations (5.13)–(5.15). The other relations, (5.54)–(5.56), are easy consequences of $y(P^*) = -y(P)$, (5.47), and (5.48). By (5.50)–(5.52), Ψ is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ away from the poles $\hat{\mu}_j(x')$ of $\phi(\,\cdot\,, x')$. By (5.13), (5.45), and (5.47),

$$-z^{-1}\phi(P, x') \underset{P \to \hat{\mu}_j(x')}{=} \partial_{x'} \ln(F_n(z, x')) + O(1) \text{ as } z \to \mu_j(x'), \tag{5.64}$$

and hence ψ_1 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ by (5.51) as long as the zeros of $F_n(\,\cdot\,, x)$ are all simple. This follows from (5.51) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j$ of $\{\hat{\mu}_j(x') \in \mathcal{K}_n \mid x' \in \Omega,\ x' \in [x_0, x]\}$ such that $\hat{\mu}_k(x')$

$\notin \mathcal{U}_j$ for all $x' \in [x_0, x]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$. Since ϕ is meromorphic on $\mathcal{K}_n$ by (5.47), ψ_2 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ by (5.52). The remaining properties of Ψ can be verified by using the definition (5.50)–(5.52) as well as relations (5.53)–(5.56). In particular, equation (5.59) follows by inserting the definition of ϕ, (5.47), into (5.51), using (5.13). $\square$

Equations (5.60)–(5.63) show that the basic identity (5.18), $z^2 G_n^2 + zF_n H_n = R_{2n+2}$, is equivalent to the elementary fact

$$(\psi_{1,+}\psi_{2,-} + \psi_{1,-}\psi_{2,+})^2 - 4\psi_{1,+}\psi_{1,-}\psi_{2,+}\psi_{2,-} = (\psi_{1,+}\psi_{2,-} - \psi_{1,-}\psi_{2,+})^2,$$

identifying $\psi_1(P) = \psi_{1,+}$, $\psi_1(P^*) = \psi_{1,-}$, $\psi_2(P) = \psi_{2,+}$, $\psi_2(P^*) = \psi_{2,-}$. This provides the intimate link between our approach and the squared function systems also employed in the literature in connection with algebro-geometric solutions of hierarchies of soliton equations.

Next, we derive Dubrovin-type equations, that is, first-order systems of nonlinear differential equations that govern the dynamics of μ_j and ν_j with respect to variations of x. Since, in the remainder of this section, we will frequently assume the affine part of $\mathcal{K}_n$ to be nonsingular, we list all restrictions on $\mathcal{K}_n$ in this case,

$$E_0 = 0,\ E_m \in \mathbb{C} \setminus \{0\},\ E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 1, \dots, 2n+1. \tag{5.65}$$

Lemma 5.3 *Suppose* (5.41) *and assume that u satisfies the nth stationary CH equation* (5.25) *subject to the constraint* (5.65) *on an open interval $\widetilde{\Omega}_\mu \subseteq \mathbb{R}$. Moreover, suppose that the zeros μ_j, $j = 1, \dots, n$, of $F_n(\,\cdot\,)$ remain distinct and nonzero on $\widetilde{\Omega}_\mu$. Then $\{\hat{\mu}_j\}_{j=1,\dots,n}$, defined by* (5.45)*, satisfies the following first-order system of differential equations on $\widetilde{\Omega}_\mu$*

$$\mu_{j,x} = 2\frac{y(\hat{\mu}_j)}{\mu_j} \prod_{\substack{k=1 \\ k\neq j}}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, n. \tag{5.66}$$

Next, assume the affine part of $\mathcal{K}_n$ to be nonsingular and introduce the initial condition

$$\{\hat{\mu}_j(x_0)\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{5.67}$$

for some $x_0 \in \mathbb{R}$, where $\mu_j(x_0) \neq 0$, $j = 1, \dots, n$, are assumed to be distinct. Then there exists an open interval $\Omega_\mu \subseteq \mathbb{R}$, with $x_0 \in \Omega_\mu$, such that the initial value problem (5.66)*,* (5.67) *has a unique solution $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{5.68}$$

and μ_j, $j = 1, \dots, n$, remain distinct and nonzero on Ω_μ.
For the zeros $\{\nu_j\}_{j=1,\dots,n}$ of $H_n(\,\cdot\,)$ similar statements hold with μ_j and Ω_μ replaced

by ν_j *and* Ω_ν, *etc. In particular,* $\{\hat{\nu}_j\}_{j=1,\dots,n}$, *defined by* (5.46), *satisfies the system*

$$\nu_{j,x} = 2\frac{(4u - u_{xx})y(\hat{\nu}_j)}{(4u + 2u_x)\nu_j} \prod_{\substack{k=1 \\ k\neq j}}^{n} (\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, n. \tag{5.69}$$

Proof We only prove equation (5.66) since the proof of (5.69) follows in an identical manner. Inserting $z = \mu_j$ into equation (5.13), one concludes from (5.45),

$$F_{n,x}(\mu_j) = -\mu_{j,x} \prod_{\substack{k=1 \\ k\neq j}}^{n} (\mu_j - \mu_k) = 2G_n(\mu_j) = -2y(\hat{\mu}_j)/\mu_j,$$

proving (5.66). The smoothness assertion (5.68) is clear as long as $\hat{\mu}_j$ stays away from the branch points $(E_m, 0)$. In case $\hat{\mu}_j$ hits such a branch point, one can use the local chart around $(E_m, 0)$ (with local coordinate $\zeta = \sigma(z - E_m)^{1/2}$, $\sigma = \pm 1$) to verify (5.68) as in the proof of Lemma 1.10. $\square$

Combining the polynomial approach in Section 5.2 with (5.44) readily yields trace formulas for the CH invariants, that is, expressions of f_ℓ and h_ℓ in terms of symmetric functions of the zeros μ_j and ν_j of F_n and H_n, respectively. For simplicity we just record the simplest case.

Lemma 5.4 *Suppose* (5.41) *and assume that* u *satisfies the* n*th stationary CH equation* (5.25). *Then,*

$$u = -\frac{1}{4} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=1}^{n} \mu_j. \tag{5.70}$$

Proof Equation (5.70) follows by considering the coefficient of z^{n-1} in F_n in (5.20), which yields

$$u = \frac{1}{2} \sum_{j=1}^{n} \mu_j + \frac{c_1}{2}.$$

The constant c_1 can be determined by considering the coefficient of the term z^{2n+1} in (5.18), which results in

$$c_1 = -\frac{1}{2} \sum_{m=0}^{2n+1} E_m.$$

$\square$

Next we turn to asymptotic properties of ϕ and ψ_j, $j = 1, 2$.

Lemma 5.5 *Suppose* (5.41) *and assume that* u *satisfies the* n*th stationary CH equation* (5.25). *Moreover, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$, $(x, x_0) \in \mathbb{R}^2$.

Then,

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} -2\zeta^{-1} - 2u + u_x + O(\zeta), & P \to P_{\infty_+}, \\ 2u + u_x + O(\zeta), & P \to P_{\infty_-}, \end{cases} \quad \zeta = z^{-1}, \quad (5.71)$$

$$\phi(P) \underset{\zeta\to 0}{=} \bigg(\prod_{m=1}^{2n+1} E_m\bigg)^{1/2} f_n^{-1}\zeta + O(\zeta^2), \quad P \to P_0, \quad \zeta = z^{1/2} \quad (5.72)$$

and

$$\psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \exp(\pm(x-x_0))(1+O(\zeta)), \quad P \to P_{\infty_\pm}, \quad \zeta = 1/z, \quad (5.73)$$

$$\psi_2(P,x,x_0) \underset{\zeta\to 0}{=} \exp(\pm(x-x_0)) \begin{cases} -2 + O(\zeta), & P \to P_{\infty_+}, \\ (2u(x)+u_x(x))\zeta + O(\zeta^2), & P \to P_{\infty_-}, \end{cases}$$
$$\zeta = 1/z, \quad (5.74)$$

$$\psi_1(P,x,x_0) \underset{\zeta\to 0}{=} \exp\bigg(-\frac{1}{\zeta}\int_{x_0}^{x} dx' \bigg(\prod_{m=1}^{2n+1} E_m\bigg)^{1/2} f_n(x')^{-1} + O(1)\bigg), \quad (5.75)$$
$$P \to P_0, \quad \zeta = z^{1/2},$$

$$\psi_2(P,x,x_0) \underset{\zeta\to 0}{=} O\big(\zeta^{-1}\big) \exp\bigg(-\frac{1}{\zeta}\int_{x_0}^{x} dx' \bigg(\prod_{m=1}^{2n+1} E_m\bigg)^{1/2} f_n(x')^{-1} + O(1)\bigg),$$
$$P \to P_0, \quad \zeta = z^{1/2}. \quad (5.76)$$

Proof The existence of the asymptotic expansions of ϕ in terms of the appropriate local coordinates $\zeta = 1/z$ near $P_{\infty_\pm}$ and $\zeta = z^{1/2}$ near P_0 is clear from the explicit form of ϕ in (5.47). Insertion of the polynomials F_n, G_n, and H_n into (5.47) then, in principle, yields the explicit expansion coefficients in (5.71) and (5.72). However, a more efficient way to compute these coefficients consists in utilizing the Riccati-type equation (5.53). Indeed, inserting the ansatz

$$\phi \underset{z\to\infty}{=} \phi_1 z + \phi_0 + O(z^{-1})$$

into (5.53) and comparing the leading powers of $1/z$ immediately yields the first line in (5.71). Similarly, the ansatz

$$\phi \underset{z\to\infty}{=} \phi_0 + \phi_1 z^{-1} + O(z^{-2})$$

inserted into (5.53) then yields the second line in (5.71). Finally, the ansatz

$$\phi \underset{z\to 0}{=} \phi_1 z^{1/2} + \phi_2 z + O(z^{3/2})$$

inserted into (5.53) yields (5.72). Expansions (5.73)–(5.76) then follow from (5.51), (5.52), (5.71), and (5.72). $\square$

Remark 5.6 We note the unusual fact that P_0, as opposed to $P_{\infty_\pm}$, is the essential singularity of ψ_j, $j = 1, 2$. What makes matters worse is the intricate x-dependence of the leading-order exponential term in ψ_j, $j = 1, 2$, near P_0, as displayed in (5.75), (5.76). This is in sharp contrast to standard Baker–Akhiezer functions that typically feature a linear behavior with respect to x in connection with their essential singularities of the type $\exp(c(x - x_0)\zeta^{-1})$ near $\zeta = 0$.

Next, we introduce

$$\begin{aligned}
&\underline{\widehat{B}}_{Q_0} : \widehat{\mathcal{K}}_n \setminus \{P_{\infty_+}, P_{\infty_-}\} \to \mathbb{C}^n, \\
&P \mapsto \underline{\widehat{B}}_{Q_0}(P) = \big(\widehat{B}_{Q_0,1}, \dots, \widehat{B}_{Q_0,n}\big) \\
&\qquad = \begin{cases} \int_{Q_0}^{P} \tilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}}, & n = 1, \\ \left(\int_{Q_0}^{P} \eta_2, \dots, \int_{Q_0}^{P} \eta_n, \int_{Q_0}^{P} \tilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}} \right), & n \geq 2, \end{cases}
\end{aligned} \tag{5.77}$$

where $\tilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}} = z^n dz/y$ (cf. (F.53)) and

$$\begin{aligned}
&\underline{\hat{\beta}}_{Q_0} : \operatorname{Sym}^n \big(\widehat{\mathcal{K}}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}\big) \to \mathbb{C}^n, \\
&\mathcal{D}_{\underline{Q}} \mapsto \underline{\hat{\beta}}_{Q_0}(\mathcal{D}_{\underline{Q}}) = \sum_{j=1}^{n} \underline{\widehat{B}}_{Q_0}(Q_j), \\
&\underline{Q} = \{Q_1, \dots, Q_n\} \in \operatorname{Sym}^n \big(\widehat{\mathcal{K}}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}\big),
\end{aligned} \tag{5.78}$$

choosing identical paths of integration from Q_0 to P in all integrals in (5.77) and (5.78). Then one obtains the following result, which indicates a characteristic difference between the CH hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

Lemma 5.7 *Assume* (5.65) *and suppose that* $\{\hat{\mu}_j\}_{j=1,\dots,n}$ *satisfies the stationary Dubrovin equations* (5.66) *on an open interval* $\Omega_\mu \subseteq \mathbb{R}$ *such that* μ_j, $j = 1, \dots, n$, *remain distinct and nonzero on* Ω_μ. *Introducing the associated divisor* $\mathcal{D}_{\underline{\hat{\mu}}} \in \operatorname{Sym}^n(\widehat{\mathcal{K}}_n)$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \operatorname{Sym}^n(\widehat{\mathcal{K}}_n)$, *one computes*

$$\partial_x \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = -\frac{2}{\Psi_n(\underline{\mu}(x))} \underline{c}(1), \quad x \in \Omega_\mu. \tag{5.79}$$

In particular, the Abel map does not linearize the divisor $\mathcal{D}_{\underline{\hat{\mu}}(\cdot)}$ *on* Ω_μ. *In addition,*

$$\partial_x \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(x)} \eta_1 = -\frac{2}{\Psi_n(\underline{\mu}(x))}, \quad x \in \Omega_\mu, \tag{5.80}$$

$$\partial_x \underline{\hat{\beta}}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \begin{cases} 2, & n = 1, \\ 2(0, \dots, 0, 1), & n \geq 2, \end{cases} \quad x \in \Omega_\mu. \tag{5.81}$$

Proof Let $x \in \Omega_\mu$. Then, using

$$\frac{1}{\mu_j} = \frac{\prod_{\substack{p=1\\p\neq j}}^n \mu_p}{\prod_{m=1}^n \mu_m} = -\frac{\Phi_{n-1}^{(j)}(\underline{\mu})}{\Psi_n(\underline{\mu})}, \quad j = 1, \dots, n, \tag{5.82}$$

(cf. (E.1), (E.2)) one obtains

$$\begin{aligned}
\partial_x\bigg(\sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j} \underline{\omega}\bigg) &= \sum_{j=1}^n \mu_{j,x} \sum_{k=1}^n \underline{c}(k)\frac{\mu_j^{k-1}}{y(\hat{\mu}_j)} = 2\sum_{j=1}^n\sum_{k=1}^n \underline{c}(k)\frac{\mu_j^{k-2}}{\prod_{\substack{\ell=1\\\ell\neq j}}^n(\mu_j-\mu_\ell)} \\
&= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{j=1}^n\sum_{k=1}^n \underline{c}(k)\frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1\\\ell\neq j}}^n(\mu_j-\mu_\ell)}\Phi_{n-1}^{(j)}(\underline{\mu}) \\
&= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{j=1}^n\sum_{k=1}^n \underline{c}(k)(U_n(\underline{\mu}))_{k,j}(U_n(\underline{\mu}))_{j,1}^{-1} \\
&= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{k=1}^n \underline{c}(k)\delta_{k,1} = -\frac{2}{\Psi_n(\underline{\mu})}\underline{c}(1),
\end{aligned} \tag{5.83}$$

using (E.13) and (E.14). Equation (5.80) is just a special case of (5.79), and (5.81) follows as in (5.83), using (E.8). $\square$

The analogous results hold for the corresponding divisor $\mathcal{D}_{\hat{\underline{\nu}}(x)}$ associated with $\phi(\,\cdot\,, x)$.

That the Abel map does not provide the proper change of variables to linearize the divisor $\mathcal{D}_{\hat{\underline{\mu}}(x)}$ in the CH context is in sharp contrast to standard integrable soliton equations such as the KdV and AKNS hierarchies (cf. also Remark 5.6). The change of variables

$$x \mapsto \tilde{x} = \int^x dx'\, \Psi_n(\underline{\mu}(x'))^{-1} \tag{5.84}$$

linearizes the Abel map $\underline{A}_{Q_0}(\mathcal{D}_{\hat{\tilde{\underline{\mu}}}(\tilde{x})})$, $\tilde{\mu}_j(\tilde{x}) = \mu_j(x)$, $j = 1, \dots, n$. The intricate relation between the variables x and $\tilde{x}$ is detailed in (5.93).

Next we turn to representations of ϕ and u in terms of the Riemann theta function associated with $\mathcal{K}_n$, assuming the affine part of $\mathcal{K}_n$ to be nonsingular. Since the Abel map fails to linearize the divisor $\mathcal{D}_{\hat{\underline{\mu}}(x)}$, one could argue that the theta function representations of ϕ and u are not particularly useful and therefore restrict the discussion to the Dubrovin equations (5.66), (5.67) and reconstruct u from the trace formula (5.70). However, we feel it is of some value to demonstrate the sharp contrast to all other hierarchies discussed in this volume explicitly. In the following, the notation established in Appendices A and C will be freely employed. In fact, given the preparatory work collected in Appendices A, C, E, and F, the proof of Theorem 5.8 below will be reduced to a few lines. To avoid the trivial case $n = 0$ (considered in Example 5.13), we assume $n \in \mathbb{N}$ for the remainder of this argument.

We choose a fixed base point Q_0 on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_0\}$. Let $\omega^{(3)}_{P_{\infty_+},P_0}$ be a normal differential of the third kind holomorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_0\}$ with simple poles at P_∞ and P_0 and residues 1 and -1, respectively (cf. (A.23), (A.26), (C.45)–(C.48)),

$$\omega^{(3)}_{P_{\infty_+},P_0} = \frac{1}{y}\prod_{j=1}^{n}(z-\lambda_j)dz \underset{\zeta\to 0}{=} \begin{cases} (\zeta^{-1}+O(1))d\zeta & \text{as } P \to P_{\infty_+}, \\ (-\zeta^{-1}+O(1))d\zeta & \text{as } P \to P_0, \end{cases} \tag{5.85}$$

where the local coordinates are given by

$$\zeta = 1/z \text{ for } P \text{ near } P_{\infty_+}, \quad \zeta = \sigma z^{1/2} \text{ for } P \text{ near } P_0,\ \sigma = \pm 1.$$

Moreover,

$$\int_{a_j} \omega^{(3)}_{P_{\infty_+},P_0} = 0, \quad j = 1, \dots, n, \tag{5.86}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_+},P_0} \underset{\zeta\to 0}{=} \ln(\zeta) + e_0 + O(\zeta) \text{ as } P \to P_{\infty_+}, \tag{5.87}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_{\infty_+},P_0} \underset{\zeta\to 0}{=} -\ln(\zeta) + d_0 + O(\zeta) \text{ as } P \to P_0 \tag{5.88}$$

for some constants $e_0, d_0 \in \mathbb{C}$. We also record

$$\underline{A}_{Q_0}(P) - \underline{A}_{Q_0}\big(P_{\infty_\pm}\big) \underset{\zeta\to 0}{=} \pm \underline{U}\zeta + O(\zeta^2) \text{ as } P \to P_{\infty_\pm}, \quad \underline{U} = \underline{c}(n).$$

In the following it will be convenient to introduce the abbreviations

$$\begin{aligned} &\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ &P \in \mathcal{K}_n,\ \underline{Q} = \{Q_1, \dots, Q_n\} \in \operatorname{Sym}^n(\mathcal{K}_n), \end{aligned} \tag{5.89}$$

and analogously,

$$\begin{aligned} &\hat{\underline{z}}(P,\underline{Q}) = \widehat{\underline{\Xi}}_{Q_0} - \widehat{\underline{A}}_{Q_0}(P) + \hat{\underline{\alpha}}_{Q_0}(\mathcal{D}_{\underline{Q}}), \\ &P \in \widehat{\mathcal{K}}_n,\ \underline{Q} = \{Q_1, \dots, Q_n\} \in \operatorname{Sym}^n\big(\widehat{\mathcal{K}}_n\big). \end{aligned} \tag{5.90}$$

Theorem 5.8 *Suppose $u \in C^\infty(\Omega)$, $u^{(m)} \in L^\infty(\Omega)$, $m \in \mathbb{N}_0$, and assume that u satisfies the nth stationary CH equation* (5.25) *on Ω subject to the constraint* (5.65). *Moreover, let $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_0\}$ and $x \in \Omega$, where $\Omega \subseteq \mathbb{R}$ is an open interval. In addition, suppose that $\mathcal{D}_{\underline{\hat{\mu}}(x)}$, or equivalently, $\mathcal{D}_{\underline{\hat{\nu}}(x)}$ is nonspecial for $x \in \Omega$. Then ϕ and u admit the representations*[1]

$$\phi(P,x) = -2\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x)))\theta(\underline{z}(P,\underline{\hat{\nu}}(x)))}{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\nu}}(x)))\theta(\underline{z}(P,\underline{\hat{\mu}}(x)))} \exp\left(-\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_+},P_0} + e_0\right), \tag{5.91}$$

[1] To avoid multi-valued expressions in formulas such as (5.91), etc., we agree always to choose the same path of integration connecting Q_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

$$u(x) = -\frac{1}{4}\sum_{m=0}^{2n+1} E_m + \frac{1}{2}\sum_{j=1}^{n} \lambda_j$$
$$+ \frac{1}{2}\sum_{j=1}^{n} U_j \partial_{w_j} \ln\left(\frac{\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x)) + \underline{w}\big)}{\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x)) + \underline{w}\big)}\right)\bigg|_{\underline{w}=0}. \tag{5.92}$$

Moreover, let $\widetilde{\Omega} \subseteq \Omega$ be such that μ_j, $j = 1, \dots, n$, are nonvanishing on $\widetilde{\Omega}$. Then, the constraint

$$2(x - x_0) = -2\int_{x_0}^{x} \frac{dx'}{\prod_{k=1}^{n} \mu_k(x')} \sum_{j=1}^{n}\left(\int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}}\right) c_j(1)$$
$$+ \ln\left(\frac{\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x))\big)\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x_0))\big)}{\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x))\big)\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x_0))\big)}\right), \quad x, x_0 \in \widetilde{\Omega} \tag{5.93}$$

holds, with

$$\underline{\hat{z}}(P_{\infty_\pm}, \underline{\hat{\mu}}(x)) = \underline{\widehat{\Xi}}_{Q_0} - \underline{\widehat{A}}_{Q_0}(P_{\infty_\pm}) + \underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) \tag{5.94}$$
$$= \underline{\widehat{\Xi}}_{Q_0} - \underline{\widehat{A}}_{Q_0}(P_{\infty_\pm}) + \underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) - 2\int_{x_0}^{x} \frac{dx'}{\Psi_n(\underline{\mu}(x'))}\,\underline{c}(1), \quad x \in \widetilde{\Omega}.$$

Proof First we temporarily assume that

$$\mu_j(x) \neq \mu_{j'}(x), \ \ \nu_k(x) \neq \nu_{k'}(x) \text{ for } j \neq j', k \neq k' \text{ and } x \in \widetilde{\Omega} \tag{5.95}$$

for appropriate $\widetilde{\Omega} \subseteq \Omega$. Since by (5.49), $\mathcal{D}_{P_0\underline{\hat{\nu}}} \sim \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}}$, and $P_{\infty_-} = (P_{\infty_+})^* \notin \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ by hypothesis, one can apply Theorem A.31 to conclude that $\mathcal{D}_{\underline{\hat{\nu}}} \in \operatorname{Sym}^n(\mathcal{K}_n)$ is nonspecial. This argument is of course symmetric with respect to $\underline{\hat{\mu}}$ and $\underline{\hat{\nu}}$. Thus, $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial if and only if $\mathcal{D}_{\underline{\hat{\nu}}}$ is. The representation (5.91) for ϕ, subject to (5.95), then follows by combining (5.49), (5.71), (5.72), and Theorem A.26 since $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ are nonspecial. The representation (5.92) for u on $\widetilde{\Omega}$ follows from the trace formula (5.70) and (F.59) (taking $k = 1$). By continuity, (5.91) and (5.92) extend from $\widetilde{\Omega}$ to Ω. Assuming $\mu_j \neq 0$, $j = 1, \dots, n$, in addition to (5.95), the constraint (5.93) follows by combining (5.80), (5.81), and (F.58). Equation (5.94) is clear from (5.79). Again, the extra assumption (5.95) can be removed by continuity, and hence (5.93) and (5.94) extend to $\widetilde{\Omega}$. $\square$

Remark 5.9 Since by (5.49) $\mathcal{D}_{P_0\underline{\hat{\nu}}}$ and $\mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}}$ are linearly equivalent, one infers

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{\Delta}, \quad \underline{\Delta} = \underline{A}_{P_0}(P_{\infty_+}).$$

Hence, one can eliminate $\mathcal{D}_{\underline{\hat{\nu}}}$ in (5.91), in terms of $\mathcal{D}_{\underline{\hat{\mu}}}$ using

$$\underline{z}(P, \underline{\hat{\nu}}) = \underline{z}(P, \underline{\hat{\mu}}) + \underline{\Delta}, \quad P \in \mathcal{K}_n.$$

Remark 5.10 Although the stationary CH solution u in (5.92) is of course a meromorphic quasi-periodic function with respect to the new variable $\tilde{x}$ in (5.84), u may exhibit a rather intricate behavior with respect to the original variable x. Generically, u has an infinite number of branch points of the type

$$u(x) \underset{x\to x_0}{=} O((x-x_0)^{2/3}) \tag{5.96}$$

and

$$\tilde{x} - \tilde{x}_0 \underset{x\to x_0}{=} O((x-x_0)^{1/3}). \tag{5.97}$$

Moreover, real-valued bounded stationary CH solutions fall into two categories and are either smooth quasi-periodic functions in x or else (5.96) and (5.97) hold at infinitely many points (depending on whether or not $\Psi_n(\underline{\mu})$ is zero-free, cf. (5.84)). We note that (5.93) relates the variables x and $\tilde{x}$.

Remark 5.11 We emphasized in Remark 5.6 that Ψ in (5.50)–(5.52) markedly differs from standard Baker–Akhiezer vectors. Hence, one cannot expect the usual theta function representation of ψ_j, $j = 1, 2$, in terms of ratios of theta functions times an exponential term containing a meromorphic differential with a pole at the essential singularity of ψ_j multiplied by $(x - x_0)$. However, combining (E.3) and (F.59), one computes

$$\begin{aligned} F_n(z) &= z^n + \sum_{\ell=0}^{n-1} \Psi_{n-\ell}(\underline{\mu}) z^\ell = z^n + \sum_{k=1}^{n} \Bigg(\Psi_{n+1-k}(\underline{\lambda}) \\ &\quad - \sum_{j=1}^{n} c_j(k)\partial_{w_j} \ln\left(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}) + \underline{w})}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}) + \underline{w})} \right)\Bigg|_{\underline{w}=0} z^{k-1} \Bigg) \\ &= \prod_{j=1}^{n}(z-\lambda_j) - \sum_{j=1}^{n}\sum_{k=1}^{n} c_j(k)\partial_{w_j} \ln\left(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}) + \underline{w})}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}) + \underline{w})} \right)\Bigg|_{\underline{w}=0} z^{k-1}, \end{aligned} \tag{5.98}$$

and hence obtains a theta function representation of ψ_1 upon inserting (5.98) into (5.59). The corresponding theta function representation of ψ_2 is then clear from (5.52) and (5.91).

Remark 5.12 The algebro-geometric CH potential u described in (5.92) is complex-valued in general. To obtain real-valued solutions one needs to impose symmetry restrictions on $\mathcal{K}_n$. The zero-curvature equation (5.57), $\Psi_x(P) = U(z)\Psi(P)$, $\Psi = (\psi_1, \psi_2)^\top$, can be rewritten as

$$-\psi_{1,xx} + z^{-1}(4u - u_{xx})\psi_1 = -\psi_1, \tag{5.99}$$

and with introduction of $f = (-d^2/dx^2 + 1)^{1/2}\psi$, (5.99) can be formally rewritten as

$$(-d^2/dx^2+1)^{-1/2}(4u-u_{xx})(-d^2/dx^2+1)^{-1/2}f=-zf.$$

If u is real-valued and $u, u_{xx} \in L^\infty(\mathbb{R})$ (the latter condition can easily be improved but this is not the point of this remark),

$$A=(-d^2/dx^2+1)^{-1/2}(4u-u_{xx})(-d^2/dx^2+1)^{-1/2} \tag{5.100}$$

extends to a bounded self-adjoint operator in $L^2(\mathbb{R})$; hence, the reality constraints on $\mathcal{K}_n$ become,

$$E_0=0, \quad E_m \in \mathbb{R}, \ m=1,\dots,2n+1,$$

that is, all branch points of $\mathcal{K}_n$ are assumed to be in real position. In the special periodic case, where A becomes a compact operator when restricted to the corresponding L^2-space over the periodicity interval, it has been shown that the projections $\mu_j(x)$ of the associated Dirichlet divisors remain in appropriate spectral gaps as x varies throughout the periodicity interval and an isopsectral torus picture familiar from the KdV context emerges. This picture extends to the present algebro-geometric setting (cf. the references provided in the notes to this section).

Next we briefly consider the trivial case $n=0$ excluded in Theorem 5.8.

Example 5.13 Assume $n=0$, $P=(z,y)\in\mathcal{K}_0\setminus\{P_{\infty_+},P_{\infty_-},P_0\}$, and let $(x,x_0)\in\mathbb{R}^2$. Then

$$\begin{aligned}
&\mathcal{K}_0\colon \mathcal{F}_0(z,y)=y^2-R_2(z)=y^2-z(z-E_1)=0, \quad E_0=0, \ E_1\in\mathbb{C},\\
&u(x)=-E_1/4, \qquad\qquad (5.101)\\
&\phi(P,x)=y-z=-\frac{E_1 z}{y+z},\\
&\psi_1(P,x,x_0)=\exp(-(y/z)(x-x_0)),\\
&\psi_2(P,x,x_0)=(1-(y/z))\exp(-(y/z)(x-x_0)).
\end{aligned}$$

Actually, the general solution of s-$\mathrm{CH}_0(u)=u_{xxx}-4u_x=0$ is given by

$$u(x)=a_1e^{2x}+a_2e^{-2x}-(E_1/4), \quad a_j\in\mathbb{C}, \ j=1,2.$$

However, the requirement $u^{(m)}\in L^\infty(\mathbb{R})$, $m\in\mathbb{N}_0$, according to (5.41), necessitates the choice $a_1=a_2=0$ and hence yields (5.101). The latter corresponds to the trace formula (5.70) in the special case $n=0$.

Up to this point we assumed that $u\in C^\infty(\mathbb{R})$ satisfies the stationary CH equation (5.25) for some fixed $n\in\mathbb{N}_0$. Next, we show that solvability of the Dubrovin equations (5.66) on $\Omega_\mu\subseteq\mathbb{R}$ in fact implies equation (5.25) on Ω_μ. As pointed out in Remark 5.15, this amounts to solving the algebro-geometric initial value problem in the stationary case.

Theorem 5.14 *Fix $n \in \mathbb{N}$, assume* (5.65)*, and suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the stationary Dubrovin equations* (5.66) *on an open interval $\Omega_\mu \subseteq \mathbb{R}$ such that μ_j, $j = 1, \dots, n$, remain distinct and nonzero on Ω_μ. Then $u \in C^\infty(\Omega_\mu)$, defined by*

$$u = -\frac{1}{4}\sum_{m=0}^{2n+1} E_m + \frac{1}{2}\sum_{j=1}^{n} \mu_j, \tag{5.102}$$

satisfies the nth stationary CH equation (5.25)*, that is,*

$$\text{s-CH}_n(u) = 0 \text{ on } \Omega_\mu. \tag{5.103}$$

Proof Given the solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (5.66), we introduce

$$F_n(z) = \prod_{j=1}^{n}(z - \mu_j), \tag{5.104}$$

$$G_n(z) = F_n(z) + (1/2)F_{n,x}(z) \tag{5.105}$$

on $\mathbb{C} \times \Omega_\mu$. The Dubrovin equations imply

$$y(\hat{\mu}_j) = (1/2)\mu_j\mu_{j,x}\prod_{\substack{k=1\\k\neq j}}^{n}(\mu_j - \mu_k) = -\mu_j F_{n,x}(\mu_j)/2 = -\mu_j G_n(\mu_j).$$

Thus,

$$R_{2n+2}(\mu_j) - \mu_j^2 G_n(\mu_j)^2 = 0, \quad j = 1, \dots, n.$$

Furthermore, $R_{2n+2}(0) = 0$, and hence there exists a polynomial H_n such that

$$R_{2n+2}(z) - z^2 G_n(z)^2 = zF_n(z)H_n(z). \tag{5.106}$$

Computing the coefficient of the term z^{2n+1} in (5.106), one finds

$$H_n(z) = (4u + 2u_x)z^n + O(z^{n-1}) \text{ as } |z| \to \infty. \tag{5.107}$$

Next, one defines a polynomial P_{n-1} by

$$P_{n-1}(z) = (4u - u_{xx})F_n(z) - H_n(z) - zG_{n,x}(z). \tag{5.108}$$

Using (5.102), (5.104), (5.105), and (5.107), one infers that indeed P_{n-1} has degree at most $n-1$. Multiplying (5.108) by G_n and replacing the term $G_nG_{n,x}$ with the result obtained upon differentiating (5.106) with respect to x yield

$$\begin{aligned} G_n(z)P_{n-1}(z) = {} & F_n(z)\big((4u - u_{xx})G_n(z) + (1/2)H_{n,x}(z)\big) \\ & + \big((1/2)F_{n,x}(z) - G_n(z)\big)H_n(z), \end{aligned}$$

and hence

$$G_n(\mu_j)P_{n-1}(\mu_j) = 0, \quad j = 1, \dots, n$$

on Ω_μ. Restricting $x \in \Omega_\mu$ temporarily to $x \in \widetilde{\Omega}_\mu$, where

$$\begin{aligned}\widetilde{\Omega}_\mu &= \{x \in \Omega_\mu \mid F_{n,x}(\mu_j(x), x) = 2iy(\hat{\mu}_j(x))/\mu_j(x) \neq 0,\ j = 1, \dots, n\}\\ &= \{x \in \Omega_\mu \mid \mu_j(x) \notin \{E_0, \dots, E_{2n+1}\},\ j = 1, \dots, n\},\end{aligned}$$

one infers that

$$P_{n-1}(\mu_j) = 0, \quad j = 1, \dots, n$$

on $\mathbb{C} \times \widetilde{\Omega}_\mu$. Since $P_{n-1}(z)$ has degree at most $n-1$, one concludes

$$P_{n-1} = 0 \text{ on } \mathbb{C} \times \widetilde{\Omega}_\mu, \tag{5.109}$$

and hence (5.14), that is,

$$zG_{n,x}(z) = (4u - u_{xx})F_n(z) - H_n(z) \tag{5.110}$$

on $\mathbb{C} \times \widetilde{\Omega}_\mu$. Differentiating (5.106) with respect to x and using equations (5.110) and (5.105), one finds

$$H_{n,x}(z) = 2F_n(z) - 2(4u - u_{xx})G_n(z) \tag{5.111}$$

on $\mathbb{C} \times \widetilde{\Omega}_\mu$. To extend these results to Ω_μ, we next investigate the case in which $\hat{\mu}_j$ hits a branch point $(E_m, 0)$, $m \neq 0$. Hence, we suppose

$$\mu_{j_0}(x) \to E_{m_0} \text{ as } x \to x_0 \in \Omega_\mu$$

for some $j_0 \in \{1, \dots, n\}$, $m_0 \in \{1, \dots, 2n+1\}$. By introducing

$$\zeta_{j_0}(x) = \sigma(\mu_{j_0}(x) - E_{m_0})^{1/2}, \quad \sigma = \pm 1, \quad \mu_{j_0}(x) = E_{m_0} + \zeta_{j_0}(x)^2,$$

for some x in an open interval centered around x_0, the Dubrovin equation (5.66) for μ_{j_0} becomes

$$\begin{aligned}\zeta_{j_0,x}(x) \underset{x \to x_0}{=} \frac{c(\sigma)}{E_{m_0}} &\Bigg(\prod_{\substack{m=0\\ m \neq m_0}}^{2n+1} (E_{m_0} - E_m)\Bigg)^{1/2} \prod_{\substack{k=1\\ k \neq j_0}}^{n} \big(E_{m_0} - \mu_k(x)\big)^{-1}\\ &\times \big(1 + O\big(\zeta_{j_0}(x)^2\big)\big)\end{aligned}$$

for some $|c(\sigma)| = 1$, and hence relations (5.109)–(5.111) extend to Ω_μ. We have now established relations (5.13)–(5.15) on $\mathbb{C} \times \Omega_\mu$, and one can now proceed, as in Section 5.2, to obtain (5.103). $\square$

Remark 5.15 A closer look at Theorem 5.14 reveals that u is uniquely determined in an open neighborhood Ω of x_0 by $\mathcal{K}_n$ and the initial condition $\underline{\hat{\mu}}(x_0) = (\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)) \in \mathrm{Sym}^n(\mathcal{K}_n)$, or equivalently, by the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0)} \in$

$\mathrm{Sym}^n(\mathcal{K}_n)$ at $x = x_0$. Conversely, given $\mathcal{K}_n$ and u in an open neighborhood Ω of x_0, one can construct the corresponding polynomial $F_n(\,\cdot\,, x)$, $G_n(\,\cdot\,, x)$, $H_n(\,\cdot\,, x)$ for $x \in \Omega$ (using the recursion relations (5.3), (5.5), (5.6) to determine the homogeneous elements $\hat{f}_\ell$, $\hat{g}_\ell$, $\hat{h}_\ell$, and (D.59) to determine $c_\ell = c_\ell(\underline{E})$, $\ell = 0, \dots, n$) and then recover the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ for $x \in \Omega$ from the zeros of $F_n(\,\cdot\,, x)$ and from (5.45). This remark is of relevance in connection with determining the isospectral set of CH potentials u in the sense that once the curve $\mathcal{K}_n$ is fixed, elements of the isospectral class of potentials are parametrized by (nonspecial) auxiliary divisors $\mathcal{D}_{\underline{\hat{\mu}}(x)}$.

5.4 The Time-Dependent CH Formalism

In this section we extend the algebro-geometric analysis of Section 5.3 to the time-dependent CH hierarchy.

For most of this section we assume the following hypothesis.

Hypothesis 5.16 *Suppose that* $u\colon \mathbb{R}^2 \to \mathbb{C}$ *satisfies*

$$\begin{aligned} &u(\,\cdot\,, t) \in C^\infty(\mathbb{R}),\ \partial_x^m u(\,\cdot\,, t) \in L^\infty(\mathbb{R}),\ m \in \mathbb{N}_0, t \in \mathbb{R}, \\ &u(x, \,\cdot\,), u_{xx}(x, \,\cdot\,) \in C^1(\mathbb{R}),\ x \in \mathbb{R}. \end{aligned} \tag{5.112}$$

The basic problem in the analysis of algebro-geometric solutions of the CH hierarchy consists in solving the time-dependent rth CH flow with initial data a stationary solution of the nth equation in the hierarchy. More precisely, given $n \in \mathbb{N}_0$, consider a solution $u^{(0)}$ of the nth stationary CH equation $\text{s-CH}_n(u^{(0)}) = 0$ associated with $\mathcal{K}_n$ and a given set of integration constants $\{c_\ell\}_{\ell=1,\dots,n} \subset \mathbb{C}$. Next, let $r \in \mathbb{N}_0$; we intend to construct a solution u of the rth CH flow $\mathrm{CH}_r(u) = 0$ with $u(t_{0,r}) = u^{(0)}$ for some $t_{0,r} \in \mathbb{R}$. To emphasize that the integration constants in the definitions of the stationary and the time-dependent CH equations are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence, we employ the notation $\widetilde{V}_r$, $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, $\tilde{f}_s$, $\tilde{g}_s$, $\tilde{h}_s$, $\tilde{c}_s$ to distinguish them from V_n, F_n, G_n, H_n, f_ℓ, g_ℓ, h_ℓ, c_ℓ in the following. In addition, we follow a more elaborate notation inspired by Hirota's τ-function approach and indicate the individual rth CH flow by a separate time variable $t_r \in \mathbb{R}$.

Summing up, we are seeking a solution u of the time-dependent algebro-geometric initial value problem

$$\begin{aligned} &\widetilde{\mathrm{CH}}_r(u) = 4u_{t_r} - u_{xxt_r} + (u_{xxx} - 4u_x)\tilde{f}_r(u) - 2(4u - u_{xx})\tilde{f}_{r,x}(u) = 0, \\ &u\big|_{t_r = t_{0,r}} = u^{(0)}, \end{aligned} \tag{5.113}$$

$$\text{s-CH}_n(u^{(0)}) = (u_{xxx} - 4u_x)f_n(u^{(0)}) - 2(4u - u_{xx})f_{n,x}(u^{(0)}) = 0, \tag{5.114}$$

for some $t_{0,r} \in \mathbb{R}$, $n, r \in \mathbb{N}_0$, where $u = u(x, t_r)$ satisfies (5.112) and a fixed curve

$\mathcal{K}_n$ is associated with the stationary solution $u^{(0)}$ in (5.114). Actually, relying on the isospectral property of the CH flows, we will go a step further and assume (5.114) not only at $t_r = t_{0,r}$ but for all $t_r \in \mathbb{R}$. Hence, we start with

$$U_{t_r} - \widetilde{V}_{r,x} + [U, \widetilde{V}_r] = 0, \tag{5.115}$$

$$-V_{n,x} + [U, V_n] = 0, \tag{5.116}$$

where (cf. (5.20)–(5.22))

$$\begin{aligned} U(z) &= \begin{pmatrix} -1 & 1 \\ z^{-1}(4u - u_{xx}) & 1 \end{pmatrix}, \\ \widetilde{V}_n(z) &= \begin{pmatrix} -G_n(z) & F_n(z) \\ z^{-1}H_n(z) & G_n(z) \end{pmatrix}, \\ \widetilde{V}_r(z) &= \begin{pmatrix} -\widetilde{G}_r(z) & \widetilde{F}_r(z) \\ z^{-1}\widetilde{H}_r(z) & \widetilde{G}_r(z) \end{pmatrix}, \end{aligned} \tag{5.117}$$

and

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^{\ell} = \prod_{j=1}^{n} (z - \mu_j), \tag{5.118}$$

$$G_n(z) = \sum_{\ell=0}^{n} g_{n-\ell} z^{\ell}, \tag{5.119}$$

$$H_n(z) = \sum_{\ell=0}^{n} h_{n-\ell} z^{\ell} = h_0 \prod_{j=1}^{n} (z - \nu_j), \quad h_0 = 4u + 2u_x, \tag{5.120}$$

$$\widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s} z^{s}, \tag{5.121}$$

$$\widetilde{G}_r(z) = \sum_{s=0}^{r} \tilde{g}_{r-s} z^{s}, \tag{5.122}$$

$$\widetilde{H}_r(z) = \sum_{s=0}^{r} \tilde{h}_{r-s} z^{s}, \quad \tilde{h}_0 = 4u + 2u_x \tag{5.123}$$

for fixed $n, r \in \mathbb{N}_0$. Here $f_\ell, \tilde{f}_s, g_\ell, \tilde{g}_s, h_\ell$, and $\tilde{h}_s$, $\ell = 0, \dots, n$, $s = 0, \dots, r$, are defined as in (5.3), (5.5), and (5.6) with appropriate sets of integration constants. Explicitly, (5.115), (5.116) are equivalent to

$$4u_{t_r} - u_{xxt_r} - \widetilde{H}_{r,x} + 2\widetilde{H}_r - 2(4u - u_{xx})\widetilde{G}_r = 0, \tag{5.124}$$

$$\widetilde{F}_{r,x} = 2(\widetilde{G}_r - \widetilde{F}_r), \tag{5.125}$$

$$z\widetilde{G}_{r,x} = (4u - u_{xx})\widetilde{F}_r - \widetilde{H}_r \tag{5.126}$$

and

$$F_{n,x} = 2(G_n - F_n), \tag{5.127}$$

$$H_{n,x} = 2H_n - 2(4u - u_{xx})G_n, \tag{5.128}$$

$$zG_{n,x} = (4u - u_{xx})F_n - H_n. \tag{5.129}$$

First we will assume the existence of a solution u of equations (5.124)–(5.129) and derive an explicit formula for u in terms of Riemann theta functions. In addition, we will show in Theorem 5.25 that (5.124)–(5.129) and hence the algebro-geometric initial value problem (5.113), (5.114) has a solution at least locally, that is, for $(x, t_r) \in \Omega$ for some open and connected set $\Omega \subseteq \mathbb{R}^2$.

One observes that equations (5.3)–(5.31) apply to F_n, G_n, H_n, f_ℓ, g_ℓ, and h_ℓ and (5.3)–(5.6), (5.20)–(5.22), with n replaced by r and c_ℓ replaced by $\tilde{c}_\ell$, apply to $\widetilde{F}_r$, $\widetilde{G}_r$, $\widetilde{H}_r$, $\tilde{f}_\ell$, $\tilde{g}_\ell$, and $\tilde{h}_\ell$. In particular, the fundamental identity (5.18) holds,

$$z^2 G_n^2 + zF_n H_n = R_{2n+2},$$

and the hyperelliptic curve $\mathcal{K}_n$ is still given by (5.42) assuming (5.43) for the remainder of this section, that is,

$$E_0 = 0, \ E_1, \dots, E_{2n+1} \in \mathbb{C} \setminus \{0\}. \tag{5.130}$$

In analogy to equations (5.45), (5.46) we define

$$\hat{\mu}_j(x, t_r) = (\mu_j(x, t_r), -\mu_j(x, t_r)G_n(\mu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \tag{5.131}$$
$$j = 1, \dots, n, \ (x, t_r) \in \mathbb{R}^2,$$

$$\hat{\nu}_j(x, t_r) = (\nu_j(x, t_r), \nu_j(x, t_r)G_n(\nu_j(x, t_r), x, t_r)) \in \mathcal{K}_n, \tag{5.132}$$
$$j = 1, \dots, n, \ (x, t_r) \in \mathbb{R}^2.$$

As in Section 5.3, the regularity assumptions (5.112) on u imply analogous regularity properties of F_n, H_n, μ_j, and ν_k.

Next, one defines the meromorphic function $\phi(\,\cdot\,, x, t_r)$ on $\mathcal{K}_n$ by

$$\phi(P, x, t_r) = \frac{y - zG_n(z, x, t_r)}{F_n(z, x, t_r)} \tag{5.133}$$

$$= \frac{zH_n(z, x, t_r)}{y + zG_n(z, x, t_r)}, \tag{5.134}$$

$$P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_\pm}\}, \ (x, t_r) \in \mathbb{R}^2.$$

Assuming (5.130), the divisor $(\phi(\,\cdot\,, x, t_r))$ of $\phi(\,\cdot\,, x, t_r)$ reads

$$(\phi(\,\cdot\,, x, t_r)) = \mathcal{D}_{P_0 \underline{\hat{\nu}}(x, t_r)} - \mathcal{D}_{P_{\infty_+} \underline{\hat{\mu}}(x, t_r)} \tag{5.135}$$

with

$$\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\},\ \underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\} \in \operatorname{Sym}^n(\mathcal{K}_n).$$

The corresponding time-dependent vector Ψ,

$$\Psi(P, x, x_0, t_r, t_{0,r}) = \begin{pmatrix} \psi_1(P, x, x_0, t_r, t_{0,r}) \\ \psi_2(P, x, x_0, t_r, t_{0,r}) \end{pmatrix}, \tag{5.136}$$

$$P \in \mathcal{K}_n \setminus \{P_{\infty_\pm}\},\ (x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$$

is defined by

$$\psi_1(P, x, x_0, t_r, t_{0,r}) = \exp\bigg(-\int_{t_{0,r}}^{t_r} ds\, \big(z^{-1}\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) \tag{5.137}$$

$$+ \widetilde{G}_r(z, x_0, s)\big) - z^{-1}\int_{x_0}^{x} dx'\, \phi(P, x', t_r) - (x - x_0)\bigg),$$

$$\psi_2(P, x, x_0, t_r, t_{0,r}) = -\psi_1(P, x, x_0, t_r, t_{0,r})\phi(P, x, t_r)/z. \tag{5.138}$$

Basic properties of ϕ can now be summarized as follows.

Lemma 5.17 *Assume Hypothesis* 5.16 *and suppose that* (5.115), (5.116) *hold. Moreover, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$ *and* $(x, t_r) \in \mathbb{R}^2$. *Then* ϕ *satisfies*

$$\phi_x(P) - z^{-1}\phi(P)^2 - 2\phi(P) = u_{xx} - 4u, \tag{5.139}$$

$$\phi_{t_r}(P) = (4u - u_{xx})\widetilde{F}_r(z) - \widetilde{H}_r(z) + 2(\widetilde{F}_r(z)\phi(P))_x \tag{5.140}$$

$$= (1/z)\widetilde{F}_r(z)\phi(P)^2 + 2\widetilde{G}_r(z)\phi(P) - \widetilde{H}_r(z), \tag{5.141}$$

$$\phi(P)\phi(P^*) = -\frac{zH_n(z)}{F_n(z)}, \tag{5.142}$$

$$\phi(P) + \phi(P^*) = -2\frac{zG_n(z)}{F_n(z)}, \tag{5.143}$$

$$\phi(P) - \phi(P^*) = \frac{2y}{F_n(z)}. \tag{5.144}$$

Proof Equations (5.139) and (5.142)–(5.144) are proved as in Lemma 5.2. To prove (5.141), one first observes that

$$\big(\partial_x - 2(z^{-1}\phi + 1)\big)\big(\phi_{t_r} - z^{-1}\widetilde{F}_r\phi^2 - 2\widetilde{G}_r\phi + \widetilde{H}_r\big) = 0,$$

using (5.139) and relations (5.124)–(5.126) repeatedly. Thus,

$$\phi_{t_r} - z^{-1}\widetilde{F}_r\phi^2 - 2\widetilde{G}_r\phi + \widetilde{H}_r = C\exp\bigg(2\int^x dx'\,\big(z^{-1}\phi + 1\big)\bigg),$$

where the left-hand side is meromorphic in a neighborhood of P_{∞_-}, whereas the right-hand side is meromorphic near P_{∞_-} only if $C = 0$. This proves (5.141).

Using (5.125) and (5.139), one obtains

$$(4u - u_{xx})\widetilde{F}_r + 2\big(\widetilde{F}_r\phi\big)_x = 2\widetilde{G}_r\phi + z^{-1}\phi^2\widetilde{F}_r.$$

Combining this result with (5.141), one concludes that (5.140) holds. □

Using relations (5.127)–(5.129) and (5.124)–(5.126), we next determine the time evolution of F_n, G_n, and H_n.

Lemma 5.18 *Assume Hypothesis* 5.16 *and suppose that* (5.115), (5.116) *hold. Then,*

$$F_{n,t_r} = 2\big(G_n\widetilde{F}_r - F_n\widetilde{G}_r\big), \tag{5.145}$$

$$zG_{n,t_r} = F_n\widetilde{H}_r - H_n\widetilde{F}_r, \tag{5.146}$$

$$H_{n,t_r} = 2\big(H_n\widetilde{G}_r - G_n\widetilde{H}_r\big). \tag{5.147}$$

Equations (5.145)–(5.147) *are equivalent to*

$$-V_{n,t_r} + \big[\widetilde{V}_r, V_n\big] = 0.$$

Proof We prove (5.145) by using (5.144) which shows that

$$(\phi(P) - \phi(P^*))_{t_r} = -2yF_n^{-2}F_{n,t_r}. \tag{5.148}$$

However, the left-hand side of (5.148) also equals

$$\phi(P)_{t_r} - \phi(P^*)_{t_r} = 4yF_n^{-2}\big(\widetilde{G}_rF_n - \widetilde{F}_rG_n\big) \tag{5.149}$$

by means of (5.141), (5.143), and (5.144). Combining (5.148) and (5.149) proves (5.145). Similarly, to prove (5.146), we use (5.143) to write

$$(\phi(P) + \phi(P^*))_{t_r} = -2zF_n^{-2}\big(G_{n,t_r}F_n - G_nF_{n,t_r}\big). \tag{5.150}$$

Here the left-hand side can be expressed as

$$\phi(P)_{t_r} + \phi(P^*)_{t_r} = 2zG_nF_n^{-2}F_{n,t_r} + 2F_n^{-1}\big(\widetilde{F}_rH_n - \widetilde{H}_rF_n\big), \tag{5.151}$$

using (5.141), (5.142), and (5.143). Combining (5.150) and (5.151), using (5.145), proves (5.146). Finally, (5.147) follows by differentiating (5.18), that is, $(zG_n)^2 + zF_nH_n = R_{2n+2}$, with respect to t_r, and using (5.145) and (5.146). □

Lemmas 5.17 and 5.18 permit one to characterize Ψ.

Lemma 5.19 *Assume Hypothesis* 5.16 *and suppose that* (5.115), (5.116) *hold. Moreover, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$ *and* $(x, x_0, t_r, t_{0,r}) \in \mathbb{R}^4$. *Then* *the Baker–Akhiezer vector* Ψ *satisfies*

$$\Psi_x(P) = U(z)\Psi(P), \tag{5.152}$$

$$-\, y\Psi(P) = zV_n(z)\Psi(P), \tag{5.153}$$

$$\Psi_{t_r}(P) = \widetilde{V}_r(z)\Psi(P), \tag{5.154}$$

$$\begin{aligned}\psi_1(P,x,x_0,t_r,t_{0,r}) = \bigg(\frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}\bigg)^{1/2} \\ \times \exp\bigg(-(y/z)\int_{t_{0,r}}^{t_r} ds\, \widetilde{F}_r(z,x_0,s)F_n(z,x_0,s)^{-1} \\ -(y/z)\int_{x_0}^{x} dx'\, F_n(z,x',t_r)^{-1}\bigg),\end{aligned} \tag{5.155}$$

$$\psi_1(P,x,x_0,t_r,t_{0,r})\psi_1(P^*,x,x_0,t_r,t_{0,r}) = \frac{F_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})}, \tag{5.156}$$

$$\psi_2(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) = -\frac{H_n(z,x,t_r)}{zF_n(z,x_0,t_{0,r})}, \tag{5.157}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) \\ &\quad + \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = 2\frac{G_n(z,x,t_r)}{F_n(z,x_0,t_{0,r})},\end{aligned} \tag{5.158}$$

$$\begin{aligned}&\psi_1(P,x,x_0,t_r,t_{0,r})\psi_2(P^*,x,x_0,t_r,t_{0,r}) \\ &\quad - \psi_1(P^*,x,x_0,t_r,t_{0,r})\psi_2(P,x,x_0,t_r,t_{0,r}) = \frac{2y}{zF_n(z,x_0,t_{0,r})}.\end{aligned} \tag{5.159}$$

In addition, as long as the zeros of $F_n(\,\cdot\,,x,t_r)$ *are all simple for* $(x,t_r), (x_0,t_{0,r}) \in \Omega$, $\Omega \subseteq \mathbb{R}^2$ *open and connected,* $\Psi(\,\cdot\,,x,x_0,t_r,t_{0,r})$ *is meromorphic on* $\mathcal{K}_n \setminus \{P_0, P_{\infty_\pm}\}$ *for* $(x,t_r), (x_0,t_{0,r}) \in \Omega$.

Proof By (5.137), $\psi_1(\,\cdot\,,x,x_0,t_r,t_{0,r})$ is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ away from the poles $\hat{\mu}_j(x_0,s)$ of $\phi(\,\cdot\,,x_0,s)$ and $\hat{\mu}_k(x',t_r)$ of $\phi(\,\cdot\,,x',t_r)$. That $\psi_1(\,\cdot\,,x,x_0,t_r,t_{0,r})$ is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ if $F_n(\,\cdot\,,x,t_r)$ has only simple zeros is a consequence of (cf. (5.64))

$$-\, z^{-1}\phi(P,x',t_r) \underset{P\to\hat{\mu}_j(x',t_r)}{=} \partial_{x'} \ln\big(F_n(z,x',t_r)\big) + O(1) \text{ as } z \to \mu_j(x',t_r),$$

$$\begin{aligned}-\, z^{-1}\widetilde{F}_r(z,x_0,s)\phi(P,x_0,s) \underset{P\to\hat{\mu}_j(x_0,s)}{=} \partial_s \ln\big(F_n(z,x_0,s)\big) + O(1) \\ \text{as } z \to \mu_j(x_0,s),\end{aligned}$$

using (5.131), (5.133), and (5.145). This follows from (5.137) by restricting P to a sufficiently small neighborhood $\mathcal{U}_j(x_0)$ of $\{\hat{\mu}_j(x_0,s) \in \mathcal{K}_n \mid (x_0,s) \in \Omega,\ s \in [t_{0,r},t_r]\}$ such that $\hat{\mu}_k(x_0,s) \notin \mathcal{U}_j(x_0)$ for all $s \in [t_{0,r},t_r]$ and all

$k \in \{1, \dots, n\} \setminus \{j\}$ and by simultaneously restricting P to a sufficiently small neighborhood $\mathcal{U}_j(t_r)$ of $\{\hat{\mu}_j(x', t_r) \in \mathcal{K}_n \mid (x', t_r) \in \Omega,\ x' \in [x_0, x]\}$ such that $\hat{\mu}_k(x', t_r) \notin \mathcal{U}_j(t_r)$ for all $x' \in [x_0, x]$ and all $k \in \{1, \dots, n\} \setminus \{j\}$. By (5.138) and since ϕ is meromorphic on $\mathcal{K}_n$ one concludes that ψ_2 is meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$ as well. Relations (5.152) and (5.153) follow as in Lemma 5.2, whereas the time evolution (5.154) is a consequence of the definition of Ψ in (5.137), (5.138) as well as (5.141) by rewriting

$$(1/z)\phi_{t_r} = \big(z^{-1}2\phi \widetilde{F}_r + \widetilde{G}_r\big)_x$$

using (5.126) and (5.140). To prove (5.155), we recall the definition (5.137), that is,

$$\begin{aligned}
\psi_1(P, x, x_0, t_r, t_{0,r}) &= \exp\bigg(-(x - x_0) - z^{-1} \int_{x_0}^{x} dx'\, \phi(P, x', t_r) \\
&\qquad - \int_{t_{0,r}}^{t_r} ds\, \big(z^{-1}\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \widetilde{G}_r(z, x_0, s)\big)\bigg) \\
&= \bigg(\frac{F_n(z, x, t_r)}{F_n(z, x_0, t_r)}\bigg)^{1/2} \exp\bigg(-(y/z) \int_{x_0}^{x} dx' F_n(z, x', t_r)^{-1} \\
&\qquad - \int_{t_{0,r}}^{t_r} ds\, \big(z^{-1}\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \widetilde{G}_r(z, x_0, s)\big)\bigg),
\end{aligned} \tag{5.160}$$

using the calculation leading to (5.59). Equations (5.133) and (5.145) show that

$$\frac{1}{z}\widetilde{F}_r(z, x_0, s)\phi(P, x_0, s) + \widetilde{G}_r(z, x_0, s) = \frac{y}{z}\,\frac{\widetilde{F}_r(z, x_0, s)}{F_n(z, x_0, s)} - \frac{1}{2}\,\frac{F_{n,t_r}(z, x_0, s)}{F_n(z, x_0, s)},$$

which, inserted into (5.160), yields (5.155). Evaluating (5.155) at the points P and P^* and multiplying the resulting expressions yield (5.156). The remaining statements are direct consequences of (5.142)–(5.144) and (5.155). □

The stationary Dubrovin-type equations in Lemma 5.3 have analogs for each CH_r flow (indexed by the parameter t_r) that govern the dynamics of μ_j and ν_j with respect to variations of x and t_r. In this context the stationary case simply corresponds to the special case $r = 0$, as described in the following result. We assume (5.65), that is,

$$E_0 = 0,\ E_m \in \mathbb{C} \setminus \{0\},\ E_m \neq E_{m'} \text{ for } m \neq m',\ m, m' = 1, \dots, 2n+1. \tag{5.161}$$

Lemma 5.20 *Assume Hypothesis* 5.16, (5.161) *and suppose* (5.115), (5.116) *hold on an open and connected set* $\widetilde{\Omega}_\mu \subseteq \mathbb{R}^2$. *Moreover, suppose that the zeros* μ_j, $j = 1, \dots, n$, *of* $F_n(\,\cdot\,)$ *remain distinct and nonzero on* $\widetilde{\Omega}_\mu$. *Then* $\{\hat{\mu}_j\}_{j=1,\dots,n}$, *defined* *by* (5.131)*, satisfies the following first-order system of differential equations on* $\widetilde{\Omega}_\mu$

$$\mu_{j,x} = 2\mu_j^{-1} y(\hat{\mu}_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} (\mu_j - \mu_k)^{-1}, \tag{5.162}$$

$$\mu_{j,t_r} = 2\widetilde{F}_r(\mu_j)\mu_j^{-1} y(\hat{\mu}_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} (\mu_j - \mu_k)^{-1}, \quad j = 1, \dots, n. \tag{5.163}$$

Next, assume the affine part of $\mathcal{K}_n$ *to be nonsingular and introduce the initial condition*

$$\{\hat{\mu}_j(x_0, t_{0,r})\}_{j=1,\dots,n} \subset \mathcal{K}_n \tag{5.164}$$

for some $(x_0, t_{0,r}) \in \mathbb{R}^2$*, where* $\mu_j(x_0, t_{0,r}) \neq 0$*,* $j = 1, \dots, n$*, are assumed to be distinct. Then there exists an open and connected set* $\Omega_\mu \subseteq \mathbb{R}^2$*, with* $(x_0, t_{0,r}) \in \Omega_\mu$*, such that the initial value problem* (5.162)–(5.164) *has a unique solution* $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ *satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega_\mu, \mathcal{K}_n), \quad j = 1, \dots, n, \tag{5.165}$$

and μ_j*,* $j = 1, \dots, n$*, remain distinct and nonzero on* Ω_μ*.*
For the zeros $\{\nu_j\}_{j=1,\dots,n}$ *of* $H_n(\,\cdot\,)$ *similar statements hold with* μ_j *and* Ω_μ *replaced by* ν_j *and* Ω_ν*, etc. In particular,* $\{\hat{\nu}_j\}_{j=1,\dots,n}$*, defined by* (5.132)*, satisfies the system*

$$\nu_{j,x} = 2(4u - u_{xx})(4u + 2u_x)^{-1} \nu_j^{-1} y(\hat{\nu}_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} (\nu_j - \nu_k)^{-1}, \tag{5.166}$$

$$\nu_{j,t_r} = 2\widetilde{H}_r(\nu_j)(4u + 2u_x)^{-1} \nu_j^{-1} y(\hat{\nu}_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} (\nu_j - \nu_k)^{-1}, \quad j = 1, \dots, n. \tag{5.167}$$

Proof It suffices to prove (5.163) since the argument for (5.167) is analogous and that for (5.162) and (5.166) has been given in the proof of Lemma 5.3. Inserting $z = \mu_j$ into (5.145), observing (5.131), yields

$$F_{n,t_r}(\mu_j) = -\mu_{j,t_r} \prod_{\substack{k=1 \\ k\neq j}}^{n} (\mu_j - \mu_k) = 2\widetilde{F}_r(\mu_j) G_n(\mu_j) = -2\frac{\widetilde{F}_r(\mu_j)}{\mu_j} y(\hat{\mu}_j).$$

The smoothness assertion (5.165) is clear as long as $\hat{\mu}_j$ stays away from the branch points $(E_m, 0)$. In case $\hat{\mu}_j$ hits such a branch point, one can use the local chart around $(E_m, 0)$ (with local coordinate $\zeta = \sigma(z - E_m)^{1/2}$, $\sigma = \pm 1$) to verify (5.165), as in the proof of Lemma 1.37. $\square$

Since the stationary trace formulas for CH invariants in terms of symmetric functions of μ_j in Lemma 5.4 extend line by line to the corresponding time-dependent setting, we next record their t_r-dependent analogs without proof. For simplicity, we confine ourselves to the simplest one only.

Lemma 5.21 *Assume Hypothesis* 5.16 *and suppose that* (5.115), (5.116) *hold. Then,*

$$u = -\frac{1}{4}\sum_{m=0}^{2n+1} E_m + \frac{1}{2}\sum_{j=1}^{n}\mu_j. \tag{5.168}$$

We also record the asymptotic properties of ϕ, the time-dependent analogs of (5.71) and (5.72) in the stationary case.

Lemma 5.22 *Assume Hypothesis* 5.16 *and suppose that* (5.115), (5.116) *hold. Moreover, let* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_0\}$. *Then,*

$$\phi(P) \underset{\zeta\to 0}{=} \begin{cases} -2\zeta^{-1} - 2u + u_x + O(\zeta) & \text{as } P \to P_{\infty_+}, \\ 2u + u_x + O(\zeta) & \text{as } P \to P_{\infty_-}, \end{cases} \quad \zeta = z^{-1}, \tag{5.169}$$

$$\phi(P) \underset{\zeta\to 0}{=} \bigg(\prod_{m=1}^{2n+1} E_m\bigg)^{1/2} f_n^{-1}\zeta + O(\zeta^2) \text{ as } P \to P_0, \quad \zeta = z^{1/2}. \tag{5.170}$$

Since the proofs of Lemmas 5.21 and 5.22 are identical to the corresponding stationary results in Lemmas 5.4 and 5.5, we omit the corresponding details.

Next, recalling the definitions of $\tilde d_{r,k}$ and $\widetilde F_r(\mu_j)$ introduced in (F.16) and (F.19) and also the definition of $\widehat{\underline{B}}_{Q_0}$ and $\hat{\underline{\beta}}_{Q_0}$ in (5.77) and (5.78), respectively, we now state the analog of Lemma 5.7, thereby underscoring marked differences between the CH hierarchy and other completely integrable systems such as the KdV and AKNS hierarchies.

Lemma 5.23 *Assume* (5.161) *and suppose that* $\{\hat\mu_j\}_{j=1,\dots,n}$ *satisfies the Dubrovin equations* (5.162), (5.163) *on an open set* $\Omega_\mu \subseteq \mathbb{R}^2$ *such that* μ_j, $j = 1, \dots, n$, *remain distinct and nonzero on* Ω_μ *and that* $\widetilde F_r(\mu_j) \neq 0$ *on* Ω_μ, $j = 1, \dots, n$. *Introducing the associated divisor* $\mathcal{D}_{\underline{\hat\mu}} \in \mathrm{Sym}^n(\widehat{\mathcal{K}}_n)$, $\underline{\hat\mu} = \{\hat\mu_1, \dots, \hat\mu_n\} \in \mathrm{Sym}^n(\widehat{\mathcal{K}}_n)$, *one computes*

$$\partial_x \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat\mu}(x,t_r)}\big) = -\frac{2}{\Psi_n(\underline{\mu}(x,t_r))}\underline{c}(1), \quad (x,t_r) \in \Omega_\mu, \tag{5.171}$$

$$\begin{aligned} \partial_{t_r} \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\underline{\hat\mu}(x,t_r)}\big) &= -\frac{2}{\Psi_n(\underline{\mu}(x,t_r))}\bigg(\sum_{k=0}^{r\wedge n} \tilde d_{r,k}(\underline{E})\Psi_k(\underline{\mu}(x,t_r))\bigg)\underline{c}(1) \\ &\quad + 2\bigg(\sum_{\ell=1\vee(n+1-r)}^{n} \tilde d_{r,n+1-\ell}(\underline{E})\underline{c}(\ell)\bigg), \quad (x,t_r) \in \Omega_\mu. \end{aligned} \tag{5.172}$$

In particular, the Abel map does not linearize the divisor $\mathcal{D}_{\underline{\hat{\mu}}(\,\cdot\,,\,\cdot\,)}$ on Ω_μ. In addition,

$$\partial_x \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(x,t_r)} \eta_1 = -\frac{2}{\Psi_n(\underline{\mu}(x,t_r))}, \quad (x,t_r) \in \Omega_\mu, \tag{5.173}$$

$$\partial_x \underline{\hat{\beta}}\big(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}\big) = \begin{cases} 2, & n=1, \\ 2(0,\dots,0,1), & n \geq 2, \end{cases} \quad (x,t_r) \in \Omega_\mu, \tag{5.174}$$

$$\begin{aligned} \partial_{t_r} \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(x,t_r)} \eta_1 = &-\frac{2}{\Psi_n(\underline{\mu}(x,t_r))} \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(\underline{E}) \Psi_k(\underline{\mu}(x,t_r)) \\ &+ 2\tilde{d}_{r,n}(\underline{E})\delta_{n,r\wedge n}, \quad (x,t_r) \in \Omega_\mu, \end{aligned} \tag{5.175}$$

$$\begin{aligned} &\partial_{t_r} \underline{\hat{\beta}}\big(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}\big) \\ &\quad = 2\bigg(\sum_{s=0}^{r} \tilde{c}_{r-s}\hat{c}_{s+1-n}(\underline{E}), \dots, \sum_{s=0}^{r} \tilde{c}_{r-s}\hat{c}_{s+1}(\underline{E}), \sum_{s=0}^{r} \tilde{c}_{r-s}\hat{c}_{s}(\underline{E}) \bigg), \\ &\qquad\qquad \hat{c}_{-\ell}(\underline{E}) = 0, \ \ell \in \mathbb{N}, \ (x,t_r) \in \Omega_\mu. \end{aligned} \tag{5.176}$$

Proof Let $(x,t_r) \in \Omega_\mu$. Since (5.171), (5.173), and (5.174) are proved as in in the stationary context of Lemma 5.7, we focus on the proofs of (5.172), (5.175), and (5.176). First we note that

$$\frac{\widetilde{F}_r}{\mu_j} = -\sum_{m=0}^{r\wedge n} \tilde{d}_{r,m}(\underline{E}) \Psi_m(\underline{\mu}) \frac{\Phi_{n-1}^{(j)}(\underline{\mu})}{\Psi_n(\underline{\mu})} + \sum_{m=1}^{r\wedge n} \tilde{d}_{r,m}(\underline{E}) \Phi_{m-1}^{(j)}(\underline{\mu}) \tag{5.177}$$

by applying (F.19), (E.10), and (5.82). Then, using (5.177), (5.163), (5.82), (E.10), and (E.4), (E.13), and (E.14), one obtains[1]

$$\begin{aligned} \partial_{t_r}\bigg(\sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j} \underline{\omega} \bigg) &= \sum_{j,k=1}^{n} \mu_{j,t_r} \underline{c}(k) \frac{\mu_j^{k-1}}{y(\hat{\mu}_j)} = 2 \sum_{j,k=1}^{n} \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell\neq j}}^{n} (\mu_j - \mu_\ell)} \frac{\widetilde{F}_r(\mu_j)}{\mu_j} \\ &= 2 \sum_{j,k=1}^{n} \underline{c}(k) \frac{\mu_j^{k-1}}{\prod_{\substack{\ell=1 \\ \ell\neq j}}^{n} (\mu_j - \mu_\ell)} \bigg(-\sum_{m=0}^{r\wedge n} \tilde{d}_{r,m}(\underline{E}) \Psi_m(\underline{\mu}) \frac{\Phi_{n-1}^{(j)}(\underline{\mu})}{\Psi_n(\underline{\mu})} \\ &\qquad + \sum_{m=1}^{r\wedge n} \tilde{d}_{r,m}(\underline{E}) \Phi_{m-1}^{(j)}(\underline{\mu}) \bigg) \\ &= -2 \sum_{m=0}^{r\wedge n} \tilde{d}_{r,m}(\underline{E}) \frac{\Psi_m(\underline{\mu})}{\Psi_n(\underline{\mu})} \sum_{k=1}^{n} \sum_{j=1}^{n} \underline{c}(k) (U_n(\underline{\mu}))_{k,j} (U_n(\underline{\mu}))_{j,1}^{-1} \\ &\quad + 2 \sum_{m=1}^{r\wedge n} \tilde{d}_{r,m}(\underline{E}) \sum_{k=1}^{n} \sum_{j=1}^{n} \underline{c}(k) (U_n(\underline{\mu}))_{k,j} (U_n(\underline{\mu}))_{j,n-m+1}^{-1} \end{aligned}$$

[1] $m \wedge n = \min\{m,n\}$, $m \vee n = \max\{m,n\}$.

$$= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{m=0}^{r\wedge n}\tilde{d}_{r,m}(\underline{E})\Psi_m(\underline{\mu})\underline{c}(1) + 2\sum_{m=1}^{r\wedge n}\tilde{d}_{r.m}(\underline{E})\underline{c}(n-m+1) \tag{5.178}$$
$$= -\frac{2}{\Psi_n(\underline{\mu})}\sum_{m=0}^{r\wedge n}\tilde{d}_{r,m}(\underline{E})\Psi_m(\underline{\mu})\underline{c}(1) + 2\sum_{m=1\vee(n+1-r)}^{n}\tilde{d}_{r,n+1-m}(\underline{E})\underline{c}(m).$$

Equation (5.175) is just a special case of (5.172), and (5.176) follows as in (5.178), through use again of (E.4). □

Analogous results hold for the corresponding divisor $\mathcal{D}_{\hat{\underline{\nu}}}$ associated with ϕ.

