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Lie Symmetry Analysis of Differential Equations in Finance

R. K. GAZIZOV* and N. H. IBRAGIMOV
Institute for Symmetry Analysis and Mathematical Modelling, University of the North-West, Private Bag X2046, Mmabatho 2735, South Africa

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Abstract. Lie group theory is applied to differential equations occurring as mathematical models in financial problems. We begin with the complete symmetry analysis of the one-dimensional Black–Scholes model and show that this equation is included in Sophus Lie's classification of linear second-order partial differential equations with two independent variables. Consequently, the Black–Scholes transformation of this model into the heat transfer equation follows directly from Lie's equivalence transformation formulas. Then we carry out the classification of the two-dimensional Jacobs–Jones model equations according to their symmetry groups. The classification provides a theoretical background for constructing exact (invariant) solutions, examples of which are presented.

Keywords: Differential equations in finance, Lie group classification and symmetry analysis, group theoretic modelling, invariant solutions.

1. Introduction

The works of Merton [1, 2] and Black and Scholes [3] opened a new era in mathematical modeling of problems in finance. Originally, their models are formulated in terms of stochastic differential equations. Under certain restrictive assumptions, these models are written as linear evolutionary partial differential equations with variable coefficients.

The widely used one-dimensional model (one state variable plus time) known as the *Black–Scholes model*, is described by the equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0, \tag{1}$$

with constant coefficients A, B, C (parameters of the model). Black and Scholes reduced it to the classical heat equation and used this relation for solving Cauchy's problem with special initial data.

Along with Equation (1), more complex models aimed at explaining additional effects are discussed in the current literature (see, e.g., [4]). We will consider here the two state variable model suggested by Jacobs and Jones [5]:

$$\begin{aligned} u_t \;=\; & \frac{1}{2}A^2x^2u_{xx} + ABCxyu_{xy} + \frac{1}{2}B^2y^2u_{yy} \\ & + \left(Dx\ln\frac{y}{x} - Ex^{3/2}\right)u_x + \left(Fy\ln\frac{G}{y} - Hyx^{1/2}\right)u_y - xu, \end{aligned} \tag{2}$$

where A, B, C, D, E, F, G, H are arbitrary constant coefficients. Jacobs and Jones [5] investigate the model numerically. An analytical study of solutions of this equation as well as of

* On leave from Ufa State Aviation Technical University, 12 K. Marx st., Ufa 450025, Russia.

other complex financial mathematics models presents a challenge for mathematicians. This is due to the fact that, as a rule, these models unlike the Black–Scholes equation (1), cannot be reduced to simple equations with known solutions. Here, we demonstrate this fact for the *Jacobs–Jones equation* (2) by using methods of the Lie group analysis.

The *Lie group analysis* is a mathematical theory that synthesizes symmetry of differential equations. The founder of this theory, Sophus Lie, was the first who classified differential equations in terms of their symmetry groups, thereby identifying the set of equations which could be integrated or reduced to lower-order equations by group theoretic algorithms. In particular, Lie [6] gave the group classification of linear second-order partial differential equations with two independent variables and developed methods of their integration. According to his classification all parabolic equations admitting the symmetry group of the highest order reduce to the heat conduction equation. These and a wide variety of other results in group analysis of differential equations are to be found in [7].

This paper is aimed at Lie group analysis (symmetries, classification and invariant solutions) of the Black–Scholes (1) and the Jacobs–Jones (2) models.

The contents of the present paper is as follows. Section 2 is designed to meet the needs of beginners and contains a short account of methods of Lie group analysis.

The group analysis of the Black–Scholes model is presented in Section 3. It is shown (Section 3.2) that symmetry group of this model equation is similar to that of the classical heat equation, and hence the Black–Scholes model is contained in the Lie classification [6]. However the practical utilization of Lie's classification is not trivial. Therefore, we discuss calculations for obtaining transformations of (1) into the heat equation (Section 3.3), transformations of solutions (Section 3.4) and invariant solutions (Section 3.5). Moreover, the structure of the symmetry group of Equation (1) allows one to apply the recent method for constructing the fundamental solution based on the so-called *invariance principle* [12, 13]. This application is discussed in Section 3.6.

The Jacobs–Jones model is considered in Section 4. Section 4.2 contains the result of the Lie group classification of Equation (2) with the coefficients satisfying the restrictions $A, B \neq 0, C \neq 0, \pm 1$. It is shown that the dimension of the symmetry group depends essentially on the parameters $A, B, \ldots$, of the model and that the equations of the form (2) cannot be reduced to the classical two-dimensional heat equation. The algorithm of construction of invariant solutions under two-parameter groups and an illustration are given in Section 4.3.

2. Outline of Methods from Group Analysis

2.1. Calculation of Infinitesimal Symmetries

Consider evolutionary partial differential equations of the second order:

$$u_t - F(t, x, u, u_{(1)}, u_{(2)}) = 0, \tag{3}$$

where u is a function of independent variables t and $x = (x^1, \ldots, x^n)$, and $u_{(1)}, u_{(2)}$ are the sets of its first and second-order partial derivatives: $u_{(1)} = (u_{x^1}, \ldots, u_{x^n})$, $u_{(2)} = (u_{x^1x^1}, u_{x^1x^2}, \ldots, u_{x^nx^n})$.

Recall that invertible transformations of the variables t, x, u,

$$\bar{t} = f(t, x, u, a), \quad \bar{x}^i = g^i(t, x, u, a), \quad \bar{u} = h(t, x, u, a), \quad i = 1, \ldots, n, \tag{4}$$

depending on a continuous parameter a are said to be *symmetry transformations of Equation* (3), if Equation (3) has the same form in the new variables $\bar{t}, \bar{x}, \bar{u}$. The set G of all such transformations forms a *continuous group,* i.e., G contains the identity transformation

$$\bar{t} = t, \quad \bar{x}^i = x^i, \quad \bar{u} = u,$$

the inverse to any transformation from G and the composition of any two transformations from G. The symmetry group G is also known as the group *admitted* by Equation (3).

According to the Lie theory, the construction of the symmetry group G is equivalent to determination of its *infinitesimal transformations*:

$$\bar{t} \approx t + a\xi^0(t, x, u), \quad \bar{x}^i \approx x^i + a\xi^i(t, x, u), \quad \bar{u} \approx u + a\eta(t, x, u). \tag{5}$$

It is convenient to introduce the *symbol* (after Lie) of the infinitesimal transformation (5), i.e., the operator

$$X = \xi^0(t, x, u)\frac{\partial}{\partial t} + \xi^i(t, x, u)\frac{\partial}{\partial x^i} + \eta(t, x, u)\frac{\partial}{\partial u}. \tag{6}$$

The operator (6) is also known in the literature as the *infinitesimal operator* or *generator* of the group G. The symbol X of the group admitted by Equation (3) is called an *operator admitted by* Equation (3).

