Lecture Notes on Differential Geometry

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Lecture Notes 0

Basics of Euclidean Geometry

By $\mathbf{R}$ we shall always mean the set of real numbers. The set of all n-tuples of real numbers $\mathbf{R}^n := \{(p^1, \ldots, p^n) \mid p^i \in \mathbf{R}\}$ is called the *Euclidean n-space*. So we have

$$p \in \mathbf{R}^n \iff p = (p^1, \ldots, p^n), \quad p^i \in \mathbf{R}.$$

Let p and q be a pair of points (or vectors) in $\mathbf{R}^n$. We define $p + q := (p^1 + q^1, \ldots, p^n + q^n)$. Further, for any scalar $r \in \mathbf{R}$, we define $rp := (rp^1, \ldots, rp^n)$. It is easy to show that the operations of addition and scalar multiplication that we have defined turn $\mathbf{R}^n$ into a vector space over the field of real numbers. Next we define the standard *inner product* on $\mathbf{R}^n$ by

$$\langle p, q \rangle = p^1 q^1 + \ldots + p^n q^n.$$

Note that the mapping $\langle \cdot, \cdot \rangle \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is linear in each variable and is symmetric. The standard inner product induces a norm on $\mathbf{R}^n$ defined by

$$\|p\| := \langle p, p \rangle^{\frac{1}{2}}.$$

If $p \in \mathbf{R}$, we usually write $|p|$ instead of $\|p\|$.

The first nontrivial fact in Euclidean geometry, and an exercise which every geometer should do, is

Exercise 1. (The Cauchy-Schwartz inequality) Prove that

$$|\langle p, q \rangle| \leqslant \|p\| \, \|q\|,$$

for all p and q in $\mathbf{R}^n$ (*Hints:* If p and q are linearly dependent the solution is clear. Otherwise, let $f(\lambda) := \langle p - \lambda q, p - \lambda q \rangle$. Then $f(\lambda) > 0$. Further, note that $f(\lambda)$ may be written as a quadratic equation in λ. Hence its discriminant must be negative).

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The standard Euclidean distance in $\mathbf{R}^n$ is given by

$$\operatorname{dist}(p,q) := \|p-q\|.$$

Exercise 2. (The triangle inequality) Show that

$$\operatorname{dist}(p,q) + \operatorname{dist}(q,r) \geqslant \operatorname{dist}(p,r)$$

for all p, q in $\mathbf{R}^n$. (*Hint:* use the Cauchy-Schwartz inequality).

By a *metric* on a set X we mean a mapping $d \colon X \times X \to \mathbf{R}$ such that

1. $d(p,q) \geqslant 0$, with equality if and only if $p = q$.
2. $d(p,q) = d(q,p)$.
3. $d(p,q) + d(q,r) \geqslant d(p,r)$.

These properties are called, respectively, positive-definiteness, symmetry, and the triangle inequality. The pair (X, d) is called a *metric space*. Using the above exercise, one immediately checks that $(\mathbf{R}^n, \operatorname{dist})$ is a metric space. *Geometry*, in its broadest definition, is the study of metric spaces, and *Euclidean Geometry*, in the modern sense, is the study of the metric space $(\mathbf{R}^n, \operatorname{dist})$.

Finally, we define the *angle* between a pair of nonzero vectors in $\mathbf{R}^n$ by

$$\operatorname{angle}(p,q) := \cos^{-1} \frac{\langle p, q\rangle}{\|p\|\,\|q\|}.$$

Note that the above is well defined by the Cauchy-Schwartz inequality. Now we have all the necessary tools to prove the most famous result in all of mathematics:

Exercise 3. (The Pythagorean theorem) Show that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the length of the sides (*Hint:* First prove that whenever $\langle p, q\rangle = 0$, $\|p\|^2 + \|q\|^2 = \|p-q\|^2$. Then show that this proves the theorem.).

The next exercise is concerned with another corner stone of Euclidean Geometry; however, the proof requires the use of some trignometric identities and is computationally intensive.

Exercise* 4. (Sum of the angles in a triangle) Show that the sum of the angles in a triangle is π.

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Lecture Notes 1

1 Curves

1.1 Definition and Examples

A (parametrized) *curve* (in Euclidean space) is a mapping $\alpha\colon I \to \mathbf{R}^n$, where I is an interval in the real line. We also use the notation

$$I \ni t \overset{\alpha}{\longmapsto} \alpha(t) \in \mathbf{R}^n,$$

which emphasizes that α sends each element of the interval I to a certain point in $\mathbf{R}^n$. We say that α is (of the class of) C^k provided that it is k times continuously differentiable. We shall always assume that α is continuous (C^0), and whenever we need to differentiate it we will assume that α is differentiable up to however many orders that we may need.

Some standard examples of curves are a *line* which passes through a point $p \in \mathbf{R}^n$, is parallel to the vector $v \in \mathbf{R}^n$, and has constant speed $\|v\|$

$$[0, 2\pi] \ni t \overset{\alpha}{\longmapsto} p + tv \in \mathbf{R}^n;$$

a *circle* of radius $\mathbf{R}$ in the plane, which is oriented counterclockwise,

$$[0, 2\pi] \ni t \overset{\alpha}{\longmapsto} \big(r\cos(t), r\sin(t)\big) \in \mathbf{R}^2;$$

and the right handed *helix* (or corkscrew) given by

$$\mathbf{R} \ni t \overset{\alpha}{\longmapsto} \big(r\cos(t), r\sin(t), t\big) \in \mathbf{R}^3.$$

Other famous examples include the *figure-eight* curve

$$[0, 2\pi] \ni t \overset{\alpha}{\longmapsto} \big(\sin(t), \sin(2t)\big) \in \mathbf{R}^2,$$

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the *parabola*

$$\mathbf{R} \ni t \overset{\alpha}{\longmapsto} (t, t^2) \in \mathbf{R}^2,$$

and the *cubic curve*

$$\mathbf{R} \ni t \overset{\alpha}{\longmapsto} (t, t^2, t^3) \in \mathbf{R}^3.$$

Exercise 1. Sketch the cubic curve (*Hint:* First draw each of the projections into the xy, yz, and zx planes).

Exercise 2. Find a formula for the curve which is traced by the motion of a fixed point on a wheel of radius r rolling with constant speed on a flat surface (*Hint:* Add the formula for a circle to the formula for a line generated by the motion of the center of the wheel. You only need to make sure that the speed of the line correctly matches the speed of the circle).

Exercise 3. Let $\alpha\colon I \to \mathbf{R}^n$, and $\beta\colon J \to \mathbf{R}^n$ be a pair of differentiable *curves*. Show that

$$\Big(\big\langle \alpha(t), \beta(t) \big\rangle\Big)' = \big\langle \alpha'(t), \beta(t) \big\rangle + \big\langle \alpha(t), \beta'(t) \big\rangle$$

and

$$\Big(\|\alpha(t)\|\Big)' = \frac{\big\langle \alpha(t), \alpha'(t) \big\rangle}{\|\alpha(t)\|}.$$

(*Hint:* The first identity follows immediately from the definition of the inner-product, together with the ordinary product rule for derivatives. The second identity follows from the first once we recall that $\|\cdot\| := \langle \cdot, \cdot \rangle^{1/2}$).

Exercise 4. Show that if α has unit speed, i.e., $\|\alpha'(t)\| = 1$, then its velocity and acceleration are orthogonal, i.e., $\langle \alpha(t), \alpha''(t) \rangle = 0$.

Exercise 5. Show that if the position vector and velocity of a planar curve $\alpha\colon I \to \mathbf{R}^2$ are always perpendicular, i.e., $\langle \alpha(t), \alpha'(t) \rangle = 0$, for all $t \in I$, then $\alpha(I)$ lies on a circle centered at the origin of $\mathbf{R}^2$.

Exercise 6. Use the fundamental theorem of Calculus for real valued functions to show:

$$\alpha(b) - \alpha(a) = \int_a^b \alpha'(t)\, dt.$$

Exercise 7. Prove that

$$\|\alpha(b) - \alpha(a)\| \leq \int_a^b \|\alpha'(t)\|\, dt.$$

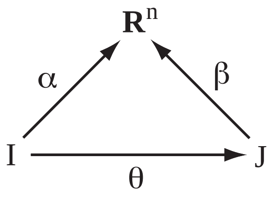
(*Hint:* Use the fundamental theorem of calculus and the Cauchy-Schwarts inequality to show that for any unit vector $u \in \mathbf{R}^n$,

$$\big\langle \alpha(b) - \alpha(a), u \big\rangle = \int_a^b \langle \alpha'(t), u\rangle dt \leq \int_a^b \|\alpha'(t)\| dt.$$

Then set $u := (\alpha(b) - \alpha(a))/\|\alpha(b) - \alpha(a)\|$.

1.2 Reparametrization

We say that $\beta\colon J \to \mathbf{R}^n$ is a *reparametrization* of $\alpha\colon I \to \mathbf{R}^n$ provided that there exists a smooth bijection $\theta\colon I \to J$ such that $\alpha(t) = \beta\big(\theta(t)\big)$. In other words, the following diagram commutes:



For instance $\beta(t) = (\cot(2t), \sin(2t))$, $0 \leq t \leq \pi$, is a reparametrization $\alpha(t) = (\sin(t), \cos(t))$, $0 \leq t \leq 2\pi$, with $\theta\colon [0, 2\pi] \to [0, \pi]$ given by $\theta(t) = t/2$.

The *geometric quantities* associated to a curve do not change under reparametrization. These include length and curvature as we define below.

1.3 Length and Arclength

By a *partition* P of an interval $[a, b]$ we mean a collection of points $\{t_0, \ldots, t_n\}$ of $[a, b]$ such that

$$a = t_0 < t_1 < \cdots < t_n = b.$$

The *approximation of the length of* α *with respect to* P is defined as

$$\text{length}[\alpha, P] := \sum_{i=1}^{n} \|\alpha(t_i) - \alpha(t_{i-1})\|,$$

and if Partition$[a, b]$ denotes the set of all partitions of $[a, b]$, then the *length* of α is given by

$$\text{length}[\alpha] := \sup \big\{ \text{ length}[\alpha, P] \mid P \in \text{Partition}[a, b] \,\big\},$$

where 'sup' denotes the supremum or the least upper bound.

Exercise 8. Show that the shortest curve between any pairs of points in $\mathbf{R}^n$ is the straight line segment joining them. (*Hint:* Use the triangle inequality).

We say that a curve is *rectifiable* if it has finite length.

Exercise* 9 (Nonrectifiable curves). Show that there exists a curve $\alpha\colon [0, 1] \to \mathbf{R}^2$ which is not rectifiable (*Hint:* One such curve, known as the *Koch curve* (Figure 1), may be obtained as the limit of a sequence of curves $\alpha_i\colon [0, 1] \to \mathbf{R}$ defined as follows. Let α_0 trace the line segment $[0, 1]$. Consider an equilateral triangle of sides $1/3$ whose base rests on the middle third of $[0, 1]$. Deleting this middle third from the interval and the triangle yields the curve traced by α_1. Repeating this procedure on each of the 4

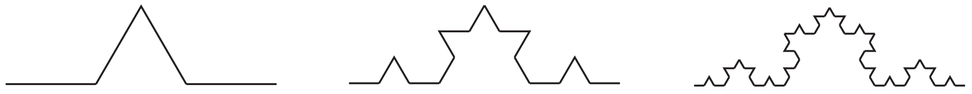


Figure 1:

subsegments of α_1 yields α_2. Similarly α_{i+1} is obtained from α_i. You need to show that α_i converge to a (continuos) curve, which may be done using the Arzela-Ascoli theorem. It is easy to see that this limit has infinite length, because the length of α_i is $(4/3)^i$. Another example of a nonrectifiable curve $\alpha\colon [0, 1] \to \mathbf{R}^2$ is given by $\alpha(t) := (t, t\sin(\pi/t))$, when $t \neq 0$, and $\alpha(t) := (0, 0)$ otherwise. The difficulty here is to show that the length is infinite.)

If a curve is C^1, then its length may be computed as the following exercise shows. Note also that the following exercise shows that a C^1 curve over a compact domain is rectifiable.

Exercise* 10 (Length of C^1 curves). Show that if $\alpha\colon I \to \mathbf{R}^n$ is a C^1 curve, then

$$\text{length}[\alpha] = \int_I \|\alpha'(t)\|\, dt.$$

(*Hints:* It suffices to show that (i) length$[\alpha, P]$ is less than the above integral, for all $P \in$ Partition$[a, b]$, and (ii) there exists a sequence P_n of partitions such that $\lim_{n\to\infty}$ length$[\alpha, P_n]$ is equal to the integral. The first part follows quickly from Exercise 7. To prove the second part, let P_n be a partition given by $t_i := a+i(b-a)/n$. Any other partition where the maximum length of the subsegments converges to zero would do as well. Then apply the mean value theorem, and recall the definition of the integral as the limit of Riemann sums. See do Carmo Exc. 8, Sec. 1-3, and the solution on p. 475.)

Exercise 11. Compute the length of a circle of radius r, and the length of one cycle of the curve traced by a point on a circle of radius r rolling on a straight line.

Exercise 12 (Invariance of length under reparametrization). Show that if β is a reparametrization of α, then length$[\beta]$ = length$[\alpha]$, i.e., length is invariant under reparametrization (*Hint*: you only need to recall the chain rule together with the integration by substitution.)

Let $L :=$ length$[\alpha]$. The *arclength* function of α is a mapping $s\colon [a, b] \to [0, L]$ given by

$$s(t) := \int_a^t \|\alpha'(u)\|\, du.$$

Thus $s(t)$ is the length of the subsegment of α which stretches from the initial time a to time t.

Exercise 13 (Regular curves). Show that if α is a *regular* curve, i.e., $\|\alpha'(t)\| \neq 0$ for all $t \in I$, then $s(t)$ is an invertible function, i.e., it is one-to-one (*Hint*: compute $s'(t)$).

Exercise 14 (Reparametrization by arclength). Show that every regular curve $\alpha\colon [a, b] \to \mathbf{R}^n$, may be reparametrized by arclength (*Hint:* Define $\beta\colon [0, L] \to \mathbf{R}^n$ by $\beta(t) := \alpha(s^{-1}(t))$, and use the chain rule to show that $\|\beta'\| = 1$; you also need to recall that since $f(f^{-1}(t)) = t$, then, again by chain rule, we have $(f^{-1})'(t) = 1/f'(f^{-1}(t))$ for any smooth function f with nonvanishing derivative.)

1.4 Cauchy's integral formula and curves of constant width

Let $\alpha\colon \to \mathbf{R}^2$ be a curve and $u(\theta) := (\cos(\theta), \sin(\theta))$ be a unit vector. The projection of α into the line passing through the origin and parallel to u is given by $\alpha_u(t) := \langle \alpha(t), u \rangle u$.

Exercise 15 (Cauchy's integral formula). Show that if $\alpha\colon I \to \mathbf{R}^2$ has length L, then the average length of the projections α_u, over all directions, is $2L/\pi$, i.e.,

$$\frac{1}{2\pi}\int_0^{2\pi} \text{length}[\alpha_{u(\theta)}]\, d\theta = \frac{2L}{\pi}.$$

(*Hint*: First prove this fact for the case when α traces a line segment. Then a limiting argument settles the general case, once you recall the definition of length. Also, there is a purely analytic and more elegant proof for the case when α is C^1. In this case the integrand is simply $\int_I \|\alpha'(t)\|\,|\cos(\theta)|dt$.)

As an application of the above formula we may obtain a sharp inequality involving *width* of *closed* curves. The width of a set $X \subset \mathbf{R}^2$ is the distance between the closest pairs of parallel lines which contain X in between them. For instance the width of a circle of radius r is $2r$. A curve $\alpha\colon [a,b] \to \mathbf{R}^2$ is said to be closed provided that $\alpha(a) = \alpha(b)$. We should also mention that α is a C^k closed curve provided that the (one-sided) derivatives of α match up at a and b.

Exercise 16 (Width and length). Show that if $\alpha\colon [a,b] \to \mathbf{R}^2$ is a closed curve with width w and length L, then

$$w \leq \frac{L}{\pi}.$$

Note that the above inequality is sharp, since for circles $w = L/\pi$. Are there other curves satisfying this property? The answer may surprise you. For any unit vector $u(\theta)$, the width of a set $X \subset \mathbf{R}^2$ in the direction u, w_u, is defined as the distance between the closest pairs of lines which contain X in between them. We say that a closed curve in the plane has *constant width* provided that w_u is constant in all directions.

Exercise 17. Show that if the equality in Exercise 16 holds then α is a curve of constant width.

The last exercise would have been insignificant if circles were the only curves of constant width, but that is not the case:

Exercise 18 (Reuleaux triangle). Consider three disks of radius r whose centers are on an equilateral triangle of sides r, see Figure 2. Show that the curve which bounds the intersection of these disks has constant width. Also show that similar constructions for any regular polygon yield curves of constant width.

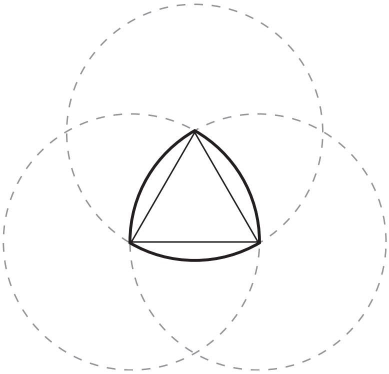


Figure 2:

It can be shown that of all curves of constant width w, Reuleaux triangle has the least are. This is known as the Blaschke-Lebesque theorem. A recent proof of this result has been obtained by Evans Harrell.

Note that the Reuleaux triangle is not a C^1 regular curve for it has sharp corners. To obtain a C^1 example of a curve of constant width, we may take a curve which is a constant distance away from the Reuleaux triangle. Further, a C^∞ example may be constructed by taking an *evolute* of a *deltoid*, see Gray p. 177.

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Lecture Notes 2

1.5 Isometries of the Euclidean Space

Let M_1 and M_2 be a pair of metric space and d_1 and d_2 be their respective metrics. We say that a mapping $f\colon M_1 \to M_2$ is an *isometry* provided that

$$d_1(p,q) = d_2\big(f(p), f(q)\big),$$

for all pairs of points in p, $q \in M_1$. An *orthogonal transformation* $A\colon \mathbf{R}^n \to \mathbf{R}^n$ is a linear map which preserves the inner product, i.e.,

$$\big\langle A(p), A(q)\big\rangle = \langle p, q\rangle$$

for all p, $q \in \mathbf{R}^n$. One may immediately check that an orthogonal transformation is an isometry. Conversely, we have:

Theorem 1. *If $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is an isometry, then*

$$f(p) = f(o) + A(p),$$

where o is the origin of $\mathbf{R}^n$ and A is an orthogonal transformation.

Proof. Let

$$\overline{f}(p) := f(p) - f(o).$$

We need to show that $\overline{f}$ is a linear and $\langle \overline{f}(p), \overline{f}(q)\rangle = \langle p, q\rangle$. To see the latter note that

$$\langle x-y, x-y\rangle = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle.$$

Thus, using the definition of $\overline{f}$, and the assumption that f is an isometry, we obtain

$$\begin{aligned}
2\big\langle \overline{f}(p), \overline{f}(q)\big\rangle &= \|\overline{f}(p)\|^2 + \|\overline{f}(q)\|^2 - \|\overline{f}(p) - \overline{f}(q)\|^2 \\
&= \|f(p) - f(o)\|^2 + \|f(q) - f(o)\|^2 - \|f(p) - f(q)\|^2 \\
&= \|p\|^2 + \|q\|^2 - \|p - q\|^2 \\
&= 2\langle p, q\rangle.
\end{aligned}$$

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Next note that, since $\overline{f}$ preserves the inner product, if e_i, $i = 1 \dots n$, is an orthonormal basis for $\mathbf{R}^n$, then so is $\overline{f}(e_i)$. Further,

$$\begin{aligned}
\langle \overline{f}(p+q), \overline{f}(e_i) \rangle &= \langle p+q, e_i \rangle = \langle p, e_i \rangle + \langle q, e_i \rangle \\
&= \langle \overline{f}(p), \overline{f}(e_i) \rangle + \langle \overline{f}(q), \overline{f}(e_i) \rangle \\
&= \langle \overline{f}(p) + \overline{f}(q), \overline{f}(e_i) \rangle.
\end{aligned}$$

Thus if follows that

$$\overline{f}(p+q) = \overline{f}(p) + \overline{f}(q).$$

Similarly, for any constant c,

$$\langle \overline{f}(cp), \overline{f}(e_i) \rangle = \langle cp, e_i \rangle = \langle c\overline{f}(p), \overline{f}(e_i) \rangle,$$

which in turn yields that $\overline{f}(cp) = \overline{f}(p)$, and completes the proof $\overline{f}$ is linear. □

If $f \colon \mathbf{R}^n \to \mathbf{R}^n$ is an isometry with $f(o) = o$ we say that it is a *rotation*, and if $A = f - f(o)$ is identity we say that f is a *translation*. Thus another way to state the above theorem is that an isometry of the Euclidean space is the composition of a rotation and a translation.

