Notes on Differential Geometry

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May 12, 2003

Abstract

These notes present various concepts in differential geometry from the elegant and unifying point of view of principal bundles and their associated vector bundles.

Contents

Notation: Throughout we consistently use hats and overbars on symbols as follows: hats for 'raised' (i.e. lifted) maps of the base or 'raised' sections (equivariant functions on total space), overbars for 'projected' maps or local representatives of sections. The differential of a map f is denoted by f_*, its evaluation at $p \in M$ by f_{*p}, and its application to a tangent vector $v_p \in T_p(M)$ by $f_{*p}(v_p)$. If v is a vector field with $v(p) = v_p$, this expression is also written as $f_*v(p)$.

1 The principal bundle

Let M be an n-dimensional manifold which we think of as representing space-time. Most of the differential geometric structures on M can be understood as originating from the structures encoded in the various principal fiber bundles over M. On of the most important examples of a principal bundle over M is the bundle of linear frames, denoted by $L(M)$, where we associate to each point $x \in M$ the set of all bases (here called 'frames') of the tangent space $T_x(M)$. This set is in bijective correspondence to the group $GL(n, \mathbb{R})$ which acts simply transitively on the set of bases of any n-dimensional real vector space.

The explicit construction of a principal fiber bundle in terms of an open covering of M and transition functions on overlaps (which we call the coordinate construction) is e.g. described in chapter I, section 5 of the book by Kobayashi and Nomizu [5]. It should be familiar. For the convenience of the reader we present in a self contained fashion the details of this construction in the Appendix to these notes. Those not familiar with the notion of principal bundles should definitely read the Appendix first.

Here we only remind that the fiber bundle consists of the total space P, with a free action of a Lie group G, such that the quotient with respect to this group action is given by the base manifold M; the projection map is denoted by π. This structure is pictorially represented by the following arrangements of spaces and maps:

$$\begin{array}{ccc} P & \xleftarrow{\;i\;} & G \\ \Big\downarrow{\scriptstyle \pi} & & \\ M & & \end{array} \tag{1}$$

We write G's action on P on the right (which means that in the coordinate construction of P the transition functions act from the left):

$$G \times P \to P\,, \quad (g,p) \mapsto p \cdot g =: R_g(p) \tag{2}$$

$$\text{where} \quad \pi \circ R_g = \pi \quad \forall g \in G \tag{3}$$

The set $p \cdot G := \{p \cdot g \mid g \in G\}$ is called the fiber through p, and the projection map $\pi : P \to M$ collapses each G-orbit to a point in M.

1.1 Sections in P

A map $\sigma : M \supset U \to P$ obeying $\pi \circ \sigma = id|_U$ is called a *local section* of P (over U) if U is a proper subset. It is called a *global section* iff $U = M$. If a section over $U \subseteq M$ exists the set $\pi^{-1}(U)$ is diffeomorphic to the product $U \times G$; the diffeomorphism is given by

$$T_\sigma : U \times G \to \pi^{-1}(U), \qquad (x,g) \mapsto \sigma(x) \cdot g \tag{4}$$

This map is said to locally trivialize (in the sense of making it a product) P over U. By definition of a bundle, any $x \in M$ lies in an open U over which a local sections exist. In contrast, global sections exist iff P is diffeomorphic to $M \times G$, which is usually a very strong restriction.

Given two sections σ and σ' over U, they are necessarily related by

$$\sigma'(x) = \sigma(x) \cdot \beta(x) \,, \tag{5}$$

for some function $\beta : M \to G$. The two local trivializations are then related by

$$T_\sigma^{-1} \circ T_{\sigma'} : (x, g) \mapsto (x, \beta(x)g) \,. \tag{6}$$

1.2 Fundamental vector fields and vertical subspaces

The Lie algebra $\mathfrak{g}$ of G induces vector fields on P, the so-called *fundamental vector fields*, as follows: Let exp be the exponential map on G and $\xi \in \mathfrak{g}$; then the fundamental vector field, $\xi^\#$, corresponding to ξ, is defined through

$$\xi^\#(p) := \frac{d}{dt}\Big|_{t=0} (p \cdot \exp(t\xi)) \ . \tag{7}$$

From (7) it follows that the fundamental vector fields are ad-equivariant, with respect to right multiplication, in the following sense:

$$R_{g*}(\xi^\#(p)) = \left(ad(g^{-1})\xi\right)^\# (p \cdot g) \,. \tag{8}$$

Clearly, π_{*p} annihilates all $\xi^\#(p)$ and by a simple counting argument they actually span $\mathrm{kernel}(\pi_{*\mathrm{p}}) \subset \mathrm{T_p(P)}$ of π_{*p} at each point $p \in P$. We call

$$V_p := span\{\xi^\#(p) \subset T_p(P) \ | \ \xi \in \mathfrak{g}\} = \mathrm{kernel}(\pi_{*\mathrm{p}}) \tag{9}$$

the *vertical subspace* at p.

Since, by definition, $F_t^{\xi^\#} : p \mapsto R_{\exp(t\xi)}(p)$ is just the flow for the vector field $\xi^\#$, it is easy to calculate the commutator of two vertical vector fields by using (8):

$$\begin{aligned} [\xi^\#, \lambda^\#] \ &= \ \frac{d}{dt}\Big|_{t=0} F_{t*}^{\xi^\#} \lambda^\# = \frac{d}{dt}\Big|_{t=0} R_{\exp(-t\xi)\,*}\lambda^\# \\ &= \ \frac{d}{dt}\Big|_{t=0} [ad(\exp(-t\xi))(\lambda)]^\# \ = \ [\xi, \lambda]^\# \,, \end{aligned} \tag{10}$$

which means that the assignment $\xi \mapsto \xi^\#$ is a Lie homomorphism from the Lie algebra of G to the Lie algebra of vector fields on P

1.3 Bundle automorphisms and gauge transformations

A *bundle automorphism* of P is a diffeomorphism $f : P \to P$ such that $f(p \cdot g) = f(p) \cdot g$. Such diffeomorphisms form a subgroup of $\mathrm{Diff}(P)$ which we call $\mathrm{Aut}(P)$:

$$\mathrm{Aut}(P) := \{f \in \mathrm{Diff}(P) \mid f \circ R_g = R_g \circ f, \ \forall g \in G\} \,. \tag{11}$$

Equation (7) and the defining equation in (11) immediately imply that each fundamental vector field is invariant under bundle automorphisms, i.e.

$$f_{*p}(\xi^{\#}(p)) = \xi^{\#}(f(p)) \tag{12}$$

By definition, automorphisms map P fiberwise, i.e., any two points in the fiber $\pi^{-1}(x)$, $x \in M$, get mapped to another single fiber $\pi^{-1}(x')$, where $x' = \bar{f}(x)$ and the diffeomorphism $\bar{f} : M \to M$ is implicitly determined through $\pi \circ f = \bar{f} \circ \pi$. This can be conveniently expressed by the commutativity of the following diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & P \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\bar{f}]{} & M \end{array} \tag{13}$$

We say that f *covers* $\bar{f}$. Alternatively, given $\phi \in \mathrm{Diff}(M)$, we say that f_ϕ is a *lift of* ϕ in $\mathrm{Aut}(P)$ iff $f_\phi \in \mathrm{Aut}(P)$ and $\bar{f}_\phi = \phi$. Occasionally we also write $\hat{\phi}$ for f_ϕ. Clearly $\bar{f}$ is uniquely determined by f. Moreover, given two automorphisms f_1, f_2, their composition clearly satisfies $\pi \circ f_1 \circ f_2 = \bar{f}_1 \circ \bar{f}_2 \circ \pi$, which means that we have a group homomorphism

$$\mathfrak{P} : \mathrm{Aut}(P) \to \mathrm{Diff}(M), \quad f \mapsto \bar{f}\,. \tag{14}$$

A *gauge transformation* is a bundle automorphism f which covers the identity, i.e., for which $\bar{f} = id_M$. We define

$$\mathrm{Gau}(P) := \{f \in \mathrm{Aut}(P) \mid \pi \circ f = \pi\}\,, \tag{15}$$

which is called the *group of gauge transformations* (not to be confused with the gauge group G). Being the kernel of $\mathfrak{P}$ it is a normal subgroup of $\mathrm{Aut}(P)$. Each gauge transformation f uniquely determines a function $f_G : P \to G$ through $f(p) =: p \cdot f_G(p)$, satisfying $f_G(p \cdot g) = g^{-1} f_G(p) g$. Hence $\mathrm{Gau}(P)$ is isomorphic to the group of mappings $P \to G$, which are *Ad*-equivariant in the following sense:

$$f_G \circ R_g = Ad_{g^{-1}} \circ f_G \tag{16}$$

and whose group multiplication is just pointwise multiplication in G. We shall address f_G as the *G-form* of $f \in \mathrm{Gau}(P)$.