The group transformations (4) corresponding to the infinitesimal transformations with the symbol (6) are found by solving the *Lie equations*

$$\frac{\mathrm{d}\bar{t}}{\mathrm{d}a} = \xi^0(\bar{t}, \bar{x}, \bar{u}), \quad \frac{\mathrm{d}\bar{x}^i}{\mathrm{d}a} = \xi^i(\bar{t}, \bar{x}, \bar{u}), \quad \frac{\mathrm{d}\bar{u}}{\mathrm{d}a} = \eta(\bar{t}, \bar{x}, \bar{u}), \tag{7}$$

with the initial conditions:

$$\bar{t}\big|_{a=0} = t, \quad \bar{x}^i\big|_{a=0} = x^i, \quad \bar{u}\big|_{a=0} = u.$$

By definition, the transformations (4) form a symmetry group G of Equation (3) if the function $\bar{u} = \bar{u}(\bar{t}, \bar{x})$ satisfies the equation

$$\bar{u}_{\bar{t}} - F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}) = 0, \tag{8}$$

whenever the function $u = u(t, x)$ satisfies Equation (3). Here $\bar{u}_{\bar{t}}$, $\bar{u}_{(1)}$, $\bar{u}_{(2)}$ are obtained from Equation (4) according to the usual formulas of change of variables in derivatives. The infinitesimal form of these formulas are written:

$$\begin{aligned} \bar{u}_{\bar{t}} &\approx u_t + a\,\zeta_0(t, x, u, u_t, u_{(1)}), \quad \bar{u}_{\bar{x}^i} \approx u_{x^i} + a\,\zeta_i(t, x, u, u_t, u_{(1)}), \\ \bar{u}_{\bar{x}^i\bar{x}^j} &\approx u_{x^ix^j} + a\,\zeta_{ij}(t, x, u, u_t, u_{(1)}, u_{tx^k}, u_{(2)}), \end{aligned} \tag{9}$$

where the functions $\zeta_0, \zeta_1, \zeta_{ij}$ are obtained by differentiation of ξ^0, ξ^i, η and are given by the *prolongation formulas*:

$$\begin{aligned} \zeta_0 &= D_t(\eta) - u_t D_t(\xi^0) - u_{x^i} D_t(\xi^i), \quad \zeta_i = D_i(\eta) - u_t D_i(\xi^0) - u_{x^j} D_i(\xi^j), \\ \zeta_{ij} &= D_j(\zeta_i) - u_{x^ix^k} D_j(\xi^k) - u_{tx^i} D_j(\xi^0). \end{aligned} \tag{10}$$

Here D_t and D_i denote the total differentiations with respect to t and x^i:

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx^k} \frac{\partial}{\partial u_{x^k}} + \cdots,$$

$$D_i = \frac{\partial}{\partial x^i} + u_{x^i} \frac{\partial}{\partial u} + u_{tx^i} \frac{\partial}{\partial u_t} + u_{x^i x^k} \frac{\partial}{\partial u_{x^k}} + \cdots.$$

Substitution of Equations (5) and (9) into the left-hand side of Equation (8) yields:

$$\bar{u}_{\bar{t}} - F(\bar{t}, \bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}) \approx u_t - F(t, x, u, u_{(1)}, u_{(2)}) + a\left(\zeta_0 - \frac{\partial F}{\partial u_{x^i x^j}} \zeta_{ij} - \frac{\partial F}{\partial u_{x^i}} \zeta_i - \frac{\partial F}{\partial u} \eta - \frac{\partial F}{\partial x^i} \xi^i - \frac{\partial F}{\partial t} \xi^0\right).$$

Therefore, by virtue of Equation (3), Equation (8) yields

$$\zeta_0 - \frac{\partial F}{\partial u_{x^i x^j}} \zeta_{ij} - \frac{\partial F}{\partial u_{x^i}} \zeta_i - \frac{\partial F}{\partial u} \eta - \frac{\partial F}{\partial x^i} \xi^i - \frac{\partial F}{\partial t} \xi^0 = 0, \tag{11}$$

where u_t is replaced by $F(t, x, u, u_{(1)}, u_{(2)})$ in $\zeta_0, \zeta_i, \zeta_{ij}$.

Equation (11) defines all infinitesimal symmetries of Equation (3) and, therefore, it is called the *determining equation*. Conventionally, it is written in the compact form

$$X(u_t - F(t, x, u, u_{(1)}, u_{(2)}))\big|_{(3)} = 0. \tag{12}$$

Here X denotes the *prolongation* of the operator (6) to the first and second-order derivatives:

$$X = \xi^0(t, x, u)\frac{\partial}{\partial t} + \xi^i(t, x, u)\frac{\partial}{\partial x^i} + \eta(t, x, u)\frac{\partial}{\partial u} + \zeta_0 \frac{\partial}{\partial u_t} + \zeta_i \frac{\partial}{\partial u_{x^i}} + \zeta_{ij} \frac{\partial}{\partial u_{x^i x^j}},$$

and the notation $\big|_{(3)}$ means evaluated on Equation (3).

The determining equation (11) (or its equivalent, Equation (12)) is a linear homogeneous partial differential equation of the second order for unknown functions $\xi^0(t, x, u)$, $\xi^i(t, x, u)$, $\eta(t, x, u)$ of the 'independent variables' t, x, u. At first glance, this equation seems to be more complicated than the original differential equation (3). However, this is an apparent complexity. Indeed, the left-hand side of the determining equation involves the derivatives u_{x^i}, $u_{x^i x^j}$, along with the variables t, x, u and functions ξ^0, ξ^i, η of these variables. Since Equation (11) is valid identically with respect to all the variables involved, the variables $t, x, u, u_{x^i}, u_{x^i x^j}$ are treated as 'independent' ones. It follows that the determining equation decomposes into a system of several equations. As a rule, this is an overdetermined system (it contains more equations than a number $n+2$ of the unknown functions ξ^o, ξ^i, η). Therefore, in practical applications, the determining equation can be solved analytically, unlike the original differential equation (3). The solution of the determining equation can be carried out either 'by hand' or, in simple cases, by using modern symbolic manipulation programs. Unfortunately, the existing software packages for symbolic manipulations do not provide solutions for complex determining equations, while a group theorist can solve the problem 'by hand' (the disbelieving reader can try, for example, to obtain the result of the group classification of the Jacobs–Jones model (2) by computer). The reader interested in learning more about the calculation of symmetries by hand in complicated situations is referred to the classical book in this field by Ovsyannikov [8] containing the best presentation of the topic.

2.2. EXACT SOLUTIONS PROVIDED BY SYMMETRY GROUPS

Group analysis provides two basic ways for construction of exact solutions: *group transformations* of known solutions and construction of *invariant solutions*.

2.2.1. *Group Transformations of Known Solutions*

The first way is based on the fact that a symmetry group transforms any solutions of the equation in question into solution of the same equation. Namely, let (4) be a symmetry transformation group of Equation (3), and let a function

$$u = \phi(t, x) \tag{13}$$

solve Equation (3). Since (4) is a symmetry transformation, the solution (13) can be also written using the new variables:

$$\bar{u} = \phi(\bar{t}, \bar{x}). \tag{14}$$

Replacing here $\bar{u}, \bar{t}, \bar{x}$ from Equations (4), we get

$$h(t, x, u, a) = \phi(f(t, x, u, a), g(t, x, u, a)).$$

Having solved this equation with respect to u, we arrive at the following one-parameter family (with the parameter a) of new solutions of Equation (3):

$$u = \psi_a(t, x). \tag{15}$$

Consequently, any known solution is a source of a multi-parameter class of new solutions provided that the differential equation considered admits a multi-parameter symmetry group. An example is given in Section 3.4, where the procedure is applied to the Black–Scholes equation.