Any mapping $f \colon \mathbf{R}^n \to \mathbf{R}^m$ given by $f(p) = q + A(p)$, where $q \in \mathbf{R}^n$, and A is any linear transformation, is called an *affine map* with translation part q and linear part A. Thus yet another way to state the above theorem is that any isometry $f \colon \mathbf{R}^n \to \mathbf{R}^n$ is an affine map whose linear part is orthogonal

An isometry of Euclidean space is also referred to as a *rigid motion*. Recall that if A^T denotes the transpose of matrix A, then

$$\langle A^T(p), q \rangle = \langle p, A(q) \rangle.$$

This yields that if A is an orthogonal transformation, then $A^T A$ is the identity matrix. In particular

$$1 = \det(A^T A) = \det(A^T)\det(A) = \det(A)^2.$$

So $\det(A) = \pm 1$. If $\det(A) = 1$, then we say that A is a *special* orthogonal transformation, $A \colon \mathbf{R}^n \to \mathbf{R}^n$ is a *proper* rotation, and any isometry $f \colon \mathbf{R}^n \to \mathbf{R}^n$ given by $f(p) = q + A(p)$ is a *proper* rigid motion.

Exercise 2 (Isometries of $\mathbf{R}^2$). Show that if $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a proper rotation, then it may be represented by a matrix of the form

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Further, any improper rotation is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Conclude then that any isometry of $\mathbf{R}^2$ is a composition of a translation, a proper rotation, and possibly a reflection with respect to the y-axis.

In the following exercise you may use the following fact: any continuous mapping of $f\colon \mathbf{S}^2 \to \mathbf{S}^2$ of the sphere to itself has a fixed point or else sends some point to its antipodal reflection. Alternatively you may show that every 3×3 orthogonal matrix has a nonzero real eigenvalue.

Exercise 3 (Isometries of $\mathbf{R}^3$). (a) Show that any proper rotation $A\colon \mathbf{R}^3 \to \mathbf{R}^3$ fixes a line ℓ through the origin. Further if Π is a plane which is orthogonal to ℓ, then A maps Π to itself by rotating it around the point $\ell \cap \Pi$ by an angle which is the same for all such planes. (b) Show that any rotation of $\mathbf{R}^3$ is a composition of rotations about the x, and y-axis. (c) Find a pair of proper rotations A_1, A_2 which do not commute, i.e., $A_1 \circ A_2 \neq A_2 \circ A_1$. (d) Note that any improper rotation becomes proper after multiplication by an orthogonal matrix with negative determinant. Use this fact to show that any rotation of $\mathbf{R}^3$ is the composition of a proper rotation with reflection through the origin, or reflection through the xy-plane. (e) Conclude that any isometry of $\mathbf{R}^3$ is a composition of the following isometries: translations, rotations about the x, or y-axis, reflections through the origin, and reflections through the xy-plane.

Exercise 4. Show that if $\alpha\colon I \to \mathbf{R}^2$ is a C^1 curve, then for any $p \in I$ there exists and open neighborhood U of p in I and a rigid motion $f\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that α restricted to U has a reparametrization $\beta\colon J \to \mathbf{R}^2$, where $J \subset \mathbf{R}$ is a neighborhood of the origin, and $B(t) = (t, h(t))$ for some C^1 function $f\colon J \to \mathbf{R}$ with $h(0) = h'(0) = 0$.

1.6 Invariance of length under isometries

Recalling the definition of length as the limit of polygonal approximations, one immediately sees that

Exercise 5. Show that if $\alpha\colon [a,b] \to \mathbf{R}^n$ is a rectifiable curve, and $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is an isometry, then $\text{length}[\alpha] = \text{length}[f \circ \alpha]$.

Recall that earlier we had shown that the length of a curve was invariant under reparametrization. The above exercise further confirms that length is indeed a 'geometric quantity'. In the case where α is C^1, it is useful to give also an analytic proof for the above exercise, mainly as an excuse to recall and apply some basic concepts from multivariable calculus.

Let $U \subset \mathbf{R}^n$ be an open subset, and $f\colon U \to \mathbf{R}^m$ be a map. Note that f is a list of m functions of n variables:

$$f(p) = f(p^1, \dots, p^n) = \big(f^1(p^1, \dots, p^n), \dots, f^m(p^1, \dots, p^n)\big).$$

The first order partial derivatives of f are given by

$$D_j f^i(p) := \lim_{h \to 0} \frac{f^i(p^1, \dots, p^j + h, \dots, p^n) - f^i(p^1, \dots, p^j, \dots, p^n)}{h}.$$

If all the functions $D_j f^i\colon U \to \mathbf{R}$ exist and are continuous, then we say that f is C^1. The *Jacobian* of f at p is the $m \times n$ matrix defined by

$$J_p(f) := \begin{pmatrix} D_1 f^1(p) & \cdots & D_n f^1(p) \\ \vdots & & \vdots \\ D_1 f^m(p) & \cdots & D_n f^m(p) \end{pmatrix}.$$

The *derivative* of f at p is the linear transformation $Df(p)\colon \mathbf{R}^n \to \mathbf{R}^m$ given by the above matrix, i.e.,

$$\Big(Df(p)\Big)(x) := \Big(J_p(f)\Big)(x).$$

Exercise 6 (Derivative of linear maps). Show that if $A\colon \mathbf{R}^n \to \mathbf{R}^m$ is a linear map, then

$$DA(p) = A$$

for all $p \in \mathbf{R}^n$. In other words, for each $p \in \mathbf{R}^n$, $(DA(p))(x) = A(x)$, for all $x \in \mathbf{R}^n$.(*Hint:* Let a_{ij}, $i = 1 \dots n$, and $j = 1 \dots m$, be the coefficients of the matrix representation of A. Then $A^j(p) = \sum_{i=1}^n a_{ij} p_i$.)

Another basic fact is the chain rule which states that if $g\colon \mathbf{R}^m \to \mathbf{R}^\ell$ is a differentiable function, then

$$D(f \circ g)(p) = Df\big(g(p)\big) \circ Dg(p).$$

Now let $\alpha\colon I \to \mathbf{R}^n$ be a C^1 curve and $f\colon \mathbf{R}^n \to \mathbf{R}^n$, given by $f(p) = f(o) + A(p)$ be an isometry. Then

$$\begin{aligned}
\text{length}[f \circ \alpha] &= \int_I \|D(f \circ \alpha)(t)\|\, dt && (1)\\
&= \int_I \|Df\big(\alpha(t)\big) \circ D\alpha(t)\|\, dt && (2)\\
&= \int_I \|DA\big(\alpha(t)\big) \circ D\alpha(t)\|\, dt && (3)\\
&= \int_I \|A\big(D\alpha(t)\big)\|\, dt && (4)\\
&= \int_I \|D\alpha(t)\|\, dt && (5)\\
&= \text{length}[\alpha] && (6)
\end{aligned}$$

The six equalities above are due respectively to (1) definition of length, (2) the chain rule, (3) definition of isometry f, (4) Exercise 6, (5) definition of orthogonal transformation, and (6) finally definition of length applied again.

1.7 Curvature of $\mathbf{C}^2$ regular curves

The curvature of a curve is a measure of how fast it is turning. More precisely, it is the speed, with respect to the arclength parameter, of the unit tangent vector of the curve. The unit tangent vector, a.k.a. *tangential indicatrix*, or *tantrix* for short, of a regular curve $\alpha\colon I \to \mathbf{R}^n$ is defined as

$$T(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}.$$

Note that the tantrix is itself a curve with parameter ranging in I and image lying on the unit sphere $\mathbf{S}^{n-1} := \{x \in \mathbf{R}^n \mid \|x\| = 1\}$. If α is parametrized with respect to arclength, i.e., $\|\alpha'(t)\| = 1$, then the curvature is given by

$$\kappa(t) = \|T'(t)\| = \|\alpha''(t)\| \quad (\text{provided } \|\alpha'\| = 1).$$

Thus the curvature of a road is the amount of centripetal force which you would feel, if you traveled on it in a car which has unit speed; the tighter the turn, the higher the curvature, as is affirmed by the following exercise:

Exercise 7. Show that the curvature of a circle of radius r is $\frac{1}{r}$, and the curvature of the line is zero (First you need to find arclength parametrizations for these curves).

Recall that, as we showed earlier, there exists a unique way to reparametrize a curve $\alpha\colon [a,b] \to \mathbf{R}^n$ by arclength (given by $\alpha \circ s^{-1}(t)$). Thus the curvature does not depend on parametrizations. This together with the following exercise shows that κ is indeed a 'geometric quantity'.

Exercise 8. Show that κ is invariant under isometries of the Euclidean space (*Hint:* See the computation at the end of the last subsection).

As a practical matter, we need to have a definition for curvature which works for all curves (not just those already parametrized by arclength), because it is often very difficult, or even impossible, to find explicit formulas for unit speed curves.

To find a general formula for curvature of C^2 regular curve $\alpha\colon I \to \mathbf{R}^n$, let $T\colon I \to \mathbf{S}^{n-1}$ be its tantrix. Let $s\colon I \to [0,L]$ be the arclength function. Since, as we discussed earlier s is invertible, we may define

$$\overline{T} := T \circ s^{-1}$$

to be a reparametrization of T. Then curvature may be defined as

$$\kappa(t) := \|\overline{T}'(s(t))\|.$$

By the chain rule,

$$\overline{T}'(t) = T'\Big(s^{-1}(t)\Big) \cdot \Big(s^{-1}\Big)'(t).$$

Further recall that $(s^{-1})'(t) = 1/\|\alpha'(s^{-1}(t))\|$. Thus

$$\kappa(t) = \frac{\|T'(t)\|}{\|\alpha'(t)\|}.$$

Exercise 9. Use the above formula, together with definition of T, to show that

$$\kappa(t) = \frac{\sqrt{\|\alpha'(t)\|^2\|\alpha''(t)\|^2 - \langle\alpha'(t), \alpha''(t)\rangle^2}}{\|\alpha'(t)\|^3}.$$

In particular, in $\mathbf{R}^3$, we have

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}.$$

(*Hint*: The first identity follows from a straight forward computation. The second identity is an immediate result of the first via the identity $\|v \times w\|^2 = \|v\|^2\|w\|^2 - \langle v, w\rangle^2$.)

Exercise 10. Show that the curvature of a planar curve which satisfies the equation $y = f(x)$ is given by

$$\kappa(x) = \frac{|f''(x)|}{\left(\sqrt{1 + \left(f'(x)\right)^2}\right)^3}.$$

(*Hint:* Use the parametrization $\alpha(t) = (t, f(t), 0)$, and use the formula in previous exercise.) Compute the curvatures of $y = x$, x^2, x^3, and x^4.

Exercise 11. Let α, β: $(-1, 1) \to \mathbf{R}^2$ be a pair of C^2 curves with $\alpha(0) = \beta(0) = (0, 0)$. Further suppose that α and β both lie on or above the x-axis, and β lies higher than or at the same height as α. Show that the curvature of β at $t = 0$ is not smaller than that of α at $t = 0$(*Hint:* Use exercise 4, and a Taylor expansion).

Exercise 12. Show that if $\alpha\colon I \to \mathbf{R}^2$ is a C^2 closed curve which is contained in a circle of radius r, then the curvature of α has to be bigger than $1/r$ at some point. In particular, closed curves have a point of nonzero curvature. (*Hint*: Shrink the circle until it contacts the curve, and use Exercise 11).

Exercise 13. Let $\alpha\colon I \to \mathbf{R}^2$ be a closed planar curve, show that

$$\text{length}[\alpha] \geq \frac{2\pi}{\max \kappa}.$$

(*Hint:* Recall that the width w of α is smaller than or equal to its length divided by π to show that a piece of α should lie inside a circle of diameter at least w).

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Lecture Notes 3

1.8 The general definition of curvature; Fox-Milnor's Theorem

Let $\alpha\colon [a,b] \to \mathbf{R}^n$ be a curve and $P = \{t_0, \dots, t_n\}$ be a partition of $[a,b]$, then (the approximation of) the total curvature of α with respect to P is defined as

$$\mathit{total}\,\kappa[\alpha, P] := \sum_{i=1}^{n-1} \text{angle}\Big(\alpha(t_i) - \alpha(t_{i-1})\ ,\ \alpha(t_{i+1}) - \alpha(t_i)\Big),$$

and the *total curvature* of α is given by

$$\mathit{total}\,\kappa[\alpha] := \sup\{\,\kappa[\alpha, P] \mid P \in \mathit{Partition}[a,b]\,\}.$$

Our main aim here is to prove the following observation due to Ralph Fox and John Milnor:

Theorem 1 (Fox-Minor). *If $\alpha\colon [a,b] \to \mathbf{R}^n$ is a C^2 unit speed curve, then*

$$\mathit{total}\,\kappa[\alpha] = \int_a^b \|\alpha''(t)\|\,dt.$$

This theorem implies, by the mean value theorem for integrals, that for any $t \in (a,b)$,

$$\kappa(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \mathit{total}\,\kappa\Big[\alpha\big|_{t-\epsilon}^{t+\epsilon}\Big].$$

The above formula may be taken as the definition of curvature for general (not necessarily C^2) curves. To prove the above theorem first we need to develop some basic spherical geometry. Let

$$\mathbf{S}^n := \{p \in \mathbf{R}^{n+1} \mid \|p\| = 1\}.$$

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denote the n-dimensional unit sphere in $\mathbf{R}^{n+1}$. Define a mapping from $\mathbf{S}^n \times \mathbf{S}^n$ to $\mathbf{R}$ by

$$\operatorname{dist}_{\mathbf{S}^n}(p,q) := \operatorname{angle}(p,q).$$

Exercise 2. Show that $(\mathbf{S}^n, \operatorname{dist}_{\mathbf{S}^n})$ is a metric space.

The above metric has a simple geometric interpretation described as follows. By *a great circle* $C \subset \mathbf{S}^n$ we mean the intersection of $\mathbf{S}^n$ with a two dimensional plane which passes through the origin o of $\mathbf{R}^{n+1}$. For any pair of points p, $q \in \mathbf{S}^2$, there exists a plane passing through them and the origin. When $p \neq \pm q$ this plane is given by the linear combinations of p and q and thus is unique; otherwise, p, q and o lie on a line and there exists infinitely many two dimensional planes passing through them. Thus through every pairs of points of $\mathbf{S}^n$ there passes a great circle, which is unique whenever $p \neq \pm q$.

Exercise 3. For any pairs of points p, $q \in \mathbf{S}^n$, let C be a great circle passing through them. If $p \neq q$, let ℓ_1 and ℓ_2 denote the length of the two segments in C determined by p and q, then $\operatorname{dist}_{\mathbf{S}^n}(p,q) = \min\{\ell_1, \ell_2\}$. (*Hint:* Let $p^\perp \in C$ be a vector orthogonal to p, then C may be parametrized as the set of points traced by the curve $p\cos(t) + p^\perp \sin(t)$.)

Let $\alpha\colon [a,b] \to \mathbf{S}^n$ be a spherical curve, i.e., a Euclidean curve $\alpha\colon [a,b] \to \mathbf{R}^{n+1}$ with $\|\alpha\| = 1$. For any partition $P = \{t_0, \dots, t_n\}$ of $[a,b]$, the spherical length of α with respect the partition P is defined as

$$\operatorname{length}_{\mathbf{S}^n}[\alpha, P] = \sum_{i=1}^{n} \operatorname{dist}_{\mathbf{S}^n}\big(\alpha(t_i), \alpha(t_{i-1})\big).$$

The norm of any partition P of $[a,b]$ is defined as

$$|P| := \max\{t_i - t_{i-1} \mid 1 \leq i \leq n\}.$$

If P^1 and P^2 are partions of $[a,b]$, we say that P^2 is a *refinement* of P^1 provided that $P^1 \subset P^2$.

Exercise 4. Show that if P^2 is a refinement of P^1, then

$$\operatorname{length}_{\mathbf{S}^n}[\alpha, P^2] \geq \operatorname{length}_{\mathbf{S}^n}[\alpha, P^1].$$

(*Hint:*Use the fact that $\operatorname{dist}_{\mathbf{S}^n}$ satisfies the triangle inequality, see Exc. 2).

The spherical length of α is defined by

$$\text{length}_{\mathbf{S}^n}[\alpha] = \sup\{\, \text{length}_{\mathbf{S}^n}[\alpha, P] \mid P \in \mathit{Partition}[a,b]\,\}.$$

Lemma 5. *If $\alpha\colon [a,b] \to \mathbf{S}^n$ is a unit speed spherical curve, then*

$$\text{length}_{\mathbf{S}^n}[\alpha] = \text{length}[\alpha].$$

Proof. Let $P^k := \{t_0^k, \dots, t_n^k\}$ be a sequence of partitions of $[a,b]$ with

$$\lim_{k\to\infty} |P^k| = 0,$$

and

$$\theta_i^k := \text{dist}_{\mathbf{S}^n}\big(\alpha^k(t_i), \alpha^k(t_{i-1})\big) = \text{angle}\big(\alpha^k(t_i), \alpha^k(t_{i-1})\big)$$

be the corresponding spherical distances. Then, since α has unit speed,

$$2\sin\left(\frac{\theta_i^k}{2}\right) = \|\alpha(t_i^k) - \alpha(t_{i-1}^k)\| \le t_i^k - t_{i-1}^k \le |P^k|.$$

In particular,

$$\lim_{k\to\infty} 2\sin\left(\frac{\theta_i^k}{2}\right) = 0.$$

Now, since $\lim_{x\to 0}\sin(x)/x = 1$, it follows that, for any $\epsilon > 0$, there exists $N > 0$, depending only on $|P^k|$, such that if $k > N$, then

$$(1-\epsilon)\theta_i^k \le 2\sin\left(\frac{\theta_i^k}{2}\right) \le (1+\epsilon)\theta_i^k,$$

which yields that

$$(1-\epsilon)\,\text{length}_{\mathbf{S}^n}[\alpha, P^k] \le \text{length}[\alpha, P^k] \le (1+\epsilon)\,\text{length}_{\mathbf{S}^n}[\alpha, P^k].$$

The above inequalities are satisfied by any $\epsilon > 0$ provided that k is large enough. Thus

$$\lim_{k\to\infty} \text{length}_{\mathbf{S}^n}[\alpha, P^k] = \text{length}[\alpha].$$

Further, note that if P is any partitions of $[a,b]$ we may construct a sequnce of partitions by successive refinements of P so that $\lim_{k\to\infty}|P^k| = 0$. By Exercise 4, $\text{length}_{\mathbf{S}^n}[\alpha, P^k] \le \text{length}_{\mathbf{S}^n}[\alpha, P^{k+1}]$. Thus the above expression shows that, for any partition P of $[a,b]$,

$$\text{length}_{\mathbf{S}^n}[\alpha, P] \le \text{length}[\alpha].$$

The last two expressions now yied that

$$\sup\{\,\mathrm{length}_{\mathbf{S}^n}[\alpha, P] \mid P \in Partition[a,b]\,\} = \mathrm{length}[\alpha],$$

which completes the proof. $\square$

Exercise 6. Show that if P^2 is a refinement of P^1, then

$$total\kappa[\alpha, P^2] \geq total\kappa[\alpha, P^1].$$

Now we are ready to prove the theorem of Fox-Milnor:

Proof of Theorem 1. As in the proof of the previous lemma, let $P^k = \{t_0^k, \ldots, t_n^k\}$ be a sequence of partitions of $[a,b]$ with $\lim_{k\to\infty} |P^k| = 0$. Set

$$\theta_i^k := \mathrm{angle}\Big(\alpha(t_i^k) - \alpha(t_{i-1}^k)\ ,\ \alpha(t_{i+1}^k) - \alpha(t_i^k)\Big),$$

where $i = 1, \ldots, n-1$. Further, set

$$\bar{t}_i^k := \frac{t_i^k + t_{i-1}^k}{2}$$

and

$$\phi_i^k := \mathrm{angle}\Big(\alpha'(\bar{t}_i^k), \alpha'(\bar{t}_{i+1}^k)\Big).$$

Recall that, by the previous lemma,

$$\lim_{k\to\infty} \sum_i \phi_i^k = \mathrm{length}_{\mathbf{S}^{n-1}}[\alpha'] = \mathrm{length}[\alpha'] = \int_a^b \|\alpha''(t)\|\, dt.$$

Thus to complete the proof it suffices to show that, for every $\epsilon > 0$, there exists N such that for all $k \geq N$,

$$|\theta_i^k - \phi_i^k| \leq \epsilon(t_{i+1}^k - t_{i-1}^k); \qquad (1)$$

for then it would follow that

$$2\epsilon[a,b] \leq \sum_i \theta_i^k - \sum_i \phi_i^k \leq 2\epsilon[a,b],$$

which would in turn yield

$$\lim_{k\to\infty} total\,\kappa[\alpha, P^k] = \lim_{k\to\infty} \sum_i \theta_i^k = \lim_{k\to\infty} \sum_i \phi_i^k = \int_a^b \|\alpha''(t)\| dt.$$

Now, similar to the proof of Lemma 5, note that given any partition P of $[a,b]$, we may construct by subsequent refinements a sequnce of partitions P^k, with $P^0 = P$, such that $\lim_{k\to\infty} |P^k| = 0$. Thus the last expression, together with Excercise 6, yields that

$$total\kappa[\alpha, P] \leq \int_a^b \|\alpha''(t)\|\, dt.$$

The last two expressions complete the proof; so it remains to establish (1). To this end let

$$\beta_i^k := \text{angle}\Big(\alpha'(\bar{t}_i^k),\, \alpha(t_i^k) - \alpha(t_{i-1}^k)\Big).$$

By the triangle inequality for angles (Exercise 2).

$$\phi_i^k \leq \beta_i^k + \theta_i^k + \beta_{i+1}^k\,, \quad \text{and} \quad \theta_i^k \leq \beta_i^k + \phi_i^k + \beta_{i+1}^k,$$

which yields

$$|\phi_i^k - \theta_i^k| \leq \beta_i^k + \beta_{i+1}^k.$$

So to prove (1) it is enough to show that for every $\epsilon > 0$

$$\beta_i^k \leq \frac{\epsilon}{2}(t_i - t_{i-1})$$

provided that k is large enough. See Exercise 7. □

Exercise* 7. Let $\alpha\colon [a,b] \to \mathbf{R}^n$ be a C^2 curve. For every t, $s \in [a,b]$, $t \neq s$, define

$$f(t,s) := \text{angle}\left(\alpha'\left(\frac{t+s}{2}\right),\, \alpha(t) - \alpha(s)\right).$$

Show that

$$\lim_{t\to s} \frac{f(t,s)}{t-s} = 0.$$

In particular, if we set $f(t,t) = 0$, then the resulting function $f\colon [a,b] \times [a,b] \to \mathbf{R}$ is continuous. So, since $[a,b]$ is compact, f is uniformly continuous, i.e., for every $\epsilon > 0$, there is a δ such that $\|f(t) - f(s)\| \leq \epsilon$, whever $|t-s| \leq \delta$. Does this result hold for C^1 curves as well?

