UNIVERSITY OF WARSAW
FACULTY OF PHYSICS
INSTITUTE OF THEORETICAL PHYSICS

Geometry of Third-Order Ordinary Differential Equations and Its Applications in General Relativity

Michał Godliński

PhD thesis written
at the Chair of Theory of Relativity and Gravitation
under supervision of
Dr. hab. Paweł Nurowski

WARSAW 2008

Contents

Introduction

This thesis addresses the problems of equivalence and geometry of third order ordinary differential equations (ODEs) which are stated as follows.

THE EQUIVALENCE PROBLEM. *Given two real differential equations*

$$y''' = F(x, y, y', y'') \tag{1}$$

and

$$y''' = \bar{F}(x, y, y', y'') \tag{2}$$

for a real function $y = y(x)$, establish whether or not there exists a local transformation of variables of a suitable type that transforms (1) *into* (2).

THE GEOMETRY PROBLEM. *Determine geometric structures defined by a class of equations $y''' = F(x, y, y', y'')$ equivalent under certain type of transformations. Find relations between invariants of the ODEs and invariants of the geometric structures.*

One may consider equivalence with respect to several types of transformations, in this work we focus on three best known types: contact, point and fibre-preserving transformations. The fibre-preserving transformations are those which transform the independent variable x and the dependent variable y in such a way that the notion of the independent variable is retained, that is the transformation of x is a function of x only:

$$x \mapsto \bar{x} = \chi(x), \qquad y \mapsto \bar{y} = \phi(x, y). \tag{3}$$

The transformation rules for the derivatives are already uniquely defined by above formulae. Let us define the total derivative to be

$$\mathfrak{D} = \partial_x + y'\partial_y + y''\partial_{y'} + y'''\partial_{y''}.$$

Then

$$y' \mapsto \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = \frac{\mathfrak{D}\phi}{\mathfrak{D}\chi}, \tag{4a}$$

$$y'' \mapsto \frac{\mathrm{d}^2\bar{y}}{\mathrm{d}\bar{x}^2} = \frac{\mathfrak{D}}{\mathfrak{D}\chi}\left(\frac{\mathfrak{D}\phi}{\mathfrak{D}\chi}\right), \tag{4b}$$

$$y''' \mapsto \frac{\mathrm{d}^3\bar{y}}{\mathrm{d}\bar{x}^3} = \frac{\mathfrak{D}}{\mathfrak{D}\chi}\left(\frac{\mathfrak{D}}{\mathfrak{D}\chi}\left(\frac{\mathfrak{D}\phi}{\mathfrak{D}\chi}\right)\right). \tag{4c}$$

The point transformations of variables mix x and y in an arbitrary way

$$x \mapsto \bar{x} = \chi(x, y), \qquad y \mapsto \bar{y} = \phi(x, y), \tag{5}$$

with the derivatives transforming as in (4).

The contact transformations are more general yet. Not only they augment the independent and the dependent variables but also the first derivative

$$x \mapsto \bar{x} = \chi(x, y, y'),$$
$$y \mapsto \bar{y} = \phi(x, y, y'),$$
$$y' \mapsto \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = \psi(x, y, y').$$

However, the functions χ, ϕ and ψ are not arbitrary here but subjecting to (4a) which now yields two additional constraints

$$\psi = \frac{\mathfrak{D}\phi}{\mathfrak{D}\chi} \qquad \Longleftrightarrow \qquad \begin{aligned} \psi \chi_{y'} &= \phi_{y'}, \\ \psi(\chi_x + y'\chi_y) &= \phi_x + y'\phi_y, \end{aligned}$$

guaranteeing that $\mathrm{d}\bar{y}/\mathrm{d}\bar{x}$ really transforms like first derivative. With these conditions fulfilled second and third derivative transform through (4b) – (4c). Of course, fibre-preserving transformations are a subclass of point ones, as well as point transformations form a subclass within contact ones.

We always assume in this work that ODEs are defined locally by a smooth real function F and are considered apart from singularities. The transformations are always assumed to be local diffeomorphisms.

EXAMPLE 1.1. In order to illustrate the equivalence problem let us consider whether or not the equations

$$y''' = 0 \qquad \text{and} \qquad y''' = 3\frac{y''^2}{y'}$$

are equivalent. As one can easily check they are contact equivalent, since the transformation

$$x \mapsto -2y',$$
$$y \mapsto 2xy'^2 - 2yy',$$
$$y' \mapsto -2xy' + y$$

applied to $y''' = 0$ brings it to the other equation. However, they are not fibre-preserving equivalent, since the quantity

$$I(x, y, y', y'') = \frac{\partial^2}{\partial y''^2} F(x, y, y', y'')$$

vanishes for every equation which is fibre-preserving equivalent to $y''' = 0$ but does not vanish for $y''' = 3\frac{y''^2}{y'}$. In order to see this we apply a fibre-preserving transformation of general form (5) to $y''' = 0$ and check that $I = 0$ for the resulting equation.

The above example shows importance of relative invariants in the equivalence problem. A relative invariant is a function of F and its derivatives such that if it vanishes for an equation $y''' = F(x, y, y', y'')$ then it also vanishes for every equation equivalent to it, thereby each relative invariant provides us with a necessary condition for equivalence. Moreover, with the help of adequate number of relative invariants we can also formulate sufficient conditions for equivalence of ODEs, although this construction is complicated and can hardly ever be carried out to the very end because of difficult calculations. The second problem, the problem of geometry, is even more fundamental and in fact it contains the problem of equivalence, for once a geometry associated with ODEs is constructed and a relationship between local invariants of the geometry and of ODEs is found then one can study the equivalence of ODEs via objects of the associated geometry.

A pioneering work on geometry of ODEs of arbitrary order is Karl Wünschmann's PhD thesis [**1**] written under supervision of F. Engel in 1905. In this paper K. Wünschmann observed that solutions of an nth-order ODE $y^{(n)} = F(x, y, y', \ldots, y^{(n-1)})$ may be considered as both curves $y = y(x, c_0, c_1, \ldots, c_{n-1})$ in the xy space and points $c = (c_0, \ldots, c_{n-1})$ in the solution space $\mathbb{R}^n$ parameterized by values of the constants of integration c_i. He defined a relation of kth-order contact between infinitesimally close solutions considered as curves; two solutions $y(x)$ and $y(x) + \mathrm{d}y(x)$ corresponding to c and $c + \mathrm{d}c$ have the kth-order contact if their kth jets coincide at some point (x_0, y_0). Wünschmann's main question was how the property of having $(n-2)$nd contact for $n = 3, 4$ and 5 might be described in terms of the solution space. In particular he examined third-order ODEs and showed that there is a distinguished class of ODEs satisfying certain condition for the function F, which we call the Wünschmann condition. For a third-order ODE in this class, the condition of having first order contact is described by a second order Monge equation for $\mathrm{d}c$. This Monge equation is nothing but the condition that the vector defined by two infinitesimally close points c and $c + \mathrm{d}c$ is null with respect to a Lorentzian conformal metric on the solution space. The last observation, although not contained in Wünschmann's work, follows immediately from his reasoning and was later made by S.-S. Chern [**5**], who cited Wünschmann's thesis.

The main contribution to the issue of point and contact geometry of third-order ODEs was made by respectively E. Cartan and S.-S. Chern in their classical papers [**2, 3, 4**] and [**5**]. We discuss their approach and results in section 1 of Introduction; here we only mention that E. Cartan [**3**] proved that every third-order ODE modulo point transformations and satisfying two differential conditions on the function F, one of them being the Wünschmann condition, has a three-dimensional Lorentzian Einstein-Weyl geometry on its solution space. E. Cartan also showed how to construct invariants of this Weyl geometry from relative point invariants of the underlying ODE. In the same vein S.-S. Chern constructed a three-dimensional Lorentzian conformal geometry for third-order ODEs considered modulo contact transformations and satisfying the Wünschmann condition. In both these cases the conformal metric is precisely the metric appearing implicitly in K. Wünschmann's thesis. Later H. Sato and A. Yoshikawa [**52**] applying N. Tanaka's theory [**55**] constructed a Cartan normal connection for arbitrary third-order ODEs (not only of the Wünschmann type) and showed how its curvature is expressed by the contact relative invariants.

Geometry of third-order ODEs appears in General Relativity and the theory of integrable systems. E.T. Newman et al [**38, 22, 23**] devised the Null Surface Formulation (NSF), a description of General Relativity in terms of families of null hypersurface generated by a Lorentzian metric. The $2 + 1$-dimensional version of this formalism [**20**] is equivalent to Chern's conformal geometry for third-order ODEs [**26, 25, 27**], which was noticed by P. Tod [**56**] for the first time. We encapsulate results of NSF in section 2. Three-dimensional Einstein-Weyl geometry was studied mainly from the perspective of the theory of twistors and integrable systems by N. Hitchin [**34**], R. Ward [**58**], C. LeBrun [**39**] and P. Tod, M. Dunajski et al [**36, 17, 18**]; for discussion of the link between the Einstein-Weyl spaces and third-order ODEs see [**46**] and [**56**].

P. Nurowski, following the ideas of E. Cartan, proposed a programme of systematic study of geometries related to differential equations, including second- and third-order ODEs. In this programme, [**27, 31, 32, 46, 48, 49**], both new and already known geometries associated with differential equations are supposed to be constructed by the Cartan equivalence method and are to be characterized in the language of Cartan connections associated with them. In particular in [**46**] P.

Nurowski provided new examples of geometries associated with ordinary differential equations including a conformal geometry with special holonomy G_2 from ODEs of the Monge type. Partial results on geometries of third-order ODEs were given in [**46, 48, 27, 31**] but the full analysis of these geometries has not been published so far and this thesis, which is a part of the programme, aims to fill this gap.

Geometry of third-order ODEs is a part of broader issue of geometry of differential equations in general. Regarding ODEs of order two, we owe classical results including construction of point invariants to S. Lie [**40**] and M. Tresse [**57**]. In particular E. Cartan [**10**] constructed a two-dimensional projective differential geometry on the solution spaces of some second-order ODEs. This geometry was further studied in [**44**] and [**49**], the latter paper pursues the analogy between geometry of three-dimensional CR structures and second-order ODEs and provides a construction of counterparts of the Fefferman metrics for the ODEs. Classification of second-order ODEs possessing Lie groups of fibre-preserving symmetries was done by L. Hsu and N. Kamran [**35**]. Geometry on the solution space of certain four-order ODEs (satisfying two differential conditions), which is given by the four-dimensional irreducible representation of $GL(2,\mathbb{R})$ and has exotic $GL(2,\mathbb{R})$ holonomy was discovered and studied by R. Bryant [**6**], see also [**47**]. The $GL(2,\mathbb{R})$ geometry of fifth-order ODEs has been recently studied by M. Godliński and P. Nurowski [**32**].

The more general problems yet are existence and properties of geometry on solution spaces of arbitrary ODEs. The problem of existence was solved by B. Doubrov [**14**], who proved that an nth-order ODE $n \geq 3$, modulo contact transformations, has a geometry based on the irreducible n-dimensional representation of $GL(2,\mathbb{R})$ provided that it satisfies $n-2$ scalar differential conditions. An implicit method of constructing these conditions was given in [**13**]. Properties of the $GL(2,\mathbb{R})$ geometries of ODEs are still an open problem; they were studied in [**32**], where the Doubrov conditions were interpreted as higher order counterparts of the Wünschmann condition, and by M. Dunajski and P. Tod [**19**].

Almost all the above papers deal with geometries on solution spaces but one can also consider other geometries, including those defined on various jet spaces. The most general result on such geometries [**16**] comes from T. Morimoto's nilpotent geometry [**42, 43**]. It concludes that with a system of ODEs there is associated a filtration on a suitable jet space together with a canonical Cartan connection.

1. Geometry of third-order ODEs — the present status

We recapitulate the classical results on geometry of third-order ODEs. We do this mostly in the original spirit of E. Cartan, emphasizing the role of systems of one-forms and Cartan connections. The version of Cartan's equivalence method [**9**] we employ below is explained in books by R. Gardner [**29**] and P. Olver [**51**]. Some its aspects are also discussed by S. Sternberg [**53**] and S. Kobayashi [**37**].

E. Cartan's and S.-S. Chern's approach to ODEs. They began with the space $\mathcal{J}^2$ of second jets of curves in $\mathbb{R}^2$ (see [**50**] and [**51**] for extensive description of jet spaces) with coordinate system (x, y, p, q) where p and q denote the first and second derivative y' and y'' for a curve $x \mapsto (x, y(x))$ in $\mathbb{R}^2$, so that this curve lifts to a curve $x \mapsto (x, y(x), y'(x), y''(x))$ in $\mathcal{J}^2$. Any solution $y = f(x)$ of $y''' = F(x, y, y', y'')$ is uniquely defined by a choice of $f(x_0)$, $f'(x_0)$ and $f''(x_0)$ at some x_0. Since that choice is equivalent to a choice of a point in $\mathcal{J}^2$, there passes exactly one solution $(x, f(x), f'(x), f''(x))$ through any point of $\mathcal{J}^2$. Therefore the solutions form a (local) congruence on $\mathcal{J}^2$, which can be described by its annihilating simple

ideal. Let us choose a coframe (ω^i) on $\mathcal{J}^2$:

$$
\begin{aligned}
\omega^1 &= \mathrm{d}y - p\mathrm{d}x, \\
\omega^2 &= \mathrm{d}p - q\mathrm{d}x, \\
\omega^3 &= \mathrm{d}q - F(x, y, p, q)\mathrm{d}x, \\
\omega^4 &= \mathrm{d}x.
\end{aligned}
$$

(6)

Each solution $y = f(x)$ is fully described by the two conditions: forms ω^1, ω^2, ω^3 vanish on the curve $t \mapsto (t, f(t), f'(t), f''(t))$ and, since this defines a solution modulo transformations of x, $\omega^4 = \mathrm{d}t$ on this curve.

Suppose now that equation (1) undergoes a contact, point or fibre-preserving transformation.[1] Then (6) transform by

$$
\begin{aligned}
\omega^1 &\mapsto \bar{\omega}^1 = u_1\omega^1, \\
\omega^2 &\mapsto \bar{\omega}^2 = u_2\omega^1 + u_3\omega^2, \\
\omega^3 &\mapsto \bar{\omega}^3 = u_4\omega^1 + u_5\omega^2 + u_6\omega^3, \\
\omega^4 &\mapsto \bar{\omega}^4 = u_8\omega^1 + u_9\omega^2 + u_7\omega^4,
\end{aligned}
$$

(7)

with some functions $u_1, \ldots, u_9$ defined on $\mathcal{J}^2$ and determined by a particular choice of transformation, for instance

$$
u_1 = \phi_y - \psi\chi_y,
$$
$$
u_7 = \mathfrak{D}\chi, \quad u_8 = \chi_y, \quad u_9 = \chi_p.
$$

In particular, $u_9 = 0$ in the point case and $u_8 = u_9 = 0$ in the fibre-preserving case. Since the transformations are non-degenerate, the condition $u_1 u_3 u_5 u_7 \neq 0$ is always satisfied and transformations (7) form groups. Thus a class of contact equivalent third-order ODEs is a local G-structure[2] $G_c \times \mathcal{J}^2$, that is a local subbundle of the bundle of linear frames on $\mathcal{J}^2$, defined by the property that the coframe $(\omega^1, \omega^2, \omega^3, \omega^4)$ belongs to it and the structural group is given by

$$
(8) \qquad G_c = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & u_5 & u_6 & 0 \\ u_8 & u_9 & 0 & u_7 \end{pmatrix}.
$$

In a like manner, for point and fibre-preserving transformation we have G-structures with groups $G_p \subset G_c$ given by $u_9 = 0$ and $G_f \subset G_p$ given by $u_8 = u_9 = 0$.

Thereby the problem of equivalence has been formulated; two ODEs are contact/point/fibre-preserving equivalent if and only if the $G_c/G_p/G_f$-structures which correspond to them are equivalent. In other words, it means that there exists a diffeomorphism which transforms one G-structure into the other G-structure.

On the bundles $G_c \times \mathcal{J}^2$, $G_p \times \mathcal{J}^2$ or $G_f \times \mathcal{J}^2$ there are four fixed, well defined one-forms $(\theta^1, \theta^2, \theta^3, \theta^4)$, the components of the canonical $\mathbb{R}^4$-valued form θ existing on the frame bundle of $\mathcal{J}^2$. Let us consider the contact case as an example. Let (x) denote x, y, p, q and (g) be coordinates in G_c given by (8). Let us choose a coordinate system (x, g) on $G_c \times \mathcal{J}^2$ compatible with the local trivialization. Then

---

[1]Although E.Cartan and S.-S. Chern did not examine fibre-preserving transformations we treat them on equal footing with the others for future references.

[2]We work in the local trivializations of the G-structure

θ^i at the point (x, g^{-1}) read

$$
\begin{aligned}
\theta^1 &= u_1 \omega^1, \\
\theta^2 &= u_2 \omega^1 + u_3 \omega^2, \\
\theta^3 &= u_4 \omega^1 + u_5 \omega^2 + u_6 \omega^3, \\
\theta^4 &= u_8 \omega^1 + u_9 \omega^2 + u_7 \omega^4.
\end{aligned}
$$
(9)

The idea of Cartan's method is the following. Starting from $G_c \times \mathcal{J}^2$ (or $G_p \times \mathcal{J}^2$ or $G_f \times \mathcal{J}^2$) one constructs a new principal bundle $\mathcal{P} = H \times \mathcal{J}^2$ (with a new group H) equipped with one fixed coframe (θ^i, Ω_μ) built in some geometric way and such that it encodes all the local invariant information about $G_c \times \mathcal{J}^2$ (or $G_p \times \mathcal{J}^2$ or $G_f \times \mathcal{J}^2$ respectively) through its structural equations.

REMARK 1.2. As an illustration of this idea consider its very special application to a metric structure of signature (k, l) on a manifold $\mathcal{M}$. The structure defines locally the bundle $O(k, l) \times \mathcal{M}$, an $O(k, l)$-reduction of the frame bundle. If (ω^i) is an orthonormal coframe on $\mathcal{M}$, then the forms (θ^i) at a point (x, g) are given by $\theta^i = (g^{-1})^i{}_j \omega^j$, $g \in O(k, l)$. Moreover, associated to (ω^i) there are the Levi-Civita connection one-forms $\Gamma^i{}_j$, which when lifted to $O(k, l) \times \mathcal{M}$ define $\frac{1}{2} n(n-1)$ connection forms $\Omega^i{}_j$ by

$$
\Omega^i{}_j(x, g) = (g^{-1})^i{}_k \Gamma^k{}_l(x) \, g^l{}_j + (g^{-1})^i{}_k \mathrm{d} g^k{}_j.
$$

Clearly $(\theta^i, \Omega^j{}_k)$ is a coframe on $O(k, l) \times \mathcal{M}$. It is the Cartan coframe for the metric structure. The Cartan invariants are the Riemannian curvature R given by the equations

$$
\begin{aligned}
\mathrm{d}\theta^i + \Omega^i{}_j \wedge \theta^j &= 0, \\
\mathrm{d}\Omega^i{}_j + \Omega^j{}_k \wedge \Omega^k{}_j &= \tfrac{1}{2} R^i_{jkl} \theta^k \wedge \theta^l
\end{aligned}
$$

and its consecutive covariant derivatives. Here it is easy to construct the desired coframe because there is a distinguished $\mathfrak{o}(k, l)$-valued connection — the Levi-Civita connection.

In our cases situation is more complicated. There are no natural candidates for the forms Ω_μ since there is no distinguished connection on the bundles $G_. \times J^2$. In order to resolve such situations E. Cartan developed the method of constructing the desired bundle $\mathcal{P}$ through a series of reductions and prolongations of an initial structural group. This procedure is too complicated to be included in Introduction; we postpone the full discussion to section 2 of chapter 2 where we follow E. Cartan's and S.-S. Chern's reasoning. Here we only formulate main conclusions.

Point equivalence. Examining the problem of point equivalence E. Cartan constructed for every ODE a local seven-dimensional bundle $\mathcal{P}$ over J^2, a reduction of $G_p \times \mathcal{J}^2$, together with a coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3)$ on $\mathcal{P}$. The coframe contains all the point invariant information on the original ODE due to the following theorem, which is the application of the equivalence method to third-order ODEs.

THEOREM 1.3 (E. Cartan). *Equations* $y''' = F(x, y, y', y'')$ *and* $\bar{y}''' = \bar{F}(\bar{x}, \bar{y}, \bar{y}', \bar{y}'')$ *with smooth* F *and* $\bar{F}$ *are locally point equivalent if and only if there exists a local diffeomorphism* $\Phi \colon \mathcal{P} \to \bar{\mathcal{P}}$ *which maps the Cartan coframe* $(\bar{\theta}^1, \ldots, \bar{\Omega}_3)$ *of* $\bar{F}$ *to the coframe* $(\theta^1, \ldots, \Omega^3)$ *of* F,

$$
\Phi^* \bar{\theta}^i = \theta^i, \qquad \Phi^* \bar{\Omega}_\mu = \Omega_\mu, \qquad i = 1, \ldots, 4, \quad \mu = 1, 2, 3.
$$

Φ *projects to a point transformation* $(x, y) \mapsto (\bar{x}(x, y), \bar{y}(x, y))$ *which maps* $\bar{F}$ *to* F.

The general idea of determining whether or not Φ exist is as follows, see[**51**]. The structural equations for the Cartan coframe are

$$
\begin{aligned}
\mathrm{d}\theta^1 =& \Omega_1 {\scriptstyle\wedge}\, \theta^1 + \theta^4 {\scriptstyle\wedge}\, \theta^2, \\
\mathrm{d}\theta^2 =& \Omega_2 {\scriptstyle\wedge}\, \theta^1 + \Omega_3 {\scriptstyle\wedge}\, \theta^2 + \theta^4 {\scriptstyle\wedge}\, \theta^3, \\
\mathrm{d}\theta^3 =& \Omega_2 {\scriptstyle\wedge}\, \theta^2 + (2\Omega_3 - \Omega_1) {\scriptstyle\wedge}\, \theta^3 + \mathbf{A_1}\theta^4 {\scriptstyle\wedge}\, \theta^1, \\
\mathrm{d}\theta^4 =& (\Omega_1 - \Omega_3) {\scriptstyle\wedge}\, \theta^4 + \mathbf{B_1}\theta^2 {\scriptstyle\wedge}\, \theta^1 + \mathbf{B_2}\theta^3 {\scriptstyle\wedge}\, \theta^1,
\end{aligned}
$$

$$
\begin{aligned}
(10) \qquad \mathrm{d}\Omega_1 =& -\Omega_2 {\scriptstyle\wedge}\, \theta^4 + (\mathbf{D_1} + 3\mathbf{B_3})\theta^1 {\scriptstyle\wedge}\, \theta^2 + (3\mathbf{B_4} - 2\mathbf{B_1})\theta^1 {\scriptstyle\wedge}\, \theta^3 \\
& + (2\mathbf{C_1} - \mathbf{A_2})\theta^1 {\scriptstyle\wedge}\, \theta^4 - \mathbf{B_2}\theta^2 {\scriptstyle\wedge}\, \theta^3, \\
\mathrm{d}\Omega_2 =& (\Omega_3 - \Omega_1) {\scriptstyle\wedge}\, \Omega_2 + \mathbf{D_2}\theta^1 {\scriptstyle\wedge}\, \theta^2 + (\mathbf{D_1} + \mathbf{B_3})\theta^1 {\scriptstyle\wedge}\, \theta^3 + \mathbf{A_3}\theta^1 {\scriptstyle\wedge}\, \theta^4 \\
& + (2\mathbf{B_4} - \mathbf{B_1})\theta^2 {\scriptstyle\wedge}\, \theta^3 + \mathbf{C_1}\theta^2 {\scriptstyle\wedge}\, \theta^4, \\
\mathrm{d}\Omega_3 =& (\mathbf{D_1} + 2\mathbf{B_3})\theta^1 {\scriptstyle\wedge}\, \theta^2 + 2(\mathbf{B_4} - \mathbf{B_1})\theta^1 {\scriptstyle\wedge}\, \theta^3 + \mathbf{C_1}\theta^1 {\scriptstyle\wedge}\, \theta^4 + \mathbf{B_2}\theta^2 {\scriptstyle\wedge}\, \theta^3,
\end{aligned}
$$

with some explicitly given functions $\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{B_4}, \mathbf{C_1}, \mathbf{D_1}, \mathbf{D_2}$ on $\mathcal{P}$. Let $(X_1, \ldots, X_7)$ denotes the frame dual to $(\theta^1, \ldots, \Omega_3)$. Since pull-back commutes with exterior differentiation each of these functions is a point relative invariant of the underlying ODE. Furthermore, the coframe derivatives $X_1(\mathbf{A_1}), \ldots, X_7(\mathbf{D_2})$, $X_1(X_1(\mathbf{A_1})), \ldots, X_7(X_7(\mathbf{D_2})), \ldots$ of arbitrary order are also point relative invariants. We calculate all relevant coframe derivatives for F and $\bar{F}$ up to some finite order n and gather them into respective functions $\mathbf{T}\colon \mathcal{P} \to \mathbb{R}^N$ and $\bar{\mathbf{T}}\colon \bar{\mathcal{P}} \to \mathbb{R}^N$ with the same target space of dimension N equal to the number of the invariants. Finally, we examine whether or not the graphs of $\mathbf{T}$ and $\bar{\mathbf{T}}$ overlap as manifolds in $\mathbb{R}^N$. If they overlap, then the sought diffeomorphism Φ exists between certain nonempty open sets $U \subset \mathbf{T}^{-1}(\mathcal{O})$ and $\bar{U} \subset \bar{\mathbf{T}}^{-1}(\mathcal{O})$, where $\mathcal{O}$ is the overlap. A detailed explanation of this procedure is given at the beginning of chapter 4; at this stage we only need to know that the coframe solves the point equivalence problem for ODEs. What joins this problem with the domain of differential geometry is the fact that the coframe is a geometric object — a Cartan connection — on $\mathcal{P} \to \mathcal{J}^2$.

In order to introduce the notion of Cartan connection we first show how to read the structure of principal bundle on $\mathcal{P} \to \mathcal{J}^2$ from (10). Fibres of the projection $\mathcal{P} \to J^2$ are annihilated by the forms $\theta^1, \theta^2, \theta^3, \theta^4$, and the vector fields X_5, X_6, X_7 are tangent to the fibres. Simultaneously, the commutation relations of these fields are isomorphic to commutators of the three-dimensional algebra[3] $\mathfrak{h} = \mathbb{R} \oplus (\mathbb{R} \oplus_\cdot \mathbb{R})$ and they define a local action of the group $H = \mathbb{R} \times (\mathbb{R} \ltimes \mathbb{R})$ on $\mathcal{P}$ for which X_5, X_6, X_7 are the fundamental fields.

Now, let us focus on the most symmetric situation when all the relative invariants $\mathbf{A_1}, \ldots, \mathbf{D_2}$ vanish. This case corresponds to an ODE which is point equivalent to the trivial $y''' = 0$. For such an ODE the equations (10) become the Maurer-Cartan equations for the algebra $\mathfrak{co}(2,1) \oplus_\cdot \mathbb{R}^3$, where $\mathfrak{co}(2,1) = \mathbb{R} \oplus \mathfrak{so}(2,1)$ is the orthogonal algebra centrally extended by dilatations generated by the identity matrix. As a consequence, $\mathcal{P}$ becomes locally the Lie group $CO(2,1) \ltimes \mathbb{R}^3$, with the left-invariant fields $X_1, \ldots, X_7$. Therefore $\mathcal{J}^2$ is the homogeneous space $H \to CO(2,1) \ltimes \mathbb{R}^3 \to CO(2,1) \ltimes \mathbb{R}^3/H$ acted upon by $CO(2,1) \ltimes \mathbb{R}^3$. This group acts on $\mathcal{J}^2$ as the group of point symmetries of $y''' = 0$, that is for any solution $y = f(x)$ of $y''' = 0$ its graph $x \to (x, y(x), y'(x), y''(x))$ in $\mathcal{J}^2$ is transformed into the graph of other solution of $y''' = 0$. The coframe (θ^i, Ω_μ) can be arranged into

---

[3]The symbol $\mathfrak{g} \oplus_\cdot \mathfrak{h}$ denotes a semidirect product of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$

the following matrix

(11)
$$\widehat{\omega} = \begin{pmatrix} \Omega_3 & 0 & 0 & 0 & 0 \\ \theta^1 & \Omega_3 - \Omega_1 & -\theta^4 & 0 & 0 \\ \theta^2 & -\Omega_2 & 0 & -\theta^4 & 0 \\ \theta^3 & 0 & -\Omega_2 & \Omega_1 - \Omega_3 & 0 \\ 0 & \theta^3 & -\theta^2 & \theta^1 & -\Omega_3 \end{pmatrix},$$

which is the Maurer-Cartan one-form of $CO(2,1) \ltimes \mathbb{R}^3$. In the language of $\widehat{\omega}$ the equations (10) read

$$\mathrm{d}\widehat{\omega} + \tfrac{1}{2}[\widehat{\omega}, \widehat{\omega}] = 0.$$

Turning to an arbitrary situation, we see that non-trivial cases can not be described in this language, since the invariants $\mathbf{A_1}, \ldots, \mathbf{D_2}$ do not vanish in general and the last equation does not hold. We need a new object, the Cartan connection, defined here after [37].

DEFINITION 1.4. Let $H \to \mathcal{P} \to \mathcal{M}$ be a principal bundle and let G be a Lie group such that H is its closed subgroup and $\dim G = \dim \mathcal{P}$. A Cartan connection of type (G, H) on $\mathcal{P}$ is a one-form $\widehat{\omega}$ taking values in the Lie algebra $\mathfrak{g}$ of G and satisfying the following conditions:

  i) $\widehat{\omega}_u : T_u \mathcal{P} \to \mathfrak{g}$ for every $u \in \mathcal{P}$ is an isomorphism of vector spaces
  ii) $A^* \lrcorner \widehat{\omega} = A$ for every $A \in \mathfrak{h}$ and the corresponding fundamental field A^*
  iii) $R_h^* \widehat{\omega} = \mathrm{Ad}(h^{-1})\widehat{\omega}$ for $h \in H$.

A Cartan connection is then an object that generalizes the notion of the Maurer-Cartan form on a Lie group. The curvature of a Cartan connection

$$\widehat{K} = \mathrm{d}\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}]$$

does not have to vanish any longer and measures, as it were, how far $H \to \mathcal{P} \to \mathcal{M}$ 'differs' from the homogeneous space $H \to G \to G/H$.

In our case the formula (11) defines a $\mathfrak{co}(2,1) \oplus \mathbb{R}^3$-valued Cartan connection, whose non-vanishing curvature contains basic point invariants, from which the full set of invariants can be constructed through exterior differentiation. This is the basic relation, announced at the beginning of Introduction, between classification of ODEs and their geometry. In chapter 3 we discuss properties of $\widehat{\omega}$ in full detail.

The geometry introduced above, however, is not the most interesting structure one can associate with the ODEs modulo point transformations. Indeed, the crucial observation E. Cartan made in his paper [3] was that the connection $\widehat{\omega}$ may generate a new type of geometry on the solution space of certain classes of ODEs.

In order to present the idea of this construction let us invoke once more the trivial equation $y''' = 0$ but now consider how the symmetry group act on the solutions regarded as curves in the xy plane. It is well-known that the full group of point symmetries of $y''' = 0$ is generated by the following one-parameter groups of transformations $(x, y) \mapsto \Phi_t^i(x, y)$

$$\Phi_t^1(x, y) = (x, y + t), \qquad\qquad \Phi_t^2(x, y) = (x, y + 2xt),$$
$$\Phi_t^3(x, y) = (x, y + x^2 t), \qquad\qquad \Phi_t^4(x, y) = (x, ye^{-t}),$$
$$\Phi_t^5(x, y) = (xe^t, ye^t), \qquad\qquad \Phi_t^6(x, y) = (x + t, y),$$
$$\Phi_t^7(x, y) = \left(\frac{x}{1 + xt}, \frac{y}{(1 + xt)^2} \right).$$

A solution to $y''' = 0$ is a parabola of the form

(12)
$$y(x) = c_2 x^2 + 2c_1 x + c_0$$

with three arbitrary integration constants c_2, c_1, c_0. Therefore a solution of $y''' = 0$ may be identified with a point $c = (c_2, c_1, c_0)^T$ in the solution space $\mathcal{S} \cong \mathbb{R}^3$. Transforming (12) according to the above formulae we find that $\Phi_t^1, \Phi_t^2, \Phi_t^3$ are translations in the solution space $\mathcal{S}$:

$$\Phi_t^1(c) = \begin{pmatrix} c_2 \\ c_1 \\ c_0 - t \end{pmatrix}, \quad \Phi_t^2(c) = \begin{pmatrix} c_2 \\ c_1 - t \\ c_0 \end{pmatrix}, \quad \Phi_t^3(c) = \begin{pmatrix} c_2 - t \\ c_1 \\ c_0 \end{pmatrix},$$

while transformations $\Phi_t^4, \ldots, \Phi_t^7$ generate the three-dimensional irreducible representation of $CO(2,1)$:

$$\Phi_t^4(c) = \exp t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} c, \qquad \Phi_t^5(c) = \exp t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} c,$$

$$\Phi_t^6(c) = \exp t \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} c, \qquad \Phi_t^7(c) = \exp t \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} c.$$

Thereby $\mathcal{S}$ is a homogeneous space equipped with the flat conformal metric[4] $[g]$

$$(13) \qquad\qquad g = (\mathrm{d}c_0)(\mathrm{d}c_2) - (\mathrm{d}c_1)^2$$

and preserved by $CO(2,1)$

$$M^T g M = e^\lambda g, \quad \text{for} \quad M \in CO(2,1) \quad \text{and} \quad \lambda \in \mathbb{R}.$$

Apart from $[g]$ there is another piece of structure in $\mathcal{S}$. The symmetry group acting on $\mathcal{S}$ is not the full ten-dimensional conformal symmetry group of the flat metric $Conf(2,1) \cong O(3,2)$, but merely $CO(2,1) \ltimes \mathbb{R}^3$, the Euclidean group extended by dilatations, hence another object is needed to reduce $O(3,2)$ to $CO(2,1) \ltimes \mathbb{R}^3$. This object is the Weyl one-form of the flat Weyl geometry. Let us remind that a Weyl geometry (g, ϕ) is a metric g and a one-form ϕ given modulo transformations $g \to e^{2\lambda} g$, $\phi \to \phi + \mathrm{d}\lambda$, see chapter 3 section 2.1. In our case the Weyl geometry is flat, because there is the flat representative (13) for which, in addition, $\phi = 0$.

Now a question arises: how the flat Weyl structure on $\mathcal{S}$ can be reconstructed from the Cartan coframe (θ^i, Ω_μ)? In order to answer this question we will use the method of construction through fibre bundles and Lie transport, which differs from E. Cartan's original reasoning and was introduced by P. Nurowski, cf [**45, 46**].

To begin with, we observe that $\mathcal{P}$ is always, not only in the trivial case, a principal bundle $CO(2,1) \to \mathcal{P} \to \mathcal{S}$. Indeed, as we said, $\mathcal{J}^2$ is foliated by solutions of the underlying ODE, which are curves in $\mathcal{J}^2$ annihilated by ω^1, ω^2 and ω^3. Thus we have a projection $\mathcal{J}^2 \to \mathcal{S}$ with the fibres being the solutions. As a consequence $\mathcal{P}$ is also a bundle over $\mathcal{S}$ with leaves of the projection annihilated by the ideal $(\theta^1, \theta^2, \theta^3)$. On the leaves of $\mathcal{P} \to \mathcal{S}$ there act the vector fields $X_4, \ldots, X_7$, members of the frame dual to (θ^i, Ω_μ). In view of (10) their commutation relations are isomorphic to $\mathfrak{co}(2,1)$ and turn each leaf into an orbit of the free action of $CO(2,1)$ regardless whether or not the invariants $\mathbf{A_1}, \ldots, \mathbf{D_2}$ vanish. If in addition $\mathbf{A_1} = \ldots = \mathbf{D_2} = 0$ then locally $\mathcal{P} \cong CO(2,1) \ltimes \mathbb{R}^3$, which generates the structure of homogeneous space on $\mathcal{S}$. Next, still in the trivial case, let us consider the bilinear form

$$(14) \qquad\qquad \widehat{g} = 2\theta^1 \theta^3 - (\theta^2)^2$$

---

[4]The symbol $[g]$ denotes the conformal metric, whose representative is the metric g. We also adapt the following notation: $\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$ for one-forms α and β.

and the one-form Ω_3 on $\mathcal{P}$. We calculate the Lie derivatives of these objects along the fields $X_4, \ldots, X_7$ tangent to the fibres of $\mathcal{P} \to \mathcal{S}$ and observe that

$$
(15) \qquad
\begin{aligned}
&\widehat{g}(X_j, \cdot) \equiv 0, \quad \text{for} \quad j = 4, 5, 6, 7, \\
&L_{X_4}\widehat{g} = 0, \quad L_{X_5}\widehat{g} = 0, \quad L_{X_6}\widehat{g} = 0, \quad L_{X_7}\widehat{g} = 2\widehat{g}
\end{aligned}
$$

and

$$
(16) \qquad
\begin{aligned}
&X_4 \lrcorner \Omega_3 = 0, \quad X_5 \lrcorner \Omega_3 = 0, \quad X_6 \lrcorner \Omega_3 = 0, \quad X_7 \lrcorner \Omega_3 = 1, \\
&L_{X_j}\Omega_3 = 0, \quad \text{for} \quad j = 4, 5, 6, 7.
\end{aligned}
$$

These properties allow us to project the pair $(\widehat{g}, \Omega_3)$ along $\mathcal{P} \to \mathcal{S}$ to a Weyl structure (g, ϕ) on $\mathcal{S}$. This is precisely the flat Weyl structure obtained above by action of the symmetry group.

The construction through Lie derivatives and the projection has an essential advantage in comparison to the symmetry approach, since it can be immediately generalized to non-trivial equations. In fact, the equations (15), (16) hold not only in the trivial case $y''' = 0$ but under much weaker conditions $\mathbf{A_1} = 0$ and $\mathbf{C_1} = 0$. Calculating the explicit forms of these invariants Cartan proved that if only an ODE given by a function $F(x, y, p, q)$ satisfies

$$
(17) \qquad \left(\mathcal{D} - \tfrac{2}{3}F_q\right)\left(\tfrac{1}{6}\mathcal{D}F_q - \tfrac{1}{9}F_q^2 - \tfrac{1}{2}F_p\right) + F_y = 0,
$$

$$
(18) \qquad \mathcal{D}^2 F_{qq} - \mathcal{D}F_{qp} + F_{qy} = 0,
$$

where

$$
\mathcal{D} = \partial_x + p\partial_y + q\partial_p + F\partial_q,
$$

then it has a Weyl geometry on its solution space $\mathcal{S}$. The quantity on the left hand side of the condition (17), the Wünschmann invariant, was found in [**1**] for the first time.

For the equations satisfying (17) and (18) the bundle $CO(2, 1) \to \mathcal{P} \to \mathcal{S}$ is the bundle of orthonormal frames for the conformal metric $[g]$ on $\mathcal{S}$, whereas $\widehat{\omega}$ of (11) becomes a $\mathfrak{co}(2, 1) \oplus \mathbb{R}^3$-valued Cartan connection. The $\mathfrak{co}(2, 1)$-part of $\widehat{\omega}$ is the Weyl connection, and the Weyl curvature is expressed by the invariants $\mathbf{B_1}, \mathbf{B_2}, \mathbf{B_3}, \mathbf{B_4}$, which do not vanish in general. Cartan proved that all the Weyl geometries constructed of third-order ODEs are Einstein, that is the traceless part of the Ricci tensor for their Weyl connections always vanishes. Finally, he observed [**4**] that these Einstein-Weyl geometries are of general form; for each three-dimensional Lorentzian Einstein-Weyl structure there is an ODE satisfying (17), (18), whose solution space carries this structure.