2.2.2. *Invariant Solutions*

If a group transformation maps a solution into itself, we arrive at what is called a *self-similar* or *group invariant solution*. The search of this type of solutions reduces the number of independent variables of the equation in question. Namely, the invariance with respect to one-parameter group reduces the number the variables by one. The further reduction can be achieved by considering an invariance under symmetry groups with two or more parameters.

For example, the construction of these particular solutions is reduced, in the case of Equation (1), either to ordinary differential equations (if the solution is invariant under a one-parameter group, see Section 3.5) or to an algebraic relation (if the solution is invariant with respect to a multi-parameter group, see Section 3.6).

The construction of invariant solutions under one-parameter groups is widely known in the literature. Therefore, we briefly sketch the procedure in Section 3.5 by considering one simple example only.

However, since the Jacobs–Jones equation involves three independent variables, its reduction to, e.g., ordinary differential equations requires an invariance under two-dimensional groups. Therefore, we discuss some details of the procedure in Section 4.3 for the Jacobs–Jones equation.

2.3. GROUP CLASSIFICATION OF DIFFERENTIAL EQUATIONS

Differential equations occurring in sciences as mathematical models, often involve undetermined parameters and/or arbitrary functions of certain variables. Usually, these arbitrary elements (parameters or functions) are found experimentally or chosen from a 'simplicity criterion'. Lie group theory provides a regular procedure for determining arbitrary elements from symmetry point of view. This direction of study is known today as *Lie group classification of differential equations*. For detailed presentations of methods used in Lie group classification of differential equations the reader is refereed to the first fundamental paper on this topic [6] dealing with the classification of linear second-order partial differential equations with two independent variables.

Lie group classification of differential equations provides a mathematical background for what can be called a *group theoretic modelling* (see [7, vol. 3, ch. 6]). In this approach, differential equations admitting more symmetries are considered to be 'preferable'. In this way, one often arrives at equations possessing remarkable physical properties.

Given a family of differential equations, the procedure of Lie group classification begins with determining the so-called *principal Lie group* of this family of equations. This is the group admitted by any equation of the family in question. The Lie algebra of the principal Lie group is called the *principal Lie algebra* of the equations and is denoted by $L_{\mathcal{P}}$ (see, e.g., [7]). It may happen that for particular choice of arbitrary elements of the family the corresponding equation admits, along with the principal Lie group, additional symmetry transformations. Determination of all distinctly different particular cases when an extension of $L_{\mathcal{P}}$ occurs is the problem of the group classification.

3. The Black–Scholes Model

3.1. THE BASIC EQUATION

For mathematical modeling stock option pricing, Black and Scholes [3] proposed the partial differential equation

$$u_t + \frac{1}{2}A^2x^2u_{xx} + Bxu_x - Cu = 0 \tag{1}$$

with constant coefficients A, B, C (parameters of the model). It is shown in [3] that Equation (1) is transformable into the classical heat equation

$$v_\tau = v_{yy}, \tag{16}$$

provided that $A \neq 0$, $\mathcal{D} \equiv B - A^2/2 \neq 0$. Using the connection between Equations (1) and (16), they give an explicit formula for the solution, defined in the interval $-\infty < t < t^*$, of the Cauchy problem with a special initial data at $t = t^*$.

3.2. SYMMETRIES

For the Black–Scholes model (1), $n = 1, x^1 = x$ and the symbol of the infinitesimal symmetries has the form

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}\,.$$

In this case, the determining equation (11) is written:

$$\zeta_0 + \frac{1}{2} A^2 x^2 \zeta_{11} + Bx\zeta_1 - C\eta + A^2 x u_{xx} \xi^1 + B u_x \xi^1 = 0, \tag{17}$$

where according to the prolongation formulas (10), the functions $\zeta_0, \zeta_1, \zeta_{ij}$ are given by

$$\begin{aligned}
\zeta_0 &= \eta_t + u_t \eta_u - u_t \xi^0_t - u_t^2 \xi^0_u - u_x \xi^1_t - u_t u_x \xi^1_u, \\
\zeta_1 &= \eta_x + u_x \eta_u - u_t \xi^0_x - u_t u_x \xi^0_u - u_x \xi^1_x - u_x^2 \xi^1_u, \\
\zeta_{11} &= \eta_{xx} + 2u_x \eta_{xu} + u_{xx} \eta_u + u_x^2 \eta_{uu} \\
&\quad - 2u_{tx} \xi^0_x - u_t \xi^0_{xx} - 2u_t u_x \xi^0_{xu} - (u_t u_{xx} + 2u_x u_{tx}) \xi^0_u - u_t u_x^2 \xi^0_{uu} \\
&\quad - 2u_{xx} \xi^1_x - u_x \xi^1_{xx} - 2u_x^2 \xi^1_{xu} - 3u_x u_{xx} \xi^1_u - u_x^3 \xi^1_{uu}.
\end{aligned}$$

The solution of the determining equation (17) provides the infinite dimensional vector space of the infinitesimal symmetries of Equation (1) spanned by following operators:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = x\frac{\partial}{\partial x}, \\
X_3 &= 2t\frac{\partial}{\partial t} + (\ln x + \mathcal{D}t)x\frac{\partial}{\partial x} + 2Ctu\frac{\partial}{\partial u}, \\
X_4 &= A^2 tx\frac{\partial}{\partial x} + (\ln x - \mathcal{D}t)u\frac{\partial}{\partial u}, \\
X_5 &= 2A^2t^2\frac{\partial}{\partial t} + 2A^2 tx \ln x\frac{\partial}{\partial x} + ((\ln x - \mathcal{D}t)^2 + 2A^2Ct^2 - A^2 t)u\frac{\partial}{\partial u},
\end{aligned} \tag{18}$$

and

$$X_6 = u\frac{\partial}{\partial u}, \quad X_\phi = \phi(t,x)\frac{\partial}{\partial u}, \tag{19}$$

where $\mathcal{D} \equiv B - A^2/2$ and $\phi(t,x)$ is an arbitrary solution of Equation (1).

The finite symmetry transformations (4),

$$\bar{t} = f(t,x,u,a), \quad \bar{x} = g(t,x,u,a), \quad \bar{u} = h(t,x,u,a),$$

corresponding to the basic generators (18) and (19), are obtained by solving the *Lie equations* (7). The result is as follows:

$$\begin{aligned}
X_1 :&\quad \bar{t} = t + a_1, \quad \bar{x} = x, \quad \bar{u} = u; \\
X_2 :&\quad \bar{t} = t, \quad \bar{x} = xa_2, \quad \bar{u} = u, \quad a_2 \neq 0; \\
X_3 :&\quad \bar{t} = ta_3^2, \quad \bar{x} = x^{a_3}\, \mathrm{e}^{\mathcal{D}(a_3^2 - a_3)t}, \quad \bar{u} = u\, \mathrm{e}^{C(a_3^2-1)t}, \quad a_3 \neq 0; \\
X_4 :&\quad \bar{t} = t, \quad \bar{x} = x\, \mathrm{e}^{A^2 t a_4}, \quad \bar{u} = u x^{a_4}\, \mathrm{e}^{((1/2)A^2 a_4^2 - \mathcal{D}a_4)t}; \\
X_5 :&\quad \bar{t} = \frac{t}{1 - 2A^2 a_5 t}, \quad \bar{x} = x^{t/(1-2A^2 a_5 t)}, \\
&\quad \bar{u} = u\sqrt{1 - 2A^2 a_5 t}\, \exp\left(\frac{[(\ln x - \mathcal{D}t)^2 + 2A^2Ct^2]a_5}{1 - 2A^2 a_5 t}\right),
\end{aligned}$$

and

$$X_6: \qquad \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = ua_6, \quad a_6 \neq 0;$$
$$X_\phi: \qquad \bar{t} = t, \quad \bar{x} = x, \quad \bar{u} = u + \phi(t, x).$$

Here $a_1, \dots, a_6$ are the parameters of the one-parameter groups generated by $X_1, \dots, X_6$, respectively, and $\phi(t, x)$ is an arbitrary solution of Equation (1). Consequently, the operators $X_1, \dots, X_6$ generate a six-parameter group and X_ϕ generates an infinite group. The general symmetry group is the composition of the above transformations.