One confirms again that the Abel map does not effect a linearization of the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ in the time-dependent CH context.

Next we turn to the representations of ϕ and u in terms of the Riemann theta function associated with $\mathcal{K}_n$, assuming the affine part of $\mathcal{K}_n$ to be nonsingular. Again, one could argue it suffices to consider the Dubrovin equations (5.162)–(5.164) and reconstruct u from the trace formula (5.168) since the Abel map fails to linearize the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$. However, as in the stationary context, we decided to present the whole formalism to enable a comparison with all other hierarchies studied in this volume. Recalling (5.85)–(5.90), the analog of Theorem 5.8 in the stationary case then reads as follows.

Theorem 5.24 *Suppose Hypothesis* 5.16 *and suppose* (5.113), (5.114) *hold on* Ω *subject to the constraint* (5.161). *In addition, let* $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_0\}$ *and* $(x, t_r), (x_0, t_{0,r}) \in \Omega$, *where* $\Omega \subseteq \mathbb{R}^2$ *is open and connected. Moreover, suppose that* $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$, *or equivalently,* $\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}$ *is nonspecial for* $(x, t_r) \in \Omega$. *Then* ϕ *and* u *admit the representations*[1]

$$\phi(P, x, t_r) = -2\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\nu}}(x, t_r)))}{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\nu}}(x, t_r)))\theta(\underline{z}(P, \underline{\hat{\mu}}(x, t_r)))} \times \exp\bigg(-\int_{Q_0}^{P}\omega^{(3)}_{P_{\infty_+},P_0} + e_0\bigg), \tag{5.179}$$

$$u(x, t_r) = -\frac{1}{4}\sum_{m=0}^{2n+1}E_m + \frac{1}{2}\sum_{j=1}^{n}\lambda_j + \frac{1}{2}\sum_{j=1}^{n}U_j\partial_{w_j}\ln\bigg(\frac{\theta\big(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}(x, t_r)) + \underline{w}\big)}{\theta\big(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}(x, t_r)) + \underline{w}\big)}\bigg)\bigg|_{\underline{w}=0}. \tag{5.180}$$

Moreover, let $\widetilde{\Omega} \subseteq \Omega$ *be such that* μ_j, $j = 1, \dots, n$, *are nonvanishing on* $\widetilde{\Omega}$. *Then,*

[1] To avoid multi-valued expressions in formulas such as (5.179), etc., we agree always to choose the same path of integration connecting Q_0 and P and refer to Remark A.28 for additional tacitly assumed conventions.

the constraint

$$
\begin{aligned}
&2(x-x_0)+2(t_r-t_{0,r})\sum_{s=0}^{r}\tilde{c}_{r-s}\hat{c}_s(\underline{E})\\
&\quad=\bigg(-2\int_{x_0}^{x}\frac{dx'}{\prod_{k=1}^{n}\mu_k(x',t_r)}\\
&\qquad\quad-2\sum_{k=0}^{r\wedge n}\tilde{d}_{r,k}(\underline{E})\int_{t_{0,r}}^{t_r}\frac{\Psi_k(\underline{\mu}(x_0,t'))}{\Psi_n(\underline{\mu}(x_0,t'))}dt'\bigg)\sum_{j=1}^{n}\bigg(\int_{a_j}\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)c_j(1)\\
&\qquad\quad+2(t_r-t_{0,r})\sum_{\ell=1\vee(n+1-\ell)}^{n}\tilde{d}_{r,n+1-\ell}(\underline{E})\sum_{j=1}^{n}\bigg(\int_{a_j}\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)c_j(\ell)\\
&\qquad\quad+\ln\left(\frac{\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x,t_r))\big)\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x_0,t_{0,r}))\big)}{\theta\big(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(x,t_r))\big)\theta\big(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(x_0,t_{0,r}))\big)}\right), \qquad (5.181)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,t_r),(x_0,t_{0,r})\in\widetilde{\Omega}
\end{aligned}
$$

holds, with

$$
\begin{aligned}
\underline{\hat{z}}(P_{\infty_\pm},\underline{\hat{\mu}}(x,t_r))&=\underline{\widehat{\Xi}}_{Q_0}-\underline{\widehat{A}}_{Q_0}(P_{\infty_\pm})+\underline{\hat{\alpha}}_{Q_0}\big(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}\big)\\
&=\underline{\widehat{\Xi}}_{Q_0}-\underline{\widehat{A}}_{Q_0}(P_{\infty_\pm})+\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_r)})-2\bigg(\int_{x_0}^{x}\frac{dx'}{\Psi_n(\underline{\mu}(x',t_r))}\bigg)\underline{c}(1) \qquad (5.182)\\
&=\underline{\widehat{\Xi}}_{Q_0}-\underline{\widehat{A}}_{Q_0}(P_{\infty_\pm})+\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_{0,r})})\\
&\quad-2\bigg(\sum_{k=0}^{r\wedge n}\tilde{d}_{r,k}(\underline{E})\int_{t_{0,r}}^{t_r}\frac{\Psi_k(\underline{\mu}(x,t'))}{\Psi_n(\underline{\mu}(x,t'))}dt'\bigg)\underline{c}(1) \qquad (5.183)\\
&\quad+2(t_r-t_{0,r})\bigg(\sum_{\ell=1\vee(n+1-r)}^{n}\tilde{d}_{r,n+1-\ell}(\underline{E})\underline{c}(\ell)\bigg), \quad (x,t_r),(x_0,t_{0,r})\in\widetilde{\Omega}.
\end{aligned}
$$

Proof First, let $\widetilde{\widetilde{\Omega}}\subseteq\Omega$ be defined by requiring that μ_j, $j=1,\dots,n$, are distinct and nonvanishing on $\widetilde{\widetilde{\Omega}}$ and $\widetilde{F}_r(\mu_j)\neq 0$ on $\widetilde{\widetilde{\Omega}}$, $j=1,\dots,n$. The representation (5.179) for ϕ on $\widetilde{\widetilde{\Omega}}$ then follows by combining (5.135), (5.169), (5.170), and Theorem A.26 since $\mathcal{D}_{\underline{\hat{\mu}}}$ and $\mathcal{D}_{\underline{\hat{\nu}}}$ are simultaneously nonspecial, as discussed in the proof of Theorem 5.8. The representation (5.180) for u on $\widetilde{\widetilde{\Omega}}$ follows from the trace formula (5.168) and (F.59) (taking $k=1$). By continuity, (5.179) and (5.180) extend from $\widetilde{\widetilde{\Omega}}$ to Ω. The constraint (5.181) then holds on $\widetilde{\widetilde{\Omega}}$ by combining (5.173)–(5.176) and (F.58). Equations (5.182) and (5.183) are clear from (5.171) and (5.172). Again by continuity, (5.181)–(5.183) extend from $\widetilde{\widetilde{\Omega}}$ to $\widetilde{\Omega}$. □

Of course, Remark 5.9 applies in the present time-dependent context.

The algebro-geometric CH solution u in (5.180) is not meromorphic with respect to x, t_r in general. In more geometrical terms, the CH$_r$ flows evolve on a nonlinear subvariety (corresponding to the constraint (5.181)) of a generalized Jacobian topologically given by $J(\mathcal{K}_n) \times \mathbb{C}^*$ ($\mathbb{C}^* = \mathbb{C} \setminus \{0\}$), as discussed in the references mentioned in the notes to Section 5.4.

Expressing $\widetilde{F}_r$ in terms of $\Psi_k(\underline{\mu})$ and hence in terms of the theta function associated with $\mathcal{K}_n$, one can use (5.155) to derive a theta function representation of ψ_j, $j = 1, 2$, in analogy to the stationary case discussed in Remark 5.11. We omit further details.

Up to this point we assumed Hypothesis 5.16 together with the basic equations (5.115) and (5.116). Next, we will show that solvability of the Dubrovin equations (5.162) and (5.163) on $\Omega_\mu \subseteq \mathbb{R}^2$ in fact implies equations (5.115) and (5.116) on Ω_μ. In complete analogy to our discussion in Section 5.3 (cf. Remark 5.15), this amounts to solving the time-dependent algebro-geometric initial value problem (5.113), (5.114) on Ω_μ. In this context we recall the definition of $\widetilde{F}_r(\mu_j)$ in terms of $\mu_1, \dots, \mu_n$, introduced in (F.16), (F.19),

$$\widetilde{F}_r(\mu_j) = \sum_{k=0}^{r \wedge n} \tilde{d}_{r,k}(\underline{E}) \Phi_k^{(j)}(\underline{\mu}), \quad r \in \mathbb{N}_0, \ \tilde{c}_0 = 1, \tag{5.184}$$

$$\tilde{d}_{r,k}(\underline{E}) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s} \hat{c}_s(\underline{E}), \quad k = 0, \dots, r \wedge n, \tag{5.185}$$

in terms of a given set of integration constants $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$.

Theorem 5.25 *Fix $n \in \mathbb{N}$ and assume* (5.161). *Suppose that $\{\hat{\mu}_j\}_{j=1,\dots,n}$ satisfies the Dubrovin equations* (5.162), (5.163) *on an open and connected set $\Omega_\mu \subseteq \mathbb{R}^2$, with $\widetilde{F}_r(\mu_j)$ in* (5.163) *expressed in terms of μ_k, $k = 1, \dots, n$, by* (5.184), (5.185). *Moreover, assume that μ_j, $j = 1, \dots, n$, remain distinct and nonzero on Ω_μ. Then $u \in C^\infty(\Omega_\mu)$, defined by*

$$u = -\frac{1}{4} \sum_{m=0}^{2n+1} E_m + \frac{1}{2} \sum_{j=1}^{n} \mu_j, \tag{5.186}$$

satisfies the rth CH equation (5.113)*, that is,*

$$\widetilde{\mathrm{CH}}_r(u) = 0 \text{ on } \Omega_\mu, \tag{5.187}$$

with initial values satisfying the nth stationary CH equation (5.114).

Proof Given solutions $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in C^\infty(\Omega_\mu, \mathcal{K}_n)$, $j = 1, \dots, n$ of (5.162) and (5.163), we define polynomials F_n, G_n, and H_n on Ω_μ as in the stationary case (cf. Theorem 5.14) with properties

$$F_n(z) = \prod_{j=1}^{n} (z - \mu_j), \tag{5.188}$$

$$G_n = F_n + (1/2)F_{n,x}, \tag{5.189}$$

$$zG_{n,x} = (4u - u_{xx})F_n - H_n, \tag{5.190}$$

$$H_{n,x} = 2H_n - 2(4u - u_{xx})G_n, \tag{5.191}$$

$$R_{2n+2} = z^2 G_n^2 + zF_n H_n, \tag{5.192}$$

treating t_r as a parameter. Define polynomials $\widetilde{G}_r$ and $\widetilde{H}_r$ by

$$\widetilde{G}_r(z) = \widetilde{F}_r(z) + (1/2)\widetilde{F}_{r,x}(z), \tag{5.193}$$

$$\widetilde{H}_r(z) = (4u - u_{xx})\widetilde{F}_r(z) - z\widetilde{G}_{r,x}(z), \tag{5.194}$$

respectively. We claim that

$$F_{n,t_r} = 2\big(G_n\widetilde{F}_r - F_n\widetilde{G}_r\big). \tag{5.195}$$

To prove (5.195), one computes from (5.162) and (5.163) that

$$F_{n,t_r}(z) = -F_n(z)\sum_{j=1}^{n}\widetilde{F}_r(\mu_j)\mu_{j,x}(z-\mu_j)^{-1},$$
$$F_{n,x}(z) = -F_n(z)\sum_{j=1}^{n}\mu_{j,x}(z-\mu_j)^{-1}.$$

Using (5.189) and (5.193), one concludes that (5.195) is equivalent to

$$\widetilde{F}_{r,x}(z) = \sum_{j=1}^{n}(\widetilde{F}_r(z) - \widetilde{F}_r(\mu_j))\mu_{j,x}(z-\mu_j)^{-1}. \tag{5.196}$$

Equation (5.196) is proved in Lemma F.9. This in turn proves (5.195). Next, taking the derivative of (5.195) with respect to x and inserting (5.189) and (5.190), yield

$$\begin{aligned} F_{n,t_r x} = 2\big(&z^{-1}(4u - u_{xx})F_n\widetilde{F}_r - z^{-1}H_n\widetilde{F}_r + G_n\widetilde{F}_{r,x} \\ &- 2(G_n - F_n)\widetilde{G}_r - F_n\widetilde{G}_{r,x}\big). \end{aligned} \tag{5.197}$$

On the other hand, by differentiating (5.189) with respect to t_r, using (5.195), one obtains

$$F_{n,t_r x} = 2\big(G_{n,t_r} - 2\big(G_n\widetilde{F}_r - F_n\widetilde{G}_r\big)\big). \tag{5.198}$$

Combining (5.189), (5.193), (5.197), and (5.198), one concludes

$$zG_{n,t_r} = F_n\widetilde{H}_r - \widetilde{F}_r H_n. \tag{5.199}$$

Next, taking the derivative of (5.192) with respect to t_r and using the expressions (5.195) and (5.199) for F_{n,t_r} and G_{n,t_r}, respectively, one obtains

$$H_{n,t_r} = 2\big(\widetilde{G}_r H_n - G_n\widetilde{H}_r\big). \tag{5.200}$$

Finally, we compute G_{n,xt_r} in two different ways. Differentiating (5.199) with respect to x, using (5.189), (5.193), and (5.191), one finds

$$zG_{n,xt_r} = \widetilde{H}_{r,x} F_n + 2(G_n \widetilde{H}_r - \widetilde{G}_r H_n) + 2(4u - u_{xx}) G_n \widetilde{F}_r - 2F_n \widetilde{H}_r. \quad (5.201)$$

Differentiating (5.190) with respect to t_r, using (5.195) and (5.200), results in

$$\begin{aligned} zG_{n,xt_r} &= (u_{t_r} - u_{xxt_r}) F_n - 2(\widetilde{G}_r H_n - G_n \widetilde{H}_r) \\ &\quad + 2(4u - u_{xx})(G_n \widetilde{F}_r - F_n \widetilde{G}_r). \end{aligned} \quad (5.202)$$

Combining (5.201) and (5.202), one concludes

$$u_{t_r} - u_{xxt_r} = \widetilde{H}_r + 2(4u - u_{xx})\widetilde{G}_r - \widetilde{H}_r,$$

which is equivalent to (5.187). □

The analog of Remark 5.15 directly extends to the current time-dependent setting.

5.5 Notes

This chapter follows Gesztesy and Holden (to appear, a).

Section 5.1. The Camassa–Holm (CH) equation, also known as the dispersive shallow water equation,

$$4u_t - u_{xxt} - 2uu_{xxx} - 4u_x u_{xx} + 24uu_x = 0 \quad (5.203)$$

(choosing a scaling of x, t that is convenient for our purpose), with u representing the fluid velocity in x-direction, was introduced[1] in Camassa and Holm (1993) and Camassa et al. (1994). Actually, (5.203) represents the limiting case $\kappa \to 0$ of the general Camassa–Holm equation

$$4v_t - v_{xxt} - 2vv_{xxx} - 4v_x v_{xx} + 24vv_x + 4\kappa v_x = 0, \quad \kappa \in \mathbb{R}. \quad (5.204)$$

However, in our formalism the general Camassa–Holm equation (5.204) just represents a linear combination of the first two equations in the CH hierarchy, and hence we consider without loss of generality (5.203) as the first nontrivial element of the Camassa–Holm hierarchy. Alternatively, one can transform

$$v(x,t) \mapsto u(x,t) = v(x - (\kappa/2)t, t) + (\kappa/4)$$

and thereby reduce (5.204) to (5.203).

Various aspects of local existence, global existence, and uniqueness of solutions of (5.203) are treated in Constantin and Escher (1998a,b,d), Constantin and

[1] The equation appeared already in Fuchssteiner and Fokas (1981) on a list of equations with a bi-Hamiltonian structure.

Molinet (2000), Danchin (2001), and Xin and Zhang (2000), and wave-breaking phenomena are discussed in Constantin (2000), Constantin and Escher (1998c), McKean (1998), and Constantin and Escher (2000). Soliton-type solutions (called "peakons") were extensively studied due to their unusual nonmeromorphic (peak-type) behavior, which features a discontinuity in the x-derivative of u with existing left and right derivatives of opposite sign at the peak. In this context, we refer, for instance, to Alber et al. (1994; 2001), Alber and Fedorov (2000; 2001), Alber and Miller (2001), Beals et al. (1999; 2000; 2001), Camassa and Holm (1993), and Camassa et al. (1994). Integrability aspects such as infinitely many conservation laws, (bi-)Hamiltonian formalism, Bäcklund transformations, infinite dimensional symmetry groups, etc., are discussed, for instance, in Camassa and Holm (1993), Camassa et al. (1994), Constantin (1997b), Fisher and Schiff (1999), Fuchssteiner (1996) (see also Fuchssteiner and Fokas (1981)), and Schiff (1996). The CH equation is derived as a model for water waves in Johnson (2002). The general CH equation (5.204) is shown to give rise to a geodesic flow of a certain right invariant metric on the Bott–Virasoro group in Misiołek (1998). In the case $\kappa = 0$, the CH equation (5.203) corresponds to the geodesic flow on the group of orientation preserving diffeomorphisms of the circle. This follows from the Lie–Poisson structure established in Camassa et al. (1994) and is also remarked upon in Misiołek (1998). Scattering data and their evolution under the CH flow are determined in Beals et al. (1998), and intimate relations with the classical moment problem and the finite Toda lattice are worked out in Beals et al. (1999; 2000; 2001).

Section 5.2. The basic polynomial recursion formalism presented is essentially taken from Alber et al. (1994). We note that our zero-curvature approach is similar (but not identical) to that sketched in Schiff (1996). For yet another zero-curvature approach, see Holm and Qiao (2002). In other treatments of the CH equation, a Lax equation approach appears to be preferred.

Up to inessential scaling of the (x, t_1) variables, $\widehat{\mathrm{CH}}_1(u) = 0$ represents the *Camassa–Holm* equation, as discussed in Camassa and Holm (1993) and Camassa et al. (1994).

The polynomial recursion formalism was introduced under the assumption of a sufficiently smooth function u in Hypothesis 5.1. The actual existence of smooth global solutions of the initial value problem associated with the CH hierarchy (5.40) is a nontrivial issue, and various aspects of it are discussed, for instance, in Constantin (2000), Constantin and Escher (1998a,b,c), Constantin and Molinet (2000), Danchin (2001), and Xin and Zhang (2000).

Section 5.3. As in all other chapters, the fundamental meromorphic function ϕ on $\mathcal{K}_n$ defined in (5.47) is still the key object of our algebro-geometric formalism. By (5.47)–(5.49), ϕ again links the auxiliary divisor $\mathcal{D}_{\underline{\hat{\mu}}}$ and its counterpart $\mathcal{D}_{\underline{\hat{\nu}}}$. This is of course a direct consequence of the identity (5.18) together with the factorizations of F_n and H_n in (5.44). Thus, our construction of positive divisors of degree n (respectively $n+1$ since the points P_0 and P_{∞_+} are also involved) on the hyperelliptic curve $\mathcal{K}_n$ of genus n again follows the recipe of Jacobi (1846), Mumford (1984, Sect. III a).1), and McKean (1985).

Dubrovin equations of the type (5.66) were first discussed by Constantin (1998a,b) and Constantin and McKean (1999) in the spatially periodic case, and by Alber and Fedorov (2000; 2001) in the algebro-geometric context.

That the Abel map does not provide the proper change of variables to linearize the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x)}$ in the CH context, as proven in (5.79), is in sharp contrast to standard integrable soliton equations such as the KdV and AKNS hierarchies and comes, perhaps, as a surprise. The change of variables in (5.84) then linearizes the Abel map $\underline{A}_{Q_0}(\mathcal{D}_{\underline{\hat{\tilde{\mu}}}(\tilde{x})})$, $\tilde{\mu}_j(\tilde{x}) = \mu_j(x)$, $j = 1, \dots, n$. These facts are well-known and discussed (by different methods) by Constantin and McKean (1999), Alber (2000), Alber and Fedorov (2000; 2001), and Alber et al. (2001).

Theta function representations for u analogous to (5.92) were first studied by Alber and Fedorov (2000; 2001) and Alber et al. (2001). These references also discuss the nature of real-valued bounded solutions that are either smooth and quasi-periodic with respect to x or exhibit an infinite number of branch points as alluded to in Remark 5.10. Additional studies of algebro-geometric solutions of (5.203) and their properties are made in Alber (2000), Alber et al. (1999; 1994; 1995; 2000a). Our own approach to algebro-geometric solutions of the CH hierarchy differs from the ones just mentioned in several aspects, and we will outline some of the differences next. In contrast to the treatments in Alber et al. (2001), Alber and Fedorov (2000; 2001), we rely on a zero-curvature approach $U_t - V_x = [V, U]$ as opposed to their Lax formalism. However, we incorporate important features of the recursion formalism developed in Alber et al. (1994) into our zero-curvature approach. Our treatment is comprehensive and self-contained in the sense that it includes Dubrovin-type equations for auxiliary divisors on the associated compact hyperelliptic curve, trace formulas, and theta function representations of solutions, which are the usual ingredients of such a formalism. Moreover, while Alber et al. (2001), Alber and Fedorov (2000; 2001) focus on solutions of the CH equation itself, we simultaneously derive theta function formulas for solutions of any equation of the CH hierarchy. Explicit theta function representations for symmetric functions of (projections of) auxiliary divisors then yield the theta function representations for any algebro-geometric solution u of the CH hierarchy. Here our strategy again differs somewhat from that employed in Alber et al. (2001), Alber and Fedorov (2000; 2001) for the CH equation. Although the latter references also employ the trace formula for u in terms of (projections of) auxiliary divisors, they subsequently rely on generalized theta functions and generalized Jacobians (going back to investigations of Clebsch and Gordan (1866)), whereas we stay within the traditional framework familiar from the KdV, AKNS, Toda hierarchies, etc., following a route similar to the treatment of the Dym equation in Novikov (1999).

The stationary algebro-geometric initial value problem in Theorem 5.14 was first discussed in Gesztesy and Holden (to appear, a).

The case of spatially periodic solutions, the corresponding inverse spectral problem, and isospectral classes of solutions were previously discussed in Constantin (1997a,c,d; 1998a,b) and Constantin and McKean (1999) under certain smoothness assumptions on u. We note that the integral kernel of $(-d^2/dx^2+1)^{-1/2}$ in $L^2(\mathbb{R})$ used in (5.100) is of the type

$$(-d^2/dx^2+1)^{-1/2}(x,x') = \pi^{-1}K_0(|x-x'|), \quad x,x'\in\mathbb{R},\ x\neq x',$$

where $K_0(\,\cdot\,)$ denotes the modified Bessel function of order zero (cf. Abramowitz and Stegun (1972, Sec. 9.6).) The problem of characterizing real-valued algebro-geometric CH solutions has been solved in Gesztesy and Holden (to appear, b).

Section 5.4. Since almost all references cited in the notes to Section 5.3 also treat time-dependent aspects, we can be brief in connection with Section 5.4.

As described in the notes to Section 5.3, the fundamental meromorphic function ϕ on $\mathcal{K}_n$ defined in (5.133), is still the key object of our algebro-geometric formalism, and the facts recorded in the stationary context also apply to the time-dependent setting.

That the Abel map does not effect a linearization of the divisor $\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}$ in the CH context is well-known and discussed (using different approaches) by Constantin and McKean (1999), Alber and Fedorov (2000; 2001), and Alber et al. (2001). A change of the variable t_1 in analogy to that in (5.84) in the stationary context, which avoids the use of a meromorphic differential (cf. (5.77), (5.78)) and linearizes the Abel map when considering the CH_1 flow, is discussed in Alber (2000) and Alber et al. (2001). That change of variables corresponds to the case $r=1$ in (5.181).

Theta function representations for u analogous to (5.180) were first studied by Alber and Fedorov (2000; 2001).

In analogy to the stationary case, the algebro-geometric CH solution u in (5.180) is not meromorphic with respect to x, t_r, in general, as discussed by Alber and Fedorov (2000; 2001), and Alber et al. (2001). In more geometrical terms, the CH_r flows evolve on a nonlinear subvariety (corresponding to the constraint (5.181)) of a generalized Jacobian topologically given by $J(\mathcal{K}_n)\times\mathbb{C}^*$ ($\mathbb{C}^*=\mathbb{C}\setminus\{0\}$). For discussions of generalized Jacobians in this context we refer, for instance, to Fedorov (1999), Gagnon et al. (1985), and Gavrilov (1999). That smooth (i.e., C^1 with respect to t_1 and C^3 and hence C^∞ with respect to x), spatially periodic CH_1 solutions u are quasi-periodic in t_1 was shown in Constantin (1998b).

The general time-dependent algebro-geometric initial value problem in Theorem 5.25 was first discussed in Gesztesy and Holden (to appear, a).

Quasi-periodicity of solutions with respect to time is discussed in Constantin (1998b).

Without going into further details, we mention that our approach extends in a straightforward manner to the Dym-type equation,

$$v_{xxt} + 2vv_{xxx} + 4v_x v_{xx} - 4\kappa v = 0, \quad \kappa \in \mathbb{R}, \tag{5.205}$$

and its hierarchy. The corresponding zero-curvature formalism leads to a trace formula analogous to (5.168) (cf. Alber and Fedorov (2000; 2001)). One needs to replace the polynomial R_{2n+2} by $R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m)$, which results in a branch point P_∞ at infinity and replaces the (nonnormalized) differential $\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}$ of the third kind by the (nonnormalized) differential $\tilde{\omega}^{(2)}_{P_\infty} = z^n dz/y$ of the second kind, etc. This approach (applied to the Dym equation $4\rho_t = \rho^3 \rho_{xxx}$, related to (5.205) by proper variable transformations) was first realized by Novikov (1999) and influenced our treatment of the CH hierarchy. For different approaches to algebro-geometric solutions of the Dym hierarchy, we also refer to Dmitrieva (1993) and Alber et al. (1995; 2000b; 2001).

Appendix A

Algebraic Curves and Their Theta Functions in a Nutshell

> It is assumed that the reader will not have a heart failure at the mention of a Riemann surface.
>
> *Julian L. Coolidge*[1]

This appendix treats some of the basic aspects of complex algebraic curves and their theta functions as used at various places in this monograph. The material presented is standard, and we include it for two major reasons: On one hand it allows us to summarize a variety of facts and explicit formulas needed in connection with the construction of algebro-geometric solutions of completely integrable equations, and, on the other hand, it will simultaneously enable us to introduce a large part of the notation used throughout this volume. We emphasize that the summary presented in this appendix is not intended as a substitute for textbook consultations. Relevant literature in this context is mentioned in the notes to this appendix.

Definition A.1 An affine plane (complex) algebraic curve $\mathcal{K}$ is the locus of zeros in $\mathbb{C}^2$ of a (nonconstant) polynomial $\mathcal{F}$ in two variables. The polynomial $\mathcal{F}$ is called nonsingular at a root (z_0, y_0) if

$$\nabla\mathcal{F}(z_0, y_0) = (\mathcal{F}_z(z_0, y_0), \mathcal{F}_y(z_0, y_0)) \neq 0.$$

The affine plane curve $\mathcal{K}$ of roots of $\mathcal{F}$ is called nonsingular at $P_0 = (z_0, y_0)$ if $\mathcal{F}$ is nonsingular at P_0. The curve $\mathcal{K}$ is called nonsingular, or smooth, if it is nonsingular at each of its points (otherwise, it is called singular).

The implicit function theorem allows one to conclude that a smooth affine curve $\mathcal{K}$ is locally a graph and to introduce complex charts on $\mathcal{K}$ as follows. If $\mathcal{F}(P_0) = 0$ with $\mathcal{F}_y(P_0) \neq 0$, there is a holomorphic function g_{P_0} such that in a neighborhood U_{P_0} of P_0, the curve $\mathcal{K}$ is characterized by the graph $y = g_{P_0}(z)$. Hence, the projection

$$\tilde{\pi}_z \colon U_{P_0} \to \tilde{\pi}_z(U_{P_0}) \subset \mathbb{C}, \quad (z, y) \mapsto z \tag{A.1}$$

[1] *A Treatise on Algebraic Plane Curves*, Dover, New York, 1959, p. x.

yields a complex chart on $\mathcal{K}$. If, on the other hand, $\mathcal{F}(P_0) = 0$ with $\mathcal{F}_z(P_0) \neq 0$, then the projection

$$\tilde{\pi}_y\colon U_{P_0} \to \tilde{\pi}_y(U_{P_0}) \subset \mathbb{C}, \quad (z, y) \mapsto y \tag{A.2}$$

defines a chart on $\mathcal{K}$. In this way, as long as $\mathcal{K}$ is nonsingular, one arrives at a complex atlas on $\mathcal{K}$. The space $\mathcal{K} \subset \mathbb{C}^2$ is second countable and Hausdorff. To obtain a Riemann surface, one needs connectedness of $\mathcal{K}$, which is implied by adding the assumption of irreducibility[1] of the polynomial $\mathcal{F}$. Thus, $\mathcal{K}$ equipped with charts (A.1) and (A.2) is a Riemann surface if $\mathcal{F}$ is nonsingular and irreducible. Affine plane curves $\mathcal{K}$ are unbounded as subsets of $\mathbb{C}^2$ and hence are noncompact. The compactification of $\mathcal{K}$ is conveniently described in terms of the projective plane $\mathbb{CP}^2$, the set of all one-dimensional (complex) subspaces of $\mathbb{C}^3$.

To simplify notations, we temporarily abbreviate $x_1 = y$ and $x_2 = z$. Moreover, we denote the linear span of $(x_2, x_1, x_0) \in \mathbb{C}^3 \setminus \{0\}$ by $[x_2 : x_1 : x_0]$. Since the homogeneous coordinates $[x_2 : x_1 : x_0]$ satisfy

$$[x_2 : x_1 : x_0] = [cx_2 : cx_1 : cx_0], \quad c \in \mathbb{C} \setminus \{0\},$$

the space $\mathbb{CP}^2$ can be viewed as the quotient space of $\mathbb{C}^3 \setminus \{0\}$ by the multiplicative action of $\mathbb{C} \setminus \{0\}$, that is, $\mathbb{CP}^2 = (\mathbb{C}^3 \setminus \{0\})/(\mathbb{C} \setminus \{0\})$, and hence $\mathbb{CP}^2$ inherits a Hausdorff topology, which is the quotient topology induced by the natural map

$$\iota\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2, \quad (x_2, x_1, x_0) \mapsto [x_2 : x_1 : x_0].$$

Next, define the open sets

$$U^m = \{\, [x_2 : x_1 : x_0] \in \mathbb{CP}^2 \mid x_m \neq 0\}, \quad m = 0, 1, 2.$$

Then,

$$f^0\colon U^0 \to \mathbb{C}^2, \quad [x_2 : x_1 : x_0] \mapsto (x_2/x_0, x_1/x_0)$$

with inverse

$$(f^0)^{-1}\colon \mathbb{C}^2 \to U^0, \quad (x_2, x_1) \mapsto [x_2 : x_1 : 1],$$

and analogously for functions f^1 and f^2 (relative to sets U^1 and U^2, respectively), are homeomorphisms. In particular, U^0, U^1, and U^2 together cover $\mathbb{CP}^2$. Thus, $\mathbb{CP}^2$ is compact since it is covered by the closed unit (poly)disks in U^0, U^1, and U^2. The element $[x_2 : x_1 : 0] \in \mathbb{CP}^2$ represents the point at infinity along the direction $x_2 : x_1$ in $\mathbb{C}^2$ (identifying $[x_2 : x_1 : 0] \in \mathbb{CP}^2$ and $[x_2 : x_1] \in \mathbb{CP}^1$). The set of all such elements then represents the line at infinity, $L_\infty = \{[x_2 : x_1 : x_0] \in \mathbb{CP}^2 \mid x_0 = 0\}$, and yields the compactification $\mathbb{CP}^2$ of $\mathbb{C}^2$. In other words, $\mathbb{CP}^2 \cong \mathbb{C}^2 \cup L_\infty$, $\mathbb{CP}^1 \cong \mathbb{C}_\infty$, and $L_\infty \cong \mathbb{CP}^1$.

[1] The polynomial $\mathcal{F}$ in two variables is called irreducible if it cannot be factored into $\mathcal{F} = \mathcal{F}_1\mathcal{F}_2$ with $\mathcal{F}_1$ and $\mathcal{F}_2$ both nonconstant polynomials in two variables.

Let $\mathcal{P}$ be a (nonconstant) homogeneous polynomial of degree d in (x_2, x_1, x_0), that is,

$$\mathcal{P}(cx_2, cx_1, cx_0) = c^d \mathcal{P}(x_2, x_1, x_0),$$

and introduce

$$\overline{\mathcal{K}} = \{[x_2 : x_1 : x_0] \in \mathbb{CP}^2 \mid \mathcal{P}(x_2, x_1, x_0) = 0\}.$$

The set $\overline{\mathcal{K}}$ is well-defined (even though $\mathcal{P}(u, v, w)$ is not for $[u : v : w] \in \mathbb{CP}^2$) and closed in $\mathbb{CP}^2$. The intersections,

$$\mathcal{K}^m = \overline{\mathcal{K}} \cap U^m, \quad m = 0, 1, 2$$

are affine plane curves when transported to $\mathbb{C}^2$. In particular,

$$\mathcal{K}^0 \cong \{(x_2, x_1) \in \mathbb{C}^2 \mid \mathcal{P}(x_2, x_1, 1) = 0\}$$

represents the affine curve $\mathcal{K}$ defined by $\mathcal{F}(z, y) = 0$, where $\mathcal{F}(x_2, x_1) = \mathcal{P}(x_2, x_1, 1)$, that is, $\mathcal{K}^0$ represents the affine part of $\overline{\mathcal{K}}$. ($\mathcal{F}$ has degree d provided x_0 is not a factor of $\mathcal{P}$, i.e., provided $\overline{\mathcal{K}}$ does not contain the projective line L_∞).

Conversely, given the affine curve $\mathcal{K}$ defined by

$$\mathcal{F}(x_2, x_1) = \sum_{\substack{r,s=0 \\ r+s \le d}}^{d} a_{r,s} z^r y^s = 0,$$

with $\mathcal{F}$ of degree d, the associated homogeneous polynomial $\mathcal{P}$ of degree d can be obtained from

$$\mathcal{P}(x_2, x_1, x_0) = x_0^d \mathcal{F}(x_2/x_0, x_1/x_0).$$

The affine curve $\mathcal{K}$ is then the intersection of the projective curve $\overline{\mathcal{K}}$ defined by $\mathcal{P}(x_2, x_1, x_0) = 0$ with U^0, that is, $\mathcal{K} \cong \overline{\mathcal{K}} \cap U^0 = \mathcal{K}^0$. The interesection of $\overline{\mathcal{K}}$ with L_∞, the line at infinity, then consists of the finite set of points

$$\overline{\mathcal{K}} \setminus \mathcal{K} = \left\{ [x_2 : x_1 : 0] \in \mathbb{CP}^2 \,\middle|\, \sum_{r=0}^{d} a_{r,d-r} z^r y^{d-r} = 0 \right\}.$$

Definition A.2 A projective plane (complex) algebraic curve $\overline{\mathcal{K}}$ is the locus of zeros in $\mathbb{CP}^2$ of a homogeneous polynomial $\mathcal{P}$ in three variables. A homogeneous (nonconstant) polynomial $\mathcal{P}$ in three variables is called nonsingular if there are no common solutions $(x_{2,0}, x_{1,0}, x_{0,0}) \in \mathbb{C}^3 \setminus \{0\}$ of

$$\mathcal{P}(x_{2,0}, x_{1,0}, x_{0,0}) = 0,$$
$$\nabla \mathcal{P}(x_{2,0}, x_{1,0}, x_{0,0}) = (\mathcal{P}_{x_2}, \mathcal{P}_{x_1}, \mathcal{P}_{x_0})(x_{2,0}, x_{1,0}, x_{0,0}) = 0.$$

The set $\overline{\mathcal{K}}$ is called a smooth (or nonsingular) projective plane curve (of degree $d \in \mathbb{N}$) if $\mathcal{P}$ is nonsingular (and of degree $d \in \mathbb{N}$).

If $x_{0,0} \neq 0$, then $[x_{2,0} : x_{1,0} : x_{0,0}] \in \mathbb{CP}^2$ is a nonsingular point of the projective curve $\overline{\mathcal{K}}$ (defined by $\mathcal{P}(x_2, x_1, x_0) = 0$) if and only if $(x_{2,0}/x_{0,0}, x_{1,0}/x_{0,0}) \in \mathbb{C}^2$ is a nonsingular point of the affine curve $\mathcal{K}$ (defined by $\mathcal{P}(x_2, x_1, 1) = 0$).

One verifies that the homogeneous polynomial $\mathcal{P}$ is nonsingular if and only if each $\mathcal{K}^m$ is a smooth affine plane curve in $\mathbb{C}^2$. Moreover, any nonsingular homogeneous polynomial $\mathcal{P}$ is irreducible, and consequently each $\mathcal{K}^m$ is a Riemann surface for $m = 0, 1, 2$. The coordinate charts on each $\mathcal{K}^m$ are simply the projections, that is, x_2/x_0 and x_1/x_0 for $\mathcal{K}^0$, x_2/x_1 and x_0/x_1 for $\mathcal{K}^1$, and finally, x_1/x_2 and x_0/x_2 for $\mathcal{K}^2$. These separate complex structures on $\mathcal{K}^m$ are compatible on $\overline{\mathcal{K}}$ and hence induce a complex structure on $\overline{\mathcal{K}}$.

The zero locus of a nonsingular homogeneous polynomial $\mathcal{P}(x_2, x_1, x_0)$ in $\mathbb{CP}^2$ defines a smooth projective plane curve $\overline{\mathcal{K}}$, which is a compact Riemann surface. Topologically, this Riemann surface is a sphere with g handles, where

$$g = (d-1)(d-2)/2 \tag{A.3}$$

with d the degree of $\mathcal{P}$. In particular, $\overline{\mathcal{K}}$ has topological genus g, and we indicate this by writing $\overline{\mathcal{K}}_g$. However, for notational convenience we use the symbol $\mathcal{K}_g$ instead (i.e., $\mathcal{K}_g$ always denotes the corresponding compact Riemann surface). In general, the projective curve $\mathcal{K}_g$ can be singular even though the associated affine curve $\mathcal{K}_g^0$ is nonsingular. In this case one has to account for the singularities at infinity and properly amend the genus formula (A.3) according to results of Clebsch, M. Noether, and Plücker.

Next, let $\mathcal{K}_g$ be a smooth projective curve not containing the point $[0, 1, 0]$ associated with the homogeneous polynomial $\mathcal{P}$ of degree d. Then

$$P(z, y, 1) = 0$$

defines y as a multi-valued function of z such that away from ramification points there correspond precisely d values of y for each value of $z \in \mathbb{C}$. The set of finite ramification points of $\mathcal{K}_g$ is given by

$$\{\, [z : y : 1] \in \mathbb{CP}^2 \mid \mathcal{P}(z, y, 1) = \mathcal{P}_y(z, y, 1) = 0\}. \tag{A.4}$$

Similarly, ramification points at infinity are defined by

$$\{\, [1 : y : 0] \in \mathbb{CP}^2 \mid \mathcal{P}(1, y, 0) = \mathcal{P}_y(1, y, 0) = 0\}. \tag{A.5}$$

The set of ramification points of $\mathcal{K}_g$ is then the union of points in (A.4) and (A.5). Given the set of ramification points $\{P_1, \dots, P_r\}$, one can cut the complex plane along smooth nonintersecting arcs $\mathcal{C}_q$ (e.g., straight lines if $P_1, \dots, P_r$ are suitably situated) connecting P_q and P_{q+1} for $q = 1, \dots, r-1$, and define holomorphic functions $f_1, \dots, f_d$ on the cut plane $\Pi = \mathbb{C} \setminus \bigcup_{q=1}^{r-1} \mathcal{C}_q$ such that

$$\mathcal{P}(z, y, 1) = 0 \text{ for } y \in \Pi \text{ if and only if } y = f_j(z) \text{ for some } j \in \{1, \dots, d\}.$$

This yields a topological construction of $\mathcal{K}_g$ by appropriately gluing together d copies of the cut plane Π, the result being a sphere with g handles (g depending on the order of the ramification points points). If $\mathcal{K}_g$ is singular, this procedure requires appropriate modifications.

There is an alternative description of $\mathcal{K}_g$ as a (branched) covering surface of the Riemann sphere $\mathcal{K}_0 = \mathbb{C}_\infty$, which naturally leads to the notion of branch points. To begin with, we briefly consider the case of a general (not necessarily compact) Riemann surface. Starting from a (real) two-dimensional connected C^0-manifold[1] $(\mathcal{M}, \mathcal{A} = (U_\alpha, z_\alpha)_{\alpha \in I})$ and a nonconstant map $F \colon \mathcal{M} \to \mathbb{C}_\infty$ such that

$$F \circ z_\alpha^{-1} \colon z_\alpha(U_\alpha) \to \mathbb{C}_\infty \text{ is nonconstant and holomorphic for all } \alpha \in I, \quad \text{(A.6)}$$

one defines a maximal atlas $\mathcal{A}(F)$ compatible with (A.6). The Riemann surface $\mathcal{R}_F = (\mathcal{M}, \mathcal{A}(F))$ is then a covering surface[2] of $\mathbb{C}_\infty$ and branch points on $\mathcal{R}_F$ are identified with those of F. More precisely, if $z = F(P)$ has a k-fold z_0-point[3] at $P = P_0$, $P_0 \in \mathcal{R}_F$ for some $k \in \mathbb{N}$, then P_0 is called unbranched (unramified) for $k = 1$ and has a branch point (respectively ramification point)[4] of order $k-1$ (respectively k) for $k \geq 2$. The set of branch points of $\mathcal{R}_F$ will be denoted by $\mathcal{B}(\mathcal{R}_F)$. Depending on the branching behavior of $P \in \mathcal{R}_F$, one then introduces the following system of charts on $\mathcal{R}_F$.

(i) $F(P_0) = z_0 \in \mathbb{C}$: One defines for appropriate $C_0 > 0$

$$\begin{aligned}
&U_{P_0} = \{P \in \mathcal{R}_F \mid |z - z_0| < C_0\}, \quad V_{P_0} = \left\{\zeta \in \mathbb{C} \mid |\zeta| < C_0^{1/k}\right\},\\
&\zeta_{P_0} \colon U_{P_0} \to V_{P_0}, \quad P \mapsto (z - z_0)^{1/k},\\
&\zeta_{P_0}^{-1} \colon V_{P_0} \to U_{P_0}, \quad \zeta \mapsto z_0 + \zeta^k.
\end{aligned}$$

1 $(\mathcal{M}, \tau)$ is a (real) two-dimensional connected C^0-manifold if $(\mathcal{M}, \tau)$ is a second countable connected Hausdorff topological space with topology τ, $\mathcal{M} = \bigcup_{\alpha \in I} U_\alpha$, $U_\alpha \in \tau$, $z_\alpha \colon U_\alpha \to \mathbb{C}$ are homeomorphisms, $z_\alpha(U_\alpha)$ is open in $\mathbb{C}$, $\alpha \in I$ (an index set), and $\mathcal{A}$ is a maximal C^0-atlas on $\mathcal{M}$.

2 In this monograph we only deal with covering surfaces of the Riemann sphere $\mathcal{K}_0 = \mathbb{C}_\infty$. The study of special elliptic algebro-geometric solutions of integrable hierarchies, however, is most naturally connected with covers of the torus $\mathcal{K}_1$.

3 F has a k-fold z_0-point at $P = P_0$, if for some chart (U_{P_0}, ζ_{P_0}) on $\mathcal{R}_F$ at P_0 with $\zeta_{P_0}(P_0) = 0$ and some chart (V_{P_0}, w_{P_0}) on $\mathbb{C}_\infty$ at $z_0 = F(P_0)$ with $w_{P_0}(F(P_0)) = 0$, $(w_{P_0} \circ F \circ \zeta_{P_0}^{-1})(\zeta) = \zeta^k$ for all $\zeta \in \zeta_{P_0}(U_{P_0})$. This includes, of course, the possibility that $z_0 = \infty$.

4 This definition of branch points is not universally adopted. Many monographs distinguish ramification and branch points in the sense that a branch point is the image of a ramification point under the covering map. In this monograph we found it convenient to follow the convention used in Farkas and Kra (1992, Sec. I.2).

(ii) $F(P_0) = z_0 = \infty$: One defines for appropriate $C_\infty > 0$

$$U_{P_0} = \{P \in \mathcal{R}_F \mid |z| > C_\infty\}, \quad V_{P_0} = \left\{\zeta \in \mathbb{C} \mid |\zeta| < C_\infty^{-1/k}\right\},$$
$$\zeta_{P_0} : U_{P_0} \to V_{P_0}, \quad P \mapsto z^{-1/k},$$
$$\zeta_{P_0}^{-1} : V_{P_0} \to U_{P_0}, \quad \zeta \mapsto \zeta^{-k}.$$

Next, consider an open nonempty and connected subset $\mathcal{S}$ of $\mathcal{R}_F$ over $\mathbb{C}_\infty$ such that F is univalent in $\mathcal{S}$, that is, F is analytic in $\mathcal{S}$ and takes distinct values at distinct points of $\mathcal{S}$ (thus, F maps $\mathcal{S}$ onto a subset $F(\mathcal{S})$ of $\mathbb{C}_\infty$ in a one-to-one fashion). If $\mathcal{S}$ is maximal with respect to this property (i.e., $\mathcal{S}$ cannot be extended to $\widetilde{\mathcal{S}} \supsetneqq \mathcal{S}$ with F univalent on $\widetilde{\mathcal{S}}$), $\mathcal{S}$ is called a sheet of $\mathcal{R}_F$. In this manner $\mathcal{R}_F$ can be pictured as consisting of finitely many or countably infinitely many sheets over $\mathbb{C}_\infty$ that are connected along branch cuts in such a way that $\mathcal{R}_F$ can be covered locally by disks (if $k = 1$ for the center of such disks) and by k-fold disks (if the center of the disk is a branch point of order $k - 1$, $k \geq 2$). The choice of branch cuts is largely arbitrary as long as they connect branch points and are non-self-intersecting. In the special case of compact Riemann surfaces, the total number of sheets, branch cuts, and branch points is finite.

An important aspect is the possibility of analytic continuation of a given (circular) function element in $\mathbb{C}_\infty$ (i.e., a convergent power series expansion in some disk) along all possible continuous paths on a (covering) Riemann surface $\mathcal{R}_F$ in such a way that the resulting function f has algebroidal[1] behavior at any point of $\mathcal{R}_F$. More precisely, $\mathcal{R}_F$ and a function $f : \mathcal{R}_F \to \mathbb{C}_\infty$ are said to correspond to each other if f is meromorphic on $\mathcal{R}_F$, two function elements of f associated with two different points on $\mathcal{R}_F$ over the same point $z \in \mathbb{C}_\infty$ are distinct, and $\mathcal{R}_F$ is maximal in the sense that there is no $\widetilde{\mathcal{R}}_F \supsetneqq \mathcal{R}_F$ such that these properties hold with $\mathcal{R}_F$ replaced by $\widetilde{\mathcal{R}}_F$. The basic fact concerning analytic functions and corresponding covering Riemann surfaces is then the following: To every function element φ in $\mathbb{C}_\infty$ there exists a covering Riemann surface $\mathcal{R}_F$ such that $\mathcal{R}_F$ and the analytic function f obtained by analytic continuation of φ along any possible continuous path on $\mathcal{R}_F$ correspond to each other.

Finally we briefly consider the special case of compact Riemann surfaces, the case at hand in this monograph. We recall that $w = f(z)$ is called algebraic if there exists an irreducible polynomial $\mathcal{P}$ in two variables such that $\mathcal{P}(z, w) = 0$ for all $z \in \mathbb{C}$. The fundamental connection between compact Riemann surfaces and algebraic functions then reads as follows: f corresponds to a compact Riemann surface if and only if f is algebraic.

Next, we consider the notion of the meromorphic function field $\mathcal{M}(\mathcal{R}_F)$ of $\mathcal{R}_F$, which by definition consists of all analytic maps $f : \mathcal{R}_F \to \mathbb{C}_\infty$. If $\mathcal{R}_F$ corresponds to an algebraic function in the sense described above, then $g : \mathcal{R}_F \to \mathbb{C}_\infty$ belongs

[1] The function f has algebroidal behavior at $P \in \mathcal{R}_F$ if $f(P) = \sum_{n=-p}^{\infty} c_n (z - z_0)^{n/k}$ for $z_0 \in \mathbb{C}$ and $f(P) = \sum_{n=-q}^{\infty} d_n z^{-n/k}$ for $z_0 = \infty$.

to $\mathcal{M}(\mathcal{R}_F)$, that is, $g \in \mathcal{M}(\mathcal{R}_F)$, if and only if $g(z)$ is a rational function in the two variables z and $f(z)$. In addition, if $N \in \mathbb{N}$ denotes the number of sheets of $\mathcal{R}_F$, $1, f, f^2, \dots, f^{N-1}$ forms a basis in $\mathcal{M}(\mathcal{R}_F)$ and g can be uniquely represented as

$$g(z) = r_0(z) + r_1(z)f(z) + \cdots + r_{N-1}(z)f(z)^{N-1}$$

with r_j, $j = 0, \dots, N-1$ rational functions. Moreover, if $f_1, f_2 \in \mathcal{M}(\mathcal{R}_F)$, then $\mathcal{Q}(f_1(z), f_2(z)) = 0$ for all $z \in \mathbb{C}$ for some irreducible polynomial $\mathcal{Q}$ in two variables.

We also mention an alternative to (A.3) for computing the topological genus of $\mathcal{R}_F$ covering $\mathbb{C}_\infty$. Denote by N the number of sheets of $\mathcal{R}_F$, by B the total branching number of $\mathcal{R}_F$,

$$B = \sum_{P \in \mathcal{R}_F} (k(P) - 1) = \sum_{P \in \mathcal{B}(\mathcal{R}_F)} (k(P) - 1),$$

where $k(P) - 1$ denotes the branching order of $P \in \mathcal{R}_F$ (of course $k(P) = 1$ for all but finitely many $P \in \mathcal{R}_F$), and by $\mathcal{B}(\mathcal{R}_F)$ the set of branch points of $\mathcal{R}_F$. Then the topological genus g of $\mathcal{R}_F$ is given by the Riemann–Hurwitz formula

$$g = 1 - N + (B/2).$$

Since hyperelliptic Riemann surfaces are of particular importance to the main body of this monograph, we end this informal introduction with a precise definition of this special case.

Definition A.3 A compact Riemann surface is called hyperelliptic if it admits a meromorphic function of degree two (i.e., a nonconstant meromorphic function with precisely two poles counting multiplicity).

We will describe hyperelliptic Riemann surfaces $\mathcal{K}_g$ of genus $g \in \mathbb{N}$ as two-sheeted coverings of the Riemann sphere $\mathbb{C}_\infty$ branched at $2g + 2$ points in great detail in Appendices B and C. The meromorphic function of degree two alluded to in Definition A.3 is then given by the projection $\tilde{\pi}$, as defined in (B.23) and (C.26). Here we just add one more brief comment on hyperelliptic curves, the principal object in the main body of this text. The projective curve

$$x_1^2 x_0^{k-2} = \prod_{\ell=1}^{k} (x_2 - e_\ell x_0), \quad k \in \mathbb{N} \tag{A.7}$$

in $\mathbb{CP}^2$ of degree k, with $e_1, \dots, e_k$ distinct complex numbers, is called elliptic if $k = 3, 4$ and hyperelliptic if $k \geq 5$. However, to simplify matters, all curves in (A.7) are usually called hyperelliptic, and this convention is adopted in Definition A.3. The projective curve (A.7) is smooth (nonsingular) if and only if $1 \leq k \leq 3$; if $k \geq 4$ it has the unique singular point $[0, 1, 0]$.

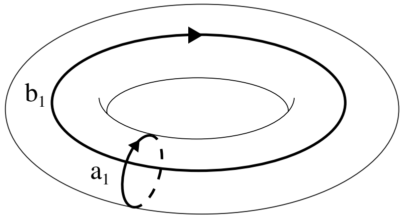

Fig. A.1. Genus $g = 1$.

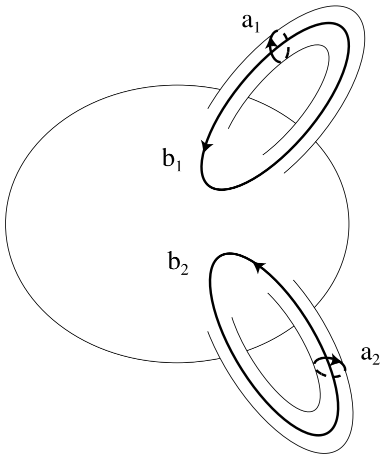

Fig. A.2. Genus $g = 2$. ($a_j \circ b_k = \delta_{jk}$)

For most of the remainder of Appendix A, we suppose that $\mathcal{K}_g$ is a compact Riemann surface of genus $g \in \mathbb{N}$ and choose a homology basis $\{a_j, b_j\}_{j=1}^g$ on $\mathcal{K}_g$ in such a way that the intersection matrix of the cycles satisfies

$$a_j \circ b_k = \delta_{j,k}, \quad a_j \circ a_k = 0, \quad b_j \circ b_k = 0, \quad j, k = 1, \dots, g \tag{A.8}$$

(with a_j and b_k intersecting to form a right-handed coordinate system, cf. Figures A.1, A.2). In particular, the first homology group of $\mathcal{K}_g$ with integer coefficients $\mathrm{H}_1(\mathcal{K}_g, \mathbb{Z})$ is the free Abelian group on the generators $[a_j]$, $[b_j]$, $j = 1, \dots, g$, where $[c]$ denotes the homology class of the cycle c.

Unless explicitly stated otherwise, it will be assumed that $g \geq 1$ for the remainder of this appendix.

Turning briefly to meromorphic differentials (1-forms) on $\mathcal{K}_g$, we state the following result.

Theorem A.4 (Riemann's period relations) *Suppose ω and ν are closed C^1 meromorphic differentials (1-forms) on $\mathcal{K}_g$. Then,*

(i)

$$\iint_{\mathcal{K}_g} \omega \wedge \nu = \sum_{j=1}^{g} \left(\left(\int_{a_j} \omega \right) \left(\int_{b_j} \nu \right) - \left(\int_{b_j} \omega \right) \left(\int_{a_j} \nu \right) \right). \quad \text{(A.9)}$$

If, in addition ω *and* ν *are holomorphic* 1*-forms on* $\mathcal{K}_g$*, then*

$$\sum_{j=1}^{g} \left(\left(\int_{a_j} \omega \right) \left(\int_{b_j} \nu \right) - \left(\int_{b_j} \omega \right) \left(\int_{a_j} \nu \right) \right) = 0. \quad \text{(A.10)}$$

(ii) If ω *is a nonzero holomorphic* 1*-form on* $\mathcal{K}_g$*, then*

$$\operatorname{Im}\left(\sum_{j=1}^{g} \left(\int_{a_j} \omega \right) \left(\int_{b_j} \omega \right) \right) > 0. \quad \text{(A.11)}$$

The proof of Theorem A.4 is usually based on Stokes' theorem and a canonical dissection of $\mathcal{K}_g$ along its cycles yielding the simply connected interior $\widehat{\mathcal{K}}_g$ of the fundamental polygon $\partial \widehat{\mathcal{K}}_g$ given by

$$\partial \widehat{\mathcal{K}}_g = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g^{-1} b_g^{-1}. \quad \text{(A.12)}$$

Given the cycles $\{a_j, b_j\}_{j=1}^{g}$, we denote by $\{\omega_j\}_{j=1}^{g}$ the corresponding normalized basis of the space of holomorphic differentials (also called Abelian differentials of the first kind) on $\mathcal{K}_g$, that is,

$$\int_{a_k} \omega_j = \delta_{j,k}, \quad j, k = 1, \dots, g. \quad \text{(A.13)}$$

The b-periods of ω_j are then defined by

$$\tau_{j,k} = \int_{b_k} \omega_j, \quad j, k = 1, \dots, g. \quad \text{(A.14)}$$

Theorem A.4 then implies the following result.

Theorem A.5 *The* $g \times g$ *matrix* $\tau = (\tau_{j,k})_{j,k=1,\dots,g}$ *is symmetric, that is,*

$$\tau_{j,k} = \tau_{k,j}, \quad j, k = 1, \dots, g \quad \text{(A.15)}$$

with a positive definite imaginary part

$$\operatorname{Im}(\tau) = \frac{1}{2i}(\tau - \tau^*) > 0. \quad \text{(A.16)}$$

Next we briefly study some consequences of a change of homology basis. Let

$$\{a_1, \dots, a_g, b_1, \dots, b_g\} \quad \text{(A.17)}$$

be a canonical homology basis on $\mathcal{K}_g$ with intersection matrix satisfying (A.8) and

$$\{a_1', \dots, a_g', b_1', \dots, b_g'\} \tag{A.18}$$

a homology basis on $\mathcal{K}_g$ related to each other by

$$\begin{pmatrix} \underline{a'}^\top \\ \underline{b'}^\top \end{pmatrix} = X \begin{pmatrix} \underline{a}^\top \\ \underline{b}^\top \end{pmatrix},$$

where

$$\begin{aligned} \underline{a}^\top &= (a_1, \dots, a_g)^\top, \quad \underline{b}^\top = (b_1, \dots, b_g)^\top, \\ \underline{a'}^\top &= (a_1', \dots, a_g')^\top, \quad \underline{b'}^\top = (b_1', \dots, b_g')^\top, \\ X &= \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \end{aligned}$$

with A, B, C, and D being $g \times g$ matrices with integer entries. Then (A.18) is also a canonical homology basis on $\mathcal{K}_g$ with an intersection matrix satisfying (A.8) if and only if

$$X \in \mathrm{Sp}(g, \mathbb{Z}),$$

where

$$\mathrm{Sp}(g, \mathbb{Z}) = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,\middle|\, X \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} X^\top = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}, \ \det(X) = 1 \right\}$$

denotes the symplectic modular group of genus $g \in \mathbb{N}$ (here A, B, C, D in X are again $g \times g$ matrices with integer entries). If $\{\omega_j\}_{j=1}^g$ and $\{\omega_j'\}_{j=1}^g$ are the normalized bases of holomorphic differentials corresponding to the canonical homology bases (A.17) and (A.18), with τ and τ' the associated b and b'-periods of $\omega_1, \dots, \omega_g$ and $\omega_1', \dots, \omega_g'$, respectively, one computes

$$\underline{\omega'} = \underline{\omega}(A + B\tau)^{-1}, \quad \tau' = (C + D\tau)(A + B\tau)^{-1}, \tag{A.19}$$

where $\underline{\omega} = (\omega_1, \dots, \omega_g)$, $\underline{\omega'} = (\omega_1', \dots, \omega_g')$.

Abelian differentials of the second kind, say $\omega^{(2)}$, are characterized by the property that all their residues vanish. They will usually be normalized by the vanishing of all their a-periods (this is achieved by adding a suitable linear combination of differentials of the first kind)

$$\int_{a_j} \omega^{(2)} = 0, \quad j = 1, \dots, g. \tag{A.20}$$

We may add in this context that the sum of the residues of any meromorphic differential ν on $\mathcal{K}_g$ vanishes, the residue at a pole $Q_0 \in \mathcal{K}_g$ of ν being defined by

$$\operatorname*{res}_{Q_0}(\nu) = \frac{1}{2\pi i} \int_{\gamma_{Q_0}} \nu,$$

where γ_{Q_0} is smooth, simple, closed contour, oriented counter-clockwise, encircling Q_0 but no other pole of ν.

Theorem A.6 *Assume $\omega^{(2)}_{Q_1,n}$ to be a differential of the second kind on $\mathcal{K}_g$ whose only pole is $Q_1 \in \widehat{\mathcal{K}}_g$ with principal part $\zeta_{Q_1}^{-n-2} d\zeta_{Q_1}$ for some $n \in \mathbb{N}_0$ and $\omega^{(1)}$ a differential of the first kind on $\mathcal{K}_g$ of the type $\omega^{(1)} = \sum_{m=0}^{\infty} c_m(Q_1)\zeta_{Q_1}^m d\zeta_{Q_1}$ near Q_1. Then*

$$\begin{aligned}\frac{1}{2\pi i}\sum_{j=1}^{g}\bigg(\bigg(\int_{a_j}\omega^{(1)}\bigg)\bigg(\int_{b_j}\omega^{(2)}_{Q_1,n}\bigg) \\ -\bigg(\int_{a_j}\omega^{(2)}_{Q_1,n}\bigg)\bigg(\int_{b_j}\omega^{(1)}\bigg)\bigg) = \frac{c_n(Q_1)}{n+1}, \quad n \in \mathbb{N}_0.\end{aligned} \tag{A.21}$$

In particular, if $\omega^{(2)}_{Q_1,n}$ is normalized and $\omega^{(1)} = \omega_j = \sum_{m=0}^{\infty} c_{j,m}(Q_1)\zeta_{Q_1}^m\, d\zeta_{Q_1}$, then the vector of b-periods of $\omega^{(2)}_{Q_1,n}/(2\pi i)$, denoted by $\underline{U}^{(2)}_n$, reads

$$\begin{aligned}\underline{U}^{(2)}_n = \big(U^{(2)}_{n,1}, \dots, U^{(2)}_{n,g}\big), \quad U^{(2)}_{n,j} = \frac{1}{2\pi i}\int_{b_j}\omega^{(2)}_{Q_1,n} = \frac{c_{j,n}(Q_1)}{n+1}, \\ n \in \mathbb{N}_0, \quad j = 1, \dots, g.\end{aligned} \tag{A.22}$$

Any meromorphic differential $\omega^{(3)}$ on $\mathcal{K}_g$ not of the first or second kind is said to be of the third kind. It is common to normalize $\omega^{(3)}$ by the vanishing of its a-periods, that is, by

$$\int_{a_j}\omega^{(3)} = 0, \quad j = 1, \dots, g. \tag{A.23}$$

A normal differential of the third kind, denoted $\omega^{(3)}_{Q_1,Q_2}$, associated with two distinct points $Q_1, Q_2 \in \widehat{\mathcal{K}}_g$ by definition has simple poles at Q_ℓ with residues $(-1)^{\ell+1}$, $\ell = 1, 2$, and vanishing a-periods.

Theorem A.7 *Suppose $\omega^{(3)}$ to be a differential of the third kind on $\mathcal{K}_g$ whose only singularities are simple poles at $Q_n \in \widehat{\mathcal{K}}_g$ with residues c_n, $n = 1, \dots, N$. Denote* *by $\omega^{(1)}$ a differential of the first kind on $\mathcal{K}_g$. Then*

$$\frac{1}{2\pi i}\sum_{j=1}^{g}\bigg(\Big(\int_{a_j}\omega^{(1)}\Big)\Big(\int_{b_j}\omega^{(3)}\Big)-\Big(\int_{b_j}\omega^{(1)}\Big)\Big(\int_{a_j}\omega^{(3)}\Big)\bigg)$$
$$=\sum_{n=1}^{N}c_n\int_{Q_0}^{Q_n}\omega^{(1)}, \tag{A.24}$$

where $Q_0\in\widehat{\mathcal{K}}_g$ is any fixed base point. In particular, if $\omega^{(3)}$ is normalized and $\omega^{(1)}=\omega_j$, then

$$\frac{1}{2\pi i}\int_{b_j}\omega^{(3)}=\sum_{n=1}^{N}c_n\int_{Q_0}^{Q_n}\omega_j,\quad j=1,\dots,g. \tag{A.25}$$

Moreover, if $\omega^{(3)}_{Q_1,Q_2}$ is a normal differential of the third kind on $\mathcal{K}_g$ holomorphic on $\mathcal{K}_g\setminus\{Q_1,Q_2\}$, then

$$\frac{1}{2\pi i}\int_{b_j}\omega^{(3)}_{Q_1,Q_2}=\int_{Q_2}^{Q_1}\omega_j,\quad j=1,\dots,g. \tag{A.26}$$

We always assume (without loss of generality) that all poles of differentials of the second and third kind on $\mathcal{K}_g$ lie on $\widehat{\mathcal{K}}_g$ (i.e., not on $\partial\widehat{\mathcal{K}}_g$). This can always be achieved by an appropriate choice of the cycles a_j and b_j. Moreover, we assume that all integration paths on the right-hand sides of (A.24)–(A.26) stay away from the cycles a_j and b_k.

Next, we turn to divisors on $\mathcal{K}_g$ and the Jacobi variety $J(\mathcal{K}_g)$ of $\mathcal{K}_g$. Let $\mathcal{M}(\mathcal{K}_g)$ and $\mathcal{M}^1(\mathcal{K}_g)$ denote the set of meromorphic functions (0-forms) and meromorphic 1-forms on $\mathcal{K}_g$, respectively, for some $g\in\mathbb{N}_0$.

Definition A.8 Let $g\in\mathbb{N}_0$. Suppose $f\in\mathcal{M}(\mathcal{K}_g)$, $\omega=h(\zeta_{Q_0})d\zeta_{Q_0}\in\mathcal{M}^1(\mathcal{K}_g)$, and (U_{Q_0},ζ_{Q_0}) is a chart near some point $Q_0\in\mathcal{K}_g$.

(i) If $(f\circ\zeta_{Q_0}^{-1})(\zeta)=\sum_{n=m_0}^{\infty}c_n(Q_0)\zeta^n$ for some $m_0\in\mathbb{Z}$ (which turns out to be independent of the chosen chart), the order $\nu_f(Q_0)$ of f at Q_0 is defined by

$$\nu_f(Q_0)=m_0.$$

One defines $\nu_f(P)=\infty$ for all $P\in\mathcal{K}_g$ if f is identically zero on $\mathcal{K}_g$.

(ii) If $h_{Q_0}(\zeta_{Q_0})=\sum_{n=m_0}^{\infty}d_n(Q_0)\zeta_{Q_0}^n$ for some $m_0\in\mathbb{Z}$ (which again is independent of the chart chosen), the order $\nu_\omega(Q_0)$ of ω at Q_0 is defined by

$$\nu_\omega(Q_0)=m_0.$$

Next, we turn to divisors and introduce some structure on the set of all divisors.

Definition A.9 Let $g \in \mathbb{N}_0$.

(i) A divisor $\mathcal{D}$ on $\mathcal{K}_g$ is a map $\mathcal{D}\colon \mathcal{K}_g \to \mathbb{Z}$, where $\mathcal{D}(P) \neq 0$ for only finitely many $P \in \mathcal{K}_g$. On the set of all divisors $\mathrm{Div}(\mathcal{K}_g)$ on $\mathcal{K}_g$ one introduces the partial ordering

$$\mathcal{D} \geq \mathcal{E} \text{ if } \mathcal{D}(P) \geq \mathcal{E}(P), \quad P \in \mathcal{K}_g.$$

(ii) The degree $\deg(\mathcal{D})$ of $\mathcal{D} \in \mathrm{Div}(\mathcal{K}_g)$ is defined by

$$\deg(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P).$$

(iii) $\mathcal{D} \in \mathrm{Div}(\mathcal{K}_g)$ is called nonnegative (or effective) if

$$\mathcal{D} \geq 0,$$

where 0 denotes the zero divisor $0(P) = 0$ for all $P \in \mathcal{K}_g$.