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Lecture Notes 4

1.9 Curves of Constant Curvature

Here we show that the only curves in the plane with constant curvature are lines and circles. The case of lines occurs precisely when the curvature is zero:

Exercise 1. Show that the only curves with constant zero curvature in $\mathbf{R}^n$ are straight lines. (*Hint*: We may assume that our curve, $\alpha\colon I \to \mathbf{R}^n$ has unit speed. Then $\kappa = \|\alpha''\|$. So zero curvature implies that $\alpha'' = 0$. Integrating the last expression twice yields the desired result.)

So it remains to consider the case where we have a planar curve whose curvature is equal to some nonzero constant c. We claim that in this case the curve has to be a circle of radius $1/c$. To this end we introduce the following definition. If a curve $\alpha\colon I \to \mathbf{R}^n$ has nonzero curvature, the *principal normal* vector field of α is defined as

$$N(t) := \frac{T'(t)}{\|T'(t)\|},$$

where $T(t) := \alpha'(t)/\|\alpha'(t)\|$ is the tantrix of α as we had defined earlier. Thus the principal normal is the tantrix of the tantrix.

Exercise 2. Show that $T(t)$ and $N(t)$ are orthogonal. (*Hint*: Differentiate both sides of the expression $\langle T(t), T(t)\rangle = 1$).

So, if α is a planar curve, $\{T(t), N(t)\}$ form a *moving frame* for $\mathbf{R}^2$, i.e., any element of $\mathbf{R}^2$ may be written as a linear combination of $T(t)$ and $N(t)$ for any choice of t. In particular, we may express the derivatives of T and

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N in terms of this frame. The definition of N already yields that, when α is parametrized by arclength,

$$T'(t) = \kappa(t)N(t).$$

To get the corresponding formula for N', first observe that

$$N'(t) = aT(t) + bN(t).$$

for some a and b. To find a note that, since $\langle T, N\rangle = 0$, $\langle T', N\rangle = -\langle T, N'\rangle$. Thus

$$\alpha = \langle N'(t), T(t)\rangle = -\langle T'(t), N(t)\rangle = -\kappa(t).$$

Exercise 3. Show that $b = 0$. (*Hint:* Differentiate $\langle N(t), N(t)\rangle = 1$).

So we conclude that

$$N'(t) = -\kappa(t)T(t),$$

where we still assume that t is the arclength parameter. The formulas for the derivative may be expressed in the matrix notation as

$$\begin{bmatrix} T(t) \\ N(t) \end{bmatrix}' = \begin{bmatrix} \kappa(t) & 0 \\ 0 & -\kappa(t) \end{bmatrix} \begin{bmatrix} T(t) \\ N(t) \end{bmatrix}.$$

Now recall that our main aim here is to classify curves of constant curvature in the plane. To this end define the *center of the osculating circle* of α as

$$p(t) := \alpha(t) + \frac{1}{\kappa(t)}N(t).$$

The circle which is centered at $p(t)$ and has radius of $1/\kappa(t)$ is called the *osculating* circle of α at time t. This is the circle which best approximates α up to the second order:

Exercise 4. Check that the osculating circle of α is tangent to α at $\alpha(t)$ and has the same curvature as α at time t.

Now note that if α is a circle, then it coincides with its own osculating circle. In particular $p(t)$ is a fixed point (the center of the circle) and $\|\alpha(t) - p(t)\|$ is constant (the radius of the circle). Conversely:

Exercise 5. Show that if α has constant curvature c, then (i) $p(t)$ is a fixed point, and (ii) $\|\alpha(t) - p(t)\| = 1/c$ (*Hint*: For part (i) differentiate $p(t)$; part (ii) follows immediately from the definition of $p(t)$).

So we conclude that a curve of constant curvature $c \neq 0$ lies on a circle of radius $1/c$.

1.10 Signed Curvature and Turning Angle

As we mentioned earlier the curvature of a curve is a measure of how fast it is turning. When the curve lies in a plane, we may assign a sign of plus or minus one to this measure depending on whether the curve is rotating clockwise or counterclockwise. Thus we arrive at a more descriptive notion of curvature for planar curves which we call *signed curvature* and denote by $\overline{\kappa}$. Then we may write

$$|\overline{\kappa}| = \kappa.$$

To obtain a formula for $\overline{\kappa}$, for any vector $v \in \mathbf{R}^2$, let iv be the counterclockwise rotation by 90 degrees. Then we may simply set

$$\overline{\kappa}(t) := \frac{\big\langle T'(t), iT(t)\big\rangle}{\|\alpha'(t)\|}.$$

Exercise 6. Show that if α is a unit speed curve then

$$\overline{\kappa}(t) = \kappa(t)\big\langle N(t), iT(t)\big\rangle.$$

In particular, $|\overline{\kappa}| = \kappa$.

Exercise 7. Compute the signed curvatures of the clockwise circle $\alpha(t) = (\cos t, \sin t)$, and the counterclockwise circle $\alpha(t) = (\cos(-t), \sin(-t))$.

Exercise 8. Show that

$$\overline{\kappa}(t) := \frac{\big\langle \gamma'(t) \times \gamma''(t), (0,0,1)\big\rangle}{\|\gamma'(t)\|^3}.$$

Another simple and useful way to define the signed curvature (and the regular curvature) of a planar curve is in terms of the *turning angle* θ, which is defined as follows. We claim that for any planar curve $\alpha\colon I \to \mathbf{R}^2$ there exists a function $\theta\colon I \to \mathbf{R}^2$ such that

$$T(t) = \big(\cos\theta(t), \sin\theta(t)\big).$$

Then, assuming that t is the arclength parameter, we have

$$\overline{\kappa}(t) = \theta'(t).$$

Exercise 9. Check the above formula.

Now we check that θ indeed exists. To this end note that T may be thought of as a mapping from I to the unit circle $\mathbf{S}^1$. Thus it suffices to show that

Proposition 10. *Show that for any continuous function $T\colon I \to \mathbf{S}^1$, where $I = [a,b]$ is a compact interval, there exists a continuous function $\theta\colon I \to \mathbf{S}^1$ such that the above formula relating T and θ holds.*

Proof. Since T is continuous and I is compact, T is *uniformly* continuous, this means that for $\epsilon > 0$, we may find a $\delta > 0$ such that $\|T(t) - T(s)\| < \epsilon$, whenever $|t - s| < \delta$. In particular, we may set δ_0 to be equal to some constant less than one, and ϵ_0 to be the corresponding constant. Now choose a partition

$$a =: x_0 \leq x_1 \leq \cdots \leq x_n := b$$

such that $|x_i - x_{i-1}| < \epsilon_0$, for $i = 1, \ldots, n$. Then T restricted to each subinterval $[x_i, x_{i-1}]$ is not unto. So we may define $\theta_i\colon [x_{i-1}, x_i] \to \mathbf{R}$ by setting $\theta_i(x)$ to be the angle in $[0, 2\pi)$, measured counterclockwise, between $T(x_{i-1})$ and $T(x)$. Finally, θ may be defined as

$$\theta(x) := \theta_0 + \sum_{i=1}^{k-1} \theta_i(x_i) + \theta_k(x), \quad \text{if} \quad x \in [x_{k-1}, x_k].$$

□

1.11 Total Signed Curvature and Winding Number

The *total signed curvature* of $\alpha\colon I \to \mathbf{R}^n$ is defined as

$$\mathit{total}\,\overline{\kappa}[\alpha] := \int_I \overline{\kappa}(t)\,dt$$

where t is the arclength parameter. Note that since $\overline{\kappa} = \theta'$, the fundamental theorem of calculus yields that, if $I = [a,b]$, then

$$\mathit{total}\,\overline{\kappa}[\alpha] = \theta(a) - \theta(b).$$

We say that $\alpha\colon [a,b] \to \mathbf{R}^2$ is a closed curve provided that $\alpha(a) = \alpha(b)$ and $T(a) = T(b)$.

Exercise 11. Show that the total signed curvature of a closed curve is a multiple of 2π.

So, if α is a closed curve,

$$\mathrm{rot}[\alpha] := \frac{1}{2\pi}\int_I \overline{\kappa}(t)\,dt$$

is an integer which we call the Hopf *rotation index* or *winding number* of α. In particular we have

$$total\,\overline{\kappa}[\alpha] = \mathrm{rot}[\alpha]2\pi.$$

Exercise 12. (i) Compute the total curvature and rotation index of a circle which has been oriented clockwise, and a circle which is oriented counterclockwise. Sketch the *figure eight curve* $(\cos t, \sin 2t)$, $0 \leq t \leq 2\pi$, and compute its total signed curvature and rotation index.

We say that α is simple if it is one-to-one in the interior of I. The following result proved by H. Hopf is one of the fundamental theorems in theory of planar curves.

Theorem 13 (Hopf). *Any simple closed planar curve has rotation index* ± 1.

Hopf proved the above result using analytic methods including the Green's theorem. Here we outline a more elementary proof which will illustrate that the above theorem is simply a generalization of one of the most famous result in classical geometry: the sum of the angles in a triangle is π, which is equivalent to the sum of the exterior angles being 2π.

First we will give another definition for $total\,\overline{\kappa}$ which will establish the connection between the total signed curvature and the sum of the exterior angles in a *polygon*. By a polygon we mean an ordered set of points

$$P := \big(p_0, \ldots, p_n\big)$$

in $\mathbf{R}^2$, where $p_n = p_0$, but $p_i \neq p_{i-1}$, for $i = 1, \ldots, n$. Each p_i is called a vertex of P. At each vertex p_i, $i = 1 \ldots n$, we define the *exterior angle* θ_i to be the angle in $[-\pi, \pi]$ determined by the vectors $p_i - p_{i-i}$, and $p_{i+1} - p_i$, and measured in the counterclockwise direction (we set $p_{n+1} := p_1$). The total curvature of P is defined as the sum of these angles:

$$total\,\overline{\kappa}[P] := \sum_{i=1}^{n} \theta_i.$$

Now let $\alpha\colon [a,b] \to \mathbf{R}^2$ be a closed planar curve. For $i = 0, \ldots, n$, set

$$t_i := a + i\frac{b-a}{n},$$

and let

$$P_n[\alpha] := \big(\alpha(t_0), \ldots, \alpha(t_n)\big)$$

be the n^{th} polygonal approximation of α. The following proposition shows that the total curvature of a closed curve is just the limit of the sum of the exterior angles of the polygonal approximations.

Proposition 14.

$$total\,\overline{\kappa}[\alpha] = \lim_{n\to\infty} total\,\overline{\kappa}\big[P_n[\alpha]\big].$$

Proof. Let θ be the rotation angle of α, and θ_i be the exterior angles of $P_n[\alpha]$. If we choose n large enough, then there exists, for $i = 0, \ldots, n$, an element $\overline{t}_i \in [t_{i-1}, t_i]$ such that $T(t_i)$ is parallel to $\alpha(t_i) - \alpha(t_{i-1})$. Consequently

$$\theta_i = \theta(\overline{t}_i) - \theta(\overline{t}_{i-1}).$$

By the mean value theorem, there exists $t_i^* \in [\overline{t}_{i-1}, \overline{t}_i]$ such that

$$\theta(\overline{t}_i) - \theta(\overline{t}_{i-1}) = \theta'(t_i^*)(\overline{t}_i - \overline{t}_{i-1}) = \kappa(t_i^*)(\overline{t}_i - \overline{t}_{i-1}).$$

So

$$\begin{aligned}\lim_{n\to\infty} total\,\overline{\kappa}\big[P_n[\alpha]\big] &= \lim_{n\to\infty}\sum_{i=1}^{n}\theta_i = \lim_{n\to\infty}\sum_{i=1}^{n}\theta'(t_i^*)(\overline{t}_i - \overline{t}_{i-1}) \\ &= \int_a^b \overline{\kappa}(t)\,dt = total\,\overline{\kappa}[\alpha].\end{aligned}$$

□

Exercise 15. Verify the second statement in the proof of the above theorem (*Hint:* Use the existence of the local representation of a piece of a curve as graph of a function, which we had proved in an earlier exercise).

Now to complete the proof of Theorem 13 we need to verify:

Exercise* 16. Show that any simple polygon with more than three vertices has a vertex such that if we delete that vertex then the remaining polygon is still simple.

Exercise* 17. Show that the operation of deleting the vertex of a polygon described above does not change the sum of the exterior angles.

Since the sum of the exterior angles in a triangle is 2π, it would follow then that the sum of the exterior angles in any simple polygon is 2π. This in turn would imply Theorem 13 via Proposition 14.

1.12 The fundamental theorem of planar curves

If $\alpha\colon [0, L] \to \mathbf{R}^2$ is a planar curve parametrized by arclength, then its signed curvature yields a function $\overline{\kappa}\colon [0, L] \to \mathbf{R}$. Now suppose that we are given a continuous function $\overline{\kappa}\colon [0, L] \to \mathbf{R}$. Is it always possible to find a unit speed curve $\alpha\colon [0, L] \to \mathbf{R}^2$ whose signed curvature is $\overline{\kappa}$? If so, to what extent is such a curve unique? In this section we show that the signed curvature does indeed determine a planar curve, and such a curve is unique up to proper rigid motions.

Recall that by a proper rigid motion we mean a composition of a translation with a proper rotation. A translation is a mapping $T\colon \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$T(p) := p + v$$

where v is a fixed vector. And a proper rotation $\rho\colon \mathbf{R}^2 \to \mathbf{R}^2$ is a linear mapping given by

$$\rho\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Exercise 18. Show that the signed curvature of a planar curve is invariant under proper rigid motions.

Exercise 19 (Local Convexity). Show that if the curvature of a planar curve $\alpha\colon I \to \mathbf{R}^2$ does not vanish at an interior point t_0 of I then there exists an open neighborhood U of t_0 in I such that $\alpha(U)$ lies on one side of the tangent line of α at t_0. (*Hint*: By the invariance of signed curvature under rigid motions, we may assume that $\alpha(t_0) = (0, 0)$ and $\alpha'(0) = (1, 0)$. Then we may reparametrize α as $(t, f(t))$ in a neighborhood of t_0. Recalling the formula for curvature for graphs, and applying the Taylor's theorem yields the desired result.)

Now suppose that we are given a function $\overline{\kappa}\colon [0, L] \to \mathbf{R}$. If there exist a curve $\alpha\colon [0, L] \to \mathbf{R}^2$ with signed curvature $\overline{\kappa}$, then

$$\theta' = \overline{\kappa}$$

where θ is the rotation angle of α. Integration yields

$$\theta(t) := \int_0^t \overline{\kappa}(s)ds + \theta(0).$$

By the definition of the turning angle

$$\alpha'(t) = \Big(\cos\theta(t), \sin\theta(t)\Big).$$

Consequently,

$$\alpha(t) = \left(\int_0^t \cos\theta(s)ds, \int_0^t \sin\theta(s)ds\right) + \alpha(0),$$

which gives an explicit formula for the desired curve.

Exercise 20 (Fundamental theorem of planar curves). Let $\alpha, \beta\colon [0, L] \to \mathbf{R}^2$ be unit speed planar curves with the same signed curvature function $\overline{\kappa}$. Show that there exists a proper rigid motion $m\colon \mathbf{R}^2 \to \mathbf{R}^2$ such that $\alpha(t) = m(\beta(t))$.

Exercise 21. Use the above formula to show that the only closed curves of constant curvature in the plane are circles.

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Lecture Notes 5

1.13 Osculating Circle and Radius of Curvature

Recall that in a previous section we defined the osculating circle of a planar curve $\alpha\colon I \to \mathbf{R}^2$ at a point a of nonvanishing curvature $t \in I$ as the circle with radius $r(t)$ and center at

$$\alpha(t) + r(t)N(t)$$

where

$$r(t) := \frac{1}{\kappa(t)}$$

is called the *radius of curvature* of α. If we had a way to define the osculating circle independently of curvature, then we could define curvature simply as the reciprocal of the radius of the osculating circle, and thus obtain a more geometric definition for curvature.

Proposition 1. *Let $r(s,t)$ be the radius of the circle which is tangent to α at $\alpha(t)$ and is also passing through $\alpha(s)$. Show that*

$$\kappa(t) = \lim_{s \to t} r(s,t).$$

Proof. Since curvature is invaraint under rigid motions, we may assume that $\alpha(t) = (0,0)$ and $\alpha'(t)$ is parallel to the x-axis. Then, we may assume that $\alpha(t) = (t, f(t))$, for some function $f\colon \mathbf{R} \to \mathbf{R}$ with $f(0) = 0$ and $f'(0) = 0$. Further, recall that

$$\kappa(t) = \frac{|f''(t)|}{(\sqrt{1 + f'(t)^2})^3}.$$

Thus

$$\kappa(0) = |f''(0)|.$$

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Next note that the center of the circle which is tangent to α at $(0,0)$ must lie on the y-axis at some point $(0,r)$, and for this circle to also pass through the point $(s, f(s))$ we must have:

$$r^2 = s^2 + (r - f(s))^2.$$

Solving the above equation for r and taking the limit as $s \to 0$, via the L'Hopital's rule, we have

$$\lim_{s\to 0} \frac{2|f(s)|}{f^2(s) + s^2} = |f''(0)| = \kappa(0),$$

which is the desired result. □

Note 2. The above limit can be used to define a notion of curvature for curves that are not twice differentiable. In this case, we may define the *upper curvature* and *lower curvature* respectively as the upper and lower limit of

$$\frac{2|f(s)|}{f^2(s) + s^2}.$$

as $s \to 0$. We may even distinguish between right handed and left handed upper or lower curvature, by taking the right handed or left handed limits respectively.

Exercise* 3. Let $\alpha\colon I \to \mathbf{R}^2$ be a planar curve and $t_0, t_1, t_2 \in I$ with $t_1 \leq t_0 \leq t_2$. Show that $\kappa(t_0)$ is the reciprocal of the limit of the radius of the circles which pass through $\alpha(t_0)$, $\alpha(t_1)$ and $\alpha(t_2)$ as $t_1, t_2 \to t_0$.

1.14 Total Curvature and Convexity

We say that a simple closed curve $\alpha\colon I \to \mathbf{R}^2$ is *convex* provided that for every $t \in I$ there exists a line $\ell \subset \mathbf{R}^2$ passing through $\alpha(t)$ such that $\alpha(I)$ lies on one side of ℓ.

The *boundary* of a subset $X \subset \mathbf{R}^n$, which we denote by $\operatorname{bd} X$, is defined as the intersection of the closure of X with the closure of its complement. A subset of $\mathbf{R}^n$ is *convex* if it contains the line segment joining each pairs of its points. Clearly the intersection of convex sets is convex.

Exercise 4. Show that if a closed planar curve $\alpha\colon I \to \mathbf{R}^2$ is convex then it lies on the boundary of a convex set. (*Hint:* Let $\Gamma := \alpha(I)$. By defintion, through each point p of Γ there passes a line ℓ_p with respect to which Γ lies on one side. Thus each ℓ_p defines a closed half plane H_p which contains Γ. Show that Γ lies on the boundary of the intersection of all these half planes).

Exercise 5 (Total curvature of convex curves). Show that the total curvature of any simple closed convex planar curve is 2π. (*Hint*: It is enough to check that the signed exterior angles of polygonal approximations of a convex curve do not change sign; because, as we showed in a previous section, the sum of these angles is the total signed curvature, and the sum of their absolute values is the total curvature by Fox-Milnor's theorem. So it would follow that the total signed curvature of α is equal to its total curvature up to a sign. Since by definition the curve is simple, however, the total signed curvature is $\pm 2\pi$ by Hopf's rotation index theorem.)

Theorem 6. *For any closed planar curve* $\alpha\colon I \to \mathbf{R}^2$,

$$\int_I \kappa(t)dt \geq 2\pi,$$

with equality if and only if α *is convex.*

First we show that the total curvature of any curve is at least 2π. To this end recall that when t is the arclength parameter $\kappa(t) = \|T'(t)\|$. Thus the total curvature is simply the total length of the tantrix curve $T\colon I \to \mathbf{S}^2$. Since T is a closed curve, to show that its total length is bigger than 2π, it suffices to check that the image of T does not lie in any semicircle.

Exercise 7. Verify the last sentence.

To see the that the image of T does not lie in any semicircle, let $u \in \mathbf{S}^1$ be a unit vector and note that

$$\int_a^b \langle T(t), u\rangle dt = \int_a^b \langle \alpha'(t), u\rangle dt = \langle \alpha(b) - \alpha(a), u\rangle = 0.$$

Since $T(t)$ is not constant (why?), it follows that the function $t \mapsto \langle T(t), u\rangle$ must change sign. So the image of T must lie on both sides of the line through the origin and orthogonal to u. Since u was chosen arbitrarily, it follows that the image of T does not lie in any semicircle, as desired.