REMARK. The group of dilations generated by the operator X_6 reflects the homogeneity of Equation (1), while the infinite group with the operator X_ϕ represents the linear superposition principle for Equation (1). These transformations are common for all linear homogeneous differential equations. Hence, the specific (non-trivial) symmetries of Equation (1) are given by the operators (18) that span a five-dimensional *Lie algebra*.

3.3. Transformation to the Heat Equation

Let us recall Lie's result of group classification of linear second-order partial differential equations with two independent variables. In the case of evolutionary parabolic equations this result is formulated as follows [6]:

Consider the family of linear parabolic equations

$$P(t, x)u_t + Q(t, x)u_x + R(t, x)u_{xx} + S(t, x)u = 0, \quad P \neq 0, \quad R \neq 0. \tag{20}$$

The principal Lie algebra $L_{\mathcal{P}}$ (i.e., the Lie algebra of operators admitted by Equation (20) with arbitrary coefficients $P(t, x)$, $Q(t, x)$, $R(t, x)$, $S(t, x)$, see Section 2.3) is spanned by the generators (19) of trivial symmetries. Any equation (20) can be reduced to the form

$$v_\tau = v_{yy} + Z(\tau, y)v \tag{21}$$

by a transformation, Lie's equivalence transformation:

$$y = \alpha(t, x), \quad \tau = \beta(t), \quad v = \gamma(t, x)u, \quad \alpha_x \neq 0, \quad \beta_t \neq 0, \tag{22}$$

obtained with the help of two quadratures.

If Equation (20) admits an extension of the principal Lie algebra $L_{\mathcal{P}}$ by one additional symmetry operator then it is reduced to the form

$$v_\tau = v_{yy} + Z(y)v \tag{23}$$

for which the additional operator is

$$X = \frac{\partial}{\partial \tau}.$$

If $L_{\mathcal{P}}$ extends by three additional operators, Equation (20) is reduced to the form

$$v_\tau = v_{yy} + \frac{A}{y^2}v, \tag{24}$$

the three additional operators being:

$$X_1 = \frac{\partial}{\partial \tau}, \quad X_2 = 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y}, \quad X_3 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \left(\frac{1}{4}y^2 + \frac{1}{2}\tau\right)v\frac{\partial}{\partial v}.$$

If $L_{\mathcal{P}}$ extends by five additional operators, Equation (20) is reduced to the heat equation

$$v_\tau = v_{yy}, \tag{25}$$

the five additional operators being:

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = \frac{\partial}{\partial \tau}, \quad X_3 = 2\tau\frac{\partial}{\partial y} - yv\frac{\partial}{\partial v}, \quad X_4 = 2\tau\frac{\partial}{\partial \tau} + y\frac{\partial}{\partial y},$$

$$X_5 = \tau^2\frac{\partial}{\partial \tau} + \tau y\frac{\partial}{\partial y} - \left(\frac{1}{4}y^2 + \frac{1}{2}\tau\right)v\frac{\partial}{\partial v}.$$

Equations (23) to (25) provide the canonical forms of all linear parabolic second-order equations (20) that admit non-trivial symmetries, i.e., extensions of the principal Lie algebra $L_{\mathcal{P}}$.

Thus, the Black–Scholes equation (1) belongs to the latter case and hence it reduces to the heat equation (25) by Lie's equivalence transformation. Let us find this transformation.

After the change of variables (22), the heat equation (25) becomes

$$u_{xx} + \left(\frac{2\gamma_x}{\gamma} + \frac{\alpha_x\alpha_t}{\beta'} - \frac{\alpha_{xx}}{\alpha_x}\right)u_x - \frac{\alpha_x^2}{\beta'}u_t + \left(\frac{\gamma_{xx}}{\gamma} + \frac{\alpha_x\alpha_t\gamma_x}{\beta'\gamma} - \frac{\alpha_x^2\gamma_t}{\beta'\gamma} - \frac{\alpha_{xx}\gamma_x}{\alpha_x\gamma}\right)u = 0,$$

where $'$ denotes the differentiation with respect to t. Comparing this equation with the Black–Scholes equation (1) rewritten in the form

$$u_{xx} + \frac{2B}{A^2x}u_x + \frac{2}{A^2x^2}u_t - \frac{2Cu}{A^2x^2} = 0$$

and equating the respective coefficients, we arrive at the following system:

$$\frac{\alpha_x^2}{\beta'} = -\frac{2}{A^2x^2}, \tag{26}$$

$$\frac{2\gamma_x}{\gamma} + \frac{\alpha_x\alpha_t}{\beta'} - \frac{\alpha_{xx}}{\alpha_x} = \frac{2B}{A^2x}, \tag{27}$$

$$\frac{\gamma_{xx}}{\gamma} + \frac{\alpha_x\alpha_t\gamma_x}{\beta'\gamma} - \frac{\alpha_x^2\gamma_t}{\beta'\gamma} - \frac{\alpha_{xx}\gamma_x}{\alpha_x\gamma} = -\frac{2C}{A^2x^2}. \tag{28}$$

It follows from Equation (26):

$$\alpha(t,x) = \frac{\varphi(t)}{A}\ln x + \psi(t), \quad \beta'(t) = -\frac{1}{2}\varphi^2(t),$$

where $\varphi(t)$ and $\psi(t)$ are arbitrary functions. Using these formulas, one obtains from Equation (27):

$$\gamma(t,x) = \nu(t)x^{(B/A^2)-(1/2)+(\psi'/A\varphi)+(\varphi'/2A^2\varphi)\ln x}$$

with an arbitrary function $\nu(t)$. After substitution of the above expressions into Equation (28), one obtains two possibilities: either

$$\varphi = \frac{1}{L - Kt}, \quad \psi = \frac{M}{L - Kt} + N, \quad K \neq 0,$$

and the function $\nu(t)$ satisfies the equation

$$\frac{\nu'}{\nu} = \frac{M^2K^2}{2(L - Kt)^2} - \frac{K}{2(L - Kt)} - \frac{A^2}{8} + \frac{B}{2} - \frac{B^2}{2A^2} - C,$$

or

$$\varphi = L, \quad \psi = Mt + N, \quad L \neq 0,$$

and

$$\frac{\nu'}{\nu} = \frac{M^2}{2L^2} - \frac{A^2}{8} + \frac{B}{2} - \frac{B^2}{2A^2} - C.$$

Here K, L, M, and N are arbitrary constants.