(iv) Let $\mathcal{D}, \mathcal{E} \in \mathrm{Div}(\mathcal{K}_g)$. Then $\mathcal{D}$ is called a multiple of $\mathcal{E}$ if

$$\mathcal{D} \geq \mathcal{E}.$$

The divisors $\mathcal{D}$ and $\mathcal{E}$ are called relatively prime if

$$\mathcal{D}(P)\mathcal{E}(P) = 0, \quad P \in \mathcal{K}_g.$$

(v) If $f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}$ and $\omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}$, then the divisor (f) of f is defined by

$$(f)\colon \mathcal{K}_g \to \mathbb{Z}, \quad P \mapsto \nu_f(P)$$

(thus, f is holomorphic if and only if $(f) \geq 0$) and the divisor of ω is defined by

$$(\omega)\colon \mathcal{K}_g \to \mathbb{Z}, \quad P \mapsto \nu_\omega(P)$$

(thus, ω is a differential of the first kind if and only if $(\omega) \geq 0$). The divisor (f) is called a principal divisor, and (ω) a canonical divisor.

(vi) The divisors $\mathcal{D}, \mathcal{E} \in \mathrm{Div}(\mathcal{K}_g)$ are called equivalent, written $\mathcal{D} \sim \mathcal{E}$, if

$$\mathcal{D} - \mathcal{E} = (f)$$

for some $f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}$. The divisor class $[\mathcal{D}]$ of $\mathcal{D}$ is defined by

$$[\mathcal{D}] = \{\mathcal{E} \in \mathrm{Div}(\mathcal{K}_g) \mid \mathcal{E} \sim \mathcal{D}\}. \tag{A.27}$$

Lemma A.10 *Let $g \in \mathbb{N}_0$. Suppose $f \in \mathcal{M}(\mathcal{K}_g)$ and $\omega \in \mathcal{M}^1(\mathcal{K}_g)$. Then*

$$\deg((f)) = 0$$

and

$$\deg((\omega)) = 2(g-1).$$

Clearly, $\mathrm{Div}(\mathcal{K}_g)$ forms an abelian group with respect to addition of divisors. The principal divisors form a subgroup $\mathrm{Div}_P(\mathcal{K}_g)$ of $\mathrm{Div}(\mathcal{K}_g)$. The quotient group $\mathrm{Div}(\mathcal{K}_g)/\mathrm{Div}_P(\mathcal{K}_g)$ consists of the cosets of divisors, the divisor classes defined in (A.27). Also, the set of divisors of degree zero, $\mathrm{Div}_0(\mathcal{K}_g)$, forms a subgroup of $\mathrm{Div}(\mathcal{K}_g)$. Since $\mathrm{Div}_P(\mathcal{K}_g) \subset \mathrm{Div}_0(\mathcal{K}_g)$, one can introduce the quotient group $\mathrm{Pic}(\mathcal{K}_g) = \mathrm{Div}_0(\mathcal{K}_g)/\mathrm{Div}_P(\mathcal{K}_g)$, which is called the Picard group of $\mathcal{K}_g$.

Definition A.11 Let $g \in \mathbb{N}_0$, and define

$$\begin{aligned} \mathcal{L}(\mathcal{D}) &= \{f \in \mathcal{M}(\mathcal{K}_g) \mid (f) \geq \mathcal{D}\}, \\ \mathcal{L}^1(\mathcal{D}) &= \{\omega \in \mathcal{M}^1(\mathcal{K}_g) \mid (\omega) \geq \mathcal{D}\}. \end{aligned} \tag{A.28}$$

Both $\mathcal{L}(\mathcal{D})$ and $\mathcal{L}^1(\mathcal{D})$ are linear spaces over $\mathbb{C}$. We denote their (complex) dimensions by

$$r(\mathcal{D}) = \dim \mathcal{L}(\mathcal{D}), \tag{A.29}$$

$$i(\mathcal{D}) = \dim \mathcal{L}^1(\mathcal{D}). \tag{A.30}$$

$i(\mathcal{D})$ is also called the index of specialty of $\mathcal{D}$ and $\mathcal{D}$ is called special (respectively nonspecial) if $i(\mathcal{D}) \geq 1$ (respectively $i(\mathcal{D}) = 0$).

Lemma A.12 *Let $g \in \mathbb{N}_0$ and $\mathcal{D} \in \mathrm{Div}(\mathcal{K}_g)$. Then $\deg(\mathcal{D})$, $r(\mathcal{D})$, and $i(\mathcal{D})$ only depend on the divisor class $[\mathcal{D}]$ of $\mathcal{D}$ (and not on the particular representative $\mathcal{D}$). Moreover, for $\omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}$ one infers*

$$i(\mathcal{D}) = r(\mathcal{D} - (\omega)), \quad \mathcal{D} \in \mathrm{Div}(\mathcal{K}_g).$$

Theorem A.13 (Riemann–Roch) *Let $g \in \mathbb{N}_0$ and $\mathcal{D} \in \mathrm{Div}(\mathcal{K}_g)$. Then $r(-\mathcal{D})$ and $i(\mathcal{D})$ are finite and*

$$r(-\mathcal{D}) = \deg(\mathcal{D}) + i(\mathcal{D}) - g + 1. \tag{A.31}$$

In particular, Riemann's inequality

$$r(-\mathcal{D}) \geq \deg(\mathcal{D}) - g + 1$$

holds.

Next we turn to Jacobi varieties and the Abel map.

Definition A.14 Define the period lattice L_g in $\mathbb{C}^g$ by

$$L_g = \{\underline{z} \in \mathbb{C}^g \mid \underline{z} = \underline{n} + \underline{m}\tau,\ \underline{n}, \underline{m} \in \mathbb{Z}^g\}. \tag{A.32}$$

Then the Jacobi variety $J(\mathcal{K}_g)$ of $\mathcal{K}_g$ is defined by

$$J(\mathcal{K}_g) = \mathbb{C}^g / L_g, \tag{A.33}$$

and the Abel maps are defined by

$$\begin{aligned} \underline{A}_{Q_0} \colon \mathcal{K}_g \to J(\mathcal{K}_g), \quad P \mapsto \underline{A}_{Q_0}(P) &= (A_{Q_0,1}(P), \dots, A_{Q_0,g}(P)) \\ &= \left(\int_{Q_0}^{P} \omega_1, \dots, \int_{Q_0}^{P} \omega_g \right) \pmod{L_g} \end{aligned} \tag{A.34}$$

and

$$\underline{\alpha}_{Q_0} \colon \operatorname{Div}(\mathcal{K}_g) \to J(\mathcal{K}_g), \quad \mathcal{D} \mapsto \underline{\alpha}_{Q_0}(\mathcal{D}) = \sum_{P \in \mathcal{K}_g} \mathcal{D}(P) \underline{A}_{Q_0}(P), \tag{A.35}$$

where $Q_0 \in \mathcal{K}_g$ is a fixed base point and (for convenience only) the same path is chosen from Q_0 to P for all $j = 1, \dots, g$ in (A.34) and (A.35)[1].

Clearly, $\underline{A}_{Q_0}$ is well-defined since changing the path from Q_0 to P amounts to adding a closed cycle whose contribution in the integral (A.34) consists in adding a vector in L_g. Moreover, $\underline{\alpha}_{Q_0}$ is a group homomorphism and $J(\mathcal{K}_g)$ is a complex torus of (complex) dimension g that depends on the choice of the homology basis $\{a_j, b_j\}_{j=1}^{g}$. However, different homology bases yield isomorphic Jacobians.

Theorem A.15 (Abel's theorem) *A divisor $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$ is principal if and only if*

$$\deg(\mathcal{D}) = 0 \textit{ and } \underline{\alpha}_{Q_0}(\mathcal{D}) = 0. \tag{A.36}$$

We note that the apparent base point dependence of the Abel map on Q_0 in (A.36) disappears if $\deg(\mathcal{D}) = 0$.

Finally, we turn to Riemann theta functions and a constructive approach to the Jacobi inversion problem. We assume $g \in \mathbb{N}$ for the remainder of this appendix.

Given the Riemann surface $\mathcal{K}_g$, the homology basis $\{a_j, b_j\}_{j=1}^{g}$, and the matrix τ of b-periods of the differentials of the first kind, $\{\omega_j\}_{j=1}^{g}$ (cf. (A.14)), the Riemann theta function associated with $\mathcal{K}_g$ and the homology basis is defined as

$$\theta(\underline{z}) = \sum_{\underline{n} \in \mathbb{Z}^g} \exp\big(2\pi i(\underline{n}, \underline{z}) + \pi i(\underline{n}, \underline{n}\tau)\big), \quad \underline{z} \in \mathbb{C}^g, \tag{A.37}$$

[1] This convention allows one to avoid the multiplicative version of the Riemann–Roch theorem at various places in this monograph.

where $(\underline{u}, \underline{v}) = \overline{\underline{u}}\, \underline{v}^\top = \sum_{j=1}^{g} \overline{u}_j v_j$ denotes the scalar product in $\mathbb{C}^g$. Because of (A.16), θ is well-defined and represents an entire function on $\mathbb{C}^g$. Elementary properties of θ are, for instance,

$$\theta(z_1, \dots, z_{j-1}, -z_j, z_{j+1}, \dots, z_n) = \theta(\underline{z}), \quad \underline{z} = (z_1, \dots, z_g) \in \mathbb{C}^g, \tag{A.38}$$

$$\theta(\underline{z} + \underline{m} + \underline{n}\tau) = \theta(\underline{z}) \exp\big(- 2\pi i (\underline{n}, \underline{z}) - \pi i (\underline{n}, \underline{n}\tau)\big), \quad \underline{m}, \underline{n} \in \mathbb{Z}^g,\ \underline{z} \in \mathbb{C}^g. \tag{A.39}$$

Lemma A.16 *Let $\underline{\xi} \in \mathbb{C}^g$ and define*

$$\widehat{F} \colon \widehat{\mathcal{K}}_g \to \mathbb{C}, \quad P \mapsto \theta(\underline{\xi} - \underline{\widehat{A}}_{Q_0}(P)), \tag{A.40}$$

where

$$\begin{aligned} &\underline{\widehat{A}}_{Q_0} \colon \widehat{\mathcal{K}}_g \to \mathbb{C}^g, \\ &\quad P \mapsto \underline{\widehat{A}}_{Q_0}(P) = \big(\widehat{A}_{Q_0,1}(P), \dots, \widehat{A}_{Q_0,g}(P)\big) = \left(\int_{Q_0}^{P} \omega_1, \dots, \int_{Q_0}^{P} \omega_g \right). \end{aligned} \tag{A.41}$$

Suppose $\widehat{F}$ is not identically zero on $\widehat{\mathcal{K}}_g$, that is, $\widehat{F} \not\equiv 0$. Then $\widehat{F}$ has precisely g zeros on $\widehat{\mathcal{K}}_g$ counting multiplicities.

Lemma A.16 can be proved by integrating $d \ln(\widehat{F})$ along $\partial \widehat{\mathcal{K}}_g$.

For subsequent use in Remark A.28 we also introduce

$$\underline{\hat{\alpha}}_{Q_0} \colon \operatorname{Div}(\widehat{\mathcal{K}}_g) \to \mathbb{C}^g, \quad \mathcal{D} \mapsto \underline{\hat{\alpha}}_{Q_0}(\mathcal{D}) = \sum_{P \in \widehat{\mathcal{K}}_g} \mathcal{D}(P) \underline{\widehat{A}}_{Q_0}(P), \tag{A.42}$$

in addition to $\underline{\widehat{A}}_{Q_0}$ in (A.41).

Theorem A.17 *Let $\underline{\xi} \in \mathbb{C}^g$ and define $\widehat{F}$ as in* (A.40)*. Assume that $\widehat{F}$ is not identically zero on $\widehat{\mathcal{K}}_g$, and let $Q_1, \dots, Q_g \in \mathcal{K}_g$ be the zeros of $\widehat{F}$ (multiplicities included) given by Lemma* A.16*. Define the corresponding positive divisor $\mathcal{D}_{\underline{Q}}$ of degree g on $\mathcal{K}_g$ by*

$$\mathcal{D}_{\underline{Q}} \colon \mathcal{K}_g \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_{\underline{Q}}(P) = \begin{cases} m & \text{if } P \text{ occurs } m \text{ times in } \{Q_1, \dots, Q_g\}, \\ 0 & \text{if } P \notin \{Q_1, \dots, Q_g\}, \end{cases}$$
$$\underline{Q} = \{Q_1, \dots, Q_g\} \in \operatorname{Sym}^g(\mathcal{K}_g), \tag{A.43}$$

and recall the Abel map $\underline{\alpha}_{Q_0}$ in (A.35)*. Then there exists a vector $\underline{\Xi}_{Q_0} \in \mathbb{C}^g$, the vector of Riemann constants, such that*

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}) = (\underline{\xi} - \underline{\Xi}_{Q_0}) \pmod{L_g}. \tag{A.44}$$

The vector $\underline{\Xi}_{Q_0} = (\Xi_{Q_{0,1}}, \dots, \Xi_{Q_{0,g}})$ *is given by*

$$\Xi_{Q_{0,j}} = (1/2)(1 + \tau_{j,j}) - \sum_{\substack{\ell=1 \\ \ell \neq j}}^{g} \int_{a_\ell} \omega_\ell(P) \int_{Q_0}^{P} \omega_j, \quad j = 1, \dots, g. \quad \text{(A.45)}$$

For the proof of Theorem A.17 one integrates $\widehat{A}_{P_{0,j}}(P) d \ln(\widehat{F}(P))$ along $\partial \widehat{\mathcal{K}}_g$.

Clearly, $\underline{\Xi}_{Q_0}$ depends on the base point Q_0 and on the choice of the homology basis $\{a_j, b_j\}_{j=1}^{g}$. In the special hyperelliptic case in which $\mathcal{K}_g$ is derived from $\mathcal{F}(z, y) = y^2 - \prod_{m=0}^{2g}(z - E_m) = 0$ and $\{E_m\}_{m=0,\dots,2g}$ are $2g + 1$ distinct points in $\mathbb{C}$, equation (A.45) simplifies to

$$\underline{\Xi}_{P_{\infty,j}} = \frac{1}{2}\left(j + \sum_{k=1}^{g} \tau_{j,k}\right), \quad j = 1, \dots, g,$$

where the base point Q_0 has been chosen to be the unique point P_∞ of $\mathcal{K}_g$ at infinity.

Remark A.18 Theorem A.17 yields a partial solution of Jacobi's inversion problem which can be stated as follows: Given $\underline{\xi} \in \mathbb{C}^g$, find a divisor $\mathcal{D}_{\underline{Q}} \in \operatorname{Div}(\mathcal{K}_g)$ such that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}) = \underline{\xi} \pmod{L_g}.$$

Indeed, if $\widetilde{F}(\,\cdot\,) = \theta(\underline{\Xi}_{Q_0} - \underline{\widehat{A}}_{Q_0}(\,\cdot\,) + \underline{\xi}) \not\equiv 0$ on $\widehat{\mathcal{K}}_g$, the zeros $Q_1, \dots, Q_g \in \widehat{\mathcal{K}}_g$ of $\widetilde{F}$ (guaranteed by Lemma A.16) satisfy Jacobi's inversion problem by (A.44). Thus, it remains to specify conditions such that $\widetilde{F} \not\equiv 0$ on $\widehat{\mathcal{K}}_g$.

Theorem A.19 *Let* $\mathcal{D} \in \operatorname{Div}(\mathcal{K}_g)$ *be of degree* $2(g-1)$, $g \in \mathbb{N}$. *Then* $\mathcal{D}$ *is a canonical divisor (i.e., the divisor of a meromorphic differential on* $\mathcal{K}_g$*) if and only if*

$$\underline{\alpha}_{Q_0}(\mathcal{D}) = -2\underline{\Xi}_{Q_0}.$$

Remark A.20 Although θ is well-defined (in fact, entire) on $\mathbb{C}^g$, it is not well-defined on $J(\mathcal{K}_g) = \mathbb{C}^g / L_g$ because of (A.39). Nevertheless, θ is a "multiplicative function" on $J(\mathcal{K}_g)$ since the multipliers in (A.39) cannot vanish. In particular, if $\underline{z}_1 = \underline{z}_2 \pmod{L_g}$, then $\theta(\underline{z}_1) = 0$ if and only if $\theta(\underline{z}_2) = 0$. Hence, it is meaningful to state that θ vanishes at points of $J(\mathcal{K}_g)$. Since the Abel map $\underline{A}_{Q_0}$ maps $\mathcal{K}_g$ into $J(\mathcal{K}_g)$, the function $\theta(\underline{\xi} - \underline{A}_{Q_0}(P))$ for $\underline{\xi} \in \mathbb{C}^g$ becomes a multiplicative function on $\mathcal{K}_g$. Again it makes sense to say that $\theta(\underline{\xi} - \underline{A}_{Q_0}(\,\cdot\,))$ vanishes at points of $\mathcal{K}_g$. In particular, Lemma A.16 and Theorem A.17 extend to the case where $\widehat{F}$ is replaced by $F\colon \mathcal{K}_g \to \mathbb{C}$, $P \mapsto \theta(\underline{\xi} - \underline{A}_{Q_0}(P))$.

In the following we use the obvious notation

$$
\begin{aligned}
X + Y &= \{(\underline{x} + \underline{y}) \in J(\mathcal{K}_g) \mid \underline{x} \in X, \underline{y} \in Y\}, \\
-X &= \{-\underline{x} \in J(\mathcal{K}_g) \mid \underline{x} \in X\}, \\
X + \underline{z} &= \{(\underline{x} + \underline{z}) \in J(\mathcal{K}_g) \mid \underline{x} \in X\},
\end{aligned} \tag{A.46}
$$

for $X, Y \subset J(\mathcal{K}_g)$ and $\underline{z} \in J(\mathcal{K}_g)$. Furthermore, we identify the mth symmetric power of $\mathcal{K}_g$, denoted $\mathrm{Sym}^m(\mathcal{K}_g)$, with the set of nonnegative divisors of degree $m \in \mathbb{N}$ on $\mathcal{K}_g$. For notational convenience, the restriction of the Abel map to $\mathrm{Sym}^m(\mathcal{K}_g)$ (i.e., its restriction to the set of nonnegative divisors of degree m) will be denoted by the same symbol $\underline{\alpha}_{Q_0}$. Moreover, we introduce the convenient notation

$$
\begin{aligned}
&\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}, \quad \mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_m}, \\
&\underline{Q} = \{Q_1, \ldots, Q_m\} \in \mathrm{Sym}^m(\mathcal{K}_g), \quad Q_0 \in \mathcal{K}_g, \ m \in \mathbb{N},
\end{aligned} \tag{A.47}
$$

where for any $Q \in \mathcal{K}_g$,

$$
\mathcal{D}_Q \colon \mathcal{K}_g \to \mathbb{N}_0, \quad P \mapsto \mathcal{D}_Q(P) = \begin{cases} 1 & \text{for } P = Q, \\ 0 & \text{for } P \in \mathcal{K}_g \setminus \{Q\}. \end{cases} \tag{A.48}
$$

Definition A.21

(i) Define

$$
\underline{W}_0 = \{0\} \subset J(\mathcal{K}_g), \quad \underline{W}_{n,Q_0} = \underline{\alpha}_{Q_0}(\mathrm{Sym}^n(\mathcal{K}_g)), \quad n \in \mathbb{N}.
$$

(ii) $Q \in \mathcal{K}_g$ is called a Weierstrass point of $\mathcal{K}_g$ if $i(g\mathcal{D}_Q) \geq 1$, where $g\mathcal{D}_Q = \sum_{j=1}^{g} \mathcal{D}_Q$.

Remark A.22

(i) Since $i(\mathcal{D}_P) = 0$ for all $P \in \mathcal{K}_1$, $\mathcal{K}_1$ has no Weierstrass points.
(ii) If $g \geq 2$ and $\mathcal{K}_g$ is hyperelliptic, $\mathcal{K}_g$ has precisely $2g + 2$ Weierstrass points. In particular, if $\mathcal{K}_g$ is given as a double cover of the Riemann sphere $\mathbb{C}_\infty$, then the Weierstrass points of $\mathcal{K}_g$ are precisely its $2g + 2$ branch points.
(iii) If $g \geq 3$ and $\mathcal{K}_g$ is not hyperelliptic, then the number N of Weierstrass points of $\mathcal{K}_g$ satisfies the inequality

$$
2g + 2 \leq N \leq g^3 - g.
$$

(iv) Special divisors $\mathcal{D}_{\underline{Q}}$ with $\deg(\underline{Q}) = N \geq g$ and $\underline{Q} = \{Q_1, \ldots, Q_N\} \in \mathrm{Sym}^N(\mathcal{K}_g)$ are the critical points of the Abel map $\underline{\alpha}_{Q_0} \colon \mathrm{Sym}^N(\mathcal{K}_g) \to J(\mathcal{K}_g)$, that is, the set of points $\mathcal{D}$ at which the rank of the differential $d\underline{\alpha}_{Q_0}$ is less than g.

(v) Although $\operatorname{Sym}^m(\mathcal{K}_g) \not\subset \operatorname{Sym}^n(\mathcal{K}_g)$ for $m < n$, one has $\underline{W}_{m,Q_0} \subseteq \underline{W}_{n,Q_0}$ for $m < n$. Thus, $\underline{W}_{n,Q_0} = J(\mathcal{K}_g)$ for $n \geq g$ by Theorem A.25 below.

Theorem A.23 *The set* $\Theta = \underline{W}_{g-1,Q_0} + \underline{\Xi}_{Q_0} \subset J(\mathcal{K}_g)$*, the so-called theta divisor, is the complete set of zeros of* θ *on* $J(\mathcal{K}_g)$*, that is,*

$$\theta(X) = 0 \text{ if and only if } X \in \Theta \tag{A.49}$$

(i.e., if and only if $X = \big(\underline{\alpha}_{Q_0}(\mathcal{D}) + \underline{\Xi}_{Q_0}\big) \pmod{L_g}$ *for some* $\mathcal{D} \in \operatorname{Sym}^{g-1}(\mathcal{K}_g)$*). The theta divisor* Θ *has complex dimension* $g-1$ *and is independent of the base point* Q_0*.*

Theorem A.24 (Riemann's vanishing theorem) *Let* $\underline{\xi} \in \mathbb{C}^g$*.*

(i) If $\theta(\underline{\xi}) \neq 0$*, then there exists a unique* $\mathcal{D} \in \operatorname{Sym}^g(\mathcal{K}_g)$ *such that*

$$\underline{\xi} = \big(\underline{\alpha}_{Q_0}(\mathcal{D}) + \underline{\Xi}_{Q_0}\big) \pmod{L_g} \tag{A.50}$$

and

$$i(\mathcal{D}) = 0.$$

(ii) If $\theta(\underline{\xi}) = 0$ *and* $g = 1$*, then*

$$\underline{\xi} = \underline{\Xi} \pmod{L_1} = (1+\tau)/2 \pmod{L_1}, \quad L_1 = \mathbb{Z} + \tau\mathbb{Z}, \ -i\tau > 0.$$

(iii) Assume $\theta(\underline{\xi}) = 0$ *and* $g \geq 2$*. Let* $s \in \mathbb{N}$ *with* $s \leq g-1$ *be the smallest integer such that* $\theta(\underline{W}_{s,Q_0} - \underline{W}_{s,Q_0} - \underline{\xi}) \neq 0$ *(i.e., there exist* $\mathcal{E}, \mathcal{F} \in \operatorname{Sym}^s(\mathcal{K}_g)$ *with* $\mathcal{E} \neq \mathcal{F}$ *such that* $\theta(\underline{\alpha}_{Q_0}(\mathcal{E}) - \underline{\alpha}_{Q_0}(\mathcal{F}) - \underline{\xi}) \neq 0$*). Then there exists a* $\mathcal{D} \in \operatorname{Sym}^{g-1}(\mathcal{K}_g)$ *such that*

$$\underline{\xi} = \big(\underline{\alpha}_{Q_0}(\mathcal{D}) + \underline{\Xi}_{Q_0}\big) \pmod{L_g} \tag{A.51}$$

and

$$i(\mathcal{D}) = s.$$

All partial derivatives of θ *with respect to* $A_{Q_0,j}$ *for* $j = 1, \dots, g$ *of order strictly less than* s *vanish at* $\underline{\xi}$*, whereas at least one partial derivative of* θ *of order* s *is nonzero at* $\underline{\xi}$*. Moreover,* $s \leq (g+1)/2$*, and the integer* s *is the same for* $\underline{\xi}$ *and* $-\underline{\xi}$*.*

Note that there is no explicit reference to the base point Q_0 in the formulation of Theorem A.24 since the set $\underline{W}_{s,Q_0} - \underline{W}_{s,Q_0} \subset J(\mathcal{K}_g)$ (cf. (A.46)) is independent of the base point Q_0, whereas $\underline{W}_{s,Q_0}$ alone is obviously not.

Theorem A.25 (Jacobi's inversion theorem) *The Abel map restricted to the set of nonnegative divisors,* $\underline{\alpha}_{Q_0}\colon \operatorname{Sym}^g(\mathcal{K}_g) \to J(\mathcal{K}_g)$ *is surjective. More precisely, given* $\underline{\tilde{\xi}} = (\underline{\xi} + \underline{\Xi}_{Q_0}) \in \mathbb{C}^g$, *the divisors* $\mathcal{D}$ *in* (A.50) *and* (A.51) *(respectively* $\mathcal{D} = \mathcal{D}_{Q_0}$ *if* $g = 1$*) solve the Jacobi inversion problem for* $\underline{\xi} \in \mathbb{C}^g$.

A special case of this analysis can be summarized as follows. Consider the function

$$G(\,\cdot\,) = \theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(\,\cdot\,) + \sum_{j=1}^{g} \underline{A}_{Q_0}(Q_j)\big), \quad Q_j \in \mathcal{K}_g, \quad j = 1, \dots, g$$

on $\mathcal{K}_g$. Then

$$\begin{aligned} G(Q_k) &= \theta(\underline{\Xi}_{Q_0} + \sum_{\substack{j=1\\ j\neq k}}^{g} \underline{A}_{Q_0}(Q_j)) \\ &= \theta\big(\underline{\Xi}_{Q_0} + \underline{\alpha}_{Q_0}(\mathcal{D}_{(Q_1,\dots,Q_{k-1},Q_{k+1},\dots,Q_g)})\big) = 0, \quad k = 1, \dots, g \end{aligned}$$

by Theorem A.23. Moreover, by Lemma A.16, Remark A.20, and Theorem A.24, the points $Q_1, \dots, Q_g$ are the only zeros of G on $\mathcal{K}_g$ if and only if $\mathcal{D}_{\underline{Q}}$ is nonspecial, that is, if and only if

$$i(\mathcal{D}_{\underline{Q}}) = 0, \quad \underline{Q} = \{Q_1, \dots, Q_g\} \in \operatorname{Sym}^g(\mathcal{K}_g).$$

Conversely, $G \equiv 0$ on $\mathcal{K}_g$ if and only if $\mathcal{D}_{\underline{Q}}$ is special, that is, if and only if $i(\mathcal{D}_{\underline{Q}}) \geq 1$. Thus, one obtains the following fact.

Theorem A.26 *Let* $\underline{Q} = \{Q_1, \dots, Q_g\} \in \operatorname{Sym}^g(\mathcal{K}_g)$ *and assume* $\mathcal{D}_{\underline{Q}}$ *to be nonspecial, that is,* $i(\mathcal{D}_{\underline{Q}}) = 0$. *Then*

$$\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}})\big) = 0 \text{ if and only if } P \in \{Q_1, \dots, Q_g\}.$$

We also mention the elementary change in the Abel map and in Riemann's vector if one changes the base point,

$$\underline{A}_{Q_1} = \big(\underline{A}_{Q_0} - \underline{A}_{Q_0}(Q_1)\big) \pmod{L_g}, \tag{A.52}$$

$$\underline{\Xi}_{Q_1} = \big(\underline{\Xi}_{Q_0} + (g-1)\underline{A}_{Q_0}(Q_1)\big) \pmod{L_g}, \quad Q_0, Q_1 \in \mathcal{K}_g. \tag{A.53}$$

Remark A.27 The L_g quasi-periodic holomorphic function θ on $\mathbb{C}^g$ can be used to construct L_g periodic, meromorphic functions f on $\mathbb{C}^g$ as follows. Either

(i)

$$f(\underline{z})=\prod_{j=1}^{N}\frac{\theta(\underline{z}+\underline{c}_j)}{\theta(\underline{z}+\underline{d}_j)},\quad \underline{z},\underline{c}_j,\underline{d}_j\in\mathbb{C}^g,\quad j=1,\dots,N,$$

where

$$\sum_{j=1}^{N}\underline{c}_j=\sum_{j=1}^{N}\underline{d}_j \pmod{\mathbb{Z}^g},$$

or

(ii)

$$f(\underline{z})=\partial_{z_j}\ln\left(\frac{\theta(\underline{z}+\underline{e})}{\theta(\underline{z}+\underline{h})}\right),\quad j=1,\dots,g,\quad \underline{z},\underline{e},\underline{h}\in\mathbb{C}^g,$$

or

(iii)

$$f(\underline{z})=\partial^2_{z_jz_k}\ln\theta(\underline{z}),\quad \underline{z}\in\mathbb{C}^g,\quad j,k=1,\dots,g.$$

Then, indeed, in all cases (i)–(iii),

$$f(\underline{z}+\underline{m}+\underline{n}\tau)=f(\underline{z}),\quad \underline{z}\in\mathbb{C}^g,\quad \underline{m},\underline{n}\in\mathbb{Z}^g$$

holds by (A.39).

Remark A.28 In the main text we frequently deal with theta function expressions of the type

$$\phi(P)=\frac{\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_2))}\exp\left(\int_{Q_0}^{P}\omega^{(3)}_{Q_1,Q_2}\right),\quad P\in\mathcal{K}_g \tag{A.54}$$

and

$$\psi(P)=\frac{\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_1))}{\theta(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P)+\underline{\alpha}_{Q_0}(\mathcal{D}_2))}\exp\left(-c\int_{Q_0}^{P}\Omega^{(2)}\right),\quad P\in\mathcal{K}_g, \tag{A.55}$$

where $\mathcal{D}_j\in\operatorname{Sym}^g(\mathcal{K}_g)$, $j=1,2$, are nonspecial positive divisors of degree g, $c\in\mathbb{C}$ is a constant, $Q_j\in\mathcal{K}_g\setminus\{P_{\infty_1},\dots,P_{\infty_N}\}$, where $\{P_{\infty_1},\dots,P_{\infty_N}\}$, $N\in\mathbb{N}$, denotes the set of points of $\mathcal{K}_g$ at infinity, $\omega^{(3)}_{Q_1,Q_2}$ is a normal differential of the third kind, and $\Omega^{(2)}$ a normalized differential of the second kind with singularities contained in $\{P_{\infty_1},\dots,P_{\infty_N}\}$. In particular, one has

$$\int_{a_j}\omega^{(3)}_{Q_1,Q_2}=\int_{a_j}\Omega^{(2)}=0,\quad j=1,\dots,g. \tag{A.56}$$

Even though we agree always to choose identical paths of integration from P_0 to P in all Abelian integrals (A.54) and (A.55), this is not sufficient to render ϕ and

ψ single-valued on $\mathcal{K}_g$. To achieve single-valuedness, one needs to replace $\mathcal{K}_g$ by its simply connected canonical dissection $\widehat{\mathcal{K}}_g$ and then replace $\underline{A}_{Q_0}, \underline{\alpha}_{Q_0}$ in (A.54) and (A.55) with $\underline{\widehat{A}}_{Q_0}, \underline{\hat{\alpha}}_{Q_0}$, as introduced in (A.41) and (A.42). In particular, one regards a_j, b_j as curves (being a part of $\partial\widehat{\mathcal{K}}_g$, cf. (A.12)) and not as homology classes $[a_j], [b_j]$ in $H_1(\mathcal{K}_g, \mathbb{Z})$. Similarly, one then replaces $\underline{\Xi}_{Q_0}$ by $\underline{\widehat{\Xi}}_{Q_0}$, etc. Moreover, to render ϕ single-valued on $\widehat{\mathcal{K}}_g$, one needs to assume in addition that

$$\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_1) - \underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_2) = 0 \tag{A.57}$$

(as opposed to merely $\underline{\alpha}_{Q_0}(\mathcal{D}_1) - \underline{\alpha}_{Q_0}(\mathcal{D}_2) = 0 \pmod{L_g}$). Similarly, in connection with ψ, one introduces the vector of b-periods $\underline{U}^{(2)}$ of $\Omega^{(2)}$ by

$$\underline{U}^{(2)} = (U_1^{(2)}, \dots, U_g^{(2)}), \quad U_j^{(2)} = \frac{1}{2\pi i} \int_{b_j} \Omega^{(2)}, \quad j = 1, \dots, g, \tag{A.58}$$

and then renders ψ single-valued on $\widehat{\mathcal{K}}_g$ by requiring

$$\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_1) - \underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_2) = c\, \underline{U}^{(2)} \tag{A.59}$$

(as opposed to merely $\underline{\alpha}_{Q_0}(\mathcal{D}_1) - \underline{\alpha}_{Q_0}(\mathcal{D}_2) = c\, \underline{U}^{(2)} \pmod{L_g}$). These statements easily follow from (A.26) and (A.39) in the case of ϕ and simply from (A.39) in the case of ψ. In fact, by (A.39),

$$\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_1 + \mathcal{D}_{Q_1}) - \underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_2 + \mathcal{D}_{Q_2}) \in \mathbb{Z}^g, \tag{A.60}$$

respectively,

$$\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_1) - \underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_2) - c\, \underline{U}^{(2)} \in \mathbb{Z}^g, \tag{A.61}$$

suffice to guarantee single-valuedness of ϕ, respectively ψ, on $\widehat{\mathcal{K}}_g$. Without the replacement of $\underline{A}_{Q_0}$ and $\underline{\alpha}_{Q_0}$ by $\underline{\widehat{A}}_{Q_0}$ and $\underline{\hat{\alpha}}_{Q_0}$ in (A.54) and (A.55) and the assumptions (A.57) and (A.59) (or (A.60) and (A.61)), ϕ and ψ are multiplicative (multi-valued) functions on $\mathcal{K}_g$ and are then most effectively discussed by introducing the notion of characters on $\mathcal{K}_g$. For simplicity, we decided to avoid the latter possibility and throughout this text will always tacitly assume (A.57) and (A.59) without particularly emphasizing this convention each time it is used.

Remark A.29 Let $\underline{\xi} \in J(\mathcal{K}_g)$ be given, assume that $\theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(\,\cdot\,) + \underline{\xi}) \not\equiv 0$ on $\mathcal{K}_g$, and suppose that $\underline{\alpha}_{Q_0}^{-1}(\underline{\xi}) = \{Q_1, \dots, Q_g\} \in \mathrm{Sym}^g(\mathcal{K}_g)$ is the unique solution of Jacobi's inversion problem. Let $f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}$, and suppose $f(Q_j) \neq \infty$ for $j = 1, \dots, g$. Then $\underline{\xi}$ uniquely determines the values $f(Q_1), \dots, f(Q_g)$. Moreover, any symmetric function of these values is a single-valued meromorphic function of $\underline{\xi} \in J(\mathcal{K}_g)$, that is, an Abelian function on $J(\mathcal{K}_g)$. Any such meromorphic function on $J(\mathcal{K}_g)$ can be expressed in terms of the Riemann theta function on $\mathcal{K}_g$. For instance, for the elementary symmetric functions of the second kind (Newton polynomials) one obtains from the residue theorem in analogy to the proof of

Lemma A.16 that

$$\sum_{j=1}^{g} f(Q_j)^n = \sum_{j=1}^{g} \int_{a_j} f(P)^n \omega_j(P) \\ - \sum_{\substack{P_r \in \mathcal{K}_g \\ f(P_r)=\infty}} \operatorname*{res}_{P=P_r} \Big(f(P)^n d \ln \big(\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}})\big)\big)\Big), \\ \underline{Q} = \{Q_1, \dots, Q_n\} \in \operatorname{Sym}^n(\mathcal{K}_g). \qquad \text{(A.62)}$$

Here an appropriate homology basis $\{a_j, b_j\}_{j=1}^{g}$, avoiding $\{Q_1, \dots, Q_g\}$ and the poles $\{P_r\}$ of f, has been chosen.

In the special case of hyperelliptic Riemann surfaces, special divisors are characterized as follows. Denote by $*\colon \mathcal{K}_g \to \mathcal{K}_g$ the sheet exchange map (involution)

$$*\colon \mathcal{K}_g \to \mathcal{K}_g, \quad [x_2 : x_1 : x_0] \mapsto [x_2 : -x_1 : x_0].$$

Theorem A.30 *Suppose $\mathcal{K}_g$ is a hyperelliptic Riemann surface of genus $g \in \mathbb{N}$ and $\mathcal{D}_{\underline{Q}} \in \operatorname{Sym}^g(\mathcal{K}_g)$ with $\underline{Q} = \{Q_1, \dots, Q_g\} \in \operatorname{Sym}^g(\mathcal{K}_g)$. Then*

$$1 \le i(\mathcal{D}_{\underline{Q}}) = s$$

if and only if there are s pairs of the type $\{P, P^*\} \subseteq \{Q_1, \dots, Q_g\}$. (This includes, of course, branch points of $\mathcal{K}_g$ for which $P = P^*$.) Obviously, one has $s \le g/2$.

We add one more result in connection with hyperelliptic Riemann surfaces.

Theorem A.31 *Suppose $\mathcal{K}_g$ is a hyperelliptic Riemann surface of genus $g \in \mathbb{N}$, $\mathcal{D}_{\underline{\hat{\mu}}} \in \operatorname{Sym}^g(\mathcal{K}_g)$ is nonspecial, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_g\}$, and $\hat{\mu}_{g+1} \in \mathcal{K}_g$ with $\hat{\mu}_{g+1}^* \notin \{\hat{\mu}_1, \dots, \hat{\mu}_g\}$. Let $\{\hat{\lambda}_1, \dots, \hat{\lambda}_{g+1}\} \subset \mathcal{K}_g$ with $\mathcal{D}_{\underline{\hat{\lambda}}\hat{\lambda}_{g+1}} \sim \mathcal{D}_{\underline{\hat{\mu}}\hat{\mu}_{g+1}}$ (i.e., $\mathcal{D}_{\underline{\hat{\lambda}}\hat{\lambda}_{g+1}} \in [\mathcal{D}_{\underline{\hat{\mu}}\hat{\mu}_{g+1}}]$). Then any g points $\hat{\nu}_j \in \{\hat{\lambda}_1, \dots, \hat{\lambda}_{g+1}\}$, $j = 1, \dots, g$ define a nonspecial divisor $\mathcal{D}_{\underline{\hat{\nu}}} \in \operatorname{Sym}^g(\mathcal{K}_g)$, $\underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_g\}$.*

Proof Since $i(\mathcal{D}_P) = 0$ for all $P \in \mathcal{K}_1$, there is nothing to be proven in the special case $g = 1$. Hence, we assume $g \ge 2$. Let $Q_0 \in \mathcal{B}(\mathcal{K}_g)$ be a fixed branch point of $\mathcal{K}_g$ and suppose that $\mathcal{D}_{\underline{\hat{\nu}}}$ is special. Then by Theorem A.30 there is a pair $\{\hat{\nu}, \hat{\nu}^*\} \subset \{\hat{\nu}_1, \dots, \hat{\nu}_g\}$ such that

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\underline{\hat{\nu}}}}),$$

where $\underline{\underline{\hat{\nu}}} = \{\hat{\nu}_1, \dots, \hat{\nu}_g\} \setminus \{\hat{\nu}, \hat{\nu}^*\} \in \operatorname{Sym}^{g-2}(\mathcal{K}_g)$. Let $\hat{\nu}_{g+1} \in \{\hat{\lambda}_1, \dots, \hat{\lambda}_{g+1}\} \setminus$

$\{\hat{\nu}_1, \ldots, \hat{\nu}_g\}$ so that $\{\hat{\nu}_1, \ldots, \hat{\nu}_{g+1}\} = \{\hat{\lambda}_1, \ldots, \hat{\lambda}_{g+1}\} \in \mathrm{Sym}^{g+1}(\mathcal{K}_g)$. Then

$$\begin{aligned} \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}\hat{\nu}_{g+1}}) &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}\hat{\nu}_{g+1}}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\lambda}}\hat{\lambda}_{g+1}}) \\ &= \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}\hat{\mu}_{g+1}}) = -\underline{A}_{Q_0}(\hat{\mu}_{g+1}^*) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}), \end{aligned} \tag{A.63}$$

and hence by Theorem A.23 and (A.63),

$$0 = \theta(\underline{\Xi}_{Q_0} + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\nu}}\hat{\nu}_{g+1}})) = \theta(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(\hat{\mu}_{g+1}^*) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}})). \tag{A.64}$$

Since by hypothesis $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial and $\hat{\mu}_{g+1}^* \notin \{\hat{\mu}_1, \ldots, \hat{\mu}_g\}$, (A.64) contradicts Theorem A.26. Thus, $\mathcal{D}_{\underline{\hat{\nu}}}$ is nonspecial. $\square$

We conclude this appendix with a brief summary of Riemann surfaces with symmetries, which is a topic of fundamental importance in characterizing real-valued algebro-geometric solutions of the KdV, sGmkdV, and CH hierarchies, as well as algebro-geometric solutions of the $\mathrm{nS}_\pm$ hierarchy and the massive Thirring system.

Assuming $\mathcal{K}_g$ to be a compact Riemann surface of genus g, let

$$\rho\colon \mathcal{K}_g \to \mathcal{K}_g$$

be an antiholomorphic involution on $\mathcal{K}_g$ (i.e., $\rho^2 = \mathrm{id}\,|_{\mathcal{K}_g}$). Moreover, let

$$\mathcal{R} = \{P \in \mathcal{K}_g \mid \rho(P) = P\}$$

be the set of fixed points of ρ (sometimes called the set of "real" points of $\mathcal{K}_g$) and denote by r the number of nontrivial connected components of $\mathcal{R}$. Topologically, these connected components are circles.

Theorem A.32 *Let ρ be an antiholomorphic involution on $\mathcal{K}_g$. Then either $\mathcal{K}_g \setminus \mathcal{R}$ is connected (and then the quotient space (the Klein surface) $\mathcal{K}_g/\rho$ is nonorientable) or $\mathcal{K}_g \setminus \mathcal{R}$ consists precisely of two connected components (and then $\mathcal{K}_g/\rho$ is orientable). In the latter case, if $\mathcal{R} \neq \emptyset$, $\mathcal{K}_g \cup \mathcal{R}$ is a bordered Riemann surface and $(\mathcal{K}_g, \rho)$ is the complex double of $\mathcal{K}_g \cup \mathcal{R}$.*

Definition A.33 Suppose ρ is an antiholomorphic involution on $\mathcal{K}_g$. Define

$$\varepsilon = \begin{cases} + & \text{if } \mathcal{K}_g \setminus \mathcal{R} \text{ is disconnected,} \\ - & \text{if } \mathcal{K}_g \setminus \mathcal{R} \text{ is connected.} \end{cases}$$

The pair $(\mathcal{K}_g, \rho)$ is called a symmetric Riemann surface; the triple (g, r, ε) denotes the type of $(\mathcal{K}_g, \mathcal{R})$. If $\varepsilon = +$, $(\mathcal{K}_g, \rho)$ is of dividing (separating) type; if $\varepsilon = -$, $(\mathcal{K}_g, \rho)$ is of nondividing (nonseparating) type.

If $r = g + 1$, then $\mathcal{K}_g$ is called an M-curve.

Theorem A.34 *Assume ρ is an antiholomorphic involution on $\mathcal{K}_g$.*

(i) If $\varepsilon = +$, then $1 \le r \le g+1$, $r = g+1 \pmod 2$, $g = r - 1 + 2k$, $0 \le k \le (g+1-r)/2$.
(ii) If $\varepsilon = -$, then $0 \le r \le g$.
(iii) If $r = 0$, then $\varepsilon = -$. If $r = g+1$, then $\varepsilon = +$.

Example A.35

(i) Consider the hyperelliptic Riemann surface $\mathcal{K}_g \colon y^2 = \prod_{m=0}^{2g+1}(z - E_m)$ with E_m,
$m = 0, \dots, 2g+1$, grouped into k real and ℓ complex conjugate pairs, $k + \ell = g+1$. Define the antiholomorphic involution $\rho_+ \colon (z, y) \mapsto (\overline{z}, \overline{y})$ on $\mathcal{K}_g$. If $(\mathcal{K}_g, \rho_+)$ is of dividing type $(g, r, +)$, then either $r = g+1 = k$ and $\ell = 0$ (if $E_m \in \mathbb{R}$, $m = 0, \dots, 2g+1$), or else, $r = 1$ if g is even and $r = 2$ if g is odd if none of the E_m are real (and hence only occur in complex conjugate pairs). In particular, if $\prod_{m=0}^{2g+1}(z - E_m)$ contains $2r > 0$ real roots, then $(\mathcal{K}_g, \rho_+)$ is of type $(g, r, +)$ if and only if $r = g+1$ and of type $(g, r, -)$ if and only if $1 \le r \le g$. If $(\mathcal{K}_g, \rho_+)$ is of nondividing type, then $r = k$ (and of course $1 \le r \le g$).
(ii) The hyperelliptic Riemann surfaces $\mathcal{K}_g \colon y^2 = \pm \prod_{m=0}^{g} |z - \widetilde{E}_m|^2$ are of the type $(g, 0, -)$ with respect to the antiholomorphic involutions $\rho_\pm \colon (z, y) \mapsto (\overline{z}, \pm\overline{y})$, respectively, since $\mathcal{R} = \emptyset$, $r = 0$ in either case.
(iii) Consider the hyperelliptic Riemann surface $\mathcal{K}_g \colon y^2 = \prod_{m=0}^{2g}(z - E_m)$ with $E_0 \in \mathbb{R}$ and E_m, $m = 1, \dots, 2g$, grouped into k real and ℓ complex conjugate pairs, $k + \ell = g$. In addition, define the antiholomorphic involution $\rho_+ \colon (z, y) \mapsto (\overline{z}, \overline{y})$ on $\mathcal{K}_g$. Then $(\mathcal{K}_g, \rho_+)$ is of dividing type $(g, r, +)$ if and only if $r = g+1 = k+1$ and $\ell = 0$ (and hence $E_m \in \mathbb{R}$, $m = 0, \dots, 2g$). If $(\mathcal{K}_g, \rho_+)$ is of nondividing type, then $r = k+1$ (and of course $1 \le r \le g$).

In the following it is convenient to abbreviate

$$\underline{\operatorname{diag}}(M) = (M_{1,1}, \dots, M_{g,g})$$

for a $g \times g$ matrix M with entries in $\mathbb{C}$.

Theorem A.36 *Let $(\mathcal{K}_g, \rho)$ be a symmetric Riemann surface.*

(i) There exists a canonical homology basis $\{a_j, b_j\}_{j=1}^{g}$ on $\mathcal{K}_g$ with intersection matrix (A.8) *and a symmetric $g \times g$ matrix R such that the $2g \times 2g$ matrix S of complex conjugation of the action of ρ on $\mathrm{H}_1(\mathcal{K}_g, \mathbb{Z})$ in this basis is*

given by

$$S = \begin{pmatrix} I_g & R \\ 0 & -I_g \end{pmatrix}, \quad R^\top = R,$$

that is,

$$\begin{aligned} &(\underline{\rho}(\underline{a}), \underline{\rho}(\underline{b})) = (\underline{a}, \underline{b}) \begin{pmatrix} I_g & R \\ 0 & -I_g \end{pmatrix} = (\underline{a}, \underline{a}R - \underline{b}), \\ &\underline{a} = (a_1, \ldots, a_g), \quad \underline{b} = (b_1, \ldots, b_g), \\ &\underline{\rho}(\underline{a}) = (\rho(a_1), \ldots, \rho(a_g)), \quad \underline{\rho}(\underline{b}) = (\rho(b_1), \ldots, \rho(b_g)), \end{aligned}$$

where R is of the following form[1].
If $\mathcal{R} \neq \emptyset$ and $\mathcal{K}_g \setminus \mathcal{R}$ is disconnected,

$$R = \begin{pmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \quad \operatorname{rank}(R) = g + 1 - r \tag{A.65}$$

(in particular, $R = 0$ if $r = g + 1$).
If $\mathcal{R} \neq \emptyset$ and $\mathcal{K}_g \setminus \mathcal{R}$ is connected,

$$R = \begin{pmatrix} I_{g+1-r} & \\ & 0 \end{pmatrix}, \quad \operatorname{rank}(R) = g + 1 - r \tag{A.66}$$

(in particular, $R = 0$ if $r = g + 1$).
If $\mathcal{R} = \emptyset$ and g is even,

$$R = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_1 \end{pmatrix}, \quad \operatorname{rank}(R) = g \tag{A.67}$$

(in particular, $R = 0$ if $g = 0$).
If $\mathcal{R} = \emptyset$ and g is odd,

$$R = \begin{pmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_1 & \\ & & & 0 \end{pmatrix}, \quad \operatorname{rank}(R) = g - 1 \tag{A.68}$$

[1] Blank entries are representing zeros in the matrices (A.65)–(A.68). Moreover, 0 in (A.65) and (A.68) denotes a 1×1 element, whereas 0 in (A.66) represents a $(r-1) \times (r-1)$ block matrix (which is absent for $r = 1$).

(in particular, $R = 0$ if $g = 1$).
Here σ_1 denotes the 2×2 Pauli matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(ii) *Given the basis of cycles $\{a_j, b_j\}_{j=1}^g$ of item (i), introduce the corresponding basis $\{\omega_j\}_{j=1}^g$ of normalized holomorphic differentials satisfying* (A.13) *and define the associated matrix τ of b-periods as in* (A.14) *and the Riemann theta function θ as in* (A.37). *Then, ω_j, $j = 1, \dots, g$, are ρ-real, that is,*

$$\rho^* \omega_j = \overline{\omega_j}, \quad j = 1, \dots, g, \tag{A.69}$$

where $\rho^*\omega$ denotes the pull back[1] *of a meromorphic differential ω by the involution ρ. Moreover,*

$$\overline{\tau} = R - \tau, \quad \mathrm{Re}(\tau) = (1/2)R, \tag{A.70}$$

$$\overline{\theta(\underline{z})} = \theta\big(\underline{\overline{z}} + (1/2)\underline{\mathrm{diag}}(R)\big), \quad \underline{z} \in \mathbb{C}^g, \tag{A.71}$$

$$\overline{\underline{\Xi}}_{Q_0} = \underline{\Xi}_{Q_0} + (1/2)\underline{\mathrm{diag}}(R) + (g-1)\underline{\alpha}_{Q_0}(\rho(Q_0)). \tag{A.72}$$

Finally, assume $\mathcal{R} \neq \emptyset$, $\mathcal{K}_g \setminus \mathcal{R}$ is disconnected (cf. (A.65)*), $\underline{x} \in \mathbb{R}^g$, $\chi_m \in \{0, 1\}$, $m = \ell + 1, \dots, g$, $\ell = \mathrm{rank}(R)$. Then*

$$\begin{aligned} &\theta(i\underline{x} + (1/2)(0, \dots, 0, \chi_{\ell+1}, \dots, \chi_g)) \neq 0 \\ &\textit{if and only if } \chi_m = 0,\ m = \ell + 1, \dots, g. \end{aligned} \tag{A.73}$$

Assuming P_0 to be a Weierstrass point of $\mathcal{K}_g$ (implying $g \geq 2$), $\underline{\Xi}_{P_0}$ is a half-period, that is,

$$\begin{aligned} &\underline{\Xi}_{P_0} = -\underline{\Xi}_{P_0} \pmod{L_g}, \\ &\underline{\Xi}_{P_0} = \underline{\beta} + \underline{\gamma}\tau, \quad \underline{\beta}, \underline{\gamma} \in (1/2)\mathbb{Z}^g. \end{aligned} \tag{A.74}$$

In this case (A.70) implies in addition to (A.72) and (A.74) that

$$\overline{\underline{\Xi}}_{P_0} = -\underline{\Xi}_{P_0} \pmod{\mathbb{Z}^g}. \tag{A.75}$$

In the context of the hyperelliptic Riemann surfaces described as two-sheeted covers of the Riemann sphere $\mathbb{C}_\infty$ in Appendices B and C, equation (A.75) remains valid if P_0 is any of the associated $2g + 2$ branch points, $g \in \mathbb{N}$.

Notes

The bulk of the material of this appendix is standard and taken from textbooks. The one we relied on most was Farkas and Kra (1992). Moreover, we used material

[1] If $\omega = f(\zeta)d\zeta$, then $\rho^*\omega = f(\rho(\zeta))d\rho(\zeta)$ and $\int_\gamma \rho^*\omega = \int_{\rho(\gamma)} \omega$, $\gamma \in \mathrm{H}_1(\mathcal{K}_g, \mathbb{Z})$. In particular, if ω is ρ-real, that is, $\rho^*(\omega) = \overline{\omega}$, then $\overline{\int_\gamma \omega} = \int_{\rho(\gamma)} \omega$, $\gamma \in \mathrm{H}_1(\mathcal{K}_g, \mathbb{Z})$.

from Mumford (1983, Ch. II; 1984, Ch. IIIa) (concerning divisors on hyperelliptic Riemann surfaces constructed originally in Jacobi (1846)) and Behnke and Sommer (1965) (in connection with covering Riemann surfaces). In addition, the following well-known sources make for great collateral reading on various topics relevant to the applications discussed in this monographs: Fay (1973), Forsyth (1965), Griffiths (1989), Griffiths and Harris (1978), Gunning (1972), Hofmann (1888), Kirwan (1992), Markushevich (1992), Miranda (1995), Narasimhan (1992), Rauch and Farkas (1974), Reyssat (1989), Rodin (1988), Schlichenmaier (1989), Shokurov (1994), Siegel (1988b), Springer (1981), and the recently reprinted classical treatise by Baker (1995). Finally, there are various reviews on compact Riemann surfaces and their associated theta functions. A masterpiece in this connection that is still of great relevance is Dubrovin (1981). In addition, we call attention to the following reviews: Bost (1992), Korotkin (1998), Lewittes (1964), Rodin (1987), Smith (1989), and Taimanov (1997).

That different homology bases yield isomorphic Jacobians, as alluded to after Definition A.14, is discussed, for instance, in Farkas and Kra (1992, p. 137) and Gunning (1966, Sec. 8(b)). For a detailed discussion of multiplicative (multi-valued) functions in connection with Remark A.28 (a topic we circumvent in this monograph), refer to Farkas and Kra (1992, Secs. III.9, VI.2). Theorem A.30 can be found in Krazer (1970, Sec. X.3).

Finally, the material on symmetric Riemann surfaces is mainly taken from Gross and Harris (1981) and Vinnikov (1993). In particular, Theorem A.36 is proved in Vinnikov (1993) (compare Alpay and Vinnikov (2002) for the proof of (A.71)). The case is $\mathcal{R} \neq \emptyset$, and $\mathcal{K}_g \setminus \mathcal{R}$ disconnected is treated in Fay (1973, Ch. VI). Classical sources for this material are Comessatti (1924), Harnack (1876), Klein (1893), and Weichold (1883). For modern treatments of this subject, refer to Fay (1973, Ch. VI), Gross and Harris (1981), Natanzon (1980; 1990), Silhol (1982), and Wilson (1978); applications to algebro-geometric solutions can be found, for instance, in Date (1982), Dubrovin (1982b; 1983), Dubrovin and Natanzon (1982), Natanzon (1995), Taimanov (1990c), and Zhivkov (1989; 1994).

Appendix B
Hyperelliptic Curves of the KdV-Type

> ... et il reste là un domaine restreint où continuent à s'exercer avec bonheur de nombreux amateurs (géométrie du triangle, du tétraèdre, des courbes et surfaces algébriques de bas degré, etc.). Mais pour le mathématicien professionnel, la mine est tarie, ...
>
> *Nicolas Bourbaki*[1]

We briefly summarize the basics of hyperelliptic KdV-type curves (i.e., those branched at infinity) as employed in Chapters 1 and 2 in connection with the KdV and sGmKdV hierarchies. We freely use the notation and results collected in Appendix A.

Fix $n \in \mathbb{N}_0$. We are going to construct the hyperelliptic Riemann surface $\mathcal{K}_n$ of (arithmetic) genus n associated with the KdV-type curve (1.20), that is, associated with the polynomial

$$\begin{aligned} &\mathcal{F}_n(z, y) = y^2 - R_{2n+1}(z) = 0, \\ &R_{2n+1}(z) = \prod_{m=0}^{2n}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}. \end{aligned} \tag{B.1}$$

At this point we explicitly permit $E_m = E_{m'}$ for some $m, m' \in \{0, \dots, 2n\}$ in order to include curves with a singular affine part. Next, we introduce an appropriate set of (at most) n (nonintersecting) cuts $\mathcal{C}_j$, joining $E_{m(j)}$ and $E_{m'(j)}$, and $\mathcal{C}_\infty$, joining E_{2n} and ∞, and denote

$$\mathcal{C} = \bigcup_{j \in J \cup \{\infty\}} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k,$$

where the finite index set $J \subseteq \{1, \dots, n\}$ has at most cardinality n. Define the cut plane

$$\Pi = \mathbb{C} \setminus \mathcal{C}$$

[1] *Éléments de Mathématique, XXIV, Algèbre, Ch. 9, Formes sesquilinéaires et formes quadratiques*, Hermann, Paris, 1973, p. 196. (" ... and there remains a restricted domain where amateurs continue to practice with happiness (geometry of the triangle, the tetrahedron, algebraic curves and surfaces of low degree, etc.). But for the professional mathematician the well is dry, ... ")

and introduce the holomorphic function

$$R_{2n+1}(\,\cdot\,)^{1/2}\colon \Pi \to \mathcal{C}, \quad z \mapsto \left(\prod_{m=0}^{2n}(z-E_m)\right)^{1/2} \tag{B.2}$$

on Π with an appropriate choice of the square root branch in (B.2). Next, define

$$\mathcal{M}_n = \{(z, \sigma R_{2n+1}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{1,-1\}\} \cup \{P_\infty\}$$

by extending $R_{2n+1}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$ and joining P_∞, the point at infinity. To describe charts on $\mathcal{M}_n$ we need to introduce more notation. Let $Q_0 \in \mathcal{M}_n$, $U_{Q_0} \subset \mathcal{M}_n$ a neighborhood of Q_0, $\zeta_{Q_0}\colon U_{Q_0} \to V_{Q_0} \subset \mathbb{C}$ a homeomorphism defined below, and write

$$\begin{aligned} Q_0 &= (z_0, \sigma_0 R_{2n+1}(z_0)^{1/2}) \text{ or } Q_0 = P_\infty, \\ Q &= (z, \sigma R_{2n+1}(z)^{1/2}) \in U_{Q_0} \subset \mathcal{M}_n, \quad V_{Q_0} = \zeta_{Q_0}(U_{Q_0}) \subset \mathbb{C}. \end{aligned}$$

Branch points and/or singular points on $\mathcal{M}_n$ are defined by

$$\mathcal{B}_s(\mathcal{K}_n) = \{(E_m, 0)\}_{m=0,\dots,2n} \cup \{P_\infty\},$$

with $(E_m, 0)$ being a branch point on $\mathcal{M}_n$ if $(E_m, 0)$ occurs an odd number of times in $\mathcal{B}_s(\mathcal{K}_n)$. If $(E_m, 0)$ occurs an even number of times in $\mathcal{B}_s(\mathcal{K}_n)$ or an odd number of times larger or equal to three, then $(E_m, 0)$ is a singular point on $\mathcal{M}_n$. The branch point P_∞ is nonsingular for $n = 0$ and 1 and singular for $n \geq 2$. Charts on $\mathcal{M}_n$ are now introduced by distinguishing three different cases; (i) $Q_0 \in \mathcal{M}_n \setminus \mathcal{B}_s(\mathcal{K}_n)$, (ii) $Q_0 = P_\infty$, and (iii) $Q_0 = (E_m, 0)$ for some $m = 0, \dots, 2n$.

(i) $Q_0 \in \mathcal{M}_n \setminus \mathcal{B}_s(\mathcal{K}_n)$: Then one defines

$$U_{Q_0} = \{Q \in \mathcal{M}_n \mid |z - z_0| < C_0\}, \quad C_0 = \min_{m=0,\dots,2n} |z_0 - E_m|, \tag{B.3}$$

where $\sigma R_{2n+1}(z)^{1/2}$ is the branch obtained by straight line analytic continuation starting from z_0,

$$V_{Q_0} = \{\zeta \in \mathbb{C} \mid |\zeta| < C_0\} \tag{B.4}$$

and

$$\zeta_{Q_0}\colon U_{Q_0} \to V_{Q_0}, \quad Q \mapsto (z - z_0) \tag{B.5}$$

with inverse

$$\zeta_{Q_0}^{-1}\colon V_{Q_0} \to U_{Q_0}, \quad \zeta \mapsto (z_0 + \zeta, \sigma R_{2n+1}(z_0 + \zeta)^{1/2}). \tag{B.6}$$

(ii) $Q_0 = P_\infty$: Here one introduces

$$U_{P_\infty} = \{Q \in \mathcal{M}_n \mid |z| > C_\infty\}, \ C_\infty = \max_{m=0,\dots,2n} |E_m|, \tag{B.7}$$

$$V_{P_\infty} = \{\zeta \in \mathbb{C} \mid |\zeta| < C_\infty^{-1/2}\}, \tag{B.8}$$

and

$$\begin{aligned} &\zeta_{P_\infty} \colon U_{P_\infty} \to V_{P_\infty}, \quad Q \mapsto \sigma/z^{1/2}, \quad P_\infty \mapsto 0, \\ &\sigma \in \{1, -1\}, \quad z^{1/2} = |z^{1/2}| \exp\big((i/2)\arg(z)\big), \quad 0 \le \arg(z) < 2\pi \end{aligned} \tag{B.9}$$

with inverse

$$\begin{aligned} \zeta_{P_\infty}^{-1} \colon V_{P_\infty} \to U_{P_\infty}, \quad \zeta \mapsto \left(\zeta^{-2}, \left(\prod_{m=0}^{2n}(1 - E_m\zeta^2)\right)^{1/2} \zeta^{-2n-1}\right), \\ 0 \mapsto P_\infty, \end{aligned} \tag{B.10}$$

where the square root is chosen such that

$$\left(\prod_{m=0}^{2n}(1 - E_m\zeta^2)\right)^{1/2} = 1 - \frac{1}{2}\left(\sum_{m=0}^{2n} E_m\right)\zeta^2 + O(\zeta^4). \tag{B.11}$$

(iii) $Q_0 = (E_{m_0}, 0)$: Here we have to distinguish two subcases: (iiia), where Q_0 is a branch point and possibly a singular point, and (iiib), where Q_0 is a singular point but not a branch point.

(iiia) Define

$$\begin{aligned} U_{Q_0} &= \{Q \in \mathcal{M}_n \mid |z - E_{m_0}| < C_{m_0}\}, \\ C_{m_0} &= \begin{cases} \min\limits_{\substack{m=0,\dots,2n \\ m \neq m_0}} |E_m - E_{m_0}|, & n \in \mathbb{N}, \\ \infty, & n = 0, \end{cases} \end{aligned} \tag{B.12}$$

$$V_{Q_0} = \left\{\zeta \in \mathbb{C} \mid |\zeta| < C_{m_0}^{1/2}\right\} \tag{B.13}$$

and

$$\zeta_{Q_0} \colon U_{Q_0} \to V_{Q_0}, \quad Q \mapsto \sigma(z - E_{m_0})^{1/2}, \quad \sigma \in \{1, -1\} \tag{B.14}$$

with inverse

$$\zeta_{Q_0}^{-1} \colon V_{Q_0} \to U_{Q_0}, \quad \zeta \mapsto \left(E_{m_0} + \zeta^2, \ \left(\prod_{\substack{m=0 \\ m \neq m_0}}^{2n} (E_{m_0} - E_m + \zeta^2)\right)^{1/2} \zeta\right) \tag{B.15}$$

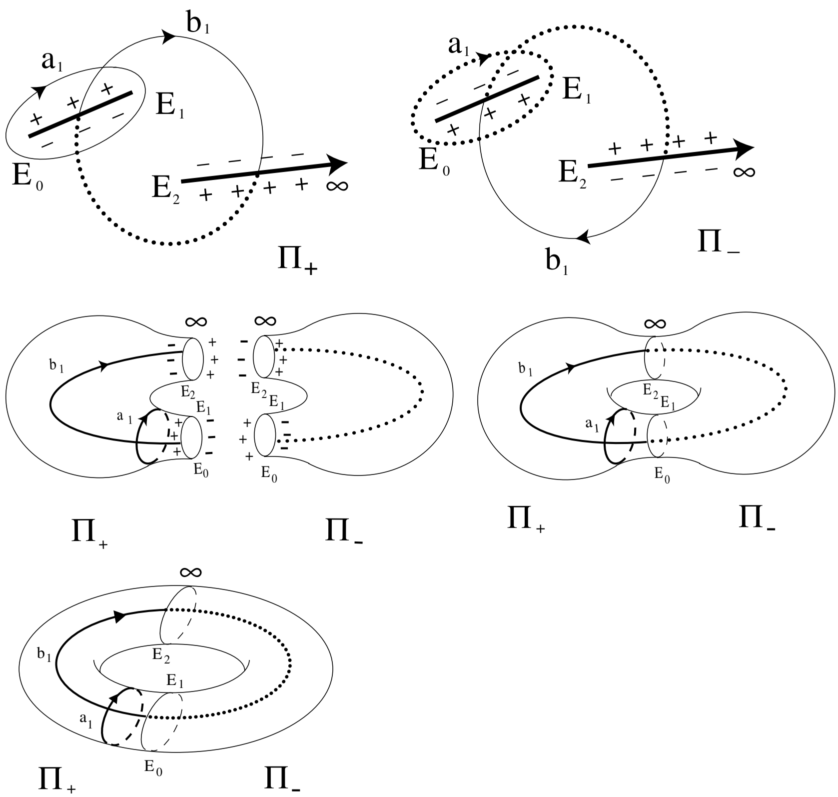

Fig. B.1. Genus $n = 1$.

for which the square root branches are chosen to yield compatibility with the charts in (B.3)–(B.15).

(iiib) Here we proceed as in case (i); the branch $\sigma R_{2n+1}(z)^{1/2}$ needs to be chosen to yield compatibility with the charts in (B.3)–(B.6) and (B.7)–(B.11).

The set $\mathcal{M}_n$ and the complex structure (B.3)–(B.15) just defined then yield a compact Riemann surface of arithmetic genus n, which we denoted by $\overline{\mathcal{K}_n}$ in Appendix A. To simplify the notation we use the symbol $\mathcal{K}_n$ to denote both the affine curve (B.1) and its projective closure $\overline{\mathcal{K}_n}$ throughout major parts of this monograph. If the zeros E_m of R_{2n+1} are all distinct,

$$E_m \neq E_{m'}, \quad m \neq m', \quad m, m' = 0, \ldots, 2n, \tag{B.16}$$

$\mathcal{K}_n$ is a compact hyperelliptic Riemann surface of topological genus n. The construction of $\mathcal{K}_n$ is sketched in Figure B.1 in the genus $n = 1$ case. A typical homology basis on $\mathcal{K}_n$ in the genus $n = 3$ case is depicted in Figure B.2.

Due to its importance in connection with spectral theoretic considerations, we now take a closer look at a particular self-adjoint case in which all zeros E_m of

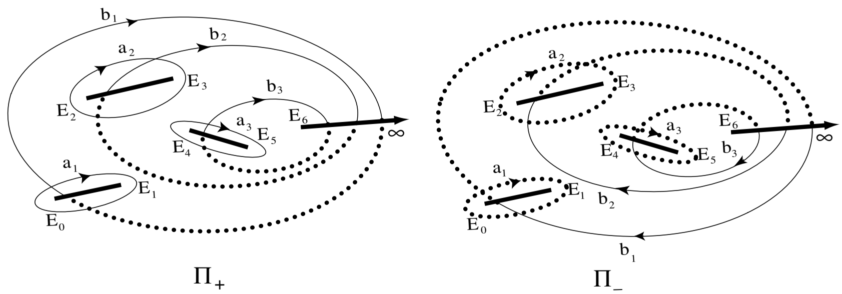

Fig. B.2. Genus $n = 3$.