Next we show that the total curvature is 2π if and only if α is convex. The "if" part has been established already in exercise 5. To prove the "only if" part, suppose that α is not convex, then there exists a tangent line ℓ_0 of α, say at $\alpha(t_0)$, with respect to which the image of α lies on both sides. Then α must have two more tangent lines parallel to ℓ_0.

Exercise 8. Verify the last sentence (*Hint*: Let u be a unit vector orthogonal to ℓ and note that the function $t \mapsto \langle \alpha(t) - \alpha(t_0), u \rangle$ must have a minimum and a maximum differerent from 0. Thus the derivative at these two points vanishes.)

Now that we have established that α has three distinct parallel lines, it follows that it must have at least two parallel tangents. This observation is worth recording:

Lemma 9. *If $\alpha \colon I \to \mathbf{R}^2$ is a closed curve which is* not *convex, then it has a pair of parallel tangent vectors wich generate disitinct parallel lines.*

Next note that

Exercise 10. If $\alpha \colon I \to \mathbf{R}^2$ is closed curve whose tantrix $T \colon I \to \mathbf{S}^1$ is not onto, then the total curvature is bigger than 2π. (*Hint*: This is an immediate consequence of the fact that T is a closed curve and it does not lie in any semicircle.)

So if T is not onto then we are done (recall that we are trying to show that if α is not convex, then its total curvature is bigger than 2π). We may assume, therefore, that T is onto. This together with the above lemma yields that the total curvature is bigger than 2π. To see this note that let t_1, $t_2 \in I$ be the two points such that $T(t_1)$ and $T(t_2)$ are parallel and the corresponding tangent lines are distict. Then T restricted to $[t_1, t_2]$ is a closed nonconstant. So either $T([t_1, t_2])$ (i) covers some open segment of the circle twice or (ii) covers the entire circle. Since we have established that T is onto, the first possibility implies that the legth of T is bigger than 2π. Further, since, T restricted to $I - (t_1, t_2)$ is not constant, the second possibility (ii) would imply the again the first case (i). Hence we conclude that if α is not convex, then its total curvature is bigger than 2π, which completes the proof of Theorem 6.

Corollary 11. *Any simple closed C^2 regular curve $\alpha \colon I \to \mathbf{R}^2$ is convex if and only if its signed curvature $\overline{\kappa}$ does not change sign. In particular, if κ never vanishes then α is convex.*

Proof. Since α is simple, its total signed curature is $\pm 2\pi$ by Hopf's theorem. After switching the orientation of α, if necessary, we may assume that the total signed curvature is 2π. Suppose, towards a contradiction, that the signed curvature does change sign. The integral of the signed curvature over the regions where its is positive must be bigger than 2π, which in turn implies that the total curvature is bigger than 2π, which contradicts the previous theorem. So if α is convex, then $\overline{\kappa}$ does not change sign.

Next suppose that $\overline{\kappa}$ does not change sign. Then the total signed curvature is equal to the total curvature (up to a sign), which, since the curve is simple, implies, via the Hopf's theorem, that the total curvature is 2π. So by the previous theorem the curve is convex. $\square$

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Lecture Notes 6

1.15 The four vertex theorem

A *vertex* of a planar curve $\alpha\colon I \to \mathbf{R}^2$ is a point where the curvature of α has a local max or min.

Exercise 1. Show that an ellipse has exactly 4 vertices, unless it is a circle.

The main aim of this section is to show that:

Theorem 2. *Any C^3 simple closed planar curve has (at least) four vertices.*

On the other hand if the curve is not simple, then the 4 vertex property may no longer be true:

Exercise 3. Sketch the limacon $\alpha\colon [0, 2\pi] \to \mathbf{R}^2$ given by

$$\alpha(t) := (2\cos t + 1)(\cos t, \sin t)$$

and show that it has only two vertices. (*Hint*: It looks like a loop with a smaller loop inside)

If the signed curvature of a closed curve changes sign, then it must have two points where κ vanishes. Since $\kappa \geq 0$, it follows then that κ has at least two local minimums. But there is a local maximum between any pairs of local minimums, so, we conclude that if the signed curvature changes sign then we have 4 vertices. It remains then to consider the case where the signed curvature does not change sign. By the result at the end of the previous section, if the signed curvature of a simple closed curve does not change sign, then the curve is convex. So we need only to prove the above theorem for convex curves.

We proceed by contradiction. Suppose that α has fewer than 4 vertices, then it must have exactly 2. Suppose that these two vertices occur at t_0

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and t_1. Then $\kappa'(t)$ will have one sign on (t_1, t_2) and the opposite sign on $I - [t_1, t_2]$. Let ℓ be the line passing through $\alpha(t_1)$ and $\alpha(t_2)$. Then, since α is convex, α restricted to (t_1, t_2) lies entirely in one of the closed half planes determined by ℓ and α restricted to $I - [t_1, t_2]$ lies in the other closed half plane.

Exercise 4. Verify the last sentence, i.e., show that if $\alpha\colon I \to \mathbf{R}^2$ is a simple closed convex planar curve, and ℓ is any line in the plane which intersects $\alpha(I)$, then ℓ intersects α in exactly two points, or $\alpha(I)$ lies on one side of ℓ.(*Hint*: Show that if α intersects ℓ at 3 points, then it lies on one side of ℓ.)

Let p be a point of ℓ and v be a vector orthogonal to ℓ, then $f\colon I \to \mathbf{R}$, given by $f(t) := \langle \alpha(t) - p, v\rangle$ has one sign on (t_1, t_2) and has the opposite sign on $I - [t_1, t_2]$. Consequently, $\kappa'(t)f(t)$ is always nonnegative. So

$$0 < \int_I \kappa'(t)\langle \alpha(t) - p, v\rangle dt.$$

On the other hand

$$\begin{aligned}
\int_I \kappa'(t)\langle \alpha(t) - p, v\rangle dt &= \kappa(t)\langle \alpha(t) - p, v\rangle\Big|_a^b - \int_I \kappa(t)\langle T(t) - p, v\rangle dt \\
&= 0 - \int_I \langle -N'(t) - p, v\rangle dt \\
&= \langle -N(t) - p, v\rangle\Big|_a^b \\
&= 0.
\end{aligned}$$

So we have a contradiction, as desired. It only remains to justify the implicit assumption above that κ is a C^1 function. In general this is not something that we can take for granted:

Exercise 5. Show that there exists a C^∞ regular planar curve whose curvature is not differentiable (*Hint:* Consider $\alpha\colon (-1, 1) \to \mathbf{R}^2$, $\alpha(t) := (t, t^3)$).

On the other hand the signed curvature is always well behaved:

Exercise 6. Show that the signed curvature of a C^3 regular curve in the plane is C^1.

So if the signed curvature does not change sign, then, either $\kappa = \overline{\kappa}$ or $\kappa = -\overline{\kappa}$, and hence, by the above exercise, κ is C^1.

The 4-vertex theorem we proved here may also be generalized to signed curvature, but the proof is more involved.

1.16 Area of planar regions and the Isoperimetric inequality

The area of a rectangle is defined as the product of lengths of two of its adjacent sides. Let $X \subset \mathbf{R}^2$ be any set, R be a collection of rectangles which cover X, and $Area(X, R)$ be the sum of the areas of all rectangles in R. Then area of X is defined as the infimum of $Area(X, R)$ where R ranges over all different ways to cover X by rectangles. In particular it follows that, if $X \subset Y$, then $Area(X) \leq Area(Y)$, and if $X = X_1 \cup X_2$, then $Area(X) = Area(X_1) + Area(X_2)$. These in turn quickly yield the areas of triangles and polygons.

Exercise 7 (Invariance under isometry and the Special linear group). Show that area is invariant under rigid motions of $\mathbf{R}^2$, and that dilation by a factor of r, i.e., multiplying each point $\mathbf{R}^2$ by r, changes the area by a factor of r^2. More generally show that any linear transformation $A\colon \mathbf{R}^2 \to \mathbf{R}^2$ changes the area by a factor of $\det(A)$.

Exercise 8 (Area of circle by plogonal approximation). Compute the area of a circle (*Hint:* For any n compute the area of regular n-gons which are inscribed in the circle, and take the limit. Each of these areas is the sum of n isoceles triangles with an angle $2\pi/n$, and adjacent sides of length equal to the radius of the circle. This gives a lower bound for the area. Un upper bound may also be obtained by taking the limit of regular polygons which circumscribe the circle.)

Recalling the defintion of Riemann sums, one quickly observes that

$$Area(X) = \int\int_X 1\, dxdy.$$

We say that a subset X of $\mathbf{R}^n$ is *connected* provided that the only subsets of X which are both open and closed in X are the X and the emptyset. Every subset of X which is connected and is different from X and the empty set is called a *component* of X.

Let $\alpha\colon I \to \mathbf{R}^2$ be a simple closed planar curve. By the Jordan curve theorem (which we will not prove here), $\mathbf{R}^2 - \alpha(I)$ consists of exactly two connected components, and the boundary of each component is $\alpha(I)$. Further, one of these components, which we call the *interior* of α, is contained in some large sphere, while the other is unbounded. By area of α we mean the area of its interior.

Theorem 9. *For any simple closed planar curve $\alpha\colon I \to \mathbf{R}^2$,*

$$Area[\alpha] \le \frac{Length[\alpha]^2}{4\pi}.$$

Equality holds only when α is a circle.

Our proof of the above theorem hinges on the following subtle fact whose proof we leave out

Lemma 10. *Of all simple closed curves of fixed length L, there exists at least one with the biggest area. Further, every such curve is C^1.*

Exercise* 11. Show that the area maximizer (for a fixed length) must be convex. (*Hint*: It is enough to show that if the maximizer, say α, is not convex, then there exist a line ℓ with respect to which $\alpha(I)$ lies on one side, and intersects $\alpha(I)$ at two points p and q but not in the intervening open segment of ℓ determined by p and q. Then reflecting one of the segments of $\alpha(I)$, determined by p and q, through ℓ increases area while leaving the length unchanged.)

We say that α is symmetric with respect to a line ℓ provided that the image of α is invariant under reflection with respect to ℓ.

Exercise 12. Show that a C^1 convex planar curve $\alpha\colon I \to \mathbf{R}^2$ is a circle, if and only if for every unit vector $u \in \mathbf{S}^1$ there exists a line perpendicular to u with respect to which α is symmetric (*Hint* Suppose that α has a line of symmetry in every direction. First show that each line of symmetry is unique in the corresponding direction. After a translation we may assume that α is symmetric with respect to both the x-axis and the y-axis. Show that this yields that α is symmetric with respect to the origin, i.e. rotation by $180°$. From this and the uniqueness of the lines of symmetry conclude that every line of symmetry passes through the origin. Finally show that each line of symmetry must meet the curve orthogonally at the intersection points. This shows that $\langle \alpha(t), \alpha'(t) \rangle = 0$, which in turn yields that $\|\alpha(t)\| = const.$)

Let $\alpha\colon I \to \mathbf{R}^2$ be an area maximizer. By Exercise 11 we may assume that α is convex. We claim that α must have a line of symmetry in every direction, which would show, by Exercise 12, that α is a circle, and hence would complete the proof.

Suppose, towards a contradiction, that there exists a direction $u \in \mathbf{S}^1$ such that α has no line of symmetry in that direction. Ater a rigid motion, we may assume that $u = (0, 1)$.

Let $[a, b]$ be the projection of $\alpha(I)$ to the x-axis. Then, since α is convex, every vertical line which passes through an interior point of (a, b) intersects $\alpha(I)$ at precisely two points. Let $f(x)$ be the y-coordinate of the higher point, and $g(x)$ be the y-coordinate of the other points. Then

$$Area[\alpha] = \int_a^b f(x) - g(x)\, dx.$$

Further note that if α is C^1 then f and g are C^1 as well, thus

$$Length[\alpha] = f(a) - g(a) + \int_a^b \sqrt{1 + f'(x)^2}\, dx + \int_a^b \sqrt{1 + g'(x)^2}\, dx + f(b) - g(b).$$

Now we are going to define a new curve $\overline{\alpha}$ which is bounded above by the graph of the function $\overline{f}\colon [a, b] \to \mathbf{R}$ given by

$$\overline{f}(x) := \frac{f(x) - g(x)}{2},$$

is bounded below by the graph of $-\overline{f}$, and is bouned on the left and right by vertical segments, which may consist only of a single point. One immdediately checks that

$$Area[\overline{\alpha}] = Area[\alpha].$$

Further, note that since by assumption α is not symmetric with respect to the x-axis, $\overline{f}$ is strictly positive on (a, b). This may be used to show that

$$Length[\overline{\alpha}] < Length[\alpha].$$

Exercise 13. Verify the last inequlaity above (*Hint*: It is enough to check that $\int_a^b \sqrt{1 + \overline{f}'(x)^2} dx$ is strictly smaller than either of the integrals in the above formula for the length of α).

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Lecture Notes 7

1.17 The Frenet-Serret Frame and Torsion

Recall that if $\alpha\colon I \to \mathbf{R}^n$ is a unit speed curve, then the unit tangent vector is defined as

$$T(t) := \alpha'(t).$$

Further, if $\kappa(t) = \|T'(t)\| \neq 0$, we may define the principal normal as

$$N(t) := \frac{T'(t)}{\kappa(t)}.$$

As we saw earlier, in $\mathbf{R}^2$, $\{T, N\}$ form a moving frame whose derivatives may be expressed in terms of $\{T, N\}$ itself. In $\mathbf{R}^3$, however, we need a third vector to form a frame. This is achieved by defining the *binormal* as

$$B(t) := T(t) \times N(t).$$

Then similar to the computations we did in finding the derivatives of $\{T, N\}$, it is easily shown that

$$\begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} = \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix},$$

where τ is the *torsion* which is defined as

$$\tau(t) := -\langle B', N \rangle.$$

Note that torsion is well defined only when $\kappa \neq 0$, so that N is defined. Torsion is a measure of how much a space curve deviates from lying in a plane:

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Exercise 1. Show that if the torsion of a curve $\alpha\colon I \to \mathbf{R}^3$ is zero everywhere then it lies in a plane. (*Hint*: We need to check that there exist a point p and a (fixed) vector v in $\mathbf{R}^3$ such that $\langle \alpha(t) - p, v\rangle = 0$. Let $v = B$, and p be any point of the curve.)

Exercise 2. Computer the curvature and torsion of the circular helix

$$(r\cos t, r\sin t, ht)$$

where r and h are constants. How does changing the values of r and h effect the curvature and torsion.

1.18 Curves of Constant Curvature and Torsion

The above exercise shows that the curvature and torsion of a circular helix are constant. The converse is also true

Theorem 3. *The only curve $\alpha\colon I \to \mathbf{R}^3$ whose curvature and torsion are nonzero constants is the circular helix.*

The rest of this section develops a number of exercises which leads to the proof of the above theorem

Exercise 4. Show that $\alpha\colon I \to \mathbf{R}^3$ is a circular helix (up to rigid motion) provided that there exists a vector v in $\mathbf{R}^3$ such that

$$\langle T, v\rangle = const,$$

and the projection of α into a plane orthogonal to v is a circle.

So first we need to show that when κ and τ are constants, v of the above exercise can be found. We do this with the aid of the Frenet-Serret frame. Since $\{T, N, B\}$ is an orthonormal frame, we may arite

$$v = a(t)T(t) + b(t)N(t) + c(t)B(t).$$

Next we need to find a, b and c subject to the condtions that v is a constant vector, i.e., $v' = 0$, and that $\langle T, v\rangle = const$. The latter implies that

$$a = const$$

because $\langle T, v\rangle = a$. In particular, we may set $a = 1$.

Exercise 5. By setting $v' = 0$ show that

$$v = T + \frac{\kappa}{\tau}B,$$

and check that v is the desired vector, i.e. $\langle T, v\rangle = const$ and $v' = 0$.

So to complete the proof of the theorem, only the following remains:

Exercise 6. Show that the projection of α into a plane orthogonal to v, i.e.,

$$\overline{\alpha}(t) := \alpha(t) - \langle \alpha(t), v\rangle \frac{v}{\|v\|^2}$$

is a circle. (*Hint*: Compute the curvature of $\overline{\alpha}$.)

1.19 Intrinsic Characterization of Spherical Curves

In this section we derive a characterization in terms of curvature and torsion for unit speed curves which lie on a shphere. Suppose $\alpha \colon I \to \mathbf{R}^3$ lies on a sphere of radius r. Then there exists a point p in $\mathbf{R}^3$ (the center of the sphere) such that

$$\|\alpha(t) - p\| = r.$$

Thus differentiation yields

$$\langle T(t), \alpha(t) - p\rangle = 0.$$

Differentiating again we obtain:

$$\langle T'(t), \alpha(t) - p\rangle + 1 = 0.$$

The above expression shows that $\kappa(t) \neq 0$. Consequently N is well defined, and we may rewrite the above expression as

$$\kappa(t)\langle N(t), \alpha(t) - p\rangle + 1 = 0.$$

Differentiating for the third time yields

$$\kappa'(t)\langle N(t), \alpha(t) - p\rangle + \kappa(t)\langle -\kappa(t)T(t) + \tau(t)B(t), \alpha(t) - p\rangle = 0,$$

which using the previous expressions above we may rewrite as

$$-\frac{\kappa'(t)}{\kappa(t)} + \kappa(t)\tau(t)\langle B(t), \alpha(t) - p\rangle = 0.$$

Next note that, since $\{T, N, B\}$ is orthonormal,

$$\begin{aligned} r^2 &= \|\alpha(t) - p\|^2 \\ &= \langle \alpha(t) - p, T(t) \rangle^2 + \langle \alpha(t) - p, N(t) \rangle^2 + \langle \alpha(t) - p, B(t) \rangle^2 \\ &= 0 + \frac{1}{\kappa^2(t)} + \langle \alpha(t) - p, B(t) \rangle^2. \end{aligned}$$

Thus, combining the previous two calculations, we obtain:

$$-\frac{\kappa'(t)}{\kappa(t)} + \kappa(t)\tau(t)\sqrt{r^2 - \frac{1}{\kappa^2(t)}} = 0.$$

Exercise 7. Check the converse, that is supposing that the curvature and torsion of some curve satisfies the above expression, verify whether the curve has to lie on a sphere of radius r.

To do the above exercise, we need to first find out where the center p of the sphere could lie. To this end we start by writing

$$p = \alpha(t) + a(t)T(t) + b(t)N(t) + c(t)B(t),$$

and try to find $a(t)$, $b(t)$ and $c(t)$ so that $p' = (0, 0, 0)$, and $\|\alpha(t) - p\| = r$. To make things easier, we may note that $\alpha(t) = 0$ (why?). Then we just need to find $b(t)$ and $c(t)$ subject to the two constraints mentioned above. We need to verify whether this is possible, when κ and τ satisfy the above expression.

Math 497C
Curves and Surfaces
Fall 2004, PSU

Oct 8, 2004[1]

Lecture Notes 8

2 Surfaces

2.1 Definition of a regular embedded surface

An n-dimensional open ball of radius r centered at p is defined by

$$B_r^n(p) := \{\, x \in \mathbf{R}^n \mid \mathrm{dist}(x,p) < r \,\}.$$

We say a subset $U \subset \mathbf{R}^n$ is *open* if for each p in U there exists an $\epsilon > 0$ such that $B_\epsilon^n(p) \subset U$. Let $A \subset \mathbf{R}^n$ be an arbitrary subset, and $U \subset A$. We say that U is open in A if there exists an open set $V \subset \mathbf{R}^n$ such that $U = A \cap V$. A mapping $f\colon A \to B$ between arbitrary subsets of $\mathbf{R}^n$ is said to be *continuous* if for every open set $U \subset B$, $f^{-1}(U)$ is open is A. Intuitively, we may think of a continuous map as one which sends nearby points to nearby points:

Exercise 1. Let A, $B \subset \mathbf{R}^n$ be arbitrary subsets, $f\colon A \to B$ be a continuous map, and $p \in A$. Show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $\mathrm{dist}(x,p) < \delta$, then $\mathrm{dist}(f(x), f(p)) < \epsilon$.

Two subsets A, $B \subset \mathbf{R}^n$ are said to be *homeomorphic*, or topologically equivalent, if there exists a mapping $f\colon A \to B$ such that f is one-to-one, onto, continuous, and has a continuous inverse. Such a mapping is called a *homeomorphism*. We say a subset $M \subset \mathbf{R}^3$ is an *embedded surface* if every point in M has an open neighborhood in M which is homeomorphic to an open subset of $\mathbf{R}^2$.

Exercise 2. (Stereographic projection) Show that the standard sphere $\mathbf{S}^2 := \{p \in \mathbf{R}^3 \mid \|p\| = 1\}$ is an embedded surface (*Hint*: Show that the stereographic projection π_+ form the north pole gives a homeomorphism between $\mathbf{R}^2$ and $\mathbf{S}^2 - (0,0,1)$. Similarly, the stereographic projection π_-

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from the south pole gives a homeomorphism between $\mathbf{R}^2$ and $\mathbf{S}^2 - (0, 0, -1)$; $\pi_+(x, y, z) := (\frac{x}{1-z}, \frac{y}{1-z}, 0)$, and $\pi_-(x, y, z) := (\frac{x}{z-1}, \frac{y}{z-1}, 0))$.

Exercise 3. (Surfaces as graphs) Let $U \subset \mathbf{R}^2$ be an open subset and $f: U \to \mathbf{R}$ be a continuous map. Then

$$\operatorname{graph}(f) := \{(x, y, f(x, y)) \mid (x, y) \in U\}$$

is a surface. (*Hint*: Show that the orthogonal projection $\pi(x, y, z) := (x, y)$ gives the desired homeomorphism).