1.4 Lifts from $\mathrm{Diff}(M)$ to $\mathrm{Aut}(P)$

Since $\mathrm{Gau}(P) = \mathrm{kernel}(\mathfrak{P})$ we have $\mathrm{image}(\mathfrak{P}) \subseteq \mathrm{Diff(M)}$ isomorphic to $\mathrm{Aut}(P)/\mathrm{Gau}(P)$. A priori $\mathfrak{P}$ need not be surjective, which means that there possibly exist diffeomorphisms of M which do not lift to automorphisms of P. However, all elements in the identity component of $\mathrm{Diff}(M)$ do have a lift, and are therefore contained in $\mathrm{image}(\mathfrak{P})$, as we will show below (c.f. 1.6). But note that even if each element of some subgroup $D' \subseteq \mathrm{Diff}(M)$ lifts to $\mathrm{Aut}(P)$, the lifting map $D' \to \mathrm{Aut}(P)$ need (and generally will) not be a group homomorphism.

There are two particular examples for lifting prescriptions $\mathrm{Diff}(M) \to \mathrm{Aut}(P)$ which frequently occur in practice and which we therefore wish to describe separately and in more detail.

1.4.1 Push-forward lift to $L(M)$

As first example consider the case $P = L(M)$, where $L(M)$ denotes the bundle of linear frames over M. Let $\phi \in \mathrm{Diff}(M)$, then a lift $f_\phi \in \mathrm{Aut}(L(M))$ of ϕ is given by 'push-forward': $f_\phi(\{e_a\}) := \{\phi_* e_a\}$ (cf. formula (70)). This clearly satisfies $\bar{f}_\phi = \phi$ and $f_\phi \circ R_g = R_g \circ f_\phi$, where $R_g : \{e_a\} \mapsto \{e_b g^b_a\}$ for $g = (g^b_a) \in GL(n, \mathbb{R})$. Moreover, the map $\mathrm{Diff}(M) \to \mathrm{Aut}(L(M))$ so defined is in fact an injective group homomorphism. We may thus regard $\mathrm{Diff}(M)$ as subgroup of $\mathrm{Aut}(L(M))$ which intersects $\mathrm{Gau}(L(M))$ only in the group identity. This implies that $\mathrm{Aut}(L(M))$ is a semi-direct product of the following form:

$$\mathrm{Aut}(L(M)) = \mathrm{Gau}(P) \rtimes_\alpha \mathrm{Diff}(M) \tag{17}$$

$$\alpha : \mathrm{Diff}(M) \to \mathrm{Aut}(\mathrm{Gau}(L(M))), \quad \alpha(\phi)(f) := f_\phi \circ f \circ (f_\phi)^{-1}. \tag{18}$$

Note that if we consider a subbundle of $L(M)$, like e.g. the bundle $F(M, \eta)$ of orthonormal frames with respect to some (pseudo-) Riemannian structure η on M, then ϕ_* no longer acts on $F(M, \eta)$ unless ϕ is an isometry of η. Hence only the subgroup of isometries of η in $\mathrm{Diff}(M)$ can now be (homomorphically) lifted by the 'push-forward'-prescription.

1.4.2 Lifts by global sections

As second example consider the class of cases where the bundle (1) is trivial. Then a global section $\sigma : M \to P$ exists giving rise to a global diffeomorphism $T_\sigma : M \times G \to P$, $T_\sigma(m, g) := \sigma(m) \cdot g$. Given $\phi \in \mathrm{Diff}(M)$, a lift $f^{(\sigma)}_\phi \in \mathrm{Aut}(P)$ of ϕ is defined by

$$T^{-1}_\sigma \circ f^{(\sigma)}_\phi \circ T_\sigma(m, g) := (\phi(m), g) \tag{19}$$

or, equivalently,

$$f^{(\sigma)}_\phi(\sigma(m) \cdot g) := \sigma(\phi(m)) \cdot g\,. \tag{20}$$

This construction clearly depends on σ. To see how, consider another section σ'. Clearly there is a unique function $\beta : M \to G$ such that $\sigma'(m) = \sigma(m) \cdot \beta(m)$. An easy computation shows that the maps $f^{(\sigma')}_\phi$ and $f^{(\sigma)}_\phi$ are then related by a gauge transformation f through $f^{(\sigma')}_\phi = f \circ f^{(\sigma)}_\phi$, whose G-form is given by

$$f_G(\sigma(m) \cdot g) := g^{-1} \beta(m) [\beta(\phi^{-1}(m))]^{-1} g\,, \tag{21}$$

which indeed defines a gauge transformation since f_G satisfies (16). Hence global sections define a map $\mathrm{Diff}(M) \to \mathrm{Aut}(P)$ (not a group homomorphism in general!) which is unique up to the obvious freedom of composing each lift with elements in $\mathrm{Gau}(P)$.

1.5 Connections

A connection on P is a smooth assignment of a so-called *horizontal subspace* H_p of $T_p(P)$ for each p, which is complementary to V_p, and right-invariant, i.e. satisfying the two conditions

$$(i) \quad : \quad H_p \oplus V_p = T_p(P) \quad \forall p \in P\,, \tag{22}$$

$$(ii) \quad : \quad R_{g*} H_p = H_{p \cdot g}\,. \tag{23}$$

Smoothness means that locally the subspaces can always be spanned by n smooth vector fields. Such horizontal subspaces can be uniquely characterized as the annihilation spaces of a $\mathfrak{g}$-valued 1-form, ω, on P, satisfying

$$(i) \; : \; \omega(\xi^{\#}) \; = \; \xi\,, \tag{24}$$

$$(ii) \; : \; R_g^* \omega \quad = \; ad(g^{-1})\omega\,. \tag{25}$$

Vertical and horizontal projectors into each V_p and H_p are then given by

$$P_V(p) \; : \quad T_p(P) \to V_p, \quad X_p \mapsto (\omega(X_p))^{\#}(p)\,, \tag{26}$$

$$P_H(p) \; : \quad T_p(P) \to H_p, \quad X_p \mapsto X_p - (\omega(X_p))^{\#}(p)\,. \tag{27}$$

Also, given $X_m \in T_m(M)$, there is a unique horizontal lift, $X_p^H \in T_p(P)$, to each $p \in \pi^{-1}(m)$, in the sense that

$$(i) \; : \; \pi_{*p}(X_p^H) \; = \; X_m\,, \tag{28}$$

$$(ii) \; : \; \omega(X_p^H) \quad = \; 0\,, \tag{29}$$

$$(iii) \; : \; R_{g*}(X_p^H) \; = \; X_{p \cdot g}^H\,, \tag{30}$$

where the right invariance, (iii), is a consequence of (i) and (ii).

1.6 Parallel transportation in P

Given a curve $\gamma : [0,1] \to M$ with $\gamma(0) = m$, $\gamma(1) = m'$ and a point $p \in \pi^{-1}(m)$. There is a unique *horizontal lift*, $\gamma_p^H : [0,1] \to P$, of γ with $\gamma_p^H(0) = p$. Horizontality means that

$$\frac{d}{dt}\gamma_p^H(t) \in H_{\gamma_p^H(t)}, \quad \forall t \in [0,1]\,. \tag{31}$$

The end point, $\gamma_p^H(1) \in \pi^{-1}(m')$, is referred to as the result of *parallelly transporting* p along γ. The horizontal lift starting from $p \cdot g$ is given by

$$\gamma_{p \cdot g}^H = R_g \circ \gamma_p^H\,, \tag{32}$$

which is horizontal due to (23). This shows that the parallel-transport map $p \mapsto \gamma_p^H(1)$ from $\pi^{-1}(m)$ to $\pi^{-1}(m')$ is R_g-equivariant.

A case of special interest is when γ is closed (a loop), so that $\gamma_p^H(0)$ and $\gamma_p^H(1)$ are both in $\pi^{-1}(m)$. Then there exists a $\mathfrak{h}_p(\gamma) \in G$ such that $\gamma_p^H(1) = p \cdot \mathfrak{h}_p(\gamma)$. The set

$$\mathfrak{H}_p := \{\mathfrak{h}_p(\gamma) \mid \gamma = \text{loop at } \pi(p)\} \tag{33}$$

is a group, the *holonomy group* at $p \in P$. From (32) we infer that the elements of the holonomy group satisfy

$$\mathfrak{h}_{p\cdot g}(\gamma) = g^{-1}\mathfrak{h}_p(\gamma)g\,, \tag{34}$$

implying that the holonomy groups at different points in the same fiber are conjugate:

$$\mathfrak{H}_{p\cdot g} = g^{-1}\mathfrak{H}_p g\,, \tag{35}$$

Finally we show how one can use parallel transport to construct a lift $\hat{\phi} \in \mathrm{Aut}(P)$ to any diffeomorphism ϕ in the identity component of $\mathrm{Diff}(M)$. By hypothesis there exists a curve $[0,1] \ni t \mapsto \phi_t \in \mathrm{Diff}(M)$ (a so-called isotopy) such that $\phi_0 = id_M$ and $\phi_1 = \phi$. Under this isotopy each point $m \in M$ traces a curve $\gamma_m(t) := \phi_t(m)$ from m to $m' = \phi(m)$. Lift this curve to a horizontal curve $\gamma_p^H(t)$, starting at $p \in \pi^{-1}(m)$. Do this for all $p \in \pi^{-1}(m)$ and for all $m \in M$. Then define $\hat{\phi}(p) := \gamma_p^H(1)$. Clearly $\hat{\phi}(p) \in \pi^{-1}(m')$ and since $\gamma_{p\cdot g}^H(t) = \gamma_p^H(t)\cdot g$ we have $\hat{\phi}(p\cdot g) = \hat{\phi}(p)\cdot g$. Hence $\hat{\phi}$ is indeed a lift of ϕ to $\mathrm{Aut}(P)$.