R_{2n+1} in (B.1) are real and distinct, that is,

$$E_0 < E_1 < \cdots < E_{2n}. \tag{B.17}$$

In this case,

$$\mathcal{C}_1 = [E_0, E_1], \ldots, \mathcal{C}_n = [E_{2n-2}, E_{2n-1}],\ \mathcal{C}_\infty = [E_{2n}, \infty),$$

and the square root branch in (B.2) is chosen according to

$$R_{2n+1}(\lambda)^{1/2} = \lim_{\varepsilon \downarrow 0} R_{2n+1}(\lambda + i\varepsilon)^{1/2}, \quad \lambda \in \mathcal{C}, \tag{B.18}$$

and

$$R_{2n+1}(\lambda)^{1/2}$$
$$= |R_{2n+1}(\lambda)^{1/2}| \begin{cases} (-1)^n i & \text{for } \lambda \in (-\infty, E_0), \\ (-1)^{n+j} i & \text{for } \lambda \in (E_{2j-1}, E_{2j}), \ \ j = 1, \ldots, n, \\ (-1)^{n+j} & \text{for } \lambda \in (E_{2j}, E_{2j+1}), \ \ j = 0, \ldots, n-1, \\ 1 & \text{for } \lambda \in (E_{2n}, \infty), \end{cases}$$
$$\lambda \in \mathbb{R}. \tag{B.19}$$

The square root branches in (B.14) and (B.15) are then defined by

$$(z - E_{m_0})^{1/2} = |(z - E_{m_0})^{1/2}| \exp\big((i/2)\arg(z - E_{m_0})\big),$$
$$\arg(z - E_{m_0}) \in \begin{cases} [0, 2\pi) & \text{for } m_0 \text{ even}, \\ (-\pi, \pi] & \text{for } m_0 \text{ odd}, \end{cases}$$
$$\left(\prod_{\substack{m=0 \\ m \neq m_0}}^{2n} (E_{m_0} - E_m + \zeta^2) \right)^{1/2} = (-1)^n i^{-m_0} \left| \left(\prod_{\substack{m=0 \\ m \neq m_0}}^{2n} (E_{m_0} - E_m) \right)^{1/2} \right|$$
$$\times \left(1 + \frac{1}{2} \left(\sum_{\substack{m=0 \\ m \neq m_0}}^{2n} (E_{m_0} - E_m)^{-1} \right) \zeta^2 + O(\zeta^4) \right) \tag{B.20}$$

to guarantee compatibility of all charts. Assuming $n \in \mathbb{N}$ for the remainder of this appendix, the homology basis $\{a_j, b_j\}_{j=1,\dots,n}$ on $\mathcal{K}_n$ is then conveniently chosen as follows. The a_j cycle encircles the interval $[E_{2j-2}, E_{2j-1}]$, $j = 1, \dots, n$, clockwise on the upper sheet Π_+. The cycle b_j starts at a point in (E_{2j-2}, E_{2j-1}), $j = 1, \dots, n$, on the upper sheet Π_+ (i.e., on the upper rim of the cut $[E_{2j-2}, E_{2j-1}]$), proceeds clockwise to intersect a_j, and then continues clockwise on Π_+ until it hits a point on the cut $[E_{2n}, \infty)$. Then b_j returns clockwise on the lower sheet to its original starting point. In doing so, b_j avoids crossing of all other cycles a_k, b_k, $k \neq j$, $k = 1, \dots, n$. (Figure B.2 shows a typical genus $n = 3$ case with E_m not necessarily in real position.)

In the following we return to the general case (B.1) (as opposed to (B.17)). Points $P \in \mathcal{K}_n \setminus \{P_\infty\}$ are denoted by

$$P = (z, \sigma R_{2n+1}(z)^{1/2}) = (z, y), \quad P \in \mathcal{K}_n \setminus \{P_\infty\},$$

where

$$y(P) \underset{\zeta \to 0}{=} \left(1 - \frac{1}{2}\left(\sum_{m=0}^{2n} E_m\right)\zeta^2 + O(\zeta^4)\right)\zeta^{-2n-1} \text{ as } P \to P_\infty, \tag{B.21}$$
$$\zeta = \sigma'/z^{1/2}, \ \sigma' \in \{1, -1\}$$

(i.e., we abbreviate $y(P) = \sigma R_{2n+1}(z)^{1/2}$). Moreover, we introduce the holomorphic sheet exchange map (involution)

$$*: \mathcal{K}_n \to \mathcal{K}_n, \quad P = (z, y) \mapsto P^* = (z, -y), \ P_\infty \mapsto P_\infty^* = P_\infty \tag{B.22}$$

and the two meromorphic projection maps

$$\tilde{\pi} : \mathcal{K}_n \to \mathbb{C} \cup \{\infty\}, \quad P = (z, y) \mapsto z, \ P_\infty \mapsto \infty \tag{B.23}$$

and

$$y : \mathcal{K}_n \to \mathbb{C} \cup \{\infty\}, \quad P = (z, y) \mapsto y, \ P_\infty \mapsto \infty. \tag{B.24}$$

The map $\tilde{\pi}$ has a pole of order 2 at P_∞, and y has a pole of order $2n + 1$ at P_∞. Moreover,

$$\tilde{\pi}(P^*) = \tilde{\pi}(P), \quad y(P^*) = -y(P), \quad P \in \mathcal{K}_n.$$

Thus, $\mathcal{K}_n$ is a two-sheeted branched covering of the Riemann sphere $\mathbb{CP}^1$ ($\cong \mathbb{C} \cup \{\infty\}$) branched at the $2n + 2$ points $\{(E_m, 0)\}_{m=0,\dots,2n}$, P_∞. Moreover, $\mathcal{K}_n$ is compact (since $\tilde{\pi}$ is open and $\mathbb{CP}^1$ is compact), and $\mathcal{K}_n$ is hyperelliptic (since it admits the meromorphic function $\tilde{\pi}$ of degree two). In this context we denote the set of branch points of $\mathcal{K}_n$ by $\mathcal{B}(\mathcal{K}_n)$. Topologically, $\mathcal{K}_n$ is a sphere with n handles and hence has genus n.

For the rest of this appendix we assume $n \in \mathbb{N}$ and that $\mathcal{K}_n$ is a compact hyperelliptic Riemann surface of (topological) genus n (cf. (B.1) and (B.16)). In this case $\tilde{\pi}$ has two simple zeros at $(0, \pm R_{2n+1}(0)^{1/2})$ if $R_{2n+1}(0) \neq 0$ or a double zero at $(0, 0)$ if $R_{2n+1}(0) = 0$ (i.e., if $0 \in \{E_m\}_{m=0,\dots,2n}$), and y has $2n+1$ simple zeros at $(E_m, 0)$ for $m = 0, \dots, 2n$.

We introduce the upper and lower sheets $\Pi_\pm$ by

$$\Pi_\pm = \{(z, \pm R_{2n+1}(z)^{1/2}) \in \mathcal{M}_n \mid z \in \Pi\} \tag{B.25}$$

and the associated charts

$$\zeta_\pm \colon \Pi_\pm \to \Pi, \quad P \mapsto z. \tag{B.26}$$

In particular, the charts in (B.3)–(B.15) are chosen to be compatible with $\zeta_\pm$ wherever they overlap.

Using the local chart near P_∞, one verifies that dz/y is a holomorphic differential on $\mathcal{K}_n$ with zeros of order $2(n-1)$ at P_∞, and hence

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j = 1, \dots, n \tag{B.27}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_n$. Upon introduction of the invertible matrix C in $\mathbb{C}^n$,

$$C = (C_{j,k})_{j,k=1,\dots,n}, \quad C_{j,k} = \int_{a_k} \eta_j, \tag{B.28}$$

$$\underline{c}(k) = (c_1(k), \dots, c_n(k)), \quad c_j(k) = \big(C^{-1}\big)_{j,k}, \quad j, k = 1, \dots, n, \tag{B.29}$$

the normalized differentials ω_j for $j = 1, \dots, n$ (cf. (A.13)),

$$\omega_j = \sum_{\ell=1}^{n} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j, k = 1, \dots, n, \tag{B.30}$$

form a canonical basis for the space of holomorphic differentials on $\mathcal{K}_n$.

In the chart $(U_{P_\infty}, \zeta_{P_\infty})$ induced by $1/\tilde{\pi}^{1/2}$ near P_∞, one infers

$$\begin{aligned}
\underline{\omega} = (\omega_1, \dots, \omega_n) &= -2\bigg(\sum_{j=1}^{n} \frac{\underline{c}(j)\zeta^{2(n-j)}}{\big(\prod_{m=0}^{2n}(1-\zeta^2 E_m)\big)^{1/2}}\bigg) d\zeta \\
&\underset{\zeta\to 0}{=} -2\bigg(\sum_{q=0}^{\infty}\sum_{k=1}^{n} \underline{c}(k)\hat{c}_{k-n+q}(\underline{E})\zeta^{2q}\bigg) d\zeta \\
&= -2\bigg(\underline{c}(n) + \bigg(\frac{1}{2}\underline{c}(n)\sum_{m=0}^{2n} E_m + \underline{c}(n-1)\bigg)\zeta^2 + O(\zeta^4)\bigg) d\zeta \text{ as } P \to P_\infty, \\
&\qquad \zeta = \sigma/z^{1/2},\ \sigma \in \{1, -1\},
\end{aligned} \tag{B.31}$$

where $\underline{E} = (E_0, \dots, E_{2n})$, and we used (B.21) and

$$\bigg(\prod_{m=0}^{2n}(1 - E_m\zeta)\bigg)^{-1/2} = \sum_{k=0}^{\infty} \hat{c}_k(\underline{E})\zeta^k$$

for $\zeta \in \mathbb{C}$ such that $|\zeta|^{-1} > \max\{|E_0|, \dots, |E_{2n}|\}$ with

$$\hat{c}_{-k}(\underline{E}) = 0, \quad k \in \mathbb{N}, \quad \hat{c}_0(\underline{E}) = 1,$$
$$\hat{c}_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{2n}=0 \\ j_0+\dots+j_{2n}=k}}^{k} \frac{(2j_0)! \cdots (2j_{2n})!}{2^{2k}(j_0!)^2 \cdots (j_{2n}!)^2} E_0^{j_0} \cdots E_{2n}^{j_{2n}}, \quad k \in \mathbb{N}. \tag{B.32}$$

Combining (A.22) and (B.31), one computes for the vector $\underline{U}_{2q}^{(2)}$ of b-periods of $\omega_{P_\infty,2q}^{(2)}/(2\pi i)$, the normalized differential of the second kind, holomorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ with principal part $\zeta^{-2q-2}d\zeta/(2\pi i)$,

$$\underline{U}_{2q}^{(2)} = \big(U_{2q,1}^{(2)}, \dots, U_{2q,n}^{(2)}\big), \tag{B.33}$$
$$U_{2q,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \omega_{P_\infty,2q}^{(2)} = -\frac{2}{2q+1}\sum_{k=1}^{n} c_j(k)\hat{c}_{k-n+q}(\underline{E}),$$
$$j = 1, \dots, n, \quad q \in \mathbb{N}_0.$$

The results of this appendix apply to the sGmKdV curves upon taking into account the additional constraint $0 \in \{E_m\}_{m=0,\dots,2n}$. Without loss of generality, we then choose

$$E_0 = 0.$$

One then computes in the sGmKdV context in the chart (U_{P_0}, ζ_{P_0}) induced by $\tilde{\pi}^{1/2}$, near the branch point $P_0 = (0, 0)$,

$$\underline{\omega} \underset{\zeta\to 0}{=} 2\bigg(\frac{\underline{c}(1)}{Q^{1/2}} + O(\zeta^2)\bigg)d\zeta \text{ as } P \to P_0, \quad Q^{1/2} = \bigg(\prod_{m=1}^{2n} E_m\bigg)^{1/2}, \tag{B.34}$$
$$\zeta = \sigma z^{1/2}, \ \sigma \in \{1, -1\}$$

with the sign of $Q^{1/2}$ determined by the compatibility of the charts and

$$y(P) \underset{\zeta\to 0}{=} Q^{1/2}\zeta + O(\zeta^3) \text{ as } P \to P_0. \tag{B.35}$$

Combining (A.22) and (B.34) then yields

$$\frac{1}{2\pi i}\int_{b_j} \omega_{P_0,0}^{(2)} = 2\frac{c_j(1)}{Q^{1/2}}, \quad j = 1, \dots, n. \tag{B.36}$$

In the special self-adjoint case (B.17), the matrix τ of b-periods (cf. (A.14)) satisfies, in addition to (A.16),

$$\tau = iT, \quad T > 0 \tag{B.37}$$

since

$$C_{j,k} = \int_{a_k} \eta_j = 2\int_{E_{2k-2}}^{E_{2k-1}} \frac{z^{j-1}dz}{R_{2n+1}(z)^{1/2}} \in \mathbb{R} \tag{B.38}$$

and

$$\int_{b_k} \eta_j = 2 \int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1} dz}{R_{2n+1}(z)^{1/2}} \in i\mathbb{R}. \tag{B.39}$$

Explicit formulas for normal differentials of the third kind, $\omega^{(3)}_{Q_1,Q_2}$, with simple poles at Q_1 and Q_2, corresponding residues $+1$ and -1, vanishing a-periods, and holomorphic on $\mathcal{K}_n \setminus \{Q_1, Q_2\}$, can easily be found. One obtains

$$\omega^{(3)}_{P_1,P_\infty} = -\frac{y+y_1}{z-z_1}\frac{dz}{2y} + \frac{\lambda_n}{y}\prod_{j=1}^{n-1}(z-\lambda_j)\,dz, \tag{B.40}$$

$$\omega^{(3)}_{P_1,P_2} = \left(\frac{y+y_1}{z-z_1} - \frac{y+y_2}{z-z_2}\right)\frac{dz}{2y} + \frac{\tilde\lambda_n}{y}\prod_{j=1}^{n-1}(z-\tilde\lambda_j)\,dz, \tag{B.41}$$

$$P_1, P_2 \in \mathcal{K}_n \setminus \{P_\infty\},$$

where $\lambda_j, \tilde\lambda_j \in \mathbb{C}$, $j = 1, \dots, n$, are uniquely determined by the requirement of vanishing a-periods, and we abbreviated $P_j = (z_j, y_j)$, $j = 1, 2$. (If $n = 1$, we use the standard convention that the product over an empty index set is replaced by 1; if $n = 0$, both products in (B.40) and (B.41) are replaced by 0.) Moreover, choosing $Q_0 = (E_{m_0}, 0)$, the following asymptotic expansions hold

$$\int_{Q_0}^{P} \omega^{(3)}_{P_1,P_\infty} \underset{\zeta\to 0}{=} \ln(\zeta) + (1/2)\ln(E_m - z_1) + O(\zeta) \text{ as } P \to P_\infty, \tag{B.42}$$

$$\int_{Q_0}^{P} \omega^{(3)}_{P_1,P_\infty} \underset{\zeta\to 0}{=} -\ln(\zeta) + (1/2)\ln(E_m - z_1) + O(\zeta) \text{ as } P \to P_1. \tag{B.43}$$

Next, we turn to the theta function representation of symmetric functions of values of a meromorphic function as discussed in Remark A.29 in the current case of KdV-type hyperelliptic Riemann surfaces. The choice $f(P) = \tilde\pi(P)$ in (A.62) then yields, after an explicit residue computation at P_∞,

$$\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \int_{a_j} \tilde\pi\,\omega_j \tag{B.44}$$
$$- \sum_{j,k=1}^{n} U^{(2)}_{0,j} U^{(2)}_{0,k} \partial^2_{w_j w_k} \ln\Big(\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat\mu}}) + \underline{w}\big)\Big)\Big|_{\underline{w}=0},$$

where $\underline{\hat\mu} = \{\hat\mu_1, \dots, \hat\mu_n\}$, $\hat\mu_j = (\mu_j, y(\hat\mu_j)) \in \mathcal{K}_n$, $j = 1, \dots, n$, assuming $\mathcal{D}_{\underline{\hat\mu}} \in \operatorname{Sym}^n(\mathcal{K}_n)$ to be nonspecial and using

$$\underline{A}_{Q_0}(P) - \underline{A}_{Q_0}(P_\infty) \underset{\zeta\to 0}{=} \underline{U}^{(2)}_0 \zeta + O(\zeta^3) \text{ as } P \to P_\infty, \quad \underline{U}^{(2)}_0 = -2\underline{c}(n) \tag{B.45}$$

according to (B.31)–(B.33). Here $Q_0 \in \mathcal{K}_n \setminus \{P_\infty\}$ denotes an appropriate base point. In the present hyperelliptic context, the constant $\sum_{j=1}^{n} \int_{a_j} \tilde{\pi}\omega_j$ can be related to the zeros $\{\lambda_j\}_{j=1,\dots,n}$ of the normalized differential of the second kind, $\omega_{P_\infty,0}^{(2)}$ as follows,

$$\sum_{j=1}^{n} \int_{a_j} \tilde{\pi}\omega_j = \sum_{j=1}^{n} \lambda_j.$$

This will be proven in Appendix F (cf. (F.48) for $k = n$ and (F.50)). Hence, one finally obtains

$$\begin{aligned} \sum_{j=1}^{n} \mu_j = &\sum_{j=1}^{n} \lambda_j \\ &- \sum_{j,k=1}^{n} U_{0,j}^{(2)} U_{0,k}^{(2)} \partial_{w_j w_k}^2 \ln\Big(\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{w}\big)\Big)\Big|_{\underline{w}=0}, \end{aligned} \tag{B.46}$$

If, in addition, $\mathcal{D}_{\underline{\hat{\mu}}}$ depends on a parameter $x \in \Omega$, $\Omega \subseteq \mathbb{R}$ an open interval, satisfying the linearization property

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) + c\, \underline{U}_0^{(2)}(x - x_0), \quad x, x_0 \in \Omega$$

for some constant $c \in \mathbb{C} \setminus \{0\}$, as verified, for instance, in the KdV context (cf. (1.106) and (1.212)), where $c = i$, one can rewrite (B.46) in the form

$$\sum_{j=1}^{n} \mu_j(x) = \sum_{j=1}^{n} \lambda_j - c^{-2}\partial_x^2 \ln\Big(\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_\infty) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})\big)\Big), \quad x \in \Omega. \tag{B.47}$$

Combined with the trace formula (1.83), (B.47) confirms the Its–Matveev formula (1.108). A systematic approach to elementary symmetric functions of $\mu_1, \dots, \mu_n$ will be discussed in Appendix F.

Next, we provide a brief illustration of the Riemann–Roch theorem in connection with KdV-type compact hyperelliptic Riemann surfaces $\mathcal{K}_n$ of genus n (cf. (B.1) and (B.16)) and explicitly determine a basis for the vector space $\mathcal{L}(-k\mathcal{D}_{P_\infty} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$ for $k \in \mathbb{N}_0$. We refer to (A.28) for the definition of $\mathcal{L}(\mathcal{D})$ and to Theorem A.13 for the Riemann–Roch theorem. In addition, we use the short-hand notation

$$k\mathcal{D}_{P_\infty} + \mathcal{D}_{\underline{\hat{\mu}}(x_0)} = \sum_{\ell=1}^{k} \mathcal{D}_{P_\infty} + \sum_{j=1}^{n} \mathcal{D}_{\hat{\mu}_j(x_0)}, \quad k \in \mathbb{N}_0,$$
$$\underline{\hat{\mu}}(x_0) = \{\hat{\mu}_1(x_0), \dots, \hat{\mu}_n(x_0)\} \in \operatorname{Sym}^n(\mathcal{K}_n)$$

and recall that

$$\mathcal{L}(-k\mathcal{D}_{P_\infty} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = \{f \in \mathcal{M}(\mathcal{K}_n) \mid (f) + k\mathcal{D}_{P_\infty} + \mathcal{D}_{\underline{\hat{\mu}}(x_0)} \geq 0\}, \quad n \in \mathbb{N}_0.$$

With $\phi(P,x)$ and $\psi(P,x,x_0)$ defined as in (1.38) and (1.41) one obtains the following result (we denote by $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}$, i.e., the largest integer less or equal to x).

Theorem B.1 *Assume $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ to be nonspecial (i.e., $i(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = 0$) and of degree $n \in \mathbb{N}$. For $k \in \mathbb{N}_0$, a basis for the vector space $\mathcal{L}(-k\mathcal{D}_{P_\infty} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$ is given by*

$$\begin{aligned}
&\{1\}, \quad k = 0, \\
&\{\tilde{\pi}^j\}_{j=0,\dots,\lfloor k/2 \rfloor} \cup \{\tilde{\pi}^j \phi(\,\cdot\,,x_0)\}_{j=0,\dots,\lfloor (k-1)/2 \rfloor}, \quad k \in \mathbb{N}.
\end{aligned} \tag{B.48}$$

Equivalently,

$$\mathcal{L}(-k\mathcal{D}_{P_\infty} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = \operatorname{span}\left\{\partial_x^j \psi(\,\cdot\,,x,x_0)\big|_{x=x_0}\right\}_{j=0,\dots,k}. \tag{B.49}$$

Proof The elements in (B.48) are easily seen to be linearly independent and belonging to $\mathcal{L}(-k\mathcal{D}_{P_\infty} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$. It remains to be shown that they are maximal. From $0 = i(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = i(\mathcal{D}_{kP_\infty} + \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$ and the Riemann–Roch Theorem A.13 one obtains $r(-k\mathcal{D}_{P_\infty} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = k+1$, proving (B.48). To prove (B.49), one repeatedly uses the Schrödinger equation (1.47) to prove inductively that

$$\begin{aligned}
&\partial_x^{2m+2}\psi(P,x,x_0) = (-\tilde{\pi})^{m+1} + R_{2m+1}(P,x), \\
&\partial_x^{2m+1}\psi(P,x,x_0) = (-\tilde{\pi})^{m}\partial_x\psi(P,x,x_0) + R_{2m}(P,x),
\end{aligned}$$

where $R_k(\,\cdot\,,x_0) \in \mathcal{L}(-k\mathcal{D}_{P_\infty} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$. $\square$

Finally, we consider a Riemann–Roch-type uniqueness result for Baker–Akhiezer functions needed in Chapters 1 and 2. Let Q_0 be an appropriate base point on $\mathcal{K}_n \setminus \{P_\infty\}$ (in the sGmKdV case we choose in addition $Q_0 \neq P_0 = (E_0, 0)$).

Lemma B.2 *Let $P \in \mathcal{K}_n \setminus \{P_\infty\}$ and $(x,t_r), (x_0,t_{0,r}) \in \Omega$ for some $\Omega \subseteq \mathbb{R}^2$. Assume $\psi(\,\cdot\,,x,t_r)$, $(x,t_r) \in \Omega$, to be meromorphic on $\mathcal{K}_n \setminus \{P_\infty\}$ with an essential singularity at P_∞ such that $\tilde{\psi}(\,\cdot\,,x,t_r)$, defined by*

$$\tilde{\psi}(P,x,t_r) = \psi(P,x,t_r)\exp\left(i(x-x_0)\int_{Q_0}^{P}\omega_{P_\infty,0}^{(2)} + i(t_r - t_{0,r})\int_{Q_0}^{P}\widetilde{\Omega}_{P_\infty,2r}^{(2)}\right), \tag{B.50}$$

is meromorphic on $\mathcal{K}_n$ and its divisor satisfies

$$(\tilde{\psi}(\,\cdot\,,x,t_r)) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}$$

*for some positive divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}$ of degree n. Here $\omega_{P_\infty,0}^{(2)}$ and $\widetilde{\Omega}_{P_\infty,2r}^{(2)}$ are defined in (*1.98*) and (*1.205*), and the path of integration in the right-hand side of* (B.50) *is chosen identical to that in the Abel maps* (A.34) *and* (A.35).[1] *Define a divisor* $\mathcal{D}_0(x, t_r)$ *by*

$$(\tilde{\psi}(\,\cdot\,, x, t_r)) = \mathcal{D}_0(x, t_r) - \mathcal{D}_{\underline{\hat{\mu}}(x_0, t_{0,r})}. \tag{B.51}$$

Then

$$\mathcal{D}_0(x, t_r) \in \mathrm{Sym}^n(\mathcal{K}_n), \quad \mathcal{D}_0(x, t_r) > 0, \quad \deg(\mathcal{D}_0(x, t_r)) = n.$$

Moreover, if $\mathcal{D}_0(x, t_r)$ *is nonspecial for all* $(x, t_r) \in \Omega$*, that is, if*

$$i(\mathcal{D}_0(x, t_r)) = 0, \quad (x, t_r) \in \Omega,$$

then $\psi(\,\cdot\,, x, t_r)$ *is unique up to a constant multiple (which may depend on the parameters* $(x, t_r), (x_0, t_{0,r}) \in \Omega$*).*

Proof By the Riemann–Roch Theorem A.13, there exists at least one such function $\tilde{\psi}(\,\cdot\,, x, t_r)$. If $\tilde{\psi}_j(\,\cdot\,, x, t_r)$ are two such functions satisfying (B.51) with corresponding divisors $\mathcal{D}_{0,j}(x, t_r)$ for $j = 1$ and 2, then one infers

$$(\tilde{\psi}_1(\,\cdot\,, x, t_r)/\tilde{\psi}_2(\,\cdot\,, x, t_r)) = \mathcal{D}_{0,1}(x, t_r) - \mathcal{D}_{0,2}(x, t_r).$$

Since $i(\mathcal{D}_{0,2}(x, t_r)) = 0$ and $\deg(\mathcal{D}_{0,2}(x, t_r)) = n$, equation (A.31) yields $r(-\mathcal{D}_{0,2}(x, t_r)) = 1$ for $(x, t_r) \in \Omega$, and hence $\tilde{\psi}_1/\tilde{\psi}_2$ is a constant on $\mathcal{K}_n$. $\square$

[1] This is to avoid multi-valued expressions and hence the use of the multiplicative Riemann–Roch theorem in the proof below.

Appendix C
Hyperelliptic Curves of the AKNS-Type

"Why," said the Dodo, "the best way to explain it is to do it."
Lewis Carroll[1]

We briefly summarize some of the basic facts on hyperelliptic AKNS-type curves (i.e., those not branched at infinity) as employed in Chapters 3–5 in connection with the AKNS and Camassa–Holm hierarchies and the classical massive Thirring system. We freely use the notation and results collected in Appendix A.

Fix $n \in \mathbb{N}_0$. We are going to construct the hyperelliptic Riemann surface $\mathcal{K}_n$ of (arithmetic) genus n associated with the AKNS-type curve (3.34), that is,

$$\begin{aligned} &\mathcal{F}_n(z, y) = y^2 - R_{2n+2}(z) = 0, \\ &R_{2n+2}(z) = \prod_{m=0}^{2n+1}(z - E_m), \quad \{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{C}. \end{aligned} \tag{C.1}$$

At this point we explicitly permit $E_m = E_{m'}$ for some $m, m' \in \{0, \dots, 2n+1\}$ in order to include curves with a singular affine part. We introduce an appropriate set of (at most) $n+1$ (nonintersecting) cuts $\mathcal{C}_j$ joining $E_{m(j)}$ and $E_{m'(j)}$ and denote

$$\mathcal{C} = \bigcup_{j \in J} \mathcal{C}_j, \quad \mathcal{C}_j \cap \mathcal{C}_k = \emptyset, \quad j \neq k,$$

where the finite index set $J \subseteq \{1, \dots, n+1\}$ has (at most) cardinality $n+1$. Define the cut plane Π,

$$\Pi = \mathbb{C} \setminus \mathcal{C},$$

and introduce the holomorphic function

$$R_{2n+2}(\,\cdot\,)^{1/2} \colon \Pi \to \mathbb{C}, \quad z \mapsto \left(\prod_{m=0}^{2n+1}(z - E_m)\right)^{1/2} \tag{C.2}$$

[1] *Alice's Adventures in Wonderland*, Puffin Books, Harmandsworth, 1962, p. 45.

on Π with an appropriate choice of the square root branch (C.2). Next, define

$$\mathcal{M}_n = \{(z, \sigma R_{2n+2}(z)^{1/2}) \mid z \in \mathbb{C},\ \sigma \in \{1, -1\}\} \cup \{P_{\infty_+}, P_{\infty_-}\}$$

by extending $R_{2n+2}(\,\cdot\,)^{1/2}$ to $\mathcal{C}$ and joining P_{∞_+} and P_{∞_-}, $P_{\infty_+} \neq P_{\infty_-}$, the two points at infinity. To describe charts on $\mathcal{M}_n$ let $Q_0 \in \mathcal{M}_n$, $U_{Q_0} \subset \mathcal{M}_n$ a neighborhood of Q_0, $\zeta_{Q_0}\colon U_{Q_0} \to V_{Q_0} \subset \mathbb{C}$ a homeomorphism defined below, and write

$$Q_0 = (z_0, \sigma_0 R_{2n+2}(z_0)^{1/2}) \text{ or } Q_0 = P_{\infty_\pm},$$
$$Q = (z, \sigma R_{2n+2}(z)^{1/2}) \in U_{Q_0} \subset \mathcal{M}_n,\ \ V_{Q_0} = \zeta_{Q_0}(U_{Q_0}) \subset \mathbb{C}.$$

Branch points and/or singular points on $\mathcal{M}_n$ are defined by

$$\mathcal{B}_s(\mathcal{K}_n) = \begin{cases} \{(E_m, 0)\}_{m=0,1}, & n = 0, \\ \{(E_m, 0)\}_{m=0,\dots,2n+1} \cup \{P_{\infty_+}, P_{\infty_-}\}, & n \in \mathbb{N}, \end{cases}$$

with $(E_m, 0)$ being a branch point on $\mathcal{M}_n$ if $(E_m, 0)$ occurs an odd number of times in $\mathcal{B}_s(\mathcal{K}_n)$. If $(E_m, 0)$ occurs an even number of times in $\mathcal{B}_s(\mathcal{K}_n)$ or an odd number of times larger or equal to three, then $(E_m, 0)$ is a singular point on $\mathcal{M}_n$. While $P_{\infty_\pm}$ are never branch points, $P_{\infty_\pm}$ are nonsingular for $n = 0$ and singular for $n \in \mathbb{N}$.

Charts on $\mathcal{M}_n$ are now introduced distinguishing three cases, (i) $Q_0 \in \mathcal{M}_n \setminus \big(\mathcal{B}_s(\mathcal{K}_n) \cup \{P_{\infty_+}, P_{\infty_-}\}\big)$, (ii) $Q_0 = P_{\infty_\pm}$, and (iii) $Q_0 = (E_m, 0)$ for some $m = 0, \dots, 2n+1$.

(i) $Q_0 \in \mathcal{M}_n \setminus \big(\mathcal{B}_s(\mathcal{K}_n) \cup \{P_{\infty_+}, P_{\infty_-}\}\big)$: Then one defines

$$U_{Q_0} = \{Q \in \mathcal{M}_n \mid |z - z_0| < C_0\}, \quad C_0 = \min_{m=0,\dots 2n+1} |z_0 - E_m|, \tag{C.3}$$

where $\sigma R_{2n+2}(z)^{1/2}$ is the branch obtained by straight line analytic continuation starting from z_0,

$$V_{Q_0} = \{\zeta \in \mathbb{C} \mid |\zeta| < C_0\} \tag{C.4}$$

and

$$\zeta_{Q_0}\colon U_{Q_0} \to V_{Q_0}, \quad Q \mapsto (z - z_0) \tag{C.5}$$

with inverse

$$\zeta_{Q_0}^{-1}\colon V_{Q_0} \to U_{Q_0}, \quad \zeta \mapsto (z_0 + \zeta, \sigma R_{2n+2}(z_0 + \zeta)^{1/2}). \tag{C.6}$$

(ii) Let $Q_0 = P_{\infty_\pm}$: Then one introduces

$$U_{P_{\infty_\pm}} = \{Q \in \mathcal{M}_n \mid |z| > C_\infty\}, \quad C_\infty = \max_{m=0,\dots 2n+1} |E_m|, \tag{C.7}$$

$$V_{P_{\infty_\pm}} = \left\{\zeta \in \mathbb{C} \mid |\zeta| < C_\infty^{-1}\right\} \tag{C.8}$$

and

$$\zeta_{P_{\infty_\pm}} : U_{P_{\infty_\pm}} \to V_{P_{\infty_\pm}}, \quad Q \mapsto z^{-1}, \quad P_{\infty_\pm} \mapsto 0 \tag{C.9}$$

with inverse

$$\begin{aligned} &\zeta_{P_{\infty_\pm}}^{-1} : V_{P_{\infty_\pm}} \to U_{P_{\infty_\pm}}, \\ &\zeta \mapsto \left(\zeta^{-1}, \mp\left(\prod_{m=0}^{2n+1}(1 - E_m\zeta)\right)^{1/2} \zeta^{-n-1}\right), \quad 0 \mapsto P_{\infty_\pm}, \end{aligned} \tag{C.10}$$

where the square root is chosen such that

$$\left(\prod_{m=0}^{2n+1}(1 - E_m\zeta)\right)^{1/2} = 1 - \frac{1}{2}\left(\sum_{m=0}^{2n+1} E_m\right)\zeta + O(\zeta^2). \tag{C.11}$$

(iii) $Q_0 = (E_m, 0)$: Here we have to distinguish two subcases: (iiia), where Q_0 is a branch point and possibly a singular point, and (iiib), where Q_0 is a singular point but not a branch point.

(iiia) Then one defines

$$U_{Q_0} = \{Q \in \mathcal{M}_n \mid |z - E_{m_0}| < C_{m_0}\}, \quad C_{m_0} = \min_{m=0,\dots,2n+1} |E_m - E_{m_0}|, \tag{C.12}$$

$$V_{Q_0} = \left\{\zeta \in \mathbb{C} \mid |\zeta| < C_{m_0}^{1/2}\right\} \tag{C.13}$$

and

$$\zeta_{Q_0} : U_{Q_0} \to V_{Q_0}, \quad Q \mapsto \sigma(z - E_{m_0})^{1/2}, \quad \sigma \in \{1, -1\} \tag{C.14}$$

with inverse

$$\zeta_{Q_0}^{-1} : V_{Q_0} \to U_{Q_0}, \quad \zeta \mapsto \left(E_{m_0} + \zeta^2, \left(\prod_{\substack{m=0 \\ m\neq m_0}}^{2n+1} (E_{m_0} - E_m + \zeta^2)\right)^{1/2} \zeta\right), \tag{C.15}$$

where the square root branches are chosen to yield compatibility with the charts in (C.3)–(C.6) and (C.7)–(C.11).

(iiib) In this case one can proceed as in case (i); the branch $\sigma R_{2n+2}(z)^{1/2}$ needs to be chosen to yield compatibility with the charts in (C.3)–(C.6) and (C.7)–(C.11).

The set $\mathcal{M}_n$ and the complex structure (C.3)–(C.15) just defined then yield a compact Riemann surface of arithmetic genus n, which we denoted by $\overline{\mathcal{K}_n}$ in Appendix A. For simplicity of notation we use the symbol $\mathcal{K}_n$ to denote both the

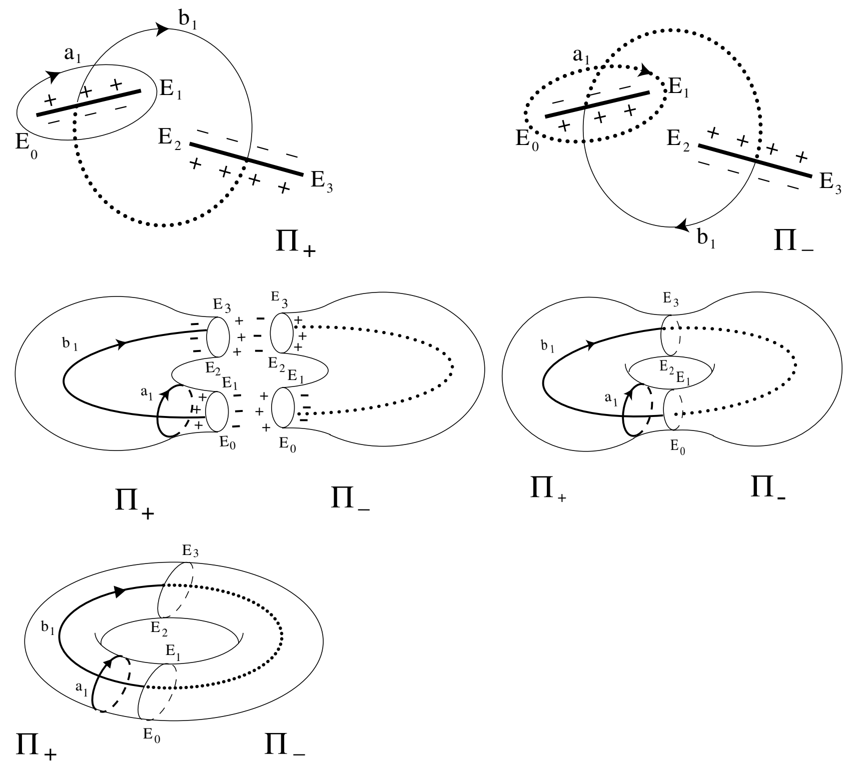

Fig. C.1. Genus $n = 1$.

affine curve (C.1) and its projective closure $\overline{\mathcal{K}}_n$ throughout major parts of this monograph. If the zeros E_m of R_{2n+2} are all distinct,

$$E_m \neq E_{m'}, \quad m \neq m', \ m, m' = 0, \dots, 2n+1, \tag{C.16}$$

$\mathcal{K}_n$ is a compact hyperelliptic Riemann surface of topological genus n. The construction of $\mathcal{K}_n$ is sketched in Figure C.1 in the genus $n = 1$ case. A typical homology basis on $\mathcal{K}_n$ in the genus $n = 3$ case is depicted in Figure C.2 (it differs from the corresponding one shown in Figure C.1).

Next, for the reader's convenience, we provide a detailed treatment of branch points in the case of a nonsingular affine part (cf. (C.16)) for the two most frequently occurring situations: the self-adjoint case, where $\{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{R}$, and the case in which $\{E_m\}_{m=0,\dots,2n+1} = \{\widetilde{E}_\ell, \overline{\widetilde{E}}\}_{\ell=0,\dots,n}$ consists of complex conjugate pairs.

Let us first consider the case with real and distinct roots, that is, $\{E_m\}_{m=0,\dots,2n+1} \subset \mathbb{R}$, and

$$E_0 < E_1 < \cdots < E_{2n+1}. \tag{C.17}$$

In this case,

$$\mathcal{C}_j = [E_{2j}, E_{2j+1}], \quad j = 0, \dots, n$$

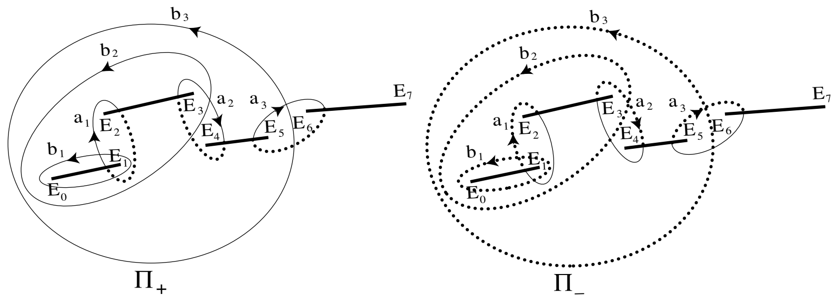

Fig. C.2. Genus $n = 3$.

and the square root branch in (C.2) is chosen according to

$$R_{2n+2}(\lambda)^{1/2} = \lim_{\varepsilon \downarrow 0} R_{2n+2}(\lambda + i\varepsilon)^{1/2}, \quad \lambda \in \mathcal{C}, \tag{C.18}$$

and

$$R_{2n+2}(\lambda)^{1/2}$$
$$= |R_{2n+2}(\lambda)^{1/2}| \begin{cases} -1 & \text{for } \lambda \in (E_{2n+1}, \infty), \\ (-1)^{n+j} & \text{for } \lambda \in (E_{2j-1}, E_{2j}),\ j = 1, \dots, n, \\ (-1)^n & \text{for } \lambda \in (-\infty, E_0), \\ i(-1)^{n+j+1} & \text{for } \lambda \in (E_{2j}, E_{2j+1}),\ j = 0, \dots, n, \end{cases}$$
$$\lambda \in \mathbb{R}. \tag{C.19}$$

The square-root branches in (C.14) and (C.15) are defined by

$$(z - E_{m_0})^{1/2} = |(z - E_{m_0})^{1/2}| \exp\big((i/2)\arg(z - E_{m_0})\big),$$
$$\arg(z - E_{m_0}) \in \begin{cases} [0, 2\pi) & m_0 \text{ even}, \\ (-\pi, \pi] & m_0 \text{ odd}, \end{cases}$$

and

$$\left(\prod_{\substack{m=0 \\ m \neq m_0}}^{2n+1} (E_{m_0} - E_m + \zeta^2)\right)^{1/2} = (-1)^n i^{-m_0-1} \left| \left(\prod_{\substack{m=0 \\ m \neq m_0}}^{2n+1} (E_{m_0} - E_m)\right)^{1/2} \right|$$
$$\times \left(1 + \frac{1}{2}\left(\sum_{\substack{m=0 \\ m \neq m_0}}^{2n+1} (E_{m_0} - E_m)^{-1}\right)\zeta^2 + O(\zeta^4)\right), \tag{C.20}$$

to guarantee compatibility of all charts.

Next we turn to the case in which the roots form complex conjugate pairs,

$$\{E_m\}_{m=0,\dots,2n+1} = \{\widetilde{E}_\ell, \overline{\widetilde{E}_\ell}\}_{\ell=0,\dots,n},$$

where we assume

$$\operatorname{Re}(\widetilde{E}_\ell) < \operatorname{Re}(\widetilde{E}_{\ell+1}),\ \ \ell = 0, \dots, n-1, \quad \operatorname{Im}(\widetilde{E}_\ell) < \operatorname{Im}(\widetilde{\widetilde{E}}_\ell), \quad \ell = 0, \dots, n.$$

In this case,

$$\mathcal{C}_\ell = \{z \in \mathbb{C} \mid z = \widetilde{E}_\ell + t(\widetilde{\widetilde{E}}_\ell - \widetilde{E}_\ell),\ 0 \le t \le 1\}, \quad \ell = 0, \dots, n$$

and the square root branch in (C.2) is chosen according to

$$R_{2n+2}(z)^{1/2} = \lim_{\varepsilon \downarrow 0} R_{2n+2}(z + (-1)^{n+\ell}\varepsilon)^{1/2}, \quad z \in \mathcal{C}_\ell, \quad \ell = 0, \dots, n, \tag{C.21}$$

where

$$R_{2n+2}(\lambda)^{1/2} = |R_{2n+2}(\lambda)^{1/2}| \times \begin{cases} -1 & \text{for } \operatorname{Re}(\lambda) \in (\widetilde{E}_n, \infty), \\ (-1)^{n+\ell+1} & \text{for } \lambda \in (\operatorname{Re}(\widetilde{E}_\ell), \operatorname{Re}(\widetilde{E}_{\ell+1})), \quad \ell = 0, \dots, n-1, \\ (-1)^n & \text{for } \lambda \in (-\infty, \operatorname{Re}(\widetilde{E}_0)). \end{cases} \tag{C.22}$$

The square root branches in (C.14) and (C.15) then are defined by

$$(z - E_{m_0})^{1/2} = |(z - E_{m_0})^{1/2}| \exp\big((i/2)\arg(z - E_{m_0})\big),$$

where, for n even,

$$\arg(z - \widetilde{E}_\ell) \in \begin{cases} (\frac{\pi}{2}, \frac{5\pi}{2}], & \ell \text{ even}, \\ [\frac{\pi}{2}, \frac{5\pi}{2}), & \ell \text{ odd}, \end{cases} \qquad \arg(z - \widetilde{\widetilde{E}}_\ell) \in \begin{cases} [-\frac{\pi}{2}, \frac{3\pi}{2}), & \ell \text{ even}, \\ (-\frac{\pi}{2}, \frac{3\pi}{2}], & \ell \text{ odd}, \end{cases}$$

and for n odd,

$$\arg(z - \widetilde{E}_\ell) \in \begin{cases} [\frac{\pi}{2}, \frac{5\pi}{2}), & \ell \text{ even}, \\ (\frac{\pi}{2}, \frac{5\pi}{2}], & \ell \text{ odd}, \end{cases} \qquad \arg(z - \widetilde{\widetilde{E}}_\ell) \in \begin{cases} (-\frac{\pi}{2}, \frac{3\pi}{2}], & \ell \text{ even}, \\ [-\frac{\pi}{2}, \frac{3\pi}{2}), & \ell \text{ odd}. \end{cases}$$

Here,

$$\begin{aligned} &\left(\prod_{\substack{m=0 \\ m \ne m_0}}^{2n+1} (E_{m_0} - E_m + \zeta^2)\right)^{1/2} \\ &\quad = \exp\left((i/2) \sum_{\substack{m=0 \\ m \ne m_0}}^{2n+1} \arg(E_{m_0} - E_m)\right) \left| \left(\prod_{\substack{m=0 \\ m \ne m_0}}^{2n+1} (E_{m_0} - E_m)\right)^{1/2} \right| \\ &\quad\quad \times \left(1 + \frac{1}{2}\left(\sum_{\substack{m=0 \\ m \ne m_0}}^{2n+1} (E_{m_0} - E_m)^{-1}\right)\zeta^2 + O(\zeta^4)\right), \end{aligned} \tag{C.23}$$

where $\exp((i/2)\sum_{\substack{m=0\\ m\neq m_0}}^{2n+1}\arg(E_{m_0}-E_m))$ is determined by analytic continuation in (C.22).

In the following we return to the general case (C.1). Points $P\in\mathcal{K}_n\setminus\{P_{\infty_-},P_{\infty_+}\}$ are denoted by

$$P=(z,\sigma R_{2n+2}(z)^{1/2})=(z,y),\quad P\in\mathcal{K}_n\setminus\{P_{\infty_-},P_{\infty_+}\}$$

where

$$y(P)\underset{\zeta\to 0}{=}\mp\left(1-\frac{1}{2}\left(\sum_{m=0}^{2n+1}E_m\right)\zeta+O(\zeta^2)\right)\zeta^{-n-1}\text{ as }P\to P_{\infty_\pm},\quad \zeta=1/z \tag{C.24}$$

(i.e., we abbreviate $y(P)=\sigma R_{2n+2}(z)^{1/2}$). Moreover, we introduce the holomorphic sheet exchange map (involution)

$$*\colon\mathcal{K}_n\to\mathcal{K}_n,\quad P=(z,y)\mapsto P^*=(z,-y),\ P_{\infty_\pm}\mapsto P^*_{\infty_\pm}=P_{\infty_\mp} \tag{C.25}$$

and the two meromorphic projection maps

$$\tilde{\pi}:\mathcal{K}_n\to\mathbb{C}\cup\{\infty\},\quad P=(z,y)\mapsto z,\ P_{\infty_\pm}\mapsto\infty \tag{C.26}$$

and

$$y\colon\mathcal{K}_n\to\mathbb{C}\cup\{\infty\},\quad P=(z,y)\mapsto y,\ P_{\infty_\pm}\mapsto\infty. \tag{C.27}$$

The map $\tilde{\pi}$ has poles of order 1 at $P_{\infty_\pm}$, and y has poles of order $n+1$ at $P_{\infty_\pm}$. Moreover,

$$\tilde{\pi}(P^*)=\tilde{\pi}(P),\quad y(P^*)=-y(P),\quad P\in\mathcal{K}_n. \tag{C.28}$$

Thus, $\mathcal{K}_n$ is a two-sheeted branched covering of the Riemann sphere $\mathbb{CP}^1$ ($\cong\mathbb{C}\cup\{\infty\}$) branched at the $2n+2$ points $\{(E_m,0)\}_{m=0,\dots,2n+1}$. Moreover, $\mathcal{K}_n$ is compact (since $\tilde{\pi}$ is open and $\mathbb{CP}^1$ is compact), and $\mathcal{K}_n$ is hyperelliptic (since it admits the meromorphic function $\tilde{\pi}$ of degree two). In this context we denote the set of branch points of $\mathcal{K}_n$ by $\mathcal{B}(\mathcal{K}_n)$. Topologically, $\mathcal{K}_n$ is a sphere with n handles and hence has genus n.

For the rest of this appendix we assume $n\in\mathbb{N}$ and that $\mathcal{K}_n$ is a compact hyperelliptic Riemann surface of (topological) genus n (cf. (C.1) and (C.16)). In this case $\tilde{\pi}$ has two simple zeros at $(0,\pm R_{2n+2}(0)^{1/2})$ if $R_{2n+2}(0)\neq 0$ or a double zero at $(0,0)$ if $R_{2n+2}(0)^{1/2}=0$ (i.e., if $0\in\{E_m\}_{m=0,\dots,2n+1}$), and y has $2n+2$ simple zeros at $(E_m,0)$ for $m=0,\dots,2n+1$.

We introduce the upper and lower sheets by

$$\Pi_\pm=\{(z,\pm R_{2n+2}(z)^{1/2})\in\mathcal{M}_n\mid z\in\Pi\} \tag{C.29}$$

and the associated charts

$$\zeta_\pm \colon \Pi_\pm \to \Pi, \quad P \mapsto z. \tag{C.30}$$

In particular, the charts in (C.3)–(C.15) are chosen to be compatible with $\zeta_\pm$ wherever they overlap.

Using the local chart near $P_{\infty_\pm}$, one verifies that dz/y is a holomorphic differential on $\mathcal{K}_n$ with zeros of order $n-1$ at $P_{\infty_\pm}$, and hence

$$\eta_j = \frac{z^{j-1}dz}{y}, \quad j = 1, \dots, n \tag{C.31}$$

form a basis for the space of holomorphic differentials on $\mathcal{K}_n$. Upon introduction of the invertible matrix C in $\mathbb{C}^n$,

$$C = (C_{j,k})_{j,k=1,\dots,n}, \quad C_{j,k} = \int_{a_k} \eta_j, \tag{C.32}$$

$$\underline{c}(k) = (c_1(k), \dots, c_n(k)), \; c_j(k) = \left(C^{-1}\right)_{j,k}, \quad j,k = 1, \dots, n, \tag{C.33}$$

the normalized holomorphic differentials ω_j for $j = 1, \dots, n$ (cf. (A.13)),

$$\omega_j = \sum_{\ell=1}^{n} c_j(\ell)\eta_\ell, \quad \int_{a_k} \omega_j = \delta_{j,k}, \quad j,k = 1, \dots, n, \tag{C.34}$$

form a canonical basis for the space of holomorphic differentials on $\mathcal{K}_n$.

In the charts $(U_{P_{\infty_\pm}}, \zeta_{P_{\infty_\pm}})$ induced by $1/\tilde{\pi}$ near $P_{\infty_\pm}$, one infers

$$\begin{aligned}
\underline{\omega} = (\omega_1, \dots, \omega_n) &= \pm \sum_{j=1}^{n} \underline{c}(j) \frac{\zeta^{n-j}\, d\zeta}{\left(\prod_{m=0}^{2n+1}(1 - E_m\zeta)\right)^{1/2}} \\
&\underset{\zeta\to 0}{=} \pm\left(\sum_{q=0}^{\infty} \sum_{k=1}^{n} \underline{c}(k)\hat{c}_{k-n+q}(\underline{E})\zeta^q \right) d\zeta \qquad \text{(C.35)} \\
&\underset{\zeta\to 0}{=} \pm\left(\underline{c}(n) + \left(\frac{1}{2}\underline{c}(n) \sum_{m=0}^{2n+1} E_m + \underline{c}(n-1)\right)\zeta + O(\zeta^2)\right) d\zeta \text{ as } P \to P_{\infty_\pm}, \\
&\qquad \zeta = 1/z,
\end{aligned}$$

where $\underline{E} = (E_0, \dots, E_{2n+1})$, and we used (C.24) and

$$\left(\prod_{m=0}^{2n+1} (1 - E_m\zeta) \right)^{-1/2} = \sum_{k=0}^{\infty} \hat{c}_k(\underline{E})\zeta^k$$

for $\zeta \in \mathbb{C}$ such that $|\zeta|^{-1} > \max\{|E_0|, \dots, |E_{2n+1}|\}$ with

$$\begin{aligned}
&\hat{c}_{-k}(\underline{E}) = 0, \quad k \in \mathbb{N}, \quad \hat{c}_0(\underline{E}) = 1, \\
&\hat{c}_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_{2n+1}=0 \\ j_0+\dots+j_{2n+1}=k}}^{k} \frac{(2j_0)! \cdots (2j_{2n+1})!}{2^{2k}(j_0!)^2 \cdots (j_{2n+1}!)^2} E_0^{j_0} \cdots E_{2n+1}^{j_{2n+1}}, \quad k \in \mathbb{N}.
\end{aligned} \tag{C.36}$$

Combining (A.22) and (C.35), one computes for the vector $\underline{U}^{(2)}_{\pm,q}$ of b-periods of $\omega^{(2)}_{P_{\infty_\pm},q}/(2\pi i)$, the normalized differential of the second kind, holomorphic on $\mathcal{K}_n \setminus \{P_{\infty_\pm}\}$, with principal part $\zeta^{-q-2}d\zeta/(2\pi i)$,

$$\underline{U}^{(2)}_{\pm,q} = \big(U^{(2)}_{\pm,q,1}, \dots, U^{(2)}_{\pm,q,n}\big), \tag{C.37}$$

$$U^{(2)}_{\pm,q,j} = \frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_{\infty_\pm},q} = \frac{\pm 1}{q+1}\sum_{k=1}^{n} c_j(k)\hat{c}_{k-n+q}(\underline{E}), \quad j = 1, \dots, n,\ q \in \mathbb{N}_0.$$

In the special self-adjoint case (C.17), the matrix τ of b-periods satisfies, in addition to (A.16),

$$\tau = iT, \quad T > 0 \tag{C.38}$$

since

$$C_{j,k} = \int_{a_k} \eta_j = 2\int_{E_{2k-2}}^{E_{2k-1}} \frac{z^{j-1}dz}{R_{2n+2}(z)^{1/2}} \in \mathbb{R} \tag{C.39}$$

and

$$\int_{b_k} \eta_j = 2\int_{E_{2k-1}}^{E_{2k}} \frac{z^{j-1}dz}{R_{2n+2}(z)^{1/2}} \in i\mathbb{R}. \tag{C.40}$$

Next, assuming $0 \notin \{E_m\}_{m=0,\dots,2n+1}$, one then computes in the charts $(U_{P_{0_\pm}}, \zeta_{P_{0_\pm}})$ induced by $\tilde{\pi}$ near $P_{0_\pm} = (0, y(P_{0_\pm}))$,

$$\underline{\omega} \underset{\zeta\to 0}{=} \pm\frac{1}{\widehat{Q}^{1/2}}\bigg(\underline{c}(1) + \bigg(\frac{1}{2}\underline{c}(1)\sum_{m=0}^{2n+1} E_m^{-1} + \underline{c}(2)\bigg)\zeta + O(\zeta^2)\bigg)d\zeta \text{ as } P \to P_{0_\pm},$$
$$\widehat{Q}^{1/2} = \bigg(\prod_{m=0}^{2n+1} E_m\bigg)^{1/2}, \quad \zeta = z, \tag{C.41}$$

using

$$y(P) \underset{\zeta\to 0}{=} \pm\widehat{Q}^{1/2} + O(\zeta) \text{ as } P \to P_{0_\pm}, \quad \zeta = z \tag{C.42}$$

with the sign of $\widehat{Q}^{1/2}$ determined by the compatibility of charts.

Finally, if $E_0 = 0$, $E_m \neq 0$, $m = 1, \dots, 2n+1$, one computes in the chart (U_{P_0}, ζ_{P_0}) induced by $\tilde{\pi}^{1/2}$ near $P_0 = (0, 0)$,

$$\underline{\omega} \underset{\zeta\to 0}{=} -2i\bigg(\frac{\underline{c}(1)}{\widetilde{Q}^{1/2}} + O(\zeta^2)\bigg)d\zeta \text{ as } P \to P_0, \quad \widetilde{Q}^{1/2} = \bigg(\prod_{m=1}^{2n+1} E_m\bigg)^{1/2}, \tag{C.43}$$
$$\zeta = \sigma z^{1/2},\ \sigma \in \{1, -1\}$$

using

$$y(P) \underset{\zeta \to 0}{=} i\widetilde{Q}^{1/2}\zeta + O(\zeta^3) \text{ as } P \to P_0, \quad \zeta = \sigma z^{1/2},\ \sigma \in \{1, -1\} \tag{C.44}$$

with the sign of $\widetilde{Q}^{1/2}$ determined by the compatibility of charts.

Explicit formulas for normal differentials of the third kind, $\omega^{(3)}{}_{Q_1,Q_2}$, with simple poles at Q_1 and Q_2, corresponding residues $+1$ and -1, vanishing a-periods, and holomorphic on $\mathcal{K}_n \setminus \{Q_1, Q_2\}$, can easily be found. One obtains

$$\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = \frac{z^n\,dz}{y} + \sum_{j=1}^{n} \gamma_j \omega_j = \frac{1}{y}\prod_{j=1}^{n}(z-\lambda_j)\,dz, \tag{C.45}$$

$$\omega^{(3)}_{P_1,P_{\infty_+}} = \frac{y+y_1}{z-z_1}\frac{dz}{2y} - \frac{1}{2y}\prod_{j=1}^{n}(z-\tilde{\lambda}_j)\,dz, \tag{C.46}$$

$$\omega^{(3)}_{P_1,P_{\infty_-}} = \frac{y+y_1}{z-z_1}\frac{dz}{2y} + \frac{1}{2y}\prod_{j=1}^{n}(z-\lambda'_j)\,dz, \tag{C.47}$$

$$\omega^{(3)}_{P_1,P_2} = \left(\frac{y+y_1}{z-z_1} - \frac{y+y_2}{z-z_2}\right)\frac{dz}{2y} + \frac{\lambda''_n}{y}\prod_{j=1}^{n-1}(z-\lambda''_j)\,dz, \tag{C.48}$$

$$P_1, P_2 \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\},$$

where $\gamma_j, \lambda_j, \tilde{\lambda}_j, \lambda'_j, \lambda''_j \in \mathbb{C}$, $j = 1, \dots, n$, are uniquely determined by the requirement of vanishing a-periods, and we abbreviated $P_j = (z_j, y_j)$, $j = 1, 2$. (If $n = 0$ in (C.45)–(C.47) and $n = 1$ in (C.48), we use the standard conventions that products and sums over empty index sets are replaced by 1 and 0, respectively; if $n = 0$, the product in (C.48) is replaced by 0.)

Next, we turn to the theta function representation of symmetric functions of values of a meromorphic function as discussed in Remark A.29 in the current case of AKNS-type hyperelliptic Riemann surfaces. The choice $f = \tilde{\pi}$ in (A.62) then yields, after a standard residue calculation at $P_{\infty_\pm}$,

$$\begin{aligned}\sum_{j=1}^{n}\mu_j &= \sum_{j=1}^{n}\int_{a_j}\tilde{\pi}\,\omega_j \\ &\quad + \sum_{j=1}^{n} U^{(2)}_{+,0,j}\partial_{w_j}\ln\left(\frac{\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_+}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{w}\big)}{\theta\big(\underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P_{\infty_-}) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}}) + \underline{w}\big)}\right)\Bigg|_{\underline{w}=0},\end{aligned} \tag{C.49}$$

where $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$, $\hat{\mu}_j = (\mu_j, y(\hat{\mu}_j)) \in \mathcal{K}_n$, $j = 1, \dots, n$, assuming $\mathcal{D}_{\underline{\hat{\mu}}} \in \operatorname{Sym}^n(\overline{\mathcal{K}_n})$ to be nonspecial and using

$$\underline{A}_{Q_0}(P) - \underline{A}_{Q_0}(P_{\infty_\pm}) \underset{\zeta\to 0}{=} \pm\underline{U}^{(2)}_{+,0}\zeta + O(\zeta^2) \text{ as } P \to P_{\infty_\pm}, \quad \underline{U}^{(2)}_{+,0} = \underline{c}(n) \tag{C.50}$$

according to (C.35)–(C.37). Here $Q_0 \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ denotes an appropriate base point. In the present hyperelliptic context, the constant $\sum_{j=1}^{n}\int_{a_j}\tilde{\pi}\,\omega_j$ can be related to the zeros $\{\lambda_j\}_{j=1,\dots,n}$ of the normal differential of the third kind, $\omega^{(3)}_{P_{\infty_+},P_{\infty_-}}$, as follows,

$$\sum_{j=1}^{n}\int_{a_j}\tilde{\pi}\,\omega_j=\sum_{j=1}^{n}\lambda_j.$$

This will be proven in Appendix F (cf. (F.68) for $k=n$ and (F.70)). Hence, one finally obtains

$$\begin{aligned}\sum_{j=1}^{n}\mu_j=\sum_{j=1}^{n}\lambda_j& \\ &+\sum_{j=1}^{n}U^{(2)}_{+,0,j}\partial_{w_j}\ln\left(\frac{\theta\big(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{\infty_+})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}})+\underline{w}\big)}{\theta\big(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{\infty_-})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}})+\underline{w}\big)}\right)\Bigg|_{\underline{w}=0}.\end{aligned}\tag{C.51}$$

If, in addition, $\mathcal{D}_{\underline{\hat{\mu}}}$ depends on a parameter $x\in\Omega$, $\Omega\subseteq\mathbb{R}$ an open interval, satisfying the linearization property

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x)})=\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(x_0)})+c\,\underline{U}^{(2)}_{+,0}(x-x_0),\quad x,x_0\in\Omega$$

for some constant $c\in\mathbb{C}\setminus\{0\}$, as verified, for instance, in the AKNS context (cf. (3.105) and (3.215)), where $c=-2i$, one can rewrite (C.51) in the form

$$\sum_{j=1}^{n}\mu_j(x)=\sum_{j=1}^{n}\lambda_j+c^{-1}\,\partial_x\ln\left(\frac{\theta\big(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{\infty_+})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}})\big)}{\theta\big(\underline{\Xi}_{Q_0}-\underline{A}_{Q_0}(P_{\infty_-})+\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}})\big)}\right),\quad x\in\Omega.\tag{C.52}$$

Combined with trace formulas of the type (3.81) and (3.83), (C.52) confirms the theta function representations (3.107) and (3.108). A systematic approach to elementary symmetric functions of $\mu_1,\dots,\mu_n$ will be discussed in Appendix F.

Next we provide a brief illustration of the Riemann–Roch theorem in connection with AKNS-type compact hyperelliptic Riemann surfaces $\mathcal{K}_n$ of genus n (cf. (C.1) and (C.16)) and explicitly determine a basis for the vector space $\mathcal{L}(-k\mathcal{D}_{P_{\infty_-}}-m(k)\mathcal{D}_{P_{\infty_+}}-\mathcal{D}_{\underline{\hat{\mu}}(x_0)})$ for $m(k)=\max(0,k-2)$ and $k\in\mathbb{N}_0$. We refer to (A.28) for the definition of $\mathcal{L}(\mathcal{D})$ and to Theorem A.13 for the Riemann–Roch theorem. In addition, we use the short-hand notation

$$\begin{aligned}k\mathcal{D}_{P_{\infty_-}}+m(k)\mathcal{D}_{P_{\infty_+}}+\mathcal{D}_{\underline{\hat{\mu}}(x_0)}&=\sum_{\ell=1}^{k}\mathcal{D}_{P_{\infty_-}}+\sum_{\ell=1}^{m(k)}\mathcal{D}_{P_{\infty_+}}+\sum_{j=1}^{n}\mathcal{D}_{\hat{\mu}_j(x_0)},\\ &k\in\mathbb{N}_0,\quad \underline{\hat{\mu}}(x_0)=\{\hat{\mu}_1(x_0),\dots,\hat{\mu}_n(x_0)\}\in\operatorname{Sym}^n(\mathcal{K}_n)\end{aligned}$$

and recall that

$$\begin{aligned}&\mathcal{L}(-k\mathcal{D}_{P_{\infty_-}}-m(k)\mathcal{D}_{P_{\infty_+}}-\mathcal{D}_{\underline{\hat{\mu}}(x_0)})\\ &\quad=\{f\in\mathcal{M}(\mathcal{K}_n)\,|\,(f)+k\mathcal{D}_{P_{\infty_-}}+m(k)\mathcal{D}_{P_{\infty_+}}+\mathcal{D}_{\underline{\hat{\mu}}(x_0)}\geq 0\},\quad k\in\mathbb{N}_0.\end{aligned}$$

With $\phi(P,x)$ defined as in (3.56) one obtains the following result.

Theorem C.1 *Assume $\mathcal{D}_{\underline{\hat{\mu}}(x_0)}$ to be nonspecial (i.e., $i(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = 0$) and of degree $n \in \mathbb{N}$. For $k \in \mathbb{N}_0$, a basis for the vector space $\mathcal{L}(-k\mathcal{D}_{P_{\infty_-}} - m(k)\mathcal{D}_{P_{\infty_+}} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$ is given by*

$$\begin{aligned} &\{1\}, \quad k = 0, \\ &\{\tilde{\pi}^\ell\}_{\ell=0,\dots,m(k)} \cup \{\tilde{\pi}^\ell \phi(\,\cdot\,, x_0)\}_{\ell=0,\dots,k-1}, \quad k \in \mathbb{N}. \end{aligned} \tag{C.53}$$

Proof The elements in (C.53) are easily seen to be linearly independent and belonging to $\mathcal{L}(-k\mathcal{D}_{P_{\infty_-}} - m(k)\mathcal{D}_{P_{\infty_+}} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)})$. It remains to be shown that they are maximal. Since $i(\mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = i(k\mathcal{D}_{\infty_-} + m(k)\mathcal{D}_{P_{\infty_+}} + \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = 0$, the Riemann–Roch Theorem A.13 implies $r(-k\mathcal{D}_{P_{\infty_-}} - m(k)\mathcal{D}_{P_{\infty_+}} - \mathcal{D}_{\underline{\hat{\mu}}(x_0)}) = k + m(k) + 1$ proving (C.53). □

Replacing ϕ by ϕ^{-1}, one can discuss $\mathcal{L}(-k\mathcal{D}_{P_{\infty_+}} - m(k)\mathcal{D}_{P_{\infty_-}} - \mathcal{D}_{\underline{\hat{\nu}}(x_0)})$ for $k \in \mathbb{N}_0$ in an analogous fashion.

Finally, we formulate the following Riemann–Roch-type uniqueness results for the Baker–Akhiezer functions needed in Chapters 3 and 4. In the following, Q_0 is an appropriate base point on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ (in the Thirring case we choose in addition $Q_0 \notin \{P_{0,+}, P_{0,-}\}$). We start with the AKNS case.

Lemma C.2 *Let $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ and $(x, t_r), (x_0, t_{0,r}) \in \Omega$ for some $\Omega \subseteq \mathbb{R}^2$. Assume $\psi(\,\cdot\,, x, t_r)$, $(x, t_r) \in \Omega$, to be meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$ with essential singularities at $P_{\infty_+}, P_{\infty_-}$ such that $\tilde{\psi}(\,\cdot\,, x, t_r)$, defined by*

$$\tilde{\psi}(P, x, t_r) = \psi(P, x, t_r) \exp\bigg(- i(x - x_0) \int_{Q_0}^{P} \Omega_0^{(2)} - i(t_r - t_{0,r}) \int_{Q_0}^{P} \widetilde{\Omega}_r^{(2)} \bigg), \tag{C.54}$$

is meromorphic on $\mathcal{K}_n$ and its divisor satisfies

$$(\tilde{\psi}(\,\cdot\,, x, t_r)) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x_0, t_{0,r})}$$

for some positive divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0, t_{0,r})}$ of degree n. Here $\Omega_0^{(2)}$ and $\widetilde{\Omega}_r^{(2)}$ are defined in (3.96) and (3.207), and the path of integration in (C.54) *is chosen identical to that in the Abel maps* (A.34) *and* (A.35).[1] *Define a divisor $\mathcal{D}_0(x, t_r)$ by*

$$(\tilde{\psi}(\,\cdot\,, x, t_r)) = \mathcal{D}_0(x, t_r) - \mathcal{D}_{\underline{\hat{\mu}}(x_0, t_{0,r})}.$$

Then

$$\mathcal{D}_0(x, t_r) \in \operatorname{Sym}^n(\mathcal{K}_n), \quad \mathcal{D}_0(x, t_r) > 0, \quad \deg(\mathcal{D}_0(x, t_r)) = n.$$

[1] This is to avoid multi-valued expressions and hence the use of the multiplicative Riemann–Roch theorem.

Moreover, if $\mathcal{D}_0(x,t_r)$ is nonspecial for all $(x,t_r) \in \Omega$, that is, if

$$i(\mathcal{D}_0(x,t_r)) = 0, \quad (x,t_r) \in \Omega,$$

then $\psi(\,\cdot\,,x,t_r)$ is unique up to a constant multiple (which may depend on the parameters $(x,t_r) \in \Omega$).

Next we turn to the case of the Thirring system.

Lemma C.3 *Let $P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ and $(x,t), (x_0,t_0) \in \Omega$ for some $\Omega \subseteq \mathbb{R}^2$. Assume $\psi(\,\cdot\,,x,t)$, $(x,t) \in \Omega$, to be meromorphic on $\mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}\}$ with essential singularities at $P_{\infty_+}, P_{\infty_-}, P_{0,+}, P_{0,-}$ such that $\tilde{\psi}(\,\cdot\,,x,t)$, defined by*

$$\tilde{\psi}(P,x,t) = \psi(P,x,t)\exp\left(i(x-x_0)\int_{Q_0}^{P}\Omega^{(2)}_{\infty,0} - i(t-t_0)\int_{Q_0}^{P}\Omega^{(2)}_{0,0}\right), \tag{C.55}$$

is meromorphic on $\mathcal{K}_n$ and its divisor satisfies

$$(\tilde{\psi}(\,\cdot\,,x,t)) \geq -\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,})}$$

*for some positive divisor $\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}$ of degree n. Here $\Omega^{(2)}_{\infty,0}$ and $\Omega^{(2)}_{0,0}$ are defined in (*4.215*), and (*4.216*), and the path of integration in* (C.55) *is chosen identical to that in the Abel maps* (A.34) *and* (A.35)*. Define a divisor $\mathcal{D}_0(x,t)$ by*

$$(\tilde{\psi}(\,\cdot\,,x,t)) = \mathcal{D}_0(x,t) - \mathcal{D}_{\underline{\hat{\mu}}(x_0,t_0)}.$$

Then

$$\mathcal{D}_0(x,t) \in \operatorname{Sym}^n(\mathcal{K}_n), \quad \mathcal{D}_0(x,t) > 0, \quad \deg(\mathcal{D}_0(x,t)) = n.$$

Moreover, if $\mathcal{D}_0(x,t)$ is nonspecial for all $(x,t) \in \Omega$, that is, if

$$i(\mathcal{D}_0(x,t)) = 0, \quad (x,t) \in \Omega,$$

then $\psi(\,\cdot\,,x,t)$ is unique up to a constant multiple (which may depend on the parameters $(x,t) \in \Omega$).

Since the proofs of Lemmas C.2 and C.3 are completely analogous to that of Lemma B.2, we omit further details.