Note that by the above exercise the cone given by $z = \sqrt{x^2 + y^2}$, and the troughlike surface $z = |x|$ are examples of embedded surfaces. These surfaces, however, are not "regular", as we will define below. From the point of view of differential geometry it is desirable that a surface be without sharp corners or vertices.

Let $U \subset \mathbf{R}^n$ be open, and $f: U \to \mathbf{R}^m$ be a map. Note that f may be regarded as a list of m functions of n variables: $f(p) = (f^1(p), \ldots, f^m(p))$, $f^i(p) = f^i(p^1, \ldots, p^n)$. The first order partial derivatives of f are given by

$$D_j f^i(p) := \lim_{h \to 0} \frac{f^i(p^1, \ldots, p^j + h, \ldots, p^n) - f^i(p^1, \ldots, p^j, \ldots, p^n)}{h}.$$

If all the functions $D_j f^i: U \to \mathbf{R}$ exist and are continuous, then we say that f is differentiable (C^1). We say that f is smooth (C^∞) if the partial derivatives of f of all order exist and are continuous. These are defined by

$$Dj_1, j_2, \ldots, j_k f^i := D_{j_1}(D_{j_2}(\cdots(D_{j_k} f^i)\cdots)).$$

Let $f: U \subset \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map, and $p \in U$. Then the Jacobian of f at p is an $m \times n$ matrix defined by

$$J_p(f) := \begin{pmatrix} D_1 f^1(p) & \cdots & D_n f^1(p) \\ \vdots & & \vdots \\ D_1 f^m(p) & \cdots & D_n f^m(p) \end{pmatrix}.$$

We say that p is a *regular point* of f if the rank of $J_p(f)$ is equal to n. If f is regular at all points $p \in U$, then we say that f is regular.

Exercise 4 (Monge Patch). Let $f: U \subset \mathbf{R}^2 \to \mathbf{R}$ be a differentiable map. Show that the mapping $X: U \to \mathbf{R}^3$, defined by $X(u^1, u^2) := (u^1, u^2, f(u^1, u^2))$ is regular (the pair (X, U) is called a *Monge Patch*).

If f is a differentiable function, then we define,

$$D_i f(p) := (D_i f^1(p), \dots D_i f^n(p)).$$

Exercise 5. Show that $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}^3$ is regular at p if and only if

$$\|D_1 f(p) \times D_2 f(p)\| \neq 0.$$

Let $f\colon U \subset \mathbf{R}^n \to \mathbf{R}^m$ be a differentiable map and $p \in U$. Then the *differential* of f at p is a mapping from $\mathbf{R}^n$ to $\mathbf{R}^m$ defined by

$$df_p(x) := \lim_{t\to 0} \frac{f(p+tx) - f(p)}{t}.$$

Exercise 6. Show that (i)

$$df_p(x) = J_p(f) \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}.$$

Conclude then that (ii) df_p is a linear map, and (iii) p is a regular value of f if and only if df_p is one-to-one. Further, (iv) show that if f is a linear map, then $df_p(x) = f(x)$, and (v) $J_p(f)$ coincides with the matrix representation of f with respect to the standard basis.

By a *regular patch* we mean a pair (U, X) where $U \subset \mathbf{R}^2$ is open and $X\colon U \to \mathbf{R}^3$ is a one-to-one, smooth, and regular mapping. Furthermore, we say that the patch is *proper* if X^{-1} is continuous. We say a subset $M \subset \mathbf{R}^3$ is a *regular embedded surface*, if for each point $p \in M$ there exists a proper regular patch (U, X) and an open set $V \subset \mathbf{R}^3$ such that $X(U) = M \cap V$. The pair (U, X) is called a *local parameterization* for M at p.

Exercise 7. Let $f\colon U \subset \mathbf{R}^2 \to \mathbf{R}$ be a smooth map. Show that graph(f) is a regular embedded surface, see Exercise 4.

Exercise 8. Show that $\mathbf{S}^2$ is a regular embedded surface (*Hint:* (Method 1) Let $p \in \mathbf{S}^2$. Then p^1, p^2, and p^3 cannot vanish simultaneously. Suppose, for instance, that $p^3 \neq 0$. Then, we may set $U := \{u \in \mathbf{R}^2 \mid \|u\| < 1\}$, and let $X(u^1, u^2) := (u^1, u^2, \pm\sqrt{1 - (u^1)^2 - (u^2)^2})$ depending on whether p^3 is positive or negative. The other cases involving p^1 and p^2 may be treated similarly. (Method 2) Write the inverse of the stereographic projection, see Exercise 2, and show that it is a regular map).

The following exercise shows that smoothness of a patch is not sufficient to ensure that the corresponding surface is without singularities (sharp edges or corners). Thus the regularity condition imposed in the definition of a regular embedded surface is not superfluous.

Exercise 9. Let $M \subset \mathbf{R}^3$ be the graph of the function $f(x, y) = |x|$. Sketch this surface, and show that there exists a smooth one-to-one map $X : \mathbf{R}^2 \to \mathbf{R}^3$ such that $X(\mathbf{R}^2) = M$ (*Hint:* Let $X(x, y) := (e^{-1/x^2}, y, e^{-1/x^2})$, if $x > 0$; $X(x, y) := (-e^{-1/x^2}, y, e^{-1/x^2})$, if $X < 0$; and, $X(x, y) := (0, 0, 0)$, if $x = 0$).

The following exercise demonstrates the significance of the requirement in the definition of a regular embedded surface that X^{-1} be continuous.

Exercise 10. Let $U := \{(u, v) \in \mathbf{R}^2 \mid -\pi < u < \pi,\ 0 < v < 1\}$, define $X : U \to \mathbf{R}^3$ by $X(u, v) := (\sin(u), \sin(2u), v)$, and set $M := X(U)$. Sketch M and show that X is smooth, one-to-one, and regular, but X^{-1} is not continuous.

Exercise 11 (Surfaces of Revolution). Let $\alpha : I \to \mathbf{R}^2$, $\alpha(t) = (x(t), y(t))$, be a regular simple closed curve. Show that the image of $X : I \times \mathbf{R} \to \mathbf{R}^3$ given by

$$X(t, \theta) := \Big(x(t)\cos\theta, x(t)\sin\theta, y(t)\Big),$$

is a regular embedded surface.

Oct 7, 2004[1]

Lecture Notes 9

2.2 Definition of Gaussian Curvature

Let $M \subset \mathbf{R}^3$ be a regular embedded surface, as we defined in the previous lecture, and let $p \in M$. By the *tangent space* of M at p, denoted by T_pM, we mean the set of all vectors v in $\mathbf{R}^3$ such that for each vector v there exists a smooth curve $\gamma\colon (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$.

Exercise 1. Let $H \subset \mathbf{R}^3$ be a plane. Show that, for all $p \in H$, T_pH is the plane parallel to H which passes through the origin.

Exercise 2. Prove that, for all $p \in M$, T_pM is a 2-dimensional vector subspace of $\mathbf{R}^3$ (*Hint:* Let (U, X) be a proper regular patch centered at p, i.e., $X(0,0) = p$. Recall that $dX_{(0,0)}$ is a linear map and has rank 2. Thus it suffices to show that $T_pM = dX_{(0,0)}(\mathbf{R}^2)$).

Exercise 3. Prove that $D_1X(0,0)$ and $D_2X(0,0)$ form a basis for T_pM (*Hint:* Show that $D_1X(0,0) = dX_{(0,0)}(1,0)$ and $D_2X(0,0) = dX_{(0,0)}(0,1)$).

By a *local gauss map* of M centered at p we mean a pair (V, n) where V is an open neighborhood of p in M and $n\colon V \to \mathbf{S}^2$ is a continuous mapping such that $n(p)$ is orthogonal to T_pM for all $p \in M$. For a more explicit formulation, let (U, X) be a proper regular patch centered at p, and define $N\colon U \to \mathbf{S}^2$ by

$$N(u_1, u_2) := \frac{D_1X(u_1,u_2) \times D_2X(u_1,u_2)}{\|D_1X(u_1,u_2) \times D_2X(u_1,u_2)\|}.$$

Set $V := X(U)$, and recall that, since (U, X) is proper, V is open in M. Now define $n\colon V \to \mathbf{S}^2$ by

$$n(p) := N \circ X^{-1}(p).$$

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Exercise 4. Check that (V, n) is indeed a local gauss map.

Exercise 5. Show that $n : \mathbf{S}^2 \to \mathbf{S}^2$, given by $n(p) := p$ is a Gauss map (*Hint:* Define $f : \mathbf{R}^3 \to \mathbf{R}$ by $f(p) := \|p\|^2$ and compute its gradient. Note that $\mathbf{S}^2$ is a level set of f. Thus the gradient of f at p must be orthogonal to $\mathbf{S}^2$).

Let M_1 and M_2 be regular embedded surfaces in $\mathbf{R}^3$ and $f : M_1 \to M_2$ be a smooth map (recall that this means that f may be extended to a smooth map in an open neighborhood of M_1 in $\mathbf{R}^3$). Then for every $p \in M_1$, we define a mapping $df_p : T_pM_1 \to T_{f(p)}M_2$, known as the *differential* of M_1 at p as follows. Let $v \in T_pM_1$ and let $\gamma_v : (-\epsilon, \epsilon) \to M_1$ be a curve such that $\gamma(0) = p$ and $\gamma_v'(0) = v$. Then we set

$$df_p(v) := (f \circ \gamma_v)'(0).$$

Exercise 6. Prove that df_p is well defined (i.e. is independent of the smooth extension) and linear (*Hint:* Let $\tilde{f}$ be a smooth extension of f to an open neighborhood of M. Then $d\tilde{f}_p$ is well defined. Show that for all $v \in T_pM$, $df_p(v) = d\tilde{f}_p(v)$.

Let (V, n) be a local gauss map centered at $p \in M$. Then the *shape operator* of M at p with respect to n is defined as

$$S_p := -dn_p.$$

Note that the shape operator is determined up to two choices depending on the local gauss map, i.e., replacing n by $-n$ switches the sign of the shape operator.

Exercise 7. Show that S_p may be viewed as a linear operator on T_pM (*Hint:* By definition, S_p is a linear map from T_pM to $T_{n(p)}\mathbf{S}^2$. Thus it suffices to show that T_pM and $T_{n(p)}\mathbf{S}^2$ coincide).

Exercise 8. A subset V of M is said to be connected if any pairs of points p and q in V may be joined by a curve in V. Suppose that V is a connected open subset of M, and, furthermore, suppose that the shape operator vanishes throughout V, i.e., for every $p \in M$ and $v \in T_pM$, $S_p(v) = 0$. Show then that V must be flat, i.e., it is a part of a plane (*Hint:* It is enough to show that the gauss map is constant on V; or, equivalently, $n(p) = n(q)$ for all pairs of points p and q in V. Since V is connected, there exists a curve $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Furthermore, since V is open, we may choose γ to be smooth as well. Define $f\colon [0,1] \to \mathbf{R}$ by $f(t) := n \circ \gamma(t)$, and differentiate. Then $f'(t) = dn_{\gamma(t)}(\gamma'(t)) = 0$. Justify the last step and conclude that $n(p) = n(q)$.

Exercise 9. Compute the shape operator of a sphere of radius r (*Hint:* Define $\pi\colon \mathbf{R}^3 - \{0\} \to \mathbf{S}^2$ by $\pi(x) := x/\|x\|$. Note that π is a smooth mapping and $\pi = n$ on $\mathbf{S}^2$. Thus, for any $v \in T_p\mathbf{S}^2$, $d\pi_p(v) = dn_p(v)$).

The *Gaussian curvature* of M at p is defined as the determinant of the shape operator:

$$K(p) := \det(S_p).$$

Exercise 10. Show that $K(p)$ does not depend on the choice of the local gauss map, i.e, replacing n by $-n$ does not effect the value of $K(p)$.

Exercise 11. Compute the curvature of a sphere of radius r (*Hint:* Use exercise 9).

Next we derive an explicit formula for $K(p)$ in terms of local coordinates. Let (U, X) be a proper regular patch centered at p. For $1 \leqslant i, j \leqslant 2$, define the functions $g_{ij}\colon U \to \mathbf{R}$ by

$$g_{ij}(u_1, u_2) := \big\langle D_i X(u_1, u_2), D_j X(u_1, u_2)\big\rangle.$$

Note that $g_{12} = g_{21}$. Thus the above expression defines three functions. These are called the *coefficients of the first fundamental form* (a.k.a. *the metric tensor*) with respect to the given patch (U, X). In the classical notation, these functions are denoted by E, F, and G ($E := g_{11}$, $F := g_{12}$, and $G := g_{22}$). Next, define $l_{ij}\colon U \to \mathbf{R}$ by

$$l_{ij}(u_1, u_2) := \big\langle D_{ij} X(u_1, u_2), N(u_1, u_2)\big\rangle.$$

Thus l_{ij} is a measure of the second derivatives of X in a normal direction. l_{ij} are known as the *coefficients of the second fundamental form* of M with respect to the local patch (U, X) (the classical notation for these functions are $L := l_{11}$, $M := l_{12}$, and $N := l_{22}$). We claim that

$$K(p) = \frac{\det(l_{ij}(0,0))}{\det(g_{ij}(0,0))}.$$

To see the above, recall that $e_i(p) := D_iX(X^{-1}(p))$ form a basis for T_pM. Thus, since S_p is linear, $S_p(e_i) = \sum_{j=1}^2 S_{ij}e_j$. This yields that $\langle S_p(e_i), e_k\rangle = \sum_{j=1}^2 S_{ij}g_{jk}$. It can be shown that that

$$\langle S_p(e_i), e_k\rangle = l_{ik},$$

see the exercise below. Then we have $[l_{ij}] = [S_{ij}][g_{ij}]$, where the symbol $[\cdot]$ denotes the matrix with the given coefficients. Thus we can write $[S_{ij}] = [g_{ij}]^{-1}[l_{ij}]$ which yields the desired result.

Exercise 12. Show that $\langle S_p(e_i(p)), e_j(p)\rangle = l_{ij}(0,0)$ (*Hints:* First note that $\langle n(p), e_j(p)\rangle = 0$ for all $p \in V$. Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$ and $\gamma'(0) = e_i(p)$. Define $f\colon (-\epsilon, \epsilon) \to M$ by $f(t) := \langle n(\gamma(t)), e_j(\gamma(t))\rangle$, and compute $f'(0)$.)

Exercise 13. Compute the Gaussian curvature of a surface of revolution, i.e., the surface covered by the patch

$$X(t,\theta) = \big(x(t)\cos\theta, x(t)\sin\theta, y(t)\big).$$

Next, letting

$$\big(x(t), y(t)\big) = (R + r\cos t, r\sin t),$$

i.e., a circle of radius r centered at $(R, 0)$, compute the curvature of a torus of revolution. Sketch the torus and indicate the regions where the curvature is postive, negative, or zero.

Exercise 14. Let (U, X) be a *Monge patch*, i.e,

$$X(u_1, u_2) := \big(u_1, u_2, f(u_1, u_2)\big),$$

centered at $p \in M$. Show that

$$K(p) := \frac{\det\big(\operatorname{Hess} f(0,0)\big)}{\big(1 + \|\operatorname{grad} f(0,0)\|^2\big)^2},$$

where $\operatorname{Hess} f := [D_{ij}f]$ is the Hessian matrix of f and $\operatorname{grad} f$ is its gradient.

Exercise 15. Compute the curvature of the graph of $z = ax^2 + by^2$, where a and b are constants. Note how the signs of a and b effect the curvature and shape of the surface. Also note the values of a and b for which the curvature is zero.

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Lecture Notes 10

2.3 Meaning of Gaussian Curvature

In the previous lecture we gave a formal definition for Gaussian curvature K in terms of the differential of the gauss map, and also derived explicit formulas for K in local coordinates. In this lecture we explore the geometric meaning of K.

2.3.1 A measure for local convexity

Let $M \subset \mathbf{R}^3$ be a regular embedded surface, $p \in M$, and H_p be hyperplane passing through p which is parallel to T_pM. We say that M is *locally convex* at p if there exists an open neighborhood V of p in M such that V lies on one side of H_p. In this section we prove:

Theorem 1. *If $K(p) > 0$ then M is locally convex at p, and if $k(p) < 0$ then M is not locally convex at p.*

When $K(p) = 0$, we cannot in general draw a conclusion with regard to the local convexity of M at p as the following two exercises demonstrate:

Exercise 2. Show that there exists a surface M and a point $p \in M$ such that M is strictly locally convex at p; however, $K(p) = 0$ (*Hint:* Let M be the graph of the equation $z = (x^2+y^2)^2$. Then may be covered by the Monge patch $X(u_1, u_2) := (u_1, u_2, ((u_1)^2 + (u_2))^2)$. Use the Monge Ampere equation derived in the previous lecture to compute the curvature at $X(0, 0)$.).

Exercise 3. Let M be the *Monkey saddle*, i.e., the graph of the equation $z = y^3 - 3yx^2$, and $p := (0, 0, 0)$. Show that $K(p) = 0$, but M is not locally convex at p.

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After a rigid motion we may assume that $p = (0,0,0)$ and T_pM is the xy-plane. Then, using the inverse function theorem , it is easy to show that there exists a Monge Patch (U, X) centered at p, as the follwing exercise demonstrates:

Exercise 4. Define $\pi\colon M \to \mathbf{R}^2$ by $\pi(q) := (q^1, q^2, 0)$. Show that $d\pi_p$ is locally one-to-one. Then, by the inverse function theorem, it follows that π is a local diffeomorphism. So there exists a neighborhood U of $(0,0)$ such that $\pi^{-1}\colon U \to M$ is one-to-one and smooth. Let $f(u_1, y^2)$ denote the z-coordinate of $\pi^{-1}(u_1, u_2)$, and set $X(u_1, u_2) := (u_1, u_2, f(u_1, u_2))$. Show that (U, X) is a proper regular patch.

The previous exercisle shows that local convexity of M at p depends on whether or not f changes sign in a neighborhood of the origin. To examine this we need to recall the Taylor's formula for functions of two variables:

$$f(u_1, u_2) = f(0,0) + \sum_{i=1}^{2} D_i f(0,0) u_i + \frac{1}{2} \sum_{i,j=1}^{2} D_{ij}(\xi_1, \xi_2) u_i u_j,$$

where (ξ_1, ξ_2) is a point on the line connecting (u_1, u_2) to $(0,0)$.

Exercise 5. Prove the Taylor's formula given above. (*Hints*: First recall Taylor's formula for functions of one variable: $g(t) = g(0) + g'(0)t + (1/2)g''(s)t^2$, where $s \in [0,t]$. Then define $\gamma(t) := (tu_1, tu_2)$, set $g(t) := f(\gamma(t))$, and apply Taylor's formula to g. Then chain rule will yield the desired result.)

Next note that, by construction, $f(0,0) = 0$. Further $D_1 f(0,0) = 0 = D_2 f(0,0)$ as well. Thus

$$f(u_1, u_2) = \frac{1}{2} \sum_{i,j=1}^{2} D_{ij}(\xi_1, \xi_2) u_i u_j.$$

Hence to complete the proof of Theorem 1, it remains to show how the quanitity on the right hand side of the above euation is influence by $K(p)$. To this end, recall the Monge-Ampere equation for curvature:

$$\det(\operatorname{Hess} f\big(\xi_1, \xi_2)\big) = K(f(\xi_1, \xi_2))\big(1 + \|\operatorname{grad} f(\xi_1, \xi_2)\|^2\big)^2.$$

Now note that $K(f(0,0)) = K(p)$. Thus, by continuity, if U is a sufficiently small neighborhood of $(0,0)$, the sign of $\det(\text{Hess}\, f)$ agrees with the sign of $K(p)$ throughout U.

Finally, we need some basic facts about quadratic forms. A *quadratic form* is a function of two variables $Q\colon \mathbf{R}^2 \to \mathbf{R}$ given by

$$Q(x,y) = ax^2 + 2bxy + cy^2,$$

where a, b, and c are constants. Q is said to be definite if $Q(x,x) \neq 0$ whenver $x \neq 0$.

Exercise 6. Show that if $ac - b^2 > 0$, then Q is definite, and if $ac - b^2 < 0$, then Q is not definite. (*Hints*: For the first part, suppose that $x \neq 0$, but $Q(x,y) = 0$. Then $ax^2+2bxy+cy^2 = 0$, which yields $a+2b(x/y)+c(x/y)^2 = 0$. Thus the discriminant of this equation must be positive, which will yield a contradiction. The proof of the second part is similar).

Theorem 1 follows from the above exercise.

2.3.2 Ratio of areas

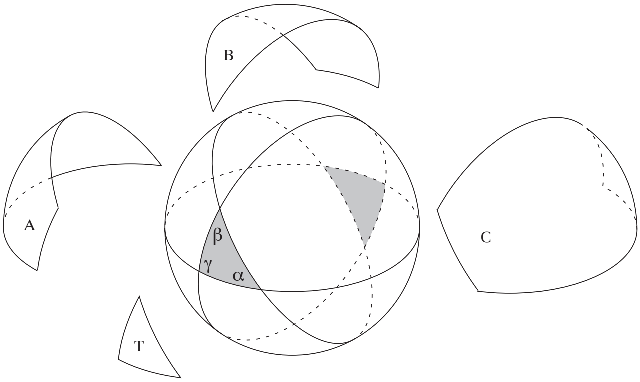
In the previous subsection we gave a geometric interpretation for the sign of Gaussian curvature. Here we describe the geometric significance of the magnitude of K.

If V is a sufficiently small neighborhood of p in M (where M, as always, denotes a regular embedded surface in $\mathbf{R}^3$), then it is easy to show that there exist a patch (U, X) centered at p such that $X(U) = V$. Area of V is then defined as follows:

$$\text{Area}(V) := \int\int_U \|D_1 X \times D_2 X\|\, du_1 du_2.$$

Using the chain rule, one can show that the above definition is independent of the the patch.