Contact equivalence. Soon after E.Cartan studied the point equivalence and geometry, the contact problem was examined by S.-S. Chern using the same method. After a slightly more complicated construction (we discuss it in chapter 2) Chern built a ten-dimensional bundle $\mathcal{P} \to \mathcal{J}^2$ equipped with the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \ldots, \Omega_6)$. For $y''' = 0$ the structural equations coincide with the Maurer-Cartan equations for the algebra $\mathfrak{o}(3, 2) \cong \mathfrak{sp}(4, \mathbb{R})$ which reflects the fact that $O(3, 2)$ is the maximal group of contact symmetries of $y''' = 0$.

Next difference between Cartan's and Chern's results is that here the coframe (θ^i, Ω_μ) for a general ODE is not a Cartan connection since it does not transform regularly along the fibres of the bundle $\mathcal{P} \to \mathcal{J}^2$, i.e. does not fulfill condition iii) of the definition. Lack of this regularity means that there are nonconstant $\Omega_\alpha \wedge \theta^k$ terms in the structural equations. However, from the point of view of S.-S. Chern,

who was interested in geometries on the solutions space analogous to the Einstein-Weyl geometries, this fault was insignificant and he did not investigate it. The Cartan connection for this problem was given later in [**52**] and in [**16**].

Leaving aside this topic let us consider S.-S. Chern's results. Given the ten-dimensional coframe he noticed that in the structural equations there appears the Wünschmann invariant and further analysis depends on whether it vanishes or not. If the Wünschmann invariant vanishes then the coframe (θ^i, Ω_μ) becomes an $\mathfrak{o}(3,2)$-valued Cartan normal conformal connection over $\mathcal{S}$. In this case S.-S. Chern gave an explicit formula for the underlying Lorentzian conformal metric $[g]$ in terms of the forms $\omega^1, \omega^2, \omega^3$ on $\mathcal{J}^2$, see eq. (51) in chapter 2. Since the contact symmetry group of $y''' = 0$ is $O(3,2)$, the full conformal group of the flat metric in $2+1$ dimensions, there is no additional geometric object on $\mathcal{S}$ and we have just the conformal geometry instead of a Weyl geometry there. Later it was shown [**48**] that the conformal geometry can be obtained from the bilinear symmetric field $\widehat{g} = (\theta^2)^2 - 2\theta^1\theta^3$ on $\mathcal{P}$ by virtue of conditions analogous to (15). The lowest-order conformal invariant in three dimensions — the Cotton tensor — was also expressed in terms of contact relative invariants for ODEs.

Turning to the equations with non-vanishing Wünschmann invariant, S.-S. Chern continued reduction of the bundle $\mathcal{P}$ according to Cartan's method and obtained a five-dimensional manifold, say $\mathcal{P}_5$, which is a line bundle over $\mathcal{J}^2$ and is furnished with a coframe $(\theta^1, \ldots, \theta^4, \Omega)$. Next, he recognized that in the homogeneous case of the equation $y''' = -y$ the bundle $\mathcal{P}_5$ is a group $\mathbb{R}^2 \ltimes \mathbb{R}^3$ and it generates a geometry of cones on the solution space. This is rather an exotic kind of structure and we will not discuss it here, referring the reader to section 7 of chapter 2.

This closes our summary of the known results on the geometry and classification of third-order ODEs.

2. Null Surface Formulation

An intriguing thing about third-order ODEs is that the Lorentzian geometry on $\mathcal{S}$ was rediscovered fifty years after S.-S. Chern from completely different perspective, the perspective of General Relativity. In a series of papers [**38, 22, 23, 24**] E.T. Newman et al proposed and developed the Null Surface Formulation (NSF), an alternate approach to General Relativity. Ideas of this approach are the following. Let $\mathcal{M}^4$ be a four-manifold coordinated by (x^μ). Usually, the basic object in General Relativity is a Lorentzian metric g on $\mathcal{M}^4$ (or the Levi-Civita connection), from which other objects are derived, the Riemann tensor, the Weyl tensor and the Einstein tensor, on which dynamics is imposed by the Einstein equations. One of many objects in Lorentzian geometry is a null hypersurface. It is by definition a hypersurface $Z(x^\mu) = const$ in $\mathcal{M}^4$ which satisfies the eiconal equation

$$(19) \qquad\qquad\qquad g(\mathrm{d}Z, \mathrm{d}Z) = 0,$$

where g is the inverse (contravariant) metric. For a given g the family of all null hypersurfaces is fully defined by the above equation.

The NSF is an alternate point of view, here one begins with $\mathcal{M}^4$ without any metric but endowed with a two-parameter family of hypersurfaces. Starting from these data a Lorentzian *conformal* metric is constructed by the property that these hypersurfaces are its null hypersurfaces; the metric is found by solving the eiconal equation (19) with respect to the components $g^{\mu\nu}$. In this approach it is the family of hypersurfaces that is a basic concept and the metric is a derived one. The family is defined by a sufficiently differentiable real function $Z(x^\mu, s, s^*)$ on $\mathcal{M} \times S^2$, where s and s^* are stereographic variables on the sphere S^2 and * denotes the complex

conjugation. Each hypersurface is a level set

$$Z(x^\mu, s, s^*) = const,$$

with some fixed s. The reason why $s \in S^2$ is that the hypersurfaces are to be null and then S^2 becomes the sphere of null directions. When x^μ are fixed and s sweeps out the sphere we obtain the corresponding hypersurfaces at x^μ orthogonal to all null directions. In order to find the metric the authors assumed that the functions

(20)
$$f^0 = Z, \quad f^+ = Z_s, \quad f^- = Z_{s^*}, \quad f^1 = Z_{ss^*}$$

form a coordinate system on $\mathcal{M}^4$ for all s, introduced the coframe $(\mathrm{d}f^0, \mathrm{d}f^+, \mathrm{d}f^-, \mathrm{d}f^1)$ and found explicit formulae for components $g^{\mu\nu}$ in this coframe. The eiconal equation implies $g^{00} = 0$ in the coframe $(\mathrm{d}f^A)$ but one must still take into account vanishing of derivatives of g^{00} with respect to s and s^*. Doing so the authors found that the conformal metric is uniquely defined by the family of hypersurfaces provided that the function Z and a real conformal factor $\Omega^2(x^\mu, s, s^*)$ for the Lorentzian conformal metric satisfy two quite complicated complex differential conditions, referred to as metricity conditions. If they are satisfied then the components of the conformal Lorentzian metric $[g]$ are products of Ω^2 and some expressions containing derivatives of Z. Moreover, both metricity conditions and the components g^{AB} only depend of Z via its derivatives Z_{ss} and $Z_{s^*s^*}$ and one can eliminate the space-time coordinates in Ω and Z_{ss} through (20) and obtain the functions

$$\Lambda(f^A, s, s^*) = Z_{ss}(x^\mu(f^A, s, s^*), s, s^*), \qquad \Omega(f^A, s, s^*) = \Omega(x^\mu(f^A, s, s^*), s, s^*).$$

Thereby information about a Lorentzian metric is encoded by two complex functions Ω and Λ of the variables $(s, s^*, Z(s, s^*), Z_s, Z_{s^*}, Z_{ss^*})$, satisfying the two metricity conditions and the integrability condition $\frac{\mathrm{d}^2}{\mathrm{d}s^{*2}}\Lambda = \frac{\mathrm{d}^2}{\mathrm{d}s^2}\Lambda^*$.

Next step was writing down Einstein equations $G^{\mu\nu} = \varkappa T^{\mu\nu}$ in the new variables. The authors proved that applying consecutive derivatives ∂_s and ∂_{s^*} to

(21)
$$G^{\mu\nu}Z_\mu Z_\nu = \varkappa T^{\mu\nu} Z_\mu Z_\nu,$$

$Z_\mu = \partial_\mu Z$, one obtains nine out of ten equations and the lacking tenth equation, the trace component, is recovered with the help of the metricity conditions. After suitable substitutions, the vacuum version of (21) reduces to one equation

$$\Omega_{f^1 f^1} - Q[\Lambda]\Omega = 0.$$

In this manner the vacuum Einstein equations were reduced to the set of four complex equations for Ω and Λ: the one above, the two metricity conditions and the integrability condition for Λ. Having proven this result in [23], the authors moved to the analysis of solutions of the Einstein equations, their linearization, perspectives for quantization and other topics we will not cover here. From our point of view the most interesting is the three-dimensional version of the NSF [20, 26, 25, 27], which leads immediately to third-order ODEs. We cite the construction in detail.

In the case of $2 + 1$-dimensional Lorentzian geometry on $\mathcal{M}^3$ the space of null directions is diffeomorphic to S^1 and a conformal class can be reconstructed from the one-parameter family of surfaces

$$Z(x^i, s) = const,$$

where $(x^i) = (x^0, x^1, x^2) \in \mathcal{M}^3$ and $s \in S^1$ real. Let us introduce the functions

(22)
$$y = Z(x^i, s), \quad p = Z_s(x^i, s), \quad q = Z_{ss}(x^i, s)$$

and the coframe

$$\sigma^0 = \mathrm{d}y, \quad \sigma^1 = \mathrm{d}p, \quad \sigma^2 = \mathrm{d}q.$$

The third derivative, $Z_{sss}(x^i, s)$, together with (22) define a real function

$$\Lambda(s, y, p, q) = Z_{sss}(x^i(s, y, p, q), s).$$

It is obvious that the variables (s, y, p, q) constitute a coordinate system on $\mathcal{J}^2$, the space of second jets of functions $S^1 \to \mathbb{R}$. Moreover, the function Λ on $\mathcal{J}^2$ defines the third-order ODE

(23)
$$Z_{sss} = \Lambda(s, Z, Z_s, Z_{ss})$$

for $Z(s)$. The function $Z(x^i, s)$, with which we have begun, can be identified with the general solution of this equation, x^i playing the role of three integration constants. In this manner $\mathcal{M}^3$ becomes the solution space of the ODE and $\mathcal{M}^3 \times S^1$ is identified with $\mathcal{J}^2$, where the projection $\mathcal{J}^2 \to \mathcal{M}^3$ is given by $(x^i, s) \mapsto (x^i)$. The fibres of this projection are solutions of (23) considered as curves in J^2. We also notice that the total derivative $\frac{\mathrm{d}}{\mathrm{d}s}$ applied to a function on $\mathcal{J}^2$ coincides with the total derivative $\mathcal{D}$

$$\frac{\mathrm{d}}{\mathrm{d}s} f(s, y, p, q) = \mathcal{D}f = (\partial_s + p\partial_y + q\partial_p + \Lambda\partial_q)f.$$

It follows that the construction of a conformal geometry from the family of surfaces is fully equivalent to Chern's construction described earlier and the metricity conditions contain the Wünschmann condition. Let us look at how this construction was done. The eiconal equation

$$g^{ij} Z_i Z_j = g^{ij} y_i y_j = 0$$

for the sought metric $g = g^{ij}\partial_{x^i} \otimes \partial_{x^j}$ implies, as before, $g^{00} = 0$ in the coframe (σ^i). Taking $\partial_s g^{00}$ gives

$$g^{ij} y_i p_j = g^{01} = 0.$$

Another derivation ∂_s yields

$$g^{ij} y_i q_j + g^{ij} p_i p_j = g^{02} + g^{11} = 0.$$

Third and fourth derivatives yield

$$g^{02}\Lambda_q + 3g^{12} = 0$$

and

$$g^{02}(\mathcal{D}\Lambda_q - \tfrac{1}{3}\Lambda_q^2 - 3\Lambda_p) + 3g^{22} = 0.$$

At this point all the components of the metric are found and are proportional to g^{02}, which becomes naturally the conformal factor $\Omega^2(x^i, s)$. The conformal metric in the basis $(\partial_y, \partial_p, \partial_q)$ is as follows

$$g^{ij} = \Omega^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -\tfrac{1}{3}\Lambda_q \\ 1 & -\tfrac{1}{3}\Lambda_q & -\tfrac{1}{3}\mathcal{D}\Lambda_q + \tfrac{1}{9}\Lambda_q^2 + \Lambda_p \end{pmatrix}.$$

Ω is not an arbitrary function of s, since $\Omega^2 = g^{02} = g^{ij} y_i q_i$ and applying $\mathcal{D}$ we get

$$\mathcal{D}\Omega = \frac{1}{3}\Omega\Lambda_q.$$

This is first of the metricity conditions, it is a constraint on Ω provided that Λ is known. Second condition is obtained by taking the fifth derivative of the eiconal equation with respect to s. It reads

$$0 = g^{ij}(5p_i\partial_s^4 Z_j + Z_i\partial_s^5 Z_j + 10\partial^2 Z_i\partial^3 Z_j) =$$
$$= 5g^{11}(\mathcal{D}\Lambda)_p + 5g^{12}(\mathcal{D}\Lambda)_q + g^{02}(\mathcal{D}^2\Lambda)_q + 10(g^{02}\Lambda_y + g^{12}\Lambda_p + g^{22}\Lambda_q).$$

Substituting the metric components by their explicit forms we obtain precisely the Wünschmann condition (17) for Λ.

The relations between third-order ODEs and $2 + 1$-dimensional conformal geometry in the NSF were further studied in [**26, 25, 27**]. In [**26**] it was shown that one can generate the conformal metric on the solution space starting from the system (6) of one-forms ω^i on $\mathcal{J}^2$ associated with an ODE. The authors defined the tensor

$$(24) \qquad\qquad 2\omega^1(\omega^3 + a\omega^1 + b\omega^2) - (\omega^2)^2$$

on $\mathcal{J}^2$ with arbitrary functions a and b. Next they imposed the condition that Lie transport of the above tensor along fibres of the projection $\mathcal{J}^2 \to solution\,space$ is conformal. The construction is parallel to (14) and the conditions for conformal Lie transport fix uniquely a and b, and yield the Wünschmann condition in a like manner to (15) and (16). In [**25**] the authors re-proved the invariance of the so obtained conformal geometry under contact transformations of the related ODE, which reflects a gauge freedom in the NSF. Finally, in [**27**] explicit formulae were given for the curvature of normal conformal connection in terms of contact invariants of third-order ODE.

Simultaneously to the research on the three-dimensional version of the NSF, much progress was made in understanding the full four-dimensional formalism in [**25, 27, 28**]. Here $(s, s^*, Z(s, s^*), Z_s, Z_{s^*}, Z_{ss^*}, Z_{ss}, Z_{s*s*})$ are coordinates on the second jet space $\mathcal{J}^2(S^2, \mathbb{R})$ of functions $S^2 \to \mathbb{R}$. Owing to this fact, the complex function Λ defines a pair of PDEs for a real function Z of the variables s and s^* through

$$Z_{ss} = \Lambda(s, s^*, Z, Z_s, Z_{s^*}, Z_{ss^*}),$$
$$Z_{s^*s^*} = \Lambda^*(s, s^*, Z, Z_s, Z_{s^*}, Z_{ss^*}),$$

while the space-time $\mathcal{M}^4$ is the space of solutions to this system, (x^μ) being constants of integrations. Following ideas of the three-dimensional construction the authors considered the six-dimensional submanifold L of $\mathcal{J}^2(\mathbb{R}^2, \mathbb{R})$ given by the PDEs. They considered the Pfaffian system on L associated with the PDEs and, following the formula (24), they built a symmetric tensor on L from the Pfaff forms. This tensor projects to a conformal geometry on $\mathcal{M}^4$ provided that the metricity conditions for Λ are satisfied, which can be now viewed as generalizations of the Wünschmann condition. The Cartan normal conformal connection for this geometry was constructed [**28**]. The Null Surface Formulation is still an ongoing project and work is being continued to obtain conformal Einstein equations in terms of invariants of the PDEs.

3. Results of the thesis

The thesis has threefold aim.

i) Constructing new geometries associated with third-order ODEs modulo contact, point and fibre-preserving transformations of variables. Considering possible applications to General Relativity.

ii) Describing new geometries together with the already known in the language of Cartan connections with curvature given by respective invariants of ODEs obtained by Cartan's method.

iii) Applying Cartan invariants to classification of certain types of third-order ODEs.

In order to construct the geometries we follow Cartan's construction outlined in section 1 of Introduction. We start from the equivalence problems formulated in terms of the G-structures (9) and apply Cartan's method to obtain manifolds $\mathcal{P}$ equipped with the coframes encoding all the invariant information about ODEs. Next we show how to read the principal bundle structures of these manifolds over

distinct bases. Usually it is the structure over $\mathcal{S}$ which is the most interesting, but we also consider structures over $\mathcal{J}^1$, $\mathcal{J}^2$ and certain six-dimensional manifold $\mathcal{M}^6$, which appears naturally. When the structure of a bundle is established then the invariant coframe defines a Cartan connection on $\mathcal{P} \to base$, usually under additional conditions playing similar role to the Wünschmann condition. In order to obtain the geometries and the Wünschmann-like conditions we often apply the P. Nurowski method of construction by Lie transport and projection.

In the most symmetric cases, when the underlying ODEs have transitive symmetry groups, the bundles $\mathcal{P}$ become locally Lie groups while their bases become homogeneous spaces, providing homogeneous models for the geometries. Since the dimension of $\mathcal{S}$ and $\mathcal{J}^1$ is three and we build geometries with at least two-dimensional structural group then the homogeneous models are given by the ODEs with at least five-dimensional symmetry group. Below we encapsulate our results. The geometries of the ODEs are also gathered in table 1 on page 19.

Unfortunately, among fourteen geometries which we consider in this work, there is no Lorentzian geometry or any new geometry which could currently be applied to General Relativity.

Contact geometries. Chapter 2 is devoted to the geometries of the ODEs modulo contact transformations. The only equations possessing at least five- dimensional contact symmetry group are the linear equations with constant coefficients[5], that is

$$y''' = 0$$

with the symmetry group $O(3,2)$ and

$$y''' = -2\mu y' + y, \qquad \mu \in \mathbb{R},$$

mutually non-equivalent for distinct μ, with the symmetry group $\mathbb{R}^2 \ltimes \mathbb{R}^3$. Sections 1 to 6 discuss geometries whose homogeneous model is generated by $y''' = 0$. In section 1 we state the main theorem in that chapter, theorem 2.1, which describes the geometry on $\mathcal{J}^2$. It can be recapitulated as follows.

THEOREM (Theorem 2.1). *The contact invariant information about an equation $y''' = F(x, y, y', y'')$ is given by the following data*
    i) *The principal fibre bundle $H_6 \to \mathcal{P} \to \mathcal{J}^2$, where $\dim \mathcal{P} = 10$ and H_6 is a six-dimensional subgroup of $O(3,2)$*
   ii) *The coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$ on $\mathcal{P}$ which defines the $\mathfrak{o}(3,2) \cong \mathfrak{sp}(4,\mathbb{R})$ Cartan normal connection $\widehat{\omega}^c$ on $\mathcal{P}$.*
The coframe and the connection $\widehat{\omega}_c$ are given explicitly in terms of F and its derivatives. There are two basic relative invariants for this geometry: the Wünschmann invariant and $F_{y''y''y''y''}$.

This theorem is almost identical to the result proved in [**52**] and the only new element we add here is the explicit formula for the connection. Section 2 contains the proof of the theorem, which repeats S.-S. Chern's construction of the coframe and the construction of the normal connection of [**52**].

Sections 4 to 6 discuss next three geometries generated by $\widehat{\omega}^c$. Section 4 contains the construction of the Lorentzian geometry on the solution space and, after [**48**], gives explicit formulae for the normal conformal connection. Section 5 studies a new type of geometry in the context of third-order ODEs, the contact-projective structure on $\mathcal{J}^1$. The idea of this geometry is the following. Consider the solutions of an ODE as a family of curves in $\mathcal{J}^1$ and ask whether these curves are among geodesics of a linear connection. The answer to this question is positive provided

---

[5]We prove this statement in chapter 4.

that $F_{y''y''y''y''} = 0$ and in this case there is a whole family of connections for which the solutions are geodesics. Such a family of connections in $\mathcal{J}^1$ is an example of a contact projective structure, see D. Fox [**21**] for the general definition of contact projective structures. Moreover, Tanaka's theory allows us to define a notion of normal Cartan connection for these structures in dimension three [**7, 8, 21**]. It is then a matter of straightforward calculations to check that $\widehat{\omega}^c$ is the normal connection for our contact projective structure. In section 6 we construct a six-dimensional split signature conformal geometry on some six-manifold $\mathcal{M}^6$ over which $\mathcal{P}$ is a bundle. Next we show that the associated normal conformal connection $\widehat{\mathbf{w}}$ has a special conformal holonomy reduced from $\mathfrak{o}(4,4)$ to $\mathfrak{o}(3,2) \oplus_. \mathbb{R}^5 \subset \mathfrak{o}(4,4)$ and, what is more, $\widehat{\omega}^c$ is precisely the $\mathfrak{o}(3,2)$ part of $\widehat{\mathbf{w}}$. These results are summarized as follows.

THEOREM. *The connection $\widehat{\omega}^c$ of theorem 2.1 has fourfold interpretation.*

1. *It is always the normal $\mathfrak{o}(3,2)$ Cartan connection on $\mathcal{J}^2$ (Chern-Sato-Yoshikawa construction.)*
2. *If the ODE has vanishing Wünschmann condition then $\widehat{\omega}^c$ is the normal Lorentzian conformal connection for the Lorentzian structure on the solution space (Chern-NSF construction.)*
3. *If the ODE satisfies $F_{y''y''y''y''} = 0$ then $\widehat{\omega}^c$ becomes the normal Cartan connection for the contact projective structure on $\mathcal{J}^1$.*
4. *$\widehat{\omega}^c$ is the $\mathfrak{o}(3,2)$-part of the $\mathfrak{o}(4,4)$ normal conformal connection for the six-dimensional split conformal geometry on $\mathcal{M}^6$ with special holonomy $\mathfrak{o}(3,2) \oplus_. \mathbb{R}^5$.*

In section 7 we turn to geometries, whose homogeneous models are provided by the equation $y''' = -2\mu y' + y$. Following Chern we reduce the bundle $\mathcal{P}$ to its five-dimensional subbundle. Then we find that

THEOREM (Theorem 2.13). *Every ODE satisfying some contact invariant condition $\mathbf{a}^c[F] = \mu = const$ has a $\mathbb{R}^2$ geometry on its solution space together with a $\mathbb{R}^2$ linear connection from the invariant coframe. The action of the algebra $\mathbb{R}^2$ on $\mathcal{S}$ is given by*

$$\begin{pmatrix} u & v & 0 \\ -\mu v & u & v \\ v & -\mu v & u \end{pmatrix}.$$

This geometry seems to be a generalization of Chern's 'cone geometry' which was associated with the equation $y''' = -y$ and briefly mentioned to exist for arbitrary ODEs. In our construction the action of $\mathbb{R}^2$ depends on the characteristic polynomial of respective linear equation, and we get a real cone geometry provided that it has three distinct roots.

Point geometries. In sections 1 to 5 of chapter 3 we study the geometries associated with the ODEs modulo point transformations. Sections 1 to 4 deal with the geometries modelled on $y''' = 0$. Our approach is analogous to the contact case and results are similar, with some caveats. The algebra $\mathfrak{co}(2,1) \oplus_. \mathbb{R}^3$ is not semisimple, hence the methods of the Tanka theory fail here. Moreover, even its generalization, the Morimoto nilpotent geometry, does not work in this case, since the point geometry of third-order ODEs on $\mathcal{J}^2$ is not a filtration. As a consequence, we do not have a general theory about existence of Cartan connections and in the case of $\mathcal{J}^1$, where we find point projective structure, certain refinement of contact projective structure, we are not able to find any connection and suppose that it does not exist in general. We also construct a six-dimensional Weyl structure in

the split signature, which is related to the six-dimensional conformal geometry of the contact case. To summarize

THEOREM. *The following statements hold*

1. *The point invariant information about $y''' = F(x, y, y', y'')$ is given by the seven-dimensional principal bundle $H_3 \to \mathcal{P} \to \mathcal{J}^2$ together with the coframe $\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3$ on $\mathcal{P}$, which defines the $\mathfrak{co}(2,1) \oplus \mathbb{R}^3$ Cartan connection $\widehat{\omega}^p$ (Cartan construction.)*
2. *If the ODE has vanishing Wünschmann (17) and Cartan (18) invariants then it has the Einstein-Weyl geometry on $\mathcal{S}$ and the Weyl connection is given by $\widehat{\omega}^p$ (Cartan construction.)*
3. *If the ODE satisfies $F_{y''y''y''} = 0$ then it has the point-projective structure on $\mathcal{J}^1$.*
4. *For any ODE there exists the split signature six-dimensional Weyl geometry, which is never Einstein.*

A new construction, which does not have a contact counterpart, is considered in section 5. This is a Lorentzian *metric structure* on the solution space $\mathcal{S}$. Its construction follows immediately from the Einstein-Weyl geometry. If the Ricci scalar of the Weyl connection is non-zero, then it is a weighted conformal function and may be fixed to a constant by an appropriate choice of the conformal gauge. The homogeneous models of this geometry are associated with

$$y''' = \frac{3\, y''^2}{2\, y'}$$

if the Ricci scalar is negative, and

$$y''' = \frac{3y''^2 y'}{y'^2 + 1}$$

if the Ricci scalar is positive. Both these equations are contact equivalent to $y''' = 0$ and their point symmetry groups are $O(2,2)$ and $O(4)$ respectively.

Fibre-preserving geometries. Sections 6 and 7 of chapter 3 are devoted to the geometries of ODEs modulo fibre-preserving transformations. We obtain a seven-dimensional bundle and the $\mathfrak{co}(2,1) \oplus \mathbb{R}^3$-valued Cartan connection $\widehat{\omega}^f$ on it. Since both $\widehat{\omega}^f$ and $\widehat{\omega}^p$ of the point case take value in the same algebra these cases are very similar to each other. Indeed, we show that one can recover $\widehat{\omega}^f$ from $\widehat{\omega}^p$ just by appending one function on the bundle. As a consequence, the geometries of the fibre-preserving case are obtained from their point counterparts by appending the object generated by the function.

We did not study obvious or not interesting geometries. In the point and fibre-preserving case geometries of $y''' = -2\mu y' + y$ are the same to what we have in the contact case, since the respective symmetry groups are the same. Also the fibre-preserving geometry on $\mathcal{J}^1$ does not seem to be worth studying.

Classification of ODEs. Chapter 4 contains the classification part of this work. We obtain two results

i) We characterize regular ODEs admitting large contact and point symmetry groups, that is the groups of dimension at least four. We give the conditions, in terms of Cartan invariants, for an ODE to posses the large symmetries. The classification is given in tables 2 and 3 on pages 72 – 75. We give the criteria for contact linearization of the ODEs. (The fibre-preserving classification of the ODEs with large symmetries is again parallel to the point classification and has been already done [**30, 33**], see also Remark 4.9.)

ii) We characterize regular ODEs fibre-preserving equivalent to II, IV, V, VI, VII and XI reduced Chazy classes, which are certain polynomial ODEs with the Painleve property. We give the explicit formulae for the transformations.

The condition of regularity assumed above is of technical nature, cf the discussion in the beginning of chapter 4, in particular definition 4.3.

To summarize, the following material contained in this work is new: sections 5 to 7 of chapter 2 excluding theorem 2.11, sections 3 to 7 of chapter 3 and the whole of chapter 4. Other sections contain a re-formulation and an extension of already known results.

All our calculations were performed or checked using the symbolic calculations program Maple.

4. Notation

In what follows we use the following symbols, in particular W denotes the Wünschmann invariant.

$$F = F(x, y, p, q),$$

$$(25) \qquad \mathcal{D} = \partial_x + p\partial_y + q\partial_p + F\partial_q,$$

$$(26) \qquad K = \tfrac{1}{6}\mathcal{D}F_q - \tfrac{1}{9}F_q^2 - \tfrac{1}{2}F_p,$$

$$(27) \qquad L = \tfrac{1}{3}F_{qq}K - \tfrac{1}{3}F_qK_q - K_p - \tfrac{1}{3}F_{qy},$$

$$(28) \qquad M = 2K_{qq}K - 2K_{qy} + \tfrac{1}{3}F_{qq}L - \tfrac{2}{3}F_qL_q - 2L_p,$$

$$(29) \qquad W = \left(\mathcal{D} - \tfrac{2}{3}F_q\right)K + F_y,$$

$$(30) \qquad Z = \frac{\mathcal{D}W}{W} - F_q.$$

Parentheses denote sets of objects:

$$(a_1, \ldots, a_k)$$

is the set consisting of $a_1, \ldots, a_k$. In particular this symbol denotes bases of vector spaces as well as coordinate systems, frames and coframes on manifolds. The linear span of vectors or covectors $a_1, \ldots, a_k$ is denoted by

$$< a_1, \ldots, a_k > .$$

If $a_1, \ldots, a_k$ are vector fields or one-forms on a manifold, then the above symbol denotes the distribution or the simple ideal generated by them. The symmetric tensor product of two one-forms or vector fields α and β is denoted by

$$\alpha\beta = \tfrac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha).$$

The symbols $A_{(\mu\nu)}$ and $A_{[\mu\nu]}$ denote symmetrization and antisymmetrization of a tensor $A_{\mu\nu}$ respectively. For a metric g of signature (k, l) the group $CO(k, l)$ is defined to be

$$CO(k, l) = \{A \in GL(k + l, \mathbb{R}) \mid \quad A^T gA = e^\lambda g, \quad \lambda \in \mathbb{R}\}.$$

Its Lie algebra

$$\mathfrak{co}(k, l) = \{a \in \mathfrak{gl}(k + l, \mathbb{R}) \mid \quad a^T g + ga = \lambda g, \quad \lambda \in \mathbb{R}\}.$$

A semidirect product of two Lie groups G H, where G acts on H is denoted by $G \ltimes H$. A semidirect product of their Lie algebras is denoted by $\mathfrak{g} \oplus \mathfrak{h}$. If the action depends of a parameter μ then we add an subscript: $\ltimes_\mu$ and $\oplus_\mu$.

TABLE 1. Geometries of third-order ODEs

Model	Manifold	Contact	Point	Fibre-preserving
$y''' = 0$	$\mathcal{J}^2$	$\mathfrak{o}(3,2)$ connection	$\mathfrak{co}(2,1) \oplus_. \mathbb{R}^3$ connection	
	$\mathcal{S}$	Lorentzian conformal	Lorentzian Einstein-Weyl	Lorentzian Einstein-Weyl with a weighted function
	$\mathcal{J}^1$	contact projective	point projective	*was not studied*
	$\mathcal{M}^6$	split conformal	split Weyl	split Weyl with a weighted function
$y''' = \frac{3y'(y'')^2}{1+(y')^2}$	$\mathcal{S}$	———	Lorentzian Ricci= *const* > 0	———
$y''' = \frac{3(y'')^2}{2y'}$	$\mathcal{S}$	———	Lorentzian Ricci= *const* < 0	Lorentzian with a one-form
$y''' = -2\mu y' + y$	$\mathcal{S}$	$\mathbb{R}^2$ geometry		

Geometries of ODEs considered modulo contact transformations of variables

1. Cartan connection on ten-dimensional bundle

We formulate the theorem about the main structure which is associated with third-order ODEs modulo contact transformations of variables, the $\mathfrak{o}(3,2)$ Cartan connection on the bundle $\mathcal{P}^c \to \mathcal{J}^2$. This structure will serve as a starting point for both analyzing of geometries of ODEs and their classification.

THEOREM 2.1. *Consider an equation $y''' = F(x, y, y', y'')$. The contact invariant information about this equation is given by the following data*

i) *The principal fibre bundle $H_6 \to \mathcal{P}^c \to \mathcal{J}^2$, where $\dim \mathcal{P}^c = 10$, $\mathcal{J}^2$ is the space of second jets of curves in the xy-plane, and H_6 is the following six-dimensional subgroup of $SP(4, \mathbb{R})$*

$$
(31) \qquad H_6 = \begin{pmatrix} \sqrt{u_1}, & \frac{1}{2}\frac{u_2}{\sqrt{u_1}}, & -\frac{1}{2}\frac{u_4}{\sqrt{u_1}}, & \frac{1}{24}\frac{u_2^2 u_5}{u_1^{3/2} u_3} - \frac{1}{2}\sqrt{u_1}\, u_6 \\[2mm] 0 & \frac{u_3}{\sqrt{u_1}}, & -\frac{u_5}{\sqrt{u_1}}, & \frac{1}{2}\frac{u_2 u_5 - u_3 u_4}{u_1^{3/2}} \\[2mm] 0 & 0 & \frac{\sqrt{u_1}}{u_3}, & -\frac{1}{2}\frac{u_2}{\sqrt{u_1}\, u_3} \\[2mm] 0 & 0 & 0 & \frac{1}{\sqrt{u_1}} \end{pmatrix}.
$$

ii) *The coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$ on $\mathcal{P}^c$, which defines the $\mathfrak{o}(3,2)$-valued Cartan normal connection $\widehat{\omega}^c$ on $\mathcal{P}^c$ by*

$$
(32) \qquad \widehat{\omega}^c = \begin{pmatrix} \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & -\frac{1}{2}\Omega_4 & -\frac{1}{4}\Omega_6 \\[2mm] \theta^4 & \Omega_3 - \frac{1}{2}\Omega_1 & -\Omega_5 & -\frac{1}{2}\Omega_4 \\[2mm] \theta^2 & \theta^3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 \\[2mm] 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Omega_1 \end{pmatrix}.
$$

Let $(x, y, p, q, u_1, u_2, u_3, u_4, u_5, u_6)$, (x^i, u_μ) for short, be a local coordinate system on $\mathcal{P}^c$, which is compatible with the local trivialization $\mathcal{P}^c = H_6 \times \mathcal{J}^2$, that is $(x^i) = (x, y, p, q)$ are coordinates in $\mathcal{J}^2$ and (u_μ) are coordinates in H_6 as in (31). Then the value of $\widehat{\omega}^c$ at the point (x^i, u_μ) in $\mathcal{P}^c$ is given by

$$
\widehat{\omega}^c(x^i, u_\mu) = u^{-1}\, \omega^c\, u + u^{-1}\mathrm{d}u
$$

where u denotes the matrix (31) *and*

$$\omega^c = \begin{pmatrix} \frac{1}{2}\Omega_1^0 & \frac{1}{2}\Omega_2^0 & -\frac{1}{2}\Omega_4^0 & -\frac{1}{4}\Omega_6^0 \\[8pt] \omega^4 & \Omega_3^0 - \frac{1}{2}\Omega_1^0 & -\Omega_5^0 & -\frac{1}{2}\Omega_4^0 \\[8pt] \omega^2 & \widetilde{\omega}^3 & \frac{1}{2}\Omega_1^0 - \Omega_3^0 & -\frac{1}{2}\Omega_2^0 \\[8pt] 2\omega^1 & \omega^2 & -\omega^4 & -\frac{1}{2}\Omega_1^0 \end{pmatrix}$$

is the connection $\widehat{\omega}^c$ calculated at the point $(x^i, u_1 = 1, u_2 = 0, u_3 = 1, u_4 = 0, u_5 = 0, u_6 = 0)$. The forms $\omega^1, \omega^2, \widetilde{\omega}^3, \omega^4$ read

(33)
$$\begin{aligned} \omega^1 &= dy - pdx, \\ \omega^2 &= dp - qdx, \\ \widetilde{\omega}^3 &= dq - Fdx - \tfrac{1}{3}F_q(dp - qdx) + K(dy - pdx), \\ \omega^4 &= dx. \end{aligned}$$

The forms $\Omega_1^0, \ldots, \Omega_6^0$ read

(34)
$$\begin{aligned} \Omega_1^0 &= - K_q\,\omega^1, \\ \Omega_2^0 &= \left(\tfrac{1}{3}W_q + L\right)\omega^1 - K_q\,\omega^2 - K\omega^4, \\ \Omega_3^0 &= - K_q\,\omega^1 + \tfrac{1}{6}F_{qq}\,\omega^2 + \tfrac{1}{3}F_q\omega^4, \\ \Omega_4^0 &= - \left(\tfrac{1}{3}W_{qq} + L_q\right)\omega^1 + \tfrac{1}{2}K_{qq}\,\omega^2, \\ \Omega_5^0 &= \tfrac{1}{2}K_{qq}\,\omega^1 - \tfrac{1}{6}F_{qqq}\,\omega^2 - \tfrac{1}{6}F_{qq}\,\omega^4, \\ \Omega_6^0 &= \left(\tfrac{1}{3}\mathcal{D}(W_{qq}) - \tfrac{4}{3}W_{qp} - \tfrac{1}{3}F_qW_{qq} + \tfrac{1}{3}F_{qqq}W + M\right)\omega^1 + \\ &\quad + \tfrac{1}{3}(F_{qqy} - F_{qqq}K - W_{qq})\,\omega^2 - K_{qq}\,\widetilde{\omega}^3 + \\ &\quad + \left(\tfrac{2}{3}F_{qy} - \tfrac{1}{3}F_{qq}K - 2L - \tfrac{4}{3}W_q\right)\omega^4. \end{aligned}$$

In above theorem we used a concept of normal Cartan connection in the sense of N. Tanaka [**55**]. A normal Cartan connection is a connection which takes value in a semisimple graded Lie algebra and whose curvature satisfies some algebraic conditions, which are, so to speak, a generalization of conditions for torsion in the case of linear connections. We explain it below.

DEFINITION 2.2. A semisimple Lie algebra $\mathfrak{g}$ is graded if it has a vector space decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$$

such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$$

and $\mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$.

Let us suppose that $\mathfrak{g}$ is a semisimple graded Lie algebra and denote $\mathfrak{m} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1}$, $\mathfrak{h} = \mathfrak{g}_0 \oplus \ldots \oplus \mathfrak{g}_k$. Let us consider a $\mathfrak{g}$ valued Cartan connection $\widehat{\omega}$ on a bundle $H \to \mathcal{P} \to \mathcal{M}$, where the Lie algebra of H is $\mathfrak{h}$. Fix a point $p \in \mathcal{P}$. The decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ defines in $T_p\mathcal{P}$ the complement $\mathcal{H}_p$ of the vertical space $\mathcal{V}_p$. Therefore we have $T_p\mathcal{P} = \mathcal{V}_p \oplus \mathcal{H}_p$, $\widehat{\omega}(\mathcal{V}_p) = \mathfrak{h}$ and $\widehat{\omega}(\mathcal{H}_p) = \mathfrak{m}$. The curvature $\widehat{K}_p = (d\widehat{\omega} + \widehat{\omega} \wedge \widehat{\omega})_p$ at p is then characterized by the tensor $\kappa_p \in \text{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$ given by

(35)
$$\kappa_p(A, B) = \widehat{K}_p(\widehat{\omega}_p^{-1}(A), \widehat{\omega}_p^{-1}(B)), \quad A, B \in \mathfrak{m}.$$

The function $\kappa \colon \mathcal{P} \to \text{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$ is called the structure function.