Thus, we arrive at the following two different transformations connecting Equations (1) and (25):

First transformation

$$\begin{aligned} y &= \frac{\ln x}{A(L - Kt)} + \frac{M}{L - Kt} + N, \quad \tau = -\frac{1}{2K(L - Kt)} + P, \\ v &= E\sqrt{L - Kt}\, \mathrm{e}^{(M^2K)/(2(L-Kt)) - (1/2)((B/A)-(A/2))^2 t - Ct} \\ &\quad \times\ x^{(B/A^2)-(1/2)+(MK/A(L-Kt))+(K\ln x/2A^2(L-Kt))}\, u, \\ K &\neq 0; \end{aligned} \tag{29}$$

Second transformation

$$\begin{aligned} y &= \frac{L}{A}\ln x + Mt + N, \quad \tau = -\frac{L^2}{2}t + P, \\ v &= E\, \mathrm{e}^{[(M^2/2L^2) - (1/2)((B/A)-(A/2))^2 - C]t}\, x^{(B/A^2)-(1/2)+(M/AL)}\, u, \quad L \neq 0. \end{aligned} \tag{30}$$

The Black–Scholes transformation (see [3, formula (9)]) is a particular case of the second transformation (30) with

$$L = \frac{2}{A}\mathcal{D}, \quad M = -\frac{2}{A^2}\mathcal{D}^2, \quad N = \frac{2}{A^2}\mathcal{D}(\mathcal{D}t^* - \ln c), \quad P = \frac{2}{A^2}\mathcal{D}^2 t^*, \quad E = \mathrm{e}^{Ct^*},$$

where t^*, c are constants involved in the initial value problem (8) of [3]. The transformation (29) is new and allows one to solve an initial value problem different from that given in [3].

3.4. Transformations of Solutions

Let

$$u = F(t, x)$$

be a known solution of Equation (1). According to Section 2.2, one can use this solution to generate families of new solutions involving the group parameters. We apply here the procedure to the transformations generated by the basic operators (18) and (19). Application of the formulas (13) to (15) yields:

$$X_1: \quad u = F(t - a_1, x);$$

$$X_2: \quad u = F(t, x\, a_2^{-1}), \quad a_2 \neq 0;$$

$$X_3: \quad u = \mathrm{e}^{C(1-a_3^{-2})t} F(t\, a_3^{-2}, x^{a_3^{-1}} \mathrm{e}^{\mathcal{D}(a_3^{-2}-a_3^{-1})t}), \quad a_3 \neq 0;$$

$$X_4: \quad u = x^{a_4} \mathrm{e}^{-((1/2)A^2 a_4^2 + \mathcal{D}a_4)t} F(t, x\mathrm{e}^{-A^2 t a_4});$$

$$X_5: \quad u = \frac{\exp\left(\frac{[(\ln x - \mathcal{D}t)^2 + 2A^2Ct^2]a_5}{1+2A^2a_5 t}\right)}{\sqrt{1+2A^2a_5t}}\, F\left(\frac{t}{1+2A^2a_5t}, x^{t/(1+2A^2a_5t)}\right);$$

and

$$X_6: \quad u = a_6 F(t, x), \quad a_6 \neq 0;$$

$$X_\phi: \quad u = F(t, x) + \phi(t, x).$$

EXAMPLE. Let us begin with the simple solution of Equation (1) depending only on t:

$$u = \mathrm{e}^{Ct}. \tag{31}$$

Using the transformation generated by X_4 we obtain the solution depending on the parameter a_4:

$$u = x^{a_4}\, \mathrm{e}^{-((1/2)A^2a_4^2 + \mathcal{D}a_4 - C)t}.$$

Letting here, for the simplicity, $a_4 = 1$ we get

$$u = x\, \mathrm{e}^{(C-B)t}.$$

If we apply to this solution the transformation generated by X_5, we get the following solution of Equation (1):

$$u = \frac{\exp\left(\frac{[(\ln x - \mathcal{D}t)^2 + 2A^2Ct^2]a_5 + (C-B)t}{1+2A^2a_5t}\right)}{\sqrt{1+2A^2a_5t}}\, x^{t/(1+2A^2a_5t)}. \tag{32}$$

Thus, beginning with the simplest solution (31) we arrive at the rather complicated solution (32). The iteration of this procedure yields more complex solutions.

Note that the solution (31) is unalterable under the transformation generated by X_2. This is an example of so-called *invariant solutions* discussed in the next subsection.

3.5. Invariant Solutions

An invariant solution with respect to a given subgroup of the symmetry group is a solution which is unalterable under the action of the transformations of the subgroup. Invariant solutions can be expressed via invariants of the subgroup (see, e.g., [7]). Here we illustrate the calculation of invariant solutions by considering the one-parameter subgroup with the generator

$$X = X_1 + X_2 + X_6 \equiv \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Invariants $I(t, x, u)$ of this group are found from the equation

$$XI = 0$$

and are given by

$$I = J(I_1, I_2),$$

where

$$I_1 = t - \ln x, \quad I_2 = \frac{u}{x}$$

are functionally independent invariants and hence form a basis of invariants. Therefore, the invariant solution can be taken in the form $I_2 = \phi(I_1)$, or

$$u = x\phi(z), \quad \text{where} \quad z = t - \ln x.$$

Substituting into Equation (1) we obtain the ordinary differential equation of the second order:

$$\frac{A^2}{2}\phi'' + \left(1 - B - \frac{A^2}{2}\right)\phi' + (B - C)\phi = 0, \quad \text{where} \quad \phi' = \frac{\mathrm{d}\phi}{\mathrm{d}z}.$$

This equation with constant coefficients can be readily solved.

The described procedure can be applied to any linear combination (with constant coefficients) of the basic generators (18) and (19). Here we apply it to the basic operators.

$$X_1: \qquad u = \phi(x), \quad \frac{1}{2}A^2x^2\phi'' + Bx\phi' - C\phi = 0,$$

this equation reduces to constant coefficients in the new independent variable $z = \ln x$;

$$X_2: \qquad u = \phi(t), \quad \phi' - C\phi = 0, \quad \text{whence} \quad u = K\,\mathrm{e}^{Ct};$$

$$X_3: \qquad u = \mathrm{e}^{Ct}\phi\left(\frac{\ln x}{\sqrt{t}} - \mathcal{D}\sqrt{t}\right), \quad A^2\phi'' - z\phi' = 0, \quad z = \frac{\ln x}{\sqrt{t}} - \mathcal{D}\sqrt{t},$$

$$\text{whence} \quad \phi(z) = K_1 \int_0^z \mathrm{e}^{\mu^2/(2A^2)}\,\mathrm{d}\mu + K_2;$$

$$X_4: \qquad u = \exp\left(\frac{(\ln x - \mathcal{D}t)^2}{2A^2t}\right)\phi(t), \quad \phi' + \left(\frac{1}{2t} - C\right)\phi = 0, \quad \text{whence}$$

$$\phi = \frac{K}{\sqrt{t}}\,\mathrm{e}^{Ct}, \quad \text{and hence} \quad u = \frac{K}{\sqrt{t}} \exp\left(\frac{(\ln x - \mathcal{D}t)^2}{2A^2 t} + Ct\right);$$

$$X_5: \qquad u = \frac{1}{\sqrt{t}} \exp\left(\frac{(\ln x - \mathcal{D}t)^2}{2A^2 t} + Ct\right)\phi\left(\frac{\ln x}{t}\right), \quad \phi'' = 0, \quad \text{hence}$$

$$u = \left(K_1 \frac{\ln x}{t^{3/2}} + \frac{K_2}{\sqrt{t}}\right) \exp\left(\frac{(\ln x - \mathcal{D}t)^2}{2A^2 t} + Ct\right).$$

Here $\mathcal{D} = B - A^2/2$ and K, K_1, K_2 are constants of integration.