2 Associated vector bundles

2.1 Definition

Given a vector space V and an action of G on V via some representation ρ:

$$G \times V \to V, \qquad (g,v) \mapsto \rho(g)v\,; \tag{36}$$

there is a free right action of G on $P \times V$:

$$G \times (P \times V) \to P \times V, \quad (g,(p,v)) \mapsto (p\cdot g, \rho(g^{-1})v)\,. \tag{37}$$

We can form the quotient

$$E(P,V,\rho) := (P \times V)/G\,, \tag{38}$$

whose elements we denote by $[p,v]$; clearly, $[p\cdot g, v] = [p, \rho(g)v]$. We shall usually omit to indicate the dependence of E on P, V and ρ, and simply write E. The projection map π of P onto M now induces a projection map:

$$\pi_E:\ E \to M, \qquad [p,v] \mapsto \pi_E([p,v]) := \pi(p)\,. \tag{39}$$

Each point $p \in \pi^{-1}(m)$ defines a map (also denoted by p):

$$p : V \to \pi_E^{-1}(m), \qquad v \mapsto [p, v], \tag{40}$$

which is a linear isomorphism between the vector spaces V and the fiber $\pi_E^{-1}(m)$ of E. In this sense the points $p \in \pi^{-1}(M)$ of P are 'frames' for the vector space $\pi_E^{-1}(m)$.

2.2 Densitized representations

Given a representations ρ, its determinant, $\det(\rho)$, features a one-dimensional representation. Any other representation ρ' on V can then be modified by multiplying it with a power of $\det(\rho)$. Representations constructed in this way occur sufficiently often to deserve a special name. We call $\rho'' = [\det(\rho)]^k \rho'$ the k-fold ρ-densitized representation ρ'. A case that frequently occurs in practice is when ρ' is a tensor-representation of ρ and its dual (=inverse adjoint). ρ'' then describes the representation of the k-fold densitized tensors. If $\dim(V) = 1$, sections in the associated vector bundle with representation $[\det(\rho)]^k$ are called scalar ρ-densities of weight k.

2.3 Gauge transformations

Any $f \in \mathrm{Aut}(P)$ defines a map $f_E : E \to E$ through

$$f_E([p, v]) := [f(p), v]. \tag{41}$$

One easily checks $f_E([p, v]) = f_E([p \cdot g, \rho(g^{-1})v])$, showing that this is indeed well defined. The association $f \mapsto f_E$ defines an action of $\mathrm{Aut}(P)$ on E. If f_G is the G-form of $f \in \mathrm{Gau}(P)$, we have

$$f_E([p, v]) := [p, \rho(f_G(p))v], \tag{42}$$

which defines an action of $\mathrm{Gau}(P)$, the group of gauge transformations, on E.

2.3.1 'Constant' gauge transformations do not exist

On P there is not only the action of the group of gauge transformations, $\mathrm{Gau}(P)$, but also the action of the gauge group G, which we denoted by $(p, g) \mapsto p \cdot g$. A corresponding action of G on E does not exist. The obvious definition $(g, [p, v]) \mapsto [p \cdot g, v]$ would only be well defined if we could replace $[p, v]$ with $[p \cdot h, \rho(h^{-1})v]$ for any $h \in G$. This leads to the condition $[p, \rho(g)v] = [p, \rho(hgh^{-1})v]$ $\forall h \in G$, which is satisfied iff g is an element of the normal subgroup $C_\rho(G) := \{g \in G \mid \rho(hgh^{-1}) = \rho(g)\, \forall h \in G\}$, which we call the *center of the representation* ρ. Hence we have an action of $C_\rho(G)$, but generally not of G.

Under the name of "constant gauge transformations" it is sometimes suggested that a well defined action of G on E (as gauge transformations) exists, at least in the case where P (and hence E) is trivial. This may sound as if for trivial bundles there exists a *natural* embedding $G \to \mathrm{Gau}(P)$, which is not correct. What is true is that for each global section $\sigma : M \to P$ there exists an injective homomorphism $G \to \mathrm{Gau}(P)$ defined through $h \mapsto f^h$, $f^h(\sigma(m) \cdot g) := \sigma(m) \cdot hg$, or simply $f^h_G(\sigma(m) \cdot g) = g^{-1}hg$, which is clearly Ad-equivariant. It is also immediate that the map $h \mapsto f^h$ is an injective homomorphism. But for a different trivializing section $\sigma' = \sigma \cdot \beta$, $\beta : M \to G$, the embedding would be $f'^h_G(\sigma(m) \cdot g) = g^{-1}(\beta(m))^{-1}h\beta(m)g$, which equals the old f^h followed by a gauge transformation f whose G-form is $f_G(\sigma(m) \cdot g) = g^{-1}[\beta(m), h]g$, where $[\cdot, \cdot]$ denotes the group commutator $[a, b] := aba^{-1}b^{-1}$. Hence there is no natural (trivialization independent) way to embed G into $\mathrm{Gau}(P)$ and hence no natural way to speak of 'constant' gauge transformations, even if the bundle is trivial.

2.4 Parallel transportation in E

The parallel transportation law for P induces one for E in the following way: Given γ and γ^H_p as before. Let $[p, v] \in \pi^{-1}_E(m)$; then we call the curve

$$\gamma^E_{[p,v]} : [0, 1] \to E, \qquad \gamma^E_{[p,v]}(t) := [\gamma^H_p(t), v] \tag{43}$$

the *parallel transportation* of $[p, v]$ along γ. Its endpoint, $\gamma^E_{[p,v]}(1) = [p, \gamma^H_p(1)]$, is called the result of parallelly transporting $[p, v]$ along γ.

2.5 Sections in E

A map $\tau : M \to E$, such that $\pi_E \circ \tau = id|_E$ is called a section in E. Since $\pi^{-1}_E(m)$ are vector spaces, such sections always exist, e.g. the "zero-section", without E being necessarily of the product form $M \times V$. A map $\hat{\tau} : P \to V$ is called ρ-equivariant iff, $\forall g \in G$,

$$\hat{\tau} \circ R_g = \rho(g^{-1}) \circ \hat{\tau}\,. \tag{44}$$

There is a bijective correspondence (compare (38)) between sections in E and ρ-equivariant, V-valued functions on P, given by

$$\tau(m) = [p, \hat{\tau}(p)] \tag{45}$$

for any $p \in \pi^{-1}(m)$ and for all m.

Given a local section σ in P, we can define a local representative $\bar{\tau}$ of τ via

$$\bar{\tau} = \hat{\tau} \circ \sigma\,. \tag{46}$$

For a different choice of local section σ' over the same subset $U \subset M$, as in (5), we obtain the relation for each $m \in U$

$$\bar{\tau}'(m) = \hat{\tau}(\sigma(m) \cdot \beta(m)) = \rho(\beta^{-1}(m))\bar{\tau}(m)\,. \tag{47}$$

As with E, an action of G on sections in E generally does not exist. Clearly, any action on E would define an action on sections in E just by composition.

A useful generalization of ρ-equivariant, V-valued functions on P is the notion of ρ-equivariant, V-valued horizontal n-forms ($0 \leq n \leq \dim(M)$), whose linear space we denote by $\Lambda^n(V,\rho)$. By definition, $\lambda \in \Lambda^n(V,\rho)$ iff

$$(i) \;:\; i_{\xi^\#}\lambda \,=\, 0\,, \tag{48}$$

$$(ii) \;:\; R_g^*\lambda \,=\, \rho(g^{-1})\lambda\,. \tag{49}$$

Condition (i) accounts for horizontality, i.e. that these forms are annihilated by vertical vectors, and (ii) is the condition of ρ-equivariance. For $n = 0$ we obtain the space of functions considered above, which we henceforth denote by $\Lambda^0(V,\rho)$.