In the space $\mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$ of $\mathfrak{g}$-valued two-forms let us define $\mathrm{Hom}^1(\wedge^2\mathfrak{m}, \mathfrak{g})$ to be the space of all $\alpha \in \mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g})$ fulfilling

$$\alpha(\mathfrak{g}_i, \mathfrak{g}_j) \subset \mathfrak{g}_{i+j+1} \oplus \ldots \oplus \mathfrak{g}_k \quad \text{for} \quad i, j < 0.$$

Since the Killing form B of $\mathfrak{g}$ is non-degenerate and satisfies $B(\mathfrak{g}_p, \mathfrak{g}_q) = 0$ for $p \neq -q$, one can identify $\mathfrak{m}^*$ with $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$. For a basis $(e_1, \ldots, e_m)$ of $\mathfrak{m}$ let $(e_1^*, \ldots, e_m^*)$ denote the unique basis of $\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ such that $B(e_i, e_j^*) = \delta_{ij}$. Tanaka considered the following complex

$$\ldots \longrightarrow \mathrm{Hom}(\wedge^{q+1}\mathfrak{m}, \mathfrak{g}) \overset{\partial^*}{\longrightarrow} \mathrm{Hom}(\wedge^q\mathfrak{m}, \mathfrak{g}) \longrightarrow \ldots$$

with $\partial^* \colon \mathrm{Hom}(\wedge^{q+1}\mathfrak{m}, \mathfrak{g}) \to \mathrm{Hom}(\wedge^q\mathfrak{m}, \mathfrak{g})$ given by the following formula

$$(\partial^*\alpha)(A_1 \wedge \ldots \wedge A_q) = \sum_i [e_i^*, \alpha(e_i \wedge A_1 \wedge \ldots \wedge A_q)]$$

$$+ \tfrac{1}{2} \sum_{i,j} \alpha([e_j^*, A_i]_\mathfrak{m} \wedge e_j \wedge A_1 \wedge \ldots \wedge \hat{A}_i \wedge \ldots \wedge A_q),$$

where $\alpha \in \mathrm{Hom}(\wedge^q\mathfrak{m}, \mathfrak{g})$, $A_1, \ldots A_q \in \mathfrak{m}$, (e_i) is any basis in $\mathfrak{m}$ and $[\,,\,]_\mathfrak{m}$ denotes the $\mathfrak{m}$-component of the bracket with respect to the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$. Finally, N. Tanaka [55] introduced the notion of normal connection, the definition below is given in the language of [7].

DEFINITION 2.3. A Cartan connection $\widehat{\omega}$ as above is normal if its structure function κ fulfills the following conditions

i) $\qquad \kappa \in \mathrm{Hom}^1(\wedge^2\mathfrak{m}, \mathfrak{g}),$

ii) $\qquad \partial^*\kappa = 0.$

N. Tanaka considered above objects in a wider context of geometric structures associated with Cartan connections. He proved the one-to-one correspondence between normal connections on $\mathcal{P} \to \mathcal{M}$ and some G-structures, so called $G_0^\#$-structures, on $\mathcal{M}$. Starting from a $G_0^\#$-structure one obtains a normal connection by a procedure of reductions and prolongations, which is a version of Cartan's equivalence method. It is the correspondence with $G_0^\#$-structures which makes the normal connections distinguished. N. Tanaka proved that a normal connection exists and is unique if only $\mathfrak{g}$ is a subalgebra of $\mathfrak{g}(\mathfrak{m}, \mathfrak{g}_0)$, so called prolongation of $\mathfrak{m}$ and $\mathfrak{g}_0$. The notion of prolongation in Tanaka's sense would lead us to his general theory, which is beyond the scope of this paper; in this work we need conditions of normality to explicitly construct the Cartan connection in the only case of theorem 2.1. For details of Tanaka's theory see [54, 55].

2. Proof of theorem 2.1

We prove theorem 2.1 by repeating Chern's construction of Cartan connection for the contact equivalence problem, supplemented later in [52].

We begin with the G_c-structure (8) on $\mathcal{J}^2$, which, according to Introduction, encodes all the contact invariant information about the underlying ODE. Let us fix on $\mathcal{J}^2 \times G_c$ a coordinate system $(x, y, p, q, u_1, \ldots, u_9)$ or (x^k, g_j^i) for short, where $(x^k) = (x, y, p, q)$ are the coordinates on $\mathcal{J}^2$ and

$$g_j^i = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & u_5 & u_6 & 0 \\ u_8 & u_9 & 0 & u_7 \end{pmatrix}$$

are coordinates on G_c. We remind that there are four well defined forms on $G_c \times \mathcal{J}^2$, the components of the canonical form θ^i:

$$\theta^i(x, g^{-1}) = g^i{}_j\, \omega^j(x), \qquad i = 1, \ldots, 4$$

with ω^i given by (6). We seek a bundle on which θ^i are supplemented to a coframe by certain new one-forms Ω_μ chosen in a well-defined geometric manner.

2.1. S.-S. Chern's construction. The construction of the bundle $\mathcal{P}^c$ and the coframe by Cartan's method is the following.

1). We calculate the exterior derivatives of θ^i on $G_c \times \mathcal{J}^2$

(36)
$$\begin{aligned}
\mathrm{d}\theta^1 &= \alpha_1 \wedge \theta^1 + T^1{}_{jk}\theta^j \wedge \theta^k, \\
\mathrm{d}\theta^2 &= \alpha_2 \wedge \theta^1 + \alpha_3 \wedge \theta^2 + T^2{}_{jk}\theta^j \wedge \theta^k, \\
\mathrm{d}\theta^3 &= \alpha_4 \wedge \theta^1 + \alpha_5 \wedge \theta^2 + \alpha_6 \wedge \theta^3 + T^3{}_{jk}\theta^j \wedge \theta^k, \\
\mathrm{d}\theta^4 &= \alpha_8 \wedge \theta^1 + \alpha_9 \wedge \theta^2 + \alpha_7 \wedge \theta^4 + T^4{}_{jk}\theta^j \wedge \theta^k,
\end{aligned}$$

where α_μ are the entries of the matrix $\mathrm{d}g^i{}_k \cdot g_j^{-1k}$ and $T^i{}_{jk}$ are some functions on $G_c \times \mathcal{J}^2$. Next we collect $T^i{}_{jk}\theta^j \wedge \theta^k$ terms

(37)
$$\begin{aligned}
\mathrm{d}\theta^1 &= \left(\alpha_1 - T^1{}_{12}\theta^2 - T^1{}_{13}\theta^3 - T^1{}_{14}\theta^4\right) \wedge \theta^1 \\
&\quad + T^1{}_{23}\theta^2 \wedge \theta^3 + T^1{}_{24}\theta^2 \wedge \theta^4 + T^1{}_{34}\theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^2 &= \left(\alpha_2 - T^2{}_{12}\theta^2 - T^2{}_{13}\theta^3 - T^2{}_{14}\theta^4\right) \wedge \theta^1 \\
&\quad + \left(\alpha_3 - T^2{}_{23}\theta^3 - T^2{}_{24}\theta^4\right) \wedge \theta^2 + T^2{}_{34}\theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^3 &= \left(\alpha_4 - T^3{}_{12}\theta^2 - T^3{}_{13}\theta^3 - T^3{}_{14}\theta^4\right) \wedge \theta^1, \\
&\quad + \left(\alpha_5 - T^3{}_{23}\theta^3 - T^3{}_{24}\theta^4\right) \wedge \theta^2 + \left(\alpha_6 - T^3{}_{34}\theta^4\right) \wedge \theta^3 \\
\mathrm{d}\theta^4 &= \left(\alpha_8 - T^4{}_{12}\theta^2 - T^4{}_{13}\theta^3 - T^4{}_{14}\theta^4\right) \wedge \theta^1, \\
&\quad + \left(\alpha_9 - T^4{}_{23}\theta^3 - T^4{}_{24}\theta^4\right) \wedge \theta^2 + \left(\alpha_7 + T^4{}_{34}\theta^3\right) \wedge \theta^4
\end{aligned}$$

and introduce new 1-forms π_μ substituting the collected terms. Eq. (36) now read

(38)
$$\begin{aligned}
\mathrm{d}\theta^1 &= \pi_1 \wedge \theta^1 + \frac{u_1}{u_3 u_7}\theta^4 \wedge \theta^2, \\
\mathrm{d}\theta^2 &= \pi_2 \wedge \theta^1 + \pi_3 \wedge \theta^2 + \frac{u_3}{u_6 u_7}\theta^4 \wedge \theta^3, \\
\mathrm{d}\theta^3 &= \pi_4 \wedge \theta^1 + \pi_5 \wedge \theta^2 + \pi_6 \wedge \theta^3, \\
\mathrm{d}\theta^4 &= \pi_8 \wedge \theta^1 + \pi_9 \wedge \theta^2 + \pi_7 \wedge \theta^4,
\end{aligned}$$

since $T^1{}_{23} = T^1{}_{34} = 0$, $T^1{}_{24} = -u_3 u_7/u_1$ and $T^2{}_{34} = -u_3/(u_6 u_7)$.

The equations (38) resemble structural equations for a linear connection very much. When θ^i are components of the canonical form and $\Gamma^j{}_k$ are components of a $\mathfrak{g}$-valued connection then we have

$$\mathrm{d}\theta^i + \Gamma^i{}_j \wedge \theta^j = \tfrac{1}{2}T^i{}_{jk}\theta^j \wedge \theta^k$$

with a torsion T. In our case π_μ are the entries of the matrix π in $\mathfrak{g}_c$

$$\pi = \begin{pmatrix} \pi_1 & 0 & 0 & 0 \\ \pi_2 & \pi_3 & 0 & 0 \\ \pi_4 & \pi_5 & \pi_6 & 0 \\ \pi_8 & \pi_9 & 0 & \pi_7 \end{pmatrix}.$$

However, π is not a linear connection. This is because in this version of Cartan's method π usually does not transform regularly along fibres of bundles ($G_c \times \mathcal{J}^2$ here) and the 'curvature' $\mathrm{d}\pi + \pi \wedge \pi$ is not necessarily a tensor, hence π is not a linear

connection in general. We may think of π as a connection in a broader meaning, that is a horizontal distribution on $G_c \times \mathcal{J}^2$, which is not necessarily right-invariant. Keeping this in mind we will refer to $T^i{}_{jk}$ as torsion. Thus π is a 'connection' chosen by the demand that its torsion is 'minimal', i.e. possesses as few terms as possible.

We observe that π_μ, which are candidates for the sought forms Ω_μ, are not uniquely defined by equations (38), for example the gauge $\pi_1 \to \pi_1 + f\theta^1$ leaves (38) unchanged. Therefore our connection is not uniquely defined by its torsion.

2). In the next step we reduce the bundle $G_c \times \mathcal{J}^2$. We choose its subbundle, say $\mathcal{P}^{(1)}$, characterized by the property that the torsion coefficients are constant on it. We choose $\mathcal{P}^{(1)}$ such that $T^1{}_{24} = -1$, $T^2{}_{34} = -1$ on it. Thus $\mathcal{P}^{(1)}$ is defined by

$$(39) \qquad u_6 = \frac{u_3^2}{u_1}, \qquad u_7 = \frac{u_1}{u_3}.$$

It is known [**29, 53**] that such a reduction preserves the equivalence, in other words, two bundles are equivalent if and only if their respective reductions are. Here $\mathcal{P}^{(1)}$ has the seven-dimensional structural group

$$G_c^{(1)} = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ u_4 & u_5 & \frac{u_3^2}{u_1} & 0 \\ u_8 & u_9 & 0 & \frac{u_1}{u_3} \end{pmatrix}.$$

3). Next we pull-back θ^i and π_μ to $\mathcal{P}^{(1)}$. But the new structural group $G_c^{(1)}$ is a seven-dimensional subgroup of G_c, so $(\theta^1, \ldots, \theta^4, \pi_1, \ldots, \pi_9)$ of (38) is not a coframe on $\mathcal{P}^{(1)}$ any longer, since

$$\pi_6 = 2\pi_3 - \pi_1 \mod (\theta^i), \qquad \pi_7 = \pi_1 - \pi_3 \mod (\theta^i).$$

Taking this into account we recalculate (38) and gather the torsion terms. We choose the new connection

$$\pi = \begin{pmatrix} \pi_1 & 0 & 0 & 0 \\ \pi_2 & \pi_3 & 0 & 0 \\ \pi_4 & \pi_5 & 2\pi_3 - \pi_1 & 0 \\ \pi_8 & \pi_9 & 0 & \pi_1 - \pi_3 \end{pmatrix}$$

so that its torsion is minimal again.

$$(40) \quad \begin{aligned} d\theta^1 &= \pi_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\ d\theta^2 &= \pi_2 \wedge \theta^1 + \pi_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\ d\theta^3 &= \pi_4 \wedge \theta^1 + \pi_5 \wedge \theta^2 + (2\pi_3 - \pi_1) \wedge \theta^3 + \left(\frac{3u_5}{u_3} - \frac{3u_2 - u_3 F_q}{u_1} \right) \theta^4 \wedge \theta^3, \\ d\theta^4 &= \pi_8 \wedge \theta^1 + \pi_9 \wedge \theta^2 + (\pi_1 - \pi_3) \wedge \theta^4. \end{aligned}$$

4). We repeat the steps **2)** and **3)**. Firstly we reduce $\mathcal{P}^{(1)}$ to the subbundle $\mathcal{P}^{(2)} \subset \mathcal{P}^{(1)}$ defined by the property that the only non-constant torsion coefficient $T^3{}_{34}$ in (40) vanishes on it,

$$(41) \qquad u_5 = \frac{u_3}{u_1} \left(u_2 - \frac{1}{3} u_3 F_q \right).$$

Next we recalculate connection, re-collect the torsion and make another reduction through the constant torsion condition (K is defined in (26).)

$$(42) \qquad u_4 = \frac{u_3^2}{u_1} K + \frac{u_2^2}{2u_1}.$$

At this stage we have reduced the frame bundle $G_c \times \mathcal{J}^2$ to the nine-dimensional subbundle $\mathcal{P}^{(3)} \to \mathcal{J}^2$, such that its structural group is the following

$$
G_c^{(3)} = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ u_2 & u_3 & 0 & 0 \\ \frac{u_2^2}{u_1} & \frac{u_2 u_3}{u_1} & \frac{u_3^3}{u_1} & 0 \\ u_8 & u_9 & 0 & \frac{u_1}{u_3} \end{pmatrix}
$$

and the frame dual to $(\omega^1, \omega^2, \omega^3 - \frac{1}{3} F_q \omega^2 + K \omega^1, \omega^4)$ belongs to $\mathcal{P}^{(3)}$. The structural equations on $\mathcal{P}^{(3)}$ read after collecting

(43)
$$
\begin{aligned}
\mathrm{d}\theta^1 &= \pi_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\
\mathrm{d}\theta^2 &= \pi_2 \wedge \theta^1 + \pi_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\
\mathrm{d}\theta^3 &= \pi_2 \wedge \theta^2 + (2\pi_3 - \pi_1) \wedge \theta^3 + \frac{u_3^3}{u_1^3} W \theta^4 \wedge \theta^1, \\
\mathrm{d}\theta^4 &= \pi_8 \wedge \theta^1 + \pi_9 \wedge \theta^2 + (\pi_1 - \pi_3) \wedge \theta^4
\end{aligned}
$$

with some one-forms $\pi_1, \pi_2, \pi_3, \pi_8, \pi_9$. The function W, defined in (29), is the Wünschmann invariant which is a contact relative invariant for third-order ODEs. It means, as we already explained in Introduction, that every contact transformation applied to an ODE preserves the condition $W = 0$ or $W \neq 0$. It follows that third-order ODEs $y''' = F(x, y, y', y'')$ and $y''' = \bar{F}(x, y, y', y'')$, satisfying $W[F] = 0$ and $W[\bar{F}] \neq 0$ respectively, are not contact equivalent. Thereby, as Chern observed, third-order ODEs fall into two main contact inequivalent branches: the ODEs satisfying $W \neq 0$, and those satisfying $W = 0$.

Equations (43) do not still define the forms π_μ uniquely but only modulo the following transformations

(44)
$$
\begin{aligned}
\pi_1 &\to \pi_1 + 2t_1 \theta^1, \\
\pi_2 &\to \pi_2 + t_1 \theta^2, \\
\pi_3 &\to \pi_3 + t_1 \theta^1, \\
\pi_8 &\to \pi_8 + t_2 \theta^1 + t_3 \theta^2 + t_1 \theta^4, \\
\pi_9 &\to \pi_9 + t_3 \theta^1 + t_4 \theta^2.
\end{aligned}
$$

That is to say, the torsion in (43) defines the $\mathfrak{g}_c^{(3)}$ connection π only up to (44).

At this point, there is no pattern of further reduction. If $W = 0$ there are only constant torsion coefficients in (43) and we do not have any conditions to define a subbundle of $\mathcal{P}^{(3)}$. In these circumstances we prolong $\mathcal{P}^{(3)}$.

5). The idea of prolongation is the following. On $\mathcal{P}^{(3)}$ there is no fixed coframe but only the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \pi_1, \pi_2, \pi_3, \pi_8, \pi_9)$ given modulo (44). But 'a coframe given modulo G' is a G-structure on $\mathcal{P}^{(3)}$. As a consequence we can deal with this new structure on $P^{(3)}$ by means of the Cartan method. Let us consider the bundle $\mathcal{P}^{(3)} \times G^{prol}$ then, where

$$
G^{prol} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}
$$

reflects the freedom (44) so that the block t reads

$$
\begin{pmatrix} 2t_1 & 0 & 0 & 0 \\ 0 & t_1 & 0 & 0 \\ t_1 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & t_1 \\ t_3 & t_4 & 0 & 0 \end{pmatrix}.
$$

On $\mathcal{P}^{(3)} \times G^{prol}$ there exist nine fixed one-forms $\theta^1, \theta^2, \theta^3, \theta^4, \Pi_1, \Pi_2, \Pi_3, \Pi_8, \Pi_9$, given by

$$\begin{pmatrix} \theta^i \\ \Pi_\mu \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \begin{pmatrix} \theta^i \\ \pi_\mu \end{pmatrix},$$

which is the canonical one-form on $\mathcal{P}^{(3)} \times G^{prol} \to \mathcal{P}^{(3)}$.

6). Now we apply the method of reductions to the above structure on $\mathcal{P}^{(3)} \times G^{prol}$. We calculate the exterior derivatives of (θ^i, Π_μ). The derivatives of θ^i take the form of (43) with π_μ replaced by Π_μ. The derivatives of Π_μ, after collecting and introducing 1-forms Λ_K containing dt_K, read

$$d\Pi_1 = \Lambda_1 \wedge \theta^1 + \Pi_8 \wedge \theta^2 - \Pi_2 \wedge \theta^4,$$

$$d\Pi_2 = \tfrac{1}{2}\Lambda_1 \wedge \theta^2 - \Pi_1 \wedge \Pi_2 - \Pi_2 \wedge \Pi_3 + \Pi_8 \wedge \theta^3 + \frac{u_3^3}{u_1^3} W \Pi_9 \wedge \theta^1$$

$$(45) \qquad + 2f_1 \theta^1 \wedge \theta^3 + f_4 \theta^1 \wedge \theta^4 + f_2 \theta^2 \wedge \theta^3 + f_5 \theta^2 \wedge \theta^4,$$

$$d\Pi_3 = \tfrac{1}{2}\Lambda_1 \wedge \theta^1 + \Pi_8 \wedge \theta^2 + \Pi_9 \wedge \theta^3 + f_1 \theta^1 \wedge \theta^2 + f_2 \theta^1 \wedge \theta^3 + f_5 \theta^1 \wedge \theta^4 + f_3 \theta^2 \wedge \theta^3,$$

$$d\Pi_8 = \Lambda_2 \wedge \theta^1 + \Lambda_3 \wedge \theta^1 + \tfrac{1}{2}\Lambda_1 \wedge \theta^4 + \Pi_9 \wedge \Pi_2 + \Pi_8 \wedge \Pi_3 + f_2 \theta^3 \wedge \theta^4,$$

$$d\Pi_9 = \Lambda_3 \wedge \theta^1 + \Lambda_4 \wedge \theta^2 + \Pi_1 \wedge \Pi_9 - 2\Pi_3 \wedge \Pi_9 + \Pi_8 \wedge \theta^4 - f_1 \theta^1 \wedge \theta^4 + f_3 \theta^3 \wedge \theta^4.$$

where f_1, f_2, f_3, f_4, f_5 are functions. This time the forms Λ_K are interpreted as connection forms. We compute $f_1, \ldots, f_5$ and choose the subbundle $\mathcal{P}^c$ of $\mathcal{P}^{(3)} \times G^{prol}$ by the condition that f_1, f_2, f_3 are equal to zero on $\mathcal{P}^c$. This is done by appropriate specifying of parameters t_2, t_3, t_4 as functions of $(x, y, p, q, u_1, u_2, u_3, u_8, u_9, t_1)$. We skip writing these complicated formulae. The structural equations on $\mathcal{P}^c$ read

$$d\theta^1 = \Pi_1 \wedge \theta^1 + \theta^4 \wedge \theta^2,$$

$$d\theta^2 = \Pi_2 \wedge \theta^1 + \Pi_3 \wedge \theta^2 + \theta^4 \wedge \theta^3,$$

$$d\theta^3 = \Pi_2 \wedge \theta^2 + (2\Pi_3 - \Pi_1) \wedge \theta^3 + A\,\theta^4 \wedge \theta^1,$$

$$d\theta^4 = \Pi_8 \wedge \theta^1 + \Pi_9 \wedge \theta^2 + (\Pi_1 - \Pi_2) \wedge \theta^4,$$

$$d\Pi_1 = \Lambda_1 \wedge \theta^1 + \Pi_8 \wedge \theta^2 - \Pi_2 \wedge \theta^4,$$

$$d\Pi_2 = (\Pi_3 - \Pi_1) \wedge \Pi_2 + A\,\Pi_9 \wedge \theta^1 + \tfrac{1}{2}\Lambda_1 \wedge \theta^2 + \Pi_8 \wedge \theta^3 + B\,\theta^1 \wedge \theta^4 + C\,\theta^2 \wedge \theta^4,$$

$$(46) \quad d\Pi_3 = \tfrac{1}{2}\Lambda_1 \wedge \theta^1 + \Pi_8 \wedge \theta^2 + \Pi_9 \wedge \theta^3 + C\,\theta^1 \wedge \theta^4,$$

$$d\Pi_8 = \Pi_9 \wedge \Pi_2 + \Pi_8 \wedge \Pi_3 - 2C\,\Pi_9 \wedge \theta^1 + \tfrac{1}{2}\Lambda_1 \wedge \theta^4 + D\,\theta^1 \wedge \theta^2 + 2E\,\theta^1 \wedge \theta^3$$
$$\qquad + G\,\theta^1 \wedge \theta^4 + H\,\theta^2 \wedge \theta^3 + J\,\theta^2 \wedge \theta^4,$$

$$d\Pi_9 = (\Pi_1 - 2\Pi_3) \wedge \Pi_9 + \Pi_8 \wedge \theta^4 + E\,\theta^1 \wedge \theta^2 + H\,\theta^1 \wedge \theta^3 + J\,\theta^1 \wedge \theta^4 + L\,\theta^2 \wedge \theta^3,$$

$$d\Lambda_1 = \Lambda_1 \wedge \Pi_1 + 2\Pi_8 \wedge \Pi_2 + 2C\,\Pi_8 \wedge \theta^1 - 2C\,\Pi_9 \wedge \theta^2 - A\,\Pi_9 \wedge \theta^4 + \widetilde{M}\,\theta^1 \wedge \theta^2$$
$$\qquad + 2(D + AL)\,\theta^1 \wedge \theta^3 + \widetilde{N}\,\theta^1 \wedge \theta^4 + 2E\,\theta^2 \wedge \theta^3 + G\,\theta^2 \wedge \theta^4$$

with certain functions $A, B, C, D, E, F, G, H, J, L, \widetilde{M}, \widetilde{N}$ on $\mathcal{P}^c$.

Above structural equations *uniquely define the only remaining auxiliary form* Λ_1. In this manner we constructed the bundle $\mathcal{P}^c \to \mathcal{J}^2$ and the fixed coframe associated to the ODEs modulo contact transformations. As we have explained in Introduction, the functions $A, \ldots, \widetilde{N}$, and their coframe derivatives are relative contact invariants for third-order ODEs.

2.2. Cartan normal connection from Tanaka's theory. The above coframe is not fully satisfactory from the geometric point of view since it does not transform equivariantly along the fibres of $\mathcal{P}^c \to \mathcal{J}^2$, that is to say, it does not define a Cartan connection.

In order to see this we consider the simplest case, related to the equation $y''' = 0$, when all the functions $A, \ldots, \widetilde{N}$ vanish. Then (46) become the Maurer-Cartan equations for the Lie algebra $\mathfrak{o}(3, 2) \cong \mathfrak{sp}(4, \mathbb{R})$ and $\mathcal{P}^c$ is locally the Lie group $O(3, 2)$. The Maurer-Cartan form on $\mathcal{P}^c$ in the four-dimensional defining representation of $\mathfrak{sp}(4, \mathbb{R})$ is given by

$$
\widetilde{\omega} = \begin{pmatrix}
\frac{1}{2}\Pi_1 & \frac{1}{2}\Pi_2 & -\frac{1}{2}\Pi_8 & -\frac{1}{4}\Lambda_1 \\[2mm]
\theta^4 & \Pi_3 - \frac{1}{2}\Pi_1 & -\Pi_9 & -\frac{1}{2}\Pi_8 \\[2mm]
\theta^2 & \theta^3 & \frac{1}{2}\Pi_1 - \Pi_3 & -\frac{1}{2}\Pi_2 \\[2mm]
2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Pi_1
\end{pmatrix}.
$$

Let $\mathfrak{h}$ be the six-dimensional subalgebra of $\mathfrak{o}(3, 2)$ annihilated by the ideal $< \theta^1, \theta^2, \theta^3, \theta^4 >$ and let H_6 be the connected simply-connected Lie subgroup of $SP(4, \mathbb{R})$ with the algebra $\mathfrak{h}$. Then $H_6 \to \mathcal{P} \to \mathcal{J}^2$ is a homogeneous space and $\widetilde{\omega}$ is a flat Cartan connection of type $(SP(4, \mathbb{R}), H_6)$ on $\mathcal{P}^c$.

However, this object is not a Cartan connection in a general case, when $A, \ldots, \widetilde{N}$ do not vanish, since its curvature $\widetilde{K} = d\widetilde{\omega} + \widetilde{\omega} \wedge \widetilde{\omega}$ is not horizontal with respect to the fibration $\mathcal{P} \to \mathcal{J}^2$, that is value of $\widetilde{K}$ on a vector tangent to a fibre of $\mathcal{P} \to \mathcal{J}^2$ is not necessarily zero; for instance $\widetilde{K}^1{}_2$ contains the term $\frac{1}{2}A\Pi_9 \wedge \theta^1$. On the other hand the horizontality of $\widetilde{K}$ is necessary and locally sufficient for $\widetilde{\omega}$ to be a Cartan connection.

In order to resolve this problem H. Sato and Y. Yoshikawa [**52**] found the structural equations for the normal connection in this problem by means of the Tanaka theory. We recalculate their result in our notation and give explicit form of the normal connection, which their paper does not contain.

Let $E^i{}_j \in \mathfrak{gl}(4, \mathbb{R})$ denotes the matrix whose (i, j)-component is equal to one and other components equal zero. We introduce the following basis in $\mathfrak{sp}(4, \mathbb{R})$

$$
\begin{aligned}
& e_1 = 2E^4{}_1, && e_2 = E^3{}_1 + E^4{}_2, && e_3 = E^3{}_2 \\
& e_4 = E^2{}_1 - E^4{}_3, && e_5 = \tfrac{1}{2}(E^1{}_1 - E^2{}_2 + E^3{}_3 - E^4{}_4), && e_6 = \tfrac{1}{2}(E^1{}_2 - E^3{}_4), \\
& e_7 = E^2{}_2 - E^3{}_3, && e_8 = -\tfrac{1}{2}(E^1{}_3 + E^4{}_2), && e_9 = -E^2{}_3, \\
& e_{10} = -\tfrac{1}{4}E^1{}_4.
\end{aligned}
$$

(47)

The form $\widetilde{\omega}$ is given by

$$
\widetilde{\omega} = \theta^1 e_1 + \theta^2 e_2 + \theta^3 e_3 + \theta^4 e_4 + \Pi_1 e_5 + \Pi_2 e_6 + \Pi_3 e_7 + \Pi_8 e_8 + \Pi_9 e_9 + \Lambda_1 e_{10}.
$$

The algebra $\mathfrak{g}$ has the following grading

$$
\mathfrak{o}(3, 2) = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3,
$$

where

$$
\begin{aligned}
& \mathfrak{g}_{-3} = < e_1 >, && \mathfrak{g}_{-2} = < e_2 >, && \mathfrak{g}_{-1} = < e_3, e_4 >, \\
& \mathfrak{g}_0 = < e_5, e_7 >, && \mathfrak{g}_1 = < e_6, e_9 >, && \mathfrak{g}_2 = < e_8 >, && \mathfrak{g}_3 = < e_{10} >
\end{aligned}
$$

and $\mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 = < e_5, \ldots, e_{10} >$. Let lower case Latin indices range from 1 to 4 and upper case Latin indices range from 1 to 10 throughout this section.

Suppose now that $\widehat{\omega}$ is an $\mathfrak{sp}(4, \mathbb{R})$ Cartan connection on $\mathcal{P}^c$. Its structure function κ, defined in (35), decomposes into

$$
\kappa = \tfrac{1}{2}\kappa^I{}_{ij}\, e_I \otimes e^i \wedge e^j,
$$

where (e^I) denotes the basis dual to (e_I) and $\kappa^I{}_{ij} = \kappa^I{}_{[ij]}$ are functions. Condition i) of definition 2.3 reads

$$\kappa^1{}_{23} = 0, \qquad \kappa^1{}_{24} = 0, \qquad \kappa^1{}_{34} = 0, \qquad \kappa^2{}_{34} = 0.$$

We read structural constants $[e_I, e_J] = c^K{}_{IJ}e_K$ for $\mathfrak{sp}(4,\mathbb{R})$, compute the Killing form B_{IJ} and its inverse B^{IJ}. The operator $\partial^* \colon \mathrm{Hom}(\wedge^2\mathfrak{m}, \mathfrak{g}) \to \mathrm{Hom}(\mathfrak{m}, \mathfrak{g})$ acts as follows

$$\partial^*(e_I \otimes e^i \wedge e^j) = \left(2\delta^{[i}{}_m B^{j]K} c^L{}_{KI} - \delta^L{}_I c^{[i}{}_{Km} B^{j]K} \right) e_L \otimes e^m.$$

We apply ∂^* to the basis $(e_I \otimes e^i \wedge e^j)$ of $\mathrm{Hom}^1(\wedge^2\mathfrak{m}, \mathfrak{g})$ and find that the condition $\partial^*\kappa = 0$ for the normality is equivalent to vanishing of the following combinations of $\kappa^I{}_{ij}$:

$$\kappa^1{}_{14}, \quad \kappa^2{}_{14}, \quad \kappa^2{}_{24}, \qquad\qquad \kappa^3{}_{24}, \quad \kappa^3{}_{34}, \quad \kappa^4{}_{23},$$

$$\kappa^1{}_{12} + \kappa^2{}_{13} + \kappa^2{}_{24}, \qquad\qquad 2\kappa^1{}_{12} - \kappa^4{}_{24} - \kappa^5{}_{34},$$

$$\kappa^1{}_{12} - \kappa^3{}_{23} - \kappa^7{}_{34}, \qquad\qquad \kappa^1{}_{13} + \kappa^2{}_{23},$$

$$\kappa^1{}_{13} - \kappa^4{}_{34}, \qquad\qquad \kappa^2{}_{12} + \kappa^4{}_{14} + \kappa^5{}_{24},$$

$$\kappa^2{}_{12} - \kappa^3{}_{13} - \kappa^7{}_{24}, \qquad\qquad \kappa^2{}_{12} + \kappa^5{}_{24} - \kappa^6{}_{34} - \kappa^7{}_{24},$$

$$\kappa^2{}_{13} + \kappa^3{}_{23} + \kappa^5{}_{34} - \kappa^7{}_{34}, \qquad\qquad \kappa^2{}_{23} + \kappa^4{}_{34},$$

$$\kappa^3{}_{12} - \kappa^5{}_{14} + \kappa^6{}_{24} + \kappa^7{}_{14}, \qquad\qquad \kappa^4{}_{12} - \kappa^5{}_{13} + 2\kappa^7{}_{13} + \kappa^9{}_{24},$$

$$\kappa^4{}_{12} - \kappa^6{}_{23} - \kappa^8{}_{34} - \kappa^9{}_{24}, \qquad\qquad \kappa^4{}_{13} + \kappa^7{}_{23} - \kappa^9{}_{34},$$

$$\kappa^4{}_{14} + \kappa^6{}_{34} + \kappa^7{}_{24}, \qquad\qquad \kappa^4{}_{24} - \kappa^5{}_{34} + 2\kappa^7{}_{34},$$

$$2\kappa^5{}_{12} - 2\kappa^8{}_{24} - \kappa^{10}{}_{34}, \qquad\qquad \kappa^5{}_{13} + \kappa^6{}_{23} - \kappa^8{}_{34},$$

$$\kappa^5{}_{14} + \kappa^6{}_{24}, \qquad\qquad \kappa^5{}_{23} - 2\kappa^7{}_{23} - \kappa^9{}_{34},$$

$$2\kappa^6{}_{12} + 2\kappa^8{}_{14} + \kappa^{10}{}_{24}, \qquad\qquad \kappa^6{}_{13} + \kappa^7{}_{12} + \kappa^8{}_{24} + \kappa^9{}_{14}.$$

We write the sought normal Cartan connection $\widehat{\omega}^c$ as follows

$$\widehat{\omega}^c = \theta^1 e_1 + \theta^2 e_2 + \theta^3 e_3 + \theta^4 e_4 + \Omega_1 e_5 + \Omega_2 e_6 + \Omega_3 e_7 + \Omega_4 e_8 + \Omega_5 e_9 + \Omega_6 e_{10}.$$

The forms Ω_μ, unknown yet, are given by

$$\Omega_1 = \Pi_1 + a_i \theta^i, \qquad \Omega_2 = \Pi_2 + b_i \theta^i, \qquad \Omega_3 = \Pi_3 + c_i \theta^i,$$

$$\Omega_4 = \Pi_8 + f_i \theta^i, \qquad \Omega_5 = \Pi_9 + g_i \theta^i, \qquad \Omega_6 = \Lambda_1 + h_i \theta^i.$$

Next, we calculate the curvature $\widehat{K}^c = \mathrm{d}\widehat{\omega}^c + \widehat{\omega}^c \wedge \widehat{\omega}^c$, find the components of the structure function and put them into the normality conditions. These are only satisfied if all the functions $a_i, b_i, c_i, e_i, f_i, g_i$ vanish except for c_1, h_1, h_2, h_3, h_4 which are arbitrary and

$$a_1 = 2c_1, \qquad b_1 = \tfrac{2}{3}C, \qquad b_2 = c_1, \qquad f_1 = \tfrac{2}{3}J, \qquad f_4 = c_1,$$

where C and J are the functions in (46). Finally, we obtain from the e_1-component of the Bianchi identity $\mathrm{d}\widehat{K}^c = \widehat{K}^c \wedge \widehat{\omega}^c - \widehat{\omega}^c \wedge \widehat{K}^c$ that

$$c_1 = 0, \qquad h_1 = \tfrac{4}{3}G - \tfrac{2}{3}\widetilde{X}_4(J), \qquad h_2 = \tfrac{2}{3}J, \qquad h_3 = 0, \qquad h_4 = -\tfrac{2}{3}C,$$

where $\widehat{X}_4$ is the vector field in the frame $(\widetilde{X}_1, \widetilde{X}_2, \widetilde{X}_3, \widetilde{X}_4, \widetilde{X}_5, \widetilde{X}_6, \widetilde{X}_7, \widetilde{X}_8, \widetilde{X}_9, \widetilde{X}_{10})$ dual to $(\theta^1, \theta^1, \theta^1, \theta^1, \Pi_1, \Pi_2, \Pi_3, \Pi_8, \Pi_9, \Lambda_1)$. The normal connection $\widehat{\omega}^c$ has been constructed. The last thing we must do is renaming the coordinates $u_8 \to u_4$, $u_9 \to u_5$ and choosing u_6 appropriately, so that formulae (31) – (34) hold. This finishes the proof of theorem 2.1.

3. Geometries on ten-dimensional bundle

3.1. Curvature and its interpretation. We turn to discussion of consequences of theorem 2.1. As we explained in Introduction, the basic object that contains the contact invariants for the ODEs is the curvature

$$\widehat{K}^c = \mathrm{d}\widehat{\omega}^c + \widehat{\omega}^c {\wedge} \widehat{\omega}^c.$$

It is given by the structural equations for the coframe $\theta^1, \ldots, \Omega_6$.

$$
\begin{aligned}
\mathrm{d}\theta^1 =& \Omega_1 {\wedge} \theta^1 + \theta^4 {\wedge} \theta^2, \\
\mathrm{d}\theta^2 =& \Omega_2 {\wedge} \theta^1 + \Omega_3 {\wedge} \theta^2 + \theta^4 {\wedge} \theta^3, \\
\mathrm{d}\theta^3 =& \Omega_2 {\wedge} \theta^2 + (2\Omega_3 - \Omega_1){\wedge} \theta^3 + \mathbf{A_2^c}\theta^2 {\wedge} \theta^1 + \mathbf{A_1^c}\theta^4 {\wedge} \theta^1, \\
\mathrm{d}\theta^4 =& \Omega_4 {\wedge} \theta^1 + \Omega_5 {\wedge} \theta^2 + (\Omega_1 - \Omega_3){\wedge} \theta^4, \\
\mathrm{d}\Omega_1 =& \Omega_6 {\wedge} \theta^1 + \Omega_4 {\wedge} \theta^2 - \Omega_2 {\wedge} \theta^4, \\
\mathrm{d}\Omega_2 =& (\Omega_3 - \Omega_1){\wedge} \Omega_2 + \tfrac{1}{2}\Omega_6 {\wedge} \theta^2 + \Omega_4 {\wedge} \theta^3 + \mathbf{A_3^c}\theta^1 {\wedge} \theta^2 + \mathbf{A_4^c}\theta^1 {\wedge} \theta^4, \\
(48) \quad \mathrm{d}\Omega_3 =& \tfrac{1}{2}\Omega_6 {\wedge} \theta^1 + \Omega_4 {\wedge} \theta^2 + \Omega_5 {\wedge} \theta^3 + \mathbf{A_5^c}\theta^1 {\wedge} \theta^2 + \mathbf{A_2^c}\theta^1 {\wedge} \theta^4, \\
\mathrm{d}\Omega_4 =& \Omega_5 {\wedge} \Omega_2 + \Omega_4 {\wedge} \Omega_3 + \tfrac{1}{2}\Omega_6 {\wedge} \theta^4 + (\mathbf{A_6^c} + \mathbf{B_2^c})\theta^1 {\wedge} \theta^2 + 2\mathbf{B_3^c}\theta^1 {\wedge} \theta^3 \\
& - \mathbf{A_3^c}\theta^1 {\wedge} \theta^4 + \mathbf{B_4^c}\theta^2 {\wedge} \theta^3 \\
\mathrm{d}\Omega_5 =& (\Omega_1 - 2\Omega_3){\wedge} \Omega_5 + \Omega_4 {\wedge} \theta^4 + (\mathbf{A_7^c} + \mathbf{B_3^c})\theta^1 {\wedge} \theta^2 + \mathbf{B_4^c}\theta^1 {\wedge} \theta^3 \\
& - \mathbf{A_5^c}\theta^1 {\wedge} \theta^4 + \mathbf{B_1^c}\theta^2 {\wedge} \theta^3, \\
\mathrm{d}\Omega_6 =& \Omega_6 {\wedge} \Omega_1 + 2\Omega_4 {\wedge} \Omega_2 + \mathbf{C_1^c}\theta^1 {\wedge} \theta^2 + 2\mathbf{B_2^c}\theta^1 {\wedge} \theta^3 + \mathbf{A_8^c}\theta^1 {\wedge} \theta^4 + 2\mathbf{B_3^c}\theta^2 {\wedge} \theta^3,
\end{aligned}
$$

where $\mathbf{A_1^c}, \ldots, \mathbf{A_8^c}, \mathbf{B_1^c}, \ldots, \mathbf{B_4^c}, \mathbf{C_1^c}$ are functions on $\mathcal{P}^c$.