Operators X_6, X_ϕ do not provide invariant solutions.

3.6. The Fundamental Solution

Investigation of initial value problems for hyperbolic and parabolic linear partial differential equations can be reduced to the construction of a particular solution with specific singularities known in the literature as *elementary* or *fundamental* solutions (see, e.g., [9–11]). Recently, it was shown [12] that for certain classes of equations, with constant and variable coefficients, admitting sufficiently wide symmetry groups, the fundamental solution is an invariant solution and it can be constructed by using the so-called *invariance principle*.

Here we find the fundamental solution for Equation (1) using the group theoretic approach presented in [13].

We can restrict ourselves by considering the fundamental solution $u = u(t, x; t_0, x_0)$ of the Cauchy problem defined as follows:

$$u_t + \frac{1}{2} A^2 x^2 u_{xx} + Bxu_x - Cu = 0, \quad t < t_0, \tag{33}$$

$$u\big|_{t\to t_0} = \delta(x - x_0). \tag{34}$$

Here $\delta(x - x_0)$ is the Dirac measure at x_0.

According to the invariance principle, we first find the subalgebra of the Lie algebra spanned by Equation (18) and $X_6 = u\frac{\partial}{\partial u}$ (for our purposes it suffices to consider this finite-dimensional algebra obtained by omitting X_ϕ) such that this subalgebra leaves invariant the *initial manifold* (i.e., the line $t = t_0$) and its restriction on $t = t_0$ conserves the *initial condition* (34). This subalgebra is the three-dimensional algebra spanned by

$$\begin{aligned}
Y_1 &= 2(t - t_0)\frac{\partial}{\partial t} + (\ln x - \ln x_0 + \mathcal{D}(t - t_0))x\frac{\partial}{\partial x} + (2C(t - t_0) - 1)u\frac{\partial}{\partial u},\\
Y_2 &= A^2(t - t_0)x\frac{\partial}{\partial x} + (\ln x - \ln x_0 - \mathcal{D}(t - t_0))u\frac{\partial}{\partial u},\\
Y_3 &= 2A^2(t - t_0)^2\frac{\partial}{\partial t} + 2A^2(t - t_0)x\ln x\frac{\partial}{\partial x}\\
&\quad + \big((\ln x - \mathcal{D}(t - t_0))^2 - \ln^2 x_0 + 2A^2C(t - t_0)^2 - A^2(t - t_0)\big)u\frac{\partial}{\partial u}.
\end{aligned}$$

Invariants are defined by the system

$$Y_1 I = 0, \quad Y_2 I = 0, \quad Y_3 I = 0.$$

Since

$$Y_3 = A^2(t - t_0)Y_1 + \left(\frac{1}{2}\,A^2(t - t_0) - B(t - t_0) + \ln x + \ln x_0\right),$$

it suffices to solve only the first two equations. Their solution is

$$I = ux^{\sigma(t)}\sqrt{t_0 - t}\,\mathrm{e}^{\omega(t,x)},$$

where

$$\sigma(t) = \frac{\mathcal{D}}{A^2} - \frac{\ln x_0}{A^2(t_0 - t)}, \quad \omega(t, x) = \frac{\ln^2 x + \ln^2 x_0}{2A^2(t_0 - t)} + \left(\frac{\mathcal{D}^2}{2A^2} + C\right)(t_0 - t). \tag{35}$$

The invariant solution is given by $I = K =$ const. and hence has the form

$$u = K\,\frac{x^{-\sigma(t)}}{\sqrt{t_0 - t}}\,\mathrm{e}^{-\omega(t,x)}, \quad t < t_0, \tag{36}$$

where $\sigma(t)$, $\omega(t, x)$ are defined by Equation (35). One can readily verify that the function (36) satisfies Equation (33). The constant coefficient K can be found from the initial condition (34).

We will use the well-known limit,

$$\lim_{s\to+0} \frac{1}{\sqrt{s}} \exp\left(-\frac{(x - x_0)^2}{4s}\right) = 2\sqrt{\pi}\,\delta(x - x_0), \tag{37}$$

and the formula of change of variables $z = z(x)$ in the Dirac measure (see, e.g., [10, p. 790]):

$$\delta(x - x_0) = \left|\frac{\partial z(x)}{\partial x}\right|_{x=x_0} \delta(z - z_0). \tag{38}$$

For the function (36), we have

$$\begin{aligned}
\lim_{t\to t_0} u &= \lim_{t\to t_0} \frac{K}{\sqrt{t_0 - t}}\,\mathrm{e}^{-\omega(t,x)-\sigma(t)\ln x}\\
&= \lim_{t\to t_0} \frac{K}{\sqrt{t_0 - t}} \exp\left(-\frac{(\ln x - \ln x_0)^2}{2A^2(t_0 - t)} - \frac{\mathcal{D}\ln x}{A^2}\right),
\end{aligned}$$

or, setting $s = t_0 - t$, $z = (\sqrt{2}/A)\ln x$,

$$\begin{aligned}
\lim_{t\to t_0} u &= K \exp\left(-\frac{\mathcal{D}}{A^2}\ln x\right) \lim_{s\to+0} \frac{1}{\sqrt{s}} \exp\left(-\frac{(z - z_0)^2}{4s}\right)\\
&= 2\sqrt{\pi}\,K \exp\left(-\frac{\mathcal{D}}{A^2}\ln x\right) \delta(z - z_0).
\end{aligned}$$

By virtue of Equation (38),

$$\delta(z - z_0) = \frac{Ax_0}{\sqrt{2}}\,\delta(x - x_0),$$

and hence

$$\lim_{t\to t_0} u = \sqrt{2\pi}\,AKx_0 \exp\left(-\frac{\mathcal{D}}{A^2}\ln x_0\right) \delta(x - x_0).$$

Therefore, the initial condition (34) yields:

$$K = \frac{1}{\sqrt{2\pi}\, A x_0} \exp\left(\frac{\mathcal{D}}{A^2} \ln x_0\right).$$

Thus, we arrive at the following fundamental solution of the Cauchy problem for Equation (1):

$$u = \frac{1}{A x_0 \sqrt{2\pi(t_0 - t)}} \exp\left(-\frac{(\ln x - \ln x_0)^2}{2A^2(t_0 - t)} - \left(\frac{\mathcal{D}^2}{2A^2} + C\right)(t_0 - t) - \frac{\mathcal{D}}{A^2}(\ln x - \ln x_0)\right).$$

REMARK. The fundamental solution can also be obtained from the fundamental solution

$$v = \frac{1}{2\sqrt{\pi\tau}} \exp\left(-\frac{y^2}{4\tau}\right)$$

of the heat equation (25) by the transformation of the form (30) with

$$M = -\frac{L}{A}\,\mathcal{D}\,, \quad N = \frac{L}{A}\,\mathcal{D} t_0 - \frac{L}{A} \ln x_0\,, \quad P = \frac{L^2}{2}\, t_0, \quad E = \frac{A x_0}{L}\, \mathrm{e}^{C t_0},$$

i.e., by the transformation

$$\tau = \frac{L^2}{2}\,(t_0 - t), \quad y = \frac{L}{A}\,\mathcal{D}(t_0 - t) + \frac{L}{A}\,(\ln x - \ln x_0), \quad v = \frac{A x_0}{L}\, \mathrm{e}^{C(t_0 - t)} u.$$

4. A Two Factor Variable Model

Methods of Lie group analysis can be successfully applied to other mathematical models used in mathematics of finance. Here we present results of calculation of symmetries for a two state variable model developed by Jacobs and Jones [5].