One immediately obtains the useful formula for the vertical Lie derivative in $\Lambda^n(V,\rho)$:

$$L_{\xi^\#}\lambda = \frac{d}{dt}\Big|_{t=0} R^*_{\exp(t\xi)}\,\lambda = -\rho(\xi)\lambda\,, \quad \forall\xi \in \mathfrak{g}\,. \tag{50}$$

2.6 Derivatives of sections in E

Consider a section τ in E and its associated function $\hat\tau \in \Lambda^0(V,\rho)$ on P. Let there also be given a vector field X on M. We wish to define derivatives of sections in E along X. This is equivalent to defining derivatives along X of functions in $\Lambda^0(V,\rho)$. It is clear that in order to do this we need to lift X to a vector field $\hat X$ on P. Assume some lifting prescription; it must be such that $\hat X(\hat\tau)$ will again be ρ-equivariant. That is, we want

$$\hat X(\hat\tau)(p\cdot g) = \rho(g^{-1})\hat X(\hat\tau)(p)\,. \tag{51}$$

Rearranging the left hand side,

$$\hat X(\hat\tau)(p\cdot g) = d\hat\tau(\hat X_{p\cdot g}) = \rho(g^{-1})(R^*_{g^{-1}}d\hat\tau)(\hat X_{p\cdot g}) = \rho(g^{-1})d\hat\tau(R_{g^{-1}*}\hat X_{p\cdot g})\,, \tag{52}$$

shows that this is true for all τ iff $\hat X$ is right invariant: $\hat X = R_{g*}\hat X$. Hence any right-invariant lifting prescription will give us a well defined derivative operation on $\Lambda^0(V,\rho)$, and hence on sections in E, by defining

$$\nabla_X\tau(m) := [p, \hat X(\hat\tau)(p)] \quad \text{for any}\, p \in \pi^{-1}(m)\,. \tag{53}$$

Local representatives can be obtained in the obvious way:

$$\nabla_X\bar\tau := \hat X(\hat\tau)\circ\sigma\,. \tag{54}$$

It is instructive to be slightly more explicit at this point. Let ϕ_t be the flow of X and $\hat\phi_t$ the flow of $\hat X$ (we only need existence for some $t \in$ some finite interval around zero.). It satisfies

$$\pi\circ\hat\phi_t = \phi_t\circ\pi\,. \tag{55}$$

With respect to the local section σ we can uniquely decompose the lifted flow $\hat{\phi}_t$,

$$\hat{\phi}_t(\sigma(m)) = \sigma(\phi_t(m)) \cdot \beta(t,m)\,, \tag{56}$$

with some function β, satisfying $\beta(t=0,\cdot) = id$ (constant map onto group identity). Here we consider $m \in U' \subset U$ and t small enough for $\phi_t(m)$ to stay inside U. We now have

$$\begin{aligned} \hat{X}(\sigma(m)) &= \frac{d}{dt}\Big|_{t=0} \hat{\phi}_t(\sigma(m)) = \frac{d}{dt}\Big|_{t=0} [\sigma(\phi_t(m)) \cdot \beta(t,m)] \\ &= R_{\beta(0,m)_*}\sigma_*(X(m)) + \frac{d}{dt}\Big|_{t=0} [\sigma(m) \cdot \beta(t,m)] \\ &= \sigma_*(X(m)) + \xi^{\#}_{\sigma(m)}\,, \end{aligned} \tag{57}$$

where

$$\xi(m) := \frac{d}{dt}\Big|_{t=0} \beta(t,m) \in \mathfrak{g}\,. \tag{58}$$

Using (57) in (54) and (50) to express the vertical (i.e. $\xi^{\#}$-) derivative yields the following general *master formula* for local representatives of derivatives:

$$\boxed{\nabla_X \bar{\tau} = X(\bar{\tau}) - \rho(\xi)\bar{\tau}}\,. \tag{59}$$

The function $\xi : U' \to \mathfrak{g}$ implicitly depends on X, on the prescription for the lifting, and on the local section σ. It does not necessarily depend only pointwise on X but may involve its derivatives, as is does for example in the case of the Lie derivative, where X is explicitly used to construct the lift (see the section on Lie derivative).

3 Derivatives

3.1 The covariant derivative

According to the foregoing section, the covariant derivative may be characterized by putting $\hat{X} = X^H$, i.e. by lifting horizontally. We call the corresponding derivative operator D_X. In order to give an expression that is independent of the vector field X with respect to which we take the derivative, we introduce the exterior covariant derivative, which is just the ordinary exterior derivative followed by projection into horizontal direction:

$$d^H \hat{\tau} := d\hat{\tau} \circ P_H = d\hat{\tau} + \rho(\omega)\hat{\tau} \tag{60}$$

where the last equality is most easily verified by applying both sides to a basis in $T_p(P)$. For horizontal vectors equality obviously holds. For a vertical vector of the form $\xi^{\#}_p$ the left hand side is zero, whereas the first term on the right gives $-\rho(\xi)\hat{\tau}$

(by (50)), which just cancels the second term on the right hand side. Since the vectors $\xi_p^\#$ span V_p, this proves (60). To obtain a local expression we set

$$\bar{\omega} := \sigma^*\omega\,, \tag{61}$$

which is usually called the 'gauge potential'. It is a local (on M) representative of the globally defined (on P) connection 1-form ω. As local representatives we now obtain

$$D\bar{\tau} \;=\; d\bar{\tau} + \rho(\bar{\omega})\bar{\tau}\,, \tag{62}$$

$$\text{or}\quad D_X\bar{\tau} \;=\; X(\bar{\tau}) + \rho(\bar{\omega}(X))\bar{\tau}\,. \tag{63}$$

Formula (63) should be read as specialization of the 'master formula' (59) to the case at hand, i.e. to $\hat{X} = X^H$. This implies $\omega(\hat{X}) = 0$ so that application of ω to (57) gives

$$\xi = -\bar{\omega}(X)\,. \tag{64}$$

If we choose a different local section σ', related to the old one as in (5), the new gauge potential would be related to the old gauge potential by:

$$\omega' = ad(\beta^{-1})\bar{\omega} + \beta^{-1}d\beta\,, \tag{65}$$

where we regarded g to be matrix valued in order simplify notation. A proof for this formula appears at the end of this section

Finally, we remark that the concept of an exterior covariant derivative extends to all $\Lambda^n(V,\rho)$. It then defines a map

$$d^H\,:\,\Lambda^n(V;\rho)\to\Lambda^{n+1}(V,\rho),\quad d^H\lambda = d\lambda + \rho(\omega)\wedge\lambda\,, \tag{66}$$

where the last equality is again most easily verified by applying both sides to n vectors of which either none, one, or more than one is vertical. Clearly, all ρ-equivariant V-valued n-forms (not necessarily horizontal) are mapped via d^H into $\Lambda^{n+1}(V,\rho)$. An important example is the curvature 2-form on P which we discuss below.

Let us now prove (65). Let $X \in T_m(M)$ and $\gamma : I \to M$ a curve such that $\gamma(0) = m$ and $\dot{\gamma}(0) = X$. Let further the two local sections σ' and σ be related as in (5). Then:

$$\begin{aligned}
\sigma'_*(X) \;&=\; \frac{d}{dt}\Big|_{t=0}\big[\sigma'(\gamma(t))\big] = \frac{d}{dt}\Big|_{t=0}\big[\sigma(\gamma(t))\cdot\beta(\gamma(t))\big]\\
&=\; R_{g(m)_*}\sigma_*(X) + \frac{d}{dt}\Big|_{t=0}\big[\sigma(m)\cdot\beta(m)\beta^{-1}(m)\beta(\gamma(t))\big]\\
&=\; R_{g(m)_*}\sigma_*(X) + \left[\frac{d}{dt}\Big|_{t=0}(\beta^{-1}(m)\beta(\gamma(t))\right]^*_{\sigma(m)\cdot\beta(m)}\\
&=\; R_{g(m)_*}\sigma_*(X) + \Big[L_{\beta^{-1}(m)_*}\beta_{*m}(X)\Big]^*_{\sigma(m)\cdot\beta(m)}
\end{aligned} \tag{67}$$

Hence,

$$(\sigma'^* \omega)(X) = \omega(\sigma'_*(X)) = \left(ad(\beta^{-1}(m))\bar{\omega} + L_{\beta^{-1}(m)_*}\beta_{*m}\right)(X)\,, \qquad (68)$$

which must be valid for all $X \in T_m(M)$, so that we can write, omitting the base point m,

$$\sigma'^* \omega = ad(\beta^{-1})\bar{\omega} + L_{\beta^{-1}{}_*}\beta_*\,. \qquad (69)$$

Taking advantage that g can be considered as map into a linear space (matrix algebra), one may write dg for g_* and g^{-1} for $L_{g^{-1}}$. This proves (65).