The functions $\mathbf{A_1^c}, \ldots, \mathbf{C_1^c}$ are contact relative invariants of the underlying ODE, that is their vanishing or not is a contact invariant property. The full set of contact invariants can be constructed by consecutive differentiation of $\mathbf{A_1^c}, \ldots, \mathbf{C_1^c}$ with respect to the frame $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10})$ dual to $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$. Utilizing the identities $\mathrm{d}^2\Omega_\mu = 0$ we compute the exterior derivatives of $\mathbf{A_i^c}, \mathbf{B_j^c}, \mathbf{C_1^c}$, for instance

$$\mathrm{d}\mathbf{B_1^c} = X_1(\mathbf{B_1^c})\theta^1 + X_2(\mathbf{B_2^c})\theta^2 + X_3(\mathbf{B_3^c})\theta^3 - 2\mathbf{B_4^c}\theta^4 + 2\mathbf{B_1^c}\Omega_1 - 5\mathbf{B_1^c}\theta^3.$$

From these formulae it follows that i) $\mathbf{A_2^c}, \ldots \mathbf{A_8^c}$ express by the coframe derivatives of $\mathbf{A_1^c}$, ii) $\mathbf{B_2^c}, \ldots \mathbf{B_4^c}$ express by coframe derivatives of $\mathbf{B_1^c}$ and iii) $\mathbf{C_1^c}$ is a function of coframe derivatives of both $\mathbf{A_1^c}$ and $\mathbf{B_1^c}$. Hence there are two basic invariants for the system (48) reading

$$\mathbf{A_1^c} = \frac{u_3^3}{u_1^3}W \qquad \mathbf{B_1^c} = \frac{u_1^2}{6u_3^5}F_{qqqq}$$

and all other invariants can be derived from them.[1]

The simplest case, in which all the contact invariants $\mathbf{A_1^c}, \ldots, \mathbf{C_1^c}$ vanish corresponds to $W = F_{qqqq} = 0$ and is characterized by

COROLLARY 2.4. *For a third-order ODE $y''' = F(x, y, y', y'')$ the following conditions are equivalent.*

   i) *The ODE is contact equivalent to $y''' = 0$.*
   ii) *It satisfies the conditions $W = 0$, and $F_{qqqq} = 0$.*
   iii) *It has the $\mathfrak{o}(3, 2)$ algebra of contact symmetry generators.*

---

[1]This property means in language of Tanaka's theory that curvature of a normal connection is generated by its harmonic part.

3.2. Structure of $\mathcal{P}^c$. The manifold $\mathcal{P}^c$ is endowed with threefold geometry of the principal bundle over the second jet space $\mathcal{J}^2$, the first jet space J^1 and the solution space $\mathcal{S}$. We discuss these structures consecutively. Let us remind that $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9, X_{10})$ denotes the dual frame to $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3, \Omega_4, \Omega_5, \Omega_6)$.

First bundle structure, $H_6 \to \mathcal{P} \to \mathcal{J}^2$, has been already introduced explicitly in theorem 2.1. Here we only show that it is actually defined by the coframe, since it can be recovered from its structural equations. Indeed, we see from (48) that the ideal spanned by $\theta^1, \theta^2, \theta^3, \theta^4$ is closed

$$\mathrm{d}\theta^i \wedge \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 = 0 \qquad \text{for} \qquad i = 1, 2, 3, 4,$$

and it follows that its annihilated distribution $< X_5, X_6, X_7, X_8, X_9, X_{10} >$ is integrable. Maximal integral leaves of this distribution are locally fibres of the projection $\mathcal{P}^c \to \mathcal{J}^2$. Furthermore, the commutation relations of these vector fields are isomorphic to the commutation relations of the six-dimensional group H_6, hence we can *define* $X_5, \ldots, X_{10}$ to be the fundamental vector fields associated to the action H_6 on $\mathcal{P}^c$.

In order to explain how $\mathcal{P}^c$ is the bundle $CO(2,1) \ltimes \mathbb{R}^3 \to \mathcal{P}^c \to \mathcal{S}$ let us first describe the space $\mathcal{S}$ itself. On $\mathcal{J}^2$ there is the congruence of solutions of the ODE. A family of solutions passing through sufficiently small open set in $\mathcal{J}^2$ is given by the mapping

$$(x; c_1, c_2, c_3) \mapsto (x, f(x; c_1, c_2, c_3), f_x(x; c_1, c_2, c_3), f_{xx}(x; c_1, c_2, c_3)),$$

where $y = f(x; c_1, c_2, c_3)$ is the general solution to $y''' = F(x, y, y', y'')$ and (c_1, c_2, c_3) are constants of integration. Thus a solution can be considered as a point in the three-dimensional real space $\mathcal{S}$ parameterized by the constants of integration. This space can be endowed with a local structure of differentiable manifold if we *choose* a parametrization $(c_1, c_2, c_3) \mapsto f(x; c_1, c_2, c_3)$ of the solutions and admit only sufficiently smooth re-parameterizations $(c_1, c_2, c_3) \mapsto (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$ of the constants. We always assume that $\mathcal{S}$ is locally a manifold. Since $\mathcal{J}^2$ is a bundle over $\mathcal{S}$ so is $\mathcal{P}^c$ and the fibres of the projection $\mathcal{P}^c \to \mathcal{S}$ are annihilated by the closed ideal $< \theta^1, \theta^2, \theta^3 >$. On the fibres there act the vector fields $X_4, X_5, X_6, X_7, X_8, X_9, X_{10}$, which form the Lie algebra $\mathfrak{co}(2,1) \oplus \mathbb{R}^3$ and thereby define the action of $CO(2,1) \ltimes \mathbb{R}^3$ on $\mathcal{P}^c$.

Apart from the projection $\mathcal{J}^2 \to \mathcal{S}$ there is also the projection $\mathcal{J}^2 \to \mathcal{J}^1$ that takes the second jet (x, y, p, q) of a curve into its first jet (x, y, p). It gives rise to the third bundle structure, $H_7 \to \mathcal{P}^c \to \mathcal{J}^1$. Here the tangent distribution is $< X_3, X_5, X_6, X_7, X_8, X_9, X_{10} >$ and it defines the action of a seven-dimensional group H_7 which of course contains H_6.

It appears that, under some conditions, $\widehat{\omega}_c$ is not only a Cartan connection over $\mathcal{J}^2$ but over $\mathcal{S}$ or $\mathcal{J}^1$ also.

4. Conformal geometry on solution space

The section on the conformal geometry on $\mathcal{S}$ only contains results of P. Nurowski [48, 46], see also [27]. Let us define on $\mathcal{P}^c$ the symmetric two-contravariant tensor field

$$\widehat{g} = (\theta^2)^2 - 2\theta^1\theta^3$$

of signature $(+ + - 0\,0\,0\,0\,0\,0\,0)$. The degenerate directions of $\widehat{g}$ are precisely those tangent to the fibres of $\mathcal{P}^c \to \mathcal{S}$

$$\widehat{g}(X_i, \cdot) = 0, \qquad \text{for} \qquad i = 4, 5, 6, 7, 8, 9, 10.$$

The Lie derivatives of $\widehat{g}$ along the degenerate directions are as follows

$$(49) \qquad L_{X_4}\widehat{g} = \frac{u_3^3}{u_1^3}W(\theta^1)^2, \qquad L_{X_7}\widehat{g} = 2\widehat{g},$$

and

$$(50) \qquad L_{X_i}\widehat{g} = 0 \qquad \text{for} \qquad i = 5, 6, 8, 9, 10.$$

Thus, if only

$$W = 0,$$

all the degenerate directions but X_7 are isometries of $\widehat{g}$ whereas X_7 is a conformal symmetry. This allows us to *project* $\widehat{g}$ to the Lorentzian conformal metric $[g]$ on the solution space $\mathcal{S}$. Since the action of $CO(2, 1) \ltimes \mathbb{R}^3$ on $\mathcal{S}$ is not given explicitly, we can not write down the explicit formula for g. We can only do this in terms of the coordinate system (x, y, p, q) on $\mathcal{J}^2$:

$$(51) \ \ g = (\omega^2)^2 - 2\omega^1\widetilde{\omega}^3 =$$
$$= (\mathrm{d}p - q\mathrm{d}x)^2 - 2(\mathrm{d}y - p\mathrm{d}x)(\mathrm{d}q - F\mathrm{d}x - \tfrac{1}{3}F_q(\mathrm{d}p - q\mathrm{d}x) + K(\mathrm{d}y - p\mathrm{d}x)).$$

In order to find the explicit expression for g in a coordinate system (c_1, c_2, c_3) on $\mathcal{S}$ we would have to find the general solution $y = f(x; c_1, c_2, c_3)$ of the underlying ODE, then solve the system

$$\begin{cases} y = f(x; c_1, c_2, c_3), \\ p = f_x(x; c_1, c_2, c_3), \\ q = f_{xx}(x; c_1, c_2, c_3) \end{cases}$$

with respect to c_i and substitute these formulae into (51).

The condition $W = 0$ means $\mathbf{A_1^c} = 0$ which causes $\mathbf{A_2^c} = \ldots = \mathbf{A_8^c} = 0$. Thus, structural equations (48) do not contain the non-constant terms proportional to θ^4 and the curvature $\widehat{K}^c$ is horizontal over $\mathcal{S}$. As a consequence, $\widehat{\omega}^c$ is a connection over $\mathcal{S}$. It appears that it is nothing but the Cartan normal conformal connection for $[g]$.

4.1. Normal conformal connection. We introduce after [**48**] the normal conformal connection and discuss its curvature. Consider $\mathbb{R}^n$ with coordinates (x^μ), $\mu = 1, \ldots, n$ equipped with the flat metric $g = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ of the signature (k, l), $n = k + l$. The group $Conf(k, l)$ of conformal symmetries of $g_{\mu\nu}$ consists of

    i) the subgroup $CO(k, l) = \mathbb{R} \times O(k, l)$ containing the group $O(k, l)$ of isometries of g and the dilatations,
    ii) the subgroup $\mathbb{R}^n$ of translations,
    iii) the subgroup $\mathbb{R}^n$ of special conformal transformations.

The stabilizer of the origin in $\mathbb{R}^n$ is the semisimple product of $CO(k, l) \ltimes \mathbb{R}^n$ of the isometries, the dilatations, and the special conformal transformations. The flat conformal space is the homogeneous space $Conf(k, l)/CO(k, l) \ltimes \mathbb{R}^n$. To this space there is associated the flat Cartan connection on the bundle $CO(k, l) \ltimes \mathbb{R}^n \to Conf(k, l) \to \mathbb{R}^n$ with values in the algebra $\mathfrak{conf}(k, l)$.

By virtue of the Möbius construction, the group $Conf(k, l)$ is isomorphic to the orthogonal group $O(k + 1, l + 1)$ preserving the metric

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & g_{\mu\nu} & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

on $\mathbb{R}^{n+2}$. This isomorphism gives rise to the following representation of $\mathfrak{conf}(k,l) \cong \mathfrak{o}(k+1,l+1)$

$$(52) \qquad \begin{pmatrix} \phi & g_{\nu\rho}\xi^\rho & 0 \\ v^\mu & \lambda^\mu{}_\nu & \xi^\mu \\ 0 & g_{\nu\rho}v^\rho & -\phi \end{pmatrix}.$$

Here the vector $(v^\mu) \in \mathbb{R}^n$ generates the translations, the matrix $(\lambda^\mu{}_\nu) \in \mathfrak{o}(k,l)$ generates the isometries, ϕ – the dilatations, and $(\xi^\mu) \in \mathbb{R}^n$ – the special conformal transformations.

Let us turn to an arbitrary case of a conformal metric $[g]$ of the signature (k,l) on a n-dimensional manifold $\mathcal{M}$, $n = k + l > 2$. Let us choose a representative g of $[g]$ and consider an orthogonal coframe (ω^μ), in which $g = g_{\mu\nu}\,\omega^\mu \otimes \omega^\nu$ with constant coefficients $g_{\mu\nu}$. We calculate the Levi-Civita connection $\Gamma^\mu{}_\nu$ for g, its Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. Next we define the following one-forms

$$P_\nu = \left(\tfrac{1}{2-n}R_{\nu\rho} + \tfrac{1}{2(n-1)(n-2)}Rg_{\nu\rho}\right)\theta^\rho.$$

Given these objects we build the following $\mathfrak{conf}(k,l)$-valued matrix ω^{conf} on M

$$(53) \qquad \omega^{conf} = \begin{pmatrix} 0 & P_\nu & 0 \\ \theta^\mu & \Gamma^\mu{}_\nu & g^{\mu\rho}P_\rho \\ 0 & g_{\nu\rho}\theta^\rho & 0 \end{pmatrix}.$$

This is the normal conformal connection on $\mathcal{M}$ in the natural gauge.[2] Consider now the conformal bundle $CO(k,l) \ltimes \mathbb{R}^n \to \mathcal{P} \to \mathcal{M}$, and choose a coordinate system (h,x) on $\mathcal{P}$ compatible with the local triviality $\mathcal{P} \cong CO(k,l) \ltimes \mathbb{R}^n \times \mathcal{M}$, where x stands for (x^μ) in $\mathcal{M}$ and the matrix $h \in CO(k,l) \ltimes \mathbb{R}^n$ reads

$$h = \begin{pmatrix} \mathrm{e}^{-\phi} & \mathrm{e}^{-\phi}g_{\nu\rho}\xi^\rho & \tfrac{1}{2}\mathrm{e}^{-\phi}\xi^\rho\xi^\sigma g_{\rho\sigma} \\ 0 & \Lambda^\mu{}_\nu & \Lambda^\mu{}_\rho\xi^\rho \\ 0 & 0 & \mathrm{e}^\phi \end{pmatrix}, \qquad \Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma g_{\mu\nu} = g_{\rho\sigma}.$$

The normal conformal connection for g is the following $\mathfrak{conf}(k,l)$-valued one-form on $\mathcal{P}$

$$\widehat{\omega}^{conf}(h,x) = h^{-1} \cdot \pi^*(\omega^{conf}(x)) \cdot h + h^{-1}\mathrm{d}h.$$

The curvature of the normal conformal connection is as follows

$$\widehat{K}^{conf}(h,x) = h^{-1} \cdot \pi^*(K^{conf}(x)) \cdot h,$$

where K^{conf} is the curvature for ω^{conf} on $\mathcal{M}$

$$K^{conf} = \begin{pmatrix} 0 & DP_\nu & 0 \\ 0 & C^\mu{}_\nu & g^{\mu\rho}DP_\rho \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$DP_\mu = \mathrm{d}P_\mu + P_\nu \wedge \Gamma^\nu{}_\mu = \tfrac{1}{2}P_{\mu\nu\rho}\omega^\nu \wedge \omega^\rho.$$

The curvature contains the lowest-order conformal invariants for g, namely

- For $n \geq 4$

$$C^\mu{}_\nu = \tfrac{1}{2}C^\mu{}_{\nu\rho\sigma}\omega^\rho \wedge \omega^\sigma$$

is the Weyl conformal tensor, while

$$P_{\mu\nu\rho} = \tfrac{1}{n-3}\nabla_\sigma C^\sigma{}_{\mu\nu\rho}$$

is its divergence.

---

[2]The gauge is natural since we have started from the Levi-Civita connection, not from any Weyl connection for g, in which case (53) contains a Weyl potential.

- For $n = 3$ the Weyl tensor identically vanishes, $C^\mu_\nu = 0$, and the lowest-order conformal invariant is the Cotton tensor $P_{\mu\nu\rho}$. It has five independent components.

The normality of conformal connections, originally defined by E. Cartan, is the following property. The algebra $\mathfrak{conf}(k,l)$ is graded: $\mathfrak{conf}(k,l) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where translations are the $\mathfrak{g}_{-1}$-part, $\mathfrak{co}(k,l)$ is the $\mathfrak{g}_0$-part and the special conformal transformations are the $\mathfrak{g}_1$-part. The normal connection for $[g]$ is the only conformal connection such that the $\mathfrak{co}(k,l)$-part of its curvature, $C^\mu_\nu = \frac{1}{2}C^\mu_{\nu\rho\sigma}\omega^\rho \wedge \omega^\sigma$, is traceless: $C^\rho_{\nu\rho\sigma} = 0$. Cartan normal conformal connections are normal in the sense of Tanaka.

4.2. Normal conformal connection from ODEs. As we mentioned, $\widehat{\omega}^c$ is the normal conformal connection over $\mathcal{S}$. In order to see this it is enough to rearrange $\widehat{\omega}^c$ according to the representation (52)

$$
(54) \qquad \widehat{\omega}^c = \begin{pmatrix} \Omega_3 & -\frac{1}{2}\Omega_6 & -\Omega_4 & -\Omega_5 & 0 \\ \theta^1 & \Omega_3 - \Omega_1 & -\theta^4 & 0 & -\Omega_5 \\ \theta^2 & -\Omega_2 & 0 & -\theta^4 & \Omega_4 \\ \theta^3 & 0 & -\Omega_2 & \Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_6 \\ 0 & \theta^3 & -\theta^2 & \theta^1 & -\Omega_3 \end{pmatrix}
$$

and calculate its curvature, which is equal to

$$
\widehat{K}^c = \begin{pmatrix} 0 & DP_1 & DP_2 & DP_3 & 0 \\ 0 & 0 & 0 & 0 & DP_3 \\ 0 & 0 & 0 & 0 & -DP_2 \\ 0 & 0 & 0 & 0 & DP_1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

with the following components of the Cotton tensor on $\mathcal{P}^c$

$$
DP_1 = -\tfrac{1}{2}\mathbf{C^c_1}\,\theta^1\wedge\theta^2 - \mathbf{B^c_2}\,\theta^1\wedge\theta^3 - \mathbf{B^c_3}\,\theta^2\wedge\theta^3,
$$
$$
DP_2 = -\mathbf{B^c_2}\,\theta^1\wedge\theta^2 - 2\mathbf{B^c_3}\,\theta^1\wedge\theta^3 - \mathbf{B^c_4}\,\theta^2\wedge\theta^3,
$$
$$
DP_3 = -\mathbf{B^c_3}\,\theta^1\wedge\theta^2 - \mathbf{B^c_4}\,\theta^1\wedge\theta^3 - \mathbf{B^c_1}\,\theta^2\wedge\theta^3.
$$

Finally, we pull-back these formula to $\mathcal{J}^2$ through $u_1 = u_3 = 1$, $u_2 = u_4 = u_5 = u_6 = 0$ and get

$$
\begin{aligned}
DP_1 =& (\tfrac{1}{2}M_p + \tfrac{1}{6}F_q M_q + \tfrac{1}{6}F_{qqq}K_y + K_q L_q - \tfrac{1}{6}K^2 F_{qqqq} + \\
& + \tfrac{1}{6}K_q F_{qqy} - \tfrac{1}{6}F_{qqyy} - \tfrac{1}{3}F_{qqq}K_q K + \tfrac{1}{3}F_{qqy}K)\omega^1\wedge\omega^2 \\
& + \tfrac{1}{2}\left(M_q - K_{qqq}K - 2K_{qq}K_q + K_{qqy}\right)\omega^1\wedge\widetilde{\omega}^3 + \\
& - \tfrac{1}{2}L_{qq}\omega^2\wedge\widetilde{\omega}^3, \\
DP_2 =& \tfrac{1}{2}\left(M_q - K_{qqq}K - 2K_{qq}K_q + K_{qqy}\right)\omega^1\wedge\omega^2 + \\
& - L_{qq}\omega^1\wedge\widetilde{\omega}^3 + \tfrac{1}{2}K_{qqq}\omega^2\wedge\widetilde{\omega}^3, \\
DP_3 =& -\tfrac{1}{2}L_{qq}\omega^1\wedge\omega^2 + \tfrac{1}{2}K_{qqq}\omega^1\wedge\widetilde{\omega}^3 - \tfrac{1}{6}F_{qqqq}\omega^2\wedge\widetilde{\omega}^3.
\end{aligned}
$$

The formulae for the conformal connection and curvature (in a slightly different notation) are given in [**48, 27**].

5. Contact projective geometry on first jet space

The connection $\widehat{\omega}^c$ gives rise to not only the above conformal structure but also a geometry on the first jet space $\mathcal{J}^1$. As we know, there are two basic contact

invariants in the curvature $\widehat{K}^c$: W and F_{qqqq}. The condition $W = 0$ yields the conformal geometry. Let us now examine the second possibility

$$F_{qqqq} = 0.$$

Quick inspection of the structural equations (48) shows that in this case the curvature does not contain $\theta^i \wedge \theta^4$ terms and thereby $\widehat{\omega}^c$ is an $\mathfrak{sp}(4, \mathbb{R})$ Cartan connection on $H_7 \to \mathcal{P}^c \to \mathcal{J}^1$.

A natural question is to what geometric structure $\widehat{\omega}^c$ is now related. In order to answer it let us look at the geometry defined on $\mathcal{J}^1$ by the solutions of the underlying ODE $y''' = F(x, y, y', y'')$. As we have already said the solutions form a congruence on $\mathcal{J}^2$ since there passes exactly one solution through a point $(x_0, y_0, p_0, q_0) \in \mathcal{J}^2$. The G_c-structure on $\mathcal{J}^2$ defined by (8) preserves this congruence and also preserves the contact invariant information about the ODE. The geometry on $\mathcal{J}^2$ is then of first order, since it is characterized by the group G_c acting on the tangent space $T\mathcal{J}^2$.

Geometry of an ODE on $\mathcal{J}^1$ is of different kind. The family of solutions is no longer a congruence. Indeed, through a fixed $(x_0, y_0, p_0) \in \mathcal{J}^1$ there pass many solutions, each of them corresponding to some value of q_0, which is given by a choice of a tangent direction at (x_0, y_0, p_0). Such a choice, however, can not be made in an arbitrary way; a solution $y = f(x)$ lifts to a curve $x \mapsto (x, f(x), f'(x))$ in $\mathcal{J}^1$, whose tangent vector field $\partial_x + f'(x)\partial_y + f''(x)\partial_p$ is always annihilated by the one-form $\omega^1 = \mathrm{d}y - p\mathrm{d}x$ on $\mathcal{J}^1$. In this manner all the admissible tangent vectors to the solutions form a rank-two distribution $\mathcal{C}$, *the contact distribution on* $\mathcal{J}^1$, which is annihilated by ω^1. This is first difference between $\mathcal{J}^2$ and $\mathcal{J}^1$: we have the rank-two distribution $\mathcal{C}$ instead of the congruence. Second and more important difference is that we still need to distinguish the family of solutions among a class of all curves with their tangent fields in $\mathcal{C}$.

There are at least two basic methods of doing this. In the first method we treat the tangent directions of curves in $\mathcal{J}^1$ as new dimensions and move to the space where over each point $(x, y, p) \in \mathcal{J}^1$ there is the fibre of all the possible tangent directions. In this manner the 'entangled' family of solutions in $\mathcal{J}^1$ would be 'stretched' to a congruence in the new space. However, the bundle of tangent directions of $\mathcal{J}^1$ is nothing but the second jet space $\mathcal{J}^2$ and thereby we would come back to the description given in the theorem 2.1.

The other way is describing the family of solutions in $\mathcal{J}^1$ as a family of unparameterized geodesics of a linear connection in $\mathcal{J}^1$. This approach leads us to the notion of projective differential geometry.

5.1. Contact projective geometry. This geometry has been exhaustively analyzed in [**21**], see also [**7, 8**]. We will not discuss the general theory here but focus on an application of the three-dimensional case to the ODEs. The definition of contact-projective geometry, see D. Fox [**21**], adapted to our situation is the following.

DEFINITION 2.5. A contact projective structure on the first jet space $\mathcal{J}^1$ is given by the following data.

i) The contact distribution $\mathcal{C}$, that is a distribution annihilated by

$$\omega^1 = \mathrm{d}y - p\mathrm{d}x.$$

ii) A family of unparameterized curves everywhere tangent to $\mathcal{C}$ and such that: a) for a given point and a direction in $\mathcal{C}$ there is exactly one curve passing through that point and tangent to that direction, b) curves of the family are among unparameterized geodesics for some linear connection on $\mathcal{J}^1$.

A contact projective structure on $\mathcal{J}^1$ is equivalently given by a family of linear connections, whose geodesic spray contains the family of curves. For ∇ to belong to this class one needs

$$(55) \qquad \nabla_V V = \lambda(V)V$$

along every curve in the family, where X denotes a tangent field to the considered curve and $\lambda(X)$ is a function. Given two such connections ∇ and $\widetilde{\nabla}$, their difference is a $(2,1)$-type tensor field

$$A(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y,$$

for all X and Y. Simultaneously we have $A(V,V) = \mu(V)V$ for $V \in \mathcal{C}$, where $\mu(V) = \widetilde{\lambda}(V) - \lambda(V)$ and μ_w at a point $w \in \mathcal{J}^1$ is a covector on the vector space $\mathcal{C}_w$. By polarization we obtain

$$(56) \qquad A(X,Y) + A(Y,X) = \mu(X)Y + \mu(Y)X, \qquad X,Y \in \mathcal{C}.$$

The connections associated to a contact projective structure, when considered at a point, form an affine space characterized by the above A.

Let us describe A more explicitly. We choose a frame $(e_1 = \partial_y, e_2 = \partial_p, e_3 = \partial_x + p\partial_y)$ and denote the dual frame by $(\sigma^1, \sigma^2, \sigma^3)$. In particular $\omega^1 = \sigma^1$ and $\mathcal{C} = <e_2, e_3>$. Let $i,j,\ldots = 1,2,3$ and $I,J,\ldots = 2,3$. Now $\nabla_j e_i = \Gamma^k{}_{ij} e_k$, $A^k{}_{ij} = \widetilde{\Gamma}^k{}_{ij} - \Gamma^k{}_{ij}$, $\mu = \mu^I e_I$ and (56) reads

$$A^k{}_{(IJ)} = \mu_{(I} \delta^k{}_{J)}.$$

Relevant components are equal to

$$(57) \qquad
\begin{array}{lll}
A^1{}_{22} = 0, & A^1{}_{(23)} = 0, & A^1{}_{33} = 0, \\
A^2{}_{22} = \mu_2, & A^2{}_{(23)} = \tfrac{1}{2}\mu_3, & A^1{}_{33} = 0, \\
A^3{}_{22} = 0, & A^3{}_{(23)} = \tfrac{1}{2}\mu_2, & A^1{}_{33} = \mu_3
\end{array}$$

and the rest of $A^k{}_{ij}$ is free. Elementary calculations assure us that the class of admissible connections is a 20-dimensional subspace of 27-dimensional space of all linear connections on $\mathcal{J}^1$. Another constraint for the connections is given by

$$(58) \qquad (\nabla_V \omega^1)V = \nabla_V(\omega^1(V)) = 0, \qquad V \in \mathcal{C}.$$

In our frame this is equivalent to $\Gamma^k_{(IJ)} = 0$. Combining eq. (57) and (58) we obtain

PROPOSITION 2.6. *The following quantities are invariant with respect to a choice of a connection in the class distinguished by a contact projective structure on $\mathcal{J}^1$*

$$(59a) \qquad \Gamma^1{}_{22} = 0, \qquad \Gamma^1{}_{(23)} = 0, \qquad \Gamma^1{}_{33} = 0,$$

$$(59b) \qquad \Gamma^3{}_{22}, \qquad 2\Gamma^3{}_{(23)} - \Gamma^2{}_{22}, \qquad \Gamma^3{}_{33} - 2\Gamma^2{}_{(23)}, \qquad \Gamma^2{}_{33}.$$

The connection coefficients are calculated in a frame (e_i) such that $\mathcal{C} = <e_2, e_3>$.

Values of four the unspecified combinations (59b) *define a contact projective structure.*

Among the above connections there is a distinguished subclass of those connections which covariantly preserve the distribution $\mathcal{C}$. We shall call them compatible connections. They satisfy not only (58) but a stronger condition

$$\nabla_X \omega^1 = \phi(X)\omega^1, \qquad \text{for all } X,$$

with some one-form ϕ. We have the following

PROPOSITION 2.7. *A compatible connection has non-vanishing torsion*

PROOF.

$$d\omega^1(X,Y) = \tfrac{1}{2}((\nabla_X\omega^1)Y - (\nabla_Y\omega^1)X + \omega^1(T(X,Y))) =$$
$$= (\phi\wedge\omega^1)(X,Y) + \tfrac{1}{2}\omega^1(T(X,Y))),$$

thus

$$(d\omega^1\wedge\omega^1)(X,Y,Z) = \tfrac{1}{2}(\omega^1(T(X,Y))\omega^1(Z) + \text{cycl.perm.}) \neq 0.$$

$\square$

5.2. Contact projective geometries from ODEs. It is obvious that the family of solutions of an arbitrary third-order ODE satisfies the conditions i) and ii a) of definition 2.5. (Condition ii a) is satisfied with the possible exception of the direction ∂_p, which belongs to $\mathcal{C}$ but it is not tangent to any solution in general. However, this exception is irrelevant since our consideration is local on $T\mathcal{J}^1$.) We ask when the solutions form a subfamily of geodesics for a linear connection.

LEMMA 2.8. *A third-order ODE $y''' = F(x,y,y',y'')$ defines a contact-projective structure on $\mathcal{J}^1$ if and only if $F_{qqqq} = 0$. Moreover, the quantities (59b) are given by*

(60)
$$\Gamma^3{}_{22} = a_3, \qquad\qquad 2\Gamma^3{}_{(23)} - \Gamma^2{}_{22} = a_2,$$
$$\Gamma^3{}_{33} - 2\Gamma^2{}_{(23)} = a_1, \quad \Gamma^2{}_{33} = -a_0,$$

where

$$F = a_3 q^3 + a_2 q^2 + a_1 q + a_0.$$

PROOF. The field $V = \frac{d}{dx}$ tangent to a solution $(x, f(x), f'(x))$ equals $V = f''e_2 + e_3$ in the frame (e_i). The geodesic equations (55) read

$$(f'')^2\,\Gamma^1{}_{22} + 2f''\Gamma^1{}_{(23)} + \Gamma^1{}_{33} = 0,$$
$$f''' + (f'')^2\,\Gamma^2{}_{22} + 2f''\Gamma^2{}_{(23)} + \Gamma^2{}_{33} = \lambda(V)f'',$$
$$(f'')^2\,\Gamma^3{}_{22} + 2f''\Gamma^3{}_{(23)} + \Gamma^3{}_{33} = \lambda(V).$$

First of these equations is equivalent to (59a). From the remaining equations we have that

$$f''' = \Gamma^3{}_{22}f''^3 + (2\Gamma^3{}_{(23)} - \Gamma^2{}_{22})f''^2 + (\Gamma^3{}_{33} - 2\Gamma^2{}_{(23)})f'' - \Gamma^2{}_{33},$$

is satisfied along every solution. $\square$

We observe that the condition $F_{qqqq} = 0$ yields $\mathbf{B^c_1} = \mathbf{B^c_2} = \mathbf{B^c_3} = \mathbf{B^c_4} = 0$, which removes all $\theta^i\wedge\theta^3$ terms in the curvature and turns $\widehat{\omega}^c$ into a connection over $\mathcal{J}^1$, since in the curvature there are only terms proportional to $\theta^1, \theta^2, \theta^4$, horizontal with respect to $\mathcal{P}^c \to \mathcal{J}^1$. Furthermore, the algebra $\mathfrak{sp}(4,\mathbb{R}) \cong \mathfrak{o}(3,2)$ has the following grading (apart from those already mentioned)

(61) $$\mathfrak{sp}(4,\mathbb{R}) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

which reads in the basis (47):

$$\mathfrak{g}_{-2} = <e_1>, \qquad \mathfrak{g}_{-1} = <e_2, e_4>,$$
$$\mathfrak{g}_0 = <e_3, e_5, e_7, e_9>,$$
$$\mathfrak{g}_1 = <e_6, e_8>, \qquad \mathfrak{g}_2 = <e_{10}>.$$

After calculating Tanaka's normality conditions by the method of chapter 2 section 2.2, we observe that $\widehat{\omega}^c$ is the normal connection with respect to the grading (61). In this manner we have re-proved the known fact that to a three-dimensional contact

projective geometry there is associated the unique normal $\mathfrak{sp}(4, \mathbb{R})$-valued Cartan connection.

PROPOSITION 2.9. *If the contact projective geometry on $\mathcal{J}^1$ exists, then $\widehat{\omega}^c$ of theorem 2.1 is the normal Cartan connection for this geometry.*

From $\widehat{\omega}^c$ one may reconstruct the compatible connections. To do this we just observe that first, second and fourth equation of (48) can be written as

$$\begin{pmatrix} \mathrm{d}\theta^1 \\ \mathrm{d}\theta^2 \\ \mathrm{d}\theta^4 \end{pmatrix} + \underbrace{\begin{pmatrix} -\Omega_1 & 0 & 0 \\ -\Omega_2 & -\Omega_3 & \theta^3 \\ -\Omega_4 & -\Omega_5 & \Omega_3 - \Omega_1 \end{pmatrix}}_{\widehat{\Gamma}} \wedge \begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^4 \end{pmatrix} = \begin{pmatrix} \theta^4 \wedge \theta^2 \\ \theta^4 \wedge \theta^3 \\ 0 \end{pmatrix}.$$

The tree by three matrix denoted by $\widehat{\Gamma}$ is the $\mathfrak{g}_0 \oplus \mathfrak{g}_1$-part of $\widehat{\omega}^c$. The following proposition holds.

PROPOSITION 2.10. *For any section $s \colon \mathcal{J}^1 \to \mathcal{P}^c$ the pull-back $s^*\widehat{\Gamma}$ written in the coframe $(s^*\theta^1, s^*\theta^2, s^*\theta^4)$ is a connection compatible with the contact projective geometry.*

PROOF. First we choose the section $s_0 \colon \mathcal{J}^1 \to \mathcal{P}^c$ given by $q = 0$, $u_1 = 1$, $u_3 = 1$ and $u_2 = u_4 = u_5 = u_6 = 0$. We denote $\Gamma = s_0^*\widehat{\Gamma}$. In the coframe $\sigma^1 = s_0^*\theta^1$, $\sigma^2 = s_0^*\theta^2$ and $\sigma^3 = s_0^*\theta^4$ we have $-s_0^*\Omega_3 = \Gamma^2{}_2 = \Gamma^2{}_{2k}\sigma^k$, $s_0^*\theta^3 = \Gamma^2{}_3 = \Gamma^2{}_{3k}\sigma^k$ and so on. Equations (60) follow from (33) and (34), provided that $F_{qqqq} = 0$.

Next we consider an arbitrary section $s \colon \mathcal{J}^1 \to \mathcal{P}^c$. In the local trivialization $\mathcal{P}^c \cong H_7 \times \mathcal{J}^1$ we have $\mathcal{P}^c \ni w = (v, x)$, where $v \in H_7$, $x \in \mathcal{J}^1$ and s is given by $x \mapsto (v(x), x)$. Now $s^*\widehat{\omega}^c(x) = v^{-1}(x)s_0^*\widehat{\omega}^c(x)v(x) + v^{-1}(x)\mathrm{d}v(x)$, and $s^*\widehat{\Gamma}$ is the $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ part of $s^*\widehat{\omega}^c$. Since the Lie algebra of H_7 is $\mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$, every $v(x)$ in the connected component of the identity may be written in the form $v(x) = v_2(x)v_1(x) = \exp(t_2(x)A_2(x))\exp(t_1(x)A_1(x))$ with $A_2(x) \in \mathfrak{g}_2$ and $A_1(x) \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$. It follows that

$$s^*\widehat{\omega}^c(x) = v^{-1}(x)_2 \left\{ v_1^{-1}(x)s_0^*\widehat{\omega}^c(x)v_1(x) + v_1^{-1}(x)\mathrm{d}v_1(x) \right\} v_2(x) + v_2^{-1}(x)\mathrm{d}v_2(x)$$

But the $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ part of the quantity in the curly brackets is the connection $\Gamma = s_0^*\widehat{\Gamma}$ written in the coframe $(s^*\theta^1, s^*\theta^2, s^*\theta^4)$ and $\mathrm{ad}\, v^{-1}(x)$ transforms it into other compatible connection, according to (57). $\qquad \square$

6. Six-dimensional conformal geometry in the split signature

Until now we have not proposed any geometric structure, apart from $\widehat{\omega}^c$, that could be associated with an ODE of generic type. Motivated by S.-S. Chern's construction we would like to build some kind of conformal geometry starting from an arbitrary ODE, which does not necessarily satisfy the Wünschmann condition.

Let us define the 'inverse' of the symmetric tensor field $\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$ of section 4 to be $\widehat{g}_{inv} = \widehat{g}^{ij}X_i \otimes X_j = 2X_1X_3 - (X_2)^2$. We take the $\mathfrak{o}(2, 1)$-part of the connection $\widehat{\omega}^c$

$$\Gamma = \begin{pmatrix} \Omega_3 - \Omega_1 & -\theta^4 & 0 \\ -\Omega_2 & 0 & -\theta^4 \\ 0 & -\Omega_2 & \Omega_1 - \Omega_3 \end{pmatrix},$$

and the Levi-Civita symbol ϵ_{ijk} in three dimensions. Next we define a new bilinear form $\widehat{g}$ on $\mathcal{P}^c$

$$\widehat{g} = \epsilon_{ijk}\, \widehat{g}^{kl}\, \theta^i\, \Gamma^j{}_l.$$

The above method of obtaining $\widehat{\mathbf{g}}$ of the split degenerate signature from $\widehat{g}$ is called the Sparling procedure [**46**]. The new metric reads

$$\widehat{\mathbf{g}} = 2(\Omega_1 - \Omega_3)\theta^2 - 2\Omega_2\theta^1 + 2\theta^4\theta^3 \tag{62}$$

and was given in [**46**] in a slightly different context for the first time. We easily find that its degenerated directions X_8, X_9, X_{10}, and $X_5 + X_7$ form an integrable distribution, so that one can consider the six-dimensional space $\mathcal{M}^6$ of its integral leaves. The degenerated directions X_8, X_9, and X_{10} are isometries

$$L_{X_6}\widehat{\mathbf{g}} = L_{X_8}\widehat{\mathbf{g}} = L_{X_9}\widehat{\mathbf{g}} = 0,$$

whereas the fourth direction, $X_5 + X_7$, is a conformal transformation

$$L_{(X_5+X_7)}\widehat{\mathbf{g}} = \widehat{\mathbf{g}}.$$

This allows us to project $\widehat{\mathbf{g}}$ to the split signature conformal metric $[\mathbf{g}]$ on $\mathcal{M}^6$ without any assumptions about the underlying ODE.