4.1. The Jacobs–Jones Equation

The Jacobs–Jones model is described by the linear partial differential equation

$$\begin{aligned} u_t \;=\; & \frac{1}{2} A^2 x^2 u_{xx} + ABCxy u_{xy} + \frac{1}{2} B^2 y^2 u_{yy} \\ & + \left(Dx \ln \frac{y}{x} - E x^{3/2}\right) u_x + \left(Fy \ln \frac{G}{y} - H y x^{1/2}\right) u_y - xu \end{aligned} \tag{2}$$

with constant coefficients A, B, C, D, E, F, G, H.

4.2. The Group Classification

Equation (2) contains parameters $A, B, \ldots, H$. These parameters are ‘arbitrary elements’ mentioned in Section 2.3. According to Section 2.3, it may happen that the Lie algebra of operators admitted by Equation (2) with arbitrary coefficients (i.e., principal Lie algebra) extends for particular choices of the coefficients $A, B, \ldots, H$. It is shown here that the dimension of the symmetry Lie algebra for the model (2), unlike the Black–Scholes model, essentially depends on choice of the coefficients $A, B, C, \ldots, H$.

4.2.1. *Result of the Group Classification*
The principal Lie algebra $L_{\mathcal{P}}$ is infinite-dimensional and spanned by

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = u\frac{\partial}{\partial u}, \quad X_\omega = \omega(t, x, y)\frac{\partial}{\partial u},$$

where $\omega(t, x, y)$ satisfies Equation (2).

We consider all possible extensions of $L_{\mathcal{P}}$ for *non-degenerate* equations (2), namely those satisfying the conditions

$$AB \neq 0, \quad C \neq \pm 1. \tag{39}$$

Moreover, we simplify calculations by imposing the additional restriction

$$C \neq 0. \tag{40}$$

4.2.2. *Extensions of $L_{\mathcal{P}}$*
The algebra $L_{\mathcal{P}}$ extends in the following cases:

1. $D = 0$,

$$X_3 = \mathrm{e}^{Ft} y\frac{\partial}{\partial y}.$$

Subcase: $AH - BCE = 0$ and $F = 0$. There is an additional extension

$$X_4 = 2AB^2(1 - C^2)ty\frac{\partial}{\partial y} + (2BC \ln x - 2A \ln y + (B - AC)ABt)u\frac{\partial}{\partial u}.$$

2. $D \neq 0$, $F = -(BD/2AC)$, $H = 0$,

$$X_3 = \exp\left(\frac{BD}{2AC}t\right) y\frac{\partial}{\partial y} + \left(\frac{D}{ABC} \ln \frac{G}{y} + 1\right) \exp\left(\frac{BD}{2AC}\, t\right) u\frac{\partial}{\partial u}.$$

3. $D \neq 0$, F is defined from the equation

$$A^2F^2 - A^2D^2 + 2ABCDF + B^2D^2 = 0,$$

and the constants E and H are connected by the relation

$$BE(ACF + ACD + BD) = AH(AF + AD + BCD),$$

$$\begin{aligned} X_3 \;=\; & \mathrm{e}^{-Dt} y\frac{\partial}{\partial y} - \left(\frac{ACF + ACD + BD}{A^2B(1 - C^2)} \ln x - \frac{AF + AD + BCD}{AB^2(1 - C^2)} \ln y \right. \\ & \left. + \frac{A^2CF + A^2CD - B^2CD - ABF}{2ABD(1 - C^2)} + \frac{F \ln G(BCD + AF + AD)}{AB^2D(1 - C^2)}\right) \mathrm{e}^{-Dt} u\frac{\partial}{\partial u}. \end{aligned}$$

Subcase: $B = 2AC$, $F = -D$, $H = 0$. There is an additional extension

$$X_4 = \mathrm{e}^{Dt} y\frac{\partial}{\partial y} + \left(\frac{D}{2A^2C^2} \ln \frac{G}{y} + 1\right) \mathrm{e}^{Dt} u\frac{\partial}{\partial u}.$$

REMARK 1. Most likely, the restriction (40) is not essential for the group classification. For example, one of the simplest equations of the form (39),

$$u_t = x^2 u_{xx} + y^2 u_{yy} - xu,$$

admits two additional operators to $L_{\mathscr{P}}$ and is included in the subcase of case 1 of the classification.

REMARK 2. The classification result shows that Equation (2) cannot be transformed, for any choice of its coefficients, into the heat equation

$$v_\tau = v_{ss} + v_{zz}.$$

Indeed, the heat equation admits an extension of $L_{\mathscr{P}}$ by seven additional operators (see, e.g., [7, vol. 2, section 7.2]) while Equation (2) can admit a maximum extension by two operators.

4.3. Invariant Solutions

The above results can be used for the construction of exact (invariant) solutions of Equation (2). We consider here examples of solutions invariant under two-dimensional subalgebras of the symmetry Lie algebra. Then a solution of Equation (2) is obtained from a linear second-order ordinary differential equation and hence the problem is reduced to a Riccati equation. The examples illustrate the general algorithm and can easily be adopted by the reader in other cases.

To construct a solution invariant under a two-dimensional symmetry algebra, one chooses two operators

$$\begin{aligned} Y_1 &= \xi_1^0(t,x,y,u)\frac{\partial}{\partial t} + \xi_1^1(t,x,y,u)\frac{\partial}{\partial x} + \xi_1^2(t,x,y,u)\frac{\partial}{\partial y} + \eta_1(t,x,y,u)\frac{\partial}{\partial u}, \\ Y_2 &= \xi_2^0(t,x,y,u)\frac{\partial}{\partial t} + \xi_2^1(t,x,y,u)\frac{\partial}{\partial x} + \xi_2^2(t,x,y,u)\frac{\partial}{\partial y} + \eta_2(t,x,y,u)\frac{\partial}{\partial u} \end{aligned}$$

that are admitted by Equation (2) and obey the Lie algebra relation:

$$[Y_1, Y_2] = \lambda_1 Y_1 + \lambda_2 Y_2, \quad \lambda_1, \lambda_2 = \text{const}.$$

The two-dimensional Lie subalgebra spanned by Y_1, Y_2 will be denoted by

$$\langle Y_1, Y_2 \rangle.$$

This algebra has two functionally independent invariants $I_1(t,x,z,u)$, $I_2(t,x,z,u)$ provided that

$$\text{rank}\begin{pmatrix} \xi_1^0 & \xi_1^1 & \xi_1^2 & \eta_1 \\ \xi_2^0 & \xi_2^1 & \xi_2^2 & \eta_2 \end{pmatrix} = 2.$$

Under these conditions, the invariants are determined by the system of differential equations

$$Y_1 I = 0, \quad Y_2 I = 0.$$

The invariants solution exists if

$$\text{rank}\left(\frac{\partial I_1}{\partial u}, \frac{\partial I_2}{\partial u}\right) = 1.$$

Then the invariant solution has the form

$$I_2 = \phi(I_1). \tag{41}$$

Substituting Equation (41) into Equation (2), one arrives at an ordinary differential equation for the function ϕ.