Appendix D
Asymptotic Spectral Parameter Expansions and Nonlinear Recursion Relations

Theorems are stable; proofs are not.
Raphael Høegh-Krohn

In this appendix we discuss asymptotic spectral parameter expansions for quantities such as F_n/y as well as nonlinear recursion relations for the corresponding homogeneous coefficients $\hat{f}_\ell$ and analogous quantities fundamental to the polynomial recursion formalism used for all $(1+1)$-dimensional integrable models.

Before individually discussing several completely integrable systems, we start with the following elementary results (which are consequences of the binomial expansion). Let

$$\begin{aligned}&\{E_m\}_{m=0,\dots,N} \subset \mathbb{C} \text{ for some } N \in \mathbb{N}_0 \\ &\text{and } \eta \in \mathbb{C} \text{ such that } |\eta| < \min\{|E_0|^{-1}, \dots, |E_N|^{-1}\}.\end{aligned}$$

Then

$$\left(\prod_{m=0}^{N}\big(1 - E_m\eta\big)\right)^{-1/2} = \sum_{k=0}^{\infty} \hat{c}_k(\underline{E})\eta^k, \tag{D.1}$$

where

$$\begin{aligned}&\hat{c}_0(\underline{E}) = 1, \\ &\hat{c}_k(\underline{E}) = \sum_{\substack{j_0,\dots,j_N=0 \\ j_0+\cdots+j_N=k}}^{k} \frac{(2j_0)!\cdots(2j_N)!}{2^{2k}(j_0!)^2\cdots(j_N!)^2} E_0^{j_0}\cdots E_N^{j_N}, \quad k \in \mathbb{N}.\end{aligned} \tag{D.2}$$

The first few coefficients explicitly read

$$\begin{aligned}&\hat{c}_0(\underline{E}) = 1, \quad \hat{c}_1(\underline{E}) = \frac{1}{2}\sum_{m=0}^{N} E_m, \\ &\hat{c}_2(\underline{E}) = \frac{1}{4}\sum_{\substack{m_1,m_2=0 \\ m_1<m_2}}^{N} E_{m_1}E_{m_2} + \frac{3}{8}\sum_{m=0}^{N} E_m^2, \quad \text{etc.}\end{aligned} \tag{D.3}$$

Similarly,

$$\left(\prod_{m=0}^{N}\big(1-E_m\eta\big)\right)^{1/2}=\sum_{k=0}^{\infty}c_k(\underline{E})\eta^k, \tag{D.4}$$

where

$$\begin{aligned} &c_0(\underline{E})=1,\\ &c_k(\underline{E})=(-1)^{N+1}\sum_{\substack{j_0,\dots,j_N=0\\ j_0+\cdots+j_N=k}}^{k}\frac{(2j_0)!\cdots(2j_N)!}{2^{2k}(j_0!)^2\cdots(j_N!)^2(2j_0-1)\cdots(2j_N-1)}\\ &\qquad\qquad\times E_0^{j_0}\cdots E_N^{j_N},\quad k\in\mathbb{N}. \end{aligned} \tag{D.5}$$

The first few coefficients explicitly are given by

$$\begin{aligned} &c_0(\underline{E})=1,\quad c_1(\underline{E})=-\frac{1}{2}\sum_{m=0}^{N}E_m,\\ &c_2(\underline{E})=\frac{1}{4}\sum_{\substack{m_1,m_2=0\\ m_1<m_2}}^{N}E_{m_1}E_{m_2}-\frac{1}{8}\sum_{m=0}^{N}E_m^2,\quad \text{etc.} \end{aligned} \tag{D.6}$$

We start with the KdV case, where $N=2n$ for some $n\in\mathbb{N}_0$.

Theorem D.1 *Assume* $u\in C^\infty(\mathbb{R})$*,* $\text{s-KdV}_n(u)=0$*, and suppose* $P=(z,y)\in\mathcal{K}_n\setminus\{P_\infty\}$*. Then* F_n/y *has the following convergent expansion as* $P\to P_\infty$*,*

$$\frac{F_n(z)}{y}\underset{\zeta\to 0}{=}\sum_{\ell=0}^{\infty}\hat{f}_\ell\,\zeta^{2\ell+1}, \tag{D.7}$$

where $\zeta=\sigma/z^{1/2}$ *is the local coordinate near* P_∞ *described in* (B.7)–(B.10) *and* $\hat{f}_\ell$ *are the homogeneous coefficients* f_ℓ *in (*1.6*). In particular,* $\hat{f}_\ell$ *can be computed from the nonlinear recursion relation*

$$\begin{aligned} &\hat{f}_0=1,\quad \hat{f}_1=\tfrac{1}{2}u,\\ &\hat{f}_{\ell+1}=-\tfrac{1}{2}\sum_{k=1}^{\ell}\hat{f}_k\hat{f}_{\ell+1-k}+\tfrac{1}{2}\sum_{k=0}^{\ell}\big(u\hat{f}_k\hat{f}_{\ell-k}+\tfrac{1}{4}\hat{f}_{k,x}\hat{f}_{\ell-k,x}-\tfrac{1}{2}\hat{f}_{k,xx}\hat{f}_{\ell-k}\big),\ \ \ell\in\mathbb{N}. \end{aligned} \tag{D.8}$$

Moreover, one infers for the E_m*-dependent integration constants* c_ℓ*,* $\ell=0,\dots,n$*, in* F_n *that*

$$c_\ell=c_\ell(\underline{E}),\quad \ell=0,\dots,n \tag{D.9}$$

and[1]

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E})\hat{f}_k, \quad \ell = 0, \dots, n, \tag{D.10}$$

$$\hat{f}_\ell = \sum_{k=0}^{\ell\wedge n} \hat{c}_{\ell-k}(\underline{E}) f_k, \quad \ell \in \mathbb{N}_0. \tag{D.11}$$

Proof In the course of this proof it will be convenient to introduce the notion of a degree, $\deg(\,\cdot\,)$, to effectively distinguish between homogeneous and nonhomogeneous quantities. Thus, assuming (1.1), one defines

$$\deg(u) = 2, \quad \deg(\partial_x) = 1,$$

implying

$$\deg(\hat{f}_\ell) = 2\ell, \quad \ell \in \mathbb{N}_0 \tag{D.12}$$

using the linear recursion relation (1.4) and induction on ℓ. Next, dividing F_n by $R_{2n+1}^{1/2}$ (temporarily fixing the branch of $R_{2n+1}(z)^{1/2}$ as $z^{n+(1/2)}$ near infinity), one obtains

$$\frac{F_n(z)}{R_{2n+1}(z)^{1/2}} \underset{|z|\to\infty}{=} \left(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\right)\left(\sum_{\ell=0}^{n} f_\ell z^{-\ell-(1/2)}\right) = \sum_{\ell=0}^{\infty} \check{f}_\ell z^{-\ell-(1/2)} \tag{D.13}$$

for some coefficients $\check{f}_\ell$ to be determined next. Dividing (1.13) by R_{2n+1} and inserting the expansion (D.13) into the resulting equation then yield the recursion relation (D.8) (with $\hat{f}_\ell$ replaced by $\check{f}_\ell$). The sign of $\check{f}_0$ has been chosen such that $\check{f}_0 = \hat{f}_0 = 1$. Moreover, one confirms inductively using the nonlinear recursion relation (D.8) satisfied by $\check{f}_\ell$ that

$$\deg(\check{f}_\ell) = 2\ell, \quad \ell \in \mathbb{N}_0. \tag{D.14}$$

Differentiating $\check{f}_\ell$ with respect to x (using (D.8)), one proves inductively that $\check{f}_\ell$ also satisfy the linear recursion relation (1.4). Hence, (D.12) and (D.14) imply

$$\check{f}_\ell = \hat{f}_\ell \text{ for all } \ell \in \mathbb{N}_0.$$

Thus, we proved

$$\frac{F_n(z)}{R_{2n+1}(z)^{1/2}} \underset{|z|\to\infty}{=} \left(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\right)\left(\sum_{\ell=0}^{n} f_\ell z^{-\ell-(1/2)}\right) = \sum_{\ell=0}^{\infty} \hat{f}_\ell z^{-\ell-(1/2)} \tag{D.15}$$

[1] $m \wedge n = \min\{m, n\}$.

and hence (D.7). A comparison of coefficients in (D.15) then proves (D.11). Next, multiplying (D.1) and (D.4), a comparison of coefficients of η^{-k} yields

$$\sum_{\ell=0}^{k} \hat{c}_{k-\ell}(\underline{E})c_\ell(\underline{E}) = \delta_{k,0}, \quad k \in \mathbb{N}_0. \tag{D.16}$$

Hence, one computes

$$\begin{aligned}\sum_{m=0}^{\ell} c_{\ell-m}(\underline{E})\hat{f}_m &= \sum_{m=0}^{\ell}\sum_{k=0}^{m} c_{\ell-m}(\underline{E})\hat{c}_{m-k}(\underline{E})f_k = \sum_{k=0}^{\ell}\sum_{p=k}^{\ell} c_{\ell-p}(\underline{E})\hat{c}_{p-k}(\underline{E})f_k \\ &= \sum_{k=0}^{\ell}\left(\sum_{m=0}^{\ell-k} c_{\ell-k-m}(\underline{E})\hat{c}_m(\underline{E})\right)f_k = f_\ell, \quad \ell = 0,\dots,n,\end{aligned}$$

applying (D.16). Hence one obtains (D.10) and thus (D.9). □

Theorem D.1 has interesting consequences for the asymptotic spectral parameter expansion of the Green's function $G(z,x,x')$ of Schrödinger operators L as described in the following remark.

Remark D.2 Let $L = -d^2/dx^2 + u$ (with $u \in L^\infty(\mathbb{R})$ not necessarily an algebro-geometric potential) and assume that for some $z \in \mathbb{C}$, and all $x_0 \in \mathbb{R}$, $\psi_\pm(z,\,\cdot\,) \in L^2((x_0,\pm\infty))$ satisfy $(L-z)\psi_\pm = 0$. Then the Green's function $G(z,x,x')$ of L (i.e., the integral kernel of the resolvent $(L-z)^{-1}$, identifying L and its $L^2(\mathbb{R})$-realization as a closed linear operator for simplicity) is given by

$$G(z,x,x') = \frac{1}{W(\psi_+(z),\psi_-(z))}\begin{cases}\psi_+(z,x)\psi_-(z,x'), & x \ge x',\\ \psi_-(z,x)\psi_+(z,x'), & x \le x',\end{cases} \tag{D.17}$$

where the Wronskian

$$W(\psi_+(z),\psi_-(z)) = \psi_+(z,x)\psi'_-(z,x) - \psi_-(z,x)\psi'_+(z,x)$$

is independent of x.

In the special algebro-geometric context we next replace

$$\psi_+(z,x) \text{ by } \psi(P,x,x_0), \quad \psi_-(z,x) \text{ by } \psi(P^*,x,x_0) \tag{D.18}$$

and

$$\begin{aligned}W(\psi_+(z),\psi_-(z)) \text{ by } W(\psi(P,\,\cdot\,,x_0),\psi(P^*,\,\cdot\,,x_0)) &= \begin{vmatrix} 1 & 1 \\ \phi(P,x_0) & \phi(P^*,x_0)\end{vmatrix} \\ &= \phi(P^*,x_0) - \phi(P,x_0) = \frac{-2iy}{F_n(z,x_0)}\end{aligned} \tag{D.19}$$

since the Wronskian is x-independent. Substituting (D.18) and (D.19) into (D.17), with the result denoted by $G(P,x,x')$ then yields the following for the diagonal

Green's function $g(P,x) = G(P,x,x)$,

$$g(P,x) = \frac{iF_n(z,x)}{2y}, \quad P = (z,y) \in \mathcal{K}_n \setminus \{P_\infty\}, \tag{D.20}$$

which is the basic quantity encountered in (D.7). In particular, in the special case where u is real-valued, the branch of $g(P,x)$ for $P \in \Pi_+$ is the Green's function $G(z,x,x')$ of a closed realization of L in $L^2(\mathbb{R})$ on the diagonal, that is, for $x = x'$ (cf. Appendix J for more details.) From (1.12) we see that g satisfies the universal third-order linear differential equation

$$g_{xxx}(P) - 4(u-z)g_x(P) - 2u_x g(P) = 0 \tag{D.21}$$

as well as the universal second-order nonlinear differential equation

$$-2g_{xx}(P)g(P) + g_x(P)^2 + 4(u-z)g(P)^2 = 1. \tag{D.22}$$

The expansion (D.7) of $F_n(z)/y$ then determines the spectral parameter expansion of the diagonal Green's function $g(P,x)$ in (D.20) as $P \to P_\infty$. Even though (D.7) and (D.20) were derived in the special algebro-geometric context, we emphasize that the expansion of (D.20) as $P \to P_\infty$ only involves the homogeneous coefficients $\hat{f}_k$, which are universal differential polynomials in u. Thus, identifying $\psi_\pm(z,x)$ and $\psi(P,x,x_0)$, $\psi(P^*,x,x_0)$ as in (D.18) yields the universal asymptotic spectral parameter expansion of the diagonal Green's function $g(z,x)$ of L as $z \to \infty$ in the general (not necessarily algebro-geometric) case. In the general case, the spectral parameter expansion of $g(z,x)$ as $z \to \infty$ will only be an asymptotic expansion valid in appropriate regions exterior to the spectrum of L. Analogous results apply in the case in which u is complex-valued, but then the actual determination of branches of $g(P,x)$ is more intricate.

The spectral theoretic content of the polynomial F_n is clearly displayed in (D.20) and especially in (J.32)–(J.48) of Appendix J.

Completely analogous asymptotic spectral parameter expansions are possible for K^β_{n+1}. In particular, we refer to the corresponding diagonal Green's function Γ^β discussed in Section 1.5 and Appendix J.

The sGmKdV case, where $N = 2n$ for some $n \in \mathbb{N}_0$ follows from the KdV results in Theorem D.1, identifying the KdV potential u and the expression $-(1/4) \times (u_x^2 + 2iu_{xx})$ (cf. (2.12)) in the sGmKdV context. Hence we omit the corresponding details.

The AKNS case, where $N = 2n+1$ for some $n \in \mathbb{N}_0$, is treated next.

Theorem D.3 *Assume* $p, q \in C^\infty(\mathbb{R})$*,* $\text{s-AKNS}_n(p,q) = 0$*, and suppose* $P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$*. Then* F_n/y*,* G_{n+1}/y*, and* H_n/y *have the following* *convergent expansions as* $P \to P_{\infty_\pm}$,

$$\frac{F_n(z)}{y} \underset{\zeta\to 0}{=} \mp \sum_{\ell=0}^{\infty} \hat{f}_\ell\, \zeta^{\ell+1}, \quad \frac{G_{n+1}(z)}{y} \underset{\zeta\to 0}{=} \mp \sum_{\ell=0}^{\infty} \hat{g}_\ell\, \zeta^{\ell}, \quad \frac{H_n(z)}{y} \underset{\zeta\to 0}{=} \mp \sum_{\ell=0}^{\infty} \hat{h}_\ell\, \zeta^{\ell+1}, \tag{D.23}$$

where $\zeta = 1/z$ *is the local coordinate near* $P_{\infty_\pm}$ *described in* (C.7)–(C.11) *and* $\hat{f}_\ell$ *and* $\hat{h}_\ell$ *are the homogeneous coefficients* f_ℓ *and* h_ℓ *in (*3.9*)–(*3.11*). In particular,* $\hat{f}_\ell$ *and* $\hat{h}_\ell$ *can be computed from the nonlinear recursion relations*

$$\begin{aligned}
&\hat{f}_0 = -iq, \quad \hat{f}_1 = \frac{1}{2}q_x,\\
&\hat{f}_\ell = \sum_{k=0}^{\ell-2}\Big(-\frac{i}{4q}\hat{f}_k\hat{f}_{\ell-2-k,xx} + \frac{iq_x}{4q^2}\hat{f}_k\hat{f}_{\ell-2-k,x} + \frac{i}{8q}\hat{f}_{k,x}\hat{f}_{\ell-2-k,x}\\
&\qquad + \frac{ip}{2}\hat{f}_k\hat{f}_{\ell-2-k}\Big) - \frac{q_x}{2q^2}\sum_{k=0}^{\ell-1}\hat{f}_k\hat{f}_{\ell-1-k} - \frac{i}{2q}\sum_{k=1}^{\ell-1}\hat{f}_k\hat{f}_{\ell-k}, \quad \ell \geq 2,
\end{aligned} \tag{D.24}$$

and

$$\begin{aligned}
&\hat{h}_0 = ip, \quad \hat{h}_1 = \frac{1}{2}p_x,\\
&\hat{h}_\ell = \sum_{k=0}^{\ell-2}\Big(\frac{i}{4p}\hat{h}_k\hat{h}_{\ell-2-k,xx} - \frac{ip_x}{4p^2}\hat{h}_k\hat{h}_{\ell-2-k,x} - \frac{i}{8p}\hat{h}_{k,x}\hat{h}_{\ell-2-k,x}\\
&\qquad - \frac{iq}{2}\hat{h}_k\hat{h}_{\ell-2-k}\Big) - \frac{p_x}{2p^2}\sum_{k=0}^{\ell-1}\hat{h}_k\hat{h}_{\ell-1-k} + \frac{i}{2p}\sum_{k=1}^{\ell-1}\hat{h}_k\hat{h}_{\ell-k}, \quad \ell \geq 2.
\end{aligned} \tag{D.25}$$

Moreover, one infers for the E_m*-dependent integration constants* c_ℓ, $\ell = 0, \dots, n$, *in* F_n, G_{n+1}, *and* H_n *that*

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n+1 \tag{D.26}$$

and[1]

$$\begin{aligned}
&f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E})\hat{f}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E})\hat{h}_k, \quad \ell = 0, \dots, n,\\
&g_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E})\hat{g}_k, \quad \ell = 0, \dots, n+1,
\end{aligned} \tag{D.27}$$

$$\begin{aligned}
&\hat{f}_\ell = \sum_{k=0}^{\ell\wedge n} \hat{c}_{\ell-k}(\underline{E}) f_k, \quad \hat{h}_\ell = \sum_{k=0}^{\ell\wedge n} \hat{c}_{\ell-k}(\underline{E}) h_k,\\
&\hat{g}_\ell = \sum_{k=0}^{\ell\wedge(n+1)} \hat{c}_{\ell-k}(\underline{E}) g_k, \quad \ell \in \mathbb{N}_0.
\end{aligned} \tag{D.28}$$

[1] $m \wedge n = \min\{m, n\}$.

Proof Again it will be convenient to introduce the notion of a degree to effectively distinguish between homogeneous and nonhomogeneous quantities. Thus, assuming (3.1), one defines

$$\deg(p) = \deg(q) = 1, \quad \deg(\partial_x) = 1,$$

implying

$$\deg(\hat{f}_\ell) = \deg(\hat{h}_\ell) = \ell + 1, \ \ell \in \mathbb{N}_0, \quad \deg(\hat{g}_\ell) = \ell, \ \ell \geq 2 \tag{D.29}$$

using the linear recursion relation (3.5)–(3.7) and induction on ℓ. Next, dividing F_n and H_n by $R_{2n+2}^{1/2}$ (temporarily fixing the branch of $R_{2n+2}(z)^{1/2}$ as z^{n+1} near infinity), one obtains

$$\frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} f_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \check{f}_\ell z^{-\ell-1}, \tag{D.30}$$

$$\frac{H_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} h_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \check{h}_\ell z^{-\ell-1} \tag{D.31}$$

for some coefficients $\check{f}_\ell$ and $\check{h}_\ell$ to be determined next. Dividing (3.25) and (3.27) by R_{2n+2} and inserting the expansions (D.30) and (D.31) into the resulting equations then yield the recursion relations (D.24) and (D.25) (with $\hat{f}_\ell$ and $\hat{h}_\ell$ replaced by $\check{f}_\ell$ and $\check{h}_\ell$, respectively). The signs of $\check{f}_0$ and $\check{h}_0$ have been chosen such that $\check{f}_0 = \hat{f}_0 = -iq$ and $\check{h}_0 = \hat{h}_0 = ip$. Moreover, one confirms inductively using the nonlinear recursion relations (D.24) and (D.25) satisfied by $\check{f}_\ell$ and $\check{h}_\ell$ that

$$\deg(\check{f}_\ell) = \deg(\check{h}_\ell) = \ell + 1, \ \ell \in \mathbb{N}_0. \tag{D.32}$$

Differentiating $\check{f}_\ell$ and $\check{h}_\ell$ with respect to x (using (D.24) and (D.25)), one proves inductively that

$$\frac{i}{q}\check{f}_{\ell+1} + \frac{1}{2q}\check{f}_{\ell,x} = -\frac{i}{p}\check{h}_{\ell+1} + \frac{1}{2p}\check{h}_{\ell,x}, \quad \ell \in \mathbb{N}_0. \tag{D.33}$$

Defining $\check{g}_{\ell+1}$ equal to the expression (D.33), one again proves inductively that

$$\check{g}_{\ell+1,x} = p\check{f}_\ell + q\check{h}_\ell, \quad \ell \in \mathbb{N}_0.$$

Thus, $\check{f}_\ell$, $\check{g}_\ell$, and $\check{h}_\ell$ also satisfy the linear recursion relation (3.5)–(3.7). Hence, (D.29) and (D.32) imply

$$\check{f}_\ell = \hat{f}_\ell \text{ and } \check{h}_\ell = \hat{h}_\ell \text{ for all } \ell \in \mathbb{N}_0.$$

Thus, we proved

$$\frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} f_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \hat{f}_\ell z^{-\ell-1}, \tag{D.34}$$

$$\frac{H_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} h_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \hat{h}_\ell z^{-\ell-1} \tag{D.35}$$

and hence (D.23) for $\hat{f}_\ell$ and $\hat{h}_\ell$. A comparison of coefficients in (D.34) and (D.35) then proves (D.28) for $\hat{f}_\ell$ and $\hat{h}_\ell$. To prove (D.28) also for $\hat{g}_\ell$ one makes the ansatz

$$\frac{G_{n+1}(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n+1} g_\ell z^{-\ell}\bigg) = \sum_{\ell=0}^{\infty} \check{g}_\ell z^{-\ell}$$

for some coefficients $\check{g}_\ell$. Next, dividing (3.23) by R_{2n+2}, that is, writing

$$\frac{G_{n+1}(z)^2}{R_{2n+2}(z)} = 1 + \frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \frac{H_n(z)}{R_{2n+2}(z)^{1/2}}, \tag{D.36}$$

and inserting expansions (D.34) and (D.35) into the right-hand side of (D.36), one proves inductively that

$$\deg(\check{g}_\ell) = \ell, \ \ \ell \geq 2.$$

Since it can easily be verified that $\check{g}_0 = \hat{g}_0 = 1$ and $\check{g}_1 = \hat{g}_1 = 0$ directly, one obtains

$$\check{g}_\ell = \hat{g}_\ell \text{ for all } \ell \in \mathbb{N}_0,$$

and hence

$$\frac{G_{n+1}(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n+1} g_\ell z^{-\ell}\bigg) = \sum_{\ell=0}^{\infty} \hat{g}_\ell z^{-\ell}. \tag{D.37}$$

This completes the proof of (D.23), and a comparison of coefficients in (D.37) then also proves (D.28) for $\hat{g}_\ell$. The proof of (D.26) and (D.27) is identical to that of (D.9) and (D.10) in Theorem D.1 and is hence omitted. $\square$

The coefficients $\hat{g}_\ell$ in (D.23) can be determined from (D.36) and (D.37). (However, it is simpler to use the linear recursion relation (3.6) instead.) Explicitly, one obtains

$$\hat{g}_0 = 1, \quad \hat{g}_1 = 0, \quad \hat{g}_2 = pq/2, \quad \hat{g}_3 = -(i/4)(p_x q - p q_x), \text{ etc.}$$

Theorem D.3 has interesting consequences for the asymptotic spectral parameter expansion of the Green's matrix $G(z, x, x')$ of the Dirac-type operator M described next.

Remark D.4 Let M be given by (3.3) (with $p, q \in L^\infty(\mathbb{R})$ not necessarily algebro-geometric potentials) and assume that for some $z \in \mathbb{C}$, and all $x_0 \in \mathbb{R}$, $\psi_{1,\pm}(z, \cdot)$, $\psi_{2,\pm}(z, \cdot) \in L^2((x_0, \pm\infty))$ satisfy $(M - z)\Psi_\pm = 0$, $\Psi_\pm = \binom{\psi_{1,\pm}}{\psi_{2,\pm}}$, that is,

$$\psi_{1,\pm,x} = -iz\psi_{1,\pm} + q\psi_{2,\pm}, \quad \psi_{2,\pm,x} = iz\psi_{2,\pm} + p\psi_{1,\pm}.$$

Then the Green's matrix $G(z, x, x')$, $x \neq x'$, of M (i.e., the integral kernel of the resolvent $(M - z)^{-1}$, identifying M and its $L^2(\mathbb{R})^2$-realization as a closed linear operator for simplicity) is given by

$$G(z,x,x') = \frac{i}{W(\Psi_-(z),\Psi_+(z))} \times \begin{cases} \begin{pmatrix} \psi_{1,+}(z,x)\psi_{2,-}(z,x') & \psi_{1,+}(z,x)\psi_{1,-}(z,x') \\ \psi_{2,+}(z,x)\psi_{2,-}(z,x') & \psi_{2,+}(z,x)\psi_{1,-}(z,x') \end{pmatrix}, & x > x', \\ \begin{pmatrix} \psi_{1,-}(z,x)\psi_{2,+}(z,x') & \psi_{1,-}(z,x)\psi_{1,+}(z,x') \\ \psi_{2,-}(z,x)\psi_{2,+}(z,x') & \psi_{2,-}(z,x)\psi_{1,+}(z,x') \end{pmatrix}, & x < x', \end{cases} \tag{D.38}$$

where the Wronskian

$$\begin{aligned} W(\Psi_-(z),\Psi_+(z)) &= \begin{vmatrix} \psi_{1,-}(z,x) & \psi_{1,+}(z,x) \\ \psi_{2,-}(z,x) & \psi_{2,+}(z,x) \end{vmatrix} \\ &= \psi_{1,-}(z,x)\psi_{2,+}(z,x) - \psi_{2,-}(z,x)\psi_{1,+}(z,x) \end{aligned}$$

is independent of x. Note that $G(z,x,x')$ is continuous at $x = x'$ in its off-diagonal elements but discontinuous on the diagonal.

In the special algebro-geometric context we next replace

$$\psi_{j,+}(z,x) \text{ by } \psi_j(P,x,x_0), \quad \psi_{j,-}(z,x) \text{ by } \psi_j(P^*,x,x_0), \quad j = 1,2 \tag{D.39}$$

and

$$\begin{aligned} W(\Psi_-(z),\Psi_+(z)) \text{ by } W(\Psi(P^*,\,\cdot\,,x_0),\Psi(P,\,\cdot\,,x_0)) &= \begin{vmatrix} 1 & 1 \\ \phi(P^*,x_0) & \phi(P,x_0) \end{vmatrix} \\ &= \phi(P,x_0) - \phi(P^*,x_0) = \frac{2y}{F_n(z,x_0)}, \end{aligned} \tag{D.40}$$

since the Wronskian is x-independent. Substituting (D.39) and (D.40) into (D.38) with the result denoted by $G(P,x,x')$ then yields

$$\begin{aligned} &(G(P,x,x+0) + G(P,x,x-0))/2 = (G(P,x-0,x) + G(P,x+0,x))/2 \\ &= \frac{i}{2y} \begin{pmatrix} G_{n+1}(z,x) & F_n(z,x) \\ H_n(z,x) & G_{n+1}(z,x) \end{pmatrix}, \quad P = (z,y) \in \mathcal{K}_n \setminus \{P_{\infty_\pm}\}, \end{aligned} \tag{D.41}$$

where F_n/y, G_{n+1}/y, and H_n/y denote the basic quantities encountered in (D.23). Since $G(P,x,x')$ is discontinuous at $x = x'$, we introduced the arithmetic mean of the corresponding one-sided limits following the usual treatment of first-order systems. In fact, in the special self-adjoint case in which $p = \overline{q}$, the arithmetic mean in (D.41) leads to the characteristic function of M, a fundamental object for studying spectral properties of M. The expansions in (D.23) then determine the off-diagonal asymptotic spectral parameter expansions of the arithmetic mean of the diagonal Green's matrix in (D.41) as $P \to P_{\infty_\pm}$. Even though (D.23) and (D.41) were derived in the special algebro-geometric context, we emphasize that the expansion of (D.41) as $P \to P_{\infty_\pm}$ only involves the homogeneous coefficients

$\hat{f}_k, \hat{h}_k$, which are universal differential polynomials in p, q. Thus, identifying $\Psi_\pm(z, x)$ and $\Psi(P, x, x_0)$, $\Psi(P^*, x, x_0)$ as in (D.39) yields the universal spectral parameter expansion of the arithmetic mean of the diagonal Green's matrix of M as $z \to \infty$ in the general (not necessarily algebro-geometric) case. In the general case, the expansion of the arithmetic mean of the diagonal Green's matrix of M as $z \to \infty$ will only be an asymptotic expansion valid in appropriate regions exterior to the spectrum of M.

Next, we turn to the case of the Thirring system, where $N = 2n + 1$ for some $n \in \mathbb{N}_0$.

Theorem D.5 *Assume* (4.2)*,* (4.15)*–*(4.17)*, and suppose* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$. *Then* F_n/y*,* G_{n+1}/y*, and* H_n/y *have the following convergent expansions as* $P \to P_{\infty_\pm}$*,*

$$\frac{F_n(z)}{y} \underset{\zeta\to 0}{=} \sum_{\ell=0}^{\infty} \hat{f}_\ell\, \zeta^{\ell+1}, \quad \frac{G_{n+1}(z)}{y} \underset{\zeta\to 0}{=} \sum_{\ell=0}^{\infty} \hat{g}_\ell\, \zeta^{\ell}, \quad \frac{H_n(z)}{y} \underset{\zeta\to 0}{=} \sum_{\ell=0}^{\infty} \hat{h}_\ell\, \zeta^{\ell+1}, \tag{D.42}$$

where $\zeta = 1/z$ *is the local coordinate near* $P_{\infty_\pm}$ *described in* (C.7)–(C.11) *and* $\hat{f}_\ell$*,* $\hat{g}_\ell$*,* $\hat{h}_\ell$ *are the homogeneous coefficients* f_ℓ*,* g_ℓ*,* h_ℓ *in* (4.9)*–*(4.11)*. In particular,* $\hat{f}_\ell$ *and* $\hat{h}_\ell$ *can be computed from the nonlinear recursion relations*

$$\begin{aligned}
&\hat{f}_0 = -2v, \quad \hat{f}_1 = i v_x + 2v^2 v^*,\\
&\hat{f}_\ell = \frac{1}{8v}\sum_{k=0}^{\ell-2}\Big(\hat{f}_{\ell-2-k,xx}\hat{f}_k - \frac{1}{2}\hat{f}_{\ell-2-k,x}\hat{f}_{k,x} - \frac{v_x}{v}\hat{f}_{\ell-2-k,x}\hat{f}_k\\
&\qquad\qquad + 2\big(v^2(v^*)^2 + i v v_x^*\big)\hat{f}_{\ell-2-k}\hat{f}_k\Big)\\
&\qquad + \frac{1}{4v}\Big(2vv^* + i\frac{v_x}{v}\Big)\sum_{k=0}^{\ell-1}\hat{f}_{\ell-1-k}\hat{f}_k + \frac{1}{4v}\sum_{k=0}^{\ell-1}\hat{f}_{\ell-k}\hat{f}_k, \quad \ell \geq 2,
\end{aligned} \tag{D.43}$$

and

$$\begin{aligned}
&\hat{h}_0 = 2v^*, \quad \hat{h}_1 = i v_x^* - 2v(v^*)^2,\\
&\hat{h}_\ell = -\frac{1}{8v^*}\sum_{k=0}^{\ell-2}\Big(\hat{h}_{\ell-2-k,xx}\hat{h}_k - \frac{1}{2}\hat{h}_{\ell-2-k,x}\hat{h}_{k,x} - \frac{v_x^*}{v^*}\hat{h}_{\ell-2-k,x}\hat{h}_k\\
&\qquad\qquad + 2\big(v^2(v^*)^2 - i v_x v^*\big)\hat{h}_{\ell-2-k}\hat{h}_k\Big)\\
&\qquad - \frac{1}{4v^*}\Big(2vv^* - i\frac{v_x^*}{v^*}\Big)\sum_{k=0}^{\ell-1}\hat{h}_{\ell-1-k}\hat{h}_k - \frac{1}{4v^*}\sum_{k=0}^{\ell-1}\hat{h}_{\ell-k}\hat{h}_k, \quad \ell \geq 2.
\end{aligned} \tag{D.44}$$

Moreover, one infers for the E_m*-dependent integration constants* c_ℓ*,* $\ell = 0, \dots, n$*,* *in F_n, G_{n+1}, and H_n that*

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n \tag{D.45}$$

and[1]

$$\begin{aligned} f_\ell &= \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{f}_k, \quad h_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{h}_k, \quad \ell = 0, \dots, n, \\ g_\ell &= \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E}) \hat{g}_k, \quad \ell = 0, \dots, n+1, \end{aligned} \tag{D.46}$$

$$\begin{aligned} \hat{f}_\ell &= \sum_{k=0}^{\ell \wedge n} \hat{c}_{\ell-k}(\underline{E}) f_k, \quad \hat{h}_\ell = \sum_{k=0}^{\ell \wedge n} \hat{c}_{\ell-k}(\underline{E}) h_k, \\ \hat{g}_\ell &= \sum_{k=0}^{\ell \wedge (n+1)} \hat{c}_{\ell-k}(\underline{E}) g_k, \quad \ell \in \mathbb{N}_0. \end{aligned} \tag{D.47}$$

Proof Again it will be convenient to introduce the notion of a degree to effectively distinguish between homogeneous and nonhomogeneous quantities. Thus, assuming (4.2), one defines

$$\deg(v) = \deg(v^*) = 1, \quad \deg(\partial_x) = 2,$$

implying

$$\deg(\hat{f}_\ell) = \deg(\hat{h}_\ell) = 2\ell + 1, \quad \deg(\hat{g}_\ell) = 2\ell, \quad \ell \in \mathbb{N}_0 \tag{D.48}$$

using the linear recursion relation (4.4)–(4.6) and induction on ℓ. Next, dividing F_n and H_n by $R_{2n+2}^{1/2}$ (temporarily fixing the branch of $R_{2n+2}(z)^{1/2}$ as z^{n+1} near infinity), one obtains

$$\frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} f_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \check{f}_\ell z^{-\ell-1}, \tag{D.49}$$

$$\frac{H_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} h_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \check{h}_\ell z^{-\ell-1} \tag{D.50}$$

for some coefficients $\check{f}_\ell$ and $\check{h}_\ell$ to be determined next. Dividing (4.25) and (4.26) by R_{2n+2} and inserting the expansions (D.49) and (D.50) into the resulting equations then yield the recursion relations (D.43) and (D.44) (with $\hat{f}_\ell$ and $\hat{h}_\ell$ replaced by $\check{f}_\ell$ and $\check{h}_\ell$, respectively). The signs of $\check{f}_0$ and $\check{h}_0$ have been chosen such that $\check{f}_0 = \hat{f}_0 = -2v$ and $\check{h}_0 = \hat{h}_0 = 2v^*$. Moreover, one confirms inductively using the nonlinear recursion relations (D.43) and (D.44) satisfied by $\check{f}_\ell$ and $\check{h}_\ell$ that

$$\deg(\check{f}_\ell) = \deg(\check{h}_\ell) = 2\ell + 1, \quad \ell \in \mathbb{N}_0. \tag{D.51}$$

[1] $m \wedge n = \min\{m, n\}$.

Differentiating $\check{f}_\ell$ and $\check{h}_\ell$ with respect to x (using (D.43) and (D.44)), one proves inductively that

$$-\frac{1}{2v}\check{f}_\ell - \frac{i}{4v}\check{f}_{\ell-1,x} + \frac{v^*}{2}\check{f}_{\ell-1} = \frac{1}{2v^*}\check{h}_\ell - \frac{i}{4v^*}\check{h}_{\ell-1,x} - \frac{v}{2}\check{h}_{\ell-1}, \quad \ell \in \mathbb{N}_0. \tag{D.52}$$

Defining $\check{g}_\ell$ equal to the expression (D.52), one again proves inductively that

$$\check{g}_{\ell,x} = 2iv^*\check{f}_\ell + 2iv\check{h}_\ell, \quad \ell \in \mathbb{N}_0.$$

Thus, $\check{f}_\ell$, $\check{g}_\ell$, and $\check{h}_\ell$ also satisfy the linear recursion relation (4.4)–(4.6). Hence, (D.48) and (D.51) imply

$$\check{f}_\ell = \hat{f}_\ell \text{ and } \check{h}_\ell = \hat{h}_\ell \text{ for all } \ell \in \mathbb{N}_0.$$

Thus, we proved

$$\frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E})z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} f_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \hat{f}_\ell z^{-\ell-1}, \tag{D.53}$$

$$\frac{H_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E})z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} h_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \hat{h}_\ell z^{-\ell-1} \tag{D.54}$$

and hence (D.42) for $\hat{f}_\ell$ and $\hat{h}_\ell$. A comparison of coefficients in (D.53) and (D.54) then proves (D.47) for $\hat{f}_\ell$ and $\hat{h}_\ell$. To prove (D.47) also for $\hat{g}_\ell$ one makes the ansatz

$$\frac{G_{n+1}(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E})z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n+1} g_\ell z^{-\ell}\bigg) = \sum_{\ell=0}^{\infty} \check{g}_\ell z^{-\ell}$$

for some coefficients $\check{g}_\ell$. Next, dividing (4.18) by R_{2n+2}, that is, writing

$$\frac{G_{n+1}(z)^2}{R_{2n+2}(z)} = 1 + z\frac{F_n(z)}{R_{2n+2}(z)^{1/2}}\frac{H_n(z)}{R_{2n+2}(z)^{1/2}}, \tag{D.55}$$

and inserting expansions (D.53) and (D.54) into the right-hand side of (D.55), one proves inductively that

$$\deg(\check{g}_\ell) = 2\ell, \quad \ell \in \mathbb{N}_0.$$

Thus,

$$\check{g}_\ell = \hat{g}_\ell \text{ for all } \ell \in \mathbb{N}_0,$$

and hence

$$\frac{G_{n+1}(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E})z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n+1} g_\ell z^{-\ell}\bigg) = \sum_{\ell=0}^{\infty} \hat{g}_\ell z^{-\ell}. \tag{D.56}$$

This completes the proof of (D.42), and a comparison of coefficients in (D.56) also proves (D.47) for $\hat{g}_\ell$. Again the proof of (D.45) and (D.46) is identical to that of (D.9) and (D.10) in Theorem D.1 and is hence omitted. □

The coefficients $\hat{g}_\ell$ in (D.42) can be determined from (D.55) and (D.56). (However, it is simpler to use the linear recursion relation (4.5) instead.) Explicitly, one obtains

$$\hat{g}_0 = 1, \quad \hat{g}_1 = -2vv^*, \text{ etc.}$$

Next, we turn to the CH system, where $N = 2n + 1$ for some $n \in \mathbb{N}_0$.

Theorem D.6 *Assume that* $u \in C^\infty(\mathbb{R})$, $d^m u/dx^{(m)} \in L^\infty(\mathbb{R})$, $m \in \mathbb{N}_0$, s-CH$_n(u) = 0$, *and suppose* $P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\}$. *Then* F_n/y *has the following convergent expansion as* $P \to P_{\infty_\pm}$,

$$\frac{F_n(z)}{y} \underset{\zeta\to 0}{=} \mp \sum_{\ell=0}^{\infty} \hat{f}_\ell \, \zeta^{\ell+1}, \tag{D.57}$$

where $\zeta = 1/z$ *is the local coordinate near* $P_{\infty_\pm}$ *described in* (C.7)–(C.11) *and* $\hat{f}_\ell$ *are the homogeneous coefficients* f_ℓ *in* (5.7). *In particular,* $\hat{f}_\ell$ *can be computed from the nonlinear recursion relation*

$$\begin{aligned}
&\hat{f}_0 = 1, \quad \hat{f}_1 = -2u, \\
&\hat{f}_{\ell+1} = \mathcal{G}\bigg(\sum_{k=1}^{\ell} \big(\hat{f}_{\ell+1-k,xx}\hat{f}_k - \tfrac{1}{2}\hat{f}_{\ell+1-k,x}\hat{f}_{k,x} - 2\hat{f}_{\ell+1-k}\hat{f}_k\big) \\
&\qquad\qquad + 2(u_{xx} - 4u)\sum_{k=0}^{\ell} \hat{f}_{\ell-k}\hat{f}_k \bigg), \quad \ell \in \mathbb{N}
\end{aligned} \tag{D.58}$$

assuming

$$\hat{f}_\ell \in L^\infty(\mathbb{R}), \quad \ell \in \mathbb{N}.$$

Moreover, one infers for the E_m*-dependent integration constants* c_ℓ, $\ell = 0, \dots, n$, *in* F_n *that*

$$c_\ell = c_\ell(\underline{E}), \quad \ell = 0, \dots, n \tag{D.59}$$

and[1]

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E})\hat{f}_k, \quad \ell = 0, \dots, n, \tag{D.60}$$

$$\hat{f}_\ell = \sum_{k=0}^{\ell\wedge n} \hat{c}_{\ell-k}(\underline{E}) f_k, \quad \ell \in \mathbb{N}_0. \tag{D.61}$$

[1] $m \wedge n = \min\{m, n\}$.

Proof Dividing F_n by $R_{2n+2}^{1/2}$ (temporarily fixing the branch of $R_{2n+2}(z)^{1/2}$ as z^{n+1} near infinity), one obtains

$$\frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \bigg(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k}\bigg)\bigg(\sum_{\ell=0}^{n} f_\ell z^{-\ell-1}\bigg) = \sum_{\ell=0}^{\infty} \check{f}_\ell z^{-\ell-1} \tag{D.62}$$

for some coefficients $\check{f}_\ell$ to be determined next. Dividing (5.31) by R_{2n+2} and inserting the expansion (D.62) into the resulting equation then yield the recursion relation (D.58) (with $\hat{f}_\ell$ replaced by $\check{f}_\ell$). More precisely, for $\check{f}_1$, one originally obtains the relation

$$-\check{f}_{1,xx} + 4\check{f}_1 = 2(u_{xx} - 4u), \text{ that is, } \bigg(-\frac{d^2}{dx^2} + 4\bigg)\big(\check{f}_1 + 2u\big) = 0.$$

Thus,

$$\check{f}_1(x) = -2u(x) + a_1 e^{2x} + b_1 e^{-2x}$$

for some $a_1, b_1 \in \mathbb{C}$, and hence the requirement $\check{f}_1 \in L^\infty(\mathbb{R})$ then yields $a_1 = b_1 = 0$. The sign of $\check{f}_0$ has been chosen such that $\check{f}_0 = \hat{f}_0 = 1$. For $\ell \geq 2$, one obtains similarly

$$\begin{aligned} -\check{f}_{\ell+1,xx} + 4\check{f}_{\ell+1} = \bigg(&\sum_{k=1}^{\ell} \big(\check{f}_{\ell+1-k,xx}\check{f}_k - \tfrac{1}{2}\check{f}_{\ell+1-k,x}\check{f}_{k,x} - 2\check{f}_{\ell+1-k}\check{f}_k\big) \\ &+ 2(u_{xx} - 4u)\sum_{k=0}^{\ell} \check{f}_{\ell-k}\check{f}_k\bigg), \quad \ell \geq 1, \end{aligned}$$

and hence,

$$\begin{aligned} \check{f}_{\ell+1} = \mathcal{G}\bigg(&\sum_{k=1}^{\ell} \big(\check{f}_{\ell+1-k,xx}\check{f}_k - \tfrac{1}{2}\check{f}_{\ell+1-k,x}\check{f}_{k,x} - 2\check{f}_{\ell+1-k}\check{f}_k\big) \\ &+ 2(u_{xx} - 4u)\sum_{k=0}^{\ell} \check{f}_{\ell-k}\check{f}_k\bigg) + a_{\ell+1}e^{2x} + b_{\ell+1}e^{-2x}, \quad \ell \geq 1 \end{aligned}$$

for some $a_{\ell+1}, b_{\ell+1} \in \mathbb{C}$. Again the requirement $\check{f}_{\ell+1} \in L^\infty(\mathbb{R})$ then yields $a_{\ell+1} = b_{\ell+1} = 0, \ell \geq 1$. If one introduces $\hat{f}_\ell$ by (5.7) with $c_k = 0, k \in \mathbb{N}$, and $\check{f}_\ell$ by (D.58), a straightforward computation shows that

$$\begin{aligned} \check{f}_{\ell,x} &= \mathcal{G}\bigg(\sum_{k=1}^{\ell-1} \big(f_{\ell-k,xxx} - 4f_{\ell-k,x}\big)f_k - \sum_{k=0}^{\ell-1} 2\big(-2(u_{xx} - 4u)f_{\ell-k-1,x} \\ &\qquad + (4u_x - u_{xxx})f_{\ell-k-1}\big)\bigg)f_k \\ &= \mathcal{G}\bigg(-\sum_{k=1}^{\ell-1} \mathcal{G}^{-1} f_{\ell-k,x} f_k + \sum_{k=0}^{\ell-1} \big(\mathcal{G}^{-1} f_{\ell-k,x}\big) f_k\bigg) \\ &= \hat{f}_{\ell,x}, \quad \ell \in \mathbb{N}. \end{aligned}$$

Hence,

$$\check{f}_\ell = \hat{f}_\ell + d_\ell, \quad \ell \in \mathbb{N} \tag{D.63}$$

for some constants $d_\ell \in \mathbb{C}$, $\ell \in \mathbb{N}$. Since $d_0 = d_1 = 0$ by inspection, we next proceed by induction on ℓ and suppose that

$$d_k = 0 \text{ and hence } \check{f}_k = \hat{f}_k \text{ for } k = 0, \dots, \ell.$$

Thus, (D.58) and (D.63) imply

$$\check{f}_{\ell+1} = \mathcal{G}\{\dots\} = \hat{f}_{\ell+1} + d_{\ell+1},$$

where $\{\dots\}$ denotes the expression on the right-hand side of (D.58) in terms of $\check{f}_k = \hat{f}_k$, $k = 0, \dots, \ell$. Hence,

$$\{\dots\} - \hat{f}_{\ell+1} + \alpha_{\ell+1} e^{2x} + \beta_{\ell+1} e^{-2x} = \mathcal{G}^{-1} d_{\ell+1} = 4 d_{\ell+1}$$

for some constants $\alpha_{\ell+1}, \beta_{\ell+1} \in \mathbb{C}$. Since $\{\dots\} - \hat{f}_{\ell+1} \in L^\infty(\mathbb{R})$, one concludes once more that $\alpha_{\ell+1} = \beta_{\ell+1} = 0$. Moreover, since $\{\dots\} - \hat{f}_{\ell+1}$ contains no constants by construction, one concludes $d_{\ell+1} = 0$, and hence

$$\check{f}_\ell = \hat{f}_\ell \text{ for all } \ell \in \mathbb{N}_0.$$

Thus, we have proved that

$$\frac{F_n(z)}{R_{2n+2}(z)^{1/2}} \underset{|z|\to\infty}{=} \left(\sum_{k=0}^{\infty} \hat{c}_k(\underline{E}) z^{-k} \right) \left(\sum_{\ell=0}^{n} f_\ell z^{-\ell-1} \right) = \sum_{\ell=0}^{\infty} \hat{f}_\ell z^{-\ell-1}, \tag{D.64}$$

and hence (D.57). A comparison of coefficients in (D.64) then proves (D.61). Next, after multiplying (D.1) and (D.4), a comparison of coefficients of z^{-k} yields

$$\sum_{\ell=0}^{k} \hat{c}_{k-\ell}(\underline{E}) c_\ell(\underline{E}) = \delta_{k,0}, \quad k \in \mathbb{N}_0. \tag{D.65}$$

Thus, one computes

$$\begin{aligned}
\sum_{m=0}^{\ell} c_{\ell-m}(\underline{E}) \hat{f}_m &= \sum_{m=0}^{\ell} \sum_{k=0}^{m} c_{\ell-m}(\underline{E}) \hat{c}_{m-k}(\underline{E}) f_k = \sum_{k=0}^{\ell} \sum_{p=k}^{\ell} c_{\ell-p}(\underline{E}) \hat{c}_{p-k}(\underline{E}) f_k \\
&= \sum_{k=0}^{\ell} \left(\sum_{m=0}^{\ell-k} c_{\ell-k-m}(\underline{E}) \hat{c}_m(\underline{E}) \right) f_k = f_\ell, \quad \ell = 0, \dots, n,
\end{aligned}$$

applying (D.65). Hence, one obtains (D.60) and thus (D.59). □

Notes

Equation (D.7) is well-known and can be found, for instance, in Gel'fand and Dikii (1975), which is the classical reference for asymptotic spectral parameter expansions of the type considered in this appendix. Explicit formulas for the coefficients $\hat{f}_\ell$ in (D.8) have been derived in Avramidi and Schimming (2000), Rosenhouse and Katriel (1987), and Schimming (1988; 1995).

For first-order Dirac-type systems and the associated Green's matrices refer, for instance, to Atkinson (1964, Sec. 9.4). The characteristic function for self-adjoint Dirac-type operators M is also discussed in Atkinson (1964, Sec. 9.5).

Appendix E
Lagrange Interpolation

A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.

Paul R. Halmos[1]

We briefly review essentials of Lagrange interpolation formulas. Assuming $n \in \mathbb{N}$ to be fixed and introducing

$$\mathcal{S}_k = \{\underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathbb{N}^k \mid \ell_1 < \cdots < \ell_k \leq n\}, \quad k = 1, \dots, n,$$
$$\mathcal{I}_k^{(j)} = \{\underline{\ell} = (\ell_1, \dots, \ell_k) \in \mathcal{S}_k \mid \ell_m \neq j\}, \quad k = 1, \dots, n-1, \;\; j = 1, \dots, n,$$

one defines the symmetric functions

$$\Psi_0(\underline{\mu}) = 1, \quad \Psi_k(\underline{\mu}) = (-1)^k \sum_{\underline{\ell} \in \mathcal{S}_k} \mu_{\ell_1} \cdots \mu_{\ell_k}, \quad k = 1, \dots, n, \tag{E.1}$$

$$\Phi_0^{(j)}(\underline{\mu}) = 1,$$
$$\Phi_k^{(j)}(\underline{\mu}) = (-1)^k \sum_{\underline{\ell} \in \mathcal{I}_k^{(j)}} \mu_{\ell_1} \cdots \mu_{\ell_k}, \quad k = 1, \dots, n-1, \;\; j = 1, \dots, n, \tag{E.2}$$
$$\Phi_n^{(j)}(\underline{\mu}) = 0, \quad j = 1, \dots, n,$$

where $\underline{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$. Explicitly, one verifies

$$\Psi_1(\underline{\mu}) = -\sum_{\ell=1}^{n} \mu_\ell, \quad \Psi_2(\underline{\mu}) = \sum_{\substack{\ell_1, \ell_2 = 1 \\ \ell_1 < \ell_2}}^{n} \mu_{\ell_1} \mu_{\ell_2}, \text{ etc.,}$$

$$\Phi_1^{(j)}(\underline{\mu}) = -\sum_{\substack{\ell=1 \\ \ell \neq j}}^{n} \mu_\ell, \quad \Phi_2^{(j)}(\underline{\mu}) = \sum_{\substack{\ell_1, \ell_2 = 1 \\ \ell_1, \ell_2 \neq j \\ \ell_1 < \ell_2}}^{n} \mu_{\ell_1} \mu_{\ell_2}, \text{ etc.}$$

[1] Quoted in J. A. Gallian, *Contemporary Abstract Algebra*, D. C. Heath, Lexington, Mass., 1990, p. 33.

Introducing

$$F_n(z) = \prod_{j=1}^{n}(z - \mu_j) = \sum_{\ell=0}^{n} \Psi_{n-\ell}(\underline{\mu}) z^{\ell}, \quad z \in \mathbb{C}, \tag{E.3}$$

one infers

$$F_{n,z}(\mu_k) = \prod_{\substack{j=1 \\ j \neq k}}^{n}(\mu_k - \mu_j).$$

The general form of Lagrange's interpolation theorem then reads as follows.

Theorem E.1 *Assume that $\mu_1, \ldots, \mu_n$ are n distinct complex numbers. Then*

$$\sum_{j=1}^{n} \frac{\mu_j^{m-1}}{F_{n,z}(\mu_j)} \Phi_k^{(j)}(\underline{\mu}) = \delta_{m,n-k} - \Psi_{k+1}(\underline{\mu})\delta_{m,n+1}, \tag{E.4}$$
$$m = 1, \ldots, n+1, \quad k = 0, \ldots, n-1.$$

Proof Let C_R be a circle with center at the origin and radius R that contains the zeros μ_j of the polynomial F_n and that is oriented counterclockwise. Cauchy's theorem then yields

$$\frac{1}{2\pi i} \oint_{C_R} d\zeta \, \frac{\zeta^{m-1}}{F_n(\zeta)(\zeta - z)} = \frac{z^{m-1}}{F_n(z)} + \sum_{k=1}^{n} \frac{\mu_k^{m-1}}{F_{n,z}(\mu_j)(\mu_j - z)},$$
$$z \neq \mu_1, \ldots, \mu_n, \quad m = 1, \ldots, n+1.$$

However, by letting $R \to \infty$, we infer that

$$\frac{1}{2\pi i} \oint_{C_R} d\zeta \, \frac{\zeta^{m-1}}{F_n(\zeta)(\zeta - z)} = \lim_{R\to\infty} \frac{R^{m-1}}{F_n(R)} = \delta_{m,n+1}, \quad m = 1, \ldots, n+1,$$

which implies

$$z^{m-1} - \sum_{k=1}^{n} \frac{\mu_k^{m-1} F_n(z)}{F_{n,z}(\mu_j)(z - \mu_j)} = F_n(z)\delta_{m,n+1}. \tag{E.5}$$

Using the symmetric functions Ψ_j, we may write

$$F_n(z) = \sum_{j=0}^{n} \Psi_j(\underline{\mu}) z^{n-j} \tag{E.6}$$

and

$$\frac{F_n(z)}{z - \mu_j} = \sum_{k=0}^{n-1} \Phi_k^{(j)}(\underline{\mu}) z^{n-1-k}. \tag{E.7}$$

Expanding both sides of equation (E.5) in powers in z, using (E.6) on the right-hand side and (E.7) on the left-hand side, proves (E.4). $\square$

The simplest Lagrange interpolation formula reads, in the case $k = 0$,

$$\sum_{j=1}^{n} \frac{\mu_j^{m-1}}{F_{n,z}(\mu_j)} = \delta_{m,n}, \quad m = 1, \ldots, n. \tag{E.8}$$

As a consequence, if Q_{n-1} denotes a polynomial of degree $n-1$, then

$$\begin{aligned} Q_{n-1}(z) &= F_n(z) \sum_{j=1}^{n} \frac{Q_{n-1}(\mu_j)}{F_{n,z}(\mu_j)(z-\mu_j)} \\ &= \sum_{j=1}^{n} Q_{n-1}(\mu_j) \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{z-\mu_k}{\mu_j - \mu_k}, \quad z \in \mathbb{C}. \end{aligned} \tag{E.9}$$

For use in the main text we finally observe the following results.

Lemma E.2 *Assume that $\mu_1, \ldots, \mu_n$ are n distinct complex numbers. Then*

(i) $$\Psi_{k+1}(\underline{\mu}) + \mu_j \Phi_k^{(j)}(\underline{\mu}) = \Phi_{k+1}^{(j)}(\underline{\mu}), \quad k = 0, \ldots, n-1, \ j = 1, \ldots, n. \tag{E.10}$$

(ii) $$\sum_{\ell=0}^{k} \Psi_{k-\ell}(\underline{\mu})\mu_j^{\ell} = \Phi_k^{(j)}(\underline{\mu}), \quad k = 0, \ldots, n, \ j = 1, \ldots, n. \tag{E.11}$$

(iii) $$\begin{aligned} \sum_{\ell=0}^{k-1} \Phi_{k-1-\ell}^{(j)}(\underline{\mu}) z^{\ell} &= \frac{1}{z-\mu_j}\left(\sum_{\ell=0}^{k} \Psi_{k-\ell}(\underline{\mu}) z^{\ell} - \Phi_k^{(j)}(\underline{\mu})\right), \\ &\qquad k = 0, \ldots, n, \ j = 1, \ldots, n. \end{aligned} \tag{E.12}$$

Proof

(i) Adding (E.6) to μ_j times (E.7), one finds

$$F_n(z) + \mu_j \frac{F_n(z)}{z-\mu_j} = \sum_{k=0}^{n-1} \big(\Psi_{k+1} + \mu_j \Phi_k^{(j)}\big) z^{n-k-1} + z^n.$$

However, one also has

$$F_n(z) + \mu_j \frac{F_n(z)}{z-\mu_j} = z \frac{F_n(z)}{z-\mu_j} = \sum_{k=0}^{n-1} \Phi_{k+1}^{(j)} z^{n-k-1} + z^n,$$

using (E.7) and recalling $\Phi_n^{(j)} = 0$. Thus, (E.10) holds.

(ii) We prove (E.11) by induction on k. Equation (E.11) clearly holds for $k = 0$; next assume that

$$\sum_{\ell=0}^{k-1} \Psi_{k-1-\ell}\mu_j^{\ell} = \Phi_{k-1}^{(j)}$$

holds. Then we find that

$$\begin{aligned}\sum_{\ell=0}^{k} \Psi_{k-\ell}\mu_j^{\ell} &= \Psi_k + \mu_j \sum_{\ell=1}^{k} \Psi_{k-\ell}\mu_j^{\ell-1} \\ &= \Psi_k + \mu_j \sum_{\ell=0}^{k-1} \Psi_{k-1-\ell}\mu_j^{\ell} = \Psi_k + \mu_j \Phi_{k-1}^{(j)} = \Phi_k^{(j)},\end{aligned}$$

using first the induction hypothesis and then (E.10).

(iii) Using (E.10) and $\Psi_0(\underline{\mu}) = \Phi_0^{(j)}(\underline{\mu}) = 1$ one computes

$$\begin{aligned}(z-\mu_j)\sum_{\ell=0}^{k-1} \Phi_{k-1-\ell}^{(j)}(\underline{\mu})z^{\ell} &= \sum_{m=1}^{k} \Phi_{k-m}^{(j)}(\underline{\mu})z^m - \sum_{\ell=0}^{k-1} \mu_j\Phi_{k-1-\ell}^{(j)}(\underline{\mu})z^{\ell} \\ &= \sum_{m=1}^{k} \Phi_{k-m}^{(j)}(\underline{\mu})z^m + \sum_{\ell=0}^{k-1} \Psi_{k-\ell}(\underline{\mu})z^{\ell} - \sum_{\ell=0}^{k-1} \Phi_{k-\ell}^{(j)}(\underline{\mu})z^{\ell} \\ &= \sum_{\ell=0}^{k-1} \Psi_{k-\ell}(\underline{\mu})z^{\ell} + \Phi_0^{(j)}(\underline{\mu})z^k - \Phi_k^{(j)}(\underline{\mu}) \\ &= \sum_{\ell=0}^{k} \Psi_{k-\ell}(\underline{\mu})z^{\ell} - \Phi_k^{(j)}(\underline{\mu}).\end{aligned}$$

□

Next, assuming $\mu_j \neq \mu_{j'}$ for $j \neq j'$, we introduce the $n \times n$ matrix $U_n(\underline{\mu})$ by

$$U_1(\underline{\mu}) = 1, \quad U_n(\underline{\mu}) = \left(\frac{\mu_k^{j-1}}{\prod_{\substack{m=1 \\ m\neq k}}^{n}(\mu_k - \mu_m)} \right)_{j,k=1}^{n}, \tag{E.13}$$

where $\underline{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$.

Lemma E.3 *Suppose $\mu_j \in \mathbb{C}$, $j = 1, \dots, n$, are n distinct complex numbers. Then*

$$U_n(\underline{\mu})^{-1} = \left(\Phi_{n-k}^{(j)}(\underline{\mu}) \right)_{j,k=1}^{n}. \tag{E.14}$$

Proof One observes that (E.14) may be written as

$$U_n(\underline{\mu}) = \left(\frac{\mu_k^{j-1}}{F_{n,z}(\mu_k)} \right)_{j,k=1}^{n}.$$

Using Lagrange's interpolation result, Theorem E.1 (replacing k by $n-k$ in (E.4)) then proves the result. $\square$

Notes

The material of this appendix is mostly taken from Gesztesy and Holden (2002; to appear, a). A proof of Lagrange's interpolation result in the simplest case $k=0$ can be found, for example, in Toda (1989b, App. E).

Appendix F

Symmetric Functions, Trace Formulas, and Dubrovin-Type Equations

> ... You should not publish this. It's a compendium of known things, written in an over-complicated and idiosyncratic notation. I think it is fair to say that ameteurs [sic!] of "KdV and all that" have known and used these things for 20 years or more.[1]

> Ich glaub von jedem Menschen das Schlechteste, selbst von mir, und ich hab mich noch selten getäuscht.
>
> *Johann Nestroy*[2]

This appendix takes a close look at Dubrovin-type equations in connection with hyperelliptic Riemann surfaces. We heavily employ elementary symmetric functions of $\mu_1, \dots, \mu_n$ and derive their theta function representations relevant in connection with trace formulas for solutions of $(1+1)$-dimensional integrable hierarchies of soliton equations. The material presented is of particular importance in connection with our discussion of algebro-geometric solutions in the time-dependent context.

First we express f_ℓ and $\widetilde{F}_r$ in terms of elementary symmetric functions of $\mu_1, \dots, \mu_n$ and recall our abbreviations $\Psi_k(\underline{\mu})$ and $\Phi_k^{(j)}(\underline{\mu})$ introduced in (E.1) and (E.2), respectively. For simplicity we will often focus on the homogeneous cases, denoted by $\hat{f}_\ell$ and $\widehat{F}_r$, where $c_k = 0$, $k = 1, \dots, \ell$. We start with $\hat{f}_\ell$.

Lemma F.1 *Let $\hat{c}_k(\underline{E})$ be defined as in* (D.2)*, with $N = 2n$ in the KdV and sGmKdV cases and $N = 2n+1$ in the AKNS and CH cases, respectively. Then one infers the following results for the KdV and CH hierarchies (cf.* (E.1)*),*[3]

$$\hat{f}_\ell = \sum_{k=0}^{\ell \wedge n} \hat{c}_{\ell-k}(\underline{E}) \Psi_k(\underline{\mu}), \quad \ell \in \mathbb{N}_0. \tag{F.1}$$

[1] A warning to the reader from an anonymous referee who clearly disliked the material in Gesztesy and Holden (2002) on which parts of this appendix are based.
[2] In *Die beiden Nachtwandler oder Das Notwendige und das Überflüssige*, first act, scene 16. ("I expect the worst of everyone, including myself, and I have seldom erred.")
[3] $m \wedge n = \min\{m, n\}$.

For the AKNS hierarchy, one obtains

$$\hat{f}_\ell = -iq \sum_{k=0}^{\ell\wedge n} \hat{c}_{\ell-k}(\underline{E})\Psi_k(\underline{\mu}), \quad \hat{h}_\ell = ip \sum_{k=0}^{\ell\wedge n} \hat{c}_{\ell-k}(\underline{E})\Psi_k(\underline{\nu}), \quad \ell \in \mathbb{N}_0, \tag{F.2}$$

where $\underline{\nu} = (\nu_1, \dots, \nu_n)$.
In the sGmKdV case, one derives

$$\hat{f}_\ell = \sum_{k=0}^{\ell\wedge n} \hat{c}_{\ell-k}(\underline{E})\Psi_k(\underline{\mu}), \quad \hat{h}_\ell = \sum_{k=0}^{\ell\wedge n} \hat{c}_{\ell-k}(\underline{E})\Psi_k(\underline{\nu}), \quad \ell \in \mathbb{N}_0, \tag{F.3}$$

and

$$\hat{f}_n = \Psi_n(\underline{\mu}), \quad \hat{h}_n = \Psi_n(\underline{\nu}). \tag{F.4}$$

Proof In the KdV and CH cases it suffices to refer to (1.11), (1.31), (5.20), (5.44) and to note

$$F_n(z) = \sum_{\ell=0}^{n} f_{n-\ell} z^\ell = \prod_{j=1}^{n} (z - \mu_j) = \sum_{\ell=0}^{n} \Psi_{n-\ell}(\underline{\mu}) z^\ell,$$

that is, $f_\ell = \Psi_\ell(\underline{\mu})$, $\ell = 0, \dots, n$. Equation (F.1) then follows from (D.11) and (D.61). The sGmKdV case (F.3), (F.4) follows in the same way using (2.29), (2.34), (2.35), (2.64), (2.91), and (2.92). In the AKNS case, one uses (3.17), (3.19), (3.51), and (3.52) and notes

$$\sum_{\ell=0}^{n} f_{n-\ell} z^\ell = -iq \prod_{j=1}^{n} (z - \mu_j) = -iq \sum_{\ell=0}^{n} \Psi_{n-\ell}(\underline{\mu}) z^\ell,$$
$$\sum_{\ell=0}^{n} h_{n-\ell} z^\ell = ip \prod_{j=1}^{n} (z - \nu_j) = ip \sum_{\ell=0}^{n} \Psi_{n-\ell}(\underline{\nu}) z^\ell,$$

that is, $f_\ell = -iq\Psi_\ell(\underline{\mu})$, $h_\ell = ip\Psi_\ell(\underline{\nu})$, $\ell = 0, \dots, n$. Equation (F.2) then follows applying (D.28). □

Theorem F.2 *Let* $r \in \mathbb{N}_0$*. In the KdV and CH cases one derives*[1] *(cf.* (E.2)*)*

$$\widehat{F}_r(\mu_j) = \sum_{s=(r-n)\vee 0}^{r} \hat{c}_s(\underline{E})\Phi_{r-s}^{(j)}(\underline{\mu}). \tag{F.5}$$

For the AKNS hierarchy, one infers

$$\widehat{F}_r(\mu_j) = -iq \sum_{s=(r-n)\vee 0}^{r} \hat{c}_s(\underline{E})\Phi_{r-s}^{(j)}(\underline{\mu}), \tag{F.6}$$

$$\widehat{H}_r(\nu_j) = ip \sum_{s=(r-n)\vee 0}^{r} \hat{c}_s(\underline{E})\Phi_{r-s}^{(j)}(\underline{\nu}). \tag{F.7}$$

[1] $m \vee n = \max\{m, n\}$.

For the sGmKdV hierarchy one concludes[1]

$$\frac{\widehat{F}_r(\mu_j)}{\mu_j} = \sum_{s=(r-1-n)\vee 0}^{r-1} \hat{c}_s(\underline{E})\Phi^{(j)}_{r-1-s}(\underline{\mu}) - \frac{\tilde{\alpha}}{\alpha}\Phi^{(j)}_{n-1}(\underline{\mu}), \tag{F.8}$$

$$\frac{\widehat{H}_r(\nu_j)}{\nu_j} = \sum_{s=(r-1-n)\vee 0}^{r-1} \hat{c}_s(\underline{E})\Phi^{(j)}_{r-1-s}(\underline{\nu}) - \frac{\tilde{\beta}}{\beta}\Phi^{(j)}_{n-1}(\underline{\nu}). \tag{F.9}$$

Proof It suffices to consider the KdV and sGmKdV cases. By definition

$$\widehat{F}_r(z) = \sum_{\ell=0}^{r} \hat{f}_{r-\ell} z^\ell = \sum_{\ell=0}^{r} z^\ell \sum_{m=0}^{(r-\ell)\wedge n} \hat{c}_{r-\ell-m}(\underline{E})\Psi_m(\underline{\mu}).$$

Consider first the case $r \leq n$. Then

$$\widehat{F}_r(z) = \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-\ell-s}(\underline{\mu}) z^\ell \tag{F.10}$$

and hence

$$\widehat{F}_r(\mu_j) = \sum_{s=0}^{r} \hat{c}_s(\underline{E})\Phi^{(j)}_{r-s}(\underline{\mu}), \tag{F.11}$$

using (E.11). In the case in which $r \geq n+1$, we find (applying (E.3))

$$\begin{aligned} \widehat{F}_r(z) &= \sum_{m=0}^{n} \Psi_m(\underline{\mu}) \sum_{s=0}^{r-m} \hat{c}_s(\underline{E}) z^{r-m-s} \\ &= \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) \bigg(\sum_{\ell=0}^{n} \Psi_\ell(\underline{\mu}) z^{n-\ell} \bigg) z^{r-n-s} + \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_\ell(\underline{\mu}) z^{r-s-\ell} \\ &= F_n(z) \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_\ell(\underline{\mu}) z^{r-s-\ell} \\ &= F_n(z) \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^{\ell}. \end{aligned} \tag{F.12}$$

Hence,

$$\widehat{F}_r(\mu_j) = \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E})\Phi^{(j)}_{r-s}(\underline{\mu}), \tag{F.13}$$

using (E.11) again.