3.2 The Lie derivative

Let P be the bundle of linear frames, $L(M)$, over M. A point $p \in \pi^{-1}(x)$ consists of n linear independent vectors $\{e_1(x), \ldots, e_n(x)\}$ (=$\{e_a(x)\}$ for abbreviation) in $T_x(M)$. As before, ϕ_t denotes the flow of X. We define the lifted flow, $\hat{\phi}_t$, by

$$\hat{\phi}_t(\{e_a\}) := \{\phi_{t*}e_a\}\,, \qquad (70)$$

which clearly commutes with the right $G = Gl(n, \mathbb{R})$ action

$$g \times \{e_a(x)\} \mapsto \{e_b(x)g^b_a\}\,. \qquad (71)$$

A local section σ assigns a basis $\{e_a(x)\}$ to each $x \in U$ with respect to which we can express the lift in the form (70):

$$\phi_{t*}e_a(\phi_t(x)) = \phi_{t*x}(e_a(x)) = e_b(\phi_t(x))(\phi_{t*x})^b_a\,, \qquad (72)$$

where $(\phi_{t*x})^b_a$ are the components of the Jacobi-matrix for ϕ_t at x with respect to the chosen frames $\{e_a(x)\}$, that is, with respect to σ. Comparison of (72) with (56) and (58) (in matrix notation) shows that

$$\beta^b_a(t,x) = (\phi_{t*x})^b_a \quad \text{and} \quad \xi^b_a(x) = \frac{d}{dt}\Big|_{t=0}(\phi_{t*x})^b_a \qquad (73)$$

respectively. The right hand side of the last equation can be explicitly calculated using (72). For this we write $y = \phi_t(x)$ and note

$$\phi_{-t*y} \circ \phi_{t*x} = id\Big|_{T_x(M)}, \quad \text{so that} \quad \phi_{t*x} = (\phi_{-t*y})^{-1}\,. \qquad (74)$$

Inserting this into (72) yields

$$\phi_{t*}e_a(y) = e_b(y)\left((\phi_{-t*y})^{-1}\right)^b_a\,. \qquad (75)$$

This is just formula (72) but written such that all quantities are evaluated at the same base point y. We can now take the derivative with respect to t at $t = 0$, keeping y fixed. Recalling that the commutator of two vector fields $[Y, X]$ is given by

$$[Y, X] = \frac{d}{dt}\Big|_{t=0} \phi_{t*} Y \,, \tag{76}$$

where ϕ_t denotes the flow of X, we have from (75)

$$[e_a, X](y) = [e_a, X]^b(y) e_b(y) = e_b(y) \frac{d}{dt}\Big|_{t=0} \left((\phi_{-t*y})^{-1} \right)^b_{\ a} = e_b(y) \xi^b_a(y) \,, \tag{77}$$

where in the last step we used that for a matrix-valued curve $A(t)$ with $A(0) = id$ we have $\frac{d}{dt}|_{t=0}(A(-t))^{-1} = \frac{d}{dt}|_{t=0} A(t)$. Hence we have shown that

$$\xi^b_a = [e_a, X]^b \,. \tag{78}$$

Given a coordinate system $\{x^\alpha\}$ on $U \subset M$ with associated frame $\{\partial/\partial x^\alpha\}$ and the decomposition $X = X^\alpha \partial/\partial x^\alpha$, one has

$$\xi^\beta_\alpha(x) = \partial_\alpha X^\beta(x) \,, \tag{79}$$

which, when used in the 'master formula' (59), gives the well known expression for the Lie derivative in a coordinate frame basis:

$$L_X \bar{\tau} = X^\alpha \partial_\alpha \bar{\tau} - \rho(\partial X)\, \bar{\tau} \,. \tag{80}$$

We can also derive a useful expression for the Lie derivative of the connection 1-form. Note that for any right-invariant vector field $\hat{X}$ we have $L_{\hat{X}} \omega \in \Lambda^1(\mathfrak{g}, ad)$ and $\omega(\hat{X}) \in \Lambda^0(\mathfrak{g}, ad)$. Then, for any right-invariant $\hat{X}$,

$$\begin{aligned} L_{\hat{X}} \omega &= (i_{\hat{X}} \circ d + d \circ i_{\hat{X}}) \omega = d(\omega(\hat{X})) + i_{\hat{X}}(\Omega - \omega \wedge \omega) \\ &= i_{\hat{X}} \Omega + d^H(\omega(\hat{X})) \,. \end{aligned} \tag{81}$$

The Lie derivative for the curvature then follows from (92):

$$L_{\hat{X}} \Omega = d L_{\hat{X}} \omega + ad(\omega) \wedge L_{\hat{X}} \omega = d^H(L_{\hat{X}} \omega) \tag{82}$$

We now pull back (81) using σ, and obtain with (57):

$$\sigma^*(L_{\hat{X}}(\omega)) = i_X \bar{\Omega} + D(\bar{\omega}(X)) + D\xi = L_X \bar{\omega} + D\xi \,, \tag{83}$$

where L_X is just the ordinary Lie derivative for (vector-valued) forms on M and where the term involving ξ has been written according to (62) in the last step. This expression is valid for any right-invariant $\hat{X}$. Specialization to the case at hand is now performed by using the Jacobi matrix for ξ, as in (73).

We find it instructive to give a second derivation. For this we again write down equation (56), using the notation $\beta(t, m) = \beta_t(m) =: \tilde{\beta}_t(\phi_t(m))$:

$$\hat{\phi}_t \circ \sigma = (\sigma \cdot \tilde{\beta}_t) \circ \phi_t \,, \tag{84}$$

and where $\beta_{t=0} = id$. Hence,

$$\sigma^*(L_{\hat{X}}\omega) = \frac{d}{dt}\Big|_{t=0} (\hat{\phi}_t \circ \sigma)^*\omega = \frac{d}{dt}\Big|_{t=0} \phi_t^* \circ (\sigma \cdot \tilde{\beta}_t)^*\omega \,. \tag{85}$$

From (65) we know that

$$(\sigma \cdot \tilde{\beta}_t)^*\omega = ad(\tilde{\beta}_t^{-1})\bar{\omega} + \tilde{\beta}_t^{-1} d\tilde{\beta}_t \,, \tag{86}$$

so that the right hand side of (85) is equal to (recall $\tilde{\beta}_t \circ \phi_t = \beta_t$)

$$\frac{d}{dt}\Big|_{t=0} \big(ad(\beta_t^{-1})(\phi_t^*\bar{\omega}) + \beta_t^{-1} d\beta_t\big) = L_X\bar{\omega} + d\xi + ad(\bar{\omega})\xi \,, \tag{87}$$

where $\xi = \dot{\beta}_{t=0}$ is given by the expression (73). The terms involving ξ can be written as $D\xi$ according to (62).

What we have said so far is restricted to the case where P is the frame bundle of M. We made use of the fact that diffeomorphisms of M have a natural lift to $L(M)$ by push-forward, as expressed in (70). This enabled us to define Lie derivatives for sections in all vector bundles associated to $L(M)$. But in the general case no such natural lift exists. As an intermediate example, let M be given a Riemannian metric, and take $P = F(M)$, where $F(M)$ denotes the bundle of orthonormal frames. Then the lifting prescription (70), applied to $F(M)$, only makes sense for isometries, so that at this level Lie derivatives can only be taken with respect to Killing vector fields. This is e.g. the case for spinors, which are sections in a vector-bundle associated to a double cover of $F(M)$. Generally, you simply cannot take Lie derivatives of spinors with respect to arbitrary vector fields. Compare [3], where the Lie derivative of spinors with respect to conformal Killing fields is explained.

If P is trivial, we already discussed in section 1.4.2 how to lift a diffeomorphism in a trivialization dependent fashion. Differentiating (20) with respect to the flow parameter s after replacing ϕ with ϕ_s, we obtain for the lifted vector field

$$\hat{X}(\sigma(m) \cdot g) := R_{g*}\sigma_*(X(m)) \,, \tag{88}$$

which is clearly right invariant and tangential to the image of σ. When σ is at the same time used to derive expressions on M, these become particularly simple due to the fact that the ξ-term in (57) is now absent. Thus (we continue our notation $\hat{\tau} \circ \sigma = \bar{\tau}$, $\sigma^*\omega = \bar{\omega}$ but keep in mind that $\sigma = \sigma$, i.e. $U = M$)

$$\sigma^*(L_{\hat{X}}\hat{\tau}) \;=\; L_X\bar{\tau} = X(\bar{\tau}) \,, \tag{89}$$

$$\sigma^*(L_{\hat{X}}\omega) \;=\; L_X\bar{\omega} \,. \tag{90}$$

By construction, these expressions are ρ-, resp. ad-equivariant, so that any other local representative with respect to $\sigma' = \sigma \cdot g$ is then given by applying $\rho(g^{-1})$, resp. $ad(g^{-1})$ to these expressions.