Exercise 7. Let $V \subset \mathbf{S}^2$ be a region bounded in between a pair of great circles meeting each other at an angle of α. Show that $\text{Area}(V) = 2\alpha$(*Hints*: Let $U := [0, \alpha] \times [0, \pi]$ and $X(\theta, \phi) := (\cos\theta \sin\phi, \sin\theta\sin\phi, \cos\phi)$. Show that $\|D_1 X \times D_2 X\| = |\sin\phi|$. Further, note that, after a rotation we may assume that $X(U) = V$. Then an integration will yield the desired result).



Exercise 8. Use the previous exercise to show that the area of a geodesic triangle $T \subset \mathbf{S}^2$ (a region bounded by three great circles) is equal to sum of its angles minus π (*Hints:* Use the picture below: $A+B+C+T=2\pi$, and $A=2\alpha-T$, $B=2\beta-T$, and $C=2\gamma-T$).

Let $V_r := B_r(p)\cap M$. Then, if r is sufficiently small, $V(r)\subset X(U)$, and, consequently, $U_r := X^{-1}(V_r)$ is well defined. In particular, we may compute the area of V_r using the patch (U_r, X). In this section we show that

$$|K(p)| = \lim_{r\to 0}\frac{\text{Area}(n(V_r))}{\text{Area}(V_r)}.$$

Exercise 9. Recall that the mean value theorem states that $\int\int_U f du_1 du_2 = f(\bar{u}^1, \bar{u}^2)\,\text{Area}(U)$, for some $(\bar{u}^1, \bar{u}^2)\in U$. Use this theorem to show that

$$\lim_{r\to 0}\frac{\text{Area}(n(V_r))}{\text{Area}(V_r)} = \frac{\|D_1N(0,0)\times D_2N(0,0)\|}{\|D_1X(0,0)\times D_2X(0,0)\|}$$

(Recall that $N := n\circ X$.)

Exercise 10. Prove Lagrange's identity: for every pair of vectors $v, w\in\mathbf{R}^3$,

$$\|v\times w\|^2 = \det\begin{vmatrix} \langle v,v\rangle & \langle v,w\rangle \\ \langle w,v\rangle & \langle w,w\rangle \end{vmatrix}.$$

Now set $g(u_1,u_2) := \det[g_{ij}(u_1,u_2)]$. Then, by the previous exercise it follows that $\|D_1X(0,0)\times D_2X(0,0)\| = \sqrt{g(0,0)}$. Hence, to complete the proof of the main result of this section it remains to show that

$$\|D_1N(0,0)\times D_2N(0,0)\| = K(p)\sqrt{g(0,0)}.$$

We prove the above formula using two different methods:
METHOD 1. Recall that $K(p) := \det(S_p)$, where $S_p := -dn_p \colon T_pM \to T_pM$ is the shape operator of M at p. Also recall that $D_iX(0,0)$, $i = 1, 2$, form a basis for T_pM. Let S_{ij} be the coefficients of the matrix representation of S_p with respect to this basis, then

$$S_p(D_iX) = \sum_{j=1}^{2} S_{ij}\, D_jX.$$

Further, recall that $N := n \circ X$. Thus the chain rule yields:

$$S_p(D_iX) = -dn(D_iX) = -D_i(n \circ X) = -D_iN.$$

Exercise 11. Verify the middle step in the above formula, i.e., show that $dn(D_iX) = D_i(n \circ X)$.

From the previous two lines of formulas, it now follows that

$$-D_iN = \sum_{j=1}^{2} S_{ij}\, D_jX.$$

Taking the inner product of both sides with D_kN, $k = 1, 2$, we get

$$\langle -D_iN, D_kN \rangle = \sum_{j=1}^{2} S_{ij} \langle D_jX, D_kN \rangle.$$

Exercise 12. Let F, $G \colon U \subset \mathbf{R}^2 \to \mathbf{R}^3$ be a pair of mappings such that $\langle F, G \rangle = 0$. Prove that $\langle D_iF, G \rangle = -\langle F, D_iG \rangle$.

Now recall that $\langle D_jX, N \rangle = 0$. Hence the previous exercise yields:

$$\langle D_jX, D_kN \rangle = -\langle D_{kj}X, N \rangle = -l_{ij}.$$

Combining the previous two lines of formulas, we get: $\langle D_iN, D_kN \rangle = \sum_{k=1}^{2} S_{ij} l_{jk}$; which in matrix notation is equivalent to

$$[\langle D_iN, D_jN \rangle] = [S_{ij}][l_{ij}].$$

Finally, recall that $\det[\langle D_iN, D_kN \rangle] = \|D_1N \times D_2N\|^2$, $\det[S_{ij}] = K$, and $\det[l_{ij}] = Kg$. Hence taking the determinant of both sides in the above equation, and then taking the square root yields the desired result.

Next, we discuss the second method for proving that $\|D_1 N \times D_2 N\| = K\sqrt{g}$.
METHOD 2. Here we work with a special patch which makes the computations easier:

Exercise 13. Show that there exist a patch (U, X) centered at p such that $[g_{ij}(0,0)]$ is the identity matrix. (*Hint:* Start with a Monge patch with respect to T_pM)

Thus, if we are working with the coordinate patch referred to in the above exercise, $g(0,0) = 1$, and, consequently, all we need is to prove that $\|D_1 N(0,0) \times D_2 N(0,0)\| = K(p)$.

Exercise 14. Let $f \colon U \subset \mathbf{R}^2 \to \mathbf{S}^2$ be a differentiable mapping. Show that $\langle D_i f(u_1, u_2), f(u_1, u_2)\rangle = 0$ (*Hints:* note that $\langle f, f\rangle = 1$ and differentiate).

It follows from the previous exercise that $\langle D_i N, N\rangle = 0$. Now recall that $N(0,0) = n \circ X(0,0) = n(p)$. Hence, we may conclude that $N(0,0) \in T_pM$. Further recall that $\{D_1 X(0,0), D_2 X(0,0)\}$ is now an orthonormal basis for T_pM (because we have chosen (U, X) so that $[g_{ij}(0,0)]$ is the identity matrix). Consequently,

$$D_i N = \sum_{k=1}^{2} \langle D_i N, D_k X\rangle D_k X,$$

where we have omitted the explicit reference to the point $(0,0)$ in the above formula in order to make the notation less cumbersome (it is important to keep in mind, however, that the above is valid only at $(0,0)$). Taking the inner product of both sides of this equation with $D_j N(0,0)$ yields:

$$\langle D_i N, D_j N\rangle = \sum_{k=1}^{2} \langle D_i N, D_k X\rangle \langle D_k X, D_j N\rangle.$$

Now recall that $\langle D_i N, D_k X\rangle = -\langle N, D_{ij} X\rangle = -l_{ij}$. Similarly, $\langle D_k X, D_j N\rangle = -l_{kj}$. Thus, in matrix notation, the above formula is equivalent to the following:

$$[\langle D_i N, D_j N\rangle] = [l_{ij}]^2$$

Finally, recall that $K(p) = \det[l_{ij}(0,0)] / \det[g_{ij}(0,0)] = \det[l_{ij}(0,0)]$. Hence, taking the determinant of both sides of the above equation yields the desired result.

2.3.3 Product of principal curvatures

For every $v \in T_pM$ with $\|v\| = 1$ we define the *normal curvature* of M at p in the direction of v by

$$k_v(p) := \langle \gamma''(0), n(p) \rangle,$$

where $\gamma\colon (-\epsilon, \epsilon) \to M$ is a curve with $\gamma(0) = p$ and $\gamma'(0) = v$.

Exercise 15. Show that $k_v(p)$ does not depend on γ.

In particular, by the above exercise, we may take γ to be a curve which lies in the intersection of M with a plane which passes through p and is normal to $n(p) \times v$. So, intuitively, $k_v(p)$ is a measure of the curvature of an orthogonal cross section of M at p.

Let $UT_pM := \{v \in T_pM \mid \|v\| = 1\}$ denote the *unit tangent space* of M at p. The *principal curvatures* of M at p are defined as

$$k_1(p) := \min_v k_v(p), \quad \text{and} \quad k_2(p) := \max_v k_v(p),$$

where v ranges over UT_pM. Our main aim in this subsection is to show that

$$K(p) = k_1(p)k_2(p).$$

Since $K(p)$ is the determinant of the shape operator S_p, to prove the above it suffices to show that $k_1(p)$ and $k_2(p)$ are the eigenvalues of S_p.

First, we need to define the *second fundamental form* of M at p. This is a bilinear map $\mathrm{II}_p\colon T_pM \times T_pM \to \mathbf{R}$ defined by

$$\mathrm{II}_p(v, w) := \langle S_p(v), w \rangle.$$

We claim that, for all $v \in UT_pM$,

$$k_v(p) = \mathrm{II}_p(v, v).$$

The above follows from the following computation

$$\begin{aligned}
\langle S_p(v), v \rangle &= -\langle dn_p(v), v \rangle \\
&= -\langle (n \circ \gamma)'(0), \gamma'(0) \rangle \\
&= \langle (n \circ \gamma)(0), \gamma''(0) \rangle \\
&= \langle n(p), \gamma''(0) \rangle
\end{aligned}$$

Exercise 16. Verify the passage from the second to the third line in the above computation, i.e., show that $-\langle (n\circ\gamma)'(0), \gamma'(0)\rangle = \langle (n\circ\gamma)(0), \gamma''(0)\rangle$ (*Hint:* Set $f(t) := \langle n(\gamma(t)), \gamma'(t)\rangle$, note that $f(t) = 0$, and differentiate.)

So we conclude that $k_i(p)$ are the minimum and maximum of $\mathrm{II}_p(v)$ over UT_pM. Hence, all we need is to show that the extrema of II_p over UT_pM coincide with the eigenvalues of S_p.

Exercise 17. Show that II_p is symmetric, i.e., $\mathrm{II}_p(v, w) = \mathrm{II}_p(w, v)$ for all v, $w \in T_pM$.

By the above exercise, S_p is a self-adjoint operator, i.e, $\langle S_p(v), w\rangle = \langle v, S_p(w)\rangle$. Hence S_p is orthogonally diagonalizable, i.e., there exist orthonormal vectors $e_i \in T_pM$, $i = 1, 2$, such that

$$S_p(e_i) = \lambda_i e_i.$$

By convention, we suppose that $\lambda_1 \leq \lambda_2$. Now note that each $v \in UT_pM$ may be represented uniquely as $v = v^1e_1 + v^2e_2$ where $(v^1)^2 + (v^2)^2 = 1$. So for each $v \in UT_pM$ there exists a unique angle $\theta \in [0, 2\pi)$ such that

$$v(\theta) := \cos\theta e_1 + \sin\theta e_2;$$

Consequently, bilinearity of II_p yields

$$\mathrm{II}_p(v(\theta), v(\theta)) = \lambda_1\cos^2\theta + \lambda_2\sin^2\theta.$$

Exercise 18. Verify the above claim, and show that minimum and maximum values of II_p are λ_1 and λ_2 respectively. Thus $k_1(p) = \lambda_1$, and $k_2(p) = \lambda_2$.

The previous exercise completes the proof that $K(p) = k_1(p)k_2(p)$, and also yields the following formula which was discovered by Euler:

$$k_v(p) = k_1(p)\cos^2\theta + k_2(p)\sin^2\theta.$$

In particular, note that by the above formula there exists always a pair of *orthogonal* directions where $k_v(p)$ achieves its maximum and minimum values. These are known as the *principal directions* of M at p.

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Lecture Notes 11

2.4 Intrinsic Metric and Isometries of Surfaces

Let $M \subset \mathbf{R}^3$ be a regular embedded surface and p, $q \in M$, then we define

$$\mathrm{dist}_M(p,q) := \inf\{\mathrm{Length}[\gamma] \mid \gamma\colon [0,1] \to M, \gamma(0) = p, \gamma(1) = q\}.$$

Exercise 1. Show that (M, dist_M) is a metric space.

Lemma 2. *Show that if M is a C^1 surface, and $X \subset M$ is compact, then for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\big|\, \mathrm{dist}_M(p,q) - \|p-q\| \,\big| \leq \epsilon \|p-q\|$$

for all p, $q \in X$, with $\mathrm{dist}_M(p,q) \leq \delta$.

Proof. Define $F\colon M \times M \to \mathbf{R}$ by $F(p,q) := \mathrm{dist}_M(p,q)/\|p-q\|$, if $p \neq q$ and $F(p,q) := 1$ otherwise. We claim that F is continuous. To see this let p_i be a sequnce of points of M which converge to a point $p \in M$. We may assume that p_i are contained in a Monge patch of M centered at p given by

$$X(u_1, u_2) = (u_1, u_2, h(u_1, u_2)).$$

Let x_i and y_i be the x and y coorindates of p_i. If p_i is sufficiently close to $p = (0,0)$, then, since $\nabla h(0,0) = 0$, we can make sure that

$$\|\nabla h(tx_i, ty_i)\|^2 \leq \epsilon,$$

for all $t \in [0,1]$ and $\epsilon > 0$. Let $\gamma\colon [0,1] \to \mathbf{R}^3$ be the curve given by

$$\gamma(t) = (tx_i, ty_i, h(tx_i, ty_i)).$$

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Then, since γ is a curve on M,

$$\begin{aligned}
\operatorname{dist}_M(p, p_i) &\leq \text{Length}[\gamma] \\
&= \int_0^1 \sqrt{x_i^2 + y_i^2 + \langle \nabla h(tx_i, ty_i), (x_i, y_i)\rangle^2}\, dt \\
&\leq \int_0^1 \sqrt{x_i^2 + y_i^2 + \epsilon(x_i^2 + y_i^2)^2}\, dt \\
&\leq \sqrt{1+\epsilon}\sqrt{x_i^2 + y_i^2} \\
&\leq (1+\epsilon)\|p - p_i\|
\end{aligned}$$

So, for any $\epsilon > 0$ we have

$$1 \leq \frac{\operatorname{dist}_M(p, p_i)}{\|p - p_i\|} \leq 1 + \epsilon$$

provided that p_i is sufficiently close to p. We conclude then that F is continuous. So $U := F^{-1}([1, 1+\epsilon])$ is an open subset of $M \times M$ which contains the diagonal $\Delta_M := \{(p,p) \mid p \in M\}$. Since $\Delta_X \subset \Delta_M$ is compact, we may then choose δ so small that $V_\delta(\Delta_X) \subset U$, where $V_\delta(\Delta_X)$ denotes the open neighborhood of Δ_X in $M \times M$ which consists of all pairs of points (p, q) with $\operatorname{dist}_M(p, q) \leq \delta$. □

Exercise 3. Does the above lemma hold also for C^0 surfaces?

If $\gamma \colon [a, b] \to M$ is any curve then we may define

$$\text{Length}_M[\gamma] := \sup\left\{ \sum_{i=1}^{k} \operatorname{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \;\middle|\; \{t_0, \dots, t_k\} \in \text{Partition}[a, b] \right\}.$$

Lemma 4. $\text{Length}_M[\gamma] = \text{Length}[\gamma]$.

Proof. Note that

$$\operatorname{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \geq \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

Thus $\text{Length}_M[\gamma] \geq \text{Length}[\gamma]$. Further, by the previous lemma, we can make sure that

$$\operatorname{dist}_M(\gamma(t_i), \gamma(t_{i-1})) \leq (1+\epsilon)\|\gamma(t_i) - \gamma(t_{i-1})\|,$$

which yields $\text{Length}_M[\gamma] \leq (1+\epsilon)\,\text{Length}[\gamma]$, for any $\epsilon > 0$. □

We say that $f\colon M \to \overline{M}$ is an *isometry* provided that

$$\operatorname{dist}_{\overline{M}}(f(p), f(q)) = \operatorname{dist}_M(p, q).$$

Lemma 5. *$f\colon M \to \overline{M}$ is an isometry, if and only if* $\text{Length}[\gamma] = \text{Length}[f \circ \gamma]$ *for all curves $\gamma\colon [a, b] \to M$.*

Proof. If f is an isometry, then, by the previous lemma,

$$\text{Length}[\gamma] = \text{Length}_M[\gamma] = \text{Length}_{\overline{M}}[f \circ \gamma] = \text{Length}_M[f \circ \gamma].$$

The converse is clear. □

Exercise 6. Justify the middle equality in the last espression displayed above.

Theorem 7. *$f\colon M \to \overline{M}$ is an* isometry *if and only if for all $p \in M$, and $v, w \in T_pM$,*

$$\big\langle df_p(v), df_p(w)\big\rangle = \langle v, w\rangle.$$

Proof. Suppose that f is an isometry. Let $\gamma\colon (-\epsilon, \epsilon) \to M$ be a curve with $\gamma(0) = p$, and $\gamma'(0) = v$. Then, by the previous lemma

$$\int_{-\epsilon}^{\epsilon} \|\gamma'(t)\|\, dt = \int_{-\epsilon}^{\epsilon} \|(f \circ \gamma)'(t)\|\, dt$$

Taking the limit of both sides as $\epsilon \to 0$ and applying the mean value theorem for integrals, yields then that

$$\|v\| = \|\gamma'(0)\| = \|(f \circ \gamma)'(0)\| = \|df_p(v)\|.$$

Thus df preserves the norm, which implies that it must preserve the inner-product as well (see the following exercise).

Conversely, suppose that $\|v\| = \|df_p(v)\|$. Then, if $\gamma\colon [a, b] \to M$ is any curve, we have

$$\int_a^b \|(f \circ \gamma)'(t)\| dt = \int_a^b \|df_{\gamma(t)}(\gamma'(t))\| dt = \int_a^b \|\gamma'(t)\| dt.$$

So f preserves the length of all curves, which, by the previous Lemma, shows that f is an isometry. □

Exercise 8. Show that a function $F\colon \mathbf{R}^n \to \mathbf{R}^n$ preserves the norm $\|\cdot\|$ if and only if it preserves the inner product $\langle \cdot, \cdot \rangle$.

2.5 Gauss's Theorema Egregium

Lemma 9. *Let $X\colon U \to M$ be a proper regular chart. Then $\overline{X} := f \circ X\colon U \to \overline{M}$ is a proper regular chart as well and $g_{ij} = \overline{g}_{ij}$ on U.*

Proof. Using the last theorem we have

$$\begin{aligned}
\overline{g}_{ij}(u_1, u_2) &= \langle D_i\overline{X}(u_1, u_2), D_j\overline{X}(u_1, u_2)\rangle \\
&= \langle D_i(f \circ X)(u_1, u_2), D_j(f \circ X)(u_1, u_2)\rangle \\
&= \langle df_{X(u_1,u_2)}(D_iX(u_1, u_2)), df_{X(u_1,u_2)}(D_jX(u_1, u_2))\rangle \\
&= \langle D_iX(u_1, u_2), D_jX(u_1, u_2)\rangle \\
&= g_{ij}(u_1, u_2).
\end{aligned}$$

□

Exercise 10. Justify the third equality in the last displayed expression above.

Let $\mathcal{F}$ denote the set of functions $f\colon U \to \mathbf{R}$ where $U \subset \mathbf{R}^2$ is an open neighborhood of the orgin.