In the sGmKdV case, one observes the identity

$$\widehat{F}_r(z) = z\widehat{F}_{r-1}(z) + \hat{f}_r$$

[1] Since r is independent of n, one obtains $\hat{f}_r = \tilde{\alpha} e^{-iu}$, $\hat{h}_r = \tilde{\beta} e^{iu}$ with $\tilde{\alpha}, \tilde{\beta} \in \mathbb{C}$ independent of α, β, and $\hat{f}_q, \hat{h}_q$, $q = 1, \ldots, r-1$ constructed as in (2.31) and (2.32).

and computes

$$\frac{\widehat{F}_r(\mu_j)}{\mu_j} = \widehat{F}_{r-1}(\mu_j) + \frac{\hat{f}_r}{\mu_j} = \sum_{s=(r-1-n)\vee 0}^{r-1} \hat{c}_s(\underline{E})\Phi^{(j)}_{r-1-s}(\underline{\mu}) - \frac{\tilde{\alpha}}{\alpha}\Phi^{(j)}_{n-1}(\underline{\mu}), \quad \text{(F.14)}$$

using $\hat{f}_r = \tilde{\alpha}e^{-iu}$ and the trace relation (2.91), (2.92). $\square$

Introducing

$$d_{\ell,k}(\underline{E}) = \sum_{m=0}^{\ell-k} c_{\ell-k-m}(\underline{E})\hat{c}_m(\underline{E}), \quad k = 0, \dots, \ell, \ \ell = 0, \dots, n, \quad \text{(F.15)}$$

$$\tilde{d}_{r,k}(\underline{E}) = \sum_{s=0}^{r-k} \tilde{c}_{r-k-s}\hat{c}_s(\underline{E}), \quad k = 0, \dots, r \wedge n, \quad \text{(F.16)}$$

for a given set of constants $\{\tilde{c}_s\}_{s=1,\dots,r} \subset \mathbb{C}$, the corresponding nonhomogeneous quantities f_ℓ, $F_n(\mu_j)$, and $\widetilde{F}_r(\mu_j)$ in the KdV, AKNS, and CH cases are then given by

$$f_\ell = \sum_{k=0}^{\ell} c_{\ell-k}(\underline{E})\hat{f}_k = \sum_{k=0}^{\ell} d_{\ell,k}(\underline{E})\Psi_k(\underline{\mu}), \quad \ell = 0, \dots, n, \quad \text{(F.17)}$$

$$F_n(\mu_j) = \sum_{\ell=0}^{n} c_{n-\ell}(\underline{E})\widehat{F}_\ell(\mu_j) = \sum_{\ell-0}^{n} d_{n,\ell}(\underline{E})\Phi^{(j)}_\ell(\underline{\mu}), \quad c_0 = 1, \quad \text{(F.18)}$$

$$\widetilde{F}_r(\mu_j) = \sum_{s=0}^{r} \tilde{c}_{r-s}\widehat{F}_s(\mu_j) = \sum_{k=0}^{r\wedge n} \tilde{d}_{r,k}(\underline{E})\Phi^{(j)}_k(\underline{\mu}), \quad r \in \mathbb{N}_0, \ \tilde{c}_0 = 1, \quad \text{(F.19)}$$

using (D.59) and (D.60). Here $c_k(\underline{E})$, $k \in \mathbb{N}_0$, is defined by (D.5).

Before we continue with a detailed discussion of the theta function representations of the elementary symmetric functions $\Psi_k(\underline{\mu})$ of $\mu_1, \dots, \mu_n$ associated with the completely integrable systems discussed in this monograph, we take a closer look at the unique local solvability question of Dubrovin-type equations. The key ingredient for such an analysis is the following classical Frobenius-type theorem.

Theorem F.3 *Let $U \times V \subseteq \mathbb{R}^m \times \mathbb{R}^n$ be open, $(t_0, \alpha_0) \in U \times V$, and let $f_k = (f_{k,1}, \dots, f_{k,n})\colon U \times V \to \mathbb{R}^n$ be C^r functions, $r \in \mathbb{N} \cup \{\infty\}$, for all $k = 1, \dots, m$.*

(i) Then there exists a neighborhood $W \subseteq U$ of t_0, and at most one C^r function $\alpha\colon W \to V$ satisfying the initial value problem

$$\begin{aligned} &\alpha_{t_k} = f_k(\,\cdot\,, \alpha) \text{ on } V, \quad k = 1, \dots, m, \\ &\alpha(t_0) = \alpha_0, \end{aligned} \quad \text{(F.20)}$$

where $t = (t_1, \dots, t_m) \in W$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in V$.

(ii) *Moreover, such a function* α *exists if and only if there is a neighborhood of* $(t_0, \alpha_0) \in U \times V$ *on which*

$$f_{k,t_\ell} - f_{\ell,t_k} + \sum_{j=1}^{n} \big((\partial_{\alpha_j} f_k) f_{\ell,j} - (\partial_{\alpha_j} f_\ell) f_{k,j} \big) = 0, \quad k, \ell = 1, \dots, m. \tag{F.21}$$

Given this result it is easy to state the basic local existence and uniqueness theorem for Dubrovin-type equations.

Theorem F.4 *Let* $\mathcal{K}_n\colon y^2 = R_{2n+p}$, $R_{2n+p} = \prod_{m=0}^{2n+p}(z - E_m)$, $p = 0, 1$, *be a hyperelliptic Riemann surface of genus n of KdV- or AKNS-type with a nonsingular affine part. Assume* $(x_0, t_0) \in \mathbb{R}^2$ *and*

$$\hat{\mu}_{0,j} \in \mathcal{K}_n, \quad \mu_{j,0} \neq \mu_{0,j'}, \ j \neq j', \ j, j' = 1, \dots, n.$$

Consider the initial value problem

$$\begin{aligned}
\mu_{j,x} &= -2iy(\hat{\mu}_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} \big(\mu_j - \mu_k\big)^{-1}, \\
\mu_{j,t} &= -2iy(\hat{\mu}_j) \widetilde{F}_j(\underline{\mu}) \prod_{\substack{k=1 \\ k \neq j}}^{n} \big(\mu_j - \mu_k\big)^{-1}, \\
\hat{\mu}_j(x_0, t_0) &= \hat{\mu}_{0,j}, \quad j = 1, \dots, n,
\end{aligned} \tag{F.22}$$

where $\underline{\mu} = (\mu_1, \dots, \mu_n)$ *and*

$$\widetilde{F}_j = \widetilde{F}_j(\underline{\mu}) = \sum_{s=0}^{r} c_s \Phi_{r-s}^{(j)}(\underline{\mu}), \quad j = 1, \dots, n$$

for some constants $c_s \in \mathbb{C}$, $s = 0, \dots, r$, *and some* $r \in \mathbb{N}_0$. *Then there exists an open and connected set* $\Omega \subseteq \mathbb{R}^2$, *with* $(x_0, t_0) \in \Omega$, *such that the initial value problem* (F.22) *has a unique solution* $\{\hat{\mu}_j\}_{j=1,\dots,n} \subset \mathcal{K}_n$ *satisfying*

$$\hat{\mu}_j \in C^\infty(\Omega, \mathcal{K}_n), \quad j = 1, \dots, n.$$

Proof Using appropriate charts on $\mathcal{K}_n$, one can apply Theorem F.3, and the proof reduces to showing that the integrability condition (F.21) is satisfied for the system (F.22). We first study the case in which each $\hat{\mu}_j$ stays away from all branch points $(E_m, 0)$, $m = 0, \dots, 2n + p$. Introducing

$$\phi_j = -2iy(\hat{\mu}_j) \prod_{\substack{k=1 \\ k \neq j}}^{n} (\mu_j - \mu_k)^{-1}, \quad \psi_j = \phi_j \widetilde{F}_j,$$

the jth component of the integrability condition (F.21) reads

$$\sum_{k=1}^{n}\big((\partial_{\mu_k}\phi_j)\phi_k\widetilde{F}_k-(\partial_{\mu_k}\phi_j\widetilde{F}_j)\phi_k\big)$$
$$=\sum_{k=1}^{n}\big(\phi_k(\partial_{\mu_k}\phi_j)(\widetilde{F}_k-\widetilde{F}_j)-\phi_k(\partial_{\mu_k}\widetilde{F}_j)\phi_j\big). \tag{F.23}$$

Using the facts

$$\partial_{\mu_j}\widetilde{F}_j=0,\quad \partial_{\mu_k}\phi_j=\phi_j(\mu_j-\mu_k)^{-1},\ j\neq k, \tag{F.24}$$

we may rewrite (F.23) as

$$\sum_{\substack{k=1\\k\neq j}}^{n}\phi_j\phi_k\sum_{s=0}^{r}c_s\big((\mu_j-\mu_k)^{-1}\big(\Phi_{r-s}^{(k)}-\Phi_{r-s}^{(j)}\big)-\partial_{\mu_k}\Phi_{r-s}^{(j)}\big)=0,$$

since

$$\partial_{\mu_k}\Phi_{r-s}^{(j)}=(\mu_j-\mu_k)^{-1}\big(\Phi_{r-s}^{(k)}-\Phi_{r-s}^{(j)}\big).$$

Next, we check the case in which one of the $\hat{\mu}_j$ hits one of the branch points $(E_m,0)\in\mathcal{B}(\mathcal{K}_n)$ and hence the right-hand sides of the system (F.22) vanish for some j. Hence, we suppose

$$\mu_{j_0}(x,t)\to E_{m_0}\text{ as }(x,t)\to(\tilde{x}_0,\tilde{t}_0)\in\Omega,$$

for some $j_0\in\{1,\dots,n\}$, $m_0\in\{0,\dots,2n+p\}$. Introducing

$$\zeta_{j_0}(x,t)=\sigma(\mu_{j_0}(x,t)-E_{m_0})^{1/2},\ \sigma=\pm1,\quad \mu_{j_0}(x,t)=E_{m_0}+\zeta_{j_0}(x,t)^2$$

for (x,t) in a neighborhood of $(\tilde{x}_0,\tilde{t}_0)$, the Dubrovin system (F.22) for μ_{j_0} becomes

$$\zeta_{j_0,x}=c(\sigma)\left(\prod_{\substack{m=0\\m\neq m_0}}^{2n+p}\big(\zeta_{j_0}^2+E_{m_0}-E_m\big)\right)^{1/2}\prod_{\substack{k=1\\k\neq j_0}}^{n}\big(\zeta_{j_0}^2+E_{m_0}-\mu_k\big)^{-1},$$

$$\zeta_{j_0,t}=c(\sigma)\widetilde{F}_j(\underline{\mu})\left(\prod_{\substack{m=0\\m\neq m_0}}^{2n+p}\big(\zeta_{j_0}^2+E_{m_0}-E_m\big)\right)^{1/2}\prod_{\substack{k=1\\k\neq j_0}}^{n}\big(\zeta_{j_0}^2+E_{m_0}-\mu_k\big)^{-1}$$

for some $|c(\sigma)|=1$. Relations (F.24) remain valid, and hence the integrability condition (F.21) holds in this case as well. □

Next we turn to theta function representation of the elementary symmetric functions $\Psi_k(\underline{\mu})$. Given a compact hyperelliptic Riemann surface $\mathcal{K}_n$ of genus n (of KdV or AKNS-type), we introduce the first-order Dubrovin-type system

(or generalized Dubrovin equations)

$$\partial_{v_k}\mu_j(\underline{v}) = \Phi^{(j)}_{n-k}(\underline{\mu}(\underline{v}))\frac{y(\hat{\mu}_j(\underline{v}))}{\prod_{\substack{m=1\\ m\neq j}}^n(\mu_j(\underline{v})-\mu_m(\underline{v}))}, \quad j,k=1,\dots,n, \quad \underline{v}=(v_1,\dots,v_n)\in\mathcal{V} \tag{F.25}$$

with initial conditions

$$\{\hat{\mu}_j(\underline{v}_0)\}_{j=1,\dots,n}\subset\mathcal{K}_n \tag{F.26}$$

for some $\underline{v}_0\in\mathcal{V}$, where $\mathcal{V}\subseteq\mathbb{C}^n$ is an open connected set such that μ_j remain distinct on $\mathcal{V}$, $\mu_j\neq\mu_{j'}$ for $j\neq j'$, $j,j'=1,\dots,n$. The obvious extension from (x,t) to $\underline{v}=(v_1,\dots,v_n)$ of Theorem F.4 clearly applies to the system (F.25), (F.26). One then obtains, using (E.4) and (F.25),

$$\begin{aligned}\partial_{v_k}\sum_{j=1}^n\int_{Q_0}^{\hat{\mu}_j(\underline{v})}\frac{z^{k-1}dz}{y} &= \sum_{j=1}^n\frac{\mu_j(\underline{v})^{k-1}}{y(\hat{\mu}_j(\underline{v}))}\partial_{v_k}\mu_j(\underline{v})\\ &= \sum_{j=1}^n\frac{\mu_j(\underline{v})^{k-1}}{y(\hat{\mu}_j(\underline{v}))}\Phi^{(j)}_{n-k}(\underline{\mu}(\underline{v}))\frac{y(\hat{\mu}_j(\underline{v}))}{\prod_{\substack{m=1\\ m\neq j}}^n(\mu_j(\underline{v})-\mu_m(\underline{v}))}\\ &= \sum_{j=1}^n\Phi^{(j)}_{n-k}(\underline{\mu}(\underline{v}))\frac{\mu_j(\underline{v})^{k-1}}{\prod_{\substack{m=1\\ m\neq j}}^n(\mu_j(\underline{v})-\mu_m(\underline{v}))}=1,\end{aligned} \tag{F.27}$$

implying

$$\sum_{j=1}^n\int_{Q_0}^{\hat{\mu}_j(\underline{v})}\frac{z^{k-1}dz}{y}-\sum_{j=1}^n\int_{Q_0}^{\hat{\mu}_j(\underline{v}_0)}\frac{z^{k-1}dz}{y}=(\underline{v})_k-(\underline{v}_0)_k, \quad k=1,\dots,n, \quad \underline{v},\underline{v}_0\in\mathcal{V}.$$

Moreover, introducing

$$v_{n+1}(\underline{v})=\sum_{j=1}^n\int_{Q_0}^{\hat{\mu}_j(\underline{v})}\frac{z^n dz}{y},$$

one then computes, as in (F.27),

$$\partial_{v_k}v_{n+1}(\underline{v})=-\Psi_{n+1-k}(\underline{\mu}(\underline{v})), \quad k=1,\dots,n, \tag{F.28}$$

using

$$\sum_{j=1}^n\Phi^{(j)}_{n-k}(\underline{\mu})\frac{\mu_j^n}{\prod_{\substack{q=1\\ q\neq j}}^n(\mu_j-\mu_q)}=-\Psi_{n+1-k}(\underline{\mu}), \quad k=1,\dots,n$$

(cf. (E.4)). Thus, one concludes

$$\prod_{j=1}^{n}(z-\mu_j(\underline{v})) = \sum_{\ell=0}^{n} \Psi_{n-\ell}(\underline{\mu}(\underline{v}))z^{\ell} = z^n - \sum_{k=1}^{n} \partial_{v_k} v_{n+1}(\underline{v})z^{k-1}, \quad \underline{v} \in \mathcal{V}, \tag{F.29}$$

whenever $\underline{\mu}$ satisfies (F.25).

Next, we recall our notation (cf. (A.34), (A.35), (A.41), (A.42), and (A.45))

$$\begin{aligned}
&\underline{z}(P,\underline{Q}) = \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{Q}}),\\
&P \in \mathcal{K}_n, \ \underline{Q} = \{Q_1, \dots, Q_n\} \in \mathrm{Sym}^n(\mathcal{K}_n),\\
&\underline{\hat{z}}(P,\underline{Q}) = \underline{\widehat{\Xi}}_{Q_0} - \underline{\widehat{A}}_{Q_0}(P) + \underline{\widehat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{Q}}),\\
&P \in \widehat{\mathcal{K}}_n, \ \underline{Q} = \{Q_1, \dots, Q_n\} \in \mathrm{Sym}^n(\widehat{\mathcal{K}}_n)
\end{aligned}$$

in connection with $\mathcal{K}_n$ and $\widehat{\mathcal{K}}_n$, respectively. Moreover, we conveniently choose $Q_0 \in \partial\widehat{\mathcal{K}}_n$ (e.g., the initial point of the curve $a_1 \subset \partial\widehat{\mathcal{K}}_n$). In addition, we recall (cf. (B.28), (B.29), (C.32), (C.33))

$$C = (C_{j,k})_{j,k=1,\dots,n}, \quad C_{j,k} = \int_{a_k} \eta_j, \tag{F.30}$$

$$\underline{c}(k) = (c_1(k), \dots, c_n(k)), \quad c_j(k) = (C^{-1})_{j,k}, \quad j,k = 1, \dots, n. \tag{F.31}$$

To derive theta function representations of the elementary symmetric functions $\Psi_k(\underline{\mu})$ of $\mu_1, \dots, \mu_n$, $k = 1, \dots, n$, we now need to distinguish between the cases of KdV- and AKNS-type curves. We start with the case of compact hyperelliptic KdV-type Riemann surfaces of genus n, where $\mathcal{K}_n$ corresponds to $y^2 = \prod_{m=0}^{2n}(z - E_m)$ with pairwise distinct $E_m \in \mathbb{C}$, $m = 0, \dots, 2n+1$ (cf. (B.1) and (B.16)). We can write

$$v_{n+1}(\underline{v}) = \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \frac{z^n dz}{y} = -2\sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \tilde{\omega}^{(2)}_{P_\infty,0}, \tag{F.32}$$

where

$$\tilde{\omega}^{(2)}_{P_\infty,0} = -z^n dz/(2y) = -\tilde{\pi}\eta_n/2 \tag{F.33}$$

represents a differential of the second kind that is not necessarily normalized, that is, the a-periods of $\tilde{\omega}^{(2)}_{P_\infty,0}$ do not necessarily vanish. We also recall the normalized differential of the second kind (cf. (1.98), (1.99)),

$$\omega^{(2)}_{P_\infty,0} = -\frac{1}{2y}\prod_{j=1}^{n}(z-\lambda_j)dz, \quad \underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n, \tag{F.34}$$

$$\int_{a_j} \omega^{(2)}_{P_\infty,0} = 0, \quad j = 1, \dots, n \tag{F.35}$$

with b-periods (cf. (1.101), (1.102)

$$U_{0,j}^{(2)} = \frac{1}{2\pi i}\int_{b_j} \omega_{P_\infty,0}^{(2)} = -2c_j(n), \quad j = 1,\dots,n. \tag{F.36}$$

Theorem F.5 *Suppose $\mathcal{D}_{\underline{\hat{\mu}}} \in \operatorname{Sym}^n(\widehat{\mathcal{K}}_n)$ is nonspecial and $\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_n\} \in \operatorname{Sym}^n(\widehat{\mathcal{K}}_n)$. Then,*

$$\begin{aligned}\sum_{j=1}^n \int_{Q_0}^{\hat{\mu}_j} \tilde{\omega}_{P_\infty,0}^{(2)} &= \sum_{j=1}^n \bigg(\int_{a_j} \tilde{\omega}_{P_\infty,0}^{(2)}\bigg) \sum_{k=1}^n \bigg(\int_{Q_0}^{\hat{\mu}_k} \omega_j - \int_{a_k} \big(\underline{\widehat{A}}_{Q_0}\big)_j \omega_k\bigg) \\ &\quad - \sum_{j=1}^n U_{0,j}^{(2)} \partial_{w_j} \ln\Big(\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}) + \underline{w})\Big)\Big|_{\underline{w}=0},\end{aligned} \tag{F.37}$$

and

$$\begin{aligned}\Psi_{n+1-k}(\underline{\mu}) &= \Psi_{n+1-k}(\underline{\lambda}) \\ &\quad - 2\sum_{j,\ell=1}^n U_{0,j}^{(2)} c_\ell(k) \partial_{w_j w_\ell}^2 \ln\Big(\theta(\underline{z}(P_\infty,\underline{\hat{\mu}}) + \underline{w})\Big)\Big|_{\underline{w}=0}, \quad k = 1,\dots,n,\end{aligned} \tag{F.38}$$

with $\underline{\lambda} = (\lambda_1,\dots,\lambda_n) \in \mathbb{C}^n$ introduced in (F.34).

Proof Let $\mathcal{D}_{\underline{\hat{\mu}}} \in \operatorname{Sym}^n(\widehat{\mathcal{K}}_n)$ be a nonspecial divisor on $\widehat{\mathcal{K}}_n$, $\underline{\hat{\mu}} = \{\hat{\mu}_1,\dots,\hat{\mu}_n\} \in \operatorname{Sym}^n(\widehat{\mathcal{K}}_n)$. Introducing

$$\widetilde{\Omega}^{(2)}(P) = \int_{Q_0}^P \tilde{\omega}_{P_\infty,0}^{(2)}, \quad P \in \mathcal{K}_n \setminus \{P_\infty\},$$

as well as the meromorphic differential

$$\nu = d\ln\big(\theta(\underline{z}(\,\cdot\,,\underline{\hat{\mu}}))\big),$$

the residue theorem applied to $\widetilde{\Omega}^{(2)}\nu$ yields

$$\begin{aligned}\int_{\partial\widehat{\mathcal{K}}_n} \widetilde{\Omega}^{(2)}\nu &= \sum_{j=1}^n \bigg(\bigg(\int_{a_j} \tilde{\omega}_{P_\infty,0}^{(2)}\bigg)\bigg(\int_{b_j}\nu\bigg) - \bigg(\int_{b_j} \tilde{\omega}_{P_\infty,0}^{(2)}\bigg)\bigg(\int_{a_j}\nu\bigg)\bigg) \\ &= 2\pi i \sum_{P\in\widehat{\mathcal{K}}_n} \operatorname*{res}_P \big(\widetilde{\Omega}^{(2)}\nu\big).\end{aligned} \tag{F.39}$$

Investigating separately the items occurring in (F.39) then yields the following facts:

$$\int_{a_j} \nu = 0, \quad j = 1,\dots,n, \tag{F.40}$$

$$\begin{aligned}\int_{b_j} \nu &= 2\pi i\Big(\big(\underline{\widehat{\Xi}}_{Q_0}\big)_j - \big(\underline{\widehat{A}}_{Q_0}(R(a_j))\big)_j + \big(\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}})\big)_j\Big) - i\pi\tau_{j,j} \\ &= \sum_{k=1}^n \bigg(\int_{Q_0}^{\hat{\mu}_k} \omega_j - \int_{a_k} \big(\underline{\widehat{A}}_{Q_0}\big)_j \omega_k\bigg), \quad j = 1,\dots,n,\end{aligned} \tag{F.41}$$

$$\sum_{P\in\widehat{\mathcal{K}}_n} \operatorname*{res}_{P}(\widetilde{\Omega}^{(2)}\nu) = \sum_{j=1}^{n} \widetilde{\Omega}^{(2)}(\hat{\mu}_j) + \operatorname*{res}_{P_\infty}(\widetilde{\Omega}^{(2)}\nu)$$
$$= \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j} \tilde{\omega}^{(2)}_{P_\infty,0} + \operatorname*{res}_{P_\infty}(\widetilde{\Omega}^{(2)}\nu), \tag{F.42}$$

by applying (A.39) in (F.40) and (F.42) and recalling the well-known results

$$\big(\underline{\widehat{A}}_{Q_0}(R(a_j))\big)_j = \frac{1}{2} + \int_{a_j} \big(\underline{\widehat{A}}_{Q_0}\big)_j \omega_j, \quad j = 1, \dots, n, \tag{F.43}$$

$$\big(\underline{\widehat{\Xi}}_{Q_0}\big)_j = \frac{1}{2}(1 + \tau_{j,j}) - \sum_{\substack{k=1\\k\neq j}}^{n} \int_{a_k} \big(\underline{\widehat{A}}_{Q_0}\big)_j \omega_k, \quad j = 1, \dots, n. \tag{F.44}$$

Here $R(a_j)$ denotes the end point of $a_j \subset \partial\widehat{\mathcal{K}}_n$, $j = 1, \dots, n$. As for the residue evaluation at P_∞ we proceed as follows. First we recall (cf. (1.100))

$$\widetilde{\Omega}^{(2)}(P) \underset{\zeta\to 0}{=} -\zeta^{-1} + O(1) \text{ as } P \to P_\infty.$$

Moreover,

$$\nu(P) = \frac{d\theta(\underline{z}(P, \underline{\hat{\mu}}))}{\theta(\underline{z}(P, \underline{\hat{\mu}}))}$$
$$\underset{\zeta\to 0}{=} \bigg(-\sum_{j=1}^{n} U^{(2)}_{0,j} \partial_{w_j} \ln\big(\theta(\underline{\hat{z}}(P_\infty, \underline{\hat{\mu}}) + \underline{w})\big)\big|_{\underline{w}=0} + O(\zeta^2)\bigg) d\zeta \text{ as } P \to P_\infty,$$

according to (1.118). Thus,

$$\big(\widetilde{\Omega}^{(2)}\nu\big)(P) \underset{\zeta\to 0}{=} \bigg(\frac{1}{\zeta} \sum_{j=1}^{n} U^{(2)}_{0,j} \partial_{w_j} \ln\big(\theta(\underline{\hat{z}}(P_\infty, \underline{\hat{\mu}}) + \underline{w})\big)\big|_{\underline{w}=0} + O(1)\bigg) d\zeta,$$
$$\text{as } P \to P_\infty,$$

implying

$$\operatorname*{res}_{P_\infty}(\widetilde{\Omega}^{(2)}\nu) = \sum_{j=1}^{n} U^{(2)}_{0,j} \partial_{w_j} \ln\big(\theta(\underline{\hat{z}}(P_\infty, \underline{\hat{\mu}}) + \underline{w})\big)\big|_{\underline{w}=0}.$$

Introducing the results above into (F.39), one concludes

$$\sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j} \tilde{\omega}^{(2)}_{P_\infty,0} = \sum_{j=1}^{n} \bigg(\int_{a_j} \tilde{\omega}^{(2)}_{P_\infty,0}\bigg) \sum_{k=1}^{n} \bigg(\int_{Q_0}^{\hat{\mu}_k} \omega_j - \int_{a_k} \big(\underline{\widehat{A}}_{Q_0}\big)_j \omega_k\bigg)$$
$$- \sum_{j=1}^{n} U^{(2)}_{0,j} \partial_{w_j} \ln\big(\theta(\underline{z}(P_\infty, \underline{\hat{\mu}}) + \underline{w})\big)\big|_{\underline{w}=0}. \tag{F.45}$$

Here we replaced $\underline{\hat{z}}$ by $\underline{z}$ to arrive at (F.45) using properties (A.39) of θ. This proves (F.37).

Next, assume that $\hat{\underline{\mu}}$ satisfies the Dubrovin system (F.25), (F.26) on some open connected set $\mathcal{V}$ such that μ_j, $j = 1, \dots, n$, remain distinct on $\mathcal{V}$. Then, by means of (B.30), (F.25), and (E.4),

$$\begin{aligned}
\partial_{v_k}\big(\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(\underline{v})})\big)_j &= \partial_{v_k}\sum_{\ell=1}^{n}\int_{Q_0}^{\hat{\mu}_\ell(\underline{v})}\omega_j = \sum_{\ell,m=1}^{n}\partial_{v_k}\int_{Q_0}^{\hat{\mu}_\ell(\underline{v})}c_j(m)\eta_m \\
&= \sum_{\ell,m=1}^{n} c_j(m)\frac{\mu_\ell(\underline{v})^{m-1}}{y(\hat{\mu}_\ell(\underline{v}))}\partial_{v_k}\mu_\ell(\underline{v}) \\
&= \sum_{\ell,m=1}^{n} c_j(m)\Phi_{n-k}^{(\ell)}\frac{\mu_\ell(\underline{v})^{m-1}}{\prod_{\substack{\ell'=1\\ \ell'\neq\ell}}^{n}(\mu_\ell(\underline{v})-\mu_{\ell'}(\underline{v}))} \\
&= c_j(k), \quad \underline{v}\in\mathcal{V}.
\end{aligned} \tag{F.46}$$

Moreover, using (F.28), (F.32), (F.37), and (F.46), one computes

$$\begin{aligned}
\Psi_{n+1-k}(\underline{\mu}(\underline{v})) &= -\partial_{v_k}v_{n+1} = 2\partial_{v_k}\sum_{j=1}^{n}\int_{Q_0}^{\hat{\mu}_j(\underline{v})}\tilde{\omega}_{P_\infty,0}^{(2)} \\
&= 2\sum_{j=1}^{n}\bigg(\int_{a_j}\tilde{\omega}_{P_\infty,0}^{(2)}\bigg)c_j(k) \\
&\quad - 2\sum_{j=1}^{n}U_{0,j}^{(2)}\partial_{v_k w_j}^2\ln\big(\theta(\underline{z}(P_\infty,\hat{\underline{\mu}}(\underline{v}))+\underline{w})\big)\big|_{\underline{w}=0}.
\end{aligned}$$

Using (F.46), one obtains

$$\begin{aligned}
\underline{z}(P_\infty,\hat{\underline{\mu}}(\underline{v})) &= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\hat{\underline{\mu}}(\underline{v})}\big) \\
&= \underline{\Xi}_{Q_0} - \underline{A}_{Q_0}(P) + \underline{\alpha}_{Q_0}\big(\mathcal{D}_{\hat{\underline{\mu}}(\underline{v}_0)}\big) + C^{-1}(\underline{v}-\underline{v}_0) \\
&= \underline{z}(P_\infty,\hat{\underline{\mu}}(\underline{v}_0)) + C^{-1}(\underline{v}-\underline{v}_0),
\end{aligned}$$

where C is the matrix defined in (F.30). Thus,

$$\begin{aligned}
&\partial_{v_k w_j}^2\ln\big(\theta(\underline{z}(P_\infty,\hat{\underline{\mu}}(\underline{v}))+\underline{w})\big)\big|_{\underline{w}=0} \\
&\quad = \sum_{\ell=1}^{n}c_\ell(k)\partial_{w_j w_\ell}^2\ln\big(\theta(\underline{z}(P_\infty,\hat{\underline{\mu}}(\underline{v}))+\underline{w})\big)\big|_{\underline{w}=0}
\end{aligned}$$

and hence

$$\begin{aligned}
\Psi_{n+1-k}(\underline{\mu}(\underline{v})) &= 2\sum_{j=1}^{n}\bigg(\int_{a_j}\tilde{\omega}_{P_\infty,0}^{(2)}\bigg)c_j(k) \\
&\quad - 2\sum_{j,\ell=1}^{n}U_{0,j}^{(2)}c_\ell(k)\partial_{w_j w_\ell}^2\ln\big(\theta(\underline{z}(P_\infty,\hat{\underline{\mu}}(\underline{v}))+\underline{w})\big)\big|_{\underline{w}=0}.
\end{aligned} \tag{F.47}$$

Since

$$\omega^{(2)}_{P_\infty,0} = -\frac{1}{2y}\prod_{j=1}^{n}(z-\lambda_j)dz = -\frac{1}{2y}\sum_{\ell=0}^{n}\Psi_{n-\ell}(\underline{\lambda})z^\ell dz$$
$$= \tilde\omega^{(2)}_{P_\infty,0} - \frac{1}{2}\sum_{\ell=0}^{n-1}\Psi_{n-\ell}(\underline{\lambda})\eta_{\ell+1}, \quad \underline{\lambda} = (\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n,$$

one concludes

$$0 = \int_{a_j}\omega^{(2)}_{P_\infty,0} = \int_{a_j}\tilde\omega^{(2)}_{P_\infty,0} - \frac{1}{2}\sum_{\ell=1}^{n-1}\Psi_{n-\ell}(\underline{\lambda})\int_{a_j}\eta_{\ell+1}$$
$$= \int_{a_j}\tilde\omega^{(2)}_{P_\infty,0} - \frac{1}{2}\sum_{m=1}^{n}\Psi_{n+1-m}(\underline{\lambda})C_{m,j},$$

and hence

$$\sum_{j=1}^{n} c_j(k)\int_{a_j}\tilde\omega^{(2)}_{P_\infty,0} = \frac{1}{2}\sum_{j,m=1}^{n}c_j(k)\Psi_{n+1-m}(\underline{\lambda})C_{m,j} = \frac{1}{2}\Psi_{n+1-k}(\underline{\lambda}). \quad \text{(F.48)}$$

This proves (F.38). $\square$

Remark F.6 The special case $k=n$ in (F.47) yields

$$\sum_{j=1}^{n}\mu_j = \sum_{j=1}^{n}c_j(n)\bigg(\int_{a_j}\tilde\omega^{(2)}_{P_\infty,0}\bigg)$$
$$- \sum_{j,\ell=1}^{n}U^{(2)}_{0,j}U^{(2)}_{0,\ell}\partial^2_{w_j w_\ell}\ln\Big(\theta(\underline{z}(P_\infty,\underline{\hat\mu})+\underline{w})\Big)\Big|_{\underline{w}=0}. \quad \text{(F.49)}$$

To reconcile (F.49) and (B.44), one computes

$$\sum_{j=1}^{n}\int_{a_j}\tilde\pi\omega_j = \sum_{j,k=1}^{n}c_j(k)\int_{a_j}\tilde\pi\eta_k$$
$$= \sum_{j=1}^{n}\bigg(c_j(n)\int_{a_j}\tilde\pi\eta_n + \sum_{k=1}^{n-1}c_j(k)\int_{a_j}\eta_{k+1}\bigg)$$
$$= \sum_{j=1}^{n}c_j(n)\int_{a_j}\tilde\pi\eta_n + \sum_{k=1}^{n-1}\sum_{j=1}^{n}C_{k+1,j}(C^{-1})_{j,k}$$
$$= \sum_{j=1}^{n}c_j(n)\int_{a_j}\tilde\pi\eta_n = \sum_{j=1}^{n}c_j(n)\bigg(\int_{a_j}\tilde\omega^{(2)}_{P_\infty,0}\bigg). \quad \text{(F.50)}$$

Thus, recalling $\underline{c}(n) = -\underline{U}^{(2)}_0/2$ (cf. (F.36)),

$$\sum_{j=1}^{n}\mu_j = \sum_{j=1}^{n}\int_{a_j}\tilde\pi\omega_j - \sum_{j,\ell=1}^{n}U^{(2)}_{0,j}U^{(2)}_{0,\ell}\partial^2_{w_j w_\ell}\ln\Big(\theta(\underline{z}(P_\infty,\underline{\hat\mu})+\underline{w})\Big)\Big|_{\underline{w}=0}, \quad \text{(F.51)}$$

which is in agreement with (B.44). Moreover, taking $k = n$ in (F.38) yields agreement with (B.46).

Next, we turn to the case of compact hyperelliptic AKNS-type Riemann surfaces of genus n, where $\mathcal{K}_n$ corresponds to $y^2 = \prod_{m=0}^{2n+1}(z - E_m)$ with pairwise distinct $E_m \in \mathbb{C}$, $m = 0, \dots, 2n+1$ (cf. (C.1) and (C.16)). In this context $v_{n+1}(\underline{v})$ can be written as

$$v_{n+1}(\underline{v}) = \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \frac{z^n dz}{y} = \sum_{j=1}^{n} \int_{Q_0}^{\hat{\mu}_j(\underline{v})} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}, \tag{F.52}$$

where

$$\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} = z^n dz/y = \tilde{\pi}\eta_n \tag{F.53}$$

represents a differential of the third kind with simple poles at P_{∞_+} and P_{∞_-} and corresponding residues $+1$ and -1, respectively. This differential is not normalized, that is, the a-periods of $\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}$ do not necessarily vanish. We also recall the normal differential of the third kind (cf. (3.91), (3.92)),

$$\omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = \frac{1}{y}\prod_{j=1}^{n}(z - \lambda_j)dz, \quad \underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n, \tag{F.54}$$

$$\int_{a_j} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = 0, \quad j = 1, \dots, n, \tag{F.55}$$

and the normalized differentials $\omega^{(2)}_{P_{\infty_\pm},0}$ of the second kind (cf. (3.94))

$$\int_{a_j} \omega^{(2)}_{P_{\infty_\pm},0} = 0, \quad j = 1, \dots, n \tag{F.56}$$

with b-periods (cf. (C.37))

$$U^{(2)}_{\pm,0,j} = \frac{1}{2\pi i}\int_{b_j} \omega^{(2)}_{P_{\infty_\pm},0} = \pm c_j(n), \quad j = 1, \dots, n. \tag{F.57}$$

Theorem F.7 *Suppose* $\mathcal{D}_{\underline{\hat{\mu}}} \in \operatorname{Sym}^n\left(\widehat{\mathcal{K}}_n\right)$ *is nonspecial and* $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \operatorname{Sym}^n\left(\widehat{\mathcal{K}}_n\right)$. *Then,*

$$\begin{aligned}\sum_{j=1}^{n}\int_{Q_0}^{\hat{\mu}_j} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} &= \sum_{j=1}^{n}\left(\int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\right)\left(\sum_{k=1}^{n}\int_{Q_0}^{\hat{\mu}_k} \omega_j\right. \\ &\quad \left. - \sum_{k=1}^{n}\int_{a_k}\left(\underline{\widehat{A}}_{Q_0}\right)_j \omega_k\right) + \ln\left(\frac{\theta(\underline{\hat{z}}(P_{\infty_+}, \underline{\hat{\mu}}))}{\theta(\underline{\hat{z}}(P_{\infty_-}, \underline{\hat{\mu}}))}\right)\end{aligned} \tag{F.58}$$

and

$$\Psi_{n+1-k}(\underline{\mu}) = \Psi_{n+1-k}(\underline{\lambda}) - \sum_{j=1}^{n} c_j(k)\partial_{w_j} \ln\left(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}) + \underline{w})}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}) + \underline{w})}\right)\bigg|_{\underline{w}=0},$$
$$k = 1, \dots, n \quad \text{(F.59)}$$

with $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ *introduced in* (C.45).

Proof Let $\mathcal{D}_{\underline{\hat{\mu}}} \in \mathrm{Sym}^n\left(\widehat{\mathcal{K}}_n\right)$ be a nonspecial divisor on $\widehat{\mathcal{K}}_n$, $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\} \in \mathrm{Sym}^n\left(\widehat{\mathcal{K}}_n\right)$. Introducing

$$\widetilde{\Omega}^{(3)}(P) = \int_{Q_0}^{P} \tilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}}, \qquad P \in \mathcal{K}_n \setminus \{P_{\infty_+}, P_{\infty_-}\},$$

we can render $\widetilde{\Omega}^{(3)}(\,\cdot\,)$ single-valued on

$$\widehat{\widehat{\mathcal{K}}}_n = \widehat{\mathcal{K}}_n \setminus \Sigma,$$

where Σ denotes the union of cuts

$$\Sigma = \Sigma(P_{\infty_+}) \cup \Sigma(P_{\infty_-}), \quad \Sigma(P_{\infty_+}) \cap \Sigma(P_{\infty_-}) = \{Q_0\}$$

with $\Sigma(P_{\infty_+})$ (respectively $\Sigma(P_{\infty_-})$) a cut connecting Q_0 and P_{∞_+} (respectively P_{∞_-}) through the open interior $\widehat{\mathcal{K}}_n$ (i.e., avoiding all curves $a_j, b_j, a_j^{-1}, b_j^{-1}$, $j = 1, \dots, n$, with the exception of the point $Q_0 \in \partial\widehat{\mathcal{K}}_n$), avoiding $\hat{\mu}_j$, $j = 1, \dots, n$. The left and right side of the cut $\Sigma(P_{\infty_\pm})$ is denoted by $\Sigma(P_{\infty_\pm})_\ell$ and $\Sigma(P_{\infty_\pm})_r$. The oriented boundary $\partial\widehat{\widehat{\mathcal{K}}}_n$ of $\widehat{\widehat{\mathcal{K}}}_n$, in obvious notation, is then given by

$$\partial\widehat{\widehat{\mathcal{K}}}_n = \Sigma(P_{\infty_+})_\ell \cup \Sigma(P_{\infty_+})_r \cup \Sigma(P_{\infty_-})_\ell \cup \Sigma(P_{\infty_-})_r \cup \partial\widehat{\mathcal{K}}_n,$$

that is, it consists of $\partial\widehat{\mathcal{K}}_n$ together with the piece from Q_0 to P_{∞_+} along the left side of the cut $\Sigma(P_{\infty_+})$ and then back to Q_0 along the right side of $\Sigma(P_{\infty_+})$, plus the corresponding pieces from Q_0 to P_{∞_-} and back to Q_0 along the cut $\Sigma(P_{\infty_-})$, preserving orientation. Introducing the meromorphic differential,

$$\nu = d\ln\left(\theta(\underline{z}(\,\cdot\,, \underline{\hat{\mu}}))\right),$$

the residue theorem applied to $\widetilde{\Omega}^{(3)}\nu$ yields

$$\int_{\partial\widehat{\widehat{\mathcal{K}}}_n} \widetilde{\Omega}^{(3)}\nu = \sum_{j=1}^{n}\left(\left(\int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}}\right)\left(\int_{b_j} \nu\right) - \left(\int_{b_j} \tilde{\omega}^{(3)}_{P_{\infty_+}, P_{\infty_-}}\right)\left(\int_{a_j} \nu\right)\right)$$
$$+ \int_{\Sigma} \widetilde{\Omega}^{(3)}\nu = 2\pi i \sum_{P \in \widehat{\widehat{\mathcal{K}}}_n} \operatorname*{res}_{P}\left(\widetilde{\Omega}^{(3)}\nu\right). \quad \text{(F.60)}$$

Investigating separately the items occurring in (F.60) then yields the following facts:

$$\sum_{P\in\widehat{\widehat{\mathcal{K}}}_n}\operatorname*{res}_P(\widetilde{\Omega}^{(3)}\nu)=\sum_{j=1}^n\widetilde{\Omega}^{(3)}(\hat{\mu}_j)=\sum_{j=1}^n\int_{Q_0}^{\hat{\mu}_j}\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}, \tag{F.61}$$

$$\int_{a_j}\nu=0,\quad j=1,\dots,n, \tag{F.62}$$

$$\int_{b_j}\nu=2\pi i\Big(\big(\underline{\widehat{\Xi}}_{Q_0}\big)_j-\big(\underline{\widehat{A}}_{Q_0}(R(a_j))\big)_j+\big(\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}})\big)_j\Big)-i\pi\tau_{j,j}, \tag{F.63}$$
$$j=1,\dots,n,$$

applying (A.39) in (F.62) and (F.63). Here $R(a_j)$ denotes the end point of $a_j\subset\partial\widehat{\mathcal{K}}_n$, $j=1,\dots,n$. In addition, the cut Σ produces the contribution

$$\begin{aligned}\int_\Sigma\widetilde{\Omega}^{(3)}\nu&=2\pi i\bigg(\int_{Q_0}^{P_{\infty_+}}\nu-\int_{Q_0}^{P_{\infty_-}}\nu\bigg)=2\pi i\int_{P_{\infty_-}}^{P_{\infty_+}}\nu\\&=2\pi i\ln\bigg(\frac{\theta(\underline{\hat{z}}(P_{\infty_+},\underline{\hat{\mu}}))}{\theta(\underline{\hat{z}}(P_{\infty_-},\underline{\hat{\mu}}))}\bigg),\end{aligned}$$

since (by an application of the residue theorem)

$$\widetilde{\Omega}^{(3)}(\hat{\mu}_\ell)-\widetilde{\Omega}^{(3)}(\hat{\mu}_r)=\pm2\pi i,\quad\hat{\mu}_\ell\in\Sigma(P_{\infty_\pm})_\ell,\ \hat{\mu}_r\in\Sigma(P_{\infty_\pm})_r, \tag{F.64}$$

where $\hat{\mu}_\ell\in\Sigma(P_{\infty_\pm})_\ell$ and $\hat{\mu}_r\in\Sigma(P_{\infty_\pm})_r$ are on opposite sides of the cut $\Sigma(P_{\infty_\pm})$. Recalling the well-known results (F.43), (F.44), equations (F.60)–(F.64) imply

$$\begin{aligned}\sum_{j=1}^n\int_{Q_0}^{\hat{\mu}_j}\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}&=\sum_{j=1}^n\bigg(\int_{a_j}\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)\bigg(\sum_{k=1}^n\int_{Q_0}^{\hat{\mu}_k}\omega_j\\&\quad-\sum_{k=1}^n\int_{a_k}\big(\underline{\widehat{A}}_{Q_0}\big)_j\omega_k\bigg)+\ln\bigg(\frac{\theta(\underline{\hat{z}}(P_{\infty_+},\underline{\hat{\mu}}))}{\theta(\underline{\hat{z}}(P_{\infty_-},\underline{\hat{\mu}}))}\bigg).\end{aligned} \tag{F.65}$$

This proves (F.58).

In the following we will apply (F.65) assuming that $\underline{\hat{\mu}}=(\hat{\mu}_1,\dots,\hat{\mu}_n)$ satisfies the first-order system (F.25), (F.26) on some open, connected set $\mathcal{V}$ such that μ_j, $j=1,\dots,n$, remain distinct on $\mathcal{V}$ and $\Phi^{(j)}_{n-k}(\underline{\mu})\neq0$ on $\mathcal{V}$, $j,k=1,\dots,n$. As in (F.46) one computes

$$\partial_{v_k}\big(\underline{\hat{\alpha}}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(\underline{v})})\big)_j=c_j(k),\quad\underline{v}\in\mathcal{V}. \tag{F.66}$$

Thus, (F.29) and (F.66) imply

$$\begin{aligned}\Psi_{n+1-k}(\underline{\mu}(\underline{v}))&=-\partial_{v_k}v_{n+1}(\underline{v})=-\sum_{j=1}^n c_j(k)\bigg(\int_{a_j}\tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}}\bigg)\\&\quad-\sum_{j=1}^n c_j(k)\partial_{w_j}\ln\bigg(\frac{\theta(\underline{z}(P_{\infty_+},\underline{\hat{\mu}}(\underline{v}))+\underline{w})}{\theta(\underline{z}(P_{\infty_-},\underline{\hat{\mu}}(\underline{v}))+\underline{w})}\bigg)\bigg|_{\underline{w}=0},\\&\qquad\underline{v}\in\mathcal{V},\ k=1,\dots,n.\end{aligned} \tag{F.67}$$

Here we replaced $\underline{\hat{z}}$ by $\underline{z}$ to arrive at (F.67) using properties (A.39) of θ. If $\hat{\mu}_j$, $j = 1, \dots, n$, are distinct and $\Phi^{(j)}_{n-k}(\underline{\mu}) \neq 0$, $j, k = 1, \dots, n$, we can choose $\hat{\mu}_j(\underline{v}_0) = \hat{\mu}_j$, $j = 1, \dots, n$ and obtain (F.59). The general case in which $\mathcal{D}_{\underline{\hat{\mu}}}$ is nonspecial then follows from (F.67) by continuity with $\mathcal{V}$ chosen such that there exists a sequence $\underline{v}_n \in \mathcal{V}$ with $\underline{\hat{\mu}}(\underline{v}_n) \to \underline{\hat{\mu}}$ as $n \to \infty$. Finally, combining (C.31), (C.32), (C.33), (C.45), and the normalization $\int_{a_j} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} = 0$, $j = 1, \dots, n$, one computes

$$
\begin{aligned}
0 = \int_{a_j} \omega^{(3)}_{P_{\infty_+},P_{\infty_-}} &= \int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} + \sum_{\ell=0}^{n-1} \Psi_{n-\ell}(\underline{\lambda}) \int_{a_j} \frac{z^\ell dz}{y} \\
&= \int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} + \sum_{m=1}^{n} \Psi_{n+1-m}(\underline{\lambda}) \int_{a_j} \eta_m \\
&= \int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} + \sum_{m=1}^{n} \Psi_{n+1-m}(\underline{\lambda}) C_{m,j}, \quad j = 1, \dots, n,
\end{aligned}
$$

and thus

$$
\sum_{j=1}^{n} c_j(k) \bigg(\int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} \bigg) = - \sum_{j,m=1}^{n} c_j(k) \Psi_{n+1-m}(\underline{\lambda}) C_{m,j} = -\Psi_{n+1-k}(\underline{\lambda}), \tag{F.68}
$$

proving (F.59). $\square$

Remark F.8 The special case $k = n$ in (F.67) yields

$$
\begin{aligned}
\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} c_j(n) \bigg(\int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} \bigg) \\
+ \sum_{j=1}^{n} c_j(n) \partial_{w_j} \ln \left(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}) + \underline{w})}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}) + \underline{w})} \right) \bigg|_{\underline{w}=0}.
\end{aligned} \tag{F.69}
$$

To reconcile (F.69) and (C.49), one computes

$$
\begin{aligned}
\sum_{j=1}^{n} \int_{a_j} \tilde{\pi} \omega_j &= \sum_{j,k=1}^{n} c_j(k) \int_{a_j} \tilde{\pi} \eta_k \\
&= \sum_{j=1}^{n} \left(c_j(n) \int_{a_j} \tilde{\pi} \eta_n + \sum_{k=1}^{n-1} c_j(k) \int_{a_j} \eta_{k+1} \right) \\
&= \sum_{j=1}^{n} c_j(n) \int_{a_j} \tilde{\pi} \eta_n + \sum_{k=1}^{n-1} \sum_{j=1}^{n} C_{k+1,j} (C^{-1})_{j,k} \\
&= \sum_{j=1}^{n} c_j(n) \int_{a_j} \tilde{\pi} \eta_n = \sum_{j=1}^{n} c_j(n) \bigg(\int_{a_j} \tilde{\omega}^{(3)}_{P_{\infty_+},P_{\infty_-}} \bigg).
\end{aligned} \tag{F.70}
$$

Thus, if one recalls that $\underline{c}(n) = \underline{U}^{(2)}_{+,0}$ (cf. (C.50)),

$$\sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \int_{a_j} \tilde{\pi}\omega_j + \sum_{j=1}^{n} U^{(2)}_{+,0,j} \partial_{w_j} \ln\left(\frac{\theta(\underline{z}(P_{\infty_+}, \underline{\hat{\mu}}) + \underline{w})}{\theta(\underline{z}(P_{\infty_-}, \underline{\hat{\mu}}) + \underline{w})} \right)\bigg|_{\underline{w}=0}, \tag{F.71}$$

which is in agreement with (C.49). Moreover, taking $k = n$ in (F.59) yields agreement with (C.51).

Next, we turn to the Dubrovin equations for auxiliary divisors. Let $n \in \mathbb{N}$. We recall the construction of the hierarchies as explained in Section 1.2 (KdV) and Section 3.2 (AKNS). In particular, we recall the function F_n in (E.3) with its zeros $\underline{\mu} = \{\mu_1, \dots, \mu_n\}$ and introduce the corresponding hyperelliptic curve $\mathcal{K}_n$. (In the AKNS case we also consider the function H_n.) Next, we fix $r \in \mathbb{N}_0$ and construct the function F_r. The integration constants in the definition of F_r are assumed to be independent of those used to construct F_n, and to emphasize this fact we denote it by $\widetilde{F}_r$ and the corresponding constants by $\tilde{c}_\ell$. The Dubrovin equations give the evolution of $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ in terms of the deformation (time) parameter t_r according to the rth equation in the hierarchy considered.

In the following we assume that $\mu_j \neq \mu_{j'}$ on Ω_μ for $j \neq j'$, where $\Omega_\mu \subseteq \mathbb{R}^2$ is open and connected, and similarly for ν_j, $j = 1, \dots, n$.

The KdV Hierarchy. The Dubrovin equations for the KdV hierarchy on Ω_μ read (cf. (1.195), (1.196))

$$\mu_{j,x} = -2i \frac{y(\hat{\mu}_j)}{\prod_{\substack{k=1\\k\neq j}}^{n} (\mu_j - \mu_k)}, \tag{F.72}$$

$$\mu_{j,t_r} = \widetilde{F}_r(\mu_j)\mu_{j,x} = -2i \frac{y(\hat{\mu}_j)}{\prod_{\substack{k=1\\k\neq j}}^{n} (\mu_j - \mu_k)} \widetilde{F}_r(\mu_j), \quad j = 1, \dots, n \tag{F.73}$$

with initial data given by $\underline{\hat{\mu}}(x_0, t_{0,r}) \in \mathrm{Sym}^n(\mathcal{K}_n)$, where

$$\hat{\mu}_j = (\mu_j, -(i/2)F_{n,x}(\mu_j)) \in \mathcal{K}_n, \quad j = 1, \dots, n.$$

The following result is used in the proof of Theorem 1.48. We recall our notation $F_n'(z) = \partial F_n(z)/\partial z$ and that $\widetilde{F}_r$ is defined as in (F.10) or (F.12) with a set of integration constants $\{\tilde{c}_1, \dots, \tilde{c}_r\} \subset \mathbb{C}$.

Lemma F.9 *Suppose $r \in \mathbb{N}_0$, $(x, t_r) \in \Omega_\mu$, where $\Omega_\mu \subseteq \mathbb{R}^2$ is open and connected, and assume $\mu_j \neq \mu_{j'}$ on Ω_μ for $j \neq j'$, $j, j' = 1, \dots, n$. Then,*

$$\widetilde{F}_{r,x}(z) = \sum_{j=1}^{n} \big(\widetilde{F}_r(\mu_j) - \widetilde{F}_r(z)\big)\mu_{j,x}(z - \mu_j)^{-1} \tag{F.74}$$

$$= \sum_{j=1}^{n} \big(\widetilde{F}_r(z) - \widetilde{F}_r(\mu_j)\big) \frac{F_{n,x}(\mu_j)}{F_{n,xz}(\mu_j)}(z - \mu_j)^{-1}. \tag{F.75}$$

Proof It suffices to prove (F.74) for the homogeneous case where $\widetilde{F}_r$ is replaced by $\widehat{F}_r$. Using

$$\Psi_{k,x}(\underline{\mu}) = -\sum_{j=1}^{n} \mu_{j,x} \Phi_{k-1}^{(j)}(\underline{\mu}), \quad k = 0, \dots, n$$

with the convention

$$\Phi_{-1}^{(j)}(\underline{\mu}) = 0, \quad j = 1, \dots, n,$$

one computes for $r \le n$,

$$\begin{aligned}
\widehat{F}_{r,x}(z) &= \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell,x}(\underline{\mu}) z^\ell \\
&= -\sum_{j=1}^{n} \mu_{j,x} \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Phi_{r-s-\ell-1}^{(j)}(\underline{\mu}) z^\ell \\
&= \sum_{j=1}^{n} \mu_{j,x} (z-\mu_j)^{-1} \sum_{s=0}^{r} \hat{c}_s(\underline{E}) \bigg(\Phi_{r-s}^{(j)}(\underline{\mu}) - \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^\ell \bigg) \\
&= \sum_{j=1}^{n} \big(\widehat{F}_r(\mu_j) - \widehat{F}_r(z) \big) \mu_{j,x} (z-\mu_j)^{-1},
\end{aligned}$$

applying (E.12), (F.10), and (F.11). For $r \ge n+1$ one obtains from (E.12), (F.12), and (F.13),

$$\begin{aligned}
\widehat{F}_{r,x}(z) &= F_{n,x}(z) \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} + \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell,x}(\underline{\mu}) z^\ell \\
&= -F_n(z) \sum_{j=1}^{n} \mu_{j,x} (z-\mu_j)^{-1} \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} \\
&\quad - \sum_{j=1}^{n} \mu_{j,x} \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \sum_{\ell=0}^{r-s} \Phi_{r-s-\ell-1}^{(j)}(\underline{\mu}) z^\ell \\
&= -F_n(z) \sum_{j=1}^{n} \mu_{j,x} (z-\mu_j)^{-1} \sum_{s=0}^{r-n} \hat{c}_s(\underline{E}) z^{r-n-s} \\
&\quad + \sum_{j=1}^{n} \mu_{j,x} (z-\mu_j)^{-1} \sum_{s=r-n+1}^{r} \hat{c}_s(\underline{E}) \bigg(\Phi_{r-s}^{(j)}(\underline{\mu}) - \sum_{\ell=0}^{r-s} \Psi_{r-s-\ell}(\underline{\mu}) z^\ell \bigg) \\
&= \sum_{j=1}^{n} \big(\widehat{F}_r(\mu_j) - \widehat{F}_r(\underline{\mu}) \big) \mu_{j,x} (z-\mu_j)^{-1}.
\end{aligned}$$

Equation (F.75) immediately follows from (F.74) and $\mu_{j,x} = F_{n,x}(\mu_j)/F_n'(\mu_j)$ (cf. (1.163)). $\square$

The sGmKdV Hierarchy. Finally, in the case of the sGmKdV hierarchy, the Dubrovin-type equations for $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ read on Ω_μ (cf. (2.186), (2.187))

$$\mu_{j,x} = -2i \frac{y(\hat{\mu}_j)}{\prod_{\substack{k=1\\k\neq j}}^n (\mu_j - \mu_k)}, \tag{F.76}$$

$$\mu_{j,t_r} = 2\frac{\widetilde{F}_r(\mu_j)}{\mu_j} \frac{y(\hat{\mu}_j)}{\prod_{\substack{k=1\\k\neq j}}^n (\mu_j - \mu_k)}, \quad j = 1, \dots, n \tag{F.77}$$

with initial data given by $\underline{\hat{\mu}}(x_0, t_{0,r}) \in \operatorname{Sym}^n(\mathcal{K}_n)$, where

$$\hat{\mu}_j = (\mu_j, -\mu_j G_{n-1}(\mu_j)) \in \mathcal{K}_n, \quad j = 1, \dots, n.$$

The corresponding equations for $\underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$ equal (cf. (2.190), (2.191))

$$\nu_{j,x} = -2i \frac{y(\hat{\nu}_j)}{\prod_{\substack{k=1\\k\neq j}}^n (\nu_j - \nu_k)}, \tag{F.78}$$

$$\nu_{j,t_r} = 2\frac{\widetilde{H}_r(\nu_j)}{\nu_j} \frac{y(\hat{\nu}_j)}{\prod_{\substack{k=1\\k\neq j}}^n (\nu_j - \nu_k)}, \quad j = 1, \dots, n \tag{F.79}$$

with initial data given by $\underline{\hat{\nu}}(x_0, t_{0,r}) \in \operatorname{Sym}^n(\mathcal{K}_n)$, where

$$\hat{\nu}_j = (\nu_j, \nu_j G_{n-1}(\nu_j)) \in \mathcal{K}_n, \quad j = 1, \dots, n.$$

The AKNS Hierarchy. In this case the Dubrovin-type equations on Ω_μ for $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ are given by (cf. (3.200), (3.201))

$$\mu_{j,x} = -2i \frac{y(\hat{\mu}_j)}{\prod_{\substack{k=1\\k\neq j}}^n (\mu_j - \mu_k)}, \tag{F.80}$$

$$\mu_{j,t_r} = -\frac{\widetilde{F}_r(\mu_j)}{iq} \mu_{j,x} = 2 \frac{y(\hat{\mu}_j)}{q \prod_{\substack{k=1\\k\neq j}}^n (\mu_j - \mu_k)} \widetilde{F}_r(\mu_j), \quad j = 1, \dots, n \tag{F.81}$$

with initial data given by $\underline{\hat{\mu}}(x_0, t_{0,r}) \in \operatorname{Sym}^n(\mathcal{K}_n)$, where

$$\hat{\mu}_j = (\mu_j, G_{n+1}(\mu_j)) \in \mathcal{K}_n, \quad j = 1, \dots, n.$$

For the corresponding evolution of $\underline{\hat{\nu}} = \{\hat{\nu}_1, \dots, \hat{\nu}_n\}$ one gets (cf. (3.204), (3.205))

$$\nu_{j,x} = -2i \frac{y(\hat{\nu}_j)}{\prod_{\substack{k=1\\k\neq j}}^n (\nu_j - \nu_k)}, \tag{F.82}$$

$$\nu_{j,t_r} = \widetilde{H}_r(\nu_j)\nu_{j,x} = -2 \frac{y(\hat{\nu}_j)}{p \prod_{\substack{k=1\\k\neq j}}^n (\nu_j - \nu_k)} \widetilde{H}_r(\nu_j), \quad j = 1, \dots, n \tag{F.83}$$

with initial data given by $\underline{\hat{\nu}}(x_0, t_{0,r}) \in \operatorname{Sym}^n(\mathcal{K}_n)$, where

$$\hat{\nu}_j = (\nu_j, -G_{n+1}(\nu_j)) \in \mathcal{K}_n, \quad j = 1, \dots, n.$$

Next we will prove that the Abel map provides a clever change of coordinates that linearizes the Dubrovin flows (we are still assuming that the affine part of the hyperelliptic curve under consideration is nonsingular). This will turn out to be a consequence of the fact that $\widetilde{F}_r(\mu_j)$ can be expressed as a linear combination of the functions $\Phi_k^{(j)}$. Using Theorem F.2 one concludes that this is not only the case for the KdV hierarchy but also for all the other hierarchies discussed in this monograph.

Theorem F.10 *Suppose* $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_n\}$ *satisfies the Dubrovin equations* (F.72), (F.73) *on* Ω_μ*, where* $\Omega_\mu \subseteq \mathbb{R}^2$ *is open and connected, assuming* $\mu_j \neq \mu_{j'}$ *on* Ω_μ *for* $j \neq j'$, $j, j' = 1, \dots, n$. *Let* $r \in \mathbb{N}_0$ *and introduce*

$$\widetilde{F}_r(\mu_j) = \sum_{k=0}^{r\wedge n} d_{r,k}\Phi_k^{(j)}(\underline{\mu}), \quad d_{r,0}, \dots, d_{r,r\wedge n} \in \mathbb{C}. \tag{F.84}$$

Then the Abel map linearizes the Dubrovin flows (F.72), (F.73) *in the sense that*

$$\partial_{t_r} \sum_{j=1}^{n} \underline{A}_{P_0,k}(\hat{\mu}_j(x, t_r)) = -2i \sum_{\ell=1\vee(n-r)}^{n} c_k(\ell) d_{r,n-\ell}, \quad (x, t_r) \in \Omega_\mu \tag{F.85}$$

and hence[1]

$$\begin{aligned}\underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\mu}}(x,t_r)}) &= \underline{\alpha}_{P_0}\big(\mathcal{D}_{\underline{\hat{\mu}}(x_0,t_{0,r})}\big) - 2i(x - x_0)c_k(n) \\ &\quad - 2i(t_r - t_{0,r}) \sum_{\ell=1\vee(n-r)}^{n} c_k(\ell) d_{r,n-\ell}, \quad (x, t_r) \in \Omega_\mu.\end{aligned} \tag{F.86}$$

Proof One computes,

$$\begin{aligned}\partial_{t_r} \sum_{j=1}^{n} \underline{A}_{P_0,k}(\hat{\mu}_j) &= \partial_{t_r} \sum_{j=1}^{n} \int_{P_0}^{\hat{\mu}_j} \omega_k \\ &= \partial_{t_r} \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_k(\ell) \int_{P_0}^{\hat{\mu}_j} \frac{z^{\ell-1}\,dz}{y(P)} \\ &= \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_k(\ell) \frac{\mu_j^{\ell-1}}{y(\hat{\mu}_j)} \mu_{j,t_r} \\ &= -2i \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_k(\ell) \frac{\mu_j^{\ell-1}}{y(\hat{\mu}_j)} \frac{y(\hat{\mu}_j)}{\prod_{m\neq j}^{n}(\mu_j - \mu_m)} \widetilde{F}_r(\mu_j) \\ &= -2i \sum_{j=1}^{n} \sum_{\ell=1}^{n} c_k(\ell) U_n(\underline{\mu})_{\ell,j} \widetilde{F}_r(\mu_j) = -2i \sum_{\ell=1\vee(n-r)}^{n} c_k(\ell) d_{r,n-\ell},\end{aligned} \tag{F.87}$$

[1] The situation here resembles the one in classical mechanics in which, by a canonical change to cyclic coordinates the momentum p_j becomes a constant of motion, and thus $q_j(t) = q_j(t_0) + p_j(t - t_0)$ is linear in time.

using Lemma E.3 in the final step. As for the x-variation, we observe that the t_0-derivative of μ_j coincides with the x-derivative in (F.72), and hence it is a special case of (F.87) with $\widetilde{F}_0 = 1$. This proves the theorem. □

Corollary F.11 *The Abel map linearizes the Dubrovin flows for the KdV, sGmKdV, and AKNS hierarchies.*

Proof Theorem F.2 shows that $\widetilde{F}_r(\mu_j)$ (and $\widetilde{H}_r(\nu_j)$ in the AKNS case) indeed satisfies the assumption (F.84) of Theorem F.10, and hence the key calculation (F.87) carries over to the sGmKdV and AKNS hierarchies. The special case $r = 0$ gives the x-variation in all but the sGmKdV case which, however, can easily be verified by explicit computation. □

Analogous considerations apply to the classical massive Thirring system.

Remark F.12 We provide a few more details in the AKNS case. Suppose $\underline{\hat{\mu}}$ satisfies (F.80), (F.81) and similarly $\underline{\hat{\nu}}$ satisfies (F.82), (F.83) with $\mu_j \neq \mu_{j'}$ and $\nu_j \neq \nu_{j'}$ for $j \neq j'$, $j, j' = 1, \ldots, n$. Let $r \in \mathbb{N}_0$ and introduce

$$\widetilde{F}_r(\mu_j) = -iq \sum_{k=0}^{r\wedge n} d_{r,k} \Phi_k^{(j)}(\underline{\mu}), \quad \widetilde{H}_r(\nu_j) = ip \sum_{k=0}^{r\wedge n} e_{r,k} \Phi_k^{(j)}(\underline{\nu}).$$

Then (F.85) and (F.86) hold. In addition, one obtains the following results for the analog of Neumann divisors $\underline{\hat{\nu}}$.

$$\partial_{t_r} \sum_{j=1}^{n} \underline{A}_{P_0,k}(\hat{\nu}_j(x, t_r)) = -2i \sum_{\ell=1\vee(n-r)}^{n} c_k(\ell) e_{r,n-\ell}$$

and hence,

$$\begin{aligned} \underline{\alpha}_{P_0}(\mathcal{D}_{\underline{\hat{\nu}}(x,t_r)}) &= \underline{\alpha}_{P_0}\big(\mathcal{D}_{\underline{\hat{\nu}}(x_0,t_{0,r})}\big) - 2i(x - x_0) c_k(n) e_{0,0} \\ &\quad - 2i(t_r - t_{0,r}) \sum_{\ell=1\vee(n-r)}^{n} c_k(\ell) e_{r,n-\ell}. \end{aligned} \tag{F.88}$$

Solving these equations we can recover the solution of the integrable equation using *trace formulas*. For the KdV hierarchy we have the classical trace formula

$$u = \sum_{m=0}^{2n} E_m - 2 \sum_{j=1}^{n} \mu_j. \tag{F.89}$$

For the AKNS hierarchy we have

$$\frac{p_x}{p} = i \sum_{m=0}^{2n+1} E_m - 2i \sum_{j=1}^{n} \nu_j, \quad \frac{q_x}{q} = -i \sum_{m=0}^{2n+1} E_m + 2i \sum_{j=1}^{n} \mu_j. \tag{F.90}$$

The "trace" relation for the sGmKdV hierarchy (perhaps, it would be more appropriate to call this a "determinant" relation) reads (cf. (2.192), (2.193))

$$u = i \ln\left((-1)^n \alpha^{-1} \prod_{j=1}^{n} \mu_j\right) = -i \ln\left((-1)^n \beta^{-1} \prod_{j=1}^{n} \nu_j\right). \tag{F.91}$$

Remark F.13 If one postulates the Dubrovin equations (F.72), (F.73), and defines u using the trace formula (F.89), it can be shown that u indeed satisfies the rth KdV equation with the correct initial condition. This is discussed in detail in connection with the algebro-geometric initial value problems solved for all models in question in the main parts of this text.

Remark F.14 For simplicity we assumed $\mu_j \neq \mu_{j'}$ on Ω_μ for $j \neq j'$ in Theorem F.10. In the self-adjoint cases, where $\{E_m\}_{m=0,\dots,N} \subset \mathbb{R}$, this condition is automatically fulfilled on $\Omega_\mu = \mathbb{R}^2$ since then all μ_j are separated from each other by spectral gaps of L or M. In the general nonself-adjoint case this is no longer true and collisions between the μ_j's become possible. Nevertheless, the Dubrovin equations, properly desingularized near such collision points, stay meaningful as long as the corresponding auxiliary divisors remain nonspecial or their specialty stems from points at infinity only, as is discussed in the references mentioned in the notes to this section.

We end this section with an example.

Example F.15 KdV and sG. Pick $E_0 = 0$ and $E_1, \dots, E_{2n} \in \mathbb{C}$, $E_m \neq E_{m'}$ for $m \neq m'$, assume $(x, t_n) \in \Omega$, where $\Omega \subseteq \mathbb{R}^2$ is open and connected, and solve

$$\mu_{j,x} = -2i \frac{y(\hat{\mu}_j)}{\prod_{\substack{k=1\\k\neq j}}^{n} (\mu_j - \mu_k)},$$

$$\mu_{j,t_n} = \left(\frac{1}{16 Q^{1/2}} \prod_{\substack{k=1\\k\neq j}}^{n} \mu_k\right) \mu_{j,x}, \quad j = 1, \dots, n,$$

with $Q = \prod_{m=1}^{2n} E_m$ and $R_{2n+1}(z) = z \prod_{m=1}^{2n} (z - E_m)$. Define

$$u = i \ln\left(Q^{-1/2} \prod_{j=1}^{n} \mu_j\right), \quad \hat{u} = \sum_{m=0}^{2n} E_m - 2 \sum_{j=1}^{n} \mu_j.$$

Then u and $\hat{u}$ satisfy the sG equation and nth KdV equation, respectively, that is,

$$4u_{xt_n} = \sin(u), \quad \mathrm{KdV}_n(\hat{u}) = 0$$

for the following choice of $\tilde{c}_\ell$,

$$\tilde{c}_0 = 1, \quad \tilde{c}_1 = \frac{(-1)^{n-1}}{16Q^{1/2}} - \hat{c}_1(\underline{E}), \quad \tilde{c}_\ell = -\sum_{p=0}^{\ell-1} \tilde{c}_p \hat{c}_{\ell-p}(\underline{E}), \quad \ell = 2, \dots, n.$$

Remark F.16 This example provides an interesting connection between the KdV_n and sG equation and illustrates the fundamental role of the Dubrovin equations as a common underlying principle for hierarchies of soliton equations in $(1+1)$-dimensions. In particular, this approach establishes an isomorphism between the classes of algebro-geometric solutions of this pair of integrable systems. Indeed, once the hyperelliptic curve $\mathcal{K}_n$ is fixed, algebro-geometric solutions of the KdV_n and sG equation are just certain symmetric functions (i.e., "trace" relations) of the solutions of the corresponding Dubrovin equations on $\mathcal{K}_n$.