4 Curvature and torsion

4.1 Curvature

The notion of curvature exits for all principal bundles with connection and accordingly also for bundles associated to them. Let P be a principal bundle and ω the connection 1-form on P; the curvature 2-form is defined as follows:

$$\Omega := d^H \omega \,, \tag{91}$$

where ω is considered as a ad-equivariant 1-form, and Ω as element in $\Lambda^2(\mathfrak{g}, ad)$. In this case it is important to note that, since ω is not horizontal, (66) may not be applied. Instead, one obtains

$$\Omega = d^H \omega = d\omega + \tfrac{1}{2} ad(\omega) \wedge \omega = d\omega + \tfrac{1}{2}[\omega, \omega] \,, \tag{92}$$

which differs by a factor of 1/2 from what a naive application of (66) would suggest. The proof runs entirely analogous as for (66). Using a local section σ and writing $\frac{1}{2}[\bar{\omega}, \bar{\omega}] = \bar{\omega} \wedge \bar{\omega}$ we get *Cartan's second structure equation* for $\bar{\Omega} := \sigma^* \Omega$:

$$\bar{\Omega} = d\bar{\omega} + \bar{\omega} \wedge \bar{\omega} \tag{93}$$

$\bar{\Omega}(X, Y) \in \mathfrak{g}$ acts via the representation ρ on V, where $\bar{\tau}$ takes its values. To save notation we momentarily write $\cdot$ instead of ρ, i.e. $\lambda \cdot v$ instead of $\rho(\lambda)v$ for $\lambda \in \mathfrak{g}$ and $v \in V$. With successive application of formula (63), e.g. $D_X \bar{\tau} = X(\bar{\tau}) + \bar{\omega}(X) \cdot \bar{\tau}$, we can write

$$\begin{aligned}
\bar{\Omega}(X, Y) \cdot \bar{\tau} &= (X(\bar{\omega}(Y)) - Y(\bar{\omega}(X)) - \bar{\omega}([X, Y]) + [\bar{\omega}(X), \bar{\omega}(Y)]) \cdot \bar{\tau} \\
&= X(\bar{\omega}(Y) \cdot \tau) - Y(\bar{\omega}(X) \cdot \tau) - \bar{\omega}([X, Y]) \cdot \bar{\tau} \\
&\quad + \bar{\omega}(X) \cdot Y(\bar{\tau}) - \bar{\omega}(Y) \cdot X(\bar{\tau}) + [\bar{\omega}(X), \bar{\omega}(Y)] \cdot \bar{\tau} \\
&= X(D_Y \bar{\tau}) - Y(D_X \bar{\tau}) - D_{[X,Y]} \bar{\tau} \\
&\quad + \bar{\omega}(X) \cdot D_Y \bar{\tau} - \bar{\omega}(Y) \cdot D_X \bar{\tau} \\
&= D_X D_Y \bar{\tau} - D_Y D_X \bar{\tau} - D_{[X,Y]} \bar{\tau} \,,
\end{aligned} \tag{94}$$

which is a useful equation in applications, in particular if X, Y refer to coordinate vector fields $\partial / \partial x^i$, whose commutators vanish.

Applying the exterior covariant derivative twice to a $\lambda \in \Lambda^n(V, \rho)$, we can use (66) and (92) to obtain

$$d^H d^H \lambda = \rho(\Omega) \wedge \lambda \,. \tag{95}$$

On the other hand, since Ω vanishes on horizontal vectors, it is immediate from the expression (92) that the d^H-derivative annihilates Ω. Hence one has the so-called *Bianchi-Identity*:

$$d^H \Omega = d\Omega + ad(\omega) \wedge \Omega = 0 \tag{96}$$

4.2 Torsion

The notion of torsion only exists for principal bundle of linear frames, i.e., if $P = L(M)$ (or subbundles thereof). Recall that a linear frame $p = \{e_a(m)\}$ at $m \in M$ is a linear map from $\mathbb{R}^n$ to $T_m(M)$:

$$f_p : \mathbb{R}^n \to T_m(M), \quad \{v^a\} \mapsto v^a e_a(m)\,, \tag{97}$$

where the map corresponding to the point p is denoted by f_p. We can now define a $\mathbb{R}^n$-valued 1-form on $L(M)$, Θ, by

$$\Theta_p := {f_p}^{-1} \circ \pi_{*p}\,. \tag{98}$$

The form obviously annihilates vertical vectors. It is also δ-equivariant, where δ denotes the defining representation of $GL(n,\mathbb{R})$ in $\mathbb{R}^n$; hence $\Theta \in \Lambda^1(\mathbb{R}^n,\delta)$. This is easy too verify:

$$R_g^*\Theta_{p\cdot g} = f_{p\cdot g}^{-1} \circ \pi_{*p\cdot g} \circ R_{g*} = f_{p\cdot g}^{-1} \circ \pi_{*p} = \delta(g^{-1})\Theta_p\,. \tag{99}$$

The torsion, $T \in \Lambda^2(\mathbb{R}^n,\delta)$, is now defined as the exterior covariant derivative of Θ:

$$T := d^H\Theta = d\Theta + \delta(\omega)\wedge\Theta\,. \tag{100}$$

For local expressions on $U \subset M$ consider $\sigma : U \to P$ and set $\bar\Theta = \sigma^*\Theta$ and $\bar T = \sigma^* T$. Note that with (98) we get $\bar\Theta(x) = f_{\sigma(x)}^{-1}$, which just means that the components $\bar\Theta^a$ of $\bar\Theta$ form the dual basis to $\{e_a\}$ which defined Σ; we have $\bar\Theta^a(e_b) = \delta^a_b$. The local expression of (100) is known as *Cartan's first structure equation*:

$$\bar T^a = D\bar\Theta^a := d\bar\Theta^a + \bar\omega^a_b \wedge \bar\Theta^b\,. \tag{101}$$

Evaluating (101) on vectors X, Y and using $\bar\Theta^a(X) = X^a$, $d\bar\Theta^a(X,Y) = X(Y^a) - Y(X^a) - [X,Y]^a$, and $X(Y^a) + [\bar\omega(X)]^a_b Y^b = (D_X Y)^a$ we get (suppressing the index a again):

$$\bar T(X,Y) = D_X Y - D_Y X - [X,Y]\,. \tag{102}$$

Finally we remark that Θ is invariant under the lifted flow (70); this is easy to prove:

$$(\hat\phi_t)^*\Theta_{\hat\phi_t(p)} = f^{-1}_{\hat\phi_t(p)} \circ \pi_{*\hat\phi_t(p)} \circ \hat\phi_{t*p} \tag{103}$$

$$= f^{-1}_{\hat\phi_t(p)} \circ \phi_{t*m} = \Theta_p\,. \tag{104}$$

For the Lie derivatives we thus obtain

$$L_{\hat X}\Theta = 0, \quad \text{and} \quad L_{\hat X} T = \delta(L_{\hat X}\omega)\wedge\Theta\,. \tag{105}$$

4.3 Relating covariant and Lie derivatives

There is an interesting relation between covariant and Lie derivatives for sections in bundles associated to $P = L(M)$. Recall that both kinds of derivatives were applications of the master formula (59) which differed in the lifting prescriptions and hence led to different expressions for ξ. With respect to a local section $\sigma : U \to L(M)$ given by the field of bases $\{e_a\}$ over U, these expressions took the form:

$$\xi^a_b = -X^c \bar{\omega}^a_{cb} \qquad \text{for covariant derivative} \tag{106}$$

$$\xi^a_b = [e_a, X]^b \qquad \text{for Lie derivative} \tag{107}$$

The local expression for the difference between covariant and Lie derivatives of a section τ is then given by

$$D_X \bar{\tau} - L_X \bar{\tau} = \rho(\bar{\omega}(X) + [e, X])\bar{\tau} \tag{108}$$

where the argument of ρ is the matrix in $\mathfrak{gl}(n, \mathbb{R})$ with components $(\bar{\omega}(X))^a_b + [e_b, X]^a$. This matrix can be given a neater form by employing (102), which, writing $D_X e_b = [\bar{\omega}(X)]^a_b\, e_a$ and $[D_{e_b} X]^a =: D_b X^a$, gives

$$T^a(X, e_b) = [\bar{\omega}(X)]^a_b + [e_b, X]^a - D_b X^a\,. \tag{109}$$

Hence the Lie derivative can be expressed in terms of the covariant derivative and the torsion:

$$L_X \bar{\tau} = D_X \bar{\tau} - \rho(DX + i_X T)\bar{\tau} \tag{110}$$

where DX and $i_X T$ stand for the component matrices $\{D_b X^a\}$ and $\{X^c T^a_{cb}\}$ with respect to $\sigma = \{e_a\}$. This expression should be compared to (80) which, for example, shows that for torsion-free connections we may just replace all partial derivatives on the right hand side of (80) with covariant ones.

5 Building new from old principal bundles

5.1 Splicing

Given two principal bundles P_1 and P_2 over M with groups G_1, G_2 and projection maps π_1 and π_2 respectively, we consider the following subset of the Cartesian product of P_1 and P_2:

$$P_1 * P_2 := \{(p_1, p_2) \in P_1 \times P_2 \mid \pi_1(p_1) = \pi_2(p_2)\}\,. \tag{111}$$

We denote points in $P_1 * P_2$ by pairs (p_1, p_2), but have it implicitly understood that they project to the same point on M. It is easy to see that $P_1 * P_2$ is itself a principal bundle over M with group $G_1 \times G_2$ and projection $\pi(p_1, p_2) := \pi_1(p_1) = \pi_2(p_2)$. It is called the splicing of bundles P_1 and P_2 [2]. But it is also a principal bundle over P_1 (P_2) with group $\{1_{G_1}\} \times G_2$ ($G_1 \times \{1_{G_2}\}$) and projection $\bar{\pi}_1(p_1, p_2) = p_1$ ($\bar{\pi}_2(p_1, p_2) = p_2$). We have thus a total of five principal bundles which can be organized in the following commuting diagram: ‘

$$\begin{array}{ccccc} P_1 & \xleftarrow{\bar{\pi}_1} & P_1 * P_2 & \xleftarrow{\bar{\pi}_2} & P_2 \\ & \searrow^{\pi_1} & \downarrow \pi & \swarrow^{\pi_2} & \\ & & M & & \end{array} \qquad (112)$$

(Technically speaking: $P_1 * P_2$ can be considered as the pull-back bundle of either P_1 via π_2 or P_2 via π_1.) It is indeed useful to think of $P_1 * P_2$ in terms of (112). Note that all the constructions in the foregoing sections apply to any of these bundles. The case of interest is where, say, P_1 is the frame bundle $L(M)$, and P_2 any other principal bundle. In physics the associated vector bundle sections of $P_1 * P_2$ then correspond to “ fields carrying different kind of indices”, “space (space-time) indices and internal indices”. Clearly, the process of splicing may be iterated, for example P_2 may itself be spliced. In this case the “internal” indices are again of “different kind”.