Lemma 11. *There exists a mapping* Briochi: $\mathcal{F} \times \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ *such that for any chart $X\colon U \to M$ centered at $p \in M$,*

$$K(p) = \text{Briochi}[g_{11}, g_{12}, g_{22}](0, 0).$$

Proof. Recall that

$$K(p) = \frac{\det l_{ij}(0, 0)}{\det g_{ij}(0, 0)},$$

and, by Lagrange's identity,

$$l_{ij} = \left\langle X_{ij}, \frac{X_1 \times X_2}{\|X_1 \times X_2\|} \right\rangle = \frac{1}{\sqrt{\det g_{ij}}} \langle X_{ij}, X_1 \times X_2\rangle,$$

where $X_{ij} := D_{ij}X$, and $X_i := D_iX$. Thus

$$K(p) = \frac{\det(\langle X_{ij}(0, 0), X_1(0, 0) \times X_2(0, 0)\rangle)}{(\det g_{ij}(0, 0))^2}.$$

Next note that

$$\det(\langle X_{ij}, X_1 \times X_2\rangle) = \langle X_{11}, X_1 \times X_2\rangle\langle X_{21}, X_1 \times X_2\rangle - \langle X_{12}, X_1 \times X_2\rangle^2$$

The right hand side of the last expression may be rewritten as

$$\det(X_{11}, X_1, X_2)\det(X_{22}, X_1, X_2) - (\det(X_{12}, X_1, X_2))^2,$$

where (u, v, w) here denotes the matrix with columns u, v, and w. Recall that if A is a square matrix with transpose A^T, then $\det A = \det A^T$. Thus the last expression displayed above is equivalent to

$$\det((X_{11}, X_1, X_2)^T(X_{22}, X_1, X_2)) - \det((X_{12}, X_1, X_2)^T(X_{12}, X_1, X_2)),$$

which in turn can be written as

$$\det\begin{pmatrix} \langle X_{11}, X_{22}\rangle & \langle X_{11}, X_1\rangle & \langle X_{11}, X_2\rangle \\ \langle X_1, X_{22}\rangle & \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_{22}\rangle & \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{pmatrix} - \det\begin{pmatrix} \langle X_{12}, X_{12}\rangle & \langle X_{12}, X_1\rangle & \langle X_{12}, X_2\rangle \\ \langle X_1, X_{12}\rangle & \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_{12}\rangle & \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{pmatrix}.$$

If we expand the above determinants along their first rows, then $\langle X_{11}, X_{22}\rangle$ and $\langle X_{12}, X_{22}\rangle$ will have the same coefficients. This implies that we can rewrite the last expression as

$$\det\begin{pmatrix} \langle X_{11}, X_{22}\rangle - \langle X_{12}, X_{12}\rangle & \langle X_{11}, X_1\rangle & \langle X_{11}, X_2\rangle \\ \langle X_1, X_{22}\rangle & \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_{22}\rangle & \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{pmatrix} - \det\begin{pmatrix} 0 & \langle X_{12}, X_1\rangle & \langle X_{12}, X_2\rangle \\ \langle X_1, X_{12}\rangle & \langle X_1, X_1\rangle & \langle X_1, X_2\rangle \\ \langle X_2, X_{12}\rangle & \langle X_2, X_1\rangle & \langle X_2, X_2\rangle \end{pmatrix}.$$

Now note that each of the entries in the above matrices can be expressed purely in terms of g_{ij}, since

$$\langle X_{ii}, X_j\rangle = \frac{1}{2}\langle X_i, X_i\rangle_j = \frac{1}{2}(g_{ii})_j,$$

$$\langle X_{ij}, X_i\rangle = \langle X_i, X_j\rangle_i - \langle X_i, X_{ji}\rangle = (g_{ij})_i - \frac{1}{2}(g_{ii})_j,$$

and

$$\begin{aligned} \langle X_{11}, X_{22}\rangle - \langle X_{12}, X_{12}\rangle &= \langle X_1, X_{22}\rangle_1 - \langle X_1, X_{12}\rangle_2 \\ &= (g_{21})_{21} - \frac{1}{2}(g_{11})_{21} - \frac{1}{2}(g_{11})_2. \end{aligned}$$

Substituting the above values in the previous matrices, we define

$$\text{Briochi}[g_{11}, g_{22}, g_{33}] :=$$

$$\frac{1}{(\det(g_{ij}))^2}\left(\det \begin{pmatrix} (g_{21})_{21} - \frac{1}{2}(g_{11})_{21} - \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{11})_1 & \frac{1}{2}(g_{11})_2 \\ (g_{21})_2 - \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{21} & g_{22} \end{pmatrix} - \det \begin{pmatrix} 0 & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{21} & g_{22} \end{pmatrix} \right).$$

Evaluating the above expression at $(0,0)$ yields that Gaussian curvature $K(p)$. $\square$

Theorem 12. *If $f\colon M \to \overline{M}$ is an isometry, then $\overline{K}(f(p)) = K(p)$, where K and $\overline{K}$ denote the Gaussian curvatures of M and $\overline{M}$ respectively.*

Proof. Let $X\colon U \to M$ be a chart centered at p, then $\overline{X} := f \circ X$ is a chart of $\overline{M}$ centered at $f(p)$. Let g_{ij} and $\overline{g}_{ij}$ denote the coefficients of the first fundemental form with respect to the chartst X and $\overline{X}$ respectively. Then, using the previous two lemmas, we have

$$\begin{aligned} \overline{K}(f(p)) &= \text{Briochi}[\overline{g}_{11}, \overline{g}_{12}, \overline{g}_{22}](0,0) \\ &= \text{Briochi}[g_{11}, g_{12}, g_{22}](0,0) \\ &= K(p). \end{aligned}$$

$\square$

Exercise 13. Let $M \subset \mathbf{R}^3$ be a regular embedded surface and $p \in M$. Suppose that $K(p) \neq 0$. Does there exist a chart $X\colon U \to M$ such that D_1X and D_2X are orthonormal at all points of U.

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Lecture Notes 12

2.6 Gauss's formulas, and Christoffel Symbols

Let $X\colon U \to \mathbf{R}^3$ be a proper regular patch for a surface M, and set $X_i := D_iX$. Then

$$\{X_1, X_2, N\}$$

may be regarded as a *moving bases* of *frame* for $\mathbf{R}^3$ similar to the Frenet Serret frames for curves. We should emphasize, however, two important differences: (i) there is no canonical choice of a moving bases for a surface or a piece of surface ($\{X_1, X_2, N\}$ depends on the choice of the chart X); (ii) in general it is not possible to choose a patch X so that $\{X_1, X_2, N\}$ is orthonormal (unless the Gaussian curvature of M vanishes everywhere).

The following equations, the first of which is known as *Gauss's formulas*, may be regarded as the analog of Frenet-Serret formulas for surfaces:

$$X_{ij} = \sum_{k=1}^{2} \Gamma_{ij}^k X_k + l_{ij} N, \qquad \text{and} \qquad N_i = -\sum_{j=1}^{2} l_i^j X_j.$$

The coefficients Γ_{ij}^k are known as the *Christoffel symbols*, and will be determined below. Recall that l_{ij} are just the coefficients of the second fundamental form. To find out what l_i^j are note that

$$-l_{ik} = -\langle N, X_{ik}\rangle = \langle N_i, X_k\rangle = -\sum_{j=1}^{2} l_i^j \langle X_j, X_k\rangle = -\sum_{j=1}^{2} l_i^j g_{jk}.$$

Thus $(l_{ij}) = (l_i^j)(g_{ij})$. So if we let $(g^{ij}) := (g_{ij})^{-1}$, then $(l_i^j) = (l_{ij})(g^{ij})$, which yields

$$l_i^j = \sum_{k=1}^{2} l_{ik} g^{kj}.$$

[1]Last revised: November 12, 2004

Exercise 1. What is $\det(l_i^j)$ equal to?.

Exercise 2. Show that $N_i = -dn(X_i) = S(X_i)$.

Next we compute the Christoffel symbols. To this end note that

$$\langle X_{ij}, X_k\rangle = \sum_{l=1}^{2} \Gamma_{ij}^l \langle X_l, X_k\rangle = \sum_{l=1}^{2} \Gamma_{ij}^l g_{lk},$$

which in matrix notation reads

$$\begin{pmatrix} \langle X_{ij}, X_1\rangle \\ \langle X_{ij}, X_2\rangle \end{pmatrix} = \begin{pmatrix} \Gamma_{ij}^1 g_{11} + \Gamma_{ij}^2 g_{21} \\ \Gamma_{ij}^2 g_{12} + \Gamma_{ij}^2 g_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{pmatrix}.$$

So

$$\begin{pmatrix} \Gamma_{ij}^1 \\ \Gamma_{ij}^2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \langle X_{ij}, X_1\rangle \\ \langle X_{ij}, X_2\rangle \end{pmatrix} = \begin{pmatrix} g^{11} & g^{21} \\ g^{12} & g^{22} \end{pmatrix} \begin{pmatrix} \langle X_{ij}, X_1\rangle \\ \langle X_{ij}, X_2\rangle \end{pmatrix},$$

which yields

$$\Gamma_{ij}^k = \sum_{l=1}^{2} \langle X_{ij}, X_l\rangle g^{lk}.$$

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$. Next note that

$$\begin{aligned} (g_{ij})_k &= \langle X_{ik}, X_j\rangle + \langle X_i, X_{jk}\rangle, \\ (g_{jk})_i &= \langle X_{ji}, X_k\rangle + \langle X_j, X_{ki}\rangle, \\ (g_{ki})_j &= \langle X_{kj}, X_i\rangle + \langle X_k, X_{ij}\rangle. \end{aligned}$$

Thus

$$\langle X_{ij}, X_k\rangle = \frac{1}{2}\big((g_{ki})_j + (g_{jk})_i - (g_{ij})_k\big).$$

So we conclude that

$$\Gamma_{ij}^k = \sum_{l=1}^{2} \frac{1}{2}\big((g_{li})_j + (g_{jl})_i - (g_{ij})_l\big) g^{lk}.$$

Note that the last equation shows that Γ_{ij}^k are *intrinsic quantities*, i.e., they depend only on g_{ij} (and derivatives of g_{ij}), and so are preserved under isometries.

Exercise 3. Compute the Christoffel symbols of a surface of revolution.

2.7 The Gauss and Codazzi-Mainardi Equations, Riemann Curvature Tensor, and a Second Proof of Gauss's Theorema Egregium

Here we shall derive some relations between l_{ij} and g_{ij}. Our point of departure is the simple observation that if $X\colon U \to \mathbf{R}^3$ is a C^3 regular patch, then, since partial derivatives commute,

$$X_{ijk} = X_{ikj}.$$

Note that

$$\begin{aligned}
X_{ijk} &= \Big(\sum_{l=1}^{2} \Gamma_{ij}^{l} X_l + l_{ij} N\Big)_k \\
&= \sum_{l=1}^{2} (\Gamma_{ij}^{l})_k X_l + \sum_{l=1}^{2} \Gamma_{ij}^{l} X_{lk} + (l_{ij})_k N + l_{ij} N_k \\
&= \sum_{l=1}^{2} (\Gamma_{ij}^{l})_k X_l + \sum_{l=1}^{2} \Gamma_{ij}^{l} \Big(\sum_{m=1}^{2} \Gamma_{lk}^{m} X_m + l_{lk} N\Big) + (l_{ij})_k N - l_{ij} \sum_{l=1}^{2} l_k^l X_l \\
&= \sum_{l=1}^{2} (\Gamma_{ij}^{l})_k X_l + \sum_{l=1}^{2}\sum_{m=1}^{2} \Gamma_{ij}^{l}\Gamma_{lk}^{m} X_m + \sum_{l=1}^{2} \Gamma_{ij}^{l} l_{lk} N + (l_{ij})_k N - \sum_{l=1}^{2} l_{ij} l_k^l X_l \\
&= \sum_{l=1}^{2} \Big((\Gamma_{ij}^{l})_k + \sum_{p=1}^{2} \Gamma_{ij}^{p}\Gamma_{pk}^{l} - l_{ij} l_k^l\Big) X_l + \Big(\sum_{l=1}^{2} \Gamma_{ij}^{l} l_{lk} + (l_{ij})_k\Big) N.
\end{aligned}$$

Switching k and j yields,

$$X_{ikj} = \sum_{l=1}^{2} \Big((\Gamma_{ik}^{l})_j + \sum_{p=1}^{2} \Gamma_{ik}^{p}\Gamma_{pj}^{l} - l_{ik} l_j^l\Big) X_l + \Big(\sum_{l=1}^{2} \Gamma_{ik}^{l} l_{lj} + (l_{ik})_j\Big) N.$$

Setting the normal and tangential components of the last two equations equal to each other we obtain

$$(\Gamma_{ij}^{l})_k + \sum_{p=1}^{2} \Gamma_{ij}^{p}\Gamma_{pk}^{l} - l_{ij} l_k^l = (\Gamma_{ik}^{l})_j + \sum_{p=1}^{2} \Gamma_{ik}^{p}\Gamma_{pj}^{l} - l_{ik} l_j^l,$$

$$\sum_{l=1}^{2} \Gamma_{ij}^{l} l_{lk} + (l_{ij})_k = \sum_{l=1}^{2} \Gamma_{ik}^{l} l_{lj} + (l_{ik})_j.$$

These equations may be rewritten as

$$(\Gamma^l_{ik})_j - (\Gamma^l_{ij})_k + \sum_{p=1}^{2} \left(\Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}\right) = l_{ik}l^l_j - l_{ij}l^l_k, \quad \text{(Gauss)}$$

$$\sum_{l=1}^{2} \left(\Gamma^l_{ik}l_{lj} - \Gamma^l_{ij}l_{lk}\right) = (l_{ij})_k - (l_{ik})_j, \quad \text{(Codazzi-Mainardi)}$$

and are known as the *Gauss's equations* and the *Codazzi-Mainardi equations* respectively. If we define the *Riemann curvature tensor* as

$$R^l_{ijk} := (\Gamma^l_{ik})_j - (\Gamma^l_{ij})_k + \sum_{p=1}^{2} \left(\Gamma^p_{ik}\Gamma^l_{pj} - \Gamma^p_{ij}\Gamma^l_{pk}\right),$$

then Gauss's equation may be rewritten as

$$R^l_{ijk} = l_{ik}l^l_j - l_{ij}l^l_k.$$

Now note that

$$\sum_{l=1}^{2} R^l_{ijk}g_{lm} = l_{ik}\sum_{l=1}^{2} l^l_j g_{lm} - l_{ij}\sum_{l=1}^{2} l^l_k g_{lm} = l_{ik}l_{jm} - l_{ij}l_{km}.$$

In particular, if $i = k = 1$ and $j = m = 2$, then

$$\sum_{l=1}^{2} R^l_{121}g_{l2} = l_{11}l_{22} - l_{12}l_{21} = \det(l_{ij}) = K\det(g_{ij}).$$

So it follows that

$$K = \frac{R^1_{121}g_{12} + R^2_{121}g_{22}}{\det(g_{ij})},$$

which shows that K is intrinsic and gives another proof of Gauss's Theorema Egregium.

Exercise 4. Show that if $M = \mathbf{R}^2$, hen $R^l_{ijk} = 0$ for all $1 \leq i, l, j, k \leq 2$ both intrinsically and extrinsically.

Exercise 5. Show that (i) $R^l_{ijk} = -R^l_{ikj}$, hence $R^l_{ijj} = 0$, and (ii) $R^l_{ijk} + R^l_{jki} + R^l_{kij} \equiv 0$.

Exercise 6. Compute the Riemann curvature tensor for $\mathbf{S}^2$ both intrinsically and extrinsically.

2.8 Fundamental Theorem of Surfaces

In the previous section we showed that if g_{ij} and l_{ij} are the coefficients of the first and second fundamental form of a patch $X\colon U \to M$, then they must satisfy the Gauss and Codazzi-Maindardi equations. These conditions turn out to be not only necessary but also sufficient in the following sense.

Theorem 7 (Fundamental Theorem of Surfaces). *Let $U \subset \mathbf{R}^2$ be an open neighborhood of the origin $(0,0)$, and $g_{ij}\colon U \to \mathbf{R}$, $l_{ij}\colon U \to \mathbf{R}$ be differentiable functions for i, $j = 1, 2$. Suppose that $g_{ij} = g_{ji}$, $l_{ij} = l_{ji}$, $g_{11} > 0$, $g_{22} > 0$ and $\det(g_{ij}) > 0$. Further suppose that g_{ij} and l_{ij} satisfy the Gauss and Codazzi-Mainardi equations. Then there exists and open set $V \subset U$, with $(0,0) \in V$ and a regular patch $X\colon V \to \mathbf{R}$ with g_{ij} and l_{ij} as its first and second fundamental forms respectively. Further, if $Y : V \to \mathbf{R}^3$ is another regular patch with first and second fundamental forms g_{ij} and l_{ij}, then Y differs from X by a rigid motion.*

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2.9 The Covariant Derivative, Lie Bracket, and Riemann Curvature Tensor of $\mathbf{R}^n$

Let $A \subset \mathbf{R}^n$, $p \in A$, and W be a *tangent vector* of A at p, i.e., suppose there exists a curve $\gamma\colon (-\epsilon, \epsilon) \to A$ with $\gamma(0) = p$ and $\gamma'(0) = W$. Then if $f\colon A \to \mathbf{R}$ is a function we define the (directional) derivative of f with respect to W at p as

$$W_p f := (f \circ \gamma)'(0) = df_p(W).$$

Similarly, if V is a *vectorfield* along A, i.e., a mapping $V\colon A \to \mathbf{R}^n$, $p \mapsto V_p$, we define the *covariant derivative* of V with respect to W at p as

$$\overline{\nabla}_{W_p} V := (V \circ \gamma)'(0) = dV_p(W).$$

Note that if f and V are C^1, then by definition they may be extended to an open neighborhood of A. So df_p and dV_p, and consequently $W_p f$ and $\overline{\nabla}_{W_p} V$ are well defined. In particular, they do not depend on the choice of the curve γ or the extensions of f and V.

Exercise 1. Let E_i be the standard basis of $\mathbf{R}^n$, i.e., $E_1 := (1, 0, \dots, 0)$, $E_2 := (0, 1, 0, \dots, 0), \dots, E_n := (0, \dots, 0, 1)$. Show that for any functions $f\colon \mathbf{R}^n \to \mathbf{R}$ and vectorfield $V\colon \mathbf{R}^n \to \mathbf{R}^n$

$$(E_i)_p f = D_i f(p) \quad \text{and} \quad \overline{\nabla}_{(E_i)_p} V = D_i V(p)$$

(*Hint:* Let $u_i\colon (-\epsilon, \epsilon) \to \mathbf{R}^n$ be given by $u_i(t) := p + tE_i$, and observe that $(E_i)_p f = (f \circ u_i)'(0)$, $\overline{\nabla}_{(E_i)_p} V = (V \circ u_i)'(0)$).

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The operation $\overline{\nabla}$ is also known as the standard *Levi-Civita* connection of $\mathbf{R}^n$. If W is a tangent vectorfield of A, i.e., a mapping $W\colon A \to \mathbf{R}^n$ such that W_p is a tangent vector of A for all $p \in A$, then we set

$$Wf(p) := W_p f \quad \text{and} \quad (\overline{\nabla}_W V)_p := \overline{\nabla}_{W_p} V.$$

Note that $Wf\colon A \to \mathbf{R}$ is a function and $\overline{\nabla}_W V$ is a vectorfield. Further, we define

$$(fW)_p := f(p)W_p.$$

Thus $fW\colon A \to \mathbf{R}^n$ is a also a vector field.

Exercise 2. Show that it $V = (V^1, \dots, V^n)$, i.e., V^i are the component functions of V, then

$$\overline{\nabla}_W V = (WV^1, \dots, WV^n).$$

Exercise 3. Show that if Z is a tangent vectorfield of A and $f\colon A \to \mathbf{R}$ is a function, then

$$\overline{\nabla}_{W+Z} V = \overline{\nabla}_W V + \overline{\nabla}_Z V, \quad \text{and} \quad \overline{\nabla}_{fW} V = f\overline{\nabla}_W V.$$

Further if $Z\colon A \to \mathbf{R}^n$ is any vectorfield, then

$$\overline{\nabla}_W (V + Z) = \overline{\nabla}_W V + \overline{\nabla}_W Z, \quad \text{and} \quad \overline{\nabla}_W (fV) = (Wf)V + f\overline{\nabla}_W V.$$

Exercise 4. Note that if V and W are a pair of vectorfields on A then $\langle V, W\rangle\colon A \to \mathbf{R}$ defined by $\langle V, W\rangle_p := \langle V_p, W_p\rangle$ is a function on A, and show that

$$Z\langle V, W\rangle = \big\langle \overline{\nabla}_Z V, W\big\rangle + \big\langle V, \overline{\nabla}_Z W\big\rangle.$$

If $V, W\colon A \to \mathbf{R}^n$ are a pair of vector fields, then their *Lie bracket* is the vector filed on A defined by

$$[V, W]_p := \overline{\nabla}_{V_p} W - \overline{\nabla}_{W_p} V.$$

Exercise 5. Show that if $A \subset \mathbf{R}^n$ is open, $V, W\colon A \to \mathbf{R}^n$ are a pair of vector fields and $f\colon A \to \mathbf{R}$ is a scalar, then

$$[V, W]f = V(Wf) - W(Vf).$$

(*Hint:* First show that $Vf = \langle V, \operatorname{grad} f\rangle$ and $Wf = \langle W, \operatorname{grad} f\rangle$ where

$$\operatorname{grad} f := (D_1 f, \dots, D_n f).$$

Next define

$$\operatorname{Hess} f(V, W) := \langle V, \nabla_W \operatorname{grad} f\rangle,$$

and show that $\operatorname{Hess} f(V, W) = \operatorname{Hess} f(W, V)$. In particular, it is enough to show that $\operatorname{Hess} f(E_i, E_j) = D_{ij} f$, where $\{E_1, \dots, E_n\}$ is the standard basis for $\mathbf{R}^n$. Then Leibnitz rule yields that

$$\begin{aligned} &V(Wf) - W(Vf) \\ &= V\langle W, \operatorname{grad} f\rangle - W\langle V, \operatorname{grad} f\rangle \\ &= \langle \nabla_V W, \operatorname{grad} f\rangle + \langle W, \nabla_V \operatorname{grad} f\rangle - \langle \nabla_W V, \operatorname{grad} f\rangle - \langle V, \nabla_W \operatorname{grad} f\rangle \\ &= \langle [V, W], \operatorname{grad} f\rangle + \operatorname{Hess} f(W, V) - \operatorname{Hess} f(V, W) \\ &= [V, W]f, \end{aligned}$$

as desired.)

If V and W are tangent vectorfields on an open set $A \subset \mathbf{R}^n$, and $Z\colon A \to \mathbf{R}^n$ is any vectorfield, then

$$\overline{R}(V, W)Z := \overline{\nabla}_V \overline{\nabla}_W Z - \overline{\nabla}_W \overline{\nabla}_V Z - \overline{\nabla}_{[V,W]} Z$$

defines a vectorfield on A. If Y is another vectorfield on A, then we may also define an associated scalar quantity by

$$\overline{R}(V, W, Z, Y) := \big\langle \overline{R}(V, W)Z, Y \big\rangle,$$

which is known as the *Riemann curvature tensor* of $\mathbf{R}^n$.

Exercise 6. Show that $\overline{R} \equiv 0$.