It is interesting to study the normal conformal connection associated to this geometry. Since $\mathcal{P}^c$ is a subbundle of the conformal bundle over M^6, we can calculate the $\mathfrak{o}(4,4)$-valued normal conformal connection (53) at once on $\mathcal{P}^c$. It is as follows.

$$\widehat{\mathbf{w}} = \begin{pmatrix}
\frac{1}{2}\Omega_1 & 0 & 0 & \frac{1}{2}\Omega_2 & -\frac{1}{2}\Omega_4 & -\frac{1}{2}\Omega_6 & 0 & 0 \\
\Omega_1 - \Omega_3 & \Omega_3 - \frac{1}{2}\Omega_1 & \frac{1}{2}\theta_4 & \frac{1}{2}\Omega_2 & 0 & -\mathrm{w}^3{}_5 & \Omega_5 & -\frac{1}{2}\Omega_4 \\
-\Omega_2 & \Omega_2 & \frac{1}{2}\Omega_1 & \mathrm{w}^3{}_4 & \mathrm{w}^3{}_5 & 0 & \Omega_4 & -\frac{1}{2}\Omega_6 \\
\theta_4 & 0 & 0 & \Omega_3 - \frac{1}{2}\Omega_1 & -\Omega_5 & -\Omega_4 & 0 & 0 \\
\theta_2 & 0 & 0 & \theta_3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\Omega_2 & 0 & 0 \\
\theta_1 & 0 & 0 & \frac{1}{2}\theta_2 & -\frac{1}{2}\theta_4 & -\frac{1}{2}\Omega_1 & 0 & 0 \\
\theta_3 & -\theta_3 & -\frac{1}{2}\theta_2 & 0 & -\frac{1}{2}\Omega_2 & -\mathrm{w}^3{}_4 & \frac{1}{2}\Omega_1 - \Omega_3 & \frac{1}{2}\Omega_2 \\
0 & \theta_2 & \theta_1 & \theta_3 & \Omega_1 - \Omega_3 & -\Omega_2 & \theta_4 & -\frac{1}{2}\Omega_1
\end{pmatrix},$$

where

$$\mathrm{w}^3{}_4 = \mathbf{A^c_4}\theta^1 + \mathbf{A^c_2}\theta^2 + \mathbf{A^c_1}\theta^4,$$
$$\mathrm{w}^3{}_5 = \tfrac{1}{2}\Omega_6 + \mathbf{A^c_3}\theta^1 + \mathbf{A^c_5}\theta^2 + \mathbf{A^c_2}\theta^4.$$

It appears that this connection is of very special form. We show that the algebra of its holonomy group is reduced to $\mathfrak{o}(3,2) \oplus \mathbb{R}^5$. Let us write down the connection as

$$\widehat{\mathbf{w}} = (\Omega_1 - \Omega_3)e_1 - \Omega_2 e_2 + \theta^4 e_3 + \theta^2 e_4 + \theta^1 e_5 + \theta^3 e_6 +$$
$$+ \Omega_1 e_7 + \Omega_4 e_8 + \Omega_5 e_9 + \Omega_6 e_{10} + \mathrm{w}^3{}_5 e_{11} + \mathrm{w}^3{}_4 e_{12},$$

where $e_1, \ldots, e_{12}$ are appropriate matrices in $\mathfrak{o}(4,4)$. The space

$$V = < e_1, \ldots, e_{12} > \subset \mathfrak{o}(4,4)$$

is not closed under the commutation relations, hence V is not a Lie subalgebra. However, if we extend V so that it contains three commutators $e_{13} = [e_3, e_{12}]$, $e_{14} = [e_5, e_{10}]$ and $e_{15} = [e_5, e_{12}]$ then $< e_1, \ldots, e_{15} >$ is a Lie algebra, a certain semidirect sum of $\mathfrak{o}(3,2)$ and $\mathbb{R}^5$. Bases of the factors are the following:

$$\mathbb{R}^5 = < e_1 + 2e_7 - 2e_{14}, e_{11}, e_{12}, e_{13}, e_{15} >,$$

$$\mathfrak{o}(3,2) =< e_2 + e_{13}, e_3, e_4, e_5, e_6 - e_{15}, e_7, e_8, e_9, e_{10}, e_{14} > .$$

The matrix of $\widehat{\mathbf{w}}$ can be transformed to the following conjugated representation, which reveals its structure well

$$\begin{pmatrix}
\frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & -\frac{1}{2}\Omega_4 & -\frac{1}{4}\Omega_6 & 2\Omega_2 & -2\mathrm{w}^3{}_4 & 2\mathrm{w}^3{}_5 & 0 \\
\theta^4 & \Omega_3 - \frac{1}{2}\Omega_1 & -\Omega_5 & -\frac{1}{2}\Omega_4 & 4\Omega_3 - 4\Omega_1 & -2\Omega_2 & 0 & -2\mathrm{w}^3{}_5 \\
\theta^2 & \theta^3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 & 4\theta^3 & 0 & 2\Omega_2 & 2\mathrm{w}^3{}_4 \\
2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Omega_1 & 0 & -4\theta^3 & 4\Omega_1 - 4\Omega_3 & -2\Omega_2 \\
0 & 0 & 0 & 0 & \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & \frac{1}{2}\Omega_4 & \frac{1}{4}\Omega_6 \\
0 & 0 & 0 & 0 & \theta^4 & \Omega_3 - \frac{1}{2}\Omega_1 & \Omega_5 & \frac{1}{2}\Omega_4 \\
0 & 0 & 0 & 0 & -\theta^2 & -\theta^3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 \\
0 & 0 & 0 & 0 & -2\theta^1 & -\theta^2 & -\theta^4 & -\frac{1}{2}\Omega_1
\end{pmatrix}.$$

$\widehat{\mathbf{w}}$ has the following block structure in this representation

$$\widehat{\mathbf{w}} = \begin{pmatrix} \widehat{\omega}^c & \widehat{\tau} \\ 0 & -\sigma\widehat{\omega}^c\sigma \end{pmatrix},$$

where

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Surprisingly enough the $\mathfrak{o}(3,2)$-part of $\widehat{\mathbf{w}}$, given by the diagonal blocks, is totally determined by the $\mathfrak{o}(3,2)$ connection $\widehat{\omega}^c$. In particular, this relation holds when $W = 0$ and $\widehat{\omega}^c$ is a conformal connection itself. In this case we have rather an unexpected link between conformal connections in dimensions three and six.

7. Further reduction and geometry on five-dimensional bundle

Theorem 2.1 is a starting point for further reduction of the structural group since one can use the non-constant invariants in (48) to eliminate more variables u_μ. From this point of view third-order ODEs fall into three main classes:

i) $W = 0$, $F_{qqqq} = 0$,
ii) $W = 0$, $F_{qqqq} \neq 0$,
iii) $W \neq 0$.

Class i) contains the equations equivalent to $y''' = 0$ and is fully characterized by the corollary 2.4. Further reduction cannot be done due to lack of non-constant structural functions.

Class ii) is not interesting from the geometric point of view since it does not contain equations with five-dimensional or larger symmetry groups, as it is proved in theorems 4.2 and 4.7 of chapter 4.

Class iii), which leads to a Cartan connection on a five-dimensional bundle, is studied below. Owing to $W \neq 0$ we continue reduction by setting $\mathbf{A_1^c} = 1$, $\mathbf{A_2^c} = 0$, which gives

$$u_1 = \sqrt[3]{W}u_3, \qquad u_5 = \frac{1}{3}\frac{W_q}{\sqrt[3]{W^2}}.$$

At this moment the auxiliary variable u_6 which was introduced by the prolongation becomes irrelevant and may be set equal to zero

$$u_6 = 0.$$

In second step we choose

$$u_2 = \frac{1}{3} Z u_3$$

and finally

$$u_4 = \frac{1}{9} \frac{W_q}{\sqrt[3]{W^2}} M - \frac{1}{3} \sqrt[3]{W} Z_q.$$

The coframe and the underlying bundle $\mathcal{P}^c$ of the theorem 2.1 have been reduced to dimension five according to the following

THEOREM 2.11 (S.-S. Chern). *A third-order ODE $y''' = F(x, y, y', y'')$ satisfying the contact invariant condition $W \neq 0$ and considered modulo contact transformations of variables, uniquely defines a 5-dimensional bundle $\mathcal{P}_5^c$ over $\mathcal{J}^2$ and an invariant coframe $(\theta^1, \dots, \theta^4, \Omega)$ on it. In local coordinates (x, y, p, q, u) this coframe is given by*

$$\theta^1 = \sqrt[3]{W} u \omega^1,$$

$$\theta^2 = \frac{1}{3} Z u \omega^1 + u \omega^2,$$

$$\theta^3 = \frac{u}{\sqrt[3]{W}} \left(K + \frac{1}{18} Z^2 \right) \omega^1 + \frac{u}{3\sqrt[3]{W}} (Z - F_q) \omega^2 + \frac{u}{\sqrt[3]{W}} \omega^3,$$

(63)
$$\theta^4 = \left(\frac{1}{9} \frac{W_q}{\sqrt[3]{W^2}} Z - \frac{1}{3} \sqrt[3]{W} Z_q \right) \omega^1 + \frac{W_q}{3\sqrt[3]{W^2}} \omega^2 + \sqrt[3]{W} \omega^4,$$

$$\Omega = \left(\left(\frac{1}{9} W_q \mathcal{D} Z - \frac{1}{27} W_q Z^2 + \frac{1}{9} W_p Z \right) \frac{1}{W} - \frac{1}{3} Z_p - \frac{1}{9} F_q Z_q \right) \omega^1$$

$$+ \left(\frac{W_p}{3W} - \frac{1}{3} Z_q \right) \omega^2 + \frac{W_q}{3W} \omega^3 + \frac{1}{3} F_q \omega^4 + \frac{du}{u}.$$

where ω^i are defined by the ODE via (6). The exterior derivatives of these forms read

$$d\theta^1 = \Omega \wedge \theta^1 - \theta^2 \wedge \theta^4,$$

$$d\theta^2 = \Omega \wedge \theta^2 + \mathbf{a}^c \, \theta^1 \wedge \theta^4 - \theta^3 \wedge \theta^4,$$

(64)
$$d\theta^3 = \Omega \wedge \theta^3 + \mathbf{b}^c \, \theta^1 \wedge \theta^2 + \mathbf{c}^c \, \theta^1 \wedge \theta^3 - \theta^1 \wedge \theta^4 + \mathbf{e}^c \theta^2 \wedge \theta^3 + \mathbf{a}^c \, \theta^2 \wedge \theta^4,$$

$$d\theta^4 = \mathbf{f}^c \, \theta^1 \wedge \theta^2 + \mathbf{g}^c \, \theta^1 \wedge \theta^3 + \mathbf{h}^c \, \theta^1 \wedge \theta^4 + \mathbf{k}^c \, \theta^2 \wedge \theta^3 - \mathbf{e}^c \, \theta^2 \wedge \theta^4,$$

$$d\Omega = \mathbf{l}^c \, \theta^1 \wedge \theta^2 + (\mathbf{f}^c - \mathbf{a}^c \mathbf{k}^c) \, \theta^1 \wedge \theta^3 + \mathbf{m}^c \, \theta^1 \wedge \theta^4 + \mathbf{g}^c \, \theta^2 \wedge \theta^3 + \mathbf{h}^c \, \theta^2 \wedge \theta^4.$$

The basic functions for (64) (i.e. generating the full set of functions by consecutive taking of coframe derivatives) are $\mathbf{a}^c, \mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c, \mathbf{k}^c$:

$$\mathbf{a}^c = \frac{1}{\sqrt[3]{W^2}} \left(K + \frac{1}{18} Z^2 + \frac{1}{9} Z F_q - \frac{1}{3} \mathcal{D} Z \right),$$

$$\mathbf{b}^c = \frac{1}{3u\sqrt[3]{W^2}} \left(\frac{1}{27} F_{qq} Z^2 + \left(K_q - \frac{1}{3} Z_p - \frac{2}{9} F_q Z_q \right) Z + \right.$$

$$\left. + \left(\frac{1}{3} \mathcal{D} Z - 2K \right) Z_q + Z_y + F_{qq} K - 3 K_p - K_q F_q - F_{qy} + W_q \right),$$

(65)
$$\mathbf{e}^c = \frac{1}{u} \left(\frac{1}{3} F_{qq} + \frac{1}{W} \left(\frac{2}{9} W_q Z - \frac{2}{3} W_p - \frac{2}{9} W_q F_q \right) \right),$$

$$\mathbf{h}^c = \frac{1}{3u\sqrt[3]{W}}\left(\left(\frac{1}{9}W_q Z^2 - \frac{1}{3}W_p Z + W_y - \frac{1}{3}W_q \mathcal{D}Z\right)\frac{1}{W} + \right.$$
$$\left. + \mathcal{D}Z_q + \frac{1}{3}F_q Z_q\right),$$

$$\mathbf{k}^c = \frac{1}{u^2\sqrt[3]{W}}\left(\frac{2W_q^2}{9W} - \frac{W_{qq}}{3}\right).$$

Our next aim is to obtain a Cartan connection. First of all we study the most symmetric case to find the Lie algebra of a connection. We assume that all the functions $\mathbf{a}^c, \ldots, \mathbf{m}^c$ are constant. Having applied the exterior derivative to (64) we get that $\mathbf{b}^c, \ldots, \mathbf{m}^c$ are equal to zero and $\mathbf{a}^c$ is an arbitrary real constant μ. In this case the equations (64) become the Maurer-Cartan equations for the algebra $\mathbb{R}^2 \oplus_\mu \mathbb{R}^3$. Straightforward calculations show that this case corresponds to a general linear equation with constant coefficients.

COROLLARY 2.12. *A third-order ODE is contact equivalent to*
$$y''' = -2\mu y' + y,$$
where μ is an arbitrary constant if and only if it satisfies

(66)

    1) $W \neq 0$,

    2) $\dfrac{1}{\sqrt[3]{W^2}}\left(K + \dfrac{1}{18}Z^2 + \dfrac{1}{9}ZF_q - \dfrac{1}{3}\mathcal{D}Z\right) = \mu$

    3) $2W_q^2 - 3W_{qq}W = 0$.

Such an equation has the five-dimensional algebra $\mathbb{R}^2 \oplus_\mu \mathbb{R}^3$ of infinitesimal contact symmetries. The equations with different constants μ_1 and μ_2 are non-equivalent.

PROOF. Assume that $\mathbf{a}^c = \mu$, $\mathbf{k}^c = 0$. It follows from $\mathrm{d}^2\theta^i = 0$ and $\mathrm{d}^2\Omega = 0$, that this assumption makes other functions in (64) vanish. Put $y''' = -2\mu y' + y$ into the formulae of theorem 2.11 and check that it satisfies $\mathbf{a}^c = \mu$, $\mathbf{k}^c = 0$. Every equation satisfying $\mathbf{a}^c = \mu$, $\mathbf{k}^c = 0$ is contact equivalent to it by virtue of theorem 1.3 adapted to this situation. $\square$

Now we immediately find a family of Cartan connections

THEOREM 2.13. *An ODE which satisfies the condition*
$$\frac{1}{\sqrt[3]{W^2}}\left(K + \frac{1}{18}Z^2 + \frac{1}{9}ZF_q - \frac{1}{3}\mathcal{D}Z\right) = \lambda$$
has the solution space equipped with the following $\mathbb{R}^2$-valued linear torsion-free connection
$$\widehat{\omega}_\lambda = \begin{pmatrix} -\Omega & -\theta^4 & 0 \\ \lambda\theta^4 & -\Omega & -\theta^4 \\ -\theta^4 & \lambda\theta^4 & -\Omega \end{pmatrix}.$$
Its curvature reads
$$\begin{pmatrix} R^1{}_1 & R^1{}_2 & 0 \\ -\lambda R^1{}_2 & R^1{}_1 & R^1{}_2 \\ R^1{}_2 & -\lambda R^1{}_2 & R^1{}_1 \end{pmatrix}$$
with
$$R^1{}_1 = (\lambda\mathbf{g}^c + \mathbf{k}^c)\theta^1 \wedge \theta^2 + (\lambda\mathbf{k}^c - \mathbf{g}^c)\theta^1 \wedge \theta^3 - \mathbf{g}^c\theta^2 \wedge \theta^3,$$
$$R^1{}_2 = -\mathbf{f}^c\theta^1 \wedge \theta^2 - \mathbf{g}^c\theta^1 \wedge \theta^3 - \mathbf{k}^c\theta^2 \wedge \theta^3.$$

The connection is flat if and only if the related ODE is contact equivalent to $y''' = -2\lambda y' + y$

PROOF. The condition $\mathbf{a}^c = \lambda$ together with its differential consequences $\mathbf{b}^c = \mathbf{c}^c = \mathbf{e}^c = \mathbf{h}^c = \mathbf{m}^c = 0$ and $\mathbf{l}^c = -\mathbf{k}^c - \lambda \mathbf{g}^c$ is the necessary and sufficient condition for the curvature of $\widehat{\omega}_\lambda$ to be horizontal. □

It follows that every ODE as above has its space of solution equipped with a geometric structure consisting of

  i) Reduction of $\mathfrak{gl}(3, \mathbb{R})$ to $\mathbb{R}^2$ represented by

$$\begin{pmatrix} a_1 & a_2 & 0 \\ -\lambda a_2 & a_1 & a_2 \\ a_2 & -\lambda a_2 & a_1 \end{pmatrix}.$$

  ii) A linear torsion-free connection Γ taking values in this $\mathbb{R}^2$.

The structure is an example of a geometry with special holonomy. The algebra $\mathbb{R}^2$ is spanned by the unit matrix and

$$\mathrm{m}(\lambda) = \begin{pmatrix} 0 & 1 & 0 \\ -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \end{pmatrix},$$

whose action on $\mathcal{S}$ is more complicated. Its eigenvalue equation

$$\det(u\mathbf{1} - \mathrm{m}(\lambda)) = u^3 + 2\lambda u - 1$$

is the characteristic polynomial of the linear ODE $y''' = -2\lambda y' + y$. If $\lambda < \frac{3}{4}\sqrt[3]{2}$ the polynomial has three distinct roots and m is a generator of non-isotropic dilatations acting along the eigenspaces. If $\lambda = \frac{3}{4}\sqrt[3]{2}$ the characteristic polynomial has two roots, one of them double, for other λs there is one eigenvalue. The action is diagonalizable only in the case of three distinct eigenvalues.

CHAPTER 3

Geometries of ODEs considered modulo point and fibre-preserving transformations of variables

1. Point case: Cartan connection on seven-dimensional bundle

Following the scheme of reduction given in chapter 2 we construct Cartan connection for ODEs modulo point transformations.

THEOREM 3.1 (E. Cartan). *The point invariant information about* $y''' = F(x, y, y', y'')$ *is given by the following data*

i) *The principal fibre bundle* $H_3 \to \mathcal{P}^p \to \mathcal{J}^2$, *where* $\dim \mathcal{P}^p = 7$, *and* H_3 *is the three-dimensional group*

$$
(67) \qquad H_3 = \begin{pmatrix} \sqrt{u_1}, & \frac{1}{2}\frac{u_2}{\sqrt{u_1}}, & 0 & 0 \\ 0 & \frac{u_3}{\sqrt{u_1}}, & 0 & 0 \\ 0 & 0 & \frac{\sqrt{u_1}}{u_3}, & -\frac{1}{2}\frac{u_2}{\sqrt{u_1}\,u_3} \\ 0 & 0 & 0 & \frac{1}{\sqrt{u_1}} \end{pmatrix}.
$$

ii) *The coframe* $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3)$, *which defines the* $\mathfrak{co}(2,1) \oplus \mathbb{R}^3$-*valued Cartan normal connection* $\widehat{\omega}^p$ *on* $\mathcal{P}_7^p$ *by*

$$
(68) \qquad \widehat{\omega}^p = \begin{pmatrix} \frac{1}{2}\Omega_1 & \frac{1}{2}\Omega_2 & 0 & 0 \\ \theta^4 & \Omega_3 - \frac{1}{2}\Omega_1 & 0 & 0 \\ \theta^2 & \theta^3 & \frac{1}{2}\Omega_1 - \Omega_3 & -\frac{1}{2}\Omega_2 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Omega_1 \end{pmatrix}.
$$

Let $(x, y, p, q, u_1, u_2, u_3) = (x^i, u_\mu)$ be a locally trivializing coordinate system in $\mathcal{P}^p$. Then the value of $\widehat{\omega}^p$ at the point (x^i, u_μ) in $\mathcal{P}^p$ is given by

$$
\widehat{\omega}^p(x^i, u_\mu) = u^{-1}\,\omega^p\,u + u^{-1}\mathrm{d}u
$$

where u *denotes the matrix* (67) *and*

$$
\omega^p = \begin{pmatrix} \frac{1}{2}\Omega_1^0 & \frac{1}{2}\Omega_2^0 & 0 & 0 \\ \widetilde{\omega}^4 & \Omega_3^0 - \frac{1}{2}\Omega_1^0 & 0 & 0 \\ \omega^2 & \widetilde{\omega}^3 & \frac{1}{2}\Omega_1^0 - \Omega_3^0 & -\frac{1}{2}\Omega_2^0 \\ 2\omega^1 & \omega^2 & -\widetilde{\omega}^4 & -\frac{1}{2}\Omega_1^0 \end{pmatrix}
$$

43

is the connection $\widehat{\omega}^p$ calculated at the point $(x^i, u_1 = 1, u_2 = 0, u_3 = 1)$. The forms $\omega^1, \omega^2, \widetilde{\omega}^3, \omega^4$ read

$$\omega^1 = dy - pdx,$$
$$\omega^2 = dp - qdx,$$
$$\widetilde{\omega}^3 = dq - Fdx - \tfrac{1}{3}F_q(dp - qdx) + K(dy - pdx),$$
$$\widetilde{\omega}^4 = dx + \tfrac{1}{6}F_{qq}(dy - pdx).$$

The forms $\Omega_1^0, \ldots, \Omega_6^0$ read

$$\Omega_1^0 = -\left(3K_q + \tfrac{2}{9}F_{qq}F_q + \tfrac{2}{3}F_{qp}\right)\omega^1 + \tfrac{1}{6}F_{qq}\omega^2,$$
$$\Omega_2^0 = \left(L + \tfrac{1}{6}F_{qq}K\right)\omega^1 - \left(2K_q + \tfrac{1}{9}F_{qq}F_q + \tfrac{1}{3}F_{qp}\right)\omega^2 + \tfrac{1}{6}F_{qq}\widetilde{\omega}^3 - K\widetilde{\omega}^4,$$
$$\Omega_3^0 = -\left(2K_q + \tfrac{1}{6}F_{qq}F_q + \tfrac{1}{3}F_{qp}\right)\omega^1 + \tfrac{1}{3}F_{qq}\omega^2 + \tfrac{1}{3}F_q\widetilde{\omega}^4.$$

PROOF. We begin with the G_p-structure on $\mathcal{J}^2$ of Introduction, which encodes an ODE up to point transformations. In the usual locally trivializing coordinate system $(x, y, p, q, u_1, \ldots, u_8)$ on $G_p \times \mathcal{J}^2$ the fundamental form θ^i is given by

$$\theta^1 = u_1\omega^1,$$
$$\theta^2 = u_2\omega^2 + u_3\omega^3,$$
$$\theta^3 = u_4\omega^1 + u_5\omega^2 + u_6\omega^3,$$
$$\theta^4 = u_8\omega^1 + u_7\omega^4.$$

We repeat the procedure of section 2 of chapter 2. We choose a connection by the minimal torsion requirement and then reduce $G_p \times \mathcal{J}^2$ using the constant torsion property. We differentiate θ^i and gather the $\theta^j \wedge \theta^k$ terms into

(69)
$$d\theta^1 = \Omega_1 \wedge \theta^1 + \frac{u_1}{u_3 u_7}\theta^4 \wedge \theta^2,$$
$$d\theta^2 = \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \frac{u_3}{u_6 u_7}\theta^4 \wedge \theta^3,$$
$$d\theta^3 = \Omega_4 \wedge \theta^1 + \Omega_5 \wedge \theta^2 + \Omega_6 \wedge \theta^3,$$
$$d\theta^4 = \Omega_8 \wedge \theta^1 + \Omega_9 \wedge \theta^2 + \Omega_7 \wedge \theta^4$$

with the auxiliary connection forms Ω_μ containing the differentials of u_μ and terms proportional to θ^i. Then we reduce $G_p \times \mathcal{J}^2$ by

$$u_6 = \frac{u_3^2}{u_1}, \qquad u_7 = \frac{u_1}{u_3}.$$

Subsequently, we get formulae identical to (41), (42):

$$u_5 = \frac{u_3}{u_1}\left(u_2 - \frac{1}{3}u_3 F_q\right),$$
$$u_4 = \frac{u_3^2}{u_1}K + \frac{u_2^2}{2u_1}$$

and also

$$u_8 = \frac{u_1}{6u_6}F_{qq}.$$

After these substitutions the structural equations for θ^i are the following

$$d\theta^1 = \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2,$$
$$d\theta^2 = \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3,$$
$$d\theta^3 = \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \mathbf{A_1^p}\theta^4 \wedge \theta^1,$$
$$d\theta^4 = (\Omega_1 - \Omega_3) \wedge \theta^4 + \mathbf{B_1^p}\theta^2 \wedge \theta^1 + \mathbf{B_2^p}\theta^3 \wedge \theta^1,$$

with some functions $\mathbf{A}_1^p, \mathbf{B}_1^p, \mathbf{B}_2^p$. But now, in contrast to the contact case, the forms $\Omega_1, \Omega_2, \Omega_3$ are defined by the above equations without any ambiguity, thus there is no need to prolong and we have the rigid coframe on the seven-dimensional bundle $\mathcal{P}^p \to \mathcal{J}^2$. $\qquad\square$

1.1. Point versus contact objects. Comparing theorem 3.1 to theorem 2.1 of the contact case, we see that the contact and point objects are related as follows. The point bundle $\mathcal{P}^p$ is a subbundle of $\mathcal{P}^c$, the embedding $\sigma \colon \mathcal{P}^p \to \mathcal{P}^c$ given by

$$(70) \qquad u_4 = \frac{u_1}{6u_3} F_{qq}, \qquad u_5 = 0, \qquad u_6 = 0.$$

In this paragraph we denote the point coframe of theorem 3.1 by $(\overset{p}{\theta^1}, \ldots, \overset{p}{\theta^3})$ in order to distinguish it from the contact coframe now denoted by $(\overset{c}{\theta^1}, \ldots, \overset{c}{\Omega_7})$. The point coframe on $\mathcal{P}^p$ can be constructed from the contact one by the following formula.

$$(71) \qquad \begin{aligned}
\overset{p}{\theta^i} &= \sigma^* \overset{c}{\theta^i}, \qquad i = 1, 2, 3, 4, \\
\overset{p}{\Omega_1} &= \sigma^* \overset{c}{\Omega_1} - 2 f_1 \, \sigma^* \overset{c}{\theta^1}, \\
\overset{p}{\Omega_2} &= \sigma^* \overset{c}{\Omega_2} - f_2 \, \sigma^* \overset{c}{\theta^1} - f_1 \, \sigma^* \overset{c}{\theta^2}, \\
\overset{p}{\Omega_3} &= \sigma^* \overset{c}{\Omega_3} - f_1 \, \sigma^* \overset{c}{\theta^1},
\end{aligned}$$

where the functions f_1 and f_2 are

$$\begin{aligned}
f_1 &= \frac{1}{u_1}(K_q + \tfrac{1}{9} F_{qq} F_q + \tfrac{1}{3} F_{qp}) + \frac{u_2}{12\, u_1 u_3} F_{qq}, \\
f_2 &= \sigma^* \mathbf{A}_2^c = \frac{u_3}{3u_1^2} W_q.
\end{aligned}$$

The mapping σ together with the functions f_1, f_2 is a new piece of structure that allows us to move from the more coarse contact coframe to the point coframe. It is obvious that F_{qq} and f_1 appearing in the above formulae are not contact invariants. What is more, they are not point invariants either, for the property $F_{qq} = 0$ or $f_1 = 0$ is not preserved under point transformations. At the level of geometric objects the passage from the contact case to the point case is given by the difference

$$\sigma^* \widehat{\omega}^c - \widehat{\omega}^p.$$

One can also proceed in the inverse direction and construct the contact bundle and coframe starting from the point case. In this approach the bundle $\mathcal{P}^c$ is the extension $\mathcal{P}^c = \mathcal{P}^p \times_{H_3} H_6$ of $\mathcal{P}^p$ and in order to get the contact coframe we must define forms $\overset{c}{\theta^1}, \ldots, \overset{c}{\Omega_6}$ on $\mathcal{P}^p$ in terms of the point coframe, then define a matrix one-form $\widehat{\omega}^c$ on P_7^p and finally lift this $\widehat{\omega}^c$ to the Cartan connection on $\mathcal{P}^c$. Since the formulae for $\overset{c}{\theta^1}, \ldots, \overset{c}{\Omega_3}$ are similar to (71) and the formulae for $\overset{c}{\Omega_4}, \overset{c}{\Omega_5}, \overset{c}{\Omega_6}$ are complicated we omit them.

1.2. Curvature. Further analysis of the coframe of theorem 3.1 is very similar to what we have done in chapter 2. The curvature of the connection is given by the following exterior differentials of the coframe, cf (10)

$$\begin{aligned}
\mathrm{d}\theta^1 &= \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2, \\
\mathrm{d}\theta^2 &= \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3, \\
\mathrm{d}\theta^3 &= \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \mathbf{A}_1^p \theta^4 \wedge \theta^1,
\end{aligned}$$

$$\mathrm{d}\theta^4 = (\Omega_1 - \Omega_3) \wedge \theta^4 + \mathbf{B_1^p}\theta^2 \wedge \theta^1 + \mathbf{B_2^p}\theta^3 \wedge \theta^1,$$

$$(72) \qquad \mathrm{d}\Omega_1 = -\Omega_2 \wedge \theta^4 + (\mathbf{D_1^p} + 3\mathbf{B_3^p})\theta^1 \wedge \theta^2 + (3\mathbf{B_4^p} - 2\mathbf{B_1^p})\theta^1 \wedge \theta^3$$

$$+ (2\mathbf{C_1^p} - \mathbf{A_2^p})\theta^1 \wedge \theta^4 - \mathbf{B_2^p}\theta^2 \wedge \theta^3,$$

$$\mathrm{d}\Omega_2 = (\Omega_3 - \Omega_1) \wedge \Omega_2 + \mathbf{D_2^p}\theta^1 \wedge \theta^2 + (\mathbf{D_1^p} + \mathbf{B_3^p})\theta^1 \wedge \theta^3 + \mathbf{A_3^p}\theta^1 \wedge \theta^4$$

$$+ (2\mathbf{B_4^p} - \mathbf{B_1^p})\theta^2 \wedge \theta^3 + \mathbf{C_1^p}\theta^2 \wedge \theta^4,$$

$$\mathrm{d}\Omega_3 = (\mathbf{D_1^p} + 2\mathbf{B_3^p})\theta^1 \wedge \theta^2 + 2(\mathbf{B_4^p} - \mathbf{B_1^p})\theta^1 \wedge \theta^3 + \mathbf{C_1^p}\theta^1 \wedge \theta^4 - 2\mathbf{B_2^p}\theta^2 \wedge \theta^3,$$

where $\mathbf{A_1^p}, \mathbf{A_2^p}, \mathbf{A_3^p}, \mathbf{B_1^p}, \mathbf{B_2^p}, \mathbf{B_3^p}, \mathbf{B_4^p}, \mathbf{C_1^p}, \mathbf{D_1^p}, \mathbf{D_2^p}$ are functions on $\mathcal{P}^p$. All these functions express by the coframe derivatives of $\mathbf{A_1^p}, \mathbf{B_1^p}, \mathbf{C_1^p}$, which therefore constitute the set of basic relative invariants for this problem and read

$$\mathbf{A_1^p} = \frac{u_3^3}{u_1^3}W,$$

$$\mathbf{B_1^p} = \frac{1}{u_3^2}\left(\frac{1}{18}F_{qqq}F_q + \frac{1}{36}F_{qq}^2 + \frac{1}{6}F_{qqp}\right) - \frac{u_2}{6u_3^3}F_{qqq},$$

$$\mathbf{C_1^p} = \frac{u_3}{u_1^2}\left(2F_{qq}K + \frac{2}{3}F_qF_{qp} - 2F_{qy} + F_{pp} + 2W_q\right).$$

$\mathbf{B_2^p}$ and $\mathbf{B_4^p}$ are also vital:

$$\mathbf{B_2^p} = \frac{u_1}{6u_3^3}F_{qqq},$$

$$\mathbf{B_4^p} = \frac{1}{u_3^2}\left(K_{qq} + \frac{1}{9}F_{qqq}F_q + \frac{1}{3}F_{qqp} + \frac{1}{12}F_{qq}^2\right).$$

COROLLARY 3.2. *For a third-order ODE $y''' = F(x, y, y', y'')$ the following conditions are equivalent.*

   i) *The ODE is point equivalent to $y''' = 0$.*
   ii) *It satisfies the conditions $W = 0$, $F_{qqq} = 0$, $F_{qq}^2 + 6F_{qqp} = 0$ and*

$$2F_{qq}K + \frac{2}{3}F_qF_{qp} - 2F_{qy} + F_{pp} = 0.$$

   iii) *It has the $\mathfrak{co}(2,1) \oplus_{\cdot} \mathbb{R}^3$ algebra of infinitesimal point symmetries.*

The manifold $\mathcal{P}^p$, like $\mathcal{P}^c$, is equipped with the threefold structure of principal bundle over $\mathcal{J}^2$, $\mathcal{J}^1$ and $\mathcal{S}$. Let $(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ be the frame dual to $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3)$.

   - $\mathcal{P}^p$ is the bundle $H_3 \to \mathcal{P}^p \to \mathcal{J}^2$ with the fundamental fields X_5, X_6, X_7.
   - It is the bundle $CO(2,1) \to \mathcal{P}^p \to \mathcal{S}$ with the fundamental fields X_4, X_5, X_6, X_7
   - It is also the bundle $H_4 \to \mathcal{P}^p \to \mathcal{J}^1$ with the fundamental fields X_3, X_5, X_6, X_7.

2. Einstein-Weyl geometry on space of solutions

We have already described the construction of the Einstein-Weyl geometry on the solution space in Introduction. Here we write it down in a more systematic manner.

2.1. Weyl geometry. A Weyl geometry on $\mathcal{M}^n$ is a pair (g, ϕ) such that g is a metric of signature (k, l), $k + l = n$ and ϕ is a one-form and they are given modulo the following transformations

$$\phi \to \phi + \mathrm{d}\lambda, \qquad\qquad g \to e^{2\lambda}g.$$

In particular $[g]$ is a conformal geometry. For any Weyl geometry there exists the Weyl connection; it is the unique torsion-free connection such that

$$\nabla g = 2\phi \otimes g.$$

The Weyl connection takes values in the algebra $\mathfrak{co}(k, l)$ of $[g]$. Let (ω^μ) be an orthonormal coframe for some g of $[g]$; $g = g_{\mu\nu}\omega^\mu \otimes \omega^\nu$ with all the coefficients $g_{\mu\nu}$ being constant. The Weyl connection one-forms Γ^μ_ν are uniquely defined by the relations

$$\mathrm{d}\omega^\mu + \Gamma^\mu_\nu \wedge \omega^\nu = 0,$$
$$\Gamma_{(\mu\nu)} = -g_{\mu\nu}\,\phi, \quad \text{where} \quad \Gamma_{ij} = g_{jk}\Gamma^k_{\ j}.$$

The curvature tensor $R^\mu_{\ \nu\rho\sigma}$, the Ricci tensor $\mathrm{Ric}_{\mu\nu}$ and the Ricci scalar R of a Weyl connection are defined as follows

$$R^\mu_\nu = \mathrm{d}\Gamma^\mu_\nu + \Gamma^\mu_\rho \wedge \Gamma^\rho_\nu = R^\mu_{\ \nu\rho\sigma}\omega^\rho \wedge \omega^\sigma$$
$$\mathrm{Ric}_{\mu\nu} = R^\rho_{\ \mu\rho\nu},$$
$$\mathrm{R} = \mathrm{Ric}_{\mu\nu}g^{\mu\nu}.$$

The Ricci scalar has the conformal weight -2, that is it transforms as $\mathrm{R} \to e^{-2\lambda}\mathrm{R}$ when $g \to e^{2\lambda}g$. Apart from these objects there is another one, which has no counterpart in the Riemannian geometry, the Maxwell two-form

$$F = \mathrm{d}\phi.$$

The Maxwell two-form $F_{\mu\nu}$ is proportional to the antisymmetric part $\mathrm{Ric}_{[\mu\nu]}$ of the Ricci tensor.

Einstein-Weyl structures are, by definition, those Weyl structures for which the symmetric trace-free part of the Ricci tensor vanishes

$$\mathrm{Ric}_{(\mu\nu)} - \tfrac{1}{n}R \cdot g_{\mu\nu} = 0.$$

2.2. Einstein-Weyl structures from ODEs. In this paragraph we follow P. Nurowski [46, 48]. One sees from the system (72) that the pair $(\widehat{g}, \Omega_3)$, where

$$\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$$

is Lie transported along the fibres of $\mathcal{P}^p \to \mathcal{S}$ in the following way

$$L_{X_4}g = \mathbf{A}^p_1(\theta^1)^2, \qquad L_{X_5}\widehat{g} = 0, \qquad L_{X_6}\widehat{g} = 0, \qquad L_{X_7}\widehat{g} = 2\widehat{g},$$

and

$$L_{X_3}\Omega_3 = \tfrac{1}{2}\mathbf{C}^p_1\theta^1,$$
$$L_{X_j}\Omega_3 = 0, \qquad \text{for} \qquad j = 5, 6, 7.$$

Due to these properties $(\widehat{g}, \Omega_3)$ descends along $\mathcal{P}^p \to \mathcal{S}$ to the Weyl structure (g, ϕ) on the solution space $\mathcal{S}$ on condition that

$$W = 0$$

and

(73) $$\left(\tfrac{1}{3}\mathcal{D}F_q - \tfrac{2}{9}F_q^2 - F_p\right)F_{qq} + \tfrac{2}{3}F_qF_{qp} - 2F_{qy} + F_{pp} = 0.$$

These conditions are equivalent to E. Cartan's original conditions (17), (18). Our conditions are even simpler, because the quantity (73) is of first order in $\mathcal{D}$.

The conformal metric of the Weyl structure (g, ϕ) coincides with the conformal metric of the contact case and is represented by

$$g = 2\omega^1\widetilde{\omega}^3 - (\omega^2)^2 =$$
$$= 2(\mathrm{d}y - p\mathrm{d}x)(\mathrm{d}q - \tfrac{1}{3}F_q\mathrm{d}p + K\mathrm{d}y + (\tfrac{1}{3}qF_q - pK - F)\mathrm{d}x) - (\mathrm{d}p - q\mathrm{d}x)^2,$$

while the Weyl potential is given by

$$\phi = -(2K_q + \tfrac{1}{9}F_{qq}F_q + \tfrac{1}{3}F_{qp})(\mathrm{d}y - p\mathrm{d}x) + \tfrac{1}{3}F_{qq}(\mathrm{d}p - q\mathrm{d}x) + \tfrac{1}{3}F_q\mathrm{d}x$$

The Weyl connection for this geometry, lifted to $CO(2,1) \to \mathcal{P}^p \to \mathcal{S}$, now the bundle of orthonormal frames, reads

$$\Gamma = \begin{pmatrix} -\Omega_1 & -\theta^4 & 0 \\ -\Omega_2 & -\Omega_3 & -\theta^4 \\ 0 & -\Omega_2 & \Omega_1 - 2\Omega_3 \end{pmatrix}.$$

The curvature is as follows

(74)
$$(R^\mu{}_\nu) = \begin{pmatrix} R^1{}_1 - F & R^1{}_2 & 0 \\ R^2{}_1 & -F & R^1{}_2 \\ 0 & R^2{}_1 & -R^1{}_1 - F \end{pmatrix}$$

with

$$F = \mathrm{d}\Omega_3 = 2\mathbf{B_3^p}\theta^1 \wedge \theta^2 + (2\mathbf{B_4^p} - 2\mathbf{B_1^p})\theta^1 \wedge \theta^3 - 2\mathbf{B_2^p}\theta^2 \wedge \theta^3,$$
$$R^1{}_1 = -\mathbf{B_3^p}\theta^1 \wedge \theta^2 - \mathbf{B_4^p}\theta^1 \wedge \theta^3 - \mathbf{B_2^p}\theta^2 \wedge \theta^3,$$
$$R^1{}_2 = \mathbf{B_1^p}\theta^1 \wedge \theta^2 + \mathbf{B_2^p}\theta^1 \wedge \theta^3,$$
$$R^2{}_1 = -\mathbf{B_3^p}\theta^1 \wedge \theta^3 + (\mathbf{B_1^p} - 2\mathbf{B_4^p})\theta^2 \wedge \theta^3.$$

The Ricci tensor reads

$$\mathrm{Ric} = \begin{pmatrix} 0 & -3\mathbf{B_3^p} & 3\mathbf{B_1^p} - 5\mathbf{B_4^p} \\ 3\mathbf{B_3^p} & 2\mathbf{B_4^p} & 3\mathbf{B_2^p} \\ -3\mathbf{B_1^p} + \mathbf{B_1^p} & -3\mathbf{B_2^p} & 0 \end{pmatrix}$$

and satisfies the Einstein-Weyl equations

$$\mathrm{Ric}_{(ij)} = \tfrac{1}{3}R \cdot g_{ij}$$

with the Ricci scalar $R = 6\mathbf{B_4^p}$. The components of the curvature in the orthogonal coframe given by $u_1 = 1, u_2 = 0, u_3 = 1$ are given by

$$\mathbf{B_1^p} = \tfrac{1}{18}F_{qqq}F_q + \tfrac{1}{6}F_{qqp} + \tfrac{1}{36}F_{qq}^2,$$
$$\mathbf{B_2^p} = \tfrac{1}{6}F_{qqq},$$
$$\mathbf{B_3^p} = \tfrac{1}{6}F_{qqy} - \tfrac{1}{3}F_{qq}K_q - \tfrac{1}{6}F_{qqq}K - \tfrac{1}{18}F_{qq}F_{qp} - \tfrac{1}{54}F_{qq}^2F_q - L_q,$$
$$\mathbf{B_4^p} = K_{qq} + \frac{1}{9}F_{qqq}F_q + \tfrac{1}{3}F_{qqp} + \frac{1}{12}F_{qq}^2.$$

3. Geometry on first jet space

In section 5 of chapter 2 we described how certain ODEs modulo contact transformations generate contact projective geometry on J^1. The fact that point transformations form a subclass within contact transformations suggests that ODEs modulo point transformations define some refined version of contact projective geometry. Indeed, the only object that is preserved by point transformations but is not preserved by contact transformations is the projection $\mathcal{J}^1 \to xy\ plane$, whose fibres are generated by ∂_p. This motivate us to propose the following

DEFINITION 3.3. A point projective structure on $\mathcal{J}^1$ is a contact projective structure, such that integral curves of the field ∂_p are geodesics of the contact projective structure.