Connections on P_1 and P_2 induce a unique connection on $P_1 * P_2$: The horizontal subspace at (p_1, p_2) is the unique subspace $H_{(p_1,p_2)} \subset T_{(p_1,p_2)}(P_1 * P_2)$, such that $\bar{\pi}_{i*} H_{(p_1,p_2)} = H_{p_i} \subset T_{p_i}(P_i)$ $(i = 1, 2)$. The connection 1-form ω on $P_1 * P_2$ is then given by

$$\omega = \bar{\pi}_1^* \omega_1 \oplus \bar{\pi}_2^* \omega_2 \qquad (113)$$

Right-invariant liftings of vector fields on M to $P_1 * P_2$ bijectively correspond to right invariant liftings to P_1 and P_2, and local sections in $P_1 * P_2$ correspond bijectively to local sections in P_1 and P_2 in the obvious way. Global sections in $P_1 * P_2$ exist iff they exist for P_1 and P_1; hence $P_1 * P_2$ is trivial iff P_1 and P_2 are trivial.

One can now define all sorts of derivatives of sections in vector bundles associated to $P_1 * P_2$, by considering right invariant liftings to P_1 and P_2 and their combinations to right invariant liftings to $P_1 * P_2$. Consider for example the Lie derivative for the case where $P_1 = L(M)$ and P_2 a trivial bundle. The vector field X on M is then lifted to P_1 according to (70) and to P_2 by brute force (88) with respect to some global section σ in P_2. This defines a right invariant lifting to $P_1 * P_2$. Given a local section $\sigma^1 : U \to P_1$ we have a local section $\sigma : U \to P_1 * P_2$, $\sigma(p_1, p_2) := (\sigma^1(m), \sigma(m))$, with respect to which we can write down local representatives. The Lie derivative then has coordinate expressions like:

$$L_X \, \tau^{A\alpha}_{B\beta} = X^\gamma \, \tau^{A\alpha}_{B\beta,\gamma} - X^\alpha_{,\gamma} \, \tau^{A\gamma}_{B\beta} + X^\gamma_{,\beta} \, \tau^{A\alpha}_{B\gamma} \qquad (114)$$

where α, β, γ are coordinate-indices (with respect to σ^1, which we have chosen to take values in coordinate frames), and the indices A, B are with respect to σ. Thus, the Lie derivative acts on coordinate indices “as usual” and does not “see” the other indices (“treats them as scalars”). But we emphasize that this latter property is due to choosing the same section σ for trivialization and for the definition of the lifting of the vector field X.

5.2 Reduction

Reduction and extension are methods to either diminish or enlarge the structure group, keeping the base fixed. In the process of reduction the bundle is reduced to a have a structure group that is a subgroup of the original one. In the process of extension, one enlarges the structure group to an extension (in the technical sense of the word) of the original group, i.e. the new structure group has the old one as a factor group but not necessarily as a subgroup. We will now discuss these cases in turn, starting with reduction.

Let G' be a subgroup of G and $i : G' \to G$ the embedding homomorphism. A G'-reduction of P is a principal G'-bundle P' over M, with right action R'_g and projection π', and a map $I : P' \to P$, such that

$$(i) \quad : \quad R_{i(g')} \circ I = I \circ R'_{g'} , \quad \forall g' \in G' \tag{115}$$

$$(ii) \quad : \quad \pi \circ I = \pi' . \tag{116}$$

In other words, P' is a subbundle of P. If the only existing G'-reduction of P is for $G' = G$, P is called irreducible. If P is reducible, an interesting question is what the irreducible subbundles are. If P is trivial, the subset $M \times 1_G$ is a 1_G-reduction of P. We call it the identity reduction. A useful general result is the following: Let G' be a closed subgroup of G such that G/G' is contractible (as topological space). Then a G'-reduction always exists. This can be applied to G' being the maximal compact subgroup of the Lie group G. The example $G = GL(n, R)$, $G' = O(n)$ and $G/G' \cong \mathbb{R}^{n(n+1)/2}$ shows that the frame bundle $L(M)$ always admits a reduction to $F(M)$, the bundle of orthonormal frames. In other words, M always admits a Riemannian metric.

Clearly the inverse procedure is trivial. Given P', we can always embed G' into a larger group G and regard P' as a reduction of P in the obvious way. We then say that P is a prolongation P' (the word extension is saved for the process described below). It is important to note that the question of admittance of a reduction is dependent on the group G one starts from. Given $G'' \subset G' \subset G$ ($\subset$ = subgroup of) and where P' is a G' reduction of P (P an extension of P'). Then it might happen that P admits a G''-reduction whereas P' does not. As explicit example we take as P' the non-trivial double cover of the circle (the edge of the closed Möbius strip) regarded as a Z_2 principal bundle over S^1. Being non-trivial it does not admit an identity-reduction. Embedding Z_2 in the circle group S^1, we obtain it as a subbundle of the 2-torus which represents P. But the torus is the trivial S^1-bundle over S^1 and clearly admits an identity-reduction.

Since this is an important point, we also want present this rather trivial example in some analytic detail. The base, M, is the circle S^1, P' is the connected double cover of S^1 which is itself a circle, and P is the trivial circle bundle over M, which is the 2-torus. We represent all circles by e^{it} for $t \in [0, 2\pi]$. The generator of the structure group Z_2 of P' is written as -1. Then we have

$$\pi' \quad : \quad P' \to M, \qquad e^{it} \mapsto e^{2it} \tag{117}$$

$$R'_{-1} \quad : \quad P' \to P', \qquad e^{it} \mapsto e^{i(t+\pi)} \tag{118}$$

$$\pi \quad : \quad P \to M, \qquad (e^{it}, e^{is}) \mapsto e^{it} \tag{119}$$

$$R_{e^{iu}} \quad : \quad P \to P, \qquad (e^{it}, e^{is}) \mapsto (e^{it}, e^{i(s+u)}) \tag{120}$$

$$i \quad : \quad Z_2 \to S^1, \qquad -1 \mapsto e^{i\pi} \tag{121}$$

$$I \quad : \quad P' \to P, \qquad e^{it} \mapsto (e^{2it}, e^{it}) \tag{122}$$

which satisfies (115,115). The identity reduction of P to the identity-bundle P'' over M (which is a circle) is then given by

$$I' : P'' \to P, \quad e^{it} \mapsto \left(e^{it}, 1\right) . \tag{123}$$

5.3 Extension

Let $\bar{G}$ be a group and $\lambda : \bar{G} \to G$ a surjective homomorphism. We call K the kernel of λ and have $G \cong \bar{G}/K$. A $\bar{G}$-extension of P is a principal $\bar{G}$-bundle $\bar{P}$ over M, with right action $\bar{R}_{\bar{g}}$ and projection $\bar{\pi}$, and a map $\Lambda : \bar{P} \to P$, such that

$$(i) \quad : \quad R_{\lambda(\bar{g})} \circ \Lambda = \Lambda \circ \bar{R}_{\bar{g}} \quad \forall \bar{g} \in \bar{G}\,, \tag{124}$$

$$(ii) \quad : \quad \pi \circ \Lambda = \bar{\pi}\,. \tag{125}$$

If P is trivial, $P = M \times G$, a $\bar{G}$-extension clearly always exists: just set $\bar{P} = M \times \bar{G}$ and $\Lambda(m, \bar{g}) = (m, \lambda(\bar{g}))$.

An important example is given for $P = F(M)$, $G = SO(n)$ and $\bar{G} = \mathrm{Spin}(n)$ (a double cover of $SO(n)$). In this case the $\mathrm{Spin}(n)$-extension of $F(M)$ is also called a spin-structure. If M represents a 4-dimensional space-time one has instead $G = SO(1, 3)$ and $\bar{G} = \mathrm{Spin}(1, 3)$. Since 3-dimensional orientable manifolds have always trivial $F(M)$, they in particular always have a spin-structure. For non-compact 4-dimensional space-times it is known that they admit a spin structure if and only if $F(M)$ is trivial [4].