2.10 The Induced Covariant Derivative on Surfaces; Gauss's Formulas revisited

Note that if $M \subset \mathbf{R}^3$ is a regular embedded surface and $V, W\colon M \to \mathbf{R}^3$ are vectorfields on M. Then $\overline{\nabla}_W V$ may no longer be tangent to M. Rather, in general we have

$$\overline{\nabla}_W V = \big(\overline{\nabla}_W V\big)^{\top} + \big(\overline{\nabla}_W V\big)^{\perp},$$

where $(\overline{\nabla}_W V)^\top$ and $(\overline{\nabla}_W V)^\perp$ respectively denote the tangential and normal components of $\overline{\nabla}_W V$ with resect to M. More explicitly, if for each $p \in M$ we let $n(p)$ be a unit normal vector to T_pM, then

$$\big(\overline{\nabla}_W V\big)_p^\perp := \langle \overline{\nabla}_{W_p} V, n(p)\rangle n(p) \quad \text{and} \quad \big(\overline{\nabla}_W V\big)^\top := \overline{\nabla}_W V - \big(\overline{\nabla}_W V\big)^\perp.$$

Let $\mathcal{X}(M)$ denote the space of tangent vectorfield on M. Then We define the *(induced) covariant derivative* on M as the mapping $\nabla\colon \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$\nabla_W V := \big(\overline{\nabla}_W V\big)^\top.$$

Exercise 7. Show that, with respect to tangent vectorfields on M, ∇ satisfies all the properties which were listed for $\overline{\nabla}$ in Exercises 3 and 4.

Next we derive an explicit expression for ∇ in terms of local coordinates. Let $X\colon U \to M$ be a proper regular patch centered at a point $p \in M$, i.e., $X(0,0) = p$, and set

$$\overline{X}_i := X_i \circ X^{-1}.$$

Then $\overline{X}_i$ are vectorfields on $X(U)$, and for each $q \in X(U)$, $(\overline{X}_i)_q$ forms a basis for T_qM. Thus on $X(U)$ we have

$$V = \sum_i V^i \overline{X}_i, \quad \text{and} \quad W = \sum_i W^i \overline{X}_i$$

for some functions V^i, $W^i\colon X(U) \to \mathbf{R}$. Consequently, on $X(U)$,

$$\begin{aligned}
\nabla_W V &= \nabla_{(\sum_j W^j \overline{X}_j)}\Big(\sum_i V^i \overline{X}_i\Big) \\
&= \sum_j \bigg(W^j \nabla_{\overline{X}_j}\Big(\sum_i V^i \overline{X}_i\Big)\bigg) \\
&= \sum_j \bigg(W^j \sum_i \Big(\overline{X}_j V^i + V^i \nabla_{\overline{X}_j} \overline{X}_i\Big)\bigg) \\
&= \sum_j \sum_i \Big(W^j(\overline{X}_j V^i) + W^j V^i \nabla_{\overline{X}_j} \overline{X}_i\Big).
\end{aligned}$$

Next note that if we define $u_j\colon (-\epsilon, \epsilon) \to \mathbf{R}^2$ by $u_j(t) := tE_j$, where $E_1 := (1,0)$ and $E_2 := (0,1)$. Then $X \circ u_i\colon (-\epsilon, \epsilon) \to M$ are curves with $X \circ u_i(0) =$

p and $(X \circ u_i)'(0) = X_i(0,0) = \overline{X}_i(p)$. Thus by the definitions of ∇ and $\overline{\nabla}$ we have

$$\begin{aligned}
\nabla_{(\overline{X}_j)_p}\overline{X}_i &= \Big(\overline{\nabla}_{(\overline{X}_j)_p}\overline{X}_i\Big)^{\top} \\
&= \Big(\big(\overline{X}_i \circ (X \circ u_j)\big)'(0)\Big)^{\top} \\
&= \Big((X_i \circ u_j)'(0)\Big)^{\top}
\end{aligned}$$

Now note that, by the chain rule,

$$(X_i \circ u_j)'(0) = DX_i\big(u_j(0)\big)Du_j(0) = X_{ij}(0,0).$$

Exercise 8. Verify the last equality above.

Thus, by Gauss's formula,

$$\begin{aligned}
\nabla_{(\overline{X}_j)_p}\overline{X}_i &= \Big(X_{ij}(0,0)\Big)^{\top} \\
&= \Big(\sum_k \Gamma_{ij}^k(0,0)X_k(0,0) + l_{ij}(0,0)N(0,0)\Big)^{\top} \\
&= \sum_k \Gamma_{ij}^k\big(X^{-1}(p)\big)X_k\big(X^{-1}(p)\big) \\
&= \sum_k \Gamma_{ij}^k\big(X^{-1}(p)\big)\big(\overline{X}_k\big)_p.
\end{aligned}$$

In particular if we set $\overline{X}_{ij} := X_{ij} \circ X^{-1}$ and define $\overline{\Gamma}_{ij}^k \colon X(U) \to \mathbf{R}$ by $\overline{\Gamma}_{ij}^k := \Gamma_{ij}^k \circ X^{-1}$, then we have

$$\nabla_{\overline{X}_j}\overline{X}_i = \big(\overline{X}_{ij}\big)^{\top} = \sum_k \overline{\Gamma}_{ij}^k \overline{X}_k,$$

which in turn yields

$$\nabla_W V = \sum_j \sum_i \Big(W^j \overline{X}_j V^i + W^j V^i \sum_k \overline{\Gamma}_{ij}^k \overline{X}_k\Big).$$

Now recall that Γ_{ij}^k depends only on the coefficients of the first fundamental form g_{ij}. Thus it follows that ∇ is intrinsic:

Exercise 9. Show that if $f\colon M \to \widetilde{M}$ is an isometry, then

$$\widetilde{\nabla}_{df(W)} df(V) = df\big(\nabla_W V\big),$$

where $\widetilde{\nabla}$ denotes the covariant derivative on $\widetilde{M}$ (*Hint:* It is enough to show that $\langle \widetilde{\nabla}_{df(\overline{X}_i)} df(\overline{X}_j), df(\overline{X}_l)\rangle = \langle df\big(\nabla_{\overline{X}_i}\overline{X}_j\big), df(\overline{X}_l)\rangle$).

Next note that if $n\colon X(U) \to \mathbf{S}^2$ is a local Gauss map then

$$\langle \nabla_W V, n\rangle = -\langle V, \nabla_W n\rangle = -\langle V, dn(W)\rangle = \langle V, S(W)\rangle,$$

where, recall that, S is the shape operator of M. Thus

$$\big(\overline{\nabla}_{W_p} V\big)^{\perp} = \langle V, S(W_p)\rangle n(p),$$

which in turn yields

$$\overline{\nabla}_W V = \nabla_W V + \big\langle V, S(W)\big\rangle n.$$

This is Gauss's formula and implies the expression that we had derived earlier in local coordinates.

Exercise 10. Verify the last sentence.

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2.11 The Induced Lie Bracket on Surfaces; The Self-Adjointness of the Shape Operator Revisited

If V, W are tangent vectorfields on M, then we define

$$[V, W]_M := \nabla_V W - \nabla_W V,$$

which is again a tangent vector field on M. Note that since, as we had verified in an earlier exercise, S is self-adjoint, the Gauss's formula yields that

$$\begin{aligned}
[V, W] &= \overline{\nabla}_V W - \overline{\nabla}_W V \\
&= \nabla_W V - \nabla_V W + \Big(\langle V, S(W)\rangle - \langle W, S(V)\rangle\Big)n \\
&= [V, W]_M.
\end{aligned}$$

In particular if V and W are tangent vectorfields on M, then $[V, W]$ is also a tangent vectorfield.

Let us also recall here, for the sake of completeness, the proof of the self-adjointness of S. To this end it suffices to show that if E_i, $i = 1$, 2, is a basis for T_pM, then $\langle E_i, S_p(E_j)\rangle = \langle S_p(E_i), E_j\rangle$. In particular we may let $E_i = X_i(0, 0)$, where $X : U \to M$ is a regular patch of M centered at p. Now note that

$$\big\langle X_i, S_p(X_j)\big\rangle = -\big\langle X_i, dn_p(X_j)\big\rangle = -\big\langle X_i, (n \circ X)_j\big\rangle = \big\langle X_{ij}, (n \circ X)\big\rangle.$$

Since the right hand side of the above expression is symmetric with respect to i and j, the right hand side must be symmetric as well, which completes the proof that S is self-adjoint.

Note that while the above proof is short and elegant one might object to it on the ground that it uses local coordinates. On the other hand, if we can

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give an independent proof that $[V,W]_M = [V,W]$, then we would have an alternative proof that S is self-adjoint. To this end note that

$$[V,W]^\top = \big(\overline{\nabla}_V W\big)^\top - \big(\overline{\nabla}_W V\big)^\top = \nabla_V W - \nabla_W V = [V,W]_M.$$

Thus to prove that $[V,W]_M = [V,W]$ it is enough to show that $[V,W]^\top = [V,W]$, i.e., $[V,W]$ is tangent to M. To see this note that if $f\colon M \to \mathbf{R}$ is any function, and $\overline{f}\colon U \to \mathbf{R}$ denoted an extension of f to an open neighborhood of M, then

$$[V,W]\overline{f} = [V,W]^\top \overline{f} + [V,W]^\perp \overline{f} = [V,W]^\top f + [V,W]^\perp \overline{f}.$$

So if we can show that the left hand side of the above expression depends only on f (not $\overline{f}$), then it would follow that the right hand side must also be independent of $\overline{f}$, which can happen only if $[V,W]^\perp$ vanishes. Hence it remains to show that $[V,W]\overline{f} = [V,W]f$. To see this recall that by a previous exercise

$$[V,W]\overline{f} = V(W\overline{f}) - W(V\overline{f}).$$

But since V and W are tangent to M, $V\overline{f} = Vf$ and $W\overline{f} = Wf$. Thus the right hand side of the above equality depends only on f, which completes the proof.

Exercise 1. Verifythe next to last statement.

2.12 The Riemann Curvature Tensor of Surfaces; The Gauss and Codazzi Mainardi Equations, and Theorema Egregium Revisited

If V, W, Z are tangent vectorfields on M, then

$$R(V,W)Z := \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V,W]} Z$$

gives a tangent vectorfield on M. Note that this operation is well defined, because, as we verified in the previous section, $[V,W]$ is tangent to M. If Y is another tangent vectorfield on M, then we may also define an associated scalar quantity by

$$R(V,W,Z,Y) := \big\langle R(V,W)Z, Y\big\rangle,$$

which is the *Riemann curvature tensor* of M, and, as we show below, coincides with the quantity of the same name which we had defined earlier in terms of local coordinates. To this end fitst recall that

$$\overline{R}(V,W)Z := \overline{\nabla}_V\overline{\nabla}_W Z - \overline{\nabla}_W\overline{\nabla}_V Z - \overline{\nabla}_{[V,W]}Z = 0$$

as we had shown in an earlier exercise. Next note that, by Gauss's formula,

$$\begin{aligned}
\overline{\nabla}_V\overline{\nabla}_W Z &= \overline{\nabla}_V\big(\nabla_W Z + \langle S(W), Z\rangle n\big) \\
&= \overline{\nabla}_V\big(\nabla_W Z\big) + \overline{\nabla}_V\big(\langle S(W), Z\rangle n\big) \\
&= \nabla_V\nabla_W Z + \langle S(V), \nabla_W Z\rangle n + V\langle S(W), Z\rangle n + \langle S(W), Z\rangle \nabla_V n.
\end{aligned}$$

Also recal that, since $\langle n, n\rangle = 1$,

$$\nabla_V n := (\overline{\nabla}_V n)^\top = \overline{\nabla}_V n = dn(V) = S(V).$$

Thus

$$\begin{aligned}
\overline{\nabla}_V\overline{\nabla}_W Z = \nabla_V\nabla_W Z &+ \langle S(W), Z\rangle S(V) \\
&+ \Big(\langle S(V), \nabla_W Z\rangle + \langle \nabla_V S(W), Z\rangle + \langle S(W), \nabla_V Z\rangle\Big) n.
\end{aligned}$$

Simlilarly,

$$\begin{aligned}
-\overline{\nabla}_W\overline{\nabla}_V Z = -\nabla_W\nabla_V Z &- \langle S(V), Z\rangle S(W) \\
&- \Big(\langle S(W), \nabla_V Z\rangle + \langle \nabla_W S(V), Z\rangle + \langle S(V), \nabla_W Z\rangle\Big) n.
\end{aligned}$$

Also note that

$$-\overline{\nabla}_{[V,W]}Z = -\nabla_{[V,W]}Z - \langle S([V,W]), Z\rangle n.$$

Adding the last three equations yield

$$\begin{aligned}
\overline{R}(V,W)Z = R(V,W)Z &+ \langle S(W), Z\rangle S(V) - \langle S(V), Z\rangle S(W) \\
&+ \Big(\langle \nabla_V S(W), Z\rangle - \langle \nabla_W S(V), Z\rangle - \langle S([V,W]), Z\rangle\Big) n.
\end{aligned}$$

Since the left hand side of the above equation is zero, each of the tangential and normal components of the right hand side must vanish as well. These respectively yield:

$$R(V,W)Z = \langle S(W), Z\rangle S(V) - \langle S(V), Z\rangle S(W)$$

and

$$\nabla_V S(W) - \nabla_W S(V) = S([V, W]),$$

which are the Gauss and Codazzi-Mainardi equations respectively. In particular, in local coordinates they take on the forms which we had derived earlier.

Exercise 2. Verify the last sentence above.

Finally note that by Gauss's equation

$$\langle R(V, W)W, V\rangle = \langle S(V), V\rangle\langle S(W), W\rangle - \langle S(W), V\rangle\langle S(V), W\rangle$$

In particular, if V and W are orthonormal, then

$$\langle R(V, W)W, V\rangle = \det(S) = K.$$

Thus we obtain yet another proof of the Theorema Egregium, which, in this latest reincarnation, does not use local coordinates.

Exercise 3. Show that if V and W are general vectorfields (not necessarily orthonormal), then

$$K = \frac{R(V, W, W, V)}{\|V \times W\|}$$

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2.13 The Geodesic Curvature

Let $\alpha\colon I \to M$ be a unit speed curve lying on a surface $M \subset \mathbf{R}^3$. Then the *absolute geodesic curvature* of α is defined as

$$|\kappa_g| := \big\|(\alpha'')^\top\big\| = \big\|\alpha'' - \big\langle \alpha'', n(\alpha)\big\rangle n(\alpha)\big\|,$$

where n is a local Gauss map of M in a neighborhood of $\alpha(t)$. In particular note that if $M = \mathbf{R}^2$, then $|\kappa_g| = \kappa$, i.e., absolute geodesic curvature of a curve on a surface is a gneralization of the curvature of curves in the plane.

Exercise 1. Show that the absolute geodesic curvature of great circles in a sphere and helices on a cylinder are zero.

Similarly, the *(signed) geodesic curvature* generalizes the notion of the signed curvature of planar curves and may be defined as follows.

We say that a surface $M \subset \mathbf{R}^3$ is *orientable* provided that there exists a (global) Gauss map $n\colon M \to \mathbf{S}^2$, i.e., a *continuous* mapping which satisfies $n(p) \in T_pM$, for all $p \in M$. Note that if n is a global Gauss map, then so is $-n$. In particular, any orientable surface admits precisely two choices for its global Gauss map. Once we choose a Gauss map n for an orientable surface, then M is said to be *oriented*.

If M is an oriented surface (with global Gauss map n), then, for every $p \in M$, we define a mapping $J\colon T_pM \to T_pM$ by

$$JV := n \times V.$$

Exercise 2. Show that if $M = \mathbf{R}^2$, and $n = (0, 0, 1)$, then J is clockwise rotation about the origin by $\pi/2$.

[1] Last revised: December 8, 2004

Then the *geodesic curvature* of a unit speed curve $\alpha\colon I \to M$ is given by

$$\kappa_g := \big\langle \alpha'', J\alpha' \big\rangle.$$

Note that, since $J\alpha'$ is tangent to M,

$$\big\langle \alpha'', J\alpha' \big\rangle = \big\langle (\alpha'')^\top, J\alpha' \big\rangle.$$

Further, since $\|\alpha'\| = 1$, α'' is orthogonal to α', which in turn yields that the projection of α'' into the tangent plane is either parallel or antiparallel to $J\alpha'$. Thus $\kappa_g > 0$ when the projection of α'' is parallel to $J\alpha'$ and is negative otherwise.

Note that if the curvature of α does not vanish (so that the principal normal N is well defined), then

$$\kappa_g = \kappa \big\langle N, JT \big\rangle.$$

In particular geodesic curvature is invariant under reparametrizations of α.

Exercise 3. Let $\mathbf{S}^2$ be oriented by its outward unit normal, i.e., $n(p) = p$, and compute the geodesic curvature of the circles in $\mathbf{S}^2$ which lie in planes $z = h$, $-1 < h < 1$. Assume that all these circles are oriented consistenly with respect to the rotation about the z-axis.

Next we derive an expression for κ_g which does not require that α have unit speed. To this end, let $s\colon I \to [0, L]$ be the arclength function of α, and recall that $\overline{\alpha} := \alpha \circ s^{-1}\colon [0, L] \to M$ has unit speed. Thus

$$\kappa_g = \overline{\kappa}_g(s) = \big\langle \overline{\alpha}''(s), J\alpha'(s) \big\rangle.$$

Now recall that $(s^{-1})' = 1/\|\alpha'\|$. Thus by chain rule.

$$\overline{\alpha}'(t) = \alpha'\big(s^{-1}(t)\big) \cdot \frac{1}{\|\alpha'(s^{-1}(t))\|}.$$

Further, differentiating both sides of the above equation yields

$$\overline{\alpha}'' = \alpha''\big(s^{-1}\big) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'\big(s^{-1}\big) \cdot \frac{-\Big\langle \alpha'\big(s^{-1}\big), \alpha\big(s^{-1}\big) \Big\rangle}{\|\alpha'(s^{-1})\|}.$$

Substituting these values into the last expression for $\overline{\kappa}_g$ above yiels

$$\kappa_g = \frac{\big\langle \alpha'', J\alpha' \big\rangle}{\|\alpha'\|^3}.$$

Exercise 4. Verify the last two equations.

Next we show that the geodesic curvature is intrinsic, i.e., it is invariant under isometries of the surface. To this end define $\tilde{\alpha}' \colon \alpha(I) \to \mathbf{R}^3$ be the vectorfiled along $\alpha(I)$ given by

$$\tilde{\alpha}'\big(\alpha(t)\big) = \alpha'(t).$$

Then one may immediatley check that

$$\alpha''(t) = \overline{\nabla}_{\alpha'(t)}\tilde{\alpha}'.$$

Thus

$$\big\langle \alpha'', J\alpha' \big\rangle = \big\langle (\alpha'')^\top, J\alpha' \big\rangle = \big\langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \big\rangle.$$

and it follows that

$$\kappa_g = \frac{\big\langle \nabla_{\alpha'}\tilde{\alpha}', J\alpha' \big\rangle}{\|\alpha'\|^3}.$$

Now recall that ∇ is intrinic. Thus to complete the proof that κ_g is irinsic it remains to show that J is intrinsic. To see this let $X \colon U \to M$ be a local patch, then

$$JX_i = \sum_{j=1}^{2} b_{ij} X_j.$$

In particular,

$$JX_1 = b_{11}X_1 + b_{12}X_2.$$

Now note that

$$0 = \langle JX_1, X_1 \rangle = b_{11}g_{11} + b_{12}g_{21}.$$

Further,

$$g_{11} = \langle X_1, X_1 \rangle = \langle JX_1, JX_1 \rangle = b_{11}^2 g_{11} + 2b_{11}b_{12}g_{12} + b_{12}^2 g_{22}.$$

Solving for b_{21} in the next to last equation, and substituting in the last equation yields

$$g_{11} = b_{11}^2 g_{11} - 2b_{11}^2 g_{11} + b_{11}^2 \frac{g_{11}^2}{g_{21}^2} g_{22} = b_{11}^2(-g_{11} + \frac{g_{11}^2}{g_{21}^2} g_{22}).$$

Thus b_{11} may be computed in terms of g_{ij} which in turn yiels that b_{12} may be computed in terms of g_{ij} as well. So JX_1 nay be expressed intrinsically. Similarly, JX_2 may be exressed intrinsically as well. So we conclude that J is intrinsic.

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2.14 Applications of the Gauss-Bonnet theorem

We talked about the Gauss-Bonnet theorem in class, and you may find the statement and prove of it in Gray or do Carmo as well. The following are all simple consequences of the Gauss-Bonnet theorem:

Exercise 1. Show that the sum of the angles in a triangle is π.

Exercise 2. Show that the total geodesic curvature of a simple closed planar curve is 2π.

Exercise 3. Show that the Gaussian curvature of a surface which is homeomorphic to the torus must alwasy be equal to zero at some point.

Exercise 4. Show that a simple closed curve with total geodesic curvature zero on a sphere bisects the area of the sphere.

Exercise 5. Show that there exists at most one closed geodesic on a cylinder with negative curvature.

Exercise 6. Show that the area of a geodesic polygon with k vertices on a sphere of radius 1 is equal to the sum of its angles minus $(k-2)\pi$.

Exercise 7. Let p be a point of a surface M, T be a geodesic triangle which contains p, and α, β, γ be the angles of T. Show that

$$K(p) = \lim_{T\to p} \frac{\alpha+\beta+\gamma-\pi}{Area(T)}.$$

In particular, note that the above proves Gauss's Theorema Egregium.

[1]Last revised: December 8, 2004

Exercise 8. Show that the sum of the angles of a geodesic triangle on a surface of positive curvature is more than π, and on a surface of negative curvature is less than π.

Exercise 9. Show that on a simply connected surface of negative curvature two geodesics emanating from the same point will never meet.

Exercise 10. Let M be a surface homeomorphic to a sphere in $\mathbf{R}^3$, and let $\Gamma \subset M$ be a closed geodesic. Show that each of the two regions bounded by Γ have equal areas under the Gauss map.

Exercise 11. Compute the area of the pseudo-sphere, i.e. the surface of revolution obtained by rotating a tractrix.