We immediately get

LEMMA 3.4. *The field ∂_p is geodesic for the contact projective geometry generated by an ODE provided that the ODEs satisfies*

$$F_{qqq} = 0.$$

PROOF. In the notation of section 5.1 of chapter 2 we have $\partial_p = e_2$ and, from proposition 2.6, $\Gamma^1_{22} = 0$. Thus $\nabla_2 e_2 = \lambda e_2$ iff $\Gamma^3_{22} = 0$, which is equivalent to $F_{qqq} = 0$ by means of lemma 2.8. $\hfill\square$

However, the condition $F_{qqq} = 0$ is not sufficient for the form (68) to be a Cartan connection for the point projective structure and we show that there does not exist any simple way to construct a Cartan connection on $\mathcal{P}^p \to J^1$. The algebra $\mathfrak{co}(2,1) \oplus_. \mathbb{R}^3 \subset \mathfrak{o}(3,2)$ inherits the following grading from (61)

$$\mathfrak{co}(2,1) \oplus_. \mathbb{R}^3 = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

but it is not semisimple, so the Tanaka method cannot be implemented. Moreover, the broadest generalization of this method – the Morimoto nilpotent geometry handling non-semisimple groups also fails in this case. This is because the Morimoto approach requires the algebra $\mathfrak{g}$ to be equal to the prolongation of its non-positive part algebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$. The notion of prolongation as well as algorithmic procedure for calculating it has been introduced by N. Tanaka [54]. In our case the prolongation of $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ is larger then $\mathfrak{co}(2,1) \oplus_. \mathbb{R}^3$ and equals precisely $\mathfrak{o}(3,2)$, which yields a contact projective structure. Thus only the contact case is solved by the methods of the nilpotent geometry.

Lacking general theory we must search a Cartan connection in a more direct way. Consider than an ODE satisfying $F_{qqq} = 0$. It follows that $\mathbf{B}^p_2 = 0$ and $\mathbf{B}^p_4 = \mathbf{B}^p_1$ in equations (72). We seek four one-forms

$$
\begin{aligned}
\Xi_1 &= \Omega_1 + a_1\theta^1 + a_2\theta^2 + a_3\theta^4, \\
\Xi_2 &= \Omega_2 + b_1\theta^1 + b_2\theta^2 + b_3\theta^4, \\
\Xi_3 &= \Omega_3 + c_1\theta^1 + c_2\theta^2 + c_3\theta^4, \\
\Xi_4 &= \theta^3 + f_1\theta^1 + f_2\theta^2 + f_3\theta^4,
\end{aligned}
$$

with yet unknown functions $a_1, \ldots, f_3$, such that the matrix

$$
\begin{pmatrix}
\frac{1}{2}\Xi_1 & \frac{1}{2}\Xi_2 & 0 & 0 \\[2mm]
\theta^4 & \Xi_3 - \frac{1}{2}\Xi_1 & 0 & 0 \\[2mm]
\theta^2 & \Xi_4 & \frac{1}{2}\Xi_1 - \Xi_3 & -\frac{1}{2}\Xi_2 \\[2mm]
2\theta^1 & \theta^2 & -\theta^4 & -\frac{1}{2}\Xi_1
\end{pmatrix}
$$

is a Cartan connection on $\mathcal{P}^7 \to \mathcal{J}^1$. Calculating the curvature for this connection we obtain that the horizontality conditions yield

$$\mathrm{d}a_1 = X_1(a_1)\theta^1 + X_2(a_1)\theta^2 + \mathbf{B}^p_1\theta^3 + X_4(a_1)\theta^4 - a_1\Omega_1 - a_2\Omega_2.$$

Unfortunately, none combination of the structural functions $\mathbf{A}^p_1, \ldots, \mathbf{D}^p_2$ and their coframe derivatives of first order satisfies this condition. Therefore we are not able to build a Cartan connection for an arbitrary point projective structure. Moreover, since $\mathbf{B}^p_1$ is a basic point invariant together with $\mathbf{A}^p_1$ and $\mathbf{C}^p_1$, it seems to us unlikely that among the coframe derivatives of $\mathbf{A}^p_1, \ldots, \mathbf{D}^p_2$ of any order there exists a function satisfying the above condition. If such a function existed it would mean that among the derivatives there is a more fundamental function from which $\mathbf{B}^p_1$ can be obtained by differentiation.

Of course, we do have a Cartan connection for the point projective geometry provided that in addition to $F_{qqq} = 0$ the conditions $\mathbf{B}^p_1 = \mathbf{D}^p_1 = 0$ are imposed. However, the geometric interpretation of these conditions is unclear.

4. Six-dimensional Weyl geometry in the split signature

The construction of the six-dimensional split signature conformal geometry given in chapter 2 has also its Weyl counterpart in the point case. A similar construction was done by P. Nurowski, [**46**] but he considered the conformal metric, not the Weyl geometry. Here, apart from the tensor

$$\widehat{\mathbf{g}} = 2(\Omega_1 - \Omega_3)\theta^2 - 2\Omega_3\theta^1 + 2\theta^4\theta^3$$

of (62), we also have the one-form

$$\frac{1}{2}\Omega_3.$$

The Lie derivatives along the degenerate direction $X_5 + X_7$ of $\widehat{\mathbf{g}}$ are

$$L_{(X_5+X_7)}\widehat{\mathbf{g}} = \widehat{\mathbf{g}} \qquad \text{and} \qquad L_{(X_5+X_7)}\Omega_3 = 0.$$

In this manner the pair $(\widehat{\mathbf{g}}, \frac{1}{2}\Omega_3)$ generates the six-dimensional split-signature Weyl geometry $(\mathbf{g}, \phi)$ on the six-manifold $\mathcal{M}^6$ being the space of integral curves of $X_5 + X_7$. The associated Weyl connection is $\mathfrak{co}(3,3) \oplus \mathbb{R}^6$-valued and has the following form.

$$\Gamma^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \frac{1}{2}\theta^4 & \Gamma^1{}_3 & 0 & \Gamma^1{}_5 & \Gamma^1{}_6 \\ \frac{1}{2}\Omega_2 & \frac{1}{2}\Omega_1 - \frac{1}{2}\Omega_3 & \Gamma^2{}_3 & -\Gamma^1{}_5 & 0 & \Gamma^2{}_6 \\ \frac{1}{2}\theta^4 & 0 & \frac{1}{2}\theta^3 - \frac{1}{2}\theta^1 & -\Gamma^1{}_6 & -\Gamma^2{}_6 & 0 \\ 0 & -\frac{1}{2}\theta^1 & \frac{1}{2}\theta^3 & -\Omega_3 & -\frac{1}{2}\Omega_2 & -\frac{1}{2}\theta^4 \\ \frac{1}{2}\theta^1 & 0 & \frac{1}{2}\theta^2 & -\frac{1}{2}\theta^4 & -\frac{1}{2}\Omega_1 - \frac{1}{2}\Omega_3 & 0 \\ -\frac{1}{2}\theta^3 & -\frac{1}{2}\theta^2 & 0 & -\Gamma^1{}_3 & -\Gamma^2{}_3 & \frac{1}{2}\Omega_1 - \frac{3}{2}\Omega_3 \end{pmatrix},$$

where

$$\Gamma^1{}_3 = \tfrac{1}{2}\Omega_2 + \tfrac{1}{2}\mathbf{A}^p_2\theta^1,$$
$$\Gamma^2{}_3 = \mathbf{A}^p_3\theta^1 + \tfrac{1}{2}\mathbf{A}^p_2\theta^2 + \mathbf{A}^p_1\theta^4,$$
$$\Gamma^1{}_5 = \mathbf{D}^p_1\theta^1 + \mathbf{B}^p_3\theta^2 + (\tfrac{3}{2}\mathbf{B}^p_4 - \mathbf{B}^p_1)\theta^3 + (\mathbf{C}^p_1 - \tfrac{1}{2}\mathbf{A}^p_2)\theta^4,$$
$$\Gamma^1{}_6 = (\tfrac{1}{2}\mathbf{B}^p_4 - \mathbf{B}^p_1)\theta^1 - \mathbf{B}^p_2\theta^2,$$
$$\Gamma^2{}_6 = (\mathbf{B}^p_3 + \mathbf{D}^p_1)\theta^1 + \tfrac{1}{2}\mathbf{B}^p_4\theta^2 + \mathbf{B}^p_2\theta^4.$$

Contrary to section 6 of chapter 2, this connection seems to have full holonomy $CO(3,3) \ltimes \mathbb{R}^6$ and it is not generated by $\widehat{\omega}^p$ in any simple way. It is also never Einstein-Weyl.

5. Three-dimensional Lorentzian geometry on solution space

The geometries of sections 2 to 4 are counterparts of respective geometries of the contact case. The point classification, however, contains another geometry, which is new when compared to the contact case. This is owing to the fact that the Einstein-Weyl geometry of section 2 has in general the non-vanishing Ricci scalar, which is a weighted function and can be fixed to a constant by an appropriate choice of the conformal gauge. Thereby the Weyl geometry on $\mathcal{S}$ is reduced to a Lorentzian metric geometry.

These properties of the Weyl geometry are reflected at the level of the ODEs by the fact that the equations

$$
y''' = \frac{3}{2}\frac{(y'')^2}{y'}
\tag{75}
$$

and

$$
y''' = \frac{3y'(y'')^2}{y'^2 + 1}
\tag{76}
$$

are contact equivalent to the trivial $y''' = 0$ by means of corollary 2.4 but they are mutually *non-equivalent* under point transformations and possess the $\mathfrak{o}(2,2)$ and $\mathfrak{o}(4)$ algebra of point symmetries respectively. Both of them generate the same flat conformal geometry but their Weyl geometries differ. After calculating equations (74) we see that the only non-vanishing component of their curvature is the Ricci scalar, which is negative for the equation (75) and positive for (76). In this circumstances we do another reduction step in the Cartan algorithm setting the Ricci scalar equal to ± 6 respectively[1], which means $\mathbf{B}_4^p = \pm 1$, and obtain a six-dimensional subbundle $\mathcal{P}_6^p$ of $\mathcal{P}^p$. The invariant coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2)$ yields the local structure of $SO(2,2)$ or $SO(4)$ on $\mathcal{P}_6^p$ while the tensor $\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$ descends to a metric rather than a conformal class on $\mathcal{S}$ by means of conditions

$$
L_{X_5}\widehat{g} = 0, \qquad L_{X_6}\widehat{g} = 0.
\tag{77}
$$

The obtained metrics are locally diffeomorphic to the metrics on the symmetric spaces $SO(2,2)/SO(2,1)$ or $SO(4)/SO(3)$.

In order to generalize this construction to a broader class of equations we assume that the Ricci scalar of the Einstein-Weyl geometry is non-zero

$$
6K_{qq} + \frac{2}{3}F_{qqq}F_q + 2F_{qqp} + \frac{1}{2}F_{qq}^2 \neq 0
$$

and set

$$
u_3 = \sqrt{\left| 6K_{qq} + \frac{2}{3}F_{qqq}F_q + 2F_{qqp} + \frac{1}{2}F_{qq}^2 \right|}
$$

in the coframe of theorem 3.1. The tensor $\widehat{g}$ on $\mathcal{P}_6^p$ projects to the metric g on $\mathcal{S}$ provided that the conditions (77) still hold, which is equivalent to

$$
W = 0 \quad \text{and} \quad \left(\mathcal{D} + \tfrac{2}{3}F_q\right)\left(6K_{qq} + \frac{2}{3}F_{qqq}F_q + 2F_{qqp} + \frac{1}{2}F_{qq}^2\right) = 0.
$$

The Cartan coframe on $\mathcal{P}_6^p$ is then given by

$$
\begin{aligned}
\mathrm{d}\theta^1 ={}& \Omega_1 \wedge \theta^1 - \theta^2 \wedge \theta^4, \\
\mathrm{d}\theta^2 ={}& \Omega_2 \wedge \theta^1 + \mathbf{p_1}\theta^2 \wedge \theta^3 - \theta^3 \wedge \theta^4, \\
\mathrm{d}\theta^3 ={}& \Omega_2 \wedge \theta^2 - \Omega_1 \wedge \theta^3 + \mathbf{p_2}\theta^2 \wedge \theta^3, \\
\mathrm{d}\theta^4 ={}& \Omega_1 \wedge \theta^4 + \mathbf{p_3}\theta^1 \wedge \theta^2 + \mathbf{p_4}\theta^1 \wedge \theta^3 + \mathbf{p_5}\theta^1 \wedge \theta^4 - \tfrac{1}{2}\mathbf{p_2}\theta^2 \wedge \theta^4 + \mathbf{p_1}\theta^3 \wedge \theta^4, \\
\mathrm{d}\Omega_1 ={}& -\Omega_2 \wedge \theta^4 + \mathbf{p_2}\Omega_2 \wedge \theta^1 + \mathbf{p_6}\theta^1 \wedge \theta^2 + \mathbf{p_7}\theta^1 \wedge \theta^3 + \mathbf{p_4}\theta^2 \wedge \theta^3 + \mathbf{p_5}\theta^2 \wedge \theta^4, \\
\mathrm{d}\Omega_2 ={}& -\Omega_1 \wedge \Omega_2 + \mathbf{p_1}\Omega_2 \wedge \theta^3 + \mathbf{p_8}\theta^1 \wedge \theta^2 + \mathbf{p_9}\theta^1 \wedge \theta^3 + \mathbf{p_{10}}\theta^2 \wedge \theta^3 + \mathbf{p_5}\theta^3 \wedge \theta^4,
\end{aligned}
$$

with some functions $\mathbf{p_1}, \dots, \mathbf{p_{10}}$ and the Levi-Civita connection is given by

$$
\begin{pmatrix}
\Gamma^1_{\ 1} & \Gamma^1_{\ 2} & 0 \\
\Gamma^2_{\ 1} & 0 & \Gamma^1_{\ 2} \\
0 & \Gamma^2_{\ 1} & -\Gamma^1_{\ 1}
\end{pmatrix},
$$

---

[1]We choose ± 6 here to avoid large numerical factors.

where

$$\Gamma^1{}_1 = -\,\Omega_1 + \tfrac{1}{2}\mathbf{p_2}\theta^2,$$
$$\Gamma^1{}_2 = \tfrac{1}{2}\mathbf{p_2}\theta^1 - \mathbf{p_1}\theta^2 - \theta^4,$$
$$\Gamma^2{}_1 = -\,\Omega_2 + \tfrac{1}{2}\mathbf{p_2}\theta^3.$$

The curvature reads

$$\begin{pmatrix} R^1{}_1 & R^1{}_2 & 0 \\ R^2{}_1 & 0 & R^1{}_2 \\ 0 & R^2{}_1 & -R^1{}_1 \end{pmatrix},$$

$$R^1{}_1 = \tfrac{1}{2}(\mathbf{p_9} - \mathbf{p_6})\theta^1 \wedge \theta^2 + (\tfrac{1}{4}(\mathbf{p_2})^2 - \mathbf{p_7})\theta^1 \wedge \theta^3 + (\mathbf{p_4} + X_2(\mathbf{p_1}) + \tfrac{1}{2}\mathbf{p_1}\mathbf{p_2})\theta^2 \wedge \theta^3,$$

$$R^1{}_2 = (\mathbf{p_{10}} - \tfrac{1}{2}X_2(\mathbf{p_2}) - \tfrac{1}{4}(\mathbf{p_2})^2)\theta^1 \wedge \theta^2 + (\mathbf{p_4} + X_2(\mathbf{p_1}) + \tfrac{1}{2}\mathbf{p_1}\mathbf{p_2})\theta^1 \wedge \theta^3$$
$$+ ((\mathbf{p_1})^2 - X_3(\mathbf{p_1}))\theta^2 \wedge \theta^3,$$

$$R^2{}_1 = -\,\mathbf{p_8}\theta^1 \wedge \theta^2 + \tfrac{1}{2}(\mathbf{p_6} - \mathbf{p_9})\theta^1 \wedge \theta^3 + (-\mathbf{p_{10}} + \tfrac{1}{2}X_2(\mathbf{p_2}) + \tfrac{1}{4}(\mathbf{p_2})^2)\theta^2 \wedge \theta^3.$$

6. Fibre-preserving case: Cartan connection on seven-dimensional bundle

The construction of a Cartan connection for the fibre preserving case is very similar to its point counterpart. This is due to the fact that every point symmetry of $y''' = 0$ is necessarily fibre-preserving and, as a consequence, the bundle we will construct is also of dimension seven. Starting from the G_f-structure of Introduction, which is given by the forms

$$\theta^1 = u_1\omega^1,$$
$$\theta^2 = u_2\omega^1 + u_3\omega^2,$$
$$\theta^3 = u_4\omega^1 + u_5\omega^2 + u_6\omega^3,$$
$$\theta^4 = u_7\omega^4,$$

and after the substitutions

$$u_6 = \frac{u_3^2}{u_1}, \qquad u_7 = \frac{u_1}{u_3},$$
$$u_5 = \frac{u_3}{u_1}\left(u_2 - \frac{1}{3}u_3 F_q\right),$$
$$u_4 = \frac{u_3^2}{u_1}K + \frac{u_2^2}{2u_1}$$

we get the following theorem.

THEOREM 3.5. *An equation $y''' = F(x, y, y', y'')$ modulo fibre-preserving transformations is described by the coframe $(\theta^1, \theta^2, \theta^3, \theta^4, \Omega_1, \Omega_2, \Omega_3)$ which generates the following Cartan connection*

$$(78) \qquad \widehat{\omega}^f = \begin{pmatrix} \tfrac{1}{2}\Omega_1 & \tfrac{1}{2}\Omega_2 & 0 & 0 \\ \theta^4 & \Omega_3 - \tfrac{1}{2}\Omega_1 & 0 & 0 \\ \theta^2 & \theta^3 & \tfrac{1}{2}\Omega_1 - \Omega_3 & -\tfrac{1}{2}\Omega_2 \\ 2\theta^1 & \theta^2 & -\theta^4 & -\tfrac{1}{2}\Omega_1 \end{pmatrix}$$

on the seven-dimensional bundle $H_3 \to \mathcal{P}^f \to \mathcal{J}^2$. The group H_3 is the same as in the point case

$$H_3 = \begin{pmatrix} \sqrt{u_1}, & \frac{1}{2}\frac{u_2}{\sqrt{u_1}}, & 0 & 0 \\ 0 & \frac{u_3}{\sqrt{u_1}}, & 0 & 0 \\ 0 & 0 & \frac{\sqrt{u_1}}{u_3}, & -\frac{1}{2}\frac{u_2}{\sqrt{u_1}\,u_3} \\ 0 & 0 & 0 & \frac{1}{\sqrt{u_1}} \end{pmatrix},$$

and the connection is explicitly given by

$$\widehat{\omega}^f(x^i, u_\mu) = u^{-1}\,\omega^f\,u + u^{-1}\mathrm{d}u$$

where $u \in H_3$ and

$$\omega^f = \begin{pmatrix} \frac{1}{2}\Omega_1^0 & \frac{1}{2}\Omega_2^0 & 0 & 0 \\ \widetilde{\omega}^4 & \Omega_3^0 - \frac{1}{2}\Omega_1^0 & 0 & 0 \\ \omega^2 & \widetilde{\omega}^3 & \frac{1}{2}\Omega_1^0 - \Omega_3^0 & -\frac{1}{2}\Omega_2^0 \\ 2\omega^1 & \omega^2 & -\widetilde{\omega}^4 & -\frac{1}{2}\Omega_1^0 \end{pmatrix}$$

is given by

$$\omega^1 = \mathrm{d}y - p\,\mathrm{d}x,$$
$$\omega^2 = \mathrm{d}p - q\,\mathrm{d}x,$$
$$\widetilde{\omega}^3 = \mathrm{d}q - F\mathrm{d}x - \tfrac{1}{3}F_q(\mathrm{d}p - q\mathrm{d}x) + K(\mathrm{d}y - p\mathrm{d}x),$$
$$\omega^4 = \mathrm{d}x$$
$$\Omega_1^0 = -K_q\,\omega^1 + \tfrac{1}{3}F_{qq}\omega^2,$$
$$\Omega_2^0 = L\,\omega^1 - K_q\,\omega^2 + \tfrac{1}{3}F_{qq}\widetilde{\omega}^3 - K\omega^4,$$
$$\Omega_3^0 = -K_q\,\omega^1 + \tfrac{1}{3}F_{qq}\,\omega^2 + \tfrac{1}{3}F_q\omega^4.$$

The exterior differentials of the coframe are equal to

$$\mathrm{d}\theta^1 = \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2 + \mathbf{B}_1^f \theta^1 \wedge \theta^2,$$
$$\mathrm{d}\theta^2 = \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3 + \mathbf{B}_1^f \theta^1 \wedge \theta^3,$$
$$\mathrm{d}\theta^3 = \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \mathbf{A}_1^f \theta^4 \wedge \theta^1 + \mathbf{B}_1^f \theta^2 \wedge \theta^3,$$
$$\mathrm{d}\theta^4 = (\Omega_1 - \Omega_3) \wedge \theta^4,$$

(79)
$$\mathrm{d}\Omega_1 = -\Omega_2 \wedge \theta^4 + (\mathbf{D}_1^f - \mathbf{B}_2^f)\theta^1 \wedge \theta^2 + \mathbf{B}_3^f \theta^1 \wedge \theta^3 +$$
$$\qquad + (2\mathbf{C}_1^f - \mathbf{A}_2^f)\theta^1 \wedge \theta^4 + \mathbf{B}_4^f \theta^2 \wedge \theta^3 + \mathbf{B}_5^f \theta^2 \wedge \theta^4,$$
$$\mathrm{d}\Omega_2 = (\Omega_3 - \Omega_1) \wedge \Omega_2 + \mathbf{D}_2^f \theta^1 \wedge \theta^2 + (\mathbf{D}_1^f - 2\mathbf{B}_2^f)\theta^1 \wedge \theta^3 + \mathbf{A}_3^f \theta^1 \wedge \theta^4$$
$$\qquad + \mathbf{B}_6^f \theta^2 \wedge \theta^3 + (\mathbf{C}_1^f - \mathbf{A}_2^f)\theta^2 \wedge \theta^4 + \mathbf{B}_5^f \theta^3 \wedge \theta^4,$$
$$\mathrm{d}\Omega_3 = (\mathbf{D}_1^f - \mathbf{B}_2^f)\theta^1 \wedge \theta^2 + \mathbf{B}_3^f \theta^1 \wedge \theta^3 + (\mathbf{C}_1^f - \mathbf{A}_2^f)\theta^1 \wedge \theta^4$$
$$\qquad + \mathbf{B}_4^f \theta^2 \wedge \theta^3 + \tfrac{1}{2}\mathbf{B}_5^f \theta^2 \wedge \theta^4,$$

where $\mathbf{A_1^f}, \mathbf{A_2^f}, \mathbf{A_3^f}, \mathbf{B_1^f}, \mathbf{B_2^f}, \mathbf{B_3^f}, \mathbf{B_4^f}, \mathbf{B_5^f}, \mathbf{B_6^f}, \mathbf{C_1^f}, \mathbf{D_1^f}, \mathbf{D_2^f}$ are functions on $\mathcal{P}^f$. All these invariants express by the coframe derivatives of $\mathbf{A_1^f}, \mathbf{B_1^f}, \mathbf{C_1^f}$, which read

$$\mathbf{A_1^f} = \frac{u_3^3}{u_1^3} W,$$

$$\mathbf{B_1^f} = \frac{1}{3u_3} F_{qq},$$

$$\mathbf{C_1^f} = \frac{u_2}{u_1^2} \left(\tfrac{1}{9} F_{qq} F_q + \tfrac{1}{3} F_{qp} + K_q \right) +$$

$$+ \frac{u_3}{u_1^2} \left(\tfrac{2}{3} F_{qq} K - \tfrac{1}{3} K_q F_q - K_p - \tfrac{2}{3} F_{qy} \right).$$

If $\mathbf{A_1^f} = 0$ then $\mathbf{A_2^f} = \mathbf{A_3^f} = 0$ and if $\mathbf{B_1^f} = 0$ then $\mathbf{B_i^f} = 0$ for $i = 2, \ldots, 6$. In particular we have

$$d\mathbf{B_1^f} = -\mathbf{B_2^f} \theta^1 + (\mathbf{B_6^f} - \mathbf{B_3^f}) \theta^2 - \mathbf{B_4^f} \theta^3 - \mathbf{B_5^f} \theta^4 - \mathbf{B_1^f} \Omega_3.$$

The flat case is given by vanishing of $\mathbf{A_1^f}$, $\mathbf{B_1^f}$ and $\mathbf{C_1^f}$.

6.1. Fibre-preserving versus point objects. An immediate observation about the fibre-preserving objects is that they are closely related to their point counterparts of theorem 3.1. The bundle $\mathcal{P}^f$ is also, like $\mathcal{P}^p$, a subbundle of the contact bundle $\mathcal{P}^c$ given by $\tau \colon \mathcal{P}^f \to \mathcal{P}^c$

(80)
$$u_4 = 0, \qquad u_5 = 0, \qquad u_6 = 0,$$

and, as before, the forms $\overset{f}{\theta^1}, \ldots, \overset{f}{\theta^4}$ of the fibre-preserving coframe are given by pull-backs of their contact counterparts

$$\overset{f}{\theta^i} = \tau^* \overset{c}{\theta^i}, \qquad i = 1, 2, 3, 4.$$

This fact suggests that there exists a distinguished diffeomorphism of $\mathcal{P}^p$ and $\mathcal{P}^f$ given by a similar condition. Indeed, there is the unique diffeomorphism $\rho \colon \mathcal{P}^p \to \mathcal{P}^f$ such that

(81)
$$\overset{p}{\theta^1} = \rho^* \overset{f}{\theta^1}, \qquad \overset{p}{\theta^2} = \rho^* \overset{f}{\theta^2}, \qquad \overset{p}{\theta^3} = \rho^* \overset{f}{\theta^3}.$$

It is given by the identity map in the coordinate systems of theorems 3.1 and 3.5. The remaining one-forms are transported as follows.

(82)
$$\overset{p}{\theta^4} = \rho^* (\overset{f}{\theta^4} + \tfrac{1}{2} \mathbf{B_1^f} \overset{f}{\theta^1}),$$

$$\overset{p}{\Omega_1} = \rho^* (\overset{f}{\Omega_1} + \mathbf{B_5^f} \overset{f}{\theta^1} - \tfrac{1}{2} \mathbf{B_1^f} \overset{f}{\theta^2}),$$

$$\overset{p}{\Omega_2} = \rho^* (\overset{f}{\Omega_2} + \tfrac{1}{2} \mathbf{B_5^f} \overset{f}{\theta^2} - \tfrac{1}{2} \mathbf{B_1^f} \overset{f}{\theta^3}),$$

$$\overset{p}{\Omega_3} = \rho^* (\overset{f}{\Omega_3} + \tfrac{1}{2} \mathbf{B_5^f} \overset{f}{\theta^1}),$$

where

$$\mathbf{B_5^f} = -\frac{1}{3u_1} (\mathcal{D}(F_{qq}) + \tfrac{1}{3} F_{qq} F_q).$$

The above formulae enable us to transform easily the fibre-preserving coframe into the point coframe. Given the fibre-preserving coframe $(\overset{f}{\theta^1}, \ldots, \overset{f}{\Omega_3})$ we compute $d\overset{f}{\theta^1}$ and take the coefficient of the $\overset{f}{\theta^1} \wedge \overset{f}{\theta^2}$ term. This is the function $\mathbf{B_1^f}$. Next we compute $d\mathbf{B_1^f}$, decompose it in the fibre-preserving coframe, take minus function

that stands next to $\overset{f}{\theta^4}$ and this is $\mathbf{B_5^f}$. We substitute these functions together with $(\overset{f}{\theta^1},\ldots,\overset{f}{\Omega_3})$ into the right hand side of (81) and (82), where ρ is the identity transformation of $\mathcal{P}^f$ and the point coframe is explicitly constructed on $\mathcal{P}^f$.

Now let us consider the inverse construction, from the fibre-preserving case to the point case. If we have only the point coframe $(\overset{p}{\theta^1},\ldots,\overset{p}{\Omega_3})$ then we can not utilize eq. (82) since we are not able to construct the function $\mathbf{B_1^f}$, which is not a point invariant[2] and, as such, does not appear among functions $\mathbf{A_1^p},\ldots,\mathbf{D_2^p}$ in (72) or among their derivatives. However, if we consider the point coframe *and* the function $\mathbf{B_1^f}$ then the construction is possible, since $\mathbf{B_5^f}$ is given by the derivative $-X_4(\mathbf{B_1^f})$ along the field X_4 of the *point* dual frame. Therefore the passage from the point case to the fibre-preserving case is possible if we supplement the connection $\widehat{\omega}^p$ with the function $\mathbf{B_1^f}$. This fact implies that each construction of the point case has its fibre-preserving counterpart which has an additional object generated by $\mathbf{B_1^f}$.

7. Fibre-preserving geometry from point geometry

7.1. Counterpart of the Einstein-Weyl geometry on $\mathcal{S}$. This geometry is constructed in the following way. Let $(\theta^1,\ldots,\Omega_3)$ denotes again the fibre-preserving coframe. Given the objects $\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$ and

$$\widehat{\phi} = \Omega_3 + \frac{1}{2}\mathbf{B_5^f}\theta^1,$$

let us also consider the function $\mathbf{B_1^f}$, and ask under what conditions the triple $(\widehat{g},\widehat{\phi},\mathbf{B_1^f})$ can be projected to a geometry on $\mathcal{S}$. There are two possibilities here, either $\mathbf{B_1^f} = 0$ or $\mathbf{B_1^f} \neq 0$. If $\mathbf{B_1^f} = 0$ then it is easy to see that the pair $(\widehat{g},\widehat{\phi})$ generates the Einstein-Weyl geometry if only $\mathbf{A_1^f} = \mathbf{C_1^f} = 0$, which means that we are in the trivial case $y''' = 0$.

Suppose $\mathbf{B_1^f} \neq 0$ then. For the geometry on $\mathcal{S}$ to exist we need not only the conditions for the Lie transport of $\widehat{g}$ and $\widehat{\phi}$ but also

$$(83) \qquad L_{X_i}\mathbf{B_1^f} = 0, \quad \text{for} \quad i = 4, 5, 6, \quad L_{X_7}\mathbf{B_1^f} = -\mathbf{B_1^f}.$$

If all these conditions are satisfied then $(\widehat{g},\widehat{\phi},\mathbf{B_1^f})$ defines on $\mathcal{S}$ the Einstein-Weyl geometry (g,ϕ) of the point case, which is equipped with an additional object: a weighted function f which transforms $f \to e^{-\lambda}f$ when $g \to e^{2\lambda}g$ and is given by the projection of $\mathbf{B_1^f}$. The conditions for existence of this geometry are $\mathbf{A_1^f} = \mathbf{B_5^f} = 0$, that is

$$(84) \qquad W = 0 \quad \text{and} \quad \mathcal{D}(F_{qq}) + \tfrac{1}{3}F_{qq}F_q = 0.$$

As usual, the condition $W = 0$ guarantees existence of $[g]$ and the other condition yields (83). The proper Lie transport of $\widehat{\phi}$ along X_4 is already guaranteed by the above conditions as their differential consequence.

7.2. Counterpart of the Weyl geometry on $\mathcal{M}^6$. In the similar vein we show that the triple $(\widehat{\mathbf{g}}, \frac{1}{2}\widehat{\phi}, \mathbf{B_1^f})$, where

$$\widehat{\mathbf{g}} = 2(\Omega_1 - \Omega_3)\theta^2 - 2\Omega_3\theta^1 + 2\theta^4\theta^3,$$
$$\widehat{\phi} = \Omega_3 + \tfrac{1}{2}\mathbf{B_5^f}\theta^1.$$

projects to the six-dimensional split signature Weyl geometry $(\mathbf{g},\phi)$ of chapter 3 section 4 equipped with a function f of conformal weight -2.

---

[2]For example the point transformation $(x, y) \to (y, x)$ destroys the condition $F_{qq} = 0$.

7.3. Counterpart of Lorentzian geometry on $\mathcal{S}$. Given (g, ϕ, f) on $\mathcal{S}$ it is natural to fix the conformal gauge so as[3] $f = 1$. This is equivalent to another substitution

$$u_3 = \tfrac{1}{3} F_{qq}$$

in the Cartan reduction algorithm, which leads us to the bundle $\mathcal{P}_6^f$ with the following differential system

$$\mathrm{d}\theta^1 = \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2,$$

$$\mathrm{d}\theta^2 = \Omega_2 \wedge \theta^1 + \mathbf{f_1}\theta^3 \wedge \theta^2 + \mathbf{f_2}\theta^4 \wedge \theta^2 + \theta^4 \wedge \theta^3,$$

$$\mathrm{d}\theta^3 = -\Omega_1 \wedge \theta^3 + \Omega_2 \wedge \theta^2 + (2 - 2\mathbf{f_3})\theta^3 \wedge \theta^2 + \mathbf{f_4}\theta^4 \wedge \theta^1 + 2\mathbf{f_2}\theta^4 \wedge \theta^3,$$

$$(85) \quad \mathrm{d}\theta^4 = \Omega_1 \wedge \theta^4 + \mathbf{f_5}\,\theta^4 \wedge \theta^1 + (\mathbf{f_3} - 2)\,\theta^4 \wedge \theta^2 + \mathbf{f_1}\,\theta^4 \wedge \theta^3,$$

$$\mathrm{d}\Omega_1 = (2\mathbf{f_3} - 2)\Omega_2 \wedge \theta^1 - \Omega_2 \wedge \theta^4 + \mathbf{f_6}\theta^1 \wedge \theta^2 + \mathbf{f_7}\theta^1 \wedge \theta^3 + \mathbf{f_8}\theta^1 \wedge \theta^4 - \mathbf{f_5}\theta^2 \theta^4,$$

$$\mathrm{d}\Omega_2 = \Omega_2 \wedge \Omega_1 - \mathbf{f_1}\Omega_2 \wedge \theta^3 - \mathbf{f_2}\Omega_2 \wedge \theta^4 + \mathbf{f_9}\theta^1 \wedge \theta^2 + \mathbf{f_{10}}\theta^1 \wedge \theta^3 + \mathbf{f_{11}}\theta^1 \wedge \theta^4 +$$

$$+ \mathbf{f_{12}}\theta^2 \wedge \theta^3 + \mathbf{f_{13}}\theta^2 \wedge \theta^4 - \mathbf{f_5}\theta^3 \wedge \theta^4.$$

If the conditions (84), now equivalent to $\mathbf{f_4} = 0 = \mathbf{f_2}$, are satisfied then $\widehat{g} = 2\theta^1\theta^3 - (\theta^2)^2$ projects to a Lorentzian metric g and

$$\widehat{\phi} = -2\mathbf{f_5}\theta^1 + 2\mathbf{f_3}\theta^2 + 2\mathbf{f_1}\theta^3$$

projects to a one-form ϕ. With the Lorentzian metric there is associated the Levi-Civita connection $(\Gamma^\mu{}_\nu)$:

$$\Gamma^1{}_1 = -\Omega_1 + (\mathbf{f_3} - 1)\theta^2,$$

$$\Gamma^1{}_2 = (\mathbf{f_3} - 1)\theta^1 + \mathbf{f_1}\theta^2 - \theta^4,$$

$$\Gamma^2{}_1 = -\Omega_2 + (\mathbf{f_3} - 1)\theta^3.$$

The covariant derivative of ϕ with respect to $\Gamma^\mu{}_\nu$ is as follows

$$\phi_{i;j} = \begin{pmatrix} -\mathbf{f_9} - (\mathbf{f_5})^2 & \tfrac{1}{2}\mathbf{f_6} + \mathbf{f_5}(\mathbf{f_3} - 2) & \mathbf{f_{12}} - \mathbf{f_3}(\mathbf{f_3} - 3) \\ \tfrac{1}{2}\mathbf{f_6} + \mathbf{f_5}\mathbf{f_3} & 2\mathbf{f_{12}} - 2\mathbf{f_3}(\mathbf{f_3} - 2) & X_2(\mathbf{f_1}) + \mathbf{f_1} \\ \mathbf{f_{12}} - \mathbf{f_3}(\mathbf{f_3} - 1) & X_2(\mathbf{f_1}) - \mathbf{f_1} & X_3(\mathbf{f_1}) \end{pmatrix}.$$

The one-form ϕ and the Ricci tensor satisfy the following identities

$$\nabla_{(i}\phi_{j)} = -\mathrm{Ric}_{ij} - \phi_i\phi_j + (\phi^k\phi_k + 2)g_{ij},$$

$$\mathrm{R} = 2\phi^k\phi_k + 6,$$

$$\mathrm{d}\phi = -2 * \phi.$$

The homogeneous model of this geometry is associated to $y''' = \tfrac{3}{2}\dfrac{(y'')^2}{y'}$.

---

[3]With possible change of the signature to make f positive.

Classification of third-order ODEs

Previous chapters provided several geometric constructions associated to third-order ODEs, now we focus on the application of the Cartan equivalence method to the classification of ODEs. Following [51] chapters 8 – 14 we describe the procedure which was outlined in Introduction.

By means of theorems 2.1, 2.11, 3.1 and 3.5 to an ODE modulo contact/point/fibre-preserving transformations there is associated a Cartan coframe on a bundle $\mathcal{P} \to \mathcal{J}^2$. From theorem 1.3 (with obvious changes) we know, that the underlying ODEs are equivalent if and only if the coframes are. Thereby the problem of equivalence of ODEs is reduced to the equivalence problem of coframes.