EXAMPLE. Consider the equation

$$u_t = \frac{1}{2} A^2 x^2 u_{xx} + ABCxyu_{xy} + \frac{1}{2} B^2 y^2 u_{yy} - Ex^{\frac{3}{2}} u_x - \frac{BCE}{A} yx^{1/2} u_y - xu. \tag{42}$$

According to the above group classification, Equation (42) admits the operators

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \quad X_2 = u\frac{\partial}{\partial u}, \quad X_3 = y\frac{\partial}{\partial y},\\
X_4 &= 2AB^2(1 - C^2)ty\frac{\partial}{\partial y} + (2BC \ln x - 2A \ln y + (B - AC)ABt)u\frac{\partial}{\partial u},\\
X_\omega &= \omega(t, x, y)\frac{\partial}{\partial u}, \quad \text{where} \quad \omega(t, x, y) \text{ solves Equation (42).}
\end{aligned} \tag{43}$$

Here we consider invariants solutions with respect to three different two-dimensional subalgebras of the algebra (43).

1. The subalgebra $\langle X_1, X_3\rangle$ has the independent invariants $I_1 = x$, $I_2 = u$. Hence, the invariant solution has the form

$$u = \phi(x), \tag{44}$$

and is determined by the equation

$$\frac{1}{2} A^2 x^2 \phi'' - Ex^{3/2}\phi' - x\phi = 0. \tag{45}$$

It reduces to the Riccati equation

$$\psi' + \psi^2 - \frac{2E}{A^2\sqrt{x}} \psi - \frac{2}{A^2 x} = 0$$

by the standard substitution

$$\psi = \phi'/\phi. \tag{46}$$

2. The subalgebra $\langle X_1 + X_2, X_3\rangle$ has the invariants $I_1 = x$, $I_2 = u\,\mathrm{e}^{-t}$. The corresponding invariant solution has the form

$$u = \mathrm{e}^t\phi(x). \tag{47}$$

The substitution into Equation (42) yields:

$$\frac{1}{2} A^2 x^2 \phi'' - Ex^{3/2}\phi' - (x + 1)\phi = 0. \tag{48}$$

It reduces to a Riccati equation by the substitution (46).

3. The subalgebra $\langle X_1, X_2 + X_3\rangle$ has the invariants $I_1 = x$, $I_2 = u/y$. The invariant solution has the form

$$u = y\phi(x) \tag{49}$$

with the function $\phi(x)$ defined by the equation

$$\frac{1}{2}A^2x^2\phi'' + (ABCx - Ex^{3/2})\phi' - \left(\frac{BCE}{A}x^{1/2} + x\right)\phi = 0. \tag{50}$$

It reduces to a Riccati equation by the substitution (46).

4.4. Infinite Ideal as a Generator of New Solutions

Recall that the infinite set of operators X_ω does not provide invariant solutions by the direct method (see the end of Section 3.5). However, we can use it to generate new solutions from known ones as follows. Let $u = \omega(t, x, y)$ be a known solution of Equation (2) so that the operator X_ω is admitted by Equation (2). Then, if X is any operator admitted by Equation (2), one obtains that

$$[X_\omega, X] = X_{\overline{\omega}}, \tag{51}$$

where $\overline{\omega}(t, x, y)$ is a solution (in general, it is different from $\omega(t, x, y)$) of Equation (2). The relation (51) means that the set L_ω of operators of the form X_ω is an *ideal* of the symmetry Lie algebra. Since the set of solutions $\omega(t, x, y)$ is infinite, L_ω is called an *infinite ideal*.

Thus, given a solution $\omega(t, x, y)$, the formula (51) provides a new solution $\overline{\omega}(t, x, y)$ to Equation (2). Let us apply this approach to the solutions given in the example of the previous subsection by letting $X = X_4$ from (43).

1. Starting with the solution (44), we have $\omega(t, x, y) = \phi(x)$, where $\phi(x)$ is determined by the differential equation (45). Then

$$[X_\omega, X_4] = (2BC \ln x - 2A \ln y + (B - AC)ABt)\phi(x)\frac{\partial}{\partial u}.$$

Hence, the new solution $u = \overline{\omega}(t, x, y)$ is

$$\overline{\omega}(t, x, y) = (2BC \ln x - 2A \ln y + (B - AC)ABt)\phi(x) \tag{52}$$

with the function $\phi(x)$ determined by Equation (45). Now we can repeat the procedure by taking the solution (52) as $\omega(t, x, y)$ in Equation (51). Then

$$[X_\omega, X_4] = \left[(2BC \ln x - 2A \ln y + (B - AC)ABt)^2\phi(x) + 4A^2B^2(1 - C^2)t\phi(x)\right]\frac{\partial}{\partial u}.$$

Hence, we arrive at the solution

$$u = \left[(2BC \ln x - 2A \ln y + (B - AC)ABt)^2 + 4A^2B^2(1 - C^2)t\right]\phi(x), \tag{53}$$

where $\phi(x)$ is again a solution of Equation (45). By iterating this procedure, one obtains an infinite set of distinctly different solutions to Equation (42). Further new solutions can be obtained by replacing X_4 by any linear combination of the operators (43).

2. For the solution (47), $\omega(t, x, y) = e^t \phi(x)$, where $\phi(x)$ is determined by Equation (48). In this case,

$$[X_\omega, X_4] = (2BC \ln x - 2A \ln y + (B - AC)ABt)\, e^t \phi(x) \frac{\partial}{\partial u}\,,$$

and the new solution $u = \overline{\omega}(t, x, y)$ has the form

$$\overline{\omega}(t, x, y) = (2BC \ln x - 2A \ln y + (B - AC)ABt)\, e^t \phi(x) \tag{54}$$

with the function $\phi(x)$ determined by Equation (48). One can iterate the procedure.

3. For the solution (49), $\omega(t, x, y) = y\phi(x)$, where $\phi(x)$ is determined by Equation (50). In this case,

$$[X_\omega, X_4] = (2BC \ln x - 2A \ln y + (2BC^2 - B - AC)ABt) y\phi(x) \frac{\partial}{\partial u}\,,$$

and the new solution is

$$u = (2BC \ln x - 2A \ln y + (2BC^2 - B - AC)ABt) y\phi(x), \tag{55}$$

where the function $\phi(x)$ is determined by Equation (50). The iteration of the procedure yields an infinite series of solutions.

5. Conclusion

In this paper, the Lie group analysis is applied to the Black–Scholes and Jacobs–Jones models. The approach provides a wide class of analytic solutions of the equations in question.

For the Black–Scholes equation, the most general transformation to the heat equation is derived. This allows one to solve initial value problems different from that given in [3]. Moreover, we use the invariance principle to construct the fundamental solution which can be used for general analysis of an arbitrary initial value problem.

For the Jacobs–Jones model, we present the group classification which shows that the dimension of the symmetry Lie algebra essentially depends on the parameters of the model. It also follows from this classification result that the Jacobs–Jones equation can not be transformed into the classical two-dimensional heat equation.

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