Notes

The material in this appendix is predominantly taken from Gesztesy and Holden (2002; to appear, a).

The first systematic use of symmetric functions of the auxiliary eigenvalues μ_j in connection with the KdV hierarchy has been made in McKean and van Moerbeke (1975).

A comprehensive treatment of Frobenius-type theorems such as Theorem F.3 can be found, for instance, in Narasimhan (1985, Sec. 2.11), Spivak (1979, Ch. 6).

Formulas (F.29), (F.58), and (F.59) (without detailed proofs and without explicit form of the constant terms on the right-hand sides of (F.58) and (F.59)) have been used in Novikov (1999) in the context of deriving algebro-geometric solutions of the Dym equation. The approach chosen in this appendix based on elementary symmetric functions and Dubrovin-type systems (F.25) is due to Gesztesy and Holden (to appear, a).

Necessary and sufficient conditions on Lax pairs to linearize the flow $t \to L_t$ on $J(C)$, where $\{L_t\}$ represents a dynamical system on the Jacobi variety $J(C)$ with C the underlying spectral curve, have been considered in Griffiths (1985). While this paper considers Lax equations within a cohomological framework, our present approach is much more modest in scope but in turn reduces the linearization problem to an elementary exercise in symmetric functions.

For simplicity we assumed that the auxiliary eigenvalues μ_j do not coincide (cf. Remark F.14). However, the Dubrovin equations, properly desingularized near such collision points, remain well-defined, as demonstrated by Birnir (1986a,b; 1987) in the case of complex-valued periodic KdV solutions as long as the auxiliary divisors remain nonspecial or their specialty stems from points at infinity only. In particular, (F.86) (and (F.88)) remain valid in the presence of such collisions.

The isomorphism between algebro-geometric KdV_n and sG equations, as displayed in Example F.15, has been discussed, for instance, in Al'ber and Al'ber (1987a,b). Analogous considerations apply to the nonlinear Schrödinger equation and the (continuum) Heisenberg chain (see, e.g., Elgin (1990)).

Appendix G
KdV and AKNS Darboux-Type Transformations

> ... he who seeks for methods without having a definite problem in mind seeks for the most part in vain.
>
> *David Hilbert*[1]

In this appendix we present a short summary of fundamental facts concerning Darboux transformations applied to the stationary KdV and AKNS hierarchies and their effect on the underlying hyperelliptic curves. We briefly outline the corresponding ideas in the KdV context.

First we factorize

$$L = -\frac{d^2}{dx^2} + u$$

into a product of first-order differential expressions plus a shift,

$$L = AA^+ + z_0, \quad A = \frac{d}{dx} + \phi, \quad A^+ = -\frac{d}{dx} + \phi, \quad u = \phi^2 + \phi_x + z_0. \tag{G.1}$$

Assuming u to satisfy one of the stationary KdV equations, reversing the order of the two factors A and A^+ produces a new Lax operator $\widehat{L}$,

$$\widehat{L} = A^+A + z_0 = -\frac{d^2}{dx^2} + \hat{u}, \quad \hat{u} = \phi^2 - \phi_x + z_0, \tag{G.2}$$

whose potential $\hat{u}$ is a new solution of one of the equations in the stationary KdV hierarchy. In short, the transformation

$$u \mapsto \hat{u}$$

represents a Darboux transformation, or equivalently, an auto-Bäcklund transformation of the stationary KdV hierarchy. Incidentally, $\pm\phi$ in (G.1), (G.2) represent

[1] Mathematical problems, Bull. Amer. Math. Soc. **37** (2000), 407–436. ("... wer, ohne ein bestimmtes Problem vor Augen zu haben, nach Methoden sucht, dessen Suchen is meist vergeblich." German original in Mathematische Probleme. Vortrag, gehalten auf dem internationalen Mathematiker-Congress zu Paris 1900, Gött. Nachr. (1900), 253–297.)

solutions of one of the stationary equations of the mKdV hierarchy, and hence

$$u \mapsto \phi \mapsto -\phi \mapsto \hat{u}$$

represents a Bäcklund transformation from the (stationary) KdV to the mKdV hierarchy ($u \mapsto \phi$) as well as auto-Bäcklund transformations for the KdV ($u \mapsto \hat{u}$) and mKdV hierarchy ($\phi \mapsto -\phi$). However, although ϕ and $-\phi$ satisfy the identical equation(s) within the stationary mKdV hierarchy, and hence

$$D = \begin{pmatrix} 0 & A^+ \\ A & 0 \end{pmatrix} \mapsto \widetilde{D} = \begin{pmatrix} 0 & -A \\ -A^+ & 0 \end{pmatrix}$$

represents an isospectral deformation of D, u and $\hat{u}$ in general do not necessarily satisfy the same stationary equation of the KdV hierarchy, that is,

$$L \mapsto \widehat{L},$$

in general, is not an isospectral deformation of L. More precisely, each solution u of (one of) the nth stationary KdV equations is associated with a hyperelliptic curve $\mathcal{K}_n$ (possibly with a singular affine part) of the type

$$\mathcal{K}_n : y^2 = \prod_{m=0}^{2n} (z - E_m), \quad \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}. \tag{G.3}$$

Similarly, $\hat{u}$ corresponds to a curve $\widehat{\mathcal{K}}_{\hat{n}}$ of the type

$$\widehat{\mathcal{K}}_{\hat{n}} : y^2 = \prod_{m=0}^{2\hat{n}} (z - \widehat{E}_m), \quad \{\widehat{E}_m\}_{m=0,\dots,2\hat{n}} \subset \mathbb{C} \tag{G.4}$$

and hence u and $\hat{u}$ (respectively L and $\widehat{L}$) are isospectral if and only if $\mathcal{K}_n = \widehat{\mathcal{K}}_{\hat{n}}$ (i.e., $\{E_m\}_{m=0,\dots,2n} = \{\widehat{E}_m\}_{m=0,\dots,2\hat{n}}$, $n = \hat{n}$).

The principal aim of this appendix is to re-examine the relationship between $\mathcal{K}_n$ and $\widehat{\mathcal{K}}_{\hat{n}}$, depending on various choices of the function ϕ in (G.1), and to provide a complete yet elementary solution of this problem in the KdV and AKNS contexts.

KdV Darboux-Type Transformations

Recalling the stationary Baker–Akhiezer function $\psi(P, x, x_0)$ given by (1.41), we define

$$\psi(P, x, x_0, \sigma) = \begin{cases} \frac{1}{2}(1+\sigma)\psi(P, x, x_0) + \frac{1}{2}(1-\sigma)\psi(P^*, x, x_0) & \text{for } \sigma \in \mathbb{C}, \\ \psi(P, x, x_0) - \psi(P^*, x, x_0) & \text{for } \sigma = \infty, \end{cases}$$
$$P \in \mathcal{K}_n \setminus \{P_\infty\}. \tag{G.5}$$

Pick $Q_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{P_\infty\}$ and introduce the differential expressions

$$A_\sigma(Q_0) = \frac{d}{dx} + \phi(Q_0, \cdot, \sigma), \quad A_\sigma^+(Q_0) = -\frac{d}{dx} + \phi(Q_0, \cdot, \sigma), \quad \sigma \in \mathbb{C}_\infty,$$

where

$$\phi(P, x, \sigma) = \psi_x(P, x, x_0, \sigma)/\psi(P, x, x_0, \sigma), \quad P \in \mathcal{K}_n \setminus \{P_\infty\}, \sigma \in \mathbb{C}_\infty. \tag{G.6}$$

One verifies (cf. (1.42))

$$L = A_\sigma(Q_0)A_\sigma^+(Q_0) + z_0 = -\frac{d^2}{dx^2} + u \tag{G.7}$$

with

$$u(x) = \phi(Q_0, x, \sigma)^2 + \phi_x(Q_0, x, \sigma) + z_0$$

independent of the choice of $\sigma \in \mathbb{C}_\infty$. Recall that the diagonal Green's function associated with L reads (cf. (D.20), (J.14), (J.8))

$$g(P, x) = \frac{\psi(P, x, x_0)\psi(P^*, x, x_0)}{W(\psi(P, \cdot, x_0), \psi(P^*, \cdot, x_0))} \tag{G.8}$$

$$= \frac{iF_n(z, x)}{2y}, \quad P = (z, y) \in \mathcal{K}_n \setminus \{P_\infty\}, \ x \in \mathbb{C}. \tag{G.9}$$

Interchanging the order of the differential expressions $A_\sigma(Q_0)$ and $A_\sigma^+(Q_0)$ in (G.7) then yields

$$\widehat{L}_\sigma(Q_0) = A_\sigma^+(Q_0)A_\sigma(Q_0) + z_0 = -\frac{d^2}{dx^2} + \hat{u}_\sigma(\cdot, Q_0)$$

with

$$\begin{aligned}\hat{u}_\sigma(x, Q_0) &= \phi(Q_0, x, \sigma)^2 - \phi_x(Q_0, x, \sigma) + z_0 \\ &= u(x) - 2(\ln(\psi(Q_0, x, x_0, \sigma)))_{xx}, \quad \sigma \in \mathbb{C}_\infty.\end{aligned}$$

The transformation

$$u \mapsto \hat{u}_\sigma(\cdot, Q_0), \quad Q_0 \in \mathcal{K}_n \setminus \{P_\infty\}, \sigma \in \mathbb{C}_\infty \tag{G.10}$$

is usually called the Darboux transformation (also Crum–Darboux transformation or single commutation method).

Next, assuming that $\psi \in \ker(L - z)$, one infers $A_\sigma^+(Q_0)\psi(z) \in \ker(\widehat{L}_\sigma(Q_0) - z)$, and

$$W\big(A_\sigma^+(Q_0)\psi_1(z), A_\sigma^+(Q_0)\psi_2(z)\big) = (z - z_0)W(\psi_1(z), \psi_2(z)), \tag{G.11}$$
$$\psi_1(z), \psi_2(z) \in \ker(L - z).$$

Define

$$\hat{\psi}_\sigma(P,x,x_0,Q_0) = \big(A_\sigma^+(Q_0)\psi(P,\,\cdot\,,x_0)\big)(x)$$
$$= (\phi(Q_0,x,\sigma) - \phi(P,x))\psi(P,x,x_0),$$
$$P \in \mathcal{K}_n \setminus \{Q_0, P_\infty\},\ \sigma \in \mathbb{C}_\infty.$$

Then

$$(\widehat{L}_\sigma(Q_0) - z)\hat{\psi}_\sigma(P,\,\cdot\,,x_0,Q_0)) = 0, \quad P = (z,y) \in \mathcal{K}_n \setminus \{Q_0, P_\infty\},$$

and we define in analogy to (G.8) the diagonal Green's function $\hat{g}_\sigma(P,x,Q_0)$ of $\widehat{L}_\sigma(Q_0)$ by

$$\hat{g}_\sigma(P,x,Q_0) = \frac{\hat{\psi}_\sigma(P,x,x_0,Q_0)\hat{\psi}_\sigma(P^*,x,x_0,Q_0)}{W(\hat{\psi}_\sigma(P,\,\cdot\,,x_0,Q_0),\hat{\psi}_\sigma(P^*,\,\cdot\,,x_0,Q_0))}, \tag{G.12}$$
$$P = (z,y) \in \mathcal{K}_n \setminus \{Q_0, P_\infty\}.$$

Lemma G.1 *Assume* s-KdV$_n(u) = 0$*, and let* $Q_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{P_\infty\}$*,* $P = (z,y) \in \mathcal{K}_n \setminus \{Q_0, P_\infty\}$*,* $\sigma \in \mathbb{C}_\infty$*. Then the diagonal Green's function* $\hat{g}_\sigma(P,\,\cdot\,,Q_0)$ *in* (G.12) *explicitly reads*

$$\hat{g}_\sigma(P,\,\cdot\,,Q_0)$$
$$= \frac{H_{n+1}(z) + \phi(Q_0,\,\cdot\,,\sigma)^2 F_n(z) - \phi(Q_0,\,\cdot\,,\sigma)F_{n,x}(z)}{-2i(z-z_0)y} \tag{G.13}$$
$$= \frac{(\phi(P) - \phi(Q_0,x,\sigma))(\phi(P^*) - \phi(Q_0,\sigma))F_n(z)}{-2i(z-z_0)y} \tag{G.14}$$
$$= \frac{i\widehat{F}_{\sigma,\hat{n}}(z)}{2\hat{y}}, \tag{G.15}$$

where $\hat{y}(\,\cdot\,)$ *denotes the meromorphic solution obtained upon solving* $y^2 = \widehat{R}_{\sigma,2\hat{n}+1}(z)$*,* $P = (z,y)$ *for some polynomial* $\widehat{R}_{\sigma,2\hat{n}+1}$ *of degree* $2\hat{n}+1 \in \mathbb{N}_0$*, and* $\widehat{F}_{\sigma,\hat{n}}(\,\cdot\,,x)$ *denotes a polynomial of degree* $\hat{n}$ *with* $0 \le \hat{n} \le n+1$*. In particular, the Darboux transformation* (G.10)*,* $u \mapsto \hat{u}_\sigma(\,\cdot\,,Q_0)$ *maps the class of algebro-geometric KdV potentials into itself.*

Proof Equations (G.13) and (G.14) follow upon use of $\phi(P,x) = \psi_x(P,x,x_0)/\psi(P,x,x_0)$, (1.43), (1.44), (1.48)–(1.51), and (G.11). Since the numerator in (G.13) is a polynomial in z and

$$\hat{g}_\sigma(P,x,Q_0) = \frac{iz^n}{2y} + O(|z|^{-1}) \text{ as } P = (z,y) \to P_\infty$$

again by (G.13), one concludes (G.15) and $0 \le \hat{n} \le n+1$. By inspection, $\widehat{F}_{\sigma,\hat{n}}(z,x)$ satisfies equation (1.13) with $u(x)$ replaced by $\hat{u}_\sigma(x,Q_0)$, n by $\hat{n}$, and $R_{2n+1}(z)$

by $\widehat{R}_{\sigma,2\hat{n}+1}(z)$. As a consequence, u, being an algebro-geometric KdV potential, implies that $\hat{u}_\sigma(\,\cdot\,, Q_0)$ is one as well. $\square$

The following theorem clarifies the dependence of $\hat{n} = \hat{n}(n, Q_0, \sigma)$ on its variables.

Theorem G.2 *Suppose* s-KdV$_n(u) = 0$*, let* $Q_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{P_\infty\}$*,* $n \in \mathbb{N}_0$*,* $\sigma \in \mathbb{C}_\infty$*, and* $\hat{n} = \hat{n}(n, Q_0, \sigma)$*, as in* (G.15)*. Then*

$$\hat{n}(n, Q_0, \sigma) = \begin{cases} n+1 & \text{for } \sigma \in \mathbb{C}_\infty \setminus \{-1, 1\} \text{ and } y_0 \neq 0, \\ n+1 & \text{for } \sigma = \infty \text{ and } y_0 = 0, \\ n & \text{for } \sigma \in \{-1, 1\} \text{ and } y_0 \neq 0, \\ n & \text{for } \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{2n+1,z}(z_0) \neq 0, \\ n-1 & \text{for } \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{2n+1,z}(z_0) = 0,\ n \in \mathbb{N}, \end{cases}$$

and hence the hyperelliptic curve[1] $\widehat{\mathcal{K}}_{\sigma,\hat{n}}(Q_0)$ *associated with* $\hat{u}_\sigma(\,\cdot\,, Q_0)$ *is of the type*

$$\widehat{\mathcal{K}}_{\sigma,\hat{n}}(Q_0)\colon \widehat{\mathcal{F}}_{\sigma,\hat{n}}(z, y, Q_0) = y^2 - \widehat{R}_{\sigma,2\hat{n}+1}(z, Q_0) = 0$$

with

$$\begin{aligned} &\widehat{R}_{\sigma,2\hat{n}+1}(z, Q_0) \\ &\quad = \begin{cases} (z-z_0)^2 R_{2n+1}(z) & \text{for } \sigma \in \mathbb{C}_\infty \setminus \{-1, 1\} \text{ and } y_0 \neq 0, \\ (z-z_0)^2 R_{2n+1}(z) & \text{for } \sigma = \infty \text{ and } y_0 = 0, \\ R_{2n+1}(z) & \text{for } \sigma \in \{-1, 1\} \text{ and } y_0 \neq 0, \\ R_{2n+1}(z) & \text{for } \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{2n+1,z}(z_0) \neq 0, \\ (z-z_0)^{-2} R_{2n+1}(z) & \text{for } \sigma \in \mathbb{C},\ y_0 = 0, \text{ and } R_{2n+1,z}(z_0) = 0,\ n \in \mathbb{N}. \end{cases} \end{aligned} \tag{G.16}$$

Here

$$R_{2n+1}(z) = \prod_{m=0}^{2n} (z - E_m).$$

Proof Our starting point will be (G.14) and a careful case distinction taking into account whether or not Q_0 is a branch point and distinguishing the cases $\sigma \in \mathbb{C} \setminus \{-1, 1\}$, $\sigma \in \{-1, 1\}$, and $\sigma = \infty$.

[1] We compactify $\widehat{\mathcal{K}}_{\sigma,\hat{n}}(Q_0)$ by adding the point P_∞ at infinity and still denote the compactified curve by $\widehat{\mathcal{K}}_{\sigma,\hat{n}}(Q_0)$.

Case (i). $\sigma \in \mathbb{C}_\infty \setminus \{-1, 1\}$ and $y_0 \neq 0$: One computes from (G.5) and (G.6),

$$\begin{aligned}
&\phi(Q_0, x, \sigma) \\
&\quad = \begin{cases} \dfrac{(1+\sigma)\psi_x(Q_0, x, x_0) + (1-\sigma)\psi_x(Q_0^*, x, x_0)}{(1+\sigma)\psi(Q_0, x, x_0) + (1-\sigma)\psi(Q_0^*, x, x_0)} & \text{for } \sigma \in \mathbb{C} \setminus \{-1, 1\}, \\ \dfrac{\psi_x(Q_0, x, x_0) - \psi_x(Q_0^*, x, x_0)}{\psi(Q_0, x, x_0) - \psi(Q_0^*, x, x_0)} & \text{for } \sigma = \infty, \end{cases}
\end{aligned}$$

and upon comparison with $\phi(Q_0, x) \neq \phi(Q_0^*, x)$,

$$\phi(Q_0, x) = \frac{\psi_x(Q_0, x, x_0)}{\psi(Q_0, x, x_0)}, \quad \phi\big(Q_0^*, x\big) = \frac{\psi_x\big(Q_0^*, x, x_0\big)}{\psi\big(Q_0^*, x, x_0\big)},$$

one concludes that no cancellations can occur in (G.14), proving $\hat{n}(n, Q_0, \sigma) = n + 1$ and the first statement in (G.16).

Case (ii). $\sigma = \infty$ and $y_0 = 0$: Combining (1.38), (1.41), (1.45), and (G.5), one computes

$$\begin{aligned}
\phi(Q_0, x, \infty) &= \lim_{P \to Q_0} \phi(P, x, \infty) \\
&= \lim_{P \to Q_0} \left(\frac{\phi(P, x) \exp\big(\int_{x_0}^x dx'\, \phi(P, x')\big) - \phi(P^*, x) \exp\big(\int_{x_0}^x dx'\, \phi(P^*, x')\big)}{\exp\big(\int_{x_0}^x dx'\, \phi(P, x')\big) - \exp\big(\int_{x_0}^x dx'\, \phi(P^*, x')\big)} \right) \\
&= \phi(Q_0, x) \\
&\quad + \lim_{P \to Q_0} \left(\frac{\phi(P, x) - \phi(P^*, x)}{\exp\big(\int_{x_0}^x dx'\, \phi(P, x')\big) - \exp\big(\int_{x_0}^x dx'\, \phi(P^*, x')\big)} \right. \\
&\qquad\qquad \left. \times \exp\left(\int_{x_0}^x dx'\, \phi(P^*, x') \right) \right) \\
&= \phi(Q_0, x) + \exp\left(\int_{x_0}^x dx'\, \phi(Q_0, x') \right) \\
&\quad \times \lim_{P \to Q_0} \left(\frac{2iy/F_n(z, x)}{\exp\big(iy \int_{x_0}^x \frac{dx'}{F_n(z,x')}\big) - \exp\big(-iy \int_{x_0}^x \frac{dx'}{F_n(z,x')}\big)} \right. \\
&\qquad\qquad \left. \times \exp\left(-\frac{1}{2} \int_{x_0}^x dx'\, \frac{F_{n,x}(z, x')}{F_n(z, x')} \right) \right) \\
&= \phi(Q_0, x) \\
&\quad + \psi(Q_0, x, x_0) \frac{1}{F_n(z_0, x)\psi(Q_0, x, x_0)} \lim_{P \to Q_0} \left(\frac{2iy}{2iy \int_{x_0}^x \frac{dx'}{F_n(z,x')} + O(y^2)} \right) \\
&= \phi(Q_0, x) + \left(F_n(z_0, x) \int_{x_0}^x \frac{dx'}{F_n(z, x')} \right)^{-1}, \quad x \in \mathbb{C} \setminus \{x_0\},
\end{aligned} \tag{G.17}$$

using $\lim_{P \to Q_0} y(P) = y(Q_0) = y_0 = 0$. From

$$\phi(Q_0) = \frac{1}{2} \frac{F_{n,x}(z_0)}{F_n(z_0)}$$

one concludes again that no cancellations can occur in (G.14). Thus, one concludes $\hat{n}(n, Q_0, \infty) = n+1$ and hence the second statement in (G.16).

The remainder of the proof requires a more refined argument, the basis of which will be derived next. First, replacing $u(x)$ by $u(x) - z_0$, we may assume without loss of generality that $z_0 = 0$ in the following. Writing

$$y^2 = R_{2n+1}(z) \underset{z\to 0}{=} y_0^2 + \tilde{y}_1 z + \tilde{y}_2 z^2 + O(z^3), \tag{G.18}$$

a comparison of the powers z^0 and z^1 in (1.13) yields

$$2f_{n,xx} f_n = f_{n,x}^2 + 4uf_n^2 + 4y_0^2 \tag{G.19}$$

and

$$f_{n-1,xx} f_n + f_{n,xx} f_{n-1} - f_{n,x} f_{n-1,x} - 4uf_n f_{n-1} + 2f_n^2 - 2\tilde{y}_1 = 0. \tag{G.20}$$

When one inserts (G.19) into (G.20), a little algebra proves the basic identity

$$f_n^2 (f_{n-1}/f_n)_{xx} + f_{n,x} f_n (f_{n-1}/f_n)_x + 2f_n^2 + 4y_0^2 (f_{n-1}/f_n) - 2\tilde{y}_1 = 0. \tag{G.21}$$

Case (iii). $\sigma \in \{-1, 1\}$ and $y_0 \neq 0$: Then (G.5) yields

$$\phi(Q_0, x, 1) = \phi(Q_0, x), \quad \phi(Q_0, x, -1) = \phi\big(Q_0^*, x\big),$$

with $\phi(Q_0, x) \neq \phi(Q_0^*, x)$ since $y_0 \neq 0$. In this case there is a cancellation in (G.14). For instance, choosing $\sigma = 1$, one computes from (1.11) and (1.38)

$$\begin{aligned}
\phi(P) - \phi(Q_0, \cdot, 1) &= \phi(P) - \phi(Q_0) \\
&\underset{P\to Q_0}{=} \frac{i(y - y_0)}{f_n} - iy_0 \frac{f_{n-1}}{f_n^2} z + \frac{1}{2}\left(\frac{f_{n-1}}{f_n}\right)_x z + O(z^2) \\
&\underset{P\to Q_0}{=} c_1 z + O(z^2)
\end{aligned}$$

since

$$y - y_0 \underset{P\to Q_0}{=} y_1 z + O(z^2), \quad \tilde{y}_1 = 2y_0 y_1.$$

It remains to show that c_1 does not vanish identically on $\mathbb{C}$. Arguing by contradiction, we assume

$$0 = c_1 = \frac{iy_1}{f_n} - \frac{iy_0}{f_n}\frac{f_{n-1}}{f_n} + \frac{1}{2}\left(\frac{f_{n-1}}{f_n}\right)_x, \quad x \in \mathbb{C}. \tag{G.22}$$

Differentiating (G.22) with respect to x and inserting the ensuing expression for $(f_{n-1}/f_n)_{xx}$ and the one for $(f_{n-1}/f_n)_x$ from (G.22) into (G.21) then result in the contradiction

$$0 = 2f_n(x)^2, \quad x \in \mathbb{C}.$$

Moreover, since

$$\phi(P^*) - \phi(Q_0,\,\cdot\,,1) = \phi(P^*) - \phi(Q_0) \underset{P\to Q_0}{=} -2iy_0 f_n^{-1} + O(z),$$

one concludes that precisely one factor of z cancels in (G.14). Thus, one infers $\hat{n}(n, Q_0, 1) = n$ and hence the third relation in (G.16). The case $\sigma = -1$ is treated analogously.

Case (iv). $\sigma \in \mathbb{C}$, $y_0 = 0$, and $R_{2n+1,z}(0) \neq 0$: Taking into account that $\phi(Q_0, x, \sigma) = \phi(Q_0, x)$ (using (G.6) and $Q_0 = Q_0^*$) is independent of $\sigma \in \mathbb{C}$, one observes that (1.11) and (1.38) yield

$$(\phi(P) - \phi(Q_0))(\phi(P^*) - \phi(Q_0)) \underset{P\to Q_0}{=} y_1^2 z f_n^{-2} + O(z^2) \tag{G.23}$$

since

$$y \underset{P\to Q_0}{=} y_1 z^{1/2} + O(z^{3/2}), \quad y_1 = \Bigg(\prod_{E_m \neq 0} E_m\Bigg)^{1/2}.$$

Thus, we infer again that precisely one factor of z cancels in (G.14). Hence, $\hat{n}(n, Q_0, \sigma) = n$, and the fourth relation in (G.16) is proved. Case (v). $\sigma \in \mathbb{C}$, $y_0 = \tilde{y}_1 = 0$, and $\tilde{y}_2 \neq 0$ (cf. (G.18)): One calculates as in (G.23) that

$$\begin{aligned}
&(\phi(P) - \phi(Q_0))(\phi(P^*) - \phi(Q_0)) \\
&\quad \underset{P\to Q_0}{=} \left(\frac{y_1^2}{f_n^2} + \frac{1}{4}\left(\left(\frac{f_{n-1}}{f_n}\right)_x\right)^2\right) z^2 + O(z^3) \\
&\quad \underset{P\to Q_0}{=} c_2 z^2 + O(z^3)
\end{aligned} \tag{G.24}$$

since

$$y \underset{P\to Q_0}{=} y_1 z + O(z^2), \quad y_1 = \Bigg(\prod_{E_m \neq 0} E_m\Bigg)^{1/2}.$$

Next we show that c_2 does not vanish identically on $\mathbb{C}$. Arguing again by contradiction, we suppose that

$$0 = c_2 = \frac{y_1^2}{f_n^2} + \frac{1}{4}\left(\left(\frac{f_{n-1}}{f_n}\right)_x\right)^2, \quad x \in \mathbb{C}.$$

Thus,

$$\left(\frac{f_{n-1}}{f_n}\right)_x = \frac{C}{f_n} \tag{G.25}$$

for some constant $C \in \mathbb{C}$. Insertion of (G.25) and its x-derivative into (G.21) then again yields the contradiction

$$0 = 2f_n(x)^2, \quad x \in \mathbb{C}.$$

Hence, $\hat{n}(n, Q_0, \sigma) = n - 1$ and the last relation in (G.16) holds in this case.
Case (vi). $\sigma \in \mathbb{C}$, $y_0 = \tilde{y}_1 = \tilde{y}_2 = 0$ (cf. (G.18)): As in (G.24) one obtains

$$(\phi(P) - \phi(Q_0))(\phi(P^*) - \phi(Q_0)) \underset{P \to Q_0}{=} \frac{1}{4}\left(\left(\frac{f_{n-1}}{f_n}\right)_x\right)^2 z^2 + O(z^3)$$

since

$$y \underset{P \to Q_0}{=} O(z^{3/2}).$$

The remainder of the proof of case (vi) is now a special case of case (v) (with $y_1 = C = 0$), and one concludes again that $\hat{n}(n, Q_0, \sigma) = n - 1$. $\square$

We can summarize the previous theorem in the following table.

Table G.3 *The table shows the value of the arithmetic genus $\hat{n}$ associated with the Darboux transformation. Here, $R_{2n+1}(z) = y_0^2 + \tilde{y}_1(z - z_0) + O((z - z_0)^2)$ as $z \to z_0$.*

<table>
<tr><td></td><td></td><td>$\sigma \in \mathbb{C} \setminus \{-1, 1\}$</td><td>$\sigma \in \{-1, 1\}$</td><td>$\sigma = \infty$</td></tr>
<tr><td>$y_0 \neq 0$</td><td></td><td>$n + 1$</td><td>n</td><td rowspan="3">$n + 1$</td></tr>
<tr><td rowspan="2">$y_0 = 0$</td><td>$\tilde{y}_1 \neq 0$</td><td colspan="2">n</td></tr>
<tr><td>$\tilde{y}_1 = 0$</td><td colspan="2">$n - 1$</td></tr>
</table>

These results show, in particular, that Darboux transformations do not change the local structure of the original curve $y^2 = R_{2n+1}(z)$, except, of course, near the point Q_0.

We conclude with the following elementary illustration.

Example G.4 Assume $n = 0$, $P = (z, y) \in \mathcal{K}_0 \setminus \{P_\infty\}$, and let $(x, x_0) \in \mathbb{R}^2$. Then

$$y^2 = R_1(z) = z - E_0, \quad E_0 \in \mathbb{C},$$
$$u(x) = E_0,$$
$$\phi(P, x) = iy, \quad \psi(P, x, x_0) = \exp(iy(x - x_0)),$$
$$g(P, x) = \frac{i}{2y},$$
$$\phi(P, x, \sigma) = \begin{cases} iy \dfrac{(1+\sigma)\exp(iy(x - x_0)) - (1-\sigma)\exp(-iy(x - x_0))}{(1+\sigma)\exp(iy(x - x_0)) + (1-\sigma)\exp(-iy(x - x_0))}, & \sigma \in \mathbb{C}, \\ iy \dfrac{\exp(iy(x - x_0)) - \exp(-iy(x - x_0))}{\exp(iy(x - x_0)) + \exp(-iy(x - x_0))}, & \sigma = \infty, \end{cases}$$
$$\phi((E_0, 0), x, \sigma) = \begin{cases} 0, & \sigma \in \mathbb{C}, \\ (x - x_0)^{-1}, & \sigma = \infty. \end{cases}$$

More generally, the case $u(x) = E_0$ can be associated with any curve $y^2 = R_{2n+1}(z)$ since $f_{n,x}(x) = 0$, $n \in \mathbb{N}_0$ in this special case.

AKNS Darboux-Type Transformations

We recall the Baker–Akhiezer function Ψ given by (3.59), which satisfies (cf. (3.68))

$$(M - z)\Psi(P, \cdot, x_0) = 0,$$

where (cf. (3.3))

$$M = i\begin{pmatrix} \frac{d}{dx} & -q \\ p & -\frac{d}{dx} \end{pmatrix}.$$

In the following, the 1, 2 and 2, 1 matrix elements of the 2×2 Green's matrix associated with M (cf. Remark D.4) on the diagonal are denoted by $g_{\ell,\ell'}(P, x)$, $\ell \neq \ell'$, $\ell, \ell' = 1, 2$. We also recall that[1] (cf. (D.41)),

$$g_{1,2}(P, x) = -i\frac{\psi_1(P, x, x_0)\psi_1(P^*, x, x_0)}{W(\Psi(P, \cdot, x_0), \Psi(P^*, \cdot, x_0))} \tag{G.26}$$

$$= \frac{iF_n(z, x)}{2y}, \quad P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\},\ x \in \mathbb{C}. \tag{G.27}$$

Equations (3.26) and (G.27) then yield the universal equation

$$\begin{aligned} &2g_{1,2,xx}(P)g_{1,2}(P) - 2\frac{q_x}{q}g_{1,2,x}(P)g_{1,2}(P) - g_{1,2,x}(P)^2 \\ &\quad + 2\big(2z^2 - 2iz(q_x/q) - 2pq\big)g_{1,2}(P)^2 = q^2, \\ &\qquad P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\}. \end{aligned}$$

Similarly, we find

$$g_{2,1}(P, x) = -i\frac{\psi_2(P, x, x_0)\psi_2(P^*, x, x_0)}{W(\Psi(P, \cdot, x_0), \Psi(P^*, \cdot, x_0))} \tag{G.28}$$

$$= \frac{iH_n(z, x)}{2y}, \quad P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\},\ x \in \mathbb{C}, \tag{G.29}$$

and

$$\begin{aligned} &g_{2,1,xx}(P)g_{2,1}(P) - 2\frac{p_x}{p}g_{2,1,x}(P)g_{2,1}(P) - g_{2,1,x}(P)^2 \\ &\quad + 2\big(2z^2 + 2iz(p_x/p) - 2pq\big)g_{2,1}(P)^2 = p^2, \\ &\qquad P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\}. \end{aligned}$$

[1] The off-diagonal elements of the Green's matrix $G(P, x, x')$ are continuous as $x \to x'$, whereas the diagonal elements are not.

The formal gauge (i.e., Darboux-type) transformation, with U given by (cf. (3.38))

$$U(z) = \begin{pmatrix} -iz & q \\ p & iz \end{pmatrix},$$

is defined using

$$\begin{aligned} \Psi(z) &\mapsto \widehat{\Psi}(z) = \Gamma(z)\Psi(z), \\ U(z) &\mapsto \widehat{U}(z) = \begin{pmatrix} -iz & -\hat{q} \\ \hat{p} & iz \end{pmatrix} \\ &\qquad = \Gamma(z)U(z)\Gamma(z)^{-1} + \Gamma_x(z)\Gamma(z)^{-1} \end{aligned} \tag{G.30}$$

with $\Gamma(z)$ a 2×2 matrix to be chosen later. Since (cf. (3.66))

$$\Psi_x(z) = U(z)\Psi(z),$$

we find

$$\widehat{\Psi}_x(z) = \widehat{U}(z)\widehat{\Psi}(z).$$

Hence,

$$(\widehat{M} - z)\widehat{\Psi}(z) = 0, \quad \widehat{\Psi}(z) = \begin{pmatrix} \hat{\psi}_1(z) \\ \hat{\psi}_2(z) \end{pmatrix}$$

with

$$\widehat{M} = i \begin{pmatrix} d/dx & -\hat{q} \\ \hat{p} & -d/dx \end{pmatrix}.$$

Next, introduce

$$\begin{aligned} \Psi(P, x, x_0, \sigma) &= \begin{pmatrix} \psi_1(P, x, x_0, \sigma) \\ \psi_2(P, x, x_0, \sigma) \end{pmatrix} \\ &= \begin{cases} \frac{1}{2}(1+\sigma)\Psi(P, x, x_0) + \frac{1}{2}(1-\sigma)\Psi(P^*, x, x_0) & \text{for } \sigma \in \mathbb{C}, \\ \Psi(P, x, x_0) - \Psi(P^*, x, x_0) & \text{for } \sigma = \infty, \end{cases} \\ &\qquad P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\}, \end{aligned} \tag{G.31}$$

pick $Q_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\}$, and define

$$\Gamma(z, Q_0, x, \sigma) = \begin{pmatrix} z - z_0 - \frac{i}{2}q(x)\phi(Q_0, x, \sigma) & \frac{i}{2}q(x) \\ \frac{i}{2}\phi(Q_0, x, \sigma) & -\frac{i}{2} \end{pmatrix}. \tag{G.32}$$

Here $\Psi(P, x, x_0)$ is defined in (3.59)–(3.61) and we recall (3.68) and

$$\begin{aligned} \phi(P, x, \sigma) &= \psi_2(P, x, x_0, \sigma)/\psi_1(P, x, x_0, \sigma), \\ P &= (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\}, \ \sigma \in \mathbb{C}_\infty. \end{aligned} \tag{G.33}$$

We note that

$$\det(\Gamma(z, Q_0, x, \sigma)) = -(i/2)(z - z_0).$$

According to (G.30), one then obtains

$$\begin{aligned}
&\widehat{M}_\sigma(Q_0) = i \begin{pmatrix} d/dx & -\hat{q}_\sigma(Q_0) \\ \hat{p}_\sigma(Q_0) & -d/dx \end{pmatrix}, \quad \sigma \in \mathbb{C}_\infty, \\
&\hat{p}_\sigma(x, Q_0) = \phi(Q_0, x, \sigma), \\
&\hat{q}_\sigma(x, Q_0) = -2iz_0 q(x) - q_x(x) + \phi(Q_0, x, \sigma) q(x)^2, \\
&\qquad Q_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\}, \ \sigma \in \mathbb{C}_\infty,
\end{aligned}$$

utilizing the fact that $\phi(P, x, \sigma)$ satisfies the Riccati-type equation (3.62) for all $\sigma \in \mathbb{C}_\infty$. The gauge transformation (or equivalently, Darboux transformation) reads

$$(p, q) \mapsto (\hat{p}_\sigma(\,\cdot\,, Q_0), \hat{q}_\sigma(\,\cdot\,, Q_0)). \tag{G.34}$$

Introducing

$$\begin{aligned}
&\widehat{\Psi}_\sigma(P, x, x_0, Q_0) = \Gamma(z, Q_0, x, \sigma)\Psi(P, x, x_0), \\
&\qquad P = (z, y) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\}, \ \sigma \in \mathbb{C}_\infty,
\end{aligned}$$

where $\Psi(P, \,\cdot\,, x_0) \in \ker(M - z)$ and $\Gamma(z, Q_0, x, \sigma)$ is defined in (G.32), one infers

$$(\widehat{M}_\sigma(Q_0) - z)\widehat{\Psi}_\sigma(P, \,\cdot\,, x_0, Q_0) = 0.$$

Moreover,

$$\begin{aligned}
&W(\widehat{\Psi}_{\sigma,1}(P, \,\cdot\,, x_0, Q_0), \widehat{\Psi}_{\sigma,2}(P, \,\cdot\,, x_0, Q_0)) \\
&\quad = -(i/2)(z - z_0) W(\Psi_1(P, \,\cdot\,, x_0), \Psi_2(P, \,\cdot\,, x_0)),
\end{aligned} \tag{G.35}$$

where

$$\begin{aligned}
&\widehat{\Psi}_{\sigma,j}(P, x, x_0, Q_0) = \Gamma(z, Q_0, x, \sigma)\Psi_j(P, x, x_0), \\
&\qquad \Psi_j(P, \,\cdot\,, x_0) \in \ker(D - z), \ j = 1, 2.
\end{aligned}$$

Given these facts we define, in analogy to (G.26) and (G.28), the $1, 2$ and $2, 1$ Green's matrix elements associated with $\widehat{M}_\sigma(Q_0)$ on the diagonal by

$$\begin{aligned}
&\hat{g}_{\sigma,1,2}(P, x, Q_0) = -i \frac{\hat{\psi}_{\sigma,1}(P, x, x_0, Q_0)\hat{\psi}_{\sigma,1}(P^*, x, x_0, Q_0)}{W(\widehat{\Psi}_\sigma(P, \,\cdot\,, x_0, Q_0), \widehat{\Psi}_\sigma(P^*, \,\cdot\,, x_0, Q_0))}, \\
&\hat{g}_{\sigma,2,1}(P, x, Q_0) = -i \frac{\hat{\psi}_{\sigma,2}(P, x, x_0, Q_0)\hat{\psi}_{\sigma,2}(P^*, x, x_0, Q_0)}{W(\widehat{\Psi}_\sigma(P, \,\cdot\,, x_0, Q_0), \widehat{\Psi}_\sigma(P^*, \,\cdot\,, x_0, Q_0))}, \\
&\qquad P = (z, y) \in \mathcal{K}_n \setminus \{Q_0, P_{\infty_-}, P_{\infty_+}\}, \ \sigma \in \mathbb{C}_\infty.
\end{aligned}$$

Lemma G.5 *Assume* s-AKNS$_n(p, q) = 0$ *and let* $Q_0 = (z_0, y_0) \in \mathcal{K}_n \setminus \{P_{\infty_-}, P_{\infty_+}\}$, $P = (z, y) \in \mathcal{K}_n \setminus \{Q_0, P_{\infty_-}, P_{\infty_+}\}$, $\sigma \in \mathbb{C}_\infty$. *Then the off-diagonal elements of the Green's matrix of* $\widehat{M}_\sigma(Q_0)$ *read*

$$\begin{aligned}
&\hat{g}_{\sigma,1,2}(P, \cdot, Q_0) \\
&\quad = i((i(z-z_0)q + (1/2)q^2\phi(Q_0, \cdot, \sigma))G_{n+1}(z) - (1/4)q^2 H_n(z) + ((z-z_0)^2 \\
&\qquad - i(z-z_0)q\phi(Q_0, \cdot, \sigma) - (1/4)q^2\phi(Q_0, \cdot, \sigma)^2)F_n(z))(-(z-z_0)y)^{-1} \qquad \text{(G.36)} \\
&\quad = i((z-z_0) + (i/2)q(\phi(P) - \phi(Q_0, \cdot, \sigma))) \\
&\qquad \times ((z-z_0) + (i/2)q(\phi(P^*) - \phi(Q_0, \cdot, \sigma)))F_n(z)(-(z-z_0)y)^{-1} \qquad \text{(G.37)} \\
&\quad = \frac{i\widehat{F}_{\sigma,\hat{n}}(z)}{2\hat{y}} \qquad \text{(G.38)}
\end{aligned}$$

and

$$\begin{aligned}
&\widehat{g}_{\sigma,2,1}(P, \cdot, Q_0) \\
&\quad = (i/4)(\phi(Q_0, \cdot, \sigma)^2 F_n(z) - 2\phi(Q_0, \cdot, \sigma)G_{n+1}(z) + H_n(z))((z-z_0)y)^{-1} \qquad \text{(G.39)} \\
&\quad = (i/4)(\phi(P) - \phi(Q_0, \cdot, \sigma))(\phi(P^*) - \phi(Q_0, \cdot, \sigma))F_n(z)((z-z_0)y)^{-1} \qquad \text{(G.40)} \\
&\quad = \frac{i\widehat{H}_{\sigma,\hat{n}}(z)}{2\hat{y}}, \qquad \text{(G.41)}
\end{aligned}$$

where $\hat{y}(\cdot)$ *denotes the meromorphic solution obtained upon solving* $y^2 = \widehat{R}_{\sigma,2\hat{n}+2}(z)$, $P = (z, y)$ *for some polynomial* $\widehat{R}_{\sigma,2\hat{n}+2}$ *of degree* $2\hat{n} + 2 \in \mathbb{N}_0$ *and* $\widehat{F}_{\sigma,\hat{n}}(\cdot, x)$, *and* $\widehat{H}_{\sigma,\hat{n}}(\cdot, x)$, *denote polynomials of degree* $\hat{n}$ *with* $0 \le \hat{n} \le n+1$. *In particular, the Darboux transformation* (G.34), $(p, q) \mapsto (\hat{p}_\sigma(\cdot, Q_0), \hat{q}_\sigma(\cdot, Q_0))$ *maps the class of algebro-geometric AKNS potentials into itself.*

Proof We present the argument for $\hat{g}_{\sigma,1,2}$ only; the case $\hat{g}_{\sigma,2,1}$ follows similarly. As in Lemma G.1, $\phi(P, x) = \psi_2(P, x, x_0)/\psi_1(P, x, x_0)$, (3.63), (3.64), (3.70)–(3.73), and (G.35) imply equations (G.36) and (G.37). Since the numerator in (G.36) is a polynomial in z and

$$\hat{g}_{\sigma,1,2}(P, x, Q_0) = \frac{\hat{q}(x)z^n}{2y} + O(|z|^{-2}) \text{ as } P = (z, y) \to P_{\infty_\pm}$$

by (G.36), one infers (G.38) and $0 \le \hat{n} \le n+1$. Again, one verifies that $\widehat{F}_{\sigma,\hat{n}}(z, x)$ satisfies equation (3.26) with $p(x)$, $q(x)$ replaced by $\hat{p}_\sigma(x, Q_0)$, $\hat{q}_\sigma(x, Q_0)$, n by $\hat{n}$, and $R_{2n+2}(z)$ by $\widehat{R}_{\sigma,2\hat{n}+2}(z)$, proving that the Darboux transformation (G.34) leaves the class of algebro-geometric AKNS potentials invariant. $\square$

The following theorem, which is in complete analogy to Theorem G.2, will clarify the dependence of $\hat{n} = \hat{n}(n, Q_0, \sigma)$ on its variables.

Theorem G.6 *Suppose* s-AKNS$_n(p,q)=0$*, let* $Q_0=(z_0,y_0)\in\mathcal{K}_n\setminus\{P_{\infty_-},P_{\infty_+}\}$*,* $n\in\mathbb{N}_0$*,* $\sigma\in\mathbb{C}_\infty$*, and* $\hat{n}=\hat{n}(n,Q_0,\sigma)$ *as in* (G.38)*. Then*

$$\hat{n}(n,Q_0,\sigma)=\begin{cases} n+1 & \text{for } \sigma\in\mathbb{C}_\infty\setminus\{-1,1\} \text{ and } y_0\neq 0,\\ n+1 & \text{for } \sigma=\infty \text{ and } y_0=0,\\ n & \text{for } \sigma\in\{-1,1\} \text{ and } y_0\neq 0,\\ n & \text{for } \sigma\in\mathbb{C},\ y_0=0, \text{ and } R_{2n+2,z}(z_0)\neq 0,\\ n-1 & \text{for } \sigma\in\mathbb{C},\ y_0=0, \text{ and } R_{2n+2,z}(z_0)=0,\ n\in\mathbb{N}, \end{cases}$$

and hence the hyperelliptic curve[1] $\widehat{\mathcal{K}}_{\sigma,\hat{n}}(Q_0)$ *associated with* $(\hat{p}_\sigma(\,\cdot\,,Q_0),\hat{q}_\sigma(\,\cdot\,,Q_0))$ *is of the type*

$$\widehat{\mathcal{K}}_{\sigma,\hat{n}}\colon \widehat{\mathcal{F}}_{\sigma,\hat{n}}(z,y,Q_0)=y^2-\widehat{R}_{\sigma,2\hat{n}+2}(z,Q_0)=0$$

with

$$\widehat{R}_{\sigma,2\hat{n}+2}(z,Q_0)=\begin{cases} (z-z_0)^2R_{2n+2}(z) & \text{for } \sigma\in\mathbb{C}_\infty\setminus\{-1,1\} \text{ and } y_0\neq 0,\\ (z-z_0)^2R_{2n+2}(z) & \text{for } \sigma=\infty \text{ and } y_0=0,\\ R_{2n+2}(z) & \text{for } \sigma\in\{-1,1\} \text{ and } y_0\neq 0,\\ R_{2n+2}(z) & \text{for } \sigma\in\mathbb{C},\ y_0=0, \text{ and } R_{2n+2,z}(z_0)\neq 0,\\ (z-z_0)^{-2}R_{2n+2}(z) & \text{for } \sigma\in\mathbb{C},\ y_0=0, \text{ and } R_{2n+2,z}(z_0)=0,\ n\in\mathbb{N}. \end{cases} \tag{G.42}$$

Here

$$R_{2n+2}(z)=\prod_{m=0}^{2n+1}(z-E_m).$$

Proof The following arguments closely parallel those in the proof of Theorem G.2. Again our starting point will be (G.37) and (G.40) and a careful case distinction between $\sigma\in\mathbb{C}_\infty\setminus\{-1,1\}$, $\sigma\in\{-1,1\}$, $\sigma=\infty$, and whether or not Q_0 is a branch point.

Case (i). $\sigma\in\mathbb{C}_\infty\setminus\{-1,1\}$ and $y_0\neq 0$: One calculates using (G.31) and (G.33)

$$\phi(Q_0,x,\sigma)=\begin{cases} \dfrac{(1+\sigma)\psi_2(Q_0,x,x_0)+(1-\sigma)\psi_2(Q_0^*,x,x_0)}{(1+\sigma)\psi_1(Q_0,x,x_0)+(1-\sigma)\psi_1(Q_0^*,x,x_0)} & \text{for } \sigma\in\mathbb{C}\setminus\{-1,1\},\\ \dfrac{\psi_2(Q_0,x,x_0)-\psi_2(Q_0^*,x,x_0)}{\psi_1(Q_0,x,x_0)-\psi_1(Q_0^*,x,x_0)} & \text{for } \sigma=\infty, \end{cases}$$

[1] We compactify $\widehat{\mathcal{K}}_{\sigma,\hat{n}}(Q_0)$ by adding the points P_{∞_+}, P_{∞_-} at infinity and still denote the compactified curve by $\widehat{\mathcal{K}}_{\sigma,\hat{n}}(Q_0)$.

and since

$$\phi(Q_0, x) = \frac{\psi_1(Q_0, x, x_0)}{\psi_2(Q_0, x, x_0)} \neq \phi(Q_0^*, x) = \frac{\psi_1(Q_0^*, x, x_0)}{\psi_2(Q_0^*, x, x_0)},$$

one concludes that no cancellations can occur in (G.37) or (G.40) and hence $\hat{n}(n, Q_0, \sigma) = n + 1$.
Case (ii). $\sigma = \infty$ and $y_0 = 0$: Then (3.59), (3.56), (3.57), (3.69), (3.65), and (G.31) imply

$$\begin{aligned}
\phi(Q_0, x, \infty) &= \lim_{P \to Q_0} \phi(P, x, \infty) \\
&= \lim_{P \to Q_0} \bigg(\bigg(\phi(P, x) \exp\bigg(\int_{x_0}^{x} dx' \, (-iz + q(x')\phi(P, x')) \bigg) \\
&\quad - \phi(P^*, x) \exp\bigg(\int_{x_0}^{x} dx' \, (-iz + q(x')\phi(P^*, x')) \bigg) \bigg) \\
&\quad \times \bigg(\exp\bigg(\int_{x_0}^{x} dx' \, (-iz + q(x')\phi(P, x')) \bigg) \\
&\quad - \exp\bigg(\int_{x_0}^{x} dx' \, (-iz + q(x')\phi(P^*, x')) \bigg) \bigg)^{-1} \bigg) \\
&= \phi(Q_0, x) \\
&\quad + \lim_{P \to Q_0} \bigg(\frac{\phi(P, x) - \phi(P^*, x)}{\exp(\int_{x_0}^{x} dx' \, q(x')\phi(P, x')) - \exp(\int_{x_0}^{x} dx' \, q(x')\phi(P^*, x'))} \\
&\quad \times \exp\bigg(\int_{x_0}^{x} dx' \, q(x')\phi(P^*, x') \bigg) \bigg) \\
&= \phi(Q_0, x) + \exp\bigg(\int_{x_0}^{x} dx' \, q(x')\phi(Q_0, x') \bigg) \\
&\quad \times \lim_{P \to Q_0} \bigg(\bigg(\bigg(\exp\bigg(y \int_{x_0}^{x} dx' \, \frac{q(x')}{F_n(z, x')} \bigg) - \exp\bigg(- y \int_{x_0}^{x} dx' \, \frac{q(x')}{F_n(z, x')} \bigg) \bigg) \\
&\quad \times \exp\bigg(\int_{x_0}^{x} dx' \, \frac{q(x')G_{n+1}(z, x')}{F_n(z, x')} \bigg) \bigg)^{-1} \frac{2y}{F_n(z, x)} \bigg) \\
&= \phi(Q_0, x) + \bigg(F_n(z_0, x) \int_{x_0}^{x} dx' \, \frac{q(x')}{F_n(z, x')} \bigg)^{-1}, \quad x \in \mathbb{C} \setminus \{x_0\},
\end{aligned}$$

by means of $\lim_{P \to Q_0} y(P) = y(Q_0) = y_0 = 0$. Since by (3.21)

$$\phi(Q_0) = \frac{G_{n+1,x}(z_0)}{F_n(z_0)} = \frac{1}{q} \bigg(\frac{F_{n,x}(z_0)}{2F_n(z_0)} + iz_0 \bigg),$$

one again concludes that no cancellation occurs in (G.37) or (G.40), and hence $\hat{n}(n, Q_0, \infty) = n + 1$.

The rest of the proof relies on some additional arguments to be discussed next. First, we will assume without loss of generality that $z_0 = 0$. This can be achieved by noticing that

$$M = i\begin{pmatrix} d/dx & -q \\ p & -d/dx \end{pmatrix} \text{ and } M_a = i\begin{pmatrix} d/dx & -qe^{-2ax} \\ pe^{2ax} & -d/dx \end{pmatrix}$$

are related by

$$UM_aU^{-1} = M - iaI, \quad U = \begin{pmatrix} e^{2ax} & 0 \\ 0 & e^{-2ax} \end{pmatrix}.$$

In the following we abbreviate

$$y^2 = R_{2n+2}(z) \underset{z\to 0}{=} y_0^2 + \tilde{y}_1 z + \tilde{y}_2 z^2 + O(z^3). \tag{G.43}$$

Case (iii). $\sigma \in \{-1, 1\}$ and $y_0 \neq 0$: Then (G.31) yields

$$\phi(Q_0, x, 1) = \phi(Q_0, x), \quad \phi(Q_0, x, -1) = \phi(Q_0^*, x),$$

with $\phi(Q_0, x) \neq \phi(Q_0^*, x)$ since $y_0 \neq 0$. In this case there is a cancellation in (G.37) and (G.40). Choosing $\sigma = 1$, one computes from (3.17), (3.18), and (3.61)

$$\phi(P^*) - \phi(Q_0, \cdot, 1) = \phi(P^*) - \phi(Q_0) \underset{P\to Q_0}{=} -2y_0 f_n^{-1} + O(z).$$

Furthermore,

$$\begin{aligned} \phi(P) - \phi(Q_0, \cdot, 1) &= \phi(P) - \phi(Q_0) \\ &\underset{P\to Q_0}{=} \frac{y - y_0}{f_n} - \left(y_0 \frac{f_{n-1}}{f_n^2} - \frac{g_n}{f_n} + \frac{g_{n+1} f_{n-1}}{f_n^2}\right) z + O(z^2) \\ &\underset{P\to Q_0}{=} \left(\frac{y_1}{f_n} - \frac{y_0 f_{n-1}}{f_n^2} + \frac{g_n}{f_n} - \frac{g_{n+1} f_{n-1}}{f_n^2}\right) z + O(z^2) \\ &\underset{P\to Q_0}{=} c_1 z + O(z^2) \end{aligned} \tag{G.44}$$

since

$$y - y_0 \underset{P\to Q_0}{=} y_1 z + O(z^2), \quad \tilde{y}_1 = 2y_0 y_1.$$

Similarly, we find

$$z + (i/2)q(\phi(P) - \phi(Q_0, \cdot, 1)) \underset{P\to Q_0}{=} (1 + (i/2)qc_1)z + O(z^2). \tag{G.45}$$

It remains to show that c_1 does not vanish identically. We assume temporarily that $\hat{g}_{\sigma,1,2}$ and $\hat{g}_{\sigma,2,1}$ have cancellations of the same order as $z \to 0$. Arguing by contradiction, we suppose that c_1 vanishes identically. But (G.44) and (G.45) then show that $\hat{g}_{\sigma,1,2}$ and $\hat{g}_{\sigma,2,1}$ would have cancellations of different order, which is a contradiction. We conclude that precisely one factor of z cancels in (G.37) and (G.40), and hence $\hat{n}(n, Q_0, 1) = n$. The case $\sigma = -1$ is treated analogously. It remains to show that $\hat{g}_{\sigma,1,2}$ and $\hat{g}_{\sigma,2,1}$ necessarily have cancellations of the same order as $z \to 0$. A comparison of $\widehat{M}_\sigma = i\left(\begin{smallmatrix} d/dx & -\hat{q}_\sigma \\ \hat{p}_\sigma & -d/dx \end{smallmatrix}\right)$ and its formal adjoint $\widehat{M}_\sigma^* = i\left(\begin{smallmatrix} d/dx & -\overline{\hat{p}_\sigma} \\ \overline{\hat{q}_\sigma} & -d/dx \end{smallmatrix}\right)$ yields the replacement of $(\hat{p}_\sigma, \hat{q}_\sigma)$ by $(\overline{\hat{q}_\sigma}, \overline{\hat{p}_\sigma})$ and hence the corresponding replacements of $(\widehat{F}_{\sigma,\hat{n}}(z,x), \widehat{G}_{\sigma,\hat{n}+1}(z,x), \widehat{H}_{\sigma,\hat{n}}(z,x))$ by $(\overline{\widehat{F}_{\sigma,\hat{n}}(\overline{z},x)}, \overline{\widehat{G}_{\sigma,\hat{n}+1}(\overline{z},x)}, \overline{\widehat{H}_{\sigma,\hat{n}}(\overline{z},x)})$ (cf. (3.4)–(3.7) and (3.17)–(3.19)) and $\widehat{R}_{\sigma,2\hat{n}+2}(z)$ by $\overline{\widehat{R}_{\sigma,2\hat{n}+2}(\overline{z})}$ (cf. the notation employed in Lemma G.5). This fact has two consequences: Firstly, from relation (3.23) we infer that the corresponding algebraic curves associated with $\widehat{M}_\sigma$ and $\widehat{M}_\sigma^*$ have complex conjugate branch points, that is, if $\{\widehat{E}_{\sigma,m}\}_{m=0,\dots,2\hat{n}+1}$ corresponds to $\widehat{M}_\sigma$, then $\{\overline{\widehat{E}_{\sigma,m}}\}_{m=0,\dots,2\hat{n}+1}$ corresponds to $\widehat{M}_\sigma^*$, where $\widehat{R}_{\sigma,2\hat{n}+2}(z) = \prod_{m=0}^{2\hat{n}+1}(z - \widehat{E}_{\sigma,m})$. Secondly, we infer

$$\hat{g}_{\sigma,2,1}(P,x) = \overline{\hat{g}^*_{\sigma,2,1}(\overline{P},x)},$$

where $P(z,y)$, $\overline{P} = (\overline{z}, \overline{y})$, and $\hat{g}^*_{\sigma,j,k}(P,x)$ denotes the Green's matrix elements associated with $\widehat{M}_\sigma^*$. This shows that any cancellations in $\widehat{g}_{\sigma,1,2}$ and $\widehat{g}_{\sigma,2,1}$ as $z \to 0$ are necessarily of identical order.

Case (iv). $\sigma \in \mathbb{C}$, $y_0 = 0$, and $R_{2n+1,z}(0) \neq 0$: Using $\phi(Q_0,x,\sigma) = \phi(Q_0,x)$ for all $\sigma \in \mathbb{C}$, (3.17), (3.18) and (3.61) yield

$$\begin{aligned} &(z + (i/2)q(\phi(P) - \phi(Q_0)))(z + (i/2)q(\phi(P^*) - \phi(Q_0))) \\ &\underset{P\to Q_0}{=} \frac{q^2 y_1^2}{4 f_n^2} z + O(z^2) \end{aligned} \tag{G.46}$$

since

$$y \underset{P\to Q_0}{=} y_1 z^{1/2} + O(z^{3/2}), \quad y_1 = \Bigg(\prod_{E_m \neq 0} E_m\Bigg)^{1/2}.$$

Thus, again precisely one factor of z cancels in (G.37) (similarly, one factor cancels in (G.40)) and hence $\hat{n}(n, Q_0, \sigma) = n$.

Case (v). $\sigma \in \mathbb{C}$, $y_0 = \tilde{y}_1 = 0$, and $\tilde{y}_2 \neq 0$ (cf. (G.43)): One computes, as in (G.46) that

$$\begin{aligned} &\big(z + (i/2)q(\phi(P) - \phi(Q_0))\big)\big(z + (i/2)q(\phi(P^*) - \phi(Q_0))\big) \\ &\underset{P\to Q_0}{=} \left(\frac{q^2 y_1^2}{4 f_n^2} + \left(1 + \frac{i}{2}q\left(\frac{g_n}{f_n} - \frac{g_{n+1} f_{n-1}}{f_n^2}\right)\right)^2\right) z^2 + O(z^3) \\ &\underset{P\to Q_0}{=} c_2 z^2 + O(z^3) \end{aligned} \tag{G.47}$$

since

$$y \underset{P\to Q_0}{=} y_1 z + O(z^2), \quad y_1 = \Bigg(\prod_{E_m \neq 0} E_m\Bigg)^{1/2}.$$

Similarly, we find

$$\begin{aligned}
&(\phi(P)-\phi(Q_0))(\phi(P^*)-\phi(Q_0))\\
&\underset{P\to Q_0}{=} -\left(\frac{y_1^2}{f_n^2}-\left(\frac{g_n}{f_n}-\frac{g_{n+1}f_{n-1}}{f_n^2}\right)^2\right)z^2+O(z^3)\\
&\underset{P\to Q_0}{=} c_3 z^2+O(z^3).
\end{aligned}\tag{G.48}$$

We see that both c_2 and c_3 cannot vanish simultaneously, and hence precisely a factor z^2 cancels in (G.37) and (G.40). Thus, $\hat{n}(n, Q_0, \sigma) = n-1$.

Case (vi). $\sigma \in \mathbb{C}$, $y_0 = \tilde{y}_1 = \tilde{y}_2 = 0$ (cf. (G.43)): In analogy to (G.47) and (G.48), one obtains

$$\begin{aligned}
&(z+(i/2)q(\phi(P)-\phi(Q_0)))(z+(i/2)q(\phi(P^*)-\phi(Q_0)))\\
&\underset{P\to Q_0}{=} \left(1+\frac{i}{2}q\left(\frac{g_n}{f_n}-\frac{g_{n+1}f_{n-1}}{f_n^2}\right)\right)^2 z^2+O(z^3)
\end{aligned}$$

and

$$\begin{aligned}
&(\phi(P)-\phi(Q_0))(\phi(P^*)-\phi(Q_0))\\
&\underset{P\to Q_0}{=} \left(\frac{g_n}{f_n}-\frac{g_{n+1}f_{n-1}}{f_n^2}\right)^2 z^2+O(z^3)
\end{aligned}$$

respectively since

$$y \underset{P\to Q_0}{=} O(z^{3/2}).$$

Thus, this case subordinates to case (v), resulting again in $\hat{n}(n, Q_0, \sigma) = n-1$. $\square$

We emphasize that Table G.3 applies as well in the AKNS context.

We conclude this section with the elementary genus zero example.

Example G.7 Assume $n = 0$, $P \in \mathcal{K}_0 \setminus \{P_{\infty_+}, P_{\infty_-}\}$, and let $(x, x_0) \in \mathbb{R}^2$. Then

$$\begin{aligned}
&y^2 = R_2(z) = (z-E_0)(z-E_1), \quad c_1 = -(E_0+E_1)/2, \quad E_0, E_1 \in \mathbb{C},\\
&p(x) = p(x_0)\exp(-2ic_1(x-x_0)), \quad q(x) = q(x_0)\exp(2ic_1(x-x_0)),\\
&p(x)q(x) = (E_0-E_1)^2/4,\\
&\phi(P,x) = \frac{y+z+c_1}{-iq(x)} = \frac{ip(x)}{y-z-c_1},\\
&\psi_1 = \exp((i(y+c_1)(x-x_0)), \quad \psi_2 = \frac{y+z+c_1}{-iq(x_0)}\exp((i(y-c_1)(x-x_0)),
\end{aligned}$$

$$\phi(P,x,\sigma)=\frac{i}{q(x)}$$
$$\times\begin{cases}\dfrac{(1+\sigma)(y+z+c_1)\exp(iy(x-x_0))+(1-\sigma)(-y+z+c_1)\exp(-iy(x-x_0))}{(1+\sigma)\exp(iy(x-x_0))+(1-\sigma)\exp(-iy(x-x_0))}, \sigma\in\mathbb{C},\\[2ex] \dfrac{(y+z+c_1)\exp(iy(x-x_0))+(y-z-c_1)\exp(-iy(x-x_0))}{\exp(iy(x-x_0))+\exp(-iy(x-x_0))}, \sigma=\infty,\end{cases}$$
$$\phi((E_j,0),x,\sigma)=\frac{1}{2q(x)}\begin{cases}i(E_j+c_1), \sigma\in\mathbb{C},\\ i(E_j+c_1)+(x-x_0)^{-1}, \sigma=\infty,\end{cases}\qquad j=0,1.$$

Notes

The material in this appendix is taken from Gesztesy and Holden (2000c).

General commutation methods (also called Crum–Darboux transformations, transmutation methods, etc.) of the type (G.1), (G.2), effecting a transition from AA^+ to A^+A, have a rather long history and go back at least to Jacobi (1837) and Darboux (1882). Since it seems impossible to give a complete bibliography on the subject, we confine ourselves to some of the most relevant sources such as, Buys and Finkel (1984), Crum (1955), Deift (1978), Deift and Trubowitz (1979), Finkel et al. (1987), Flaschka and McLaughlin (1976b), Gesztesy (1993), Gesztesy et al. (1996b), Gesztesy and Teschl (1996), Gesztesy and Weikard (1993), McKean and van Moerbeke (1975), McKean and Trubowitz (1976), Schmincke (1978), and Wahlquist (1976). For commutation methods in connection with Bäcklund transformations in the algebro-geometric context, for instance, to Ercolani and Flaschka (1985), Flaschka (1983), Gesztesy et al. (1991), Gesztesy and Svirsky (1995), Latham and Previato (1994), Matveev and Salle (1991), McKean (1985; 1986; 1987; 1992), Ohmiya (1988a,b; 1995; 1999), Ohmiya and Mishev (1993), Previato (1993), Prikarpatskii (1981), Samoilenko and Prikarpatskii (1985), and Veselov and Shabat (1993).

Historically, the first attempts to link $\mathcal{K}_n$ and $\widehat{\mathcal{K}}_{\hat{n}}$ were made by Drach (1918; 1919a,b), who appears to have been the first to study particular aspects (the case $\hat{n}=n+1$) of Theorem G.2 around 1918. Theorem G.2 was first derived by purely algebro-geometric means in Ehlers and Knörrer (1982). An elementary but lengthy derivation of Theorem G.2 (focusing on the case where $\hat{n}(n,\sigma)=n-1$) was recently provided in Ohmiya (1999) based on two other papers: Ohmiya (1995), and Ohmiya and Mishev (1993). The proof presented in this appendix, taken from Gesztesy and Holden (2000c), seems to be the only elementary and relatively short one available at this point.

It seems worthwhile to point out that the case $\sigma=\infty$ and $y_0=0$, which leads to $\hat{n}(n,Q_0,\sigma)=n+1$, necessarily constructs an algebro-geometric KdV potential $\hat{u}_\infty(x,Q_0)$ singular at $x=x_0$ (cf. (G.17)). Moreover, the curves (G.3) and (G.4) may of course be singular, that is, some (or even all) of the E_m's may coincide.

In fact, the class of rational algebro-geometric solutions constructed in Adler and Moser (1978) (see also Ablowitz and Airault (1981), Ohmiya (1988b), and Ohmiya and Mishev (1993)) arises exactly in this manner with all E_m's vanishing. Similarly, the class of N-soliton solutions and, more generally, N solitons relative to an algebro-geometric background potential, as described, for instance, in Adler and van Moerbeke (1994), Deift (1978), Deift and Trubowitz (1979), Dubrovin et al. (1976), Gesztesy et al. (1991), Gesztesy and Svirsky (1995), Gesztesy and Teschl (1996), Gesztesy and Weikard (1993), Kay and Moses (1956), McKean (1979b; 1987; 1992), McRae and Weikard (1997), Previato (1998), and van Moerbeke (1993), results in N pairs of coinciding E_m's.

The results described in Theorem G.2 are not confined to hyperelliptic curves $\mathcal{K}_n$ of finite (arithmetic) genus n. In fact, upon shifting the emphasis from $F_n(z, x)$ to the diagonal Green's function $g(P, x)$, the results in Theorem G.2 extend to certain classes of transcendental hyperelliptic curves of infinite (arithmetic) genus $\mathcal{K}_\infty$ (including those associated with periodic potentials u), as shown in Gesztesy (2001).

In the AKNS context the gauge transformation (G.34) can be inferred from the results in Konopelchenko (1982) and Konopelchenko and Rogers (1992) with a bit of additional work, as shown in Gesztesy and Weikard (1998a); cf. also Gesztesy and Weikard (1998b). Adding solitons (i.e., inserting eigenvalues into the spectrum of M) and its effect on the Baker–Akhiezer vector Ψ has also been studied in Flaschka (1983) and Flaschka and Newell (1981). Lemma G.5 was first noted in Gesztesy and Weikard (1998a); see also Gesztesy and Weikard (1998b), and Theorem G.6 first appeared in Gesztesy and Holden (2000c).

Appendix H
Elliptic Functions

> The theory of elliptic functions is the fairyland of mathematics. The mathematician who once gazes upon this enchanting and wondrous domain crowded with the most beautiful relations and concepts is forever captivated.
>
> *Richard Bellman*[1]

In this appendix we state some fundamental definitions and special results on elliptic functions useful in connection with the elliptic algebro-geometric examples presented in Chapters 1 and 3.

Definition H.1 A function $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ with two periods a and b,

$$f(z + na + mb) = f(z), \quad z \in \mathbb{C},\ n, m \in \mathbb{Z},$$

where the ratio of a and b is not real, $\operatorname{Im}(a/b) \neq 0$, is called doubly periodic. If all its periods are of the form $m_1 a + m_2 b$, where m_1 and m_2 are integers, then a and b are called fundamental periods of f.
A doubly periodic meromorphic function is called elliptic.

It is customary to denote the fundamental periods of an elliptic function by $2\omega_1$ and $2\omega_3$ with $\operatorname{Im}(\omega_3/\omega_1) > 0$. We also introduce $\omega_2 = \omega_1 + \omega_3$ and $\omega_4 = 0$. The numbers $\omega_1, \dots, \omega_4$ are called half-periods. The fundamental period parallelogram Δ denotes the domain consisting of the line segments $[0, 2\omega_1)$, $[0, 2\omega_3)$ and the interior of the parallelogram with vertices 0, $2\omega_1$, $2\omega_2$, and $2\omega_3$.