Consider two smooth coframes (ω^i) on $\mathcal{M}^n$ and $(\bar{\omega}^i)$ on $\bar{\mathcal{M}}^n$. Let (X_i) and $(\bar{X}_i)$ be the dual frames. We ask under what conditions there exists a local diffeomorphism $\Phi \colon \mathcal{M} \supset U \to \bar{U} \subset \bar{M}$ such that $\Phi^* \bar{\omega}^i = \omega^i$. To answer this question we need to define several notions. The structural equations for (ω^i) read $\mathrm{d}\omega^i = \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k$, where $T^i_{jk} = T^i_{[jk]}$ are smooth functions since the coframe is smooth. The functions T^i_{jk} and their coframe derivatives $T^i_{jk|l} = X_l(T^i_{jk})$, $T^i_{jk|lm} = X_m(X_l(T^i_{jk}))$ etc. of any order are called structural functions of the coframe. T^i_{jk} are the structural functions of zero order, $T^i_{jk|l}$ are the structural functions of first order and so on. Since pull-back commutes with exterior differentiation all the structural functions are relative invariants of the equivalence problem of coframes. We say that smooth functions $f_1, \ldots, f_k$ are independent at a point w if $\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_k \neq 0$ at w. The rank of a coframe (ω^i) at w is defined to be the maximal number of its independent structural functions at w. The order s at w of a coframe of rank r at w is the smallest natural number such that among structural functions of order at most s there are r functions independent at w.

We call a smooth coframe regular in an open set U if for each $j \geq 0$ the number of its independent structural functions of order j is constant on U. In particular the rank and the order of a regular coframe are both constant on U. From now on we confine our considerations to coframes regular on sufficiently small topologically trivial open subsets U of $\mathcal{M}$. For a regular coframe of order s one may choose a set of r independent structural functions $I_1, \ldots, I_r$, which are of order at most s and all the remaining structural functions T_σ of any order are expressible in terms of I_j, $T_\sigma = f_\sigma(I_1, \ldots, I_r)$. Indeed, all the $(s+1)$-order functions are of this form by definition of the order of a coframe, whereas all the functions of order $s + 2$ or greater are their coframe derivatives. These derivatives depend on I_j and $I_{j|k}$ but $I_{j|k}$ are functions of order at most $s + 1$ so their are also functions of I_j.

Thereby all the structural functions are described by i) the set $(I_1, \ldots, I_r)$ and ii) the formulae which characterize how the $(s+1)$-order functions are expressed by $(I_1, \ldots, I_r)$. Both these objects may be encoded in the so called classifying function. The classifying function for a coframe of order s is a function $\mathbf{T} \colon \mathcal{M} \to \mathbb{R}^N$ given by all the structural functions of order at most $s + 1$, which are lexicographically ordered with respect to their indices, and N is the number of these structural

functions

$$\mathbf{T} : w \in U \longmapsto (T^i_{jk}, T^i_{jk|l_1}, \ldots, T^i_{jk|l_1 \ldots l_{s+1}}) \in \mathbb{R}^N.$$

By smoothness and regularity of the underlying coframe the graph $\mathbf{T}(U)$ is an r-dimensional submanifold in $\mathbb{R}^N$. The last ingredient which we need is definition of overlapping; two r-dimensional submanifolds of $\mathbb{R}^N$ overlap if their intersection is also an r-dimensional submanifold of $\mathbb{R}^N$. Now we are in position to cite the following

THEOREM 4.1 ([51], theorem 14.24). *Let (ω^i) and $(\bar{\omega}^i)$ be smooth, regular coframes defined, respectively, on $\mathcal{M}^n$ and $\bar{\mathcal{M}}^n$. There exists a local diffeomorphism $\Phi : \mathcal{M} \to \bar{\mathcal{M}}$ such that $\Phi^* \bar{\omega}^i = \omega^i$, $i = 1, \ldots, n$, if and only if the coframes have the same order $s = \bar{s}$ and the graphs $\mathbf{T}(\mathcal{M})$ and $\bar{\mathbf{T}}(\bar{\mathcal{M}})$ of their classifying functions overlap. Moreover, if $w_0 \in \mathcal{M}$ and $\bar{w}_0 \in \bar{\mathcal{M}}$ are any points mapping to the same point*

$$z_0 = \mathbf{T}(w_0) = \bar{\mathbf{T}}(\bar{w}_0) \in \mathbf{T}(\mathcal{M}) \cap \bar{\mathbf{T}}(\bar{\mathcal{M}})$$

on the overlap, then there is a unique equivalence map Φ such that $\bar{w}_0 = \Phi(w_0)$.

An important notion is a symmetry of a coframe. This is a diffeomorphism $\Phi : \mathcal{M} \to \mathcal{M}$ such that $\Phi^* \omega^i = \omega^i$. We have

THEOREM 4.2 ([51], theorem 14.26). *The symmetry group G of a regular coframe (ω^i) of rank r on $\mathcal{M}^n$ is a local Lie group of transformations of dimension $n - r$.*

There is a class of coframes which admit a particularly simple description in this language. These are coframes of rank zero, whose all structural functions T^i_{jk} are constant. These coframes are of order zero and the image of their classifying function is a point in $\mathbb{R}^N$. Thus, two rank-zero coframes are equivalent if and only if their structural constants are equal, $T^i_{jk} = \bar{T}^i_{jk}$. Furthermore, a coframe of rank zero has an n dimensional Lie symmetry group G. The algebra of this group is precisely the algebra with structural constants T^i_{jk} and the coframe may be interpreted as the left invariant coframe defining a local structure of G on $\mathcal{M}$.

Let us turn to ODEs. In order to determine whether or not two given ODEs are contact/point/fibre-preserving equivalent one constructs the respective invariant coframes for both the ODEs, calculates the structural functions, which are now relative invariants for the underlying ODEs, and applies theorem 4.1 provided that the coframes are regular. In particular, contact/point/fibre-preserving symmetry group of the underlying ODE is isomorphic to the symmetry group of the associated coframe.

In practice there are three difficulties in this approach. First difficulty is that the coframes and invariants we have built so far are defined on manifolds $\mathcal{P}$ larger then $\mathcal{J}^2$. As a consequence, the invariants contain not only x, y, p, q but the auxiliary bundle variables u_μ as well. Since it is natural to describe ODEs in terms of x, y, p, q only, first of all we must finish the reduction of the group parameters, so that finally no free u_μ remains and we obtain a coframe on $\mathcal{J}^2$ from which we compute invariants of ODEs depending on x, y, p, q only.

Second problem is that the above method requires regularity of the Cartan coframes. Thus we have to restrict our consideration to the class of regular ODEs defined as follows.

DEFINITION 4.3. An ODE $y''' = F(x, y, y', y'')$ is regular if it is given by a locally smooth function F such that the Cartan coframe obtained after maximal possible reduction of the structural group is regular.

Third and essential obstacle for the full classification is that the task of finding whether or not two graphs of functions $\mathbf{T}\colon \mathcal{J}^2 \to \mathbb{R}^N$ overlap is highly nontrivial, particularly in our cases, where the components of $\mathbf{T}$ are compound functions depending on x, y, p, q through F. Taking this into account we restrict the classification to two classes of equations for which we are able to find compact criteria for the equivalence.

i) The regular equations possessing large contact or point symmetry groups, that is symmetry groups of dimension at least four.
ii) The regular equations fibre-preserving equivalent to reduced Chazy classes II, IV – VII and XII.

For these two families we carry out the classification to the very end, however we also provide some partial result in the case of totally arbitrary ODEs. This partial result, theorems 4.5 and 4.7, is the explicit construction of the invariant coframes on $\mathcal{J}^2$ in the contact case without further analysis of the classifying function.

1. Equations with large contact symmetry group

This class of ODEs is particularly convenient for characterization since, as we shall see, to these equations we can always associate a Cartan coframe of rank zero, with the only exception of ODEs contact equivalent to general linear ODEs (88). These exceptional ODEs are characterized by the fact that their symmetry group act intransitively on $\mathcal{J}^2$, cf corollary 4.4 and proposition 4.8

Let us begin with the ten-dimensional coframe of theorem 2.1. As we said in section 7 of chapter 2, ODEs fall into three main classes:

- $W = 0$ and $F_{qqqq} = 0$,
- $W \neq 0$,
- $W = 0$ but $F_{qqqq} \neq 0$.

If $F_{qqqq} = W = 0$ then we are in the situation of corollary 2.4 and this is the only case of the ODEs with a ten-dimensional contact symmetry group, since for any other ODE there are non-constant relative invariants in equations (48) and the dimension of the symmetry group is less then ten. Below we consider the case $W \neq 0$.

1.1. ODEs with $W \neq 0$. For these equations we have the five dimensional coframe of theorem 2.11:

$$
\begin{aligned}
d\theta^1 =& \Omega \wedge \theta^1 - \theta^2 \wedge \theta^4, \\
d\theta^2 =& \Omega \wedge \theta^2 + \mathbf{a}^c\, \theta^1 \wedge \theta^4 - \theta^3 \wedge \theta^4, \\
d\theta^3 =& \Omega \wedge \theta^3 + \mathbf{b}^c\, \theta^1 \wedge \theta^2 + \mathbf{c}^c\, \theta^1 \wedge \theta^3 - \theta^1 \wedge \theta^4 + \mathbf{e}^c\, \theta^2 \wedge \theta^3 + \mathbf{a}^c \theta^2 \wedge \theta^4, \\
d\theta^4 =& \mathbf{f}^c\, \theta^1 \wedge \theta^2 + \mathbf{g}^c\, \theta^1 \wedge \theta^3 + \mathbf{h}^c\, \theta^1 \wedge \theta^4 + \mathbf{k}^c\, \theta^2 \wedge \theta^3 - \mathbf{e}^c\, \theta^2 \wedge \theta^4, \\
d\Omega =& \mathbf{l}^c\, \theta^1 \wedge \theta^2 + (\mathbf{f}^c - \mathbf{a}^c \mathbf{k}^c)\, \theta^1 \wedge \theta^3 + \mathbf{m}^c\, \theta^1 \wedge \theta^4 + \mathbf{g}^c\, \theta^2 \wedge \theta^3 + \mathbf{h}^c\, \theta^2 \wedge \theta^4.
\end{aligned}
$$
(86)

with functions $\mathbf{a}^c, \mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c, \mathbf{k}^c$ given by

$$
\begin{aligned}
\mathbf{a}^c =& \frac{1}{\sqrt[3]{W^2}} \left(K + \frac{1}{18} Z^2 + \frac{1}{9} Z F_q - \frac{1}{3} \mathcal{D} Z \right), \\
\mathbf{b}^c =& \frac{1}{3u \sqrt[3]{W^2}} \left(\frac{1}{27} F_{qq} Z^2 + \left(K_q - \frac{1}{3} Z_p - \frac{2}{9} F_q Z_q \right) Z + \right. \\
& \left. + \left(\frac{1}{3} \mathcal{D} Z - 2K \right) Z_q + Z_y + F_{qq} K - 3 K_p - K_q F_q - F_{qy} + W_q \right), \\
\mathbf{e}^c =& \frac{1}{u} \left(\frac{1}{3} F_{qq} + \frac{1}{W} \left(\frac{2}{9} W_q Z - \frac{2}{3} W_p - \frac{2}{9} W_q F_q \right) \right),
\end{aligned}
$$
(87)

$$\mathbf{h}^c = \frac{1}{3u\sqrt[3]{W}}\left(\left(\frac{1}{9}W_qZ^2 - \frac{1}{3}W_pZ + W_y - \frac{1}{3}W_q\mathcal{D}Z\right)\frac{1}{W} + \right.$$
$$\left. + \mathcal{D}Z_q + \frac{1}{3}F_qZ_q\right),$$

$$\mathbf{k}^c = \frac{1}{u^2\sqrt[3]{W}}\left(\frac{2W_q^2}{9W} - \frac{W_{qq}}{3}\right).$$

This means in view of theorem 4.2 that if $W \neq 0$ then we have symmetry groups of dimension at most five. We proved in corollary 2.12 that the assumption $\mathbf{a}^c = const$ and $\mathbf{k}^c = 0$ implies that the underlying ODE has a five-dimensional group of symmetry. Now we easily see that there are no other equations with a five-dimensional symmetry group here. In fact, this property requires that all the functions in (86) are constants, in particular $\mathbf{a}^c = const$ and $\mathbf{k}^c = const$. But $\mathbf{k}^c \sim u^{-2}$ so it is a constant function on the bundle $\mathcal{P}_5^c$ iff it vanishes and we are back in corollary 2.12. At this stage we know that all the ODEs with $W \neq 0$ other than $y''' = -2\mu y' + y$ have the symmetry group of dimension at most four. Next we reduce the last free bundle parameter u. The Cartan algorithm bifurcates at this point:

    i) The functions $\mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c$ and $\mathbf{k}^c$ in (86) vanish.

    ii) At least one function among $\mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c, \mathbf{k}^c$ does not vanish.

We will discuss both these possibilities consecutively.

    i). Utilizing the Jacobi identity for (86) we get that all the functions but $\mathbf{a}^c$ vanish, $\mathbf{a}^c = \mathbf{a}^c(x)$ and $\mathrm{d}\mathbf{a}^c = (\mathbf{a}^c)_4\theta^4$, $\mathrm{d}(\mathbf{a}^c)_4 = (\mathbf{a}^c)_{44}\theta^4$ etc, so neither $\mathbf{a}^c$ nor its derivatives of any order contain the last auxiliary variable u and the full reduction of the structural group cannot be done. We check that the ODE

(88) $$y''' = -2\mu(x)y' + (1 - \mu'(x))y,$$

with an arbitrary smooth function $\mu(x)$, satisfies $\mathbf{b}^c = \mathbf{e}^c = \mathbf{h}^c = \mathbf{k}^c = 0$ and $\mathbf{a}^c = \mu(x)$. Furthermore, a linear third-order ODE of general form satisfies either $W = 0$, in which case it is equivalent to $y''' = 0$, or $W \neq 0$, in which case $\mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c$ and $\mathbf{k}^c$ vanish. Thus case i) describes the ODEs satisfying $W \neq 0$ and linearizable through contact transformations, in particular the linear equations with constant coefficients are distinguished by the additional condition $\mathbf{a}^c = const$.

COROLLARY 4.4. *The following third-order ODEs are linearizable via contact transformations of variables*

- *The equations satisfying $W = F_{qqqq} = 0$. These are equivalent to*

$$y''' = 0$$

  and admit the group $O(3,2)$ of contact symmetries.
- *The equations satisfying $W \neq 0$, $\mathbf{b}^c = \mathbf{e}^c = \mathbf{h}^c = \mathbf{k}^c = 0$ and $\mathbf{a}^c = \mu(x)$, where $\mu(x)$ is any smooth non-constant real function. These are equivalent to the general linear equation*

$$y''' = -2\mu(x)y' + (1 - \mu'(x))y$$

  and admit a four-dimensional group of contact symmetries. The symmetry group acts on three-dimensional orbits in $\mathcal{J}^2$. If such an equation is given in the above form, then symmetries are generated by

$$V_i = f_i\partial_y + f_i'\partial_p + f_i''\partial_q, \qquad i = 1, 2, 3$$
$$V_4 = y\partial_y + p\partial_p + q\partial_q,$$

  where f_1, f_2, f_3 are any three functionally independent solutions of the ODE.

- *The equations satisfying $W \neq 0$, $\mathbf{k}^c = 0$ and $\mathbf{a}^c = \mu \in \mathbb{R}$. These are equivalent to the general linear equation with constant coefficients*

$$y''' = -2\mu y' + y$$

and admit the group $\mathbb{R}^2 \ltimes_\mu \mathbb{R}^3$ of contact symmetries.

The equations with different values of $\mu(x)$ or μ are non-equivalent.

ii). In this case we use a non-vanishing function among $\mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c, \mathbf{k}^c$ in (86) to do the last reduction and eventually obtain a coframe on $\mathcal{J}^2$. The substitution is as follows.

1. If $\mathbf{k}^c \neq 0$ then we set $\mathbf{k}^c = \epsilon = \pm 1$ depending on the sign of the quantity $\frac{1}{\sqrt[3]{W}}\left(\frac{2W_q^2}{9W} - \frac{W_{qq}}{3}\right)$, which gives the substitution

(89)
$$u = \frac{1}{\sqrt[6]{|W|}}\sqrt{\left|\frac{2W_q^2}{9W} - \frac{W_{qq}}{3}\right|}$$

in the coframe of theorem 2.11.

2. If $\mathbf{k}^c = 0$ and $\mathbf{e}^c \neq 0$, then we set $\mathbf{e}^c = 1$ and substitute

$$u = \frac{1}{3}F_{qq} + \frac{1}{W}\left(\frac{2}{9}W_q Z - \frac{2}{3}W_p - \frac{2}{9}W_q F_q\right).$$

3. If $\mathbf{k}^c = \mathbf{e}^c = 0$ and $\mathbf{h}^c \neq 0$, then we set $\mathbf{h}^c = 1$ and

$$u = \frac{1}{3\sqrt[3]{W}}\left(\left(\frac{1}{9}W_q Z^2 - \frac{1}{3}W_p Z + W_y - \frac{1}{3}W_q \mathcal{D}Z\right)\frac{1}{W} + \right.$$
$$\left. + \mathcal{D}Z_q + \frac{1}{3}F_q Z_q\right).$$

4. Finally, if $\mathbf{k}^c = \mathbf{e}^c = \mathbf{h}^c = 0$ and $\mathbf{b}^c \neq 0$, then we set $\mathbf{h}^c = 1$ and

(90)
$$u = \frac{1}{3\sqrt[3]{W^2}}\left(\frac{1}{27}F_{qq}Z^2 + \left(K_q - \frac{1}{3}Z_p - \frac{2}{9}F_q Z_q\right)Z\right.$$
$$\left. + \left(\frac{1}{3}\mathcal{D}Z - 2K\right)Z_q + Z_y + F_{qq}K - 3K_p - K_q F_q - F_{qy} + W_q\right).$$

In this manner we obtain

THEOREM 4.5. *Let $y''' = F(x, y, y', y'')$ be an ODE such that i) $W \neq 0$, ii) the functions $\mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c, \mathbf{k}^c$ in (87) do not vanish simultaneously. The contact invariant information on this ODE is given by the following Cartan coframe $(\theta^1, \theta^2, \theta^3, \theta^4)$ on $\mathcal{J}^2$:*

(91)
$$\theta^1 = u\sqrt[3]{W}\omega^1,$$
$$\theta^2 = u\left(\frac{1}{3}Z\omega^1 + \omega^2\right),$$
$$\theta^3 = \frac{u}{\sqrt[3]{W}}\left(\left(K + \frac{1}{18}Z^2\right)\omega^1 + \frac{1}{3}(Z - F_q)\omega^2 + \omega^3\right),$$
$$\theta^4 = \left(\frac{1}{9}\frac{W_q}{\sqrt[3]{W^2}}Z - \frac{1}{3}\sqrt[3]{W}Z_q\right)\omega^1 + \frac{W_q}{3\sqrt[3]{W^2}}\omega^2 + \sqrt[3]{W}\omega^4,$$

where u is given by eq. (89) – (90), depending on which functions among $\mathbf{b}^c, \mathbf{e}^c, \mathbf{h}^c, \mathbf{k}^c$ are non-zero.

The exterior derivatives of the above coframe read

$$
\begin{aligned}
d\theta^1 &= a\,\theta^1{\wedge}\theta^2 + \mathbf{I_1^c}\,\theta^1{\wedge}\theta^3 + \mathbf{I_2^c}\,\theta^1{\wedge}\theta^4 - \theta^2{\wedge}\theta^4,\\
d\theta^2 &= e\,\theta^1{\wedge}\theta^2 + \mathbf{I_3^c}\,\theta^1{\wedge}\theta^4 + \mathbf{I_1^c}\,\theta^2{\wedge}\theta^3 + \mathbf{I_2^c}\,\theta^2{\wedge}\theta^4 - \theta^3{\wedge}\theta^4,\\
d\theta^3 &= h\,\theta^1{\wedge}\theta^2 + k\,\theta^1{\wedge}\theta^3 - \theta^1{\wedge}\theta^4 + \mathbf{I_4^c}\,\theta^2{\wedge}\theta^3 + \mathbf{I_3^c}\,\theta^2{\wedge}\theta^4 + \mathbf{I_2^c}\,\theta^3{\wedge}\theta^4,\\
d\theta^4 &= m\,\theta^1{\wedge}\theta^2 + n\,\theta^1{\wedge}\theta^3 + s\,\theta^1{\wedge}\theta^4 + \epsilon_1\,\theta^2{\wedge}\theta^3 - (a + \mathbf{I_4^c})\,\theta^2{\wedge}\theta^4,
\end{aligned}
$$

(92)

where $\epsilon_1 = \pm 1, 0$ and $\mathbf{I_1^c}, \mathbf{I_2^c}, \mathbf{I_3^c}, \mathbf{I_4^c}, a, e, h, k, m, n, s$ are functions. The most important invariants read

$$
\epsilon_1 = \mathrm{sgn}\left(2W_q^2 - 3WW_{qq}\right),
$$

$$
\mathbf{I_1^c} = \frac{9W_{qqq}W^2 - 9W_{qq}W_qW + 4W_q^3}{2(2W_q^2 - 3WW_{qq})\sqrt{|2W_q^2 - 3WW_{qq}|}},
$$

$$
\begin{aligned}
\mathbf{I_2^c} = \frac{1}{6\sqrt[3]{W}(3W_{qq}W - 2W_q^2)}\Big(&(6WF_q - 6ZW)W_{qq} + 4ZW_q^2 - 4F_qW_q^2\\
&+ (3F_{qq}W - 12W_p - 6WZ_q)W_q + 18WW_{qp} - 9W^2Z_{qq} - 9W^2F_{qqq}\Big),
\end{aligned}
$$

$$
\mathbf{I_3^c} = \frac{1}{\sqrt[3]{(W^2}}\left(K - \frac{1}{3}\mathcal{D}Z + \frac{1}{18}Z^2 + \frac{1}{9}F_qZ\right),
$$

$$
\begin{aligned}
\mathbf{I_4^c} = \frac{1}{6\sqrt[3]{W}(2W_q^2 - WW_{qq})\sqrt{|2W_q^2 - WW_{qq}|}}\Big(&9F_qWW_{qqq} - 9ZWW_{qqq}\\
&+ (15F_qWW_q - 15ZWW_q - 27W_pW + 18WF_{qq} - 18WZ_q)W_{qq}\\
&+ 8F_qW_q^3 - 8ZW_q^3 + (6WZ_q + 24W_p - 9F_{qq}W)W_q^2\\
&- (9W^2Z_{qq} + 18WW_{qp} + 9W^2F_{qqq})W_q + 27W_{qqp}W^2\Big).
\end{aligned}
$$

1.2. Equations satisfying $W \neq 0$ and admitting large contact symmetry groups. Apart from the contact linearizable equations all the equations admitting contact symmetry group are characterized by the coframe (91). By virtue of theorem 4.2, an ODE associated with the coframe (91) admits a four-dimensional symmetry group if all its relative invariants in (92) are constant.

As we said, two ODEs associated with the coframe (91) and admitting a four-dimensional contact symmetry group are equivalent if and only if their respective invariants have the same constant value.

Below we describe our method of finding these equations. First we assume that the invariants $\mathbf{I_1^c}, \mathbf{I_2^c}, \mathbf{I_3^c}, \mathbf{I_4^c}$ are constant, which is a necessary condition for a large symmetry group. Then we close the system (92) and the identities $\mathrm{d}^2\theta^i = 0$ give us information about the remaining invariants, for example

$$
\begin{aligned}
\mathrm{d}^2\theta^1 = &(\mathbf{I_1^c}\mathbf{I_3^c} - \mathbf{I_2^c}\mathbf{I_4^c} + e + s)\theta^1{\wedge}\theta^4{\wedge}\theta^2 + (a\mathbf{I_1^c} + \mathbf{I_1^c}\mathbf{I_4^c} - \epsilon_1\mathbf{I_2^c} + n)\theta^1{\wedge}\theta^3{\wedge}\theta^2 +\\
&+ (a - \mathbf{I_1^c}\mathbf{I_2^c})\theta^1{\wedge}\theta^4{\wedge}\theta^3 + \mathrm{d}a{\wedge}\theta^1{\wedge}\theta^2 = 0.
\end{aligned}
$$

This equation yields $\mathrm{d}^2\theta^1{\wedge}\theta^2 = 0 = (a - \mathbf{I_1^c}\mathbf{I_2^c})\theta^1{\wedge}\theta^4{\wedge}\theta^3{\wedge}\theta^2$, hence $a = \mathbf{I_1^c}\mathbf{I_2^c} = const$ and we get that n and $e+s$ are also constants expressed by $\mathbf{I_j^c}$. Next $\mathrm{d}^2\theta^2{\wedge}\theta^2 = 0$ gives $e-k+s = 0$ and $\mathrm{d}^2\theta^3{\wedge}\theta^1 = 0$ gives $k = const$, so a, e, k, n and s are constant. Continuing this reasoning we also find that h and m are constant. As a consequence $\mathbf{I_1^c}, \mathbf{I_2^c}, \mathbf{I_3^c}, \mathbf{I_4^c} = const$ is the necessary and sufficient condition for an ODE to admit a large symmetry group.

In these circumstances the identities $\mathrm{d}^2\theta^i = 0$ become a system of quadratic algebraic equations for $\mathbf{I_1^c}, \dots, s$. We solve this system by the method of consecutive substitutions. Doing so we find that the system has no solutions if $\epsilon_1 = 0$. This fact

implies that the ODEs which we search exist only in the branch 1 above defined by the normalization (89). In the case $\epsilon_1 = \pm 1$ the system is underdetermined and we express all the invariants by ϵ_1 and $\mathbf{I}_1^c$, whose value is arbitrary except for $\mathbf{I}_1^c = 0, \pm 3/\sqrt{2}$.

$$\mathbf{I}_3^c = \frac{3(\mathbf{I}_2^c)^2}{8(\mathbf{I}_1^c)^2}(3\epsilon_1 - (\mathbf{I}_1^c)^2), \qquad\qquad \mathbf{I}_4^c = \frac{\mathbf{I}_2^c}{2\mathbf{I}_1^c}(3\epsilon_1 - 2(\mathbf{I}_1^c)^2),$$

$$a = \mathbf{I}_1^c \mathbf{I}_2^c, \qquad\qquad e = \frac{1}{8\mathbf{I}_1^c}(\mathbf{I}_2^c)^2(9\epsilon_1 - 5(\mathbf{I}_1^c)^2),$$

$$h = -\frac{3\epsilon_1}{16(\mathbf{I}_1^c)^3}(16(\mathbf{I}_1^c)^2 - 3(\mathbf{I}_2^c)^3(\mathbf{I}_1^c)^2 + 9(\mathbf{I}_2^c)^3\epsilon_1), \quad n = -\frac{1}{2}\epsilon_1 \mathbf{I}_2^c,$$

$$m = \frac{\epsilon_1(\mathbf{I}_2^c)^3}{8(\mathbf{I}_1^c)^2}((\mathbf{I}_2^c)^2 - 9\epsilon_1), \qquad\qquad k = \frac{5(\mathbf{I}_2^c)^2}{8\mathbf{I}_1^c}(3\epsilon_1 - (\mathbf{I}_1^c)^2),$$

$$s = -\frac{3\epsilon_1(\mathbf{I}_2^c)^2}{4\mathbf{I}_1^c},$$

and

$$\mathbf{I}_2^c = 2\sqrt[3]{\frac{(\mathbf{I}_1^c)^2}{(2(\mathbf{I}_1^c)^2 - 9\epsilon_1)}}.$$

Thereby we have obtained all the structural equations which *may be generated* by ODEs admitting large symmetry groups. However, it is unknown whether every admissible values of the invariants are realized by the ODEs. In order to find the equations we apply two approaches. The first approach is as follows. We choose some simple ODEs, for example $F = q^\alpha$ or $F = e^q$, calculate the equations (92) for them and check if they satisfy the large symmetry conditions. By this method we find the ODEs which realize some but not all the admissible values of ϵ_1 and $\mathbf{I}_1^c$. In order to find the remaining ODEs we use the fact that when all the invariants are constant then the coframe (θ^i) is a left-invariant coframe of a four-dimensional Lie group. This group is precisely the group of symmetry of the underlying ODE which acts on $\mathcal{J}^2$ and the frame dual to (θ^i) is a system of infinitesimal symmetry generators for the differential equation. We integrate the coframe (θ^i), that is we find the explicit formulae for the forms θ^i which satisfy the equations (92) with constant invariants $\mathbf{I}_1^c, \ldots, s$. Having found formulae for the symmetry generators we find their common invariant functions on $\mathcal{J}^2$ which are our desired ODEs. On integrating the coframe (θ^i) we used M. MacCallum's classification of real four-dimensional Lie algebras [41] to transform the frames into a simple canonical form, which simplified the calculations.

In this way we have found the full list of third-order ODEs satisfying the condition $W \neq 0$ and admitting at least four dimensional group of contact symmetries. We gather these equations in table 2 together with equations of the branch $W = 0$, which is analyzed below. Two equations of the same type given in the tables are equivalent if and only if the constants they involve are equal. For example $F = q^\mu$ and $F = q^\nu$ are equivalent provided that $\mu = \nu$. We give the necessary and sufficient conditions for any ODE to be contact equivalent to any member of the list. We also attach the description of the symmetry algebra. Our result agrees with the part of B. Doubrov and B. Komrakov's classification [15] referring to third-order ODEs, however we provide other canonical forms of the considered equations.

1.3. ODEs with $W = 0$ and $F_{qqqq} \neq 0$. We repeat the scheme explained above. This time we start from the coframe of theorem 2.1. The following substitutions

$$u_1 = u_3^2 \sqrt{\left|\frac{u_3}{F_{qqqq}}\right|}, \qquad\qquad u_2 = -3u_3 \frac{K_{qqq}}{F_{qqqq}},$$

$$u_4 = u_3 \sqrt{\left|\frac{u_3}{F_{qqqq}}\right|} \left(\frac{3K_{qqqq}}{F_{qqqq}} - \frac{12F_{qqqqq}K_{qqq}}{5F_{qqqq}^2}\right), \quad u_5 = -u_3 \sqrt{\left|\frac{u_3}{F_{qqqq}}\right|} \frac{F_{qqqqq}}{5F_{qqqq}},$$

$$u_6 = 0$$

reduce the bundle $\mathcal{P}^c$ to a five-dimensional subbundle. The last reduction is as follows.

1. If

(93)
$$2L_{qq}F_{qqqq} - 3K_{qqq}^2 \neq 0,$$

  then

(94)
$$u_3 = \sqrt[3]{9L_{qq} - \frac{27K_{qqq}^2}{2F_{qqqq}}}.$$

2. If $L_{qq}F_{qqqq} - 3K_{qqq}^2 = 0$ but

(95)
$$5F_{qqqqqq}F_{qqqq} - 6F_{qqqqq}^2 \neq 0,$$

  then

(96)
$$u_3 = \frac{25F_{qqqq}^3}{5F_{qqqqqq}F_{qqqq} - 6F_{qqqqq}^2}.$$

3. If $L_{qq}F_{qqqq} - 3K_{qqq}^2 = 0$ and $5F_{qqqqqq}F_{qqqq} - 6F_{qqqqq}^2 = 0$, but

(97)
$$\frac{1}{3}F_{qq} + \frac{1}{F_{qqqq}}\left(\frac{18}{5}K_{qqqq} + \frac{2}{5}F_{qqqqp} + \frac{2}{15}F_q F_{qqqqq}\right) - \frac{12F_{qqqqq}K_{qqq}}{5F_{qqqq}^2} \neq 0$$

  then

(98)
$$u_3 = \frac{1}{3}F_{qq} + \frac{1}{F_{qqqq}}\left(\frac{18}{5}K_{qqqq} + \frac{2}{5}F_{qqqqp} + \frac{2}{15}F_q F_{qqqqq}\right) - \frac{12F_{qqqqq}K_{qqq}}{5F_{qqqq}^2}.$$

REMARK 4.6. The three quantities defined in (93), (95), (97) cannot vanish simultaneously because it would be in contradiction to the condition $F_{qqqq} \neq 0$.

Thus we have the following

THEOREM 4.7. Let $y''' = F(x, y, y', y'')$ be an ODE satisfying $W = 0$ and $F_{qqqq} \neq 0$. The contact invariant information on the ODE is given by the following Cartan coframe $(\theta^1, \theta^2, \theta^3\theta^4)$ on $\mathcal{J}^2$:

$$\theta^1 = u^2 \sqrt{\left|\frac{u}{F_{qqqq}}\right|}\,\omega^1,$$

$$\theta^2 = u\left(-\frac{3K_{qqq}}{F_{qqqq}}\omega^1 + \omega^2\right),$$

$$\theta^3 = \sqrt{\left|\frac{F_{qqqq}}{u}\right|}\left(\left(K + \frac{9K_{qqq}^2}{2F_{qqqq}^2}\right)\omega^1 - \left(\frac{F_q}{3} + \frac{3K_{qqq}}{F_{qqqq}}\right)\omega^2 + \omega^3\right),$$

$$\theta^4 = u\sqrt{\left|\frac{u}{F_{qqqq}}\right|}\left(\left(\frac{3K_{qqq}}{F_{qqqq}} - \frac{12F_{qqqqq}K_{qqq}}{5F_{qqqq}^2}\right)\omega^1 - \frac{F_{qqqqq}}{5F_{qqqq}}\omega^2 + \omega^4\right),$$

where u is the function of x, y, p, q given by the formulae (93) – (98).

The exterior derivatives of the above forms are the following

$$d\theta^1 = a\,\theta^1 \wedge \theta^2 + \mathbf{I_5^c}\,\theta^1 \wedge \theta^3 + \mathbf{I_6^c}\,\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^4,$$

$$d\theta^2 = e\,\theta^1 \wedge \theta^2 + \epsilon_2\,\theta^1 \wedge \theta^4 + \frac{2}{5}\mathbf{I_5^c}\,\theta^2 \wedge \theta^3 + \frac{2}{5}\mathbf{I_6^c}\,\theta^2 \wedge \theta^4 - \theta^3 \wedge \theta^4,$$

$$d\theta^3 = f\,\theta^1 \wedge \theta^2 + g\,\theta^1 \wedge \theta^3 + \mathbf{I_7^c}\,\theta^2 \wedge \theta^3 + \epsilon_2^c\,\theta^2 \wedge \theta^4 - \frac{1}{5}\mathbf{I_6^c}\,\theta^3 \wedge \theta^4,$$

$$d\theta^4 = k\,\theta^1 \wedge \theta^2 + l\,\theta^1 \wedge \theta^3 + m\,\theta^1 \wedge \theta^4 + \mathbf{I_8^c}\,\theta^2 \wedge \theta^3 + s\,\theta^2 \wedge \theta^4 - \frac{3}{5}\mathbf{I_5^c}\,\theta^3 \wedge \theta^4,$$

where $\epsilon_2 = \pm 1, 0$ is defined through

$$\epsilon_2 = \mathrm{sgn}(2F_{qqqq}L_{qq} - 3K_{qqq}^3)$$

and $\mathbf{I_5^c}, \mathbf{I_6^c}, \mathbf{I_7^c}, \mathbf{I_8^c}, a, e, f, g, k, l, m, s$ are functions of x, y, p, q. We do not display these functions, since they are complicated. They can be immediately calculated from theorem 4.7. We apply the procedure of seeking the ODEs with four-dimensional symmetry group and insert the results in table 2 on pages 72 – 73.

Finally, we have the following geometric description of general linearizable ODEs.

PROPOSITION 4.8. *The only smooth third-order ODEs admitting large contact symmetry group acting intransitively on $\mathcal{J}^2$ are the equations contact equivalent to*

$$y''' = -2\mu(x)y' + (1 - \mu'(x))y,$$

with an arbitrary smooth function $\mu(x) \neq const.$

PROOF. We showed in corollary 4.4 that the above equations admit a 4-dimensional contact symmetry group acting on 3-dimensional orbits in $\mathcal{J}^2$. All other ODEs admitting a large contact symmetry group G possess Cartan coframes of rank zero. These coframes were explicitly constructed in theorem 2.1 for the case $W = 0$, $F_{qqqq} = 0$, in theorems 2.11 and 4.5 for the case $W \neq 0$ and in theorem 4.7 for the case $W = 0$, $F_{qqqq} \neq 0$. The coframe for the case $W = 0$, $F_{qqqq} = 0$ generates on $\mathcal{J}^2$ local structure of the homogeneous space $SP(4, \mathbb{R})/H_6$. The coframes of theorems 4.5 and 4.7 equip $\mathcal{J}^2$ with local structure of G. In both these cases the action of G on $\mathcal{J}^2$ is transitive. $\qquad\square$

2. Equations with large point symmetry group

Given the detailed description of the ODEs with large contact symmetry groups it is already easy to find the equations admitting large point symmetry groups. It follows from the fact that any equation possessing point symmetries has at least the same number of contact symmetries ('number of symmetries' means the dimension of the symmetry group here.) Therefore the equations with large point symmetry groups lie entirely within the classes with large contact symmetry groups. As a consequence, we must only do the full reduction for the equations contact equivalent to those in table 2 and analyze existence of point symmetries. The procedure of reduction for point transformations is fully analogous to the contact case. We have the following main branches

    i) Linear and point linearizable equations equivalent to $y''' = 0$ with the 7-dimensional algebra $\mathfrak{co}(2, 1) \oplus \mathbb{R}^3$ of point symmetries.

    ii) Non-linearizable equations admitting the 6-dimensional algebras $\mathfrak{o}(2, 2)$ or $\mathfrak{o}(4)$ of point symmetries. They are equivalent to $y''' = \frac{3y''^2}{2y'}$ or $y''' = \frac{3y'y''^2}{y'^2+1}$ respectively. These classes are new when compared to the contact classification.

iii) The linear and point linearizable equations which satisfy $W \neq 0$. They are equivalent to $y'' = -2\mu(x)y' + (1 - \mu'(x))y$ and have a 5-dimensional or a 4-dimensional group of symmetries.

iv) All the remaining equations satisfying $W \neq 0$.

v) All the remaining equations satisfying $W = 0$.

Branches i) to iii) are relatively simple to characterize in terms of point invariants. i) has been described in corollary 3.2. ii), which is contact equivalent to $y''' = 0$, has been discussed in section 5 of chapter 3, and description of iii) is based on the reduction to a five-dimensional coframe, as in section 1.2. After

$$u_1 = \sqrt[3]{W}u_3, \qquad u_2 = \tfrac{1}{3}Zu_3$$

we obtain a five dimensional coframe $(\theta^1, \ldots, \theta^4, \Omega)$ with the following basic invariants

$$\mathbf{a}^c = \frac{1}{\sqrt[3]{W^2}}\left(K + \frac{1}{18}Z^2 + \frac{1}{9}ZF_q - \frac{1}{3}\mathcal{D}Z\right),$$

$$\mathbf{b}^p = \frac{1}{3u\sqrt[3]{W^2}}\left(\left(\frac{1}{12}F_{qq} + \frac{1}{18}Z_q\right)Z^2 + \left(K_q - \frac{1}{3}Z_p - \frac{1}{9}F_qZ_q + \frac{1}{18}F_{qq}F_q\right)Z + \right.$$
$$\left. - \frac{1}{6}F_{qq}\mathcal{D}Z - KZ_q + Z_y + \frac{3}{2}F_{qq}K - 3K_p - K_qF_q - F_{qy}\right),$$

$$\mathbf{e}^p = \frac{1}{u}\left(\frac{1}{6}F_{qq} - \frac{1}{3}Z_q + \frac{1}{W}\left(\frac{2}{9}W_qZ - \frac{2}{3}W_p - \frac{2}{9}W_qF_q\right)\right),$$

$$\mathbf{h}^p = \frac{1}{3u\sqrt[3]{W}}\left(\left(\frac{1}{18}W_qZ^2 - \left(\frac{1}{3}W_p + \frac{1}{9}W_qF_q\right)Z + W_y - W_qK\right)\frac{1}{W} + \right.$$
$$\left. - 3K_q - \frac{1}{3}F_{qq}F_q - F_{qp}\right),$$

$$\mathbf{k}^p = \frac{1}{3}\frac{W_q}{\sqrt[3]{W^2}u}.$$

The linearizable equations are described by conditions expressed in terms of these invariants, as given in table 3.