A Construction of principal bundles

Let M be a manifold (later to be called the "base") and a collection of open subsets $\{U_\alpha \subset M \mid \alpha \in \mathcal{J}\}$ which cover M, i.e. $\bigcup_{\alpha \in \mathcal{J}} U_\alpha = M$; the set $\mathcal{J}$ is just some index set. Let further G be some group; we consider the collection of sets $U_\alpha \times G$, for all $\alpha \in \mathcal{J}$. For each ordered pair $\alpha, \beta \in \mathcal{J}$ for which $U_\alpha \cap U_\beta \neq \emptyset$ we are given a function $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, such that the following three conditions are satisfied (e is the identity of G):

$$\phi_{\alpha\alpha}(x) = e \qquad \forall x \in U_\alpha\,, \tag{126}$$

$$\phi_{\alpha\beta}(x) = [\phi_{\beta\alpha}(x)]^{-1} \qquad \forall x \in U_\alpha \cap U_\beta\,, \tag{127}$$

$$\phi_{\alpha\beta}(x)\phi_{\beta\gamma}(x) = \phi_{\alpha\gamma}(x) \qquad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma\,. \tag{128}$$

Next we take the disjoint union of the sets $U_\alpha \times G$:

$$\mathcal{U} := \bigcup_{\alpha \in \mathcal{J}} (\alpha, U_\alpha \times G) \tag{129}$$

and "glue" the different pieces together by using the functions $\phi_{\alpha\beta}$. The mathematical expression for "gluing" is to identify via some equivalence relation. The equivalence relation that we use on $\mathcal{U}$ is defined as follows:

$$(\alpha, x, g) \sim (\beta, y, h) \;\Leftrightarrow\; x = y \quad \text{and} \quad g = \phi_{\alpha\beta}(y)\, h\,. \tag{130}$$

That this is indeed an equivalence relation is directly implied by (126-128): Reflexivity, i.e. that $(\alpha, x, g) \sim (\alpha, x, g)$, follows from (126); symmetry, i.e. that $(\alpha, x, g) \sim (\beta, y, h)$ implies $(\beta, y, h) \sim (\alpha, x, g)$, follows from (127); and finally transitivity, i.e. that $(\alpha, x, g) \sim (\beta, y, h)$ and $(\beta, y, h) \sim (\gamma, z, k)$ imply $(\alpha, x, g) \sim (\gamma, z, k)$, follows from (128).

Now we consider the space of equivalence classes, which we call P:

$$P := \mathcal{U}/\sim\;. \tag{131}$$

We call it the *principal bundle* for the data: M (the base), G (the fiber or the group), $\{U_\alpha\}$ (the cover), and $\{\phi_{\alpha\beta}\}$ (the transition functions). To indicate this dependence, one could write $P(M, G, \{U_\alpha\}, \{\phi_{\alpha\beta}\})$.

We shall denote the equivalence class of $(\alpha, x, g) \in \mathcal{U}$ by $[(\alpha, x, g)] \in P$. There is a natural surjective map $\pi : P \to M$, given by

$$\pi([(\alpha, x, g)]) := x\,. \tag{132}$$

This is obviously well defined on equivalence classes. The preimage of some point $x \in M$ is given by

$$\pi^{-1}(x) = \bigcup_{g \in G} [(\alpha, x, g)] \tag{133}$$

if $p \in U_\alpha$. This set is homeomorphic to G; though there is no natural homomorphism, since if $p \in U_\alpha \cap U_\beta$, we could have just as well written down (133) with β instead of α. According to (130) the relation is

$$\bigcup_{g \in G} [(\alpha, x, g)] = \bigcup_{g \in G} [(\beta, x, \phi_{\alpha\beta}(x)\, g)]\,, \tag{134}$$

which says that our identification of $\pi^{-1}(x)$ with G cannot distinguish between G and $\phi_{\alpha\beta}(x)G$ for all α, β for which $x \in U_\alpha \cap U_\beta$.

The set P carries a natural right action G, given by

$$G \times P\,, (h, [(\alpha, x, g)]) \mapsto [(\alpha, x, gh)]\,. \tag{135}$$

This is well defined since for $(\alpha, x, g) \sim (\beta, y, k)$ we have $x = y$ and $g = \phi_{\alpha\beta}(y)\, k$, and hence also $gh = \phi_{\alpha\beta}(y)\, kh$ so that $(\alpha, x, gh) \sim (\beta, y, kh)$. Note that this is true because we used *left* multiplication in our definition of the "gluing maps" in (130). For the right action with $g \in G$ on P we shall also write R_g or just by juxtaposition of the point $p \in P$ with $g \in G$ and a dot in-between:

$$R_g : P \to P\,, \quad p \mapsto R_g(p) = p \cdot g \tag{136}$$

This action is free on P (i.e. $R_g(p) = p$ for some $p \in P$ implies $g = e$) and simply transitive on each fiber $\pi^{-1}(x)$ (i.e. for any two $p, p' \in \pi^{-1}(x)$ there is a unique $g \in G$ such that $R_g(p) = p'$).

Our original building blocks – the sets $U_\alpha \times G$ – can now be seen as chart images of bundle-chart maps:

$$\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G\,, \quad [(\alpha, x, g)] \mapsto (x, g) \tag{137}$$

which are equivariant under the right action of G on P and $U_\alpha \times G$ (the latter being the obvious one: $(g, (x, h)) \mapsto (x, hg)$, which we also denote by R_g):

$$\phi_\alpha \circ R_h = R_h \circ \phi_\alpha\,, \quad \forall g \in G\,. \tag{138}$$

In the overlap $\pi^{-1}(U_\alpha \cap U_\beta)$ the different chart maps are related by transition functions, which are clearly just given by the $\phi_{\alpha\beta}$-functions that we used to glue the different patches:

$$\phi_\alpha(p) = \phi_{\alpha\beta}(\pi(p))\, \phi_\beta(p)\,, \quad \forall p \in \pi^{-1}(U_\alpha \cap U_\beta)\,. \tag{139}$$

The transition functions on overlapping charts are given by left G-multiplications:

$$\begin{aligned} \phi_\alpha \circ \phi_\beta^{-1} :\ (U_\alpha \cap U_\beta) \times G \quad &\to \quad (U_\alpha \cap U_\beta) \times G \\ (x, g) \quad &\mapsto \quad (x, \phi_{\alpha\beta}(x)\, g)\,. \end{aligned} \tag{140}$$

This follows directly from (130). The converse map is given by

$$\begin{aligned} \phi_\beta^{-1} \circ \phi_\alpha :\ \pi^{-1}(U_\alpha \cap U_\beta) \ &\to \ \pi^{-1}(U_\alpha \cap U_\beta) \\ [(\alpha, x, g)] \ &\mapsto \ [(\beta, x, g)] = [(\alpha, x, \phi_{\alpha\beta}(x)\, g)]\,. \end{aligned} \tag{141}$$

A *local section* of P over U_α is a map $\sigma_\alpha : U_\alpha \to P$, such that

$$\pi \circ \sigma_\alpha = \mathrm{id}\big|_{U_\alpha}\,. \tag{142}$$

The chart maps ϕ_α define obvious local sections (e = identity in G):

$$\sigma_\alpha(x) := \phi_\alpha^{-1}(x, e) = [(\alpha, x, e)]\,. \tag{143}$$

The relation between the sections over U_α and U_β is then given by

$$\sigma_\beta(x) = \sigma_\alpha(x) \cdot \phi_{\alpha\beta}(x) \tag{144}$$

This follows from (141), since $\sigma_\beta(x) = \phi_\beta^{-1}(x,e) = \phi_\beta^{-1} \circ \phi_\alpha \circ \phi_\alpha^{-1}(x,e) = \phi_\beta^{-1} \circ \phi_\alpha[(\alpha,x,e)] = [(x,g,\phi_{\alpha\beta}(x))] = [(\alpha,x,e)] \cdot \phi_{\alpha\beta}(x) = \sigma_\alpha(x) \cdot \phi_{\alpha\beta}(x)$.

Conversely, given a local section $\sigma_\alpha : U_\alpha \to P$, any $p \in \pi^{-1}(U_\alpha)$ can be uniquely written as $\sigma_\alpha(x)g$ for $U_\alpha \ni x = \pi(p)$ and $g \in G$. Hence σ_α defines an obvious bundle-chart map:

$$\phi_\alpha(\sigma_\alpha(x)g) := (x,g)\,. \tag{145}$$

Hence we can give the following definition of a principal bundle:

Definition 1. Let M be a manifold and G a Lie group. A *principal bundle* over M with standard fiber G consists of a manifold P and a free right action $G \times P \to P$, $(g,p) \mapsto R_g(p) = p \cdot g$ such that the following conditions hold:

1. Let $p \sim p'$ iff $\exists g \in G$ s.t. $p' = p \cdot g$ be the equivalence relation defined by G, then M is diffeomorphic to $P/\sim$ (also denoted by P/G). The canonical projection map $\pi : P \to M$ is differentiable.

2. P is locally trivial in the following sense: there is a covering of M by open sets U_α, $\alpha \in \mathcal{J}$, and diffeomorphisms $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that $R_g \circ \phi_\alpha = \phi_\alpha \circ R_g$.

Remark: Sometimes it is more convenient to use the inverse chart maps $T_\alpha := \phi_\alpha^{-1}$.

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