The class of elliptic functions with fundamental periods $2\omega_1, 2\omega_3$ is closed under addition, subtraction, multiplication, division by nonzero divisors, and differentiation. If f is an entire elliptic function, then f is constant. An elliptic function f that is not constant must have at least one pole in Δ, and the total number of poles in Δ is finite. The total number of poles (counting multiplicity) of an elliptic function f in Δ is called the order of f. The sum of residues of an elliptic function

[1] *A Brief Introduction to Theta Functions*, Holt, Rinehart, and Winston, New York, 1961, p. vii.

f at all its poles in Δ equals zero. In particular, the order of a nonconstant elliptic function f is at least 2. The total number of points in Δ where the nonconstant elliptic function f assumes the value $A \in \mathbb{C}_\infty$ (counting multiplicity), denoted by $n(A)$, is equal to the order of f. In particular, $n(A) \geq 2$. Furthermore, $s(A)$, the sum of all the points in Δ where the nonconstant elliptic function f assumes the value A, is congruent to $s(\infty)$, the sum of all the points in Δ where f has a pole, that is, $s(A) = s(\infty) + 2m_1\omega_1 + 2m_3\omega_3$, where m_1 and m_3 are certain integers.

We now introduce the fundamental Weierstrass $\wp$-function.

Definition H.2 The function $\wp(\,\cdot\,|\,\omega_1, \omega_3)$ defined by

$$\wp(z|\,\omega_1, \omega_3) = \frac{1}{z^2} + \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \left(\frac{1}{(z - 2m\omega_1 - 2n\omega_3)^2} - \frac{1}{(2m\omega_1 + 2n\omega_3)^2}\right), \quad z \in \mathbb{C},\ z \neq 0 \pmod{\Delta}, \tag{H.1}$$

often simply denoted by $\wp(\,\cdot\,)$ for brevity, is an even elliptic function of order 2 with fundamental periods $2\omega_1$ and $2\omega_3$.

Every elliptic function with fundamental periods $2\omega_1$ and $2\omega_3$ may be written as $R_1(\wp) + R_2(\wp)\wp'$, where R_1 and R_2 are rational functions of $\wp(\,\cdot\,|\,\omega_1, \omega_3)$. We recall that the derivative $\wp'$ of $\wp$ is an odd elliptic function of order 3 with fundamental periods $2\omega_1$ and $2\omega_3$.

The Laurent expansions of $\wp$ and $\wp'$ at $z = 0$ are given by

$$\wp(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty} c_k z^{2k-2},$$
$$\wp'(z) = -\frac{2}{z^3} + \sum_{k=2}^{\infty}(2k-2)c_k z^{2k-3},$$

where

$$c_2 = 3\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(2m\omega_1 + 2n\omega_3)^4}, \quad c_3 = 5\sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(2m\omega_1 + 2n\omega_3)^6},$$
$$c_k = \frac{3}{(2k+1)(k-3)} \sum_{m=2}^{k-2} c_m c_{k-m}, \quad k \geq 4. \tag{H.2}$$

The numbers $g_2 = 20c_2$ and $g_3 = 28c_3$ are called invariants of $\wp$. Since $\wp(\,\cdot\,|\,\omega_1, \omega_3)$ is also uniquely characterized by its invariants g_2 and g_3, one frequently uses the notation $\wp(\,\cdot\,; g_2, g_3)$.

The function $\wp$ satisfies the first-order differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3 \tag{H.3}$$

and the second-order differential equation

$$\wp''(z) = 6\wp(z)^2 - g_2/2.$$

The function $\wp'$ being of order 3 has three zeros in Δ. Since $\wp'$ is odd and elliptic, it is obvious that these zeros are the half-periods ω_1, $\omega_2 = \omega_1 + \omega_3$ and ω_3. Denote $\wp(\omega_j) = e_j$, $j = 1, 2, 3$. Then (H.3) implies that $4e_j^3 - g_2 e_j - g_3 = 0$ for $j = 1$, 2, 3. Therefore,

$$\begin{aligned}
0 &= e_1 + e_2 + e_3, \\
g_2 &= -4(e_1e_2 + e_1e_3 + e_2e_3) = 2\left(e_1^2 + e_2^2 + e_3^2\right), \\
g_3 &= 4e_1e_2e_3 = \frac{4}{3}\left(e_1^3 + e_2^3 + e_3^3\right).
\end{aligned}$$

Weierstrass also introduced two other functions denoted by ζ and σ. The Weierstrass ζ-function, $\zeta(\,\cdot \mid \omega_1, \omega_3)$, or simply $\zeta(\,\cdot\,)$, is defined by

$$\frac{d}{dz}\zeta(z) = -\wp(z), \quad \lim_{z\to 0}\left(\zeta(z) - \frac{1}{z}\right) = 0, \quad z \in \mathbb{C},\ z \neq 0 \pmod{\Delta}.$$

ζ is a meromorphic function with simple poles at $2m\omega_1 + 2n\omega_3$ for $m, n \in \mathbb{Z}$ having residues 1. It is not periodic but satisfies

$$\zeta(z + 2\omega_j) = \zeta(z) + 2\eta_j, \quad j = 1, 2, 3, 4,$$

where $\eta_j = \zeta(\omega_j)$ for $j = 1, 2, 3$ and $\eta_4 = 0$. The Laurent expansion of ζ at $z = 0$ is given by

$$\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} \frac{c_k}{2k-1} z^{2k-1},$$

with the c_k given in (H.2).

The Weierstrass σ-function, $\sigma(\,\cdot \mid \omega_1, \omega_3)$, or simply $\sigma(\,\cdot\,)$, is defined by

$$\frac{\sigma'(z)}{\sigma(z)} = \zeta(z), \quad \lim_{z\to 0}\frac{\sigma(z)}{z} = 1, \quad z \in \mathbb{C}.$$

σ is an entire function with simple zeros at the points $2m\omega_1 + 2n\omega_3$ for $m, n \in \mathbb{Z}$. It satisfies

$$\sigma(z + 2\omega_j) = -\sigma(z)e^{2\eta_j(z+\omega_j)}, \quad j = 1, 2, 3.$$

Next we recall the following fundamental theorems.

Theorem H.3 *Given an elliptic function f with fundamental periods $2\omega_1$ and $2\omega_3$, let $b_1, \ldots, b_r$ be the poles of f in Δ. Suppose the principal part of the Laurent* *expansion near b_k is given by*

$$\sum_{\ell=1}^{\beta_k} \frac{A_{\ell,k}}{(z-b_k)^\ell}, \quad k=1,\ldots,r.$$

Then

$$f(z) = C + \sum_{k=1}^{r} \sum_{\ell=1}^{\beta_k} (-1)^{\ell-1} \frac{A_{\ell,k}}{(\ell-1)!} \zeta^{(\ell-1)}(z-b_k),$$

where C is a suitable constant and ζ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$. Conversely, every such function is an elliptic function if $\sum_{k=1}^{r} A_{1,k} = 0$.

Theorem H.4 *Given an elliptic function f of order n with fundamental periods $2\omega_1$ and $2\omega_3$, let $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ be the zeros and poles of f in Δ each counted a number of times equal to its order. Then*

$$f(z) = C \frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_{n-1})\sigma(z-b_n')},$$

where C is a suitable constant, σ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$, and where

$$b_n' - b_n = (a_1 + \cdots + a_n) - (b_1 + \cdots + b_n)$$

is a period of f. Conversely, every such function is an elliptic function.

Finally, we turn to elliptic functions of the second kind. A meromorphic function $\psi : \mathbb{C} \to \mathbb{C}_\infty$ for which there exist two complex constants ω_1 and ω_3 with nonreal ratio and two complex constants ρ_1 and ρ_3 such that for $i = 1, 3$

$$\psi(z + 2\omega_i) = \rho_i \psi(z)$$

is called elliptic of the second kind. It is common to call $2\omega_1$ and $2\omega_3$ the quasi-periods of ψ. Together with $2\omega_1$ and $2\omega_3$, $2m_1\omega_1 + 2m_3\omega_3$ are also quasi-periods of ψ if m_1 and m_3 are integers. If every quasi-period of ψ can be written as an integer linear combination of $2\omega_1$ and $2\omega_3$, then these are called fundamental quasi-periods.

Theorem H.5 *A function ψ that is elliptic of the second kind and has fundamental quasi-periods $2\omega_1$ and $2\omega_3$ can always be put in the form*

$$\psi(z) = C \exp(\lambda z) \frac{\sigma(z-a_1)\cdots\sigma(z-a_n)}{\sigma(z-b_1)\cdots\sigma(z-b_n)} \tag{H.4}$$

for suitable constants C, λ, $a_1, \ldots, a_n$, and $b_1, \ldots, b_n$. Here σ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$. Conversely, every function of the type (H.4) *is elliptic of the second kind.*

Theorem H.6 *Given numbers $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_m$ such that $\beta_k \neq \beta_\ell$* (mod Δ) *for $k \neq \ell$, the following identity holds*

$$\prod_{j=1}^{m} \frac{\sigma(z-\alpha_j)}{\sigma(z-\beta_j)} = \sum_{j=1}^{m} \frac{\prod_{k=1}^{m} \sigma(\beta_j - \alpha_k)}{\prod_{\ell=1,\ell\neq j}^{m} \sigma(\beta_j - \beta_\ell)} \frac{\sigma(z-\beta_j+\beta-\alpha)}{\sigma(z-\beta_j)\sigma(\beta-\alpha)},$$

where

$$\alpha = \sum_{j=1}^{m} \alpha_j \text{ and } \beta = \sum_{j=1}^{m} \beta_j$$

and σ is constructed from the fundamental periods $2\omega_1$ and $2\omega_3$.

Notes

For standard monographs on elliptic functions refer, for instance, to Akhiezer (1990), Burkhardt (1906), Chandrasekharan (1985), Forsyth (1965), Fricke (1913), Halphen (1886; 1888; 1891), Hancock (1958), Hurwitz and Courant (1964), Jones and Singerman (1987), Krause (1895; 1897), Markushevich (1985), McKean and Moll (1997), Rauch and Lebowitz (1973), Siegel (1988a), and Whittaker and Watson (1986). For a comprehensive summary of results, refer to Abramowitz and Stegun (1972, Ch. 18), whose notation we follow in this text.

Theorem H.6, in a somewhat different form, can be found with some effort in Krause (1895, pp. 292–296; 1897, pp. 259–264); a sketch of its proof has been given in Gesztesy and Weikard (1998b, Theorem 2.5).

Appendix I
Herglotz Functions

Ingen trykfejl; hvert ord er vigtig.
Henrik Ibsen[1]

We briefly summarize a few basic facts on Herglotz functions relevant to Green's functions and the spectral theory of one-dimensional Schrödinger operators in Chapter 1 (cf. also Appendix J).

Definition I.1 Let $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \gtrless 0\}$. Any analytic map $m\colon \mathbb{C}_+ \to \mathbb{C}_+$ extended to $\mathbb{C}_-$ by $m(\overline{z}) = \overline{m(z)}$ for $z \in \mathbb{C}_+$, is called a Herglotz function.[2]

With m a Herglotz function, one verifies that

$$\hat{m}(z) = \frac{-1 + \beta m(z)}{\beta + m(z)}, \quad \beta \in \mathbb{R} \cup \{\infty\},$$

and

$$\hat{m}(z) = \ln(m(z))$$

($\ln(re^{i\phi}) = \ln(r) + i\phi$ for $r > 0$ and $0 < \phi < \pi$) are Herglotz functions as well.

Herglotz functions admit particular representations (Borel transforms) in terms of measures on $\mathbb{R}$. Since this aspect is of fundamental importance in the context of spectral theory for Schrödinger and Jacobi operators, we recall the following classical results.

[1] In *En folkefiende* (1882), third act. ("No misprints; every word is important." *An Enemy of the People*.)

[2] There appears to be considerable confusion in the literature since Nevanlinna, Pick, Nevanlinna–Pick function, as well as R-function in addition to Herglotz function, are also in use. In part these discrepancies can be traced back to the use of the upper half-plane $\mathbb{C}_+$ versus the open unit disk D; in some cases the geographical location of the author in question determines the preferred notation.

Theorem I.2 *Let m be a Herglotz function. Then,*
(i) There exists a measure $d\omega$ on $\mathbb{R}$ and a $\xi \in L^1_{\mathrm{loc}}(\mathbb{R})$ real-valued such that

$$\begin{aligned} m(z) &= a + bz + \int_{\mathbb{R}} d\omega(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) \\ &= \exp\left(c + \int_{\mathbb{R}} d\lambda \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right) \xi(\lambda) \right), \end{aligned} \tag{I.1}$$

where

$$\int_{\mathbb{R}} \frac{d\omega(\lambda)}{1+\lambda^2} < \infty, \quad a = \mathrm{Re}(m(i)), \quad b \geq 0$$

and

$$0 \leq \xi \leq 1 \text{ a.e.}, \quad c = \mathrm{Re}(\ln(m(i))).$$

(ii) m (and hence $\ln(m)$*) have nontangential limits at almost every* $\lambda \in \mathbb{R}$*. Moreover,*

$$\omega((\lambda, \mu]) = \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \int_{\lambda+\delta}^{\mu+\delta} d\nu \, \mathrm{Im}(m(\nu + i\varepsilon)), \quad \lambda, \mu \in \mathbb{R}, \quad \lambda < \mu,$$

$$\xi(\lambda) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \mathrm{Im}\left(\ln\left(m(\lambda + i\varepsilon)\right)\right) \textit{ for } \text{a.e. } \lambda \in \mathbb{R}.$$

(iii) Let $n, p \in \mathbb{N}$ and $b = 0$. Then

$$\int_{-\infty}^{0} d\lambda \, (1+\lambda^2)^{-1} |\lambda|^p |\xi(\lambda)| + \int_0^\infty d\lambda \, (1+\lambda^2)^{-1} |\lambda|^n |\xi(\lambda)| < \infty$$

if and only if

$$\int_{-\infty}^{0} d\omega(\lambda) \, (1+\lambda^2)^{-1} |\lambda|^p + \int_0^\infty d\omega(\lambda) \, (1+\lambda^2)^{-1} |\lambda|^n < \infty$$

$$\textit{and} \quad \lim_{z \to i\infty} m(z) = a - \int_{\mathbb{R}} d\omega(\lambda) \, (1+\lambda^2)^{-1} \lambda > 0.$$

(iv) Furthermore,

$$m(z) = 1 + \int_{\mathbb{R}} d\omega(\lambda) \, (\lambda - z)^{-1}, \quad \int_{\mathbb{R}} d\omega(\lambda) < \infty$$

if and only if

$$m(z) = \exp\left(\int_{\mathbb{R}} d\lambda \, (\lambda - z)^{-1} \xi(\lambda) \right), \quad 0 \leq \xi \leq 1 \text{ a.e.}, \quad \xi \in L^1(\mathbb{R}).$$

In this case

$$\int_{\mathbb{R}} d\omega(\lambda) = \int_{\mathbb{R}} d\lambda \, \xi(\lambda).$$

(v) Local singularities and zeros of m are necessarily located on the real axis and are at most of first order in the sense that

$$\omega(\{\lambda\}) = \lim_{\varepsilon\downarrow 0}(\omega(\lambda+\varepsilon)-\omega(\lambda-\varepsilon)) = -\lim_{\varepsilon\downarrow 0} i\varepsilon\, m(\lambda+i\varepsilon) \geq 0, \quad \lambda\in\mathbb{R},$$
$$\lim_{\varepsilon\downarrow 0} i\varepsilon m(\lambda+i\varepsilon)^{-1} \geq 0, \quad \lambda\in\mathbb{R}.$$

In particular, isolated poles of m are simple and located on the real axis, the corresponding residues being negative.

Moreover, (I.1) implies

$$\frac{d}{dz}\ln(m(z)) = \int_{\mathbb{R}} d\lambda\,(\lambda-z)^{-2}\xi(\lambda), \tag{I.2}$$

which is a useful fact in connection with trace formulas.

The next result is used in the KdV context in Lemmas 1.10, 1.11, 1.37, and 1.38.

Theorem I.3 *Let* $\{E_m\}_{m=0,\dots,2n}\subset\mathbb{R}$ *with* $E_0<E_1<\cdots<E_{2n}$, $n\in\mathbb{N}$, *define* $\Sigma\subset\mathbb{R}$ *by*

$$\Sigma = \bigcup_{j=0}^{n-1}[E_{2j},E_{2j+1}]\cup[E_{2n},\infty), \tag{I.3}$$

and introduce $R_{2n+1}^{1/2}$ *as in* (B.17)–(B.20) *followed by an analytic continuation to* $\mathbb{C}\setminus\Sigma$. *Moreover let* F_n *and* H_{n+1} *be two polynomials of degree n and* $n+1$, *respectively. Then*

$$\frac{iF_n(z)}{R_{2n+1}(z)^{1/2}}$$

is a Herglotz function if and only if all zeros of F_n *are real and there is precisely one zero in each of the intervals* $[E_{2j-1},E_{2j}]$, $j=1,\dots,n$. *Moreover, if* $iF_n/R_{2n+1}^{1/2}$ *is a Herglotz function, then it can be represented in the form*

$$\frac{iF_n(z)}{R_{2n+1}(z)^{1/2}} = \frac{1}{\pi}\int_\Sigma \frac{d\lambda\, F_n(\lambda)}{R_{2n+1}(\lambda)^{1/2}}\frac{1}{\lambda-z}, \quad z\in\mathbb{C}\setminus\Sigma.$$

Similarly,

$$\frac{iH_{n+1}(z)}{R_{2n+1}(z)^{1/2}}$$

is a Herglotz function if and only if all zeros of H_{n+1} *are real and there is precisely one zero in each of the intervals* $(-\infty,E_0]$ *and* $[E_{2j-1},E_{2j}]$, $j=1,\dots,n$. *Moreover, if* $iH_{n+1}/R_{2n+1}^{1/2}$ *is a Herglotz function, then it can be represented in the* *form*

$$\frac{iH_{n+1}(z)}{R_{2n+1}(z)^{1/2}} = \mathrm{Re}\left(\frac{iH_{n+1}(i)}{R_{2n+1}(i)^{1/2}}\right) + \frac{1}{\pi}\int_{\Sigma}\frac{d\lambda\, H_{n+1}(\lambda)}{R_{2n+1}(\lambda)^{1/2}}\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \quad z \in \mathbb{C}\setminus\Sigma.$$

These results naturally extend to matrix-valued situations. Since we actually have occasion to use matrix Herglotz functions in connection with Schrödinger operators on $\mathbb{R}$, we briefly summarize a few results in this context.

Definition I.4 Denote by $M_n(\mathbb{C})$ the set of $n \times n$ matrices with entries in $\mathbb{C}$. A map $M\colon \mathbb{C}_+ \to M_n(\mathbb{C})$, extended to $\mathbb{C}_-$ by $M(\overline{z}) = M(z)^*$ for all $z \in \mathbb{C}_+$, is called an $n \times n$ Herglotz matrix if it is analytic on $\mathbb{C}_+$ and[1] $\mathrm{Im}(M(z)) \geq 0$ for all $z \in \mathbb{C}_+$.

Theorem I.5 *Let M be an $n \times n$ Herglotz matrix. Then,*

(i) There exists an $n \times n$ matrix measure $d\Omega$ on $\mathbb{R}$ and a self-adjoint matrix $\Xi \in L^1_{\mathrm{loc}}(\mathbb{R})^{n\times n}$ such that

$$\begin{aligned} M(z) &= A + Bz + \int_{\mathbb{R}} d\Omega(\lambda)\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) \\ &= \exp\left(C + \int_{\mathbb{R}} d\lambda\left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right)\Xi(\lambda)\right), \end{aligned} \tag{I.4}$$

where

$$\int_{\mathbb{R}}\frac{d\Omega(\lambda)}{1+\lambda^2} < \infty, \quad A = \mathrm{Re}(M(i)), \quad B \geq 0,$$

and[2]

$$0 \leq \Xi \leq I_n \text{ a.e.}, \quad C = \mathrm{Re}(\ln(M(i))).$$

(ii) M (and $\ln(M)$) have nontangential limits at almost every $\lambda \in \mathbb{R}$. Moreover,

$$\Omega((\lambda,\mu]) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\, \mathrm{Im}(M(\nu + i\varepsilon)), \quad \lambda,\mu \in \mathbb{R}, \quad \lambda < \mu, \tag{I.5}$$

$$\Xi(\lambda) = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\mathrm{Im}\left(\ln\left(M(\lambda + i\varepsilon)\right)\right) \text{ for a.e. } \lambda \in \mathbb{R}, \tag{I.6}$$

$$\Omega(\{\lambda\}) = \lim_{\varepsilon\downarrow 0}(\Omega(\lambda+\varepsilon) - \Omega(\lambda - \varepsilon)) = -\lim_{\varepsilon\downarrow 0} i\varepsilon\, M(\lambda + i\varepsilon) \geq 0, \quad \lambda \in \mathbb{R}, \tag{I.7}$$

[1] We denote $\mathrm{Im}(M) = (M - M^*)/2i$ and $\mathrm{Re}(M) = (M + M^*)/2$.
[2] I_n denotes the identity matrix in $M_n(\mathbb{C})$.

and

$$\lim_{\varepsilon \downarrow 0} i\varepsilon\, M(\lambda + i\varepsilon)^{-1} \geq 0, \quad \lambda \in \mathbb{R}.$$

If M is an $n \times n$ Herglotz matrix, then any diagonal element $M_{p,p} \not\equiv 0$ is a (scalar) Herglotz function (in the sense of Definition I.1) for $p = 1, \ldots, n$. Moreover, if A is a self-adjoint operator in a complex separable Hilbert space $\mathcal{H}$, then $(f, (A - \cdot)^{-1} f)$ is a Herglotz function for each $f \in \mathcal{H} \setminus \{0\}$. Finally, by a straightforward application of the first resolvent identity, diagonal Green's functions of scalar self-adjoint Schrödinger and Jacobi-type operators are all (scalar) Herglotz functions.

Notes

The fundamental results on Herglotz functions and their representations as Borel transforms, in parts, are due to Fatou, Herglotz, Luzin, Nevanlinna, Plessner, Privalov, de la Vallée Poussin, Riesz, and others. A fairly extensive list of pertinent references can be found in Gesztesy and Tsekanovskii (2000). The exponential Herglotz representation (I.1) is due to Aronszajn and Donoghue (1957) (see also Aronszajn and Donoghue (1964)). The corresponding matrix-valued case (I.4) (and more generally, the infinite-dimensional case) can be found in Carey (1976), Gesztesy et al. (1999), and Gesztesy and Tsekanovskii (2000). Details concerning Theorem I.3 (and its matrix-valued generalizations) can be found in Gesztesy and Sakhnovich (to appear). Applications of this circle of ideas to (inverse) spectral theory of (matrix-valued) Schrödinger operators can be found, for instance, in Belokolos et al. (to appear), Clark et al. (2000), Gesztesy (1995), Gesztesy and Holden (1995; 1997), Gesztesy et al. (1993; 1995a,b), Gesztesy and Makarov (2000), Gesztesy et al. (1996a; 1999), Gesztesy and Simon (1995; 1996a,b), and Gesztesy and Tsekanovskii (2000).

Appendix J

Spectral Measures and Weyl–Titchmarsh m-Functions for Schrödinger Operators

Proof is the idol before whom the pure mathematician tortures himself.

Sir Arthur Eddington[1]

In this appendix we indicate the role of Herglotz functions (cf. Appendix I) in connection with spectral theory of Schrödinger operators on the half-line $[0, \infty)$ and on all of $\mathbb{R}$. The material presented includes a discussion of (matrix-valued) spectral functions and asymptotic spectral parameter expansions of half-line Weyl–Titchmarsh m-functions for general (i.e., not necessarily Dirichlet) boundary conditions. We also treat the special case of algebro-geometric potentials and explicitly compute the corresponding spectral matrix.

We start with Schrödinger operators on the half-line $[0, \infty)$ under the following basic assumptions.

Hypothesis J.1 *Suppose*

$$u \in L^1([0, R]) \textit{ for all } R > 0, \quad u \textit{ real-valued},$$

and that the differential expression

$$L_+ = -\frac{d^2}{dx^2} + u, \quad x \geq 0$$

is in the limit point case at $+\infty$.

Associated with L_+ we introduce the following self-adjoint operator $H_{+,\alpha}$ in $L^2([0, \infty))$. Define

$$\begin{aligned} &H_{+,\alpha} f = L_+ f, \quad \alpha \in [0, \pi), \\ &f \in \operatorname{dom}(H_{+,\alpha}) = \{g \in L^2([0, \infty)) \mid g, g' \in AC([0, R]) \text{ for all } R > 0, \\ &\qquad \sin(\alpha) g'(0_+) + \cos(\alpha) g(0_+) = 0,\ L_+ g \in L^2([0, \infty))\}. \end{aligned} \tag{J.1}$$

[1] Quoted in N. J. Rose, *Mathematical Maxims and Minims*, Raleigh, NC, Rome Press, 1988, p. 130.

$H_{+,\alpha}$ is a real operator (i.e., $g \in \mathrm{dom}(H_{+,\alpha})$ implies $\overline{g} \in \mathrm{dom}(H_{+,\alpha})$ and $H_{+,\alpha}\overline{g} = \overline{H_{+,\alpha}\, g}$) with uniform spectral multiplicity one.

Next we introduce the fundamental system $\phi_\alpha(z,\,\cdot\,), \theta_\alpha(z,\,\cdot\,)$ for $z \in \mathbb{C}$ of solutions of

$$(L_+ - z)\psi(z) = 0, \quad x \geq 0 \tag{J.2}$$

satisfying

$$\phi_\alpha(z, 0_+) = -\theta'_\alpha(z, 0_+) = -\sin(\alpha), \quad \phi'_\alpha(x, 0_+) = \theta_\alpha(z, 0_+) = \cos(\alpha) \tag{J.3}$$

such that the Wronskian $W(\theta_\alpha(z), \phi_\alpha(z)) = 1$. Furthermore, let $\psi_{+,\alpha}(z,\,\cdot\,)$ for $z \in \mathbb{C} \setminus \mathbb{R}$ be the unique solution of (J.2) that satisfies

$$\psi_{+,\alpha}(z,\,\cdot\,) \in L^2([0,\infty)), \quad \sin(\alpha)\psi'_{+,\alpha}(z, 0_+) + \cos(\alpha)\psi_{+,\alpha}(z, 0_+) = 1.$$

The function $\psi_{+,\alpha}(z,\,\cdot\,)$ is of the form

$$\psi_{+,\alpha}(z, x) = \theta_\alpha(z, x) + m_{+,\alpha}(z)\phi_\alpha(z, x),$$

where $m_{+,\alpha}$ denotes Weyl–Titchmarsh's m-function, which is well-known to be a Herglotz function (cf. also the comment following (J.5)). To avoid repetitions, we list properties of $m_{+,\alpha}$ a bit later (together with those of $m_{-,\alpha}$). Here we just note that the Herglotz property of $m_{+,\alpha}$ together with the asymptotic behavior (J.12) and (J.13) yields the existence of a measure $d\omega_{+,\alpha}$ on $\mathbb{R}$, the spectral measure of $H_{+,\alpha}$, such that

$$\begin{aligned} m_{+,\alpha}(z) &= \mathrm{Re}(m_{+,\alpha}(i)) + \int_{\mathbb{R}} d\omega_{+,\alpha}(\lambda) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right), \quad \alpha \in [0,\pi) \\ &= \cot(\alpha) + \int_{\mathbb{R}} d\omega_{+,\alpha}(\lambda)\,(\lambda - z)^{-1}, \quad \alpha \in (0,\pi) \end{aligned}$$

with

$$\int_{\mathbb{R}} \frac{d\omega_{+,\alpha}(\lambda)}{1+|\lambda|} < \infty, \quad \alpha \in (0,\pi), \qquad \int_{\mathbb{R}} \frac{d\omega_{+,0}(\lambda)}{1+|\lambda|} = \infty, \qquad \int_{\mathbb{R}} \frac{d\omega_{+,0}(\lambda)}{1+\lambda^2} < \infty.$$

The Green's function $G_{+,\alpha}(z, x, x')$ of $H_{+,\alpha}$ finally reads

$$\begin{aligned} ((H_{+,\alpha} - z)^{-1} f)(x) &= \int_0^\infty dx'\, G_{+,\alpha}(z, x, x') f(x'), \\ &\qquad z \in \mathbb{C} \setminus \mathrm{spec}(H_{+,\alpha}),\ f \in L^2([0,\infty)), \end{aligned}$$

where

$$\begin{aligned} G_{+,\alpha}(z, x, x') &= \begin{cases} \phi_\alpha(z, x)\psi_{+,\alpha}(z, x') & \text{for } 0 \leq x \leq x', \\ \phi_\alpha(z, x')\psi_{+,\alpha}(z, x) & \text{for } 0 \leq x' \leq x, \end{cases} \\ &= \int_{\mathbb{R}} d\omega_{+,\alpha}(\lambda)\,(\lambda - z)^{-1}\phi_\alpha(\lambda, x)\phi_\alpha(\lambda, x'). \end{aligned}$$

In particular,

$$
\begin{aligned}
G_{+,\alpha}(z,0,0) &= -\sin(\alpha)\big(\cos(\alpha) - m_{+,\alpha}(z)\sin(\alpha)\big), \quad \alpha \in [0,\pi) \qquad \text{(J.4)}\\
&= \sin^2(\alpha)\int_{\mathbb{R}} d\omega_{+,\alpha}(\lambda)\,(\lambda - z)^{-1}, \quad \alpha \in (0,\pi)
\end{aligned}
$$

and for each $x \geq 0$, one concludes that

$$
G_{+,\alpha}(\,\cdot\,, x, x) \text{ is Herglotz} \qquad \text{(J.5)}
$$

in accordance with the paragraph following (I.7). Together with (J.4) this yields a proof that $m_{+,\alpha}$ is Herglotz too.

Next, we recall a few facts in connection with Schrödinger operators on $\mathbb{R}$. We use the following basic assumptions.

Hypothesis J.2 *Suppose*

$$
u \in L^1_{\mathrm{loc}}(\mathbb{R}), \quad u \textit{ real-valued},
$$

and that the differential expression

$$
L = -\frac{d^2}{dx^2} + u, \quad x \in \mathbb{R}
$$

is in the limit point case at $\pm\infty$.

The self-adjoint operator H in $L^2(\mathbb{R})$ associated with L is then introduced by

$$
\begin{aligned}
&Hf = Lf,\\
&f \in \mathrm{dom}(H) = \{g \in L^2(\mathbb{R}) \mid g, g' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}),\ Lg \in L^2(\mathbb{R})\}.
\end{aligned}
$$

As in the half-line case (J.1), H is a real operator. Moreover, the point spectrum $\mathrm{spec}_{\mathrm{p}}(H)$ of H is simple.

Next we define $\phi_\alpha(z,\,\cdot\,)$ and $\theta_\alpha(z,\,\cdot\,)$ as in (J.2) and (J.3) (replacing L_+ by L) and introduce the uniquely determined solutions $\psi_{\pm,\alpha}(z,\,\cdot\,)$ of

$$
(L - z)\psi(z) = 0, \quad x \in \mathbb{R}
$$

satisfying

$$
\psi_{\pm,\alpha}(z,\,\cdot\,) \in L^2([R,\pm\infty)), \quad \sin(\alpha)\psi'_{\pm,\alpha}(z,0) + \cos(\alpha)\psi_{\pm,\alpha}(z,0) = 1
$$

for all $R \in \mathbb{R}$. One infers

$$
\psi_{\pm,\alpha}(z,x) = \theta_\alpha(z,x) + m_{\pm,\alpha}(z)\phi_\alpha(z,x)
$$

in terms of the half-line Weyl–Titchmarsh m-functions $m_{\pm,\alpha}$. With our conventions

$$
\pm\, m_{\pm,\alpha} \text{ are Herglotz}, \quad \pm\mathrm{Im}(m_{\pm,\alpha}(z)) > 0, \quad \pm z \in \mathbb{C}_+, \qquad \text{(J.6)}
$$

$$
\overline{m_{\pm,\alpha}(z)} = m_{\pm,\alpha}(\overline{z}), \quad z \in \mathbb{C}\setminus\mathbb{R}, \qquad \text{(J.7)}
$$

$$
W(\psi_{+,\alpha}(z), \psi_{-,\alpha}(z)) = m_{-,\alpha}(z) - m_{+,\alpha}(z). \qquad \text{(J.8)}
$$

Moreover, in accordance with Theorem I.2, we recall the following facts:

$$\pm\lim_{\varepsilon\downarrow 0} i\varepsilon\, m_{\pm,\alpha},(\lambda+i\varepsilon) = \begin{cases} 0 & \text{for } \phi_\alpha(\lambda,\,\cdot\,)\notin L^2((0,\pm\infty)), \\ -\|\phi_\alpha(\lambda,\,\cdot\,)\|_2^{-2} & \text{for } \phi_\alpha(\lambda,\,\cdot\,)\in L^2((0,\pm\infty)), \end{cases}$$
$$\lambda\in\mathbb{R}, \tag{J.9}$$

$$m_{\pm,\alpha_1}(z) = \frac{-\sin(\alpha_1-\alpha_2)+\cos(\alpha_1-\alpha_2)m_{\pm,\alpha_2}(z)}{\cos(\alpha_1-\alpha_2)+\sin(\alpha_1-\alpha_2)m_{\pm,\alpha_2}(z)}, \quad \alpha_1,\alpha_2\in[0,\pi), \tag{J.10}$$

$$\begin{aligned} m_{\pm,\alpha}(z) &= \mathrm{Re}(m_{\pm,\alpha}(\pm i)) \pm \int_{\mathbb{R}} d\omega_{\pm,\alpha}(\lambda)\left(\frac{1}{\lambda-z}-\frac{\lambda}{1+\lambda^2}\right), \quad \alpha\in[0,\pi) \\ &= \cot(\alpha) \pm \int_{\mathbb{R}} d\omega_{\pm,\alpha}(\lambda)\,(\lambda-z)^{-1}, \quad \alpha\in(0,\pi), \end{aligned} \tag{J.11}$$

$$m_{\pm,\alpha}(z) \underset{z\to i\infty}{=} \cot(\alpha) \pm \frac{i}{\sin^2(\alpha)}\,z^{-1/2} - \frac{\cos(\alpha)}{\sin^3(\alpha)}\,z^{-1} + o(z^{-1}), \quad \alpha\in(0,\pi), \tag{J.12}$$

$$m_{\pm,0}(z) \underset{z\to i\infty}{=} \pm i z^{1/2} + o(1) \tag{J.13}$$

with

$$\int_{\mathbb{R}} \frac{d\omega_{\pm,\alpha}(\lambda)}{1+|\lambda|} < \infty, \quad \alpha\in(0,\pi), \quad \int_{\mathbb{R}} \frac{d\omega_{\pm,0}(\lambda)}{1+|\lambda|} = \infty, \quad \int_{\mathbb{R}} \frac{d\omega_{\pm,0}(\lambda)}{1+\lambda^2} < \infty,$$

$$\begin{aligned} \pm\int_0^{\pm\infty} dx\, \psi_{\pm,\alpha}(z_1,x)\psi_{\pm,\alpha}(z_2,x) &= \pm\frac{m_{\pm,\alpha}(z_1)-m_{\pm,\alpha}(z_2)}{z_1-z_2} \\ &= \int_{\mathbb{R}} (\lambda-z_1)^{-1}(\lambda-z_2)^{-1}\, d\omega_{\pm,\alpha}(\lambda). \end{aligned}$$

The Green's function $G(z,x,x')$ of H is then characterized by

$$((H-z)^{-1}f)(x) = \int_{\mathbb{R}} dx'\, G(z,x,x')f(x'), \quad z\in\mathbb{C}\setminus\mathrm{spec}(H),\ f\in L^2(\mathbb{R}),$$

$$G(z,x,x') = \frac{1}{m_{-,\alpha}(z)-m_{+,\alpha}(z)} \begin{cases} \psi_{-,\alpha}(z,x)\psi_{+,\alpha}(z,x') & \text{for } x\le x', \\ \psi_{-,\alpha}(z,x')\psi_{+,\alpha}(z,x) & \text{for } x'\le x. \end{cases} \tag{J.14}$$

Again (cf. the paragraph following (I.7)), for each $x\in\mathbb{R}$, the diagonal Green's function $g(z,x)$ of H has the Herglotz property, that is,

$$g(\,\cdot\,,x) = G(\,\cdot\,,x,x) \text{ is Herglotz.} \tag{J.15}$$

We emphasize that our choice of reference point $x_0=0$ in (J.3) was purely a matter of convenience. In Section 1.5 it turns out to be advantageous to introduce a (variable) reference point $x=x_0$ instead. Without going into further details at this point, we agree to add an additional variable x_0 in this case and hence use the notation $\theta_\alpha(z,x,x_0)$, $\phi_\alpha(z,x,x_0)$, $\psi_{\pm,\alpha}(z,x,x_0)$, $m_{\pm,\alpha}(z,x_0)$, $d\omega_{\pm,\alpha}(\lambda,x_0)$, etc.

The Weyl–Titchmarsh M-matrix for H is then defined by

$$\begin{aligned} M_\alpha(z,x_0) &= \big(M_{\alpha,p,q}(z,x_0)\big)_{p,q=1,2} \\ &= \big(m_{-,\alpha}(z,x_0) - m_{+,\alpha}(z,x_0)\big)^{-1} \\ &\quad\times \begin{pmatrix} m_{-\alpha}(z,x_0)m_{+,\alpha}(z,x_0) & \frac{1}{2}(m_{-,\alpha}(z,x_0)+m_{+,\alpha}(z,x_0)) \\ \frac{1}{2}(m_{-,\alpha}(z,x_0)+m_{+,\alpha}(z,x_0)) & 1 \end{pmatrix}. \end{aligned} \tag{J.16}$$

By inspection,

$$\begin{aligned} &\det(M_\alpha(z,x_0)) = -1/4, \quad z \in \mathbb{C}\setminus\mathbb{R}, \\ &\operatorname{Im}(M_\alpha(z,x_0)) > 0, \quad z \in \mathbb{C}_+, \end{aligned}$$

and hence

$$M_{\alpha,p,p}(\,\cdot\,,x_0) \text{ are Herglotz for } p = 1,2.$$

According to Theorem I.5, (J.12), and (J.13), there exists a matrix-valued measure $d\Omega_\alpha(\,\cdot\,,x_0)$ on $\mathbb{R}$, the matrix-valued spectral measure of H, such that

$$M_\alpha(z,x_0) \,=\, \operatorname{Re}(M_\alpha(i,x_0)) + \int_{\mathbb{R}} d\Omega_\alpha(\lambda,x_0)\left(\frac{1}{(\lambda-z)} - \frac{\lambda}{1+\lambda^2}\right), \quad \alpha \in [0,\pi), \tag{J.17}$$

$$M_\alpha(z,x_0) \underset{z\to i\infty}{=} \frac{i}{2}\begin{pmatrix} \cos^2(\alpha) & \cos(\alpha)\sin(\alpha) \\ \cos(\alpha)\sin(\alpha) & \sin^2(\alpha) \end{pmatrix} z^{1/2} + o(1), \quad \alpha(0,\pi),$$

$$M_0(z,x_0) \underset{z\to i\infty}{=} \frac{i}{2}\begin{pmatrix} z^{1/2}+o(1) & o(z^{-1/2}) \\ o(z^{-1/2}) & z^{-1/2}+o(z^{-1}) \end{pmatrix}$$

with

$$\begin{aligned} &\int_{\mathbb{R}} \frac{d\,\|\Omega_\alpha(\lambda,x_0)\|}{1+|\lambda|} = \infty, \quad \alpha \in (0,\pi), \\ &\int_{\mathbb{R}} \frac{d\,\|\Omega_{0,1,1}(\lambda,x_0)\|}{1+|\lambda|} = \infty, \quad \int_{\mathbb{R}} \frac{d\,\|\Omega_{0,1,1}(\lambda,x_0)\|}{1+\lambda^2} < \infty, \\ &\int_{\mathbb{R}} \frac{d\,\|\Omega_{0,p,q}(\lambda,x_0)\|}{1+|\lambda|} < \infty,\ \ p,q \in \{1,2\},\ \ (p,q) \neq (1,1). \end{aligned}$$

With the introduction of

$$\begin{aligned} \partial_1 G(z,x_0,x') &= \partial_{x_1} G(z,x_1,x')\big|_{x_1=x_0}, \\ \partial_2 G(z,x,x_0) &= \partial_{x_2} G(z,x,x_2)\big|_{x_2=x_0}, \\ \partial_1\partial_2 G(z,x_0,x_0) &= \partial_{x_1}\partial_{x_2} G(z,x_1,x_2)\big|_{x_1=x_0,x_2=x_0}, \quad \text{etc.,} \end{aligned}$$

the expression (J.16) for $M_\alpha(z, x_0)$ can be rewritten as

$$\begin{aligned}
&M_{\alpha,1,1}(z, x_0) = \big(-\sin(\alpha) + \cos(\alpha)\partial_1\big)\big(-\sin(\alpha) + \cos(\alpha)\partial_2\big)G(z, x_0, x_0),\\
&M_{\alpha,1,2}(z, x_0) = M_{\alpha,2,1}(z, x_0)\\
&= (1/2)\big((\cos(\alpha) + \sin(\alpha)\partial_1)(-\sin(\alpha) + \cos(\alpha)\partial_2)\\
&\quad + (-\sin(\alpha) + \cos(\alpha)\partial_1)(\cos(\alpha) + \sin(\alpha)\partial_2)\big)G(z, x_0 \pm 0, x_0 \mp 0),\\
&M_{\alpha,2,2}(z, x_0) = \big(\cos(\alpha) + \sin(\alpha)\partial_1\big)\big(\cos(\alpha) + \sin(\alpha)\partial_2\big)G(z, x_0, x_0).
\end{aligned} \tag{J.18}$$

Closely associated with H is the family of operators $H_{x_0}^\beta$ defined in $L^2(\mathbb{R})$ by

$$\begin{aligned}
&H_{x_0}^\beta f = Lf, \quad \beta \in \mathbb{R} \cup \{\infty\},\ x_0 \in \mathbb{R},\\
&f \in \operatorname{dom}(H_{x_0}^\beta) = \{g \in L^2(\mathbb{R}) \mid g, g' \in \mathrm{AC}([x_0, \pm R]) \text{ for all } R > 0,\\
&\qquad \lim_{\varepsilon \downarrow 0}\big(g'(x_0 \pm \varepsilon) + \beta g(x_0 \pm \varepsilon)\big) = 0,\ Lg \in L^2(\mathbb{R})\}.
\end{aligned}$$

Here, in obvious notation, $\beta = \infty$ denotes the Dirichlet Schrödinger operator $H_{x_0}^D = H_{x_0}^\infty$ and $\beta = 0$ the corresponding Neumann Schrödinger operator $H_{x_0}^N = H_{x_0}^0$. Moreover, $H_{x_0}^\beta$ decomposes into a direct sum of half-line operators

$$H_{x_0}^\beta = H_{-,x_0}^\beta \oplus H_{+,x_0}^\beta, \quad L^2(\mathbb{R}) = L^2((-\infty, x_0]) \oplus L^2([x_0, \infty)). \tag{J.19}$$

The resolvent of $H_{x_0}^\beta$ reads

$$\begin{aligned}
&\big((H_{x_0}^\beta - z)^{-1} f\big)(x) = \int_{\mathbb{R}} dx'\, G_{x_0}^\beta(z, x, x') f(x'),\\
&\qquad z \in \mathbb{C} \setminus \operatorname{spec}(H_{x_0}^\beta),\ f \in L^2(\mathbb{R}),
\end{aligned} \tag{J.20}$$

where

$$\begin{aligned}
&G_{x_0}^\beta(z, x, x') = G(z, x, x') - \frac{(\beta + \partial_2)G(z, x, x_0)(\beta + \partial_1)G(z, x_0, x')}{(\beta + \partial_2)(\beta + \partial_1)G(z, x_0, x_0)} \qquad \text{(J.21)}\\
&\qquad \beta \in \mathbb{R},\ z \in \mathbb{C} \setminus \big(\operatorname{spec}(H_{x_0}^\beta) \cup \operatorname{spec}(H)\big),\\
&G_{x_0}^\infty(z, x, x') = G(z, x, x') - G(z, x, x_0)G(z, x_0, x')G(z, x_0, x_0)^{-1}, \qquad \text{(J.22)}\\
&\qquad z \in \mathbb{C} \setminus \big(\operatorname{spec}(H_{x_0}^\infty) \cup \operatorname{spec}(H)\big).
\end{aligned}$$

By (J.15) and (J.18), both denominators on the right-hand side of (J.21) and (J.22) are Herglotz functions; in particular,

$$\begin{aligned}
&\Gamma^\beta(z, x_0) = (\beta + \partial_1)(\beta + \partial_2)G(z, x_0, x_0) = M_{\alpha,2,2}(z, x_0)/\sin^2(\alpha) \qquad \text{(J.23)}\\
&\qquad = (m_{-,\alpha}(z, x_0) - m_{+,\alpha}(z, x_0))^{-1}/\sin^2(\alpha), \quad \beta = \cot(\alpha) \in \mathbb{R},\\
&\Gamma^\infty(z, x_0) = G(z, x_0, x_0) = M_{0,2,2}(z, x_0) = (m_{-,0}(z, x_0) - m_{+,0}(z, x_0))^{-1}. \qquad \text{(J.24)}
\end{aligned}$$

Although the asymptotic expansions (J.12) and (J.13) are optimal under the weak Hypothesis J.2 on u, one can obtain asymptotic expansions to all orders in $z^{-1/2}$

if one assumes $u \in C^\infty(\mathbb{R})$ in addition to Hypothesis J.2. In fact, the Riccati-type equations for $m_{\pm,\alpha}(z,x)$ and $m_{\pm,0}(z,x)$, that is,

$$\begin{aligned}&(1+\beta^2)m_{\pm,\alpha,x}(z,x)+2\beta(1+u(x)-z)m_{\pm,\alpha}(z,x)\\&\quad+(\beta^2+z-u(x))m_{\pm,\alpha}(z,x)^2=\beta^2(u(x)-z)-1, \quad \alpha\in(0,\pi),\end{aligned} \tag{J.25}$$

$$m_{\pm,0,x}(z,x)+m_{\pm,0}(z,x)^2=u(x)-z, \tag{J.26}$$

imply the following recursion relations for the coefficients $m_{\pm,\alpha,j}(x)$ in the asymptotic expansion for $m_{\pm,\alpha}(z,x)$,

$$m_{\pm,\alpha}(z,x) \underset{z\to i\infty}{=} \sum_{j=0}^{\infty} m_{\pm,\alpha,j}(x)\big(z^{-1/2}\big)^j, \quad \alpha\in(0,\pi). \tag{J.27}$$

The coefficients are given by

$$\begin{aligned}&m_{\pm,\alpha,0}=\beta, \quad m_{\pm,\alpha,1}=\pm i(1+\beta^2), \quad m_{\pm,\alpha,2}=-\beta(1+\beta^2),\\&m_{\pm,\alpha,j+1}=\pm\bigg(-\frac{1}{2i}m_{\pm,\alpha,j,x}+im_{\pm,\alpha,j}+\frac{u-\beta^2}{2i(1+\beta^2)}\sum_{\ell=1}^{j-1}m_{\pm,\alpha,\ell}m_{\pm,\alpha,j-\ell}\\&\qquad-\frac{1}{2i(1+\beta^2)}\sum_{\ell=1}^{j-1}m_{\pm,\alpha,\ell+1}m_{\pm,\alpha,j+1-\ell}\bigg),\\&\qquad\qquad j=2,3,\dots,\ \alpha\in(0,\pi).\end{aligned} \tag{J.28}$$

When $\alpha=0$ one finds similarly

$$m_{\pm,0}(z,x) \underset{z\to i\infty}{=} \sum_{j=-1}^{\infty} m_{\pm,0,j}(x)\big(z^{-1/2}\big)^j \tag{J.29}$$

with coefficients given by

$$\begin{aligned}&m_{\pm,0,-1}=\pm i, \quad m_{\pm,0,0}=0,\\&m_{\pm,0,1}=\mp\frac{i}{2}u, \quad m_{\pm,0,2}=\frac{1}{4}u_x,\\&m_{\pm,0,j+1}=\pm\frac{i}{2}\bigg(m_{\pm,0,j,x}+\sum_{\ell=1}^{j-1}m_{\pm,0,\ell}m_{\pm,0,j-\ell}\bigg), \quad j=2,3,\dots\end{aligned} \tag{J.30}$$

One verifies

$$m_{-,0,j}=(-1)^j m_{+,0,j}, \quad j\in\{-1\}\cup\mathbb{N}_0. \tag{J.31}$$

Expansions (J.27) and (J.29) are uniform with respect to x as long as x varies in compact intervals. Moreover, expansions (J.12)–(J.13), (J.27), and (J.29) are valid as $|z|\to\infty$ outside any cone with apex $\inf\operatorname{spec}(H_x^\beta)$ and arbitrarily small opening angle $\varepsilon > 0$ along the real axis and are uniform with respect to $\arg(z)$ within that cone.

In the special algebro-geometric case studied in Section 1.3 the preceding formalism simplifies considerably. In particular, the main objects such as spectral and Weyl–Titchmarsh m-functions, Green's functions, etc., can be expressed directly in terms of quantities related to our recursion formalism such as the polynomials $F_n(z,x)$ and $K^{\beta}_{n+1}(z,x)$ introduced in (1.11) and (1.55). Below we record the most important of these formulas. We recall some of our conventions in the self-adjoint algebro-geometric case, such as $\{E_m\}_{m=0,\dots,2n} \subset \mathbb{R}$ with $E_0 < E_1 < \cdots < E_{2n}, n \in \mathbb{N}$, and $\Sigma = \bigcup_{j=0}^{n-1}[E_{2j}, E_{2j+1}] \cup [E_{2n}, \infty)$, as in (I.3). Moreover, we introduce $R^{1/2}_{2n+1}$ as in (B.17)–(B.20) followed by an analytic continuation to $\mathbb{C} \setminus \Sigma$.

In the following, let $P = (z, y) = (z, \sigma R_{2n+1}(z)^{1/2})$ with $\sigma \in \{-1, +1\}$. Restricting $\psi(P, x, x_0)$ and $\phi(P, x)$ to the upper and lower sheets $\Pi_\pm$ (cf. (B.25) and (B.26)) and denoting the corresponding branches by $\psi_\pm(z, x, x_0)$ and $\phi_\pm(z, x)$, equations (1.38), (1.39), and (1.41), and (B.17)–(B.19) yield for $z \in \mathbb{C} \setminus \operatorname{spec}(H)$

$$\begin{aligned} m_{\pm,0}(z, x_0) &= \psi_{\pm,x}(z, x, x_0)\big|_{x=x_0} = \phi_\pm(z, x_0) \\ &= \frac{\pm i R_{2n+1}(z)^{1/2} + \frac{1}{2} F_{n,x}(z, x_0)}{F_n(z, x_0)} &\text{(J.32)} \\ &= \frac{H_{n+1}(z, x_0)}{\mp i R_{2n+1}(z)^{1/2} + \frac{1}{2} F_{n,x}(z, x_0)}. &\text{(J.33)} \end{aligned}$$

Abbreviating $\cot(\alpha) = \beta$, equations (J.10) and (J.32) imply

$$\begin{aligned} &m_{\pm,\alpha}(z, x_0) \\ &\quad = \frac{\pm(1+\beta^2) i R_{2n+1}(z)^{1/2} + \beta(H_{n+1}(z, x_0) - F_n(z, x_0)) - (1/2)(1-\beta^2) F_{n,x}(z, x_0)}{K^{\beta}_{n+1}(z, x_{,0})} \\ &\quad = \frac{F_n(z, x_0) - \beta F_{n,x}(z, x_0) + \beta^2 H_{n+1}(z, x_0)}{\mp(1+\beta^2) i R_{2n+1}(z)^{1/2} + \beta(H_{n+1}(z, x_0) - F_n(z, x_0)) - (1/2)(1-\beta^2) F_{n,x}(z, x_0)} \end{aligned} \tag{J.34}$$

with $F_n(z, x_0)$, $H_{n+1}(z, x_0)$, and $K^{\beta}_{n+1}(z, x_{,0})$ defined in (1.11), (1.33), and (1.55). By inspection, $m_{+,\alpha}(z, x_0)$ is the analytic continuation of $m_{-,\alpha}(z, x_0)$ through the open interior of Σ, and vice versa, a fact typical for reflectionless potentials. Combining (J.16) and (J.32) then yields

$$M_0(z, x_0) = \frac{i}{2R_{2n+1}(z)^{1/2}} \begin{pmatrix} H_{n+1}(z, x_0) & F_{n,x}(z, x_0)/2 \\ F_{n,x}(z, x_0)/2 & F_n(z, x_0) \end{pmatrix}. \tag{J.35}$$

In the case where $\alpha \in (0,\pi)$, we find, using (J.16) and (J.34), that

$$\begin{aligned}
&M_{\alpha,1,1}(z,x_0)\\
&\quad = i\,\frac{(1+\beta^2)^2 R_{2n+1}(z) + \big((1/2)(\beta^2-1)F_{n,x}(z,x_0) + \beta(H_{n+1}(z,x_0)-F_n(z,x_0))\big)^2}{2(1+\beta^2)K_{n+1}^{\beta}(z,x_0)R_{2n+1}(z)^{1/2}},\\
&M_{\alpha,1,2}(z,x_0) = M_{\alpha,2,1}(z,x_0) = i\,\frac{(\beta^2-1)F_{n,x}(z,x_0) + 2\beta\big(H_{n+1}(z,x_0)-F_n(z,x_0)\big)}{4(1+\beta^2)R_{2n+1}(z)^{1/2}},\\
&M_{\alpha,2,2}(z,x_0) = i\,\frac{K_{n+1}^{\beta}(z,x_0)}{2(1+\beta^2)R_{2n+1}(z)^{1/2}}.
\end{aligned} \tag{J.36}$$

Finally, equations (I.5) and (J.17) imply

$$\begin{aligned}
&\frac{d\Omega_{\alpha,p,q}}{d\lambda}(\lambda,x_0)\\
&\quad = \begin{cases} -(i/\pi)M_{\alpha,p,q}(\lambda+i0,x_0), & \lambda\in \operatorname{spec}(H)^o,\\ 0, & \lambda \in \mathbb{R}\setminus\operatorname{spec}(H),\end{cases} \qquad p,q=1,2,
\end{aligned} \tag{J.37}$$

where

$$\operatorname{spec}(H) = \operatorname{spec}_{\mathrm{ess}}(H) = \operatorname{spec}_{\mathrm{ac}}(H) = \bigcup_{j=0}^{n-1}[E_{2j},E_{2j+1}]\cup[E_{2n},\infty), \tag{J.38}$$

$$\operatorname{spec}_{\mathrm{sc}}(H) = \operatorname{spec}_{\mathrm{p}}(H) = \emptyset, \tag{J.39}$$

$$\operatorname{spec}\left(H_{x_0}^{\beta}\right) = \operatorname{spec}(H)\cup\left\{\lambda_\ell^{\beta}(x_0)\right\}_{\ell=0,\dots,n}, \quad \beta\in\mathbb{R}, \tag{J.40}$$

$$\operatorname{spec}\left(H_{x_0}^{\infty}\right) = \operatorname{spec}(H)\cup\{\mu_j(x_0)\}_{j=1,\dots,n}, \quad \mu_j(x_0)=\lambda_j^{\infty}(x_0),\ j=1,\dots,n, \tag{J.41}$$

$$\operatorname{spec}_{\mathrm{ess}}\left(H_{x_0}^{\beta}\right) = \operatorname{spec}_{\mathrm{ac}}\left(H_{x_0}^{\beta}\right) = \operatorname{spec}(H), \quad \beta\in\mathbb{R}\cup\{\infty\}, \tag{J.42}$$

$$\operatorname{spec}_{\mathrm{sc}}\left(H_{x_0}^{\beta}\right) = \emptyset, \quad \beta\in\mathbb{R}\cup\{\infty\}, \tag{J.43}$$

and

$$\lambda_0^{\beta}(x_0)\le E_0, \quad \beta\in\mathbb{R}, \quad \lambda_\ell^{\beta}(x_0)\in[E_{2\ell-1},E_{2\ell}], \quad \ell=1,\dots,n,\ \beta\in\mathbb{R}\cup\{\infty\} \tag{J.44}$$

in the algebro-geometric case.

Introducing

$$g(P,x) = \frac{iF_n(z,x)}{2y}, \quad P=(z,y) \tag{J.45}$$

and its branches $g_\pm(z,x)$ by restricting $g(P,x)$ to the upper and lower sheet $\Pi_\pm$, a comparison of (J.14) and (J.32) then yields for the diagonal Green's function of

H (cf. (D.20)),

$$g(z,x) = g_+(z,x) = \frac{i F_n(z,x)}{2R_{2n+1}(z)^{1/2}} = \frac{i \prod_{j=1}^{n}(z-\mu_j(x))}{2R_{2n+1}(z)^{1/2}}. \tag{J.46}$$

Equation (J.46), in particular, identifies the n zeros $\mu_j(x)$ of $F_n(z,x)$ as the Dirichlet eigenvalues of H, that is, the eigenvalues of $H_x^D = H_x^\infty$. Similarly, a comparison of (J.33) and (1.34) shows that the $n+1$ zeros $\nu_\ell(x)$ of $H_{n+1}(z,x)$ are the corresponding Neumann eigenvalues of H, that is, the eigenvalues of $H_x^N = H_x^0$. Introducing

$$\Gamma^\beta(P,x) = (\beta + \partial_{x_1})(\beta + \partial_{x_2}) G(P,x_1,x_2)\big|_{x_1=x,x_2=x} = \frac{i K_{n+1}^\beta(z,x)}{2y(P)}, \quad \beta \in \mathbb{R} \tag{J.47}$$

and its branches $\Gamma_\pm^\beta(z,x) = (\beta+\partial_{x_1})(\beta+\partial_{x_2})G_\pm(z,x_1,x_2)|_{x_1=x,x_2=x}$, restricting P to the upper and lower sheets $\Pi_\pm$, one infers

$$\Gamma^\beta(z,x) = \Gamma_+^\beta(z,x) = \frac{i K_{n+1}^\beta(z,x)}{2R_{2n+1}(z)^{1/2}} = \frac{i \prod_{\ell=0}^{n}(z-\lambda_\ell^\beta(x))}{2R_{2n+1}(z)^{1/2}}. \tag{J.48}$$

In analogy to (J.46) in the context of Dirichlet eigenvalues, (J.48) identifies the $n+1$ zeros $\lambda_\ell^\beta(x)$ of $K_{n+1}^\beta(z,x)$ as the eigenvalues of H_x^β. In the Dirichlet case one analogously considers $\Gamma^\infty(P,x) = g(P,x)$.

Notes

For general Weyl–Titchmarsh theory for Schrödinger and Sturm–Liouville operators, refer, for instance, to Carmona and Lacroix (1990, Chs. III, VII), Coddington and Levinson (1985, Ch. 9), Dunford and Schwartz (1963, Ch. XIII), Eastham and Kalf (1982, Ch. 2), Levitan (1987, Chs. 2, 6–8), Levitan and Sargsjan (1975, Chs. 1, 2; 1991, Chs. 1, 2, 6), Pastur and Figotin (1992, Ch. V), Pearson (1988, Chs. 6, 7), and Titchmarsh (1962, Chs. II, III).

The special algebro-geometric case in (J.32)–(J.44) can be found in Levitan (1987) and Levitan and Savin (1988) (cf. also Gesztesy et al. (1996a)). Equations (J.45)–(J.48) are further studied in Gesztesy et al. (1995b; 1996a).

The asymptotic expansion (J.29) is derived in detail in Danielyan and Levitan (1991) (see also Atkinson (1981), Everitt (1972), and Clark and Gesztesy (2001)). Although the recursion relation (J.30) is well-known (see, e.g., Gel'fand and Dikii (1975)), the one in (J.28) appears to be new. It can, however, be derived quickly from (J.25), which in turn follows from the familiar Riccati-type equation (J.26) and (J.10) by choosing $\alpha_1 = 0$ and $\alpha_2 = \alpha$.

More on reflectionless potentials can be found, for instance, in Belokolos et al. (to appear), Clark and Gesztesy (2001), Clark et al. (2000), Craig (1989a,b), Gesztesy and Sakhnovich (to appear), Gesztesy and Simon (1996b), Gesztesy and Tsekanovskii (2000), Kotani (1984; 1985; 1986; 1987a,b; 1988), Kotani and Krishna (1988), Kotani and Simon (1988), Marchenko (1991), and Sodin and Yuditskiĭ (1995a,b; 1996).

List of Symbols

> There is nothing that can be said by mathematical symbols and relations which cannot also be said by words. The converse, however, is false. Much that can be and is said by words cannot successfully be put into equations because it is nonsense.
>
> *Clifford A. Truesdell*[1]

$\mathbb{N}$,	the natural numbers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,	the nonnegative integers
$\mathbb{Z}$,	the integers
$\mathbb{R}$,	the real numbers
$\mathbb{T}$,	the one-dimensional torus (homeomorphic to the circle S^1)
$\mathbb{C}$,	the complex numbers
$\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1$,	the Riemann sphere
$\mathbb{CP}^2 = (\mathbb{C}^3 \setminus \{0\})/(\mathbb{C} \setminus \{0\})$,	the projective plane, p. 328
$\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \gtrless 0\}$,	the open upper (lower) complex half-plane
$\lfloor x \rfloor = \sup\{n \in \mathbb{Z} \vert n \leq x\}$,	the largest integer not exceeding x
$p \vee q$,	the maximum of p and q
$p \wedge q$,	the minimum of p and q
$\operatorname{Re}(z)$, $\operatorname{Im}(z)$,	the real and imaginary part of $z \in \mathbb{C}$
$\arg(z)$,	the argument of $z \in \mathbb{C}$
$\overline{z}$,	the complex conjugate of z
I_m,	the identity matrix in $\mathbb{C}^m$, $m \geq 2$
$\underline{a} = (a_1, \ldots, a_m)$,	a row vector in $\mathbb{C}^m$, $\underline{a}^\top$ a column vector in $\mathbb{C}^m$
$M^\top$,	the transpose of the matrix M
M^*,	the adjoint (conjugate transpose) of the matrix M
$\underline{\operatorname{diag}}(M) = (M_{1,1}, \ldots, M_{m,m})$,	a row vector built of the diagonal terms of an $m \times m$ matrix M

[1] *Six Lectures on Modern Natural Philosophy*, Springer, New York, 1966, p. 35.

$\mathrm{dom}(T)$,	the domain of an operator T
$\ker(T)$,	the kernel (null space) of a linear operator T
$\mathrm{ran}(T)$,	the range of a linear operator T
$\mathrm{spec}(T)$,	the spectrum of a closed linear operator T
$\mathrm{spec}_{\mathrm{p}}(T)$,	the point spectrum (i.e., the set of eigenvalues)
$\mathrm{spec}_{\mathrm{ac}}(T)$,	the absolutely continuous spectrum
$\mathrm{spec}_{\mathrm{sc}}(T)$,	the singularly continuous spectrum
$\mathrm{spec}_{\mathrm{ess}}(T)$,	the essential spectrum of T
$\mathrm{tr}(A)$,	the trace of a trace-class operator A
$[A, B] = AB - BA$,	the commutator of A and B
$G(z, x, x')$,	the Schrödinger operator Green's function, p. 439
$g(z, x) = G(z, x, x)$,	the diagonal Green's function, p. 439
$L^p(I)$,	the set of all measurable functions f such that $\lvert f\rvert^p$ is Lebesgue integrable on I
$L^p_{\mathrm{loc}}(\mathbb{R})$,	the set of all measurable functions f such that $\lvert f\rvert^p$ is Lebesgue integrable on all compact intervals
$C^\infty(\Omega)$,	the set of all infinitely differentiable functions on an open subset $\Omega \subseteq \mathbb{R}$
$C^\infty(\Omega, \mathcal{K})$,	the set of all infinitely differentiable functions on Ω taking values in $\mathcal{K}$
$AC_{(\mathrm{loc})}(I)$,	the set of (locally) absolutely continuous functions on the interval $I \subset \mathbb{R}$
$H^{k,p}(I), k \in \mathbb{N}, p \geq 1$,	the Sobolev space of order (k, p) on the interval $I \subset \mathbb{R}$
A^o,	the open interior of $A \subset \mathbb{R}$
$g = O(f)$ as $x \to x_0$,	("big-Oh") if g/f is bounded in a neighborhood of x_0
$g = o(f)$ as $x \to x_0$,	("little-Oh") if $g(x)/f(x) \to 0$ as $x \to x_0$
$\partial_w = \frac{\partial}{\partial w}$,	the (partial) derivative with respect to w
$\partial_w^m = \frac{\partial^m}{\partial w^m}, m \in \mathbb{N}, \partial^2_{w_1 w_2} = \frac{\partial^2}{\partial_{w_1}\partial_{w_2}}$,	
$W(f, g) = fg' - f'g$,	the Wronskian of f and g
$\mathrm{Sym}^n(X) = \big\{\{x_1, \dots, x_n\} \mid x_j \in X, j = 1, \dots, n\big\}$,	the nth symmetric product of X
$\Psi_k(\underline{\mu})$,	elementary symmetric functions, $\Phi_k^{(j)}(\underline{\mu})$, p. 385
$\mathcal{K}_g$,	a compact Riemann surface of genus g, p. 329

$\mathcal{B}(\mathcal{K}_n)$,	the set of branch points of $\mathcal{K}_n$ in the hyperelliptic case, p. 355, 366
$\partial\widehat{\mathcal{K}}_g = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_1 a_g^{-1} b_g^{-1}$,	the fundamental polygon of $\mathcal{K}_g$, p. 334
$\widehat{\mathcal{K}}_g$,	the simply connected interior of the fundamental polygon $\partial\widehat{\mathcal{K}}_g$, p. 344
$\tilde{\pi}_z, \tilde{\pi}_y$,	projections, p. 327
$[x_2 : x_1 : x_0]$,	homogeneous coordinates, p. 328
$\{a_j, b_j\}_{j=1}^g$,	a homology basis on $\mathcal{K}_g$, p. 332
ω,	Abelian differential of the first kind, p. 334
$\omega^{(2)}, \Omega^{(2)}$,	Abelian differentials of the second kind, p. 335
$\omega^{(3)}, \Omega^{(3)}$,	Abelian differentials of the third kind, p. 335
$\operatorname{res}_{P=Q} f(P)$,	the residue of a meromorphic function f on a Riemann surface $\mathcal{K}_g$ at $Q \in \mathcal{K}_g$
$\mathcal{M}(\mathcal{K}_g)$,	the set of meromorphic functions (0-forms) on $\mathcal{K}_g$, p. 336
$\mathcal{M}^1(\mathcal{K}_g)$,	the set of meromorphic 1-forms on $\mathcal{K}_g$, p. 336
L_g,	the period lattice, p. 338
$J(\mathcal{K}_g) = \mathbb{C}^g/L_g$,	the Jacobi variety of $\mathcal{K}_g$, p. 338
$\mathcal{D}$,	a divisor, p. 336
$\operatorname{Div}(\mathcal{K})$,	the set of divisors, p. 336
$\deg(\mathcal{D})$,	the degree of a divisor $\mathcal{D}$, p. 337
(f),	the divisor of $f \in \mathcal{M}(\mathcal{K}_g) \setminus \{0\}$, p. 337
(ω),	the divisor of $\omega \in \mathcal{M}^1(\mathcal{K}_g) \setminus \{0\}$, p. 337
$\mathcal{D} \sim \mathcal{E}$,	the equivalence of divisors $\mathcal{D}$ and $\mathcal{E}$, p. 337
$[\mathcal{D}]$,	the divisor class of the divisor $\mathcal{D}$, p. 337
$i(\mathcal{D}) = \dim\{\omega \in \mathcal{M}^1(\mathcal{K}) \mid (\omega) \ge \mathcal{D}\}$,	the index of specialty of the divisor $\mathcal{D}$, p. 338
$r(\mathcal{D}) = \dim\{f \in \mathcal{M}(\mathcal{K}) \mid (f) \ge \mathcal{D}\}$,	p. 338
$\mathcal{D}_{\underline{Q}}$,	a nonnegative divisor of degree $m \in \mathbb{N}$, $\underline{Q} = \{Q_1, \dots, Q_m\} \in \operatorname{Sym}^m(\mathcal{K}_g)$, p. 339
$\underline{A}_Q, \underline{\alpha}_Q, \underline{\widehat{A}}_Q, \underline{\hat{\alpha}}_Q$,	Abel maps, p. 338, 339
θ,	Riemann's theta function, p. 339
$\underline{\Xi}_Q$,	the vector of Riemann constants, p. 340
$* : \mathcal{K}_n \to \mathcal{K}_n$,	the hyperelliptic involution (sheet exchange map), p. 345, 354, 365
$\Pi_\pm$,	the upper and lower sheets of $\mathcal{K}_n$ in the hyperelliptic case, p. 355, 366
$\wp, \zeta, \sigma$,	the Weierstrass $\wp$-, ζ-, and σ-function, p. 428, 429, 429
$\psi(P, x, x_0), \psi(P, x, x_0, t_r, t_{r,0})$,	Baker–Akhiezer functions, p. 31, 66

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This paper fills a much needed gap in the literature[2]

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[1] Publications with three or more authors are abbreviated with "First author et al. (year)" in the text. If more than one publication yield the same abbreviation, latin letters a,b,c, etc., are added after the year. Publications are alphabetically ordered using all authors' names and year of publication.

[2] The review that never was. For more details, see Chinese acrobatics, an old-time brewery, and the "much needed gap": The Life of Mathematical Reviews, *Notices Amer. Math. Soc.* **44** (1997), p. 332.

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