The branches iv) and v) need more thorough study. We consider iv) first. We know from section 1.2 that any equation with large point symmetry group in this branch satisfies necessarily $3WW_{qq} - 2W_q^2 \neq 0$. It implies $W_q \neq 0$, which allows us to reduce the last free group parameter via

$$u_3 = \frac{1}{3}\frac{W_q}{\sqrt[3]{W^2}}.$$

The set of basic invariants is the following

$$\mathbf{I_1^p} = -3\frac{W_{qq}W}{W_q^2},$$

$$\mathbf{I_2^p} = \frac{1}{\sqrt[3]{W}W_q}\left(3W_p + W_qF_q - W_qZ - 3WZ_q - 3F_{qq}W\right),$$

$$\mathbf{I_3^p} = -\frac{\sqrt[3]{W^2}}{2W_q}\left(2Z_q + F_{qq}\right),$$

$$\mathbf{I_4^p} = \frac{1}{18\sqrt[3]{W^2}}\left(Z^2 - 6\mathcal{D}Z + 18K + 2ZF_q\right),$$

and the above invariants are (up to constant numbers) the T_{13}^1, T_{14}^1, T_{13}^2, and T_{14}^2 coefficients in the structural equations

$$\mathrm{d}\theta^i = \frac{1}{2}T_{jk}^i\theta^j \wedge \theta^k, \qquad T_{jk}^i = -T_{kj}^i.$$

for the coframe. The ODEs with large point symmetry groups which fall into this branch are types II.2, II.3, IV, and VI of table 3.

We turn to the branch v). The coframe is given either by

$$u_1 = -\frac{3F_{qqq}^5}{4F_{qqqq}^3}, \qquad u_2 = \frac{F_{qqq}^2 N}{2F_{qqqq}}, \qquad u_3 = \frac{F_{qqq}^2}{2F_{qqqq}},$$

provided that $F_{qqqq} \neq 0$ or, if $F_{qqqq} = 0$ but $F_{qqq} \neq 0$, by

$$u_1 = -\frac{1}{36F_{qqq}^4}\left(6F_{qqqp} + 5F_{qqq}F_{qq}\right)^3,$$

$$u_2 = -\frac{1}{6F_{qqq}^2}\left(6F_{qqqp} + 5F_{qqq}F_{qq}\right)N,$$

$$u_3 = -\frac{1}{6F_{qqq}}\left(6F_{qqqp} + 5F_{qqq}F_{qq}\right),$$

where

$$N = F_{qqp} + \frac{1}{6}F_{qq}^2 + \frac{1}{3}F_{qqq}F_q.$$

For $F_{qqqq} \neq 0$ we have the following basic invariants

$$\mathbf{I}_5^p = \frac{F_{qqq}F_{qqqqq}}{F_{qqqq}^2},$$

$$\mathbf{I}_6^p = \frac{F_{qqqq}}{F_{qqq}^4}\left(\frac{8}{3}F_{qqqq} - 12F_{qqq}K_{qqq} + \frac{5}{9}F_{qqqq}F_{qq}^2 + 20F_{qqqq}K_{qq}\right),$$

$$\mathbf{I}_7^p = \frac{F_{qqqq}}{F_{qqq}^4}\left(6N_q F_{qqq} - 6N F_{qqqq} + F_{qq}F_{qqq}^2\right)$$

$$\mathbf{I}_8^p = -\frac{2}{27}\frac{F_{qqqq}^4}{F_{qqq}^8}\left(4N F_q F_{qqq} + 6\mathcal{D}N F_{qqq} + \right.$$
$$\left. - 9N^2 - F_{qq}^2 N - 36K_{qq}N - 6F_{qqq}^2 K\right),$$

which are obtained from the coefficients $T_{13}^1, T_{14}^1, T_{13}^2$ and T_{14}^2 in the structural equations.

For the branch $F_{qqqq} = 0$, $F_{qqq} \neq 0$, which contains only one class of equations with large point symmetry groups, the equations equivalent to $F = q^3$, we have the invariants

$$\mathbf{I}_9^p = T_{12}^1, \qquad \mathbf{I}_{10}^p = T_{14}^1, \qquad \mathbf{I}_{11}^p = T_{14}^2.$$

Properties of the ODEs admitting a large point symmetry group are given in table 3 on pages 74 – 75. In the point classification we also have a counterpart of proposition 4.8.

REMARK 4.9. The fibre-preserving classification of the ODEs admitting a large symmetry group is parallel to the point classification and has been already done [**30, 33**]. The main difference is that types I.3, II.3. and IX do not admit four-dimensional fibre-preserving symmetry groups.

3. Fibre-preserving equivalence to certain reduced Chazy equations

An ordinary differential equation in the *complex* domain is said to have the Painlevé property if its general solution does not have movable branch points, that is branch points whose location depends on integration constants, [**12**]. The problem of classifying the third-order Painlevé ODEs which are polynomials in y, y', and y'' and are locally analytic in x was studied by J. Chazy [**11**], who considered the polynomial equations modulo the following transformations

$$x \to \chi(x), \qquad y \to \alpha(x)y + \beta(x),$$

which are a subclass of complex analytic fibre-preserving transformations. J. Chazy found thirteen classes of the ODEs satisfying the Painleve property. Each of these classes has a particularly simple representative – the reduced Chazy class – obtained by a certain limit procedure. Here we are interested in reduced Chazy classes II, IV, V, VI, VII and XI $\sigma \neq 11$. They are as follows

$$II: \qquad F = -2yq - 2p^2,$$
$$IV: \qquad F = -3yq - 3p^2 - 3y^2p,$$
$$V: \qquad F = -2yq - 4p^2 - 2y^2p,$$
$$VI: \qquad F = -yq - 5p^2 - y^2p,$$
$$VII: \qquad F = -yq - 2p^2 + 2y^2p,$$
$$XI: \qquad F = -2yq - 2p^2 + \frac{24}{\sigma^2 - 1}\left(p + y^2\right)^2, \qquad \sigma \in \mathbf{N},\ \sigma \neq 1, 6k.$$

All of them have the form

$$(99) \qquad\qquad F = \kappa yq + \lambda p^2 + \mu y^2 p + \nu y^4,$$

with some constant numbers κ, λ, μ, ν. We exclude type XI for $\sigma = 11$ for technical reasons.

We aim to find necessary and sufficient conditions for a *regular real* third order ODE to be fibre-preserving equivalent to one of the above equations. We find the Cartan coframe for such equations and the explicit formulae for the fibre-preserving invariants. Next we find the functional relations between the invariants, which allows us in to describe the classifying function $\mathbf{T}$ explicitly and consequently describe its image.

First step is calculating the structural equations for the fibre-preserving coframe of theorem 3.5 for the Chazy types. They are as follows

$$d\theta^1 = \Omega_1 \wedge \theta^1 + \theta^4 \wedge \theta^2,$$
$$d\theta^2 = \Omega_2 \wedge \theta^1 + \Omega_3 \wedge \theta^2 + \theta^4 \wedge \theta^3,$$
$$d\theta^3 = \Omega_2 \wedge \theta^2 + (2\Omega_3 - \Omega_1) \wedge \theta^3 + \mathbf{A}_1^f \theta^4 \wedge \theta^1,$$
$$d\theta^4 = (\Omega_1 - \Omega_3) \wedge \theta^4,$$
$$d\Omega_1 = -\Omega_2 \wedge \theta^4 + (2\mathbf{C}_1^f - \mathbf{A}_2^f)\theta^1 \wedge \theta^4,$$
$$d\Omega_2 = (\Omega_3 - \Omega_1) \wedge \Omega_2 + \mathbf{A}_3^f \theta^1 \wedge \theta^4 + (\mathbf{C}_1^f - \mathbf{A}_2^f)\theta^2 \wedge \theta^4,$$
$$d\Omega_3 = (\mathbf{C}_1^f - \mathbf{A}_2^f)\theta^1 \wedge \theta^4.$$

In this system the functions $\mathbf{B}_i^p$, $i = 1 \ldots 6$, $\mathbf{D}_1^p$, and $\mathbf{D}_2^p$ vanish, which is equivalent to the following three fibre-preserving invariant conditions

$$(100) \qquad F_{qq} = 0, \qquad F_{qpp} = 0, \qquad F_{ppp} = 2F_{qpy} - \frac{2}{3}F_{qp}^2.$$

The most important non-vanishing invariants read

$$\mathbf{A}_2^f = \frac{1}{u_1^3}\left(\left(\lambda - \frac{7\kappa}{6}\right)u_2 u_3 + \left(\mu + \frac{\kappa\lambda}{3} + \frac{\kappa^2}{18}\right)yu_3^2\right),$$
$$\mathbf{C}_1^f - \mathbf{A}_2^f = \frac{\kappa u_3}{3u_1^2},$$
$$X_4(\mathbf{C}_1^f - \mathbf{A}_2^f) = -\left(\frac{2}{3}\kappa u_2 u_3 + \frac{1}{9}\kappa^2 yu_3^2\right)\frac{1}{u_1^3}.$$

For the Chazy classes one can reduce the parameters through

$$(101) \qquad\qquad \mathbf{A}_2^f = 1, \quad \mathbf{C}_1^f - \mathbf{A}_2^f = \frac{1}{3}, \quad X_4(\mathbf{C}_1^f - \mathbf{A}_2^f) = 0.$$

This leads to the four-dimensional system

$$
\begin{aligned}
(102) \quad
d\theta^1 &= \mathbf{a}\,\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^4, \\
d\theta^2 &= -2\tau\,\theta^1 \wedge \theta^2 + \mathbf{b}\,\theta^1 \wedge \theta^4 + 2\mathbf{a}\,\theta^2 \wedge \theta^4 - \theta^3 \wedge \theta^4, \\
d\theta^3 &= \left(\tfrac{\lambda}{\kappa} - \tfrac{2}{3}\right)\theta^1 \wedge \theta^2 - 3\tau\,\theta^1 \wedge \theta^3 + \mathbf{c} \in \theta^1 \wedge \theta^4 + \mathbf{b}\,\theta^2 \wedge \theta^4 + 3\mathbf{a}\,\theta^3 \wedge \theta^4, \\
d\theta^4 &= \tau\,\theta^1 \wedge \theta^4,
\end{aligned}
$$

where

$$
\tau = \frac{\mu}{\kappa^2} + \frac{\lambda}{6\kappa} + \frac{1}{4},
$$

and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are functions on $\mathcal{J}^2$. We check by direct calculations that in this case the coframe is of order one and all the invariants are generated by $\mathbf{a}$ and $\mathbf{a}_4 = X_4(\mathbf{a})$ by the following formulae

$$
\begin{aligned}
\mathbf{b} =&\, \frac{1}{\tau}\left(\frac{1}{3} - \frac{\lambda}{\kappa}\right)\mathbf{a} - \frac{1}{2\tau} + \frac{1}{12\tau^2}\left(\frac{1}{3} - \frac{\lambda}{\kappa}\right), \\
\mathbf{c} =&\, \frac{1}{\tau}\left(\frac{\lambda}{\kappa} - \frac{7}{6}\right)\mathbf{a}_4 + \frac{1}{2\tau}\left(\frac{\lambda}{\kappa} - \frac{7}{6}\right)\mathbf{a}^2 + \left(-\frac{1}{\tau} + \frac{1}{6\tau^2}\left(\frac{\lambda}{\kappa} - \frac{7}{6}\right)\right)\mathbf{a} - \frac{1}{2\tau^2} \\
&+ \frac{1}{36\tau^3}\left(\frac{\lambda}{\kappa} - \frac{144\nu}{\kappa^3} + \frac{3}{2}\right),
\end{aligned}
$$

$$
(103) \quad \mathbf{a}_1 = -2\tau\mathbf{a} - \frac{1}{6}, \qquad \mathbf{a}_2 = \tau, \qquad \mathbf{a}_3 = 0,
$$

$$
\mathbf{a}_{41} = -3\tau\mathbf{a}_4 + 2\tau\mathbf{a}^2 + \left(\frac{\lambda}{\kappa} - \frac{1}{6}\right)\mathbf{a} + \frac{1}{12\tau}\left(\frac{\lambda}{\kappa} - \frac{1}{3}\right) + \frac{1}{2},
$$

$$
\mathbf{a}_{42} = -4\tau\mathbf{a} - \frac{1}{6}, \qquad \mathbf{a}_{43} = \tau,
$$

$$
\begin{aligned}
\mathbf{a}_{44} =&\, -7\mathbf{a}_4\mathbf{a} - \frac{\mathbf{a}_4}{6\tau} - 6\mathbf{a}^3 + \frac{1}{\tau}\left(\frac{\lambda}{\kappa} - 1\right)\mathbf{a}^2 + \left(\frac{1}{\tau} + \frac{1}{6\tau^2}\left(\frac{\lambda}{\kappa} - \frac{1}{2}\right)\right)\mathbf{a} \\
&+ \frac{1}{6\tau^2} + \frac{1}{\tau^3}\left(\frac{\nu}{\kappa^3} - \frac{1}{72}\right),
\end{aligned}
$$

where, as usual, $\mathbf{a}_i = X_i(\mathbf{a})$, $\mathbf{a}_{ij} = X_j(X_i(\mathbf{a}))$. These algebraic formulae, when differentiated, enable to express all other derivatives of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in terms of $\mathbf{a}$ and $\mathbf{a}_4$. In order to do this we only must consecutively substitute the coframe derivatives of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ with the right hand side of (103), for instance

$$
\mathbf{a}_{11} = -2\tau\mathbf{a}_1 = 4\tau^2\mathbf{a} + \frac{\tau}{3}, \qquad\qquad \mathbf{a}_{12} = -2\mathbf{a}_2 = -2\tau^2,
$$

etc. Thereby the non-constant components of the classifying function $\mathbf{T}\colon \mathcal{J}^2 \to \mathbb{R}^N$ are completely characterized by

$$
(x, y, p, q) \mapsto (\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_{41}, \mathbf{a}_{42}, \mathbf{a}_{43}, \mathbf{a}_{44}).
$$

The graph of this function in $\mathbb{R}^{11}$ is parameterized by $\mathbf{a}$ and $\mathbf{a}_4$.

Let us consider an arbitrary third-order ODE. It is locally fibre-preserving equivalent to one of the Chazy classes if and only if graphs of respective classifying functions overlap. This is only possible if i) conditions (100) are satisfied, ii) the reduction defined by the conditions (101) is possible, and iii) after the reduction the equations (102) and (103) hold. The reduction (101) is possible iff

$$
\begin{aligned}
(104) \qquad
P =&\, \mathcal{D}F_{qp} - F_{qy} \neq 0, \\
Q =&\, 2W_p - \mathcal{D}W_q + F_q W_q \neq 0.
\end{aligned}
$$

After the reduction to dimension four given by

$$u_1 = \frac{2P^2}{Q}, \qquad u_3 = \frac{2F_q P^3}{3Q} - \frac{2P^2 \mathcal{D}P}{Q}, \qquad u_3 = -\frac{4P^3}{Q}$$

we get the frame

$$X_1 = \frac{Q}{P^2}\left(\frac{Q}{4W_q} - \frac{F_q}{6}\right)\partial_p + \frac{Q}{P^2}\left(\frac{Q^2}{16W_q^2} - \frac{F_q^2}{36} - \frac{K}{2}\right)\partial_q,$$

(105)
$$X_2 = -\frac{Q^2}{4P^3}\partial_p - \frac{Q^3}{8P^3 W_q}\partial_q,$$

$$X_3 = \frac{Q^3}{8P^4}\partial_q,$$

$$X_4 = -\frac{2P}{Q}\mathcal{D}.$$

Equations (102) are satisfied if and only if

$$2W_{pp} - W_{qy} + F_{qp}W_q = 0,$$
$$W_q \mathcal{D}P - P\mathcal{D}W_q = 0,$$
$$P_y + \frac{1}{3}PF_{qp} = 0,$$

(106)
$$Q_y + \frac{1}{3}QF_{qp} - 2\tau P^2 = 0,$$

$$K_p + \frac{1}{2}F_{qy} - \frac{5}{36}F_q F_{qp} + \frac{1}{W_q}\left(F_{qp}\mathcal{D}W_q - \frac{1}{12}F_q W_{qy} + \frac{1}{2}\mathcal{D}W_{qy}\right)$$

$$- \frac{3W_{qy}\mathcal{D}W_q}{4W_q^2} + \left(\frac{2}{3} - \frac{\lambda}{\kappa}\right)P = 0.$$

Finally, equations (103) must be satisfied by functions

$$\mathbf{a} = \frac{P}{W_q} + \frac{1}{Q}\left(4\mathcal{D}P - \frac{2}{3}F_q P - \frac{2PW_p}{W_q}\right) - \frac{2P\mathcal{D}Q}{Q^2},$$

$$\mathbf{b} = \left\{\frac{5}{2W_q^2} + \frac{1}{Q}\left(-\frac{10F_q}{3W_q} - \frac{10W_p}{W_q^2}\right) + \frac{1}{Q^2}\left(-4F_p - 4K - \frac{2}{9}F_q^2 + \right.\right.$$

(107)
$$\left.\left. + \frac{1}{W_q}\left(\frac{20}{3}W_p F_q - 4\mathcal{D}W_p + 2\mathcal{D}Q\right) + \frac{10W_p^2}{W_q^2}\right)\right\}P^2,$$

$$\mathbf{c} = \frac{8P^3 W}{Q^3},$$

and by their derivatives $\mathbf{a}_4, \mathbf{a}_{41},\ldots,\mathbf{a}_{44}$ with respect to the frame $X_1,\ldots,X_4$. Therefore, by means of theorem 4.1 we have

PROPOSITION 4.10. *An ODE is locally fibre-preserving equivalent to one of the Chazy classes II, IV, V, VI, VII or XI for $\sigma \neq 11$ in a neighbourhood of a point $j_0 \in \mathcal{J}^2$ if and only if i) the ODE satisfies the conditions (100), (104), (106), and (103) with the invariants $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{a}_4$, and $\mathbf{a}_{41},\ldots,\mathbf{a}_{44}$ given by (105) and (107), ii) the values of $\mathbf{a}(j_0)$ and $\mathbf{a}_4(j_0)$ for the ODE and the Chazy class coincide.*

Given these criteria for the equivalence it is interesting to find a transformation of variables transforming an ODE

$$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = F\left(x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)$$

to its equivalent Chazy class

$$\frac{\mathrm{d}^3\bar{y}}{\mathrm{d}\bar{x}^3} = \kappa\bar{y}\frac{\mathrm{d}^2\bar{y}}{\mathrm{d}\bar{x}^2} + \lambda\left(\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}\right)^2 + \mu\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}\bar{y}^2 + \nu\bar{y}^4.$$

We show that the transformation

$$\bar{y} = \phi(x,y), \qquad \bar{x} = \chi(x)$$

may be easily found.

Let us apply to a Chazy class the above (arbitrary) fibre-preserving transformation and calculate for so obtained ODE (which is a general ODE equivalent to the Chazy classes) the quantities P, Q and $F_q - pF_{qp}$:

(108)
$$P = -\kappa\chi_x\phi_y,$$
$$Q = 2\kappa^2\tau\,\chi_x^2\,\phi_y\,\phi,$$
$$F_q - pF_{qp} = \kappa\chi_x\phi + 3\frac{\chi_{xx}}{\chi_x} - 3\frac{\phi_{xy}}{\phi_y}.$$

From first and second equation we get

$$(\log|\phi|)_y = 2\tau\frac{P^2}{Q},$$
$$\chi_x = -\frac{Q}{2\tau P\phi}.$$

Putting this into third equation of (108) we obtain

$$(\log|\phi|)_x = \frac{1}{2}\left(\log\left|\frac{Q^2}{P^3}\right|\right)_x - \frac{\kappa Q}{12\tau P} + \frac{1}{6}(pF_{qp} - F_q).$$

Finally, after integration of χ_x and ϕ_x we have

(109)
$$\bar{y} = \frac{c_1 Q}{|P|^{\frac{3}{2}}}\exp\left\{\int_{x_0}^x\left(-\frac{\kappa Q}{12\tau P} + \frac{1}{6}(pF_{qp} - F_q)\right)dx + 2\tau\int_{y_0}^y\frac{P^2}{Q}\bigg|_{x=x_0}dy\right\},$$
$$\bar{x} = \frac{1}{2\tau}\int_{x_0}^x\frac{Q}{P\phi}dx + c_2.$$

We summarize this calculation as follows.

PROPOSITION 4.11. *If there exists a fibre-preserving transformation from an equation $y''' = F(x,y,y',y'')$ to a reduced Chazy type II, IV – VII or XI $\sigma \neq 11$, then it is given by the inverse of eq. (109), where P and Q are calculated for $y''' = F(x,y,y',y'')$ according to formulae (104).*

TABLE 2. Equations admitting large contact symmetry groups

	Equation		Characterization				Symmetry algebra $\mathfrak{g}$	dim $\mathfrak{g}$
I	$F = 0$		$W = 0,$ $\qquad F_{qqqq} = 0$				$\mathfrak{o}(3,2)$	10
II	$F = -2\mu p + y$	$\mu \in \mathbb{R}$	$W \neq 0,$ $\quad \mathbf{a}^c = \mu,$ $\quad \mathbf{k}^c = 0$				$[V_1, V_4] = -\mu V_2 + V_3,$ $[V_1, V_5] = V_1,$ $[V_2, V_4] = V_1 - \mu V_3,$ $[V_2, V_5] = V_2,$ $[V_3, V_4] = V_2,$ $[V_3, V_5] = V_3,$	5
III	$F = -2\mu(x)p + (1 - \mu'(x))y$		$W \neq 0,$ $\quad \mathbf{a}^c = \mu(x),$ $\quad \mathbf{b}^c = \mathbf{e}^c = \mathbf{h}^c = \mathbf{k}^c = 0$				$[V_1, V_4] = V_1,$ $[V_2, V_4] = V_2,$ $[V_3, V_4] = V_3,$	
IV	$F = q^{3/2 + 1/(2\sqrt{\mu})}$	$\mu > 0, \neq \frac{1}{9}$	$0 < 1 + 4\epsilon_{\mathbf{1}}(\mathbf{I_1^c})^{-2} \neq \frac{1}{9}$	$\mu = 1 + 4\epsilon_{\mathbf{1}}(\mathbf{I_1^c})^{-2}$	$\mathbf{I_1^c}, \mathbf{I_2^c}, \mathbf{I_3^c}, \mathbf{I_4^c}$	$W \neq 0,$ $\epsilon_{\mathbf{1}} \neq 0,$ $= const$	$[V_1, V_4] = 2V_1,$ $[V_2, V_4] = (1 + \sqrt{\mu})V_2,$ $[V_2, V_3] = V_1,$ $[V_3, V_4] = (1 - \sqrt{\mu})V_3$	4
V	$F = (q^2 + 1)^{\frac{3}{2}} \exp\left(\frac{\arctan q}{\sqrt{\mu}}\right)$	$\mu > 0$	$\epsilon_{\mathbf{1}} = -1, 1 - 4(\mathbf{I_1^c})^{-2} < 0$	$\mu = 4(\mathbf{I_1^c})^{-2} - 1$			$[V_1, V_4] = 2V_1,$ $[V_2, V_4] = V_2 - \sqrt{\mu}V_3$ $[V_2, V_3] = V_1,$ $[V_3, V_4] = \sqrt{\mu}V_2 + V_3$	
VI	$F = \exp q$		$\epsilon_{\mathbf{1}} = -1, \mathbf{I_1^c} = -2$				$[V_1, V_4] = 2V_1,$ $[V_2, V_4] = V_2 + V_3$ $[V_2, V_3] = V_1,$ $[V_3, V_4] = V_3$	

Table 2. Equations admitting large contact symmetry groups

	Equation		Characterization			Symmetry algebra $\mathfrak{g}$	dim $\mathfrak{g}$
VII	$F = \mu\left(\frac{q^2}{1-p^2} - p^2 + 1\right)^{3/2} - 3\frac{q^2 p}{1-p^2} + p^3 - p^2$	$\mu > 0$	$\epsilon_{\mathbf{2}} = 1,$ $0 < \mathbf{I^c_7} < \frac{\sqrt[3]{6}}{3},$	$\mu = \sqrt{\left\|\frac{9(\mathbf{I^c_7})^3}{9(\mathbf{I^c_7})^3 - 2}\right\|}$	$W = 0,\ F_{qqqq} \neq 0,\ \mathbf{I^c_5}, \mathbf{I^c_6}, \mathbf{I^c_7}, \mathbf{I^c_8} = const$	$\mathfrak{u}(2)$	4
VIII	$F = \mu\frac{(2qy - p^2)^{3/2}}{y^2}$	$0 < \mu < 1$	$\epsilon_{\mathbf{2}} = 1,\ \mathbf{I^c_7} < 0$				
		$\mu > 1$	$\epsilon_{\mathbf{2}} = -1,\ \mathbf{I^c_7} > \frac{\sqrt[3]{6}}{3}$			$\mathfrak{gl}(2, \mathbb{R})$	
		$\mu = 1$	$\epsilon_{\mathbf{2}} = \mathbf{I^c_7} = 0,\ \mathbf{I^c_8} = 1$				
IX	$F = 4\mu(q - p^2)^{3/2} + 6qp - 4p^3$	$\mu > 0$	$\epsilon_{\mathbf{2}} = -1,\ 0 < \mathbf{I^c_7} < \frac{\sqrt[3]{6}}{3}$	$\mu = \sqrt{\left\|\frac{9(\mathbf{I^c_7})^3}{9(\mathbf{I^c_7})^3 - 2}\right\|}$			
X	$F = \mu\left(\frac{q^2}{p^2} + p^2\right)^{3/2} + 3\frac{q^2}{p} + p^3$	$0 < \mu < 1$	$\epsilon_{\mathbf{2}} = -1,\ \mathbf{I^c_7} < 0$				
		$\mu > 1$	$\epsilon_{\mathbf{2}} = 1,\ \mathbf{I^c_7} > \frac{\sqrt[3]{6}}{3}$				
		$\mu = 1$	$\epsilon_{\mathbf{2}} = \mathbf{I^c_7} = 0,\ \mathbf{I^c_8} = -1$				
XI	$F = (q^2 + 1)^{3/2}$		$\epsilon_{\mathbf{2}} = 1,\ \mathbf{I^c_7} = \frac{\sqrt[3]{6}}{3}$			$[V_2, V_4] = -V_3,\ [V_2, V_3] = V_1,$ $[V_3, V_4] = V_2$	
XII	$F = q^{3/2}$		$\epsilon_{\mathbf{2}} = -1,\ \mathbf{I^c_7} = \frac{\sqrt[3]{6}}{3}$			$[V_2, V_4] = V_2,\ [V_2, V_3] = V_1,$ $[V_3, V_4] = -V_3$	

Table 3. Equations admitting large point symmetry groups

	Equation		Characterization			Symmetry algebra $\mathfrak{g}$	dim $\mathfrak{g}$
I.1	$F = 0$		$F_{qq}^2 + 6F_{qqp} = 0$	$F_{qq}K + \frac{1}{3}F_q F_{qp} - F_{qy} + \frac{1}{2}F_{pp} = 0$ $W = 0, \qquad F_{qqq} = 0$		$\mathfrak{co}(2,1) \oplus_. \mathbb{R}^3$	7
I.2	$F = \frac{3}{2}\frac{q^2}{p}$		$F_{qq}^2 + 6F_{qqp} < 0$			$\mathfrak{o}(2,2)$	6
I.3	$F = \frac{3q^2 p}{1+p^2}$		$F_{qq}^2 + 6F_{qqp} > 0$			$\mathfrak{o}(4)$	6
I.4	$F = q^3$		$W = F_{qqqq} = 0, \quad \mathbf{I}_9^p = -\frac{2}{5}, \quad \mathbf{I}_{10}^p = \frac{1}{25}, \quad \mathbf{I}_{11}^p = 0$			$[V_1, V_4] = 2V_1, [V_2, V_4] = \frac{4}{3}V_2,$ $[V_2, V_3] = V_1, [V_3, V_4] = -\frac{2}{3}V_3$	4
II.1	$F = -2\mu p + y$	$\mu \in \mathbb{R}$	$W \neq 0, \quad \mathbf{a}^c = \mu, \quad \mathbf{e}^p = \mathbf{k}^p = 0$			as in table 2	5
II.2	$F = \mu\frac{q^2}{p}$	$\mu > \frac{3}{2}, \neq 3$	$\mathbf{I}_2^p \notin [0, \sqrt[3]{4}]$	$\mathbf{I}_2^p = \sqrt[3]{\frac{(2\mu-3)^2}{\mu(\mu-3)}}$	$W \neq 0, \mathbf{I}_1^p = -2$ $\mathbf{I}_2^p, \mathbf{I}_3^p, \mathbf{I}_4^p = const$	$[V_1, V_2] = V_1, [V_3, V_4] = V_3$	4
II.3	$F = \frac{3p+\mu}{p^2+1}q^2$	$\mu > 0$	$\mathbf{I}_2^p \in (0, \sqrt[3]{4})$	$\mathbf{I}_2^p = \sqrt[3]{\frac{4\mu^2}{\mu^2+9}}$		$[V_1, V_2] = V_3, [V_3, V_1] = V_2$ $[V_3, V_4] = V_3, [V_2, V_4] = V_2$	4

TABLE 3. Equations admitting large point symmetry groups

	Equation		Characterization			Symmetry algebra $\mathfrak{g}$	dim $\mathfrak{g}$
III	$F = -2\mu(x)p + (1 - \mu'(x))y$		$W \neq 0, \quad \mathbf{a}^c = \mu(x), \quad \mathbf{b}^p = \mathbf{e}^p = \mathbf{h}^p = \mathbf{k}^p = 0$			as in table 2	4
IV	$F = q^\mu$	$\mu \neq 0, 1, \frac{3}{2}, 3$	$\mathbf{I_1^p} \neq -3$	$\mathbf{I_1^p} = \frac{4-3\mu}{\mu-1}$	$W \neq 0$		
VI	$F = \exp q$		$\mathbf{I_1^p} = -3$		$\mathbf{I_1^p}, \mathbf{I_2^p}, \mathbf{I_3^p}, \mathbf{I_4^p} = const$		
VIII	$F = \mu\frac{(2qy-p^2)^{3/2}}{y^2}$	$\mu > 0$	$\mathbf{I_8^p} > -\frac{3}{2}$	$\mathbf{I_8^p} = -\frac{3}{2} + \frac{2}{\mu^2}$	$F_{qqqq} \neq 0, \; W = 0$		
IX	$F = 4\mu(q - p^2)^{3/2} + 6qp - 4p^3$	$\mu > 0$	$\mathbf{I_8^p} < -\frac{3}{2}$	$\mathbf{I_8^p} = -\frac{3}{2} - \frac{2}{\mu^2}$			
XII	$F = q^{3/2}$		$\mathbf{I_8^p} = -\frac{3}{2}$		$\mathbf{I_5^p}, \mathbf{I_6^p}, \mathbf{I_7^p}, \mathbf{I_8^p} = const$		

Bibliography

[1] K. Wünschmann *Über Berührungsbedingungen bei Integralkurven von Differentialgleichungen*, Dissertation, Greiswald, 1905

[2] E. Cartan, Les espaces generalises et l'integration de certaines classes d'equations differentielles, *C. R. Acad. Sci.* **206** (1938) 1425

[3] E. Cartan, La geometria de las ecuaciones diferenciales de tercer orden, *Oeuvres Complètes*, Part III vol. 2, Paris: Gauthier-Villars, 1955 (original 1941)

[4] E. Cartan, Sur une classe d'espaces de Weyl *Ann. Sc. Ec. Norm.Sup.* **60** (1943) 1

[5] S.-S. Chern, The geometry of the differential equation $y''' = F(x, y, y', y'')$, *Selected Papers* vol. I, Springer-Verlag, 1978 (original 1940)

[6] R. Bryant, Two exotic holonomies in dimension four, path geometries, and twistor theory, *Proc. Symp. Pure Math.* **53** (1991) 33.

[7] A. Cap, Two constructions with parabolic geometries, *Rend. Circ. Mat. Palermo Suppl.* **79** (2006) 11, `arXiv:math/0504389`

[8] A. Cap and H. Schichl, Parabolic geometries and canonical Cartan connections, *Hokkaido Math. J.* **29** (2000) 453

[9] E. Cartan, Les problèmes d'équivalence, in: *Oeuvres Complètes*, Part III, vol. 2, Paris: Gauthier-Villars, 1955

[10] E. Cartan, Sur les variétés à connexion projective, *Bull. Soc. Math.* **52** (1924) 205

[11] J. Chazy, Sur les équations différentielles du troisiéme ordre et d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, *Acta Math.* **34** (1911) 317

[12] C. Cosgrove, Chazy classes IX – XI of third-order differential equations, *Studies in Applied Mathematics* **104** (2000) 171

[13] B. Doubrov, Contact trivialization of ordinary differential equations, *Differential Geometry and Its Applications* (2001) 73

[14] B. Doubrov, Generalized Wilczynski invariants for non-linear ordinary differential equations, (2007) `arXiv:math/0702251`

[15] B. Doubrov and B.Komrakov, Contact Lie algebras of vector fields on the plane, *Geometry and Topology* **3** (1999) 1

[16] B. Doubrov, B. Komrakov and T. Morimoto, Equivalence of holonomic differential equations, *Lobachevskij Journal of Mathematics* **3** (1999) 39

[17] M. Dunajski, L.J. Mason and P. Tod, Einstein-Weyl geometry, the dKP equation and twistor theory, *J. Geom. Phys.* **37** (2001) 63

[18] M. Dunajski and P. Tod, Einstein-Weyl structures from Hyper-Kähler metrics with conformal Killing vectors, *Differ. Geom. Appl.* **14** (2001) 39

[19] M. Dunajski and P. Tod, Paraconformal geometry of nth order ODEs, and exotic holonomy in dimension four, *J. Geom. Phys.* **56** (2006) 1790

[20] D. Forni, M. Iriondo and C. Kozameh, Null surfaces formulation in three dimensions, *J. Math. Phys.* **41** (2000) 5517

[21] D. Fox, Contact projective structures, *Indiana University Mathematics Journal* **54** (2005) 1547, `arXiv:math/0402332`

[22] S. Fritelli, C. Kozameh and E.T. Newman, Lorentzian metrics from characteristic surfaces, *J. Math. Phys.* **36** (1995) 4975

[23] S. Fritelli, C. Kozameh and E.T. Newman, GR via characteristic surfaces, *J. Math. Phys.* **36** (1995) 4984

[24] S. Fritelli, E.T. Newman and C. Kozameh, On the dynamics of the characteristic surfaces, *J. Math. Phys.* **36** (1995) 6397

[25] S. Fritelli, N. Kamran and E.T. Newman, Differential equations and conformal geometry, *J. Geom. Phys.* **43** (2002) 133

[26] S. Fritelli, C. Kozameh and E.T. Newman, Differential Geometry from Differential Equations, *Commun. Math. Phys.* **223** (2001) 383

[27] S. Fritelli, C. Kozameh, E.T. Newman and P. Nurowski, Cartan normal conformal connections from differential equations, *Class. Quantum Grav.* **19** (2002) 5235

[28] E. Gallo, C. Kozameh, E.T. Newman and K. Perkins, Cartan normal conformal connections from pairs of second-order PDEs, *Class. Quantum Grav.* **21** (2004) 4063

[29] R. Gardner, *The Method of Equivalence and Its Applications*, Philadelphia: SIAM, 1989

[30] M. Godliński, *Cartan's Method of Equivalence and Sasakian Structures* (in Polish), Master Thesis, University of Warsaw, 2001

[31] M. Godliński and P. Nurowski, Third-order ODEs and four-dimensional split signature Einstein metrics, *J. Geom. Phys.* **56** (2006) 344

[32] M. Godliński and P. Nurowski, $GL(2, \mathbb{R})$ geometry of ODE's, (2007) `arXiv:0710.0297`

[33] G. Grebot, The characterization of third order ordinary differential equations admitting a transitive fiber-preserving point symmetry group, *Journ. Math. Anal. Appl.* **206** (1997) 364

[34] N. J. Hitchin, Complex manifolds and Einstein's equations, *Twistor Geometry and Non-linear Systems* (*Lecture Notes in Mathematics*) vol. 970 Berlin: Springer, 1982

[35] L. Hsu and N. Kamran, Classification of second-order ordinary differential equations admitting Lie groups of fiber-preserving symmetries, *Proc. London Math. Soc.* **58** (1989) 387

[36] P. E. Jones and P. Tod, Minitwistor spaces and Einstein-Weyl spaces, *Class. Quantum Grav.* **2** (1985) 565

[37] S. Kobayashi, *Transformation Groups in Differential Geometry*, N.Y., Heidelberg, Berlin: Springer-Verlag, 1972

[38] C. Kozameh and E.T. Newman, Theory of light cone cuts of null infinity *J. Math. Phys.* **24** (1983) 2481

[39] C. LeBrun, Explicit self-dual metrics on $CP^2 \# \cdots \# CP^2$, *J. Diff. Geom.* **34** (1991) 233

[40] S. Lie, Klassifikation und Integration von gewohnlichen Differentialgleichungen zwischen x, y, die eine Gruppe von Transformationen gestatten III, *Gesammelte Abhandlungen* vol. 5, Leipzig: Treubner, 1924

[41] M.A.H. MacCallum, On the Classification of the Real Four-Dimensional Lie Algebras, *On Einstein's Path Essays in Honor of Engelbert Schucking*, New-York: Springer-Verlag, 1999

[42] T. Morimoto, Geometric structures on filtred manifolds, *Hokkaido Math. J.* **22** (1993) 263

[43] T. Morimoto, Lie Algebras, Geometric Structures and Differential Equations on Filtred Manifolds, in: *Advanced Studies in Pure Mathematics* 37, 2002, Lie Groups, Geometric Structures and Differential Equations – One Hundred Years after Sophus Lie–, 205-252

[44] E.T. Newman and P. Nurowski, Projective connections associated with second-order ODEs, *Class. Quantum Grav.* **20** (2003) 2325

[45] P. Nurowski, On a certain formulation of the Einstein equations, *J. Math. Phys.* **39** (1998) 5477

[46] P. Nurowski, Differential equations and conformal structures, *J. Geom. Phys.* **55** (2005) 19

[47] P. Nurowski, Comment on GL(2,R) geometry of 4th order ODEs, (2007) `arXiv:0710.1658`

[48] P. Nurowski, Notes on Cartan connections, unpublished

[49] P. Nurowski and G.A.J. Sparling, Three-dimensional CR structures and second-order ordinary differential equations, *Class. Quantum Grav.* **20** (2003) 4995

[50] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Second Edition, Graduate Texts in Mathematics, vol. 107, New York: Springer–Verlag, 1993

[51] P.J. Olver, *Equivalence, Invariants and Symmetry*, Cambridge: Cambridge University Press, 1995

[52] H. Sato and A.Y. Yoshikawa, Third order ordinary differential equations and Legendre connection, *J. Math. Soc. Japan* **50** No. 4 (1998) 993

[53] S. Sternberg, *Lectures on Differential Geometry*, Englewood Cliffs, N.J.: Prentice-Hall, 1965

[54] N. Tanaka, On differential systems, graded Lie algebras and pseudo-groups, *J. Math. Kyoto Univ* **10** (1970) 1

[55] N. Tanaka, On the equivalence probles associated with simple graded Lie algebras, *Hokkaido Math. J.* **8** (1979) 23

[56] K.P. Tod, Einstein-Weyl spaces and third-order differential equations, *J. Math. Phys.* **41** (2000) 5572

[57] M. A. Tresse, Determinations des invariants ponctuels de l'equation differentielle ordinaire du second ordre $y'' = \omega(x, y, y')$, Leipzig: Hirzel, 1896

[58] R.S. Ward, Einstein-Weyl spaces and $SU(\infty)$ Toda fields, *Class. Quantum Grav.* **7** (1